XXIXth Spanish Relativity Meeting (ERE 2006) IOP Publishing
Journal of Physics: Conference Series **66** (2007) 012029 doi:10.1088/1742-6596/66/1/012029

On wormholes and black holes solutions of Einstein gravity coupled to a K-massless scalar field

J Estevez-Delgado[1] **and T Zannias**[2]

[1] Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana de San Nicólas de Hidalgo, Morelia, Mich, MEXICO

E-mail: joaquin@fismat.umich.mx

[2] Ins. de Física y Matemáticas, Universidad Michoacana de San Nicólas de Hidalgo, A.P. 2-82, 58040 Morelia, Mich, MEXICO

E-mail: zannias@ginette.ifm.umich.mx

Abstract. We investigate the nature of black holes and wormholes admitted by a K-essence model involving a massless scalar field Φ, minimally coupled to gravity. Via Weyl's formalism, we show that any axial wormhole of the theory can be generated by a unique pair of harmonic functions: $U(\lambda) = -\frac{\pi}{2}C + C\arctan(\frac{\lambda}{\lambda_0}), \Phi(\lambda) = -\frac{\pi}{2}D + D\arctan(\frac{\lambda}{\lambda_0})$ where λ is one of the oblate coordinate, $\lambda_0 > 0$ and (C, D) real parameters. The properties of the wormholes depends crucially upon the values of the parameters (C, D). Whenever (C, D) are chosen so that $2C^2 - kD^2 = -2$ the wormhole is spherical, while for the case where $2C^2 - kD^2 = -4$ or $2C^2 - kD^2 = -6$ the wormhole throat possesses toroidal topology. Those two families of wormholes exhaust all regular static and axisymmetric wormholes admitted by this theory. For completeness we add that whenever (C, D) satisfy $2C^2 - kD^2 = -2l$ with $l \geq \frac{3}{2}$ one still generates a spacetime possessing two asymptotically flat but the throat connecting the two ends contains a string like singularity. For the refined case where $2C^2 - kD^2 = -2l$ with $l = 4, 5, ...$ the resulting spacetime represents a multi-sheeted configuration which even though free of curvature singularities nevertheless the spacetime topology is distinct to so far accepted wormhole topology. Spacetimes generated by the pair $(U(\lambda), \Phi(\lambda))$ and parameters (C, D) subject to $2C^2 - kD^2 = -2l$ with $l < \frac{3}{2}$ contain naked curvature singularities. For the classes of regular wormholes, the parameters (C, D) determine the ADM masses of the asymptotically flat ends and can be positive, negative or zero. Except for the cases of zero mass wormholes, the two ends possess ADM masses of opposite sign. In contrast to wormhole sector, the black hole sector of the theory is trivial. Any static, asymptotically flat solution of the theory that admits a $I\!R \times S^2$ bifurcating, regular Killing horizon necessary possesses a constant exterior scalar field. Under the assumption that the event horizon of any static black hole of this theory is a Killing horizon, the results show that the only static black hole admitted by this K-essence model, is the Schwarzschild black hole.

1. Introduction

In this work we shall discuss a few properties of static and axially symmetric wormholes and static black holes admitted by a K-essence model involving a massless scalar field Φ coupled to Einstein gravity via the action functional:

$$S[g, \Phi] = \int \left[\frac{R}{2k} + \frac{1}{2}\nabla^{\alpha}\Phi\,\nabla_{\alpha}\Phi\right]\sqrt{-g}d^4x, \qquad k = \frac{8\pi G}{c^3}, \quad \alpha = 0, 1, 2, 3,$$

The resulting field eqs are:

$$G_{\alpha\beta} = -k[\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{1}{2}g_{\alpha\beta}\nabla^\gamma\Phi\,\nabla_\gamma\Phi], \qquad \nabla^\alpha\nabla_\alpha\Phi = 0, \tag{1}$$

and except for the negative sign in the right hand side of Einsteins tensor, are identical as for the case where Φ is coupled to gravity ordinarily (conventions for signature Riemann, Ricci etc are as in ref. [1]). The system (1) is equivalent to: $R_{\alpha\beta} = -k\nabla_\alpha\Phi\nabla_\beta\Phi$, $\nabla^\alpha\nabla_\alpha\Phi = 0$, and therefore for any non zero null vector X it follows that $R_{\alpha\beta}X^\alpha X^\beta \leq 0$. However violation of the null convergence condition implies that this theory should admit a non trivial wormhole sector. By appealing to Weyl's formalism we shall show that any axial wormhole of (1) can be generated by a pair of functions (U, Φ) harmonic relative to the Euclidean flat metric. Relative to a set of oblate coordinates this pair is described by: $U(\lambda) = -\frac{\pi}{2}C + C\arctan(\frac{\lambda}{\lambda_0})$, $\Phi(\lambda) = -\frac{\pi}{2}D + D\arctan(\frac{\lambda}{\lambda_0})$ where λ is one of the oblate coordinates, $\lambda_0 > 0$ and (C, D) real parameters. Whenever (C, D) are chosen so that $2C^2 - kD^2 = -2$ we shall show that resulting wormhole is spherical, while whenever $2C^2 - kD^2 = -4$ a toroidal wormhole is generated. Such wormhole contains two asymptotically flat ends which however are connected by an $S^1 \times S^1$ throat in sharp contrast to spherical wormholes where the throat is topologically S^2. In contrast to the non trivial wormhole sector admitted by the theory, the static black hole sector appears to be trivial. Under the assumption that the event horizon of any static black hole is a Killing horizon, the only static black hole admitted is the Schwarzschild black hole.

The plan of the present paper is as follows: In the next section we shall employ the Weyl formalism to derive a few properties of static and axially symmetric solutions of (1) relative to a set of oblate coordinates and in turn those properties will be important for the construction and identification of axial wormhole spacetimes. In section 3 we shall discuss the nature of static black holes admitted by (1) and shall finish the paper with a discussion regarding the extension of the present results.

2. Axially Symmetric Wormholes and the Weyl Formalism

We begin by assuming that (g, Φ) represents a non singular, static and axially symmetric solution of (1) and let $(\xi_t,\, \xi_\varphi)$ stand for the timelike and axial Killing fields admitted by g. As long as Φ shares the same symmetries as g, it follows from (1) that: $R_{\alpha\beta}\xi^\alpha_{(t)}\xi^\beta_{(t)} + R_{\alpha\beta}\xi^\alpha_{(\varphi)}\xi^\beta_{(\varphi)} = 0$, and thus exist a local chart so that [2]:

$$g = -e^{2U}dt^2 + r^2e^{-2U}d\varphi^2 + e^{2(V-U)}(dr^2 + dz^2), \tag{2}$$

(for exceptions to this rule consult the appendix of [2]). We consider an open subset S of the Euclidean three space with a closed disk of radius r_0 deleted from its interior and coordinatize this S employing oblate coordinates (λ, μ, φ) where $\lambda \in (0, \lambda_0\epsilon)$, $\mu \in [-1, 1]$, $\epsilon > 0$ and $\lambda_0 \neq 0$. Those coordinates are related to cylindrical (r, z, φ) coordinates via: $r^2 = (\lambda^2 + \lambda_0^2)(1 - \mu^2)$, $\quad z = \lambda\mu$, $\quad \varphi = \varphi$. In the (λ, μ, φ) chart, the intersection of S with the positive z-axis is labeled by $\lambda \in (0, \lambda_0\epsilon)$ and $\mu = 1$ while the intersection of S with the negative part corresponds to $\lambda \in (0, \lambda_0\epsilon)$ and $\mu = -1$. Points with $\lambda = c$, $c > 0$ are confocal ellipsoids while $\mu = c_1$, $c_1 \in (-1, 1)$ represent hyperbola of revolution around the z-axis. In the limit $\lambda = c$ with $c \to 0$ the ellipsoids degenerate into a closed disk which coincides with the r_0 disk, provided $\lambda_0 = r_0$. Assuming here after that such choice has been made, the upper (correspondingly lower) phase of this disk have coordinates $\lambda = 0$, $\mu \in (0, 1]$ (correspondingly $\lambda = 0$, $\mu \in [-1, 0)$) while $(\lambda = 0,\ \mu = 0)$ describe the boundary of this disk. It is important to keep in mind that S contains neither interior nor boundary points of this disk.

We consider the manifold $M_1 = I\!R \times S$ and define on it the metric g:

$$g = -e^{2U}dt^2 + (\lambda^2 + \lambda_0^2)(1 - \mu^2)e^{-2U}d\varphi^2 + (\lambda^2 + \lambda_0^2\mu^2)e^{2(V-U)}\left[\frac{d\lambda^2}{\lambda^2 + \lambda_0^2} + \frac{d\mu^2}{1 - \mu^2}\right], \tag{3}$$

where $t \in (-\infty, \infty)$, $U = U(\lambda, \mu)$, $V = V(\lambda, \mu)$. This g augmented by a scalar $\Phi(\lambda, \mu)$ is a solution of (1) provided (U, Φ, V) satisfy:

$$(\lambda^2+\lambda_0^2)\Phi_{\lambda\lambda}+(1-\mu^2)\Phi_{\mu\mu}+2\lambda\Phi_\lambda-2\mu\Phi_\mu = 0, \quad (\lambda^2+\lambda_0^2)U_{\lambda\lambda}+(1-\mu^2)U_{\mu\mu}+2\lambda U_\lambda-2\mu U_\mu = 0, \tag{4}$$

$$V_\lambda = \frac{(1-\mu^2)^2}{2(\lambda^2+\lambda_0^2\mu^2)}\left[\lambda\left(k\Phi_\mu^2-2U_\mu^2\right)-\frac{\lambda^2+\lambda_0^2}{1-\mu^2}\left(k\lambda\Phi_\lambda^2-2\lambda U_\lambda^2-2k\mu\Phi_\lambda\Phi_\mu+4\mu U_\lambda U_\mu\right)\right], \tag{5}$$

$$V_\mu = -\frac{(\lambda^2+\lambda_0^2)^2}{2(\lambda^2+\lambda_0^2\mu^2)}\left[\mu\left(k\Phi_\lambda^2-2U_\lambda^2\right)-\frac{1-\mu^2}{\lambda^2+\lambda_0^2}\left(k\mu\Phi_\mu^2-2\mu U_\mu^2-2k\lambda\Phi_\lambda\Phi_\mu+4\lambda U_\lambda U_\mu\right)\right], \tag{6}$$

where in above and here after the subscripts (λ, μ) denote differentiation with respect to (λ, μ). Eqs. (4) shows that (U, Φ) are harmonic relative to the Euclidean flat metric written in oblate coordinates, while the integrability conditions for (5-6) are satisfied once (Φ, U) satisfy (4).

Primary we shall be interested to construct solutions of (4-6) satisfying:

a) elementary flatness holds true i.e., $V \equiv 0$ on the axis of the axial Killing field,
b) the spacetime $(I\!R \times S, g, \Phi)$ is singularity free.

The general solution of (4) can be represented in the form:

$$\Phi(\lambda, \mu) = \sum_{l=0}^{\infty}\left[C_l S_l(\frac{\lambda}{\lambda_0}) + D_l\left(F_{l-1}(\frac{\lambda}{\lambda_0}) + S_l(\frac{\lambda}{\lambda_0})\arctan(\frac{\lambda}{\lambda_0})\right)\right]P_l(\mu), \tag{7}$$

$$U(\lambda, \mu) = \sum_{l=0}^{\infty}\left[C_l' S_l(\frac{\lambda}{\lambda_0}) + D_l'\left(F_{l-1}(\frac{\lambda}{\lambda_0}) + S_l(\frac{\lambda}{\lambda_0})\arctan(\frac{\lambda}{\lambda_0})\right)\right]P_l(\mu), \tag{8}$$

where (S_l, F_{l-1}) are functions related to Legendre polynomials P_l with imaginary argument $\lambda/\lambda_0 i$ while the constants C_l, D_l, C_l' and D_l' have been chosen so that the right hand sides of (7, 8) are manifestly real. In particular $S_l(\lambda/\lambda_0) = P_l(\lambda/\lambda_0 i)$ while F_{l-1} is a linear combination of $P_l(\lambda/\lambda_0 i)$ of order zero up to order less than the largest integer $< l(l-1)$ (for more details regarding the representation of F_{l-1} see [3]). The functions $\Phi(\lambda, \mu)$ and $U(\lambda, \mu)$ defined by (7,8) satisfy (4) provided $-\lambda_0\epsilon < \lambda < \lambda_0\epsilon$, $\mu \in [-1, 1]$. However (7,8) can be analytically continued for all $\lambda \in (-\infty, \infty)$, $\mu \in [-1, 1]$ so that in the extended domain satisfy (4). This follows as a consequence of the fact that the operator $L = (\lambda^2+\lambda_0^2)\frac{\partial^2}{\partial\lambda^2} + (1-\mu^2)\frac{\partial^2}{\partial\mu^2} + 2\lambda\frac{\partial}{\partial\lambda} + 2\mu\frac{\partial}{\partial\mu}$ has analytic coefficients in the domain $-\lambda_0 < \lambda < \lambda_0$, $\mu \in (-1, 1)$ and by appealing to properties of analytic continuation theory. This property of (7,8) will be used further ahead.

For the moment we view (7,8) as defined on S and on this S the function $V(\lambda, \mu)$ can be obtained via:

$$V(\lambda, \mu) = \int_{(\lambda_0,\mu_0)}^{(\lambda,\mu)} V_\lambda(\lambda', \mu')d\lambda' + V_\mu(\lambda', \mu')d\mu' + C(\lambda), \qquad 0 < \lambda/\lambda_0 < \epsilon, \quad \mu \in [-1, 1], \tag{9}$$

where the line integral is taken along any path on the (λ, μ) plane connecting (λ_0, μ_0) to (λ, μ). For a path starting from a point on the positive z-axis i.e., $(\lambda, \mu = 1)$, continuing along the coordinate line $\lambda = c$ and terminating at (λ, μ) relation (9) yields:

$$V(\lambda, \mu) = \int_{(\lambda,1)}^{(\lambda,\mu)}\left[A(\lambda, \mu')\frac{d}{d\mu'}\ln(\lambda^2+\lambda_0^2\mu'^2) - \lambda_0^2 B(\lambda, \mu')\frac{d}{d\lambda}\ln(\lambda^2+\lambda_0^2\mu'^2)\right]d\mu' + C(\lambda), \tag{10}$$

where $A(\lambda,\mu)$ and $B(\lambda,\mu)$ stand for:

$$A(\lambda,\mu) := -\frac{(\lambda^2+\lambda_0^2)^2}{4\lambda_0^2}(k\Phi_\lambda^2 - 2U_\lambda^2) + \frac{1}{4\lambda_0^2}(\lambda^2+\lambda_0^2)(1-\mu^2)(k\Phi_\mu^2 - 2U_\mu^2),$$

$$B(\lambda,\mu) := \frac{(\lambda^2+\lambda_0^2)(1-\mu^2)}{2\lambda_0^2}(k\Phi_\lambda\Phi_\mu - 2U_\lambda U_\mu).$$

For any (U,Φ) described by (7,8), the right hand side of (10) defines a smooth function on S, which however diverges logarithmically as $(\lambda \to 0, \mu \to 0)$. This divergence can be seen by noticing that (7,8) imply $U_\lambda U_\mu|_{(0,0)} = \Phi_\lambda\Phi_\mu|_{(0,0)} = 0$ and thus $B(0,0) = 0$. Therefore as long as $A(0,0) \neq 0$, (10) implies: $V(\lambda,\mu) = A(0,0)\ln(\lambda^2+\lambda_0^2\mu^2) + \delta(\lambda,\mu) + C(\lambda)$, where $\delta(\lambda,\mu)$ represents the contributions in (10) of the convergent integrals as $(\lambda \to 0, \mu \to 0)$. We write the above relation in the form:

$$V(\lambda,\mu) = -\frac{1}{4}\Big[\lambda_0^2(k\Phi_\lambda(0,0)^2 - 2U_\lambda(0,0)^2) - k\Phi_\mu(0,0)^2 + 2U_\mu(0,0)^2\Big]\ln\left(\frac{\lambda^2+\lambda_0^2\mu^2}{\lambda^2+\lambda_0^2}\right) + \delta(\lambda,\mu), \quad (11)$$

where $C(\lambda)$ has been chosen so that $\delta(\lambda,\mu=1) = 0$ and $V(\lambda,\mu=1) = 0$ on a point of the positive z-axis. Since (5) implies that $V(\lambda,\mu)$ is constant along any connected component of the $z-$axis, it follows from (11) that elementary flatness holds true on the positive part of the $z-$axis. Setting:

$$A \equiv A(0,0) = -\frac{1}{4}[\lambda_0^2(k\Phi_\lambda(0,0)^2 - 2U_\lambda(0,0)^2) - k\Phi_\mu(0,0)^2 + 2U_\mu(0,0)^2], \quad (12)$$

it follows from (3) in combination with (11) that g can be written in the equivalent form:

$$g = -e^{2U(\lambda,\mu)}dt^2 + (\lambda^2+\lambda_0^2)(1-\mu^2)e^{-2U(\lambda,\mu)}d\varphi^2 + \frac{(\lambda^2+\lambda_0^2\mu^2)^{2A+1}}{(\lambda^2+\lambda_0^2)^{2A}}\times$$

$$e^{2[\delta(\lambda,\mu)-U(\lambda,\mu)]}\left[\frac{d\lambda^2}{\lambda^2+\lambda_0^2} + \frac{d\mu^2}{1-\mu^2}\right], \qquad \lambda \in (0, \lambda_0\epsilon). \quad (13)$$

In this form g is well defined for any (λ,μ) away from the coordinate ring $(\lambda = 0, \mu = 0)$.

Aided by symbolic manipulations and after considerable efforts, it was shown in [3] that the Kretchman scalar $K = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ evaluated using (13), admits the following representation:

$$(\lambda^2+\lambda_0^2)^{-4A}(\lambda^2+\lambda_0^2\mu^2)^{4A+3}e^{4\delta-4U}K = H^2(\lambda,\mu) + G(\lambda,\mu) + (\lambda^2+\lambda_0^2\mu^2)k^2Q^2(\lambda,\mu), \quad (14)$$

where:

$$H^2(\lambda,\mu) = (1-\mu^2)(\lambda^2+\lambda_0^2)[\,(\lambda^2+\lambda_0^2)U_\lambda^2 + (1-\mu^2)U_\mu^2]\Big\{\,4(\lambda^2+\lambda_0^2)(1-\mu^2)\times$$

$$(k\Phi_\lambda\Phi_\mu - 2U_\mu U_\lambda)^2 + [\,(\lambda^2+\lambda_0^2)(k\Phi_\lambda^2 - 2U_\lambda^2) - (1-\mu^2)(k\Phi_\mu^2 - 2U_\mu^2 + 2)]^2\,\Big\},$$

$$Q(\lambda,\mu) = (\lambda^2+\lambda_0^2)\Phi_\lambda^2 + (1-\mu^2)\Phi_\mu^2,$$

while the length of $G(\lambda,\mu)$ does not permit a detailed display. It does however satisfy: $G(0,0) = 0$ and vanishes identically whenever U =const and Φ =const i.e. the case of flat space time. The function $H(\lambda,\mu)$ satisfies: $H^2(0,0) = 4\lambda_0^2[\lambda_0^2U_\lambda(0,0)^2 + U_\mu(0,0)^2](2A+1)^2$, and since $\lambda_0^2U_\lambda(0,0)^2 + U_\mu(0,0)^2 \neq 0$, it follows that as long as $2A+1 \neq 0$ the invariant K has a "branch pole" at $\lambda^2+\lambda_0^2\mu^2$ with exponent $4A+3$. Therefore any g generated by a pair of harmonic functions chosen so that (U,Φ) obey:

$$4A+3=-[\lambda_0^2(k\Phi_\lambda(0,0)^2-2U_\lambda(0,0)^2)-k\Phi_\mu(0,0)^2+2U_\mu(0,0)^2]+3\leq 0, \tag{15}$$

it follows from (14) that K is bounded as the $(\lambda=0,\mu=0)$ ring is approached. In the opposite case i.e. for any g generated by a pair (U,Φ) so that $4A+3>0$ and as long as $2A+1\neq 0$, the scalar K becomes unbounded as the $(\lambda=0,\mu=0)$ ring is approached and thus it represents an irremovable curvature singularity. For the particular case where the pair (U,Φ) has been chosen so that $2A+1=0$ the scalar K may be regular on the $(\lambda=0,\mu=0)$ ring. The right hand side of (14) tends to zero as $(\lambda\to 0,\ \mu\to 0)$ and whether K is singular or not, depends upon the way $H(\lambda,\mu)$ and $G(\lambda,\mu)$ approach zero as $(\lambda\to 0,\ \mu\to 0)$. Those properties of the scalar K are sufficient for our needs.

For latter purposes, it is worth mentioning that relative to a set of spherical coordinates (R,θ,φ) defined by:

$$\lambda=\frac{R^2-\lambda_0^2}{2R},\qquad \mu=\cos\theta,\qquad \varphi=\varphi, \tag{16}$$

the metric (13) takes the form:

$$g=-e^{2U(R,\theta)}dt^2+e^{-2U(R,\theta)}\left[\frac{(R^2+\lambda_0^2)^2}{4R^2}\sin^2\theta\right]d\varphi^2+e^{2\delta(R,\theta)-2U(R,\theta)}\frac{(R^2+\lambda_0^2)^2}{4R^4}\times$$

$$\left[\frac{(R^2-\lambda_0^2)^2+4R^2\lambda_0^2\cos^2\theta}{(R^2-\lambda_0^2)^2+4\lambda_0^2R^2}\right]^{2A+1}(dR^2+R^2d\theta^2),\quad R>\lambda_0,\ \ \theta\in[0,\pi],\ \ \varphi\in[0,2\pi]. \tag{17}$$

Notice that the coordinate disk $\lambda=0$, $\mu\in[-1,0)\cup(0,1]$ is mapped via (16) into $R=\lambda_0$ $\theta\in[0,\pi/2)\cup(\pi/2,\pi]$, $\varphi\in[0,2\pi]$ while the coordinate singularity at $(\lambda=0,\mu=0)$ is mapped at: $(R=\lambda_0,\theta=\pi/2)$.

The so far established properties of local solutions of (4-6) will be used for the construction of wormhole spacetimes (M_w,g,Φ), i.e., non singular, possessing two asymptotically flat spacetimes with (g,Φ) solution of (1). In order to constructed such spacetimes let us start with a pair (U,Φ) defined on $[0,\lambda_0\epsilon)\times[-1,1]$ and chosen so that that on $(\lambda=0,\mu=0)$ obey either $4A+3\leq 0$ or $2A+1=0$. At first we extend the range of λ from $[0,\lambda_0\epsilon)$ into $(0,\infty)$ and analytically continue (U,Φ) from $[0,\lambda_0\epsilon)$ to the same domain. According to the remarks following (7,8), the so extended (U,Φ) satisfy (4) on the extended domain and on this domain we construct $V(\lambda,\mu)$ using (9). Let us denote by (M,g,Φ) the resulting spacetime. This spacetime fails to be asymptotically flat at spacelike infinity unless the pair (U,Φ) is subject to additional restrictions. In that regard one notices that as $\lambda\to\infty$, then (3) implies:

$$g=-e^{2U}dt^2+\lambda^2\sin^2\theta e^{-2U}d\varphi^2+e^{2(V-U)}(d\lambda^2+\lambda^2d\theta^2)+O(\lambda^{-2}),$$

and thus asymptotically λ behaves as a radial coordinate. Accordingly (M,g,Φ) is asymptotically flat as $\lambda\to\infty$, provided at first U satisfies:

$$\lim_{\lambda\to\infty}U(\lambda,\mu)=O(\frac{1}{\lambda}),\qquad \lim_{\lambda\to\infty}U_\mu(\lambda,\mu)=O(\frac{1}{\lambda}),\qquad \lim_{\lambda\to\infty}U_\lambda(\lambda,\mu)=O(\frac{1}{\lambda^2}). \tag{18}$$

Moreover by appealing to (10) combined with (5, 6) it is easily inferred that as long as:

$$\lim_{\lambda\to\infty}\Phi(\lambda,\mu)=O(\frac{1}{\lambda}),\qquad \lim_{\lambda\to\infty}\Phi_\mu(\lambda,\mu)=O(\frac{1}{\lambda}),\qquad \lim_{\lambda\to\infty}\Phi_\lambda(\lambda,\mu)=O(\frac{1}{\lambda^2}), \tag{19}$$

then $\lim_{\lambda\to\infty} V(\lambda,\mu) = O(\lambda^{-1})$. Thus whenever the pair (U,Φ) has been chosen so that additionally satisfies (18,19) the resulting (M,g,Φ) is asymptotically flat at the end specified by: $\lambda\to\infty$. As a byproduct of asymptotic flatness and as long as elementary flatness holds true along the positive $z-$axis, elementary flatness holds along the entire $z-$axis and details of this property can be found in [3].

Even though the pair (U,Φ) chosen so that it obey (18,19) and $4A+3\leq 0$ or $2A+1=0$, generates a spacetime (M,g,Φ) which is asymptotically flat as $\lambda\to\infty$ and the curvature is well behaved as $(\lambda\to 0,\ \mu\to 0)$, this spacetime is geodesically incomplete as $\lambda\to 0$. This incompleteness is welcomed and by suitable extension it offers the means to construct a spacetime that has a wormhole topology. In ref [3] we have constructed the wormhole spacetime by employing a two sheeted Riemannian surface. In this work we shall construct wormholes by avoiding the use of Riemann surfaces and the method is based in the following completion: For the specific pair (U,Φ) under consideration we extend the range of λ to $(-\infty,0]$ and analytically continued (U,Φ) over the same interval. For the so extended pair, the function $V(\lambda,\mu)$ is obtained from (10) choosing a contour lying on the (λ,μ) plane with $\lambda\in(-\infty,0)$. The resulting metric is form identical to (13) with the exception that $\lambda\in(-\infty,0)$. The transformation (compare to 16):

$$\lambda=\frac{R^2-\lambda_0^2}{2R},\qquad \mu=\cos\theta,\qquad \varphi=\varphi, \tag{20}$$

maps this spacetime into the interior of the $R<\lambda_0$ semiplane. This property permit us to consider (17) as defined for all $R\in(0,\infty)$ and for simplicity denoted this spacetime still by (M,g,Φ). This (M,g,Φ) has two ends specified by $\lambda\to\infty$ (or $R\to\infty$) and $\lambda\to-\infty$ (or $R\to 0$), By the choice of (U,Φ) the end defined by $\lambda\to\infty$, is asymptotically flat and below we examine whether the end defined by $\lambda\to-\infty$ $(R\to 0)$ is also asymptotically flat. For that let us assume that Φ has the form:

$$\Phi(\lambda,\mu)=\sum_{l=0}^{n}\left[C_lS_l(\frac{\lambda}{\lambda_0})+D_l\left(F_{l-1}(\frac{\lambda}{\lambda_0})+S_l(\frac{\lambda}{\lambda_0})\arctan\frac{\lambda}{\lambda_0}\right)\right]P_l(\mu). \tag{21}$$

As $\lambda\to\infty$, the $l=n$ contribution yields:

$$\hat{K}_n=C_nS_n(\frac{\lambda}{\lambda_0})+D_n\left(F_{n-1}(\frac{\lambda}{\lambda_0})+S_n(\frac{\lambda}{\lambda_0})\arctan\frac{\lambda}{\lambda_0}\right)\simeq[C_n+\frac{\pi}{2}D_n]\lambda^n+O(\lambda^{n-1}),$$

and accordingly to (18, 19), asymptotic flatness as $\lambda\to\infty$ requires: $C_n+\frac{\pi}{2}D_n=0$, while asymptotic flatness at the end specified by $\lambda\to-\infty$, requires: $C_n-\frac{\pi}{2}D_n=0$. However the above conditions are incompatible. Fixing the constant $(C_n,\ D_n)$ so that $C_n+\frac{\pi}{2}D_n=0$, holds true violates $C_n-\frac{\pi}{2}D_n=0$. Even though for simplicity we have considered only the case $l=n$ the incompatibility extends to all l. Identical conclusions holds true for the coefficient $(C'_n,\ D'_n)$ specifying the function U (see eq. 8). Thus choosing (U,Φ) so that asymptotic flatness prevails at the "upper" $\lambda\to\infty$ end, the process of analytical extension yields a spacetime that fails to be asymptotically flat at the "lower" $\lambda\to-\infty$ end. The only exception to this conclusion occurs for the $l=0$ mode and in this case (7,8) imply:

$$U(\lambda)=C_1+C_2\arctan(\frac{\lambda}{\lambda_0}),\qquad \Phi(\lambda)=D_1+D_2\arctan(\frac{\lambda}{\lambda_0}),\qquad \lambda\in(-\infty,\infty), \tag{22}$$

with (C_1,C_2,D_1,D_2) for the moment arbitrary constants. The above analysis show that if (1) admits regular axially symmetric wormholes they ought to be generated by the pair (U,Φ) described by (22). The simple form of (22) permits the explicit construction of all metrics satisfying (1). Integrating (5,6) we eventually find (for details see [3])

$$g=-e^{2U(\lambda)}dt^2+(\lambda^2+\lambda_0^2)(1-\mu^2)e^{-2U(\lambda)}d\varphi^2+\frac{(\lambda^2+\lambda_0^2\mu^2)^{C_2^2-\frac{k}{2}D_2^2+1}}{(\lambda^2+\lambda_0^2)^{C_2^2-\frac{k}{2}D_2^2}}\times$$

$$e^{-2U(\lambda)}\left[\frac{d\lambda^2}{\lambda^2+\lambda_0^2}+\frac{d\mu^2}{1-\mu^2}\right], \qquad \lambda\in(-\infty,\infty). \tag{23}$$

The parameter A constructed from (22) is: $A = C_2^2/2 - kD_2^2/4$, and thus: $2A+1 = C_2^2 - kD_2^2/2+1$, while the Kretchman scalar K of (23) take the form:

$$K = 4\lambda_0^2\frac{F_1(\lambda,\mu)}{(\lambda^2+\lambda_0^2)^5}\left(\frac{\lambda^2+\lambda_0^2\mu^2}{\lambda^2+\lambda_0^2}\right)^{-4A-3} e^{4C_1+4C_2\arctan(\frac{\lambda}{\lambda_0})}, \tag{24}$$

where $F_1(\lambda,\mu)$ is a polynomial in (λ,μ) obeying $F_1(0,0)\neq 0$. It is easily inferred from this K that whenever (C_2,D_2) have been chosen so that $A\equiv C_2^2/2 - kD_2^2/4 > -3/4$, the scalar K becomes unbounded as points on the $(\lambda=0,\mu=0)$ ring are approached and here after we shall disregard this family of metrics from further considerations. On the other hand, whenever (C_2,D_2) have been chosen so that either $A\equiv C_2^2/2 - kD_2^2/4 \leq -3/4$, or $2A+1=0$ i.e. $kD_2^2 - 2C_2^2 = 2$, the invariant K is bounded on the $(\lambda=0,\mu=0)$ ring. For the particular case where (C_2,D_2) have been chosen so that $2A+1=0$, (23) takes the form:

$$g=-e^{2U(\lambda)}dt^2+e^{-2U(\lambda)}\left[(\lambda^2+\lambda_0^2)(1-\mu^2)d\varphi^2+d\lambda^2+\frac{\lambda^2+\lambda_0^2}{1-\mu^2}d\mu^2\right], \quad \lambda\in(-\infty,\infty), \tag{25}$$

and this g represents a spherical wormhole. Indeed there exist a coordinate transformation so that (25) and (22) reduce to the form (for details see [3]):

$$g = -e^{4\gamma\arctan\frac{r}{r_0}-2\gamma\pi}dt^2+\frac{(r^2+r_0^2)^2}{r^4}e^{-4\gamma\arctan\frac{r}{r_0}+2\gamma\pi}[dr^2+r^2d\Omega], \qquad r\in(0,\infty) \tag{26}$$

$$\Phi = 2\overline{\gamma}\arctan\frac{r}{r_0}-\overline{\gamma}\pi, \qquad \overline{\gamma}=\pm\left[\frac{2}{k}(\gamma^2+1)\right]^{\frac{1}{2}}, \qquad \gamma = C_2, \qquad \overline{\gamma}=D_2. \tag{27}$$

Computing the area $A(r)$ of $SO(3)$ spheres on any $t=$const spacelike hypersurface we find:

$$A(r) = 4\pi\frac{(r^2+r_0^2)^2}{r^2}e^{-4\gamma\arctan(\frac{r}{r_0})+2\gamma\pi}, \tag{28}$$

and thus: $\lim_{r\to\infty}A(r)\to\infty$, $\lim_{r\to 0}A(r)\to\infty$. Moreover $A(r)$ has a minimum at $r=(-\gamma+\sqrt{\gamma^2+1})r_0$. Therefore any $t=$const hypersurface, besides the asymptotically flat end defined as $r\to\infty$, possesses another end corresponding to $r\to 0$. The two ends are connected via a throat located at $r=(-\gamma+\sqrt{\gamma^2+1})r_0$. It is easily inferred from (26) that the ADM mass of the two ends are $M_1 = 2\gamma r_0$ and $M_2=-2\gamma r_0 e^{\gamma\pi}$. The zero mass spherical wormhole is obtained setting $\gamma=0$ in (26).

We now consider the family of metrics g described by (23) under the assumption (C_2,D_2) subject to $4A+3\leq 0$. The issue is whether those metrics represents wormholes and in the affirmative case to examine their properties. To carry out this analysis is preferable to employ the spherical like coordinates introduced in (17). Transforming (23) in those coordinates we get:

$$g = -e^{2U(R,\theta)}dt^2+e^{-2U(R,\theta)}\left[\frac{(R^2+\lambda_0^2)^2}{4R^2}\sin^2\theta d\varphi^2+\frac{(R^2+\lambda_0^2)^2}{4R^4}\times\right.$$

$$\left.\left(\frac{(R^2-\lambda_0^2)^2+4R^2\lambda_0^2\cos^2\theta}{(R^2-\lambda_0^2)^2+4\lambda_0^2R^2}\right)^{2A+1}(dR^2+R^2d\theta^2)\right], \quad R\in(0,\infty), \quad \theta\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi] \quad \varphi\in[0,2\pi]. \tag{29}$$

Since $4A+3 \leq 0$, the curvature of g is regular at ($R=\lambda_0$,$\theta=\pi/2$) and therefore one expects that g ought to be extendible up to and including ($R=\lambda_0$,$\theta=\pi/2$). However what is not known a-priori is the nature of the extended manifold where the extended metric is regular. Since the components $g_{00}, g_{\varphi\varphi}$, the conformal factor $e^{-2U(R,\theta)}$ and the overall multiplicative factor $(R^2+\lambda_0^2)^2 R^{-4}$ of (29) are regular at ($R=\lambda_0$, $\theta=\pi/2$), it is sufficient to consider the same issue for the following Riemannian metric:

$$g_2 = \left(\frac{(R^2-\lambda_0^2)^2 + 4R^2\lambda_0^2\cos^2\theta}{(R^2+\lambda_0^2)^2}\right)^{2A+1}[dR^2+R^2d\theta^2], \quad R \in (0,\infty), \quad \theta \in [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]. \tag{30}$$

Shifting the origin of the (R,θ) coordinates to the point ($R=\lambda_0$,$\theta=\pi/2$) by introducing Cartesian coordinates (x,y) via $x=R\cos\theta$, $y=R\sin\theta$ and subsequently defining $(\hat{x},\hat{y})$ via $\hat{x}=x$, $\hat{y}=y-\lambda_0$, it follows that g_2 locally i.e., an open neighborhood $O\setminus\{\hat{x}=0,\hat{y}=0\}$ takes the form:

$$g_2 = Z(\hat{x}^2+\hat{y}^2)^{(2A+1)}[d\hat{x}^2+d\hat{y}^2] = Z\hat{R}^{2(2A+1)}[d\hat{R}^2+\hat{R}^2d\hat{\theta}^2], \qquad \hat{R}>0, \quad \hat{\theta}\in[0,2\pi], \tag{31}$$

where $(\hat{R},\hat{\theta})$ are spherical coordinates associated to $(\hat{x},\hat{y})$ and $Z(\hat{R},\hat{\theta})$ is a smooth non vanishing function as $\hat{R}\to 0$. Since any analytic function $f(z)=u(\hat{x},\hat{y})+iv(\hat{x},\hat{y})$ defined on $I\!R^2\setminus\{(0,0)\}$ generates a coordinate transformation $(\hat{x},\hat{y})\to(u,v)$ that leaves g_2 form invariant. We consider the analytic function:

$$f(z)=z^b=\hat{R}^b[\cos(b\hat{\theta})+i\sin(b\hat{\theta})] = u(\hat{x},\hat{y})+iv(\hat{x},\hat{y}), \quad b\in I\!R, \quad b\neq 0. \tag{32}$$

The transformation $(\hat{x},\hat{y})\to(u,v)$ casts g_2 in the form:

$$g_2 = Z\frac{(\hat{x}^2+\hat{y}^2)^{2A+1}}{b^2(\hat{x}^2+\hat{y}^2)^{b-1}}[du^2+dv^2], \tag{33}$$

and thus as long as $A\neq -1$ the choice $b=2(A+1)$ implies that g_2 is well defined at the origin of the $(\hat{x},\hat{y})$ plane. We shall now investigate the structure of the two-manifold where the extended g_2 is regular. For that we define spherical polar coordinates $(\overline{R},\overline{\theta})$ associated with the Cartesian coordinates (u,v) via: $\overline{R}^2=u^2+v^2$, $\tan\overline{\theta}=\tan(b\hat{\theta})$. At first a circle of coordinate radious $\hat{R}\neq 0$ in $O\setminus\{\hat{x}=0,\hat{y}=0\}$ has the angular coordinate $\hat{\theta}$ taking its values in $[0,2\pi]$. Moreover in the $(\hat{x},\hat{y})$ plain the point with coordinates $(R,0)$ and $(R,2\pi)$ represent the same point. However the transformation (32) would assign coordinates: ($\overline{R}=\hat{R}^b,\overline{\theta}=0$) respectively ($\overline{R}=\hat{R}^b,\overline{\theta}=2\pi b$) for this point and this observation provides clues regarding the nature of the manifold where g_2 is regular. Let us first assume that $b<1$. For this case since it follows from $g=Zb^{-2}(d\overline{R}^2+\overline{R}^2d\overline{\theta}^2)$ that the manifold where g_2 is regular would possesses a conical singularity at the origin. On the other hand for the case where $b>1$ the range $\overline{\theta}$ is greater than 2π. In such case the requirement that the transformation (32) is single-valued demands the manifold where g_2 is regular ought to be taken as a Riemann surface cuttet along say the $\hat{y}=0$ axis. A 2π rotation in the angle $\overline{\theta}$ transfer us to "lower" Riemann sheet. The structure of this Riemann surface depends upon the nature of $b=2(A+1)$, and below we shall briefly summarize the various possibilities. Whether b is rational or irrational we may represent it in the form: $b=n+a$, $n=\pm1,\pm2..$ and $a<1$. In this case the Riemann surface contains n complete sheets and the ray laying on the $n+1$ sheet with $\overline{\theta}=2\pi n+2\pi a$ is identified to the ray $\overline{\theta}=0$ lying on the first sheet. In that way the transformation (32) becomes single valued. For the particular case where $a=0$ the Riemann surface contains exactly $n-$sheets. For this particular case $b=2(A+1)=n$, $n=\pm1\pm2\ldots$ and since the regularity of the spacetime curvature requires either $4A+3\leq 0$ or $2A+1=0$

it follows that $4A + 3 = -(2n+1)$, $n = -1, 1, 2\ldots$ The choice $n = -1$ yields the condition :$2A + 1 = 0$ already examined. The choice $n = 1$ implies that the Riemann sheet where g_2 is regular consists of a single sheet and yields a toroidal wormhole. In terms of the constants (C_2, D_2) we recall that $A = C_2^2/2 - kD_2^2/4$ and the condition $4A + 3 = -(2n+1)$ is equivalent to: $2C_2^2 - kD_2^2 = -2l$ with $l = 1, 2, 3\ldots$ The choice $l = 1$ yields condition: $2A + 1 = 0$ the choice $l = 2$, requires $A = -1$ and this case will be examined further below while $l = 3$ yields a toroidal wormhole. For the case where $b = 2(A+1)$ is not an integer, the requirement $4A+3 \leq 0$ combined with $A = C_2^2/2 - kD_2^2/4$ can be represented in the form: $2C^2 - kD^2 = -2l$ with $l \geq \frac{3}{2}$. A finer classification dealing with conditions guaranteeing the appearance of stringy, toroidal, spherical or objects with multisheted topology occur is discussed in details [3].

The above considerations fail for the particular case where $A = -1$. In this event $b = 0$ and (32) is not any longer well defined. At first we note that for $A = -1$, (30) takes the form:

$$g_2 = \left(\frac{(R^2+\lambda_0^2)^2}{(R^2-\lambda_0^2)^2 + 4R^2\lambda_0^2\cos^2\theta}\right)[dR^2 + R^2 d\theta^2], \quad R \in (0,\infty), \quad \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi], \tag{34}$$

and in terms of the coordinates $(\hat{x}, \hat{y})$ or $(\hat{R}, \hat{\theta})$ defined earlier on reduces to:

$$g_2 = \frac{Z}{\hat{x}^2 + \hat{y}^2}[d\hat{x}^2 + d\hat{y}^2] = \frac{Z}{\hat{R}^2}[d\hat{R}^2 + \hat{R}^2 d\hat{\theta}^2], \qquad \hat{\theta} \in [0, 2\pi], \qquad \hat{R} > 0.$$

Defining a new coordinate so that $\hat{R} = \lambda_0 e^{r/\lambda_0}$, transforms g_2 into: $g_2 := Z[dr^2 + d\hat{\theta}^2]$ which is manifest regular at $\hat{x} = \hat{y} = 0$. Leaving intermediate computations aside, the spacetime metric g at $\hat{R} = 0$ takes the form:

$$g = -e^{-C_2\pi}dt^2 + e^{C_2\pi}\lambda_0^2 d\varphi^2 + e^{C_2\pi}\lambda_0^2 d\hat{\theta}^2, \qquad \varphi \in [0, 2\pi], \qquad \hat{\theta} \in [0, 2\pi], \tag{35}$$

and thus the t =const, $\hat{R} = 0$ two spaces represent a two torus, of area $A = 4\pi e^{C_2\pi}\lambda_0^2$. Since on the other hand (29) for $A = -1$ takes the form:

$$g = -e^{2U(R,\theta)}dt^2 + e^{-2U(R,\theta)}\left[\frac{(R^2+\lambda_0^2)^2}{4R^2}\sin^2\theta d\varphi^2 + \frac{(R^2+\lambda_0^2)^4(dR^2+R^2d\theta^2)}{4R^4[(R^2-\lambda_0^2)^2+4R^2\lambda_0^2\cos^2\theta]}\right], \tag{36}$$

it follows by defining a new coordinate $\rho = 2R$ that g is asymptotically flat as $\rho \to \infty$. On the other hand defining a coordinate via $\hat{\rho} = -\lambda_0^2/2R$ it also follows that g is asymptotically flat as $R \to 0$. In view of 35 the two asymptotically flat ends are connected via a throat that is topologically a two-torus. Properties of those toroidal wormholes are discussed in [3].

3. On static black holes of the theory

In this section, we turn our attention to static black holes admitted by (1). We therefore assume that (M, g, Φ) with (g, Φ) a solution of (1) describes a static black hole, whose event horizon is a bifurcate Killing horizon with a regular bifurcation surface $\mathcal{S}$ of finite area. Let ξ_t denote the hypersurface orthogonal Killing field which asymptotically approach a time translation. As it was shown in [4] the hypersurfaces orthogonal to ξ_t smoothly intersect S and moreover asymptotically they become Euclidean. Introducing coordinate (t, x^1, x^2, x^3) covering the exterior region, we represent g in the form: $g = -V^2dt^2 + \gamma_{ij}dx^i\,dx^j$, and relative to this chart (g, Φ) satisfy:

$$R_{ab} = V^{-1}D_aD_bV - kD_a\Phi D_b\Phi, \qquad D^aD_aV = 0, \qquad D^aD_a\Phi = -V^{-1}D^aVD_a\Phi, \tag{37}$$

where (R_{ab}, D) stand for the Ricci and covariant derivative operator of γ. Any solution of the above system regular on $\mathcal{S}$ and asymptotically Euclidean necessary possesses Φ =constant. In order to show this we note that the last eq. after multiplication of both sides by V, yields: $D^a(VD_a\Phi) = 0$, which implies $\Phi D^a(VD_a\Phi) = 0$. Assuming that $\lim_{r\to\infty}\Phi = O(1/r)$, then by integrating $\Phi D^a(VD_a\Phi) = 0$ over any t =const hypersurface Σ we obtain:

$$\int_\Sigma VD^a\Phi\, D_a\Phi\sqrt{\gamma}d^3x = \oint_{r\to\infty} V\Phi D_a\Phi dS^a + \lim_{r\to\infty}\oint_{\mathcal{S}} V\Phi D_a\Phi dS^a.$$

However both of the surface integral give zero contribution, the first by virtue of the asymptotic fall of rate of $(\Phi,\, V)$ and γ_{ab}, while the second one by virtue that $\lim V = 0$ as the two surface S is approached and the regular nature of $D^a\Phi D_a\Phi$ on the bifurcation sphere $\mathcal{S}$. Accordingly $\Phi = 0$ in the black hole exterior. On the other hand (1) implies that if (g, Φ) is any solution then $(g, \Phi + K)$ with K =const is also a solution. Accordingly any black hole solution of (1) would have at most a constant Φ exterior to the horizon which for all purposes is unobservable. The claim that the only black hole of $R_{ab} = V^{-1}D_aD_bV,\ D^aD_aV = 0$ is the Schwarzschild solution follows from the standard uniqueness theorems [5, 6].

4. Discussion

The seminal paper by Morris and Thorne [7] and recent cosmological observations suggesting that the universe is undergoing an accelerating expansion acted as a stimulus for the development of physics and astrophysics of wormholes. The rapid expansion of the field is summarized in the book by Visser [8] and in a relatively recent review article by Lemos et al [9]. In this work by appealing to Weyl formalism the axially symmetric wormhole sector of a particular K-essence model has been specified. The conclusion that the only axisymmetric wormholes admitted by the model considered are spherical or toroidal ones is striking. Is that conclusion a consequence of the specific model considered or is there a deeper reason for the absence of axial wormholes topologically S^2? In settling that question it may be recalled that the general form of the Lagrangian describing a K-essence [10] has the form: $L = L(\Phi, \nabla^\alpha\Phi\nabla_\alpha\Phi)$. It would be therefore of relevance to examine the structure of wormholes for a more general case of K-essence. This issue is currently under investigation. Another direction that the present results can be extended, concerns the interaction of wormholes with external matter. Since the Weyl formalism is well suited for the description of black holes distorted by external matter and fields [2, 11, 12] a suitable extension of the formalism presented in this work would describe distorted by external matter wormholes. This issue will be discussed elsewhere.

Acknowledgments

This work was supported by a grant of Coordinación Científica - UMSNH. J. Estevez-Delgado acknowledges support from SEP-PROMEP project: PTC74. Finally our thanks to U. Nucamendi, O. Sarbach and F. Guzman for fruitful discussions.

References

[1] Wald R M 1984 *General Relativity* (Chicago: Chicago University Press)
[2] Geroch R and Hartle J B 1982 *J. Math. Phys.* **23** 680
[3] Estevez-Delgado J and Zannias T 2006 Submitted to: *Class. Quantum Grav.*
[4] Racz I and Wald R M 1992 *Class. Quantum Grav.* **9** 2643
[5] Israel W 1967 *Phys. Rev.* D **164** 1776
[6] Heusler M 1996 *Black Hole Uniqueness Theorems* (Cambridge: Cambridge University Press)
[7] Morris M S and Thorne K 1988 *Am. J. Phys.* **56** 395
[8] Visser M 1995 *Lorentzian Wormholes: From Einstein to Hawking* (New York: A. Ins. of Phys.)
[9] Lemos J P S, Lobo F S N and Quinet de Oliveira S 2003 *Preprint* gr-qc/0302059
[10] Armendariz-Picon C, Damour T and Mukhanov V 1999 *Phys. Lett.* B **498** 209
Chiba T, Okabe T and Yanaguchi M 2000 *Phys. Rev.* D **62** 023511
[11] Israel W 1973 *Lett. Nuovo Cimento* **6** 267
[12] Estevez-Delgado J and Zannias T 2004 *Phys. Rev.* D **70** 064038