ПРОБЛЕМЫ СОВРЕМЕННОЙ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

Сборник научных статей посвященных
60-летию профессора И.Л. Бухбиндера

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Проблемы современной теоретической физики. Сборник научных статей, посвященных 60-летию профессора И.Л. Бухбиндера.
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Сборник содержит статьи активно работающих ученых разных стран, посвященные суперсимметричной квантовой теории поля, теории полей высших спинов, теории гравитации, космологии, теории суперструн, математической физике. В числе авторов статей – коллеги, соавторы научных работ и ученики профессора И.Л. Бухбиндера. Сборник включает, в основном, статьи обзорного характера и хорошо отражает современное состояние различных направлений теоретической физики. Представляет интерес для научных работников, аспирантов, студентов старших курсов физико-математических факультетов, специализирующихся в области теоретической физики высоких энергий.

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PROBLEMS OF MODERN THEORETICAL PHYSICS

A volume in honour of Professor I.L. Buchbinder
in the occasion of his 60th birthday.

Tomsk 2008

Problems of Modern Theoretical Physics.
A volume in honour of Professor I.L. Buchbinder in the occasion of his 60th birthday.

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Volume contains the papers on problems of supersymmetric quantum field theory, higher spin field theory, gravity, cosmology, superstring theory, mathematical physics, written by colleagues, coauthors and former students of Professor I.L. Buchbinder. Most of the papers are the reviews presenting the current state of the various trends in modern theoretical physics. Volume is intended for researchers and PhD students in area of high energy theoretical physics.

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Editor V. Epp

Portrait of I.L. Buchbinder and cover photograph by Andrey Masanov.
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Предисловие

Что виделось вчера как цель глазам твоим, –
Для завтрашнего дня – оковы;
Мысль – только пища мыслей новых,
Но голод их неутолим.

Из стихотворения Э. Верхарна "Невозможное".
Перевод с французского М. Донского

3 июля 2008 года исполняется 60 лет известному ученому в области теоретической физики, доктору физико-математических наук, профессору Иосифу Львовичу Бухбиндеру. Настоящий сборник составляют статьи его коллег, друзей, соавторов и учеников по проблемам, лежащим в сфере его научных интересов и связанных с современным развитием теоретической и математической физики.

В различные периоды творческой деятельности И.Л. Бухбиндера его научные исследования относились к неравновесной статистической механике, теории магнетизма, релятивистской квантовой механике, квантовой теории поля в искривленном пространстве-времени, квантовой гравитации, суперсимметричной квантовой теории поля, теории полей высших спинов, теории струн. По всем этим направлениям теоретической физики им получены важные научные результаты, некоторые из которых заслужили мировое признание.

Научно-педагогическая деятельность И.Л. Бухбиндера тесно связана с Томским государственным педагогическим университетом, где он прошел путь от ассистента до профессора, заведующего кафедрой теоретической физики. В течение ряда лет он читал лекции по различным специальным курсам и руководил научной работой аспирантов в Томском государственном университете. Им создана в Томске признанная научная школа по теоретической физике, подготовлено большое количество молодых ученых. Некоторые из его учеников сами получили мировую известность. Значительное место в исследовательской деятельности И.Л. Бухбиндера занимает сотрудничество с ведущими российскими и зарубежными научными организациями, особые отношения связывают его с лабораторией теоретической физики Объединенного института ядерных исследований (г. Дубна) и отделением теоретической физики Физического института РАН им. П.Н. Лебедева (г. Москва).

Достижения И.Л. Бухбиндера в развитии науки и подготовке научно-педагогических кадров И.Л. Бухбиндеру получили высокую государственную оценку. Ему присвоено почетное звание "Заслуженный деятель науки Российской Федерации он награжден медалью ордена "За заслуги перед Отечеством"II степени.

К своим 60 годам Иосиф Львович, как и в юности, влюблен в науку, полон идей и планов, является активно работающим ученым. Авторы статей рассматривают данный сборник как коллективный подарок ко дню его рождения. Вместе со многими другими его коллегами, друзьями, соавторами и учениками они желают ему крепкого здоровья, долгих лет жизни и плодотворной научной деятельности.

В.В. Обухов
Ректор Томского государственного
педагогического университета

Preface

If you can dream – and not make dreams your master;
    If you can think – and not make thoughts your aim;
If you can meet with Triumph and Disaster
    And treat those two imposters just the same;

From "IF " by Rudyard Kipling.

On July 3, 2008 Professor Ioseph Lvovich Buchbinder, a famous and internationally recognized scientist in area of theoretical physics, is celebrating his sixtieth birthday. The present volume is a collection of papers of his colleagues, friends, co-authors, and former students from all over the world, who wish to pay tribute to this significant event.

In different periods of his activity I.L. Buchbinder has done research into nonequilibrium statistical mechanics, theory of magnetism, relativistic quantum mechanics, quantum field theory in the curved space-time, quantum gravity, supersymmetric quantum field theory, string theory, and higher spin field theory. In all of these areas of theoretical physics I.L. Buchbinder received important results. Some of them became internationally recognized.

I.L. Buchbinder's research and teaching activity are closely linked to Tomsk State Pedagogical University, where he started as an assistant professor and then became full professor and head of the Department of Theoretical Physics. For many years he has been lecturing on various special courses and supervised PhD students at Tomsk State University. He has established an internationally acknowledged scientific school in theoretical physics and trained a number of young researchers, some of whom have become famous scientists. In his scientific work, I.L. Buchbibnder has been paying a great attention to research collaboration with the leading Russian and foreign scientific institutions. Of special mention are his close relations with the Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research (Dubna) and the Department of Theoretical Physics of the Lebedev Physical Institute (Moscow).

On his 60th birthday Ioseph Buchbinder is still fascinated by science just as he was in his youth He is full of new ideas and plans and continues his research activity.

The contributors present this volume from all their hearts to I.L. Buchbinder as a gift on his 60th anniversary .All of them together with many other of his colleagues, friends, co-authors and students wish him good health for many years ahead and fruitful scientific activity.

V.V. Obukhov
Rector,
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Heterotic Monad Bundles

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Abstract

We review recent progress on the construction of monad vector bundles in the context of $E_8 \times E_8$ heterotic string compactifications. In particular, we explain how stability of these bundles can be shown and how the complete resulting particle spectrum can be computed.

1 Introduction

Compactification of the $E_8 \times E_8$ heterotic string of Calabi-Yau manifolds [1] is the oldest and arguably still the most promising approach towards particle physics model building from string theory. Many of the generic features of low-energy particle physics can be obtained from such compactifications [2]. Among the virtues of such heterotic models are generic gauge unification thanks to a universal gauge kinetic function, gauge-gravity unification [3] in the strong-cpoupling limit and families contained in an underlying spinor representation of SO(10), features which do not easily arise in the context of type II model building.

Despite early advanced [4], many years of heterotic model building experience and substantial recent progress [5]–[15] the construction of a "heterotic strandard model" still remains elusive. The main obstacle is of a technical nature and comes from to the inherent mathematical difficulties in describing and understanding the holomorphic vector bundles on Calabi-Yau manifolds required for heterotic compactifications. The task becomes even more challenging since one needs to understand large classes of Calabi-Yau manifolds and associated vector bundles, as any small number of models is likely to fail when confronted with the more detailed properties of the standard model of particle physics.

In this note, we report on progress in this direction [16, 17], focusing on a particular class on Calabi-Yau manifolds and bundles over them. Specifically, we consider manifolds

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defined as complete intersections in products of projective spaces [18]–[23], complete intersection Calabi-Yaus or CICYs in short, and so-called monad bundles [24] over them. Monad bundles have been considered repeatedly in the literature over the years [25]–[11]. However, progress has been hindered because the essential property of *stability* had not been proven for monad bundles. Stability of a holomorphic vector bundle is a property which originates from the physical requirement of preserving four-dimensional $N = 1$ supersymmetry in a heterotic compactification. More specifically, only a stable holomorphic bundle allows a connection gauge field for which the 10-dimensional gaugino supersymmetry variation vanishes. One of the main purposes of this note is to explain that the stability of certain classes of monad bundles on CICYs can be proved and, hence, that they lead to viable heterotic compactifications with $N = 1$ supersymmetry. We also show that the number of so-called *positive* monad bundles is finite and present a complete classification of such bundles. It is shown how the complete spectrum of particles for such positive monads can be calculated.

2 Review of heterotic Calabi-Yau compactifications

In this section we briefly summarise the main ingredients needed for the compactification of the $E_8 \times E_8$ heterotic string on Calabi-Yau manifolds. For a more detailed review see, for example, Refs. [2].

A heterotic model is specified by four pieces of topological data: A Calabi-Yau three-fold X, two holomorphic vector bundles V and $\tilde{V}$ on X each with structure group each contained in E_8 and a class $W \in H_2(X, \mathbb{Z})$ of the second homology group of X. This data is subject to three consistency conditions.

- **Anomaly cancellation:** For an anomaly-free model (assuming bundles with $c_1(V) = c_1(\tilde{V}) = 0$) the constraint

$$c_2(TX) - c_2(V) - c_2(\tilde{V}) = W \tag{1}$$

  needs to be satisfied.

- **Effectiveness:** The class W needs to be *effective*, that is, it needs to have a representative curve $C \subset X$ which is holomorphic.

- **Stability:** The vector bundles V and $\tilde{V}$ need to be stable.

The physical interpretation of this data in terms of the bosonic fields of the 10-dimensional heterotic string is straightforward. The Calabi-Yau space X, via Yau's theorem, gives rise to a six-dimensional internal Ricci-flat metric with the external four-dimensional part simply being the Minkowski metric. The two bundles V and $\tilde{V}$, from the Donaldson-Uhlenbeck-Yau theorem [29], carry connection gauge fields which can be interpreted as the internal parts of the $E_8 \times E_8$ gauge fields. Finally, the holomorphic curve $C \subset X$ with associated class W is wrapped by a five-brane which otherwise stretches the four-dimensional uncompactified space-time. Together, this configuration subject to the above constraints then defines a consistent $N = 1$ supersymmetry compactification of the heterotic string. So far we have referred to the weakly coupled heterotic string in 10 dimensions, but the above topological data also specifies a compactifiation of the 11-dimensional strong coupling limit [30, 3, 31] of the heterotic string in an analogous way.

For this data, we will make a number of standard model-building choices. The "observable" vector bundle V should have vanishing first Chern class, $c_1(V) = 0$, and rank $n = \text{rank}(V) = 3, 4, 5$, so that the structure group G of V is $\text{SU}(n)$. The low-energy gauge group H is then given by the commutant of G within E_8. The low-energy matter multiplets follow from the decomposition of the **248** adjoint of E_8 under $G \times H$ and the number of these multiplets can be computed from the first bundle cohomology of V. The details of this are summarised in Table 1. Given the choice of a Calabi-Yau manifold X and an observable

$G \times H$	Breaking Pattern: $\mathbf{248} \to$	Particle Spectrum
$SU(3) \times E_6$	$(\mathbf{1},\mathbf{78}) \oplus (\mathbf{3},\mathbf{27}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{27}}) \oplus (\mathbf{8},\mathbf{1})$	$n_{27} = h^1(V)$ $n_{\overline{27}} = h^1(V^\star) = h^2(V)$ $n_1 = h^1(V \otimes V^\star)$
$SU(4) \times SO(10)$	$(\mathbf{1},\mathbf{45}) \oplus (\mathbf{4},\mathbf{16}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{16}}) \oplus (\mathbf{6},\mathbf{10}) \oplus (\mathbf{15},\mathbf{1})$	$n_{16} = h^1(V)$ $n_{\overline{16}} = h^1(V^\star) = h^2(V)$ $n_{10} = h^1(\wedge^2 V)$ $n_1 = h^1(V \otimes V^\star)$
$SU(5) \times SU(5)$	$(\mathbf{1},\mathbf{24}) \oplus (\mathbf{5},\mathbf{10}) \oplus (\overline{\mathbf{5}},\overline{\mathbf{10}}) \oplus (\mathbf{10},\overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}},\mathbf{5}) \oplus (\mathbf{24},\mathbf{1})$	$n_{10} = h^1(V)$ $n_{\overline{10}} = h^1(V^\star) = h^2(V)$ $n_5 = h^1(\wedge^2 V^\star)$ $n_{\overline{5}} = h^1(\wedge^2 V)$ $n_1 = h^1(V \otimes V^\star)$

Table 1: *A vector bundle V with structure group G can break the E_8 gauge group of the heterotic string into a GUT group H. The low-energy representation are found from the branching of the* **248** *adjoint of E_8 under $G \times H$ and the low-energy spectrum is obtained by computing the indicated bundle cohomology groups.*

bundle V, in order to be able to satsify the anomaly and effectiveness constraint, we demand that

$$c_2(TX) - c_2(V) \text{ is an effective class on } X \ . \tag{2}$$

In this case, there exists a holomorphic curve $C \subset X$ with second homology class $W = [C]$, such that the anomaly condition is satisfied by wrapping a five-brane on C and choosing the "hidden" bundle $\tilde{V}$ to be trivial.

It then remains to prove the stability of V which is one of the main subjects of this note. In order to define the concept of stability, we first have to introduce the *slope* $\mu(V)$ of a bundle V by

$$\mu(V) = \frac{1}{\text{rank}(V)} \int_X c_1(V) \wedge J \wedge J \ . \tag{3}$$

Given this notion, a bundle V is called *stable* if for all coherent sub-sheaves $\mathcal{F} \subset V$ with $1 \leq \text{rank}(\mathcal{F}) < \text{rank}(V)$ the condition $\mu(\mathcal{F}) < \mu(V)$ is satisfied. In other words, the slope of any coherent sub-sheaf has to be smaller than the slope of the bundle itself. Since the bundles considered here have vanishing first Chern class stability amounts to the condition that

$$\mu(\mathcal{F}) < 0 \tag{4}$$

for all coherent sub-sheaves $\mathcal{F} \subset V$ with $1 \leq \text{rank}(\mathcal{F}) < \text{rank}(V)$. It is usually hard to get a handle on all coherent sub-sheaves of a bundle V which makes stability a property difficult to prove. We will see in the following that there are certain sufficient (although not necessary) criteria which are often suitable to prove stability but do not require knowledge of all coherent sub-sheaves. Stable $SU(n)$ bundles V satisfy the properties

$$H^0(X, V) = H^3(X, V) = 0 \tag{5}$$

which are necessary (but usually not sufficient) criteria for stability. The vanishing conditions (5) mean that the index of the bundle, which can be expressed in terms of the third

Chern class using the index theorem, is given by

$$\text{ind}(V) = h^2(X,V) - h^1(X,V) = \frac{1}{2}\int_X c_3(V) \tag{6}$$

and, hence, measures the chiral asymmetry of the model. We also note that a bundle V is stable if and only if its dual $V^\star$ is stable.

Table 1 shows that our choice of bundles V leads to a GUT theory at low energy. Usually, the GUT group is further broken to the standard model group by introducing a Wilson line on the quotient space X/Γ, where Γ is a discrete symmetry of X. In the present note we will not carry this out explicitly, but merely impose two constraints which follow from this construction and the requirement of having three families in the final model. The three-family condition means that $\text{ind}(V)/|\Gamma|$ (where $|\Gamma|$ is the order of Γ) must be three and, therefore, we must have that

$$\text{ind}(V) \text{ is divisible by } 3\ . \tag{7}$$

Further, in order to be able to form the quotient $X/|\Gamma|$ the Euler number, $\chi(X)$ of X must be divisible by $|\Gamma|$, so

$$\chi(X) \text{ divisible by } \text{ind}(V)/3\ . \tag{8}$$

As we will see, these two rudimentary physical conditions already impose fairly strong constraints on our models.

3 Calabi-Yau manifolds and monad bundles

3.1 Complete intersection Calabi-Yau manifolds

In this paper, we will focus on what is perhaps the simplest class of Calabi-Yau manifolds, namely complete intersection Calabi-Yau manifolds (CICYs) (for a review see Ref. [32]). They are embedded into an ambient space $\mathcal{A} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ which is a product of m projective spaces, each with dimension n_r and Kahler form J_r. The manifold X is then defined as the common zero locus of $K = \sum_{r=1}^m n_r - 3$ polynomials p_i with multi-degree $\mathbf{q}_i = (q_i^1, \ldots, q_i^m)$ (this means the degree of the polynomial p_i in the projective coordinates of the r^{th} projective space is q_i^r). The Calabi-Yau condition, that is the vanishing of the first Chern class $c_1(TX) = 0$, then amounts to

$$\sum_{i=1}^K q_i^r = n_r + 1 \quad \forall\, r = 1, \ldots, m\ . \tag{9}$$

The degrees of the defining polynomials p_i are suitably summarised in a $m \times K$ *configuration matrix*

$$\left[\begin{array}{c|cccc} \mathbb{P}^{n_1} & q_1^1 & q_2^1 & \ldots & q_K^1 \\ \mathbb{P}^{n_2} & q_1^2 & q_2^2 & \ldots & q_K^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_1^m & q_2^m & \ldots & q_K^m \end{array}\right]_{m\times K} . \tag{10}$$

With this notation the famous quintic Calabi-Yau manifolds can be written as $[\mathbb{P}^4|5]$. It turns out that there are finitely many CICYs and their classification [18] leads to 7890 different types, each represented by its configuration matrix. Based on the results of this original classification (accessible at [33]), we have compiled a list of these configuration matrices, together with all the relevant topological data. This data includes the Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$, the second Chern classes $c_2(X) = c_(2r)(X)\nu^r$ in a basis ν^r of the fourth cohomology dual to J_r, and the triple intersection numbers d_{rst}. In this note, we will focus on a sub-class of these CICYs, namely the so-called *favourable* CICYs whose second cohomology is spanned by the (restriction of the) ambient space Kahler forms J_r and which are, hence, characterised by

$$h^{1,1}(X) = m \; . \tag{11}$$

It turns out that 4515 of the 7890 CICYs are favourable, so we are still dealing with a large class. Working with this sub-class has a number of practical advantages. First, the Kahler cone, that is, the set of allowed Kahler forms J on X, has a simple description and is given by $J = t^r J_r$, where the Kahler moduli t^r have to be positive, $t^r \geq 0$. Further, the effective classes in $H_2(X) \simeq H^4(X)$ are characterised by linear combinations $w_r\nu^r$ with positive integers w_r. This means that the anomaly/effectiveness condition (2) can be written as

$$c_{2r}(V) \leq c_{2r}(TX) \quad \forall\, r = 1, \ldots, m \; . \tag{12}$$

Finally, for favourable CICYs, all line bundles on X can be obtained as restrictions of line bundles on the ambient space $\mathcal{A}$. Specifically, it we denote by $\mathcal{O}_{\mathbb{P}^n}(k)$ the k^{th} power of the hyperplane bundle on $\mathbb{P}^n$, then the line bundles on $\mathcal{A}$ can be written as $\mathcal{O}_\mathcal{A}(\mathbf{k}) \equiv \mathcal{O}_{\mathbb{P}^{n_1}}(k^1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(k^m)$ where $\mathbf{k} = (k^1, \ldots, k^m)$ in an integer vector. The restriction of $\mathcal{O}_\mathcal{A}(\mathbf{k})$ to X is then denoted by $\mathcal{O}_X(\mathbf{k})$ and these line bundles indeed provide a complete set on X. Writing $L = \mathcal{O}_X(\mathbf{k})$, the Chern characters of these line bundles are given by

$$\begin{array}{ll} \text{ch}_1(L) & = c_1(L) = k^r J_r \\ \text{ch}_2(L) & = \frac{1}{2} k^r k^s J_r \wedge J_s \\ \text{ch}_3(L) & = \frac{1}{6} k^r k^s k^t J_r \wedge J_s \wedge J_t \; , \end{array} \tag{13}$$

From the Atiyah-Singer index theorem, the index of L can be written as

$$\begin{aligned} \text{ind}(L) \equiv \sum_{q=0}^{3} (-1)^q h^q(X, L) &= \int_X \text{ch}(L) \wedge \text{Td}(X) = \int_X \left[\text{ch}_3(L) + \frac{1}{12} \text{ch}_2(TX) \wedge c_1(L) \right] \\ &= \frac{1}{6} \left(d_{rst} k^r k^s k^t + \frac{1}{2} k^r c_{2r}(TX) \right) \; . \end{aligned} \tag{14}$$

In Ref. [34], the cohomology for all line bundles on favourable CICYs X has been computed explicitly. The results are complicated and require many case distinctions but they allow the calculation of all line bundle cohomologies on all favourable CICYs. Here, we merely mention some generic results for line bundle cohomology. For positive line bundles, that is line bundles $L = \mathcal{O}_X(\mathbf{k})$ with all $k^r > 0$ Kodaira vanishing implies that $H^q(X, L) = 0$ for all $q > 0$. Hence, for positive line bundles $H^0(X, L)$ is the only potentially non-vanishing cohomology and Eq. (14) reads $h^0(X, L) = \text{ind}(L)$. Similarly if follows that for negative line bundles, that is line bundles $L = \mathcal{O}_X(\mathbf{k})$ with all $k^r < 0$, all but the third cohomology vanishes and $h^3(X, L) = -\text{ind}(V)$. The explicit formulae for line bundle cohomology in Ref. [34] also show that semi-positive line bundles, that is line bundles $L = \mathcal{O}_X(\mathbf{k})$ with all $k^r \geq 0$, have at least one section, so $h^0(X, L) > 0$. These results will be important since line bundles are the main building blocks of our monads, as we will see momentarily.

3.2 Monads on CICYs

We define a monad bundle [35]–[39] V on a (favourable) CICY X by the short exact sequence

$$\begin{aligned} & 0 \to V \to B \xrightarrow{f} C \to 0 \ , \text{ where} \\ B &= \bigoplus_{i=1}^{r_B} \mathcal{O}_X(\mathbf{b}_i) \ , \quad C = \bigoplus_{j=1}^{r_C} \mathcal{O}_X(\mathbf{c}_j) \ . \end{aligned} \tag{15}$$

are sums of line bundles with ranks r_B and r_C, respectively. The map f can be thought of as (the restriction to X of) a $r_B \times r_C$ matrix of polynomials with degrees $\mathbf{c}_a - \mathbf{b}_i$. Provided all line bundles in C are greater than all line bundles in B (by which we mean $c_a^r \geq b_i^r$ for all a, i, r and for all a and i there exists an r such that $c_a^r > b_i^r$) this short exact sequence indeed defines a bundle V on X. In this note we will mostly consider positive monads, that is monads for which $c_a^r > 0$ and $b_i^r > 0$ for all a, i, r. This is the class of monads which has been traditionally considered in the literature and it offers significant practical advantages due to the simplicity of positive line bundle cohomology. We will argue later that the condition of positivity can be relaxed but for now we will focus on positive monads.

The Chern classes of a monad bundle V can be explicitly written as

$$\begin{aligned} \text{rk}(V) &= r_B - r_C = n \ , \\ c_1^r(V) &= \sum_{i=1}^{r_B} b_i^r - \sum_{j=1}^{r_C} c_j^r \ , \\ c_{2r}(V &= \frac{1}{2} d_{rst} \left(\sum_{j=1}^{r_C} c_j^s c_j^t - \sum_{i=1}^{r_B} b_i^s b_i^t \right) \ , \\ c_3(V) &= \frac{1}{3} d_{rst} \left(\sum_{i=1}^{r_B} b_i^r b_i^s b_i^t - \sum_{j=1}^{r_C} c_j^r c_j^s c_j^t \right) \ , \end{aligned} \tag{16}$$

where $c_1^r(V) = 0$ has been assumed to simplify the expressions for $c_{2r}(V)$ and $c_3(V)$. A number of physical constraints should be imposed on the monad construction. First, we would like the structure group of V to be $\text{SU}(n)$, where $n = 3, 4, 5$. This means that $\text{rk}(V) = n = r_B - r_C$ and $c_1^r(V) = 0$. In addition, the anomaly constraint (12) has to be satisfied.

4 Classification of positive monads on favourable CICYs

We can ask if the class of positive monads on favourable CICYs with structure groups $\text{SU}(n)$, where $n = 3, 4, 5$ and which satisfy the anomaly constraint (12) constitute a finite class. To answer this question, we define the quantities $b_{\max}^r = \max_i \{b_i^r\}$, the maximal entries in B for each projective space. Using the anomaly condition (12), the fact that $c_1^r(V) = 0$ and the explicit expressions for the Chern classes (16) one can then derive the inequality [17]

$$2c_{2r}(TX) \geq n \sum_{s,t} d_{rst} b_{\max}^t \ . \tag{17}$$

Since the matrix $\sum_s d_{rst}$ is always non-singular, the first of these inequalities provides an upper bound for $b_{\max}^r$, the maximal entries of B. More precisely, we can scan all 4515

Config	No.Bundles	Config	No.Bundles
$\begin{bmatrix} 5 \end{bmatrix}$	$(20,14,9)$	$\begin{bmatrix} 3 & 3 \end{bmatrix}$	$(5,3,2)$
$\begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}$	$(2,1,0)$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	$(611,308,56)$
$\begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix}$	$(12,5,,0)$	$\begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$	$(126,17,0)$
$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$	$(3,0,0)$	$\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$	$(5,0,0)$
$\begin{bmatrix} 0 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$	$(5,0,0)$	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	$(5,0,0)$
$\begin{bmatrix} 0 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$	$(126,17,0)$	$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	$(2,0,0)$
$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix}$	$(3,0,0)$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$	$(5,0,0)$
$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$	$(74,0,0)$	$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}$	$(9,0,0)$
$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$	$(34,0,0)$	$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$	$(3,0,0)$
Config	No.Bundles	Config	No.Bundles
$\begin{bmatrix} 4 & 2 \end{bmatrix}$	$(7,5,3)$	$\begin{bmatrix} 3 & 2 & 2 \end{bmatrix}$	$(3,2,1)$
$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$	$(62,43,14)$	$\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$	$(80,12,0)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$	$(15,8,0)$	$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$	$(153,35,19)$
$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$	$(13,2,0)$	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix}$	$(5,0,0)$
$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$	$(12,5,0)$	$\begin{bmatrix} 0 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$	$(8,0,0)$
$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	$(2,0,0)$	$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	$(1,0,0)$
$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$	$(553,232,0)$	$\begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$	$(8,0,0)$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$	$(25,0,0)$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$	$(9,0,0)$
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	$(9,0,0)$	$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$	$(3665,625,0)$

Table 2: *The 36 manifolds which admit positive monads. The No.Bundles column next to each manifold is a triple, corresponding to the numbers of respectively ranks 3,4, and 5 monads.*

favourable CICYs and find all positive integer solutions $b^r_{\max}$ of Eq. (17). It turns out that only on 63 favourable CICYs does a solution exist and the values for $b^r_{\max}$ never exceed 6.

One can also derive the upper bound

$$r_B \leq \left(1 + \sum_{r=1}^{m} b^r_{\max}\right) , \qquad (18)$$

for the rank of B and, together with the inequality (17) this shows that the class of positive monads is finite. One can now perform a systematic computer search for a complete classification. It turns out that only 36 CICYs do indeed allow for positive monad bundles consistent with all constraints and that a total of 7118 monads exist over these 36 manifolds. We have listed these 36 CICYs, together with the number of monads over each of them, in Table 2. The number of monad bundles as a function of their index, $\text{ind}(V)$, has been plotted in Fig. 1. The left plot of all 7118 monads shows a roughly Gaussian distribution with a peak at a fairly high index of around -60. For the plot on the right-hand side we have only taken into account bundles which satisfy the three-family conditions (7) and (8). It is apparent that even these two rudimentary physical constraints lead to a significant reduction in numbers. The precise figures are summarised in Table 3. It should be noted that the additional requirement of a not too large index (so that the order of the discrete group necessary to obtain three families is not too large) leaves us only with a rather small number of viable bundles, corresponding

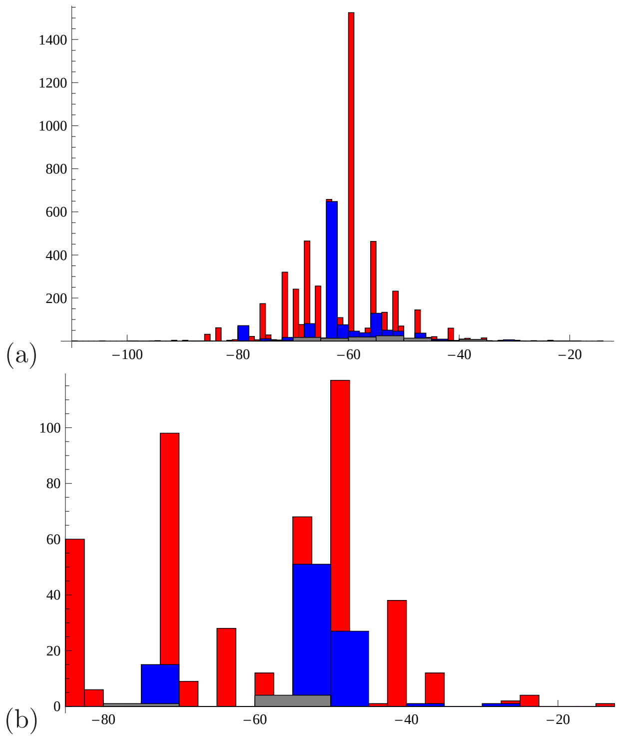

Figure 1: *(a) Histogram for the index,* ind(V)*, of the positive monads, 5680 of rank 3 (in red), 1334 of rank 4 (in blue), and 104 of rank 5 (in gray), found over 36 favourable CICYs: the horizontal axis is* ind(V) *and the vertical, the number of bundles; (b) the same data set, but only taking those monads which have* ind$(V) = 3k$ *for some positive integer* k *and such that* k *divides the Euler number of the corresponding CICY.*

to the tail in the right Fig. 1.

	Bundles	ind$(V) = 3k$	ind$(V) = 3k$ and k divides $\eta(X)$
rank 3	5680	3091	458
rank 4	1334	207	96
rank 5	104	52	5
Total	7118	3350	559

Table 3: *The number of positive monad bundles on favourable CICYs. Imposing that the third Chern class is divisible by 3 reduces the number and requiring in addition that the quotient of* $c_3(V)$ *by 3 divides the Euler number of the corresponding CICY further reduces the number.*

5 Stability

In this section, we give an informal account of the stability proof for positive monads on CICYs. Full details of the proof can be found in Refs. [34].

To start, it is useful to introduce "dual" coordinates s_r on the Kahler cone, defined by $s_r = \frac{1}{\mathrm{rk}(V)} d_{rst} t^s t^t$, so that the stability condition (4) can be written as

$$\mu(\mathcal{F}) = s_r c_1^r(\mathcal{F}) < 0 \ , \tag{19}$$

for any coherent sub-sheaf $\mathcal{F} \subset V$ with $1 \leq \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(V)$. We note that, in terms of the variables s_r the Kahler cone is contained in $s_r \geq 0$ although it does usually not cover all of the positive quadrant.

As a warm-up is it useful to discuss the case of *cyclic* CICYs, that is CICYs with $h^{1,1}(X) = 1$, first. There are five such cyclic manifolds in the CICY list with a total of 77 positive monad bundles over them. In this case, assume the existence of a de-stabilising sheaf $\mathcal{F} \subset V$, so $\mu(\mathcal{F}) \geq 0$ and $k = \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(V)$. Then define a line bundle $L = \wedge^k \mathcal{F} \subset \wedge^k V$. This line bundle still satisfies $\mu(L) \geq 0$ and given that the Kahler cone is characterised by $s \geq 0$, Eq. (19) implies that $L = \mathcal{O}_X(k)$, where $k \geq 0$. As previously discussed such semi-positive line bundles have at least one section and, hence, $\wedge^k V$ has a section. This means $h^0(X, \wedge^k V) > 0$. Turning this argument around we arrive at *Hoppe's criterion*. If $H^0(X, \wedge^k V) = 0$ for all $k = 1, \ldots, \mathrm{rk}(V) - 1$, then V is stable.

We can now apply this criterion to $V^\star$ which is stable exactly if V is. In particular, we need to show that $H^0(X, V^\star) = 0$ and $H^0(X, \wedge^{n-1} V^\star) = H^0(X, V) = 0$. To show the former we write down the long exact sequence associated to the dual of the monad sequence (15). It reads

$$\begin{array}{ccccccc} 0 & \to & H^0(X, C^\star) & \to & H^0(X, B^\star) & \to & H^0(X, V^\star) \\ & \to & H^1(X, C^\star) & \to & H^1(X, C^\star) & \to & \ldots \end{array} \tag{20}$$

Since $B^\star$ and $C^\star$ are sums of negative line bundles we know that $H^0(X, B^\star) = 0$ and $H^1(X, C^\star) = 0$ which, from the above sequence, immediately implies that $H^0(X, V^\star) = 0$. To show that $H^0(X, V) = 0$ is not quite so easy and has been explicitly carried out in Ref. [17]. This completes the proof for rank three bundles. For rank four and five bundles similar arguments as above can be used [17] to demonstrate the vanishing of $H^0(X, \wedge^k V^\star)$ for $k = 2, \ldots, n-2$. This completes the stability proof for positive monad bundles on cyclic CICYs.

What about CICYs with $h^{1,1}(X) > 1$? As before, it is useful to introduce the line bundle $L = \wedge^k \mathcal{F}$, where $k = \mathrm{rk}(\mathcal{F})$, associated to a coherent sheaf $\mathcal{F}$ and to note that $\mu(L) = \mu(\mathcal{F})$. If we can show that $\mu(L) \geq 0$ for a certain fixed patch $\{s_r\}$ in the Kahler cone for all line bundles L which cannot be excluded as sub-line bundles of $\wedge^k V$ then V is stable in this part of the Kahler cone. We can, therefore, simplify our task by proving for as many line bundles as possible that they cannot be sub line-bundles of $\wedge^k V$. It turns out that all our bundles do satisfy Hoppe's criterion $H^0(X, \wedge^k V) = 0$ for $k = 1, \ldots, \mathrm{rk}(V) - 1$ (although this is only sufficient for stability in the cyclic case). This means that L can only be a sub line bundle of $\wedge^k V$ if $H^0(X, L) = 0$. Further, $\mathrm{Hom}(L, \wedge^k V) \simeq H^0(X, L^\star \otimes \wedge^k V)$ must be non-zero for L to be a sub line bundle. It turns out that these two criteria are, in many cases, sufficient to prove stability.

Let us discuss this for the case $h^{1,1} = 2$, which covers about 1500 of our monad bundles. Line bundles $L = \mathcal{O}_X(k, l)$ with $k \geq 0$ and $l \geq 0$ have a section and can, hence, not be sub line bundles of $\wedge^k V$. Negative line bundles $L = \mathcal{O}_X(k, l)$ with $k \leq 0$ and $l \leq 0$ do not destabilise the Kahler cone which is in the positive quadrant. Mixed line bundles $L = \mathcal{O}_X(k, l)$ with k and l of different sign are potentially dangerous. However, most of them can be excluded since $H^0(X, L^\star \otimes \wedge^k V) = 0$ as shown in Ref. [34]. It turns out, the remaining mixed line bundles may destabilise some of the Kahler cone but leave an open subset stable. The situation is summarised in Fig. 2. These statements can be systematically checked by a computer scan

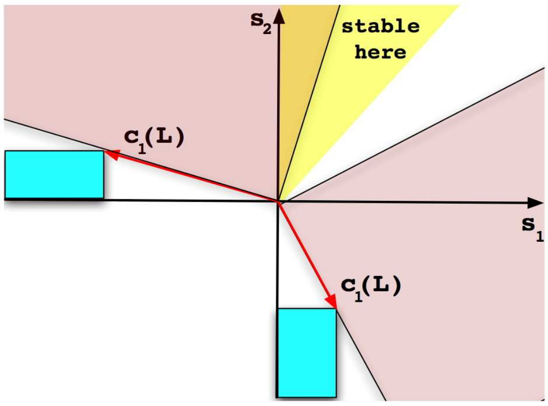

Figure 2: *Schematic plot for stability proof of monad bundles on CICYs with $h^{1,1} = 2$ in the (s_1, s_2) plane of the two Kahler moduli. The Kahler cone is indicated in yellow. Potential sub line bundles L with $c_1(L)$ in the positive quadrant are excluded by Hoppe's criterion. Subsheaft with $c_1(L)$ in the negative quadrant do not destabilise the Kahler cone. Line bundles in the mixed quadrants can be shown to satisfy $H^0(X, L^\star \otimes \wedge^k V) = 0$ except for the light blue regions. The associated destabilised regions (light red) do not cover the whole Kahler cone.*

and in this way stability can be shown for all 1200 or so rank three and many of the rank four monad bundles on CICYs with $h^{1,1}(X) = 2$.

The method of this proof can, at least in principle, be applied to CICYs with $h^{1,1}(X) > 2$. Of course, in this case, the geometry of the Kahler cone and the various destabilising regions become much more complicated. We have not yet carried this out explicitly but we believe that our proof for $h^{1,1} \leq 2$ provides an adequate basis for the conjecture that *all positive monads on favourable CICYs are stable.*

6 Particle spectrum

We have already computed the index which gives the chiral asymmetry for our models. Now, we should determine the number of families and anti-families independently. From Table 1, the number of families (anti-families) is given by the dimension of the cohomology $H^1(X, V)$ $(H^2(X, V))$. This cohomology can be obtained from the long exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(X,V) & \rightarrow & H^0(X,B) & \rightarrow & H^0(X,C) \\ & \rightarrow & H^1(X,V) & \rightarrow & H^1(X,B) & \rightarrow & H^1(X,C) \\ & \rightarrow & H^2(X,V) & \rightarrow & H^2(X,B) & \rightarrow & H^2(X,C) \\ & \rightarrow & H^3(X,V) & \rightarrow & H^3(X,B) & \rightarrow & H^3(X,C) \rightarrow 0 \end{array} \tag{21}$$

which follows from the defining short exact sequence (15) for the monad bundle V. Since both B and C are sums of positive line bundles on X we know that $H^q(X, B) = H^q(X, C) = 0$ for $q > 0$. Hence, in the above sequence $H^2(X, V)$ is "enclosed" by zeros and must vanish.

We also know from stability that $H^0(X,V) = H^3(X,V) = 0$ so that $H^1(X,V)$ is the only non-vanishing cohomology of V. Its dimension is, therefore, given by the (negative) index or, from Eq. (21), in terms of the cohomology of B and C. Specialising to the three cases for the rank of V this implies

$$\left.\begin{array}{lcl} E_6 & : & n_{27} \\ \mathrm{SO}(10) & : & n_{16} \\ \mathrm{SU}(5) & : & n_{10} \end{array}\right\} = h^0(X,C) - h^0(X,B) = -\mathrm{ind}(V)\ , \qquad \left.\begin{array}{l} n_{\overline{27}} \\ n_{\overline{16}} \\ n_{\overline{10}} \end{array}\right\} = 0\ . \tag{22}$$

Consequently, all our models have a vanishing number of anti-families and the number of families can be easily computed from the index of the bundle. In fact, we have already plotted the index in Fig. 1. The absence of vector-like pairs for all positive monads may be considered a phenomenologically attractive feature.

For rank four and five bundles we also need to calculate the number of Higgs multiplets. From Table 1 this means we should calculate $h^1(X, \wedge^2 V^\star)$. The details of this computation which is somewhat involved and requires ambient space cohomology and Koszul resolutions are given in Ref. [17]. The result, however, is simply that

$$H^1(X, \wedge^2 V^\star) = 0 \tag{23}$$

provided that the map f in the definition (15) of the monad is sufficiently generic (that is, it is made up from sufficiently generic polynomials of the appropriate degrees). For the rank four case the low energy gauge group is SO(10) and the Higgs multiplets reside in the fundamential, **10**. Since $\wedge^2 V \simeq \wedge^2 V^\star$ for SU(4) bundles, Eq. (23) implies

$$n_{10} = h^1(X, \wedge^2 V) = 0\ , \tag{24}$$

and, hence, that the number of Higgs multiplets vanishes in this case. For rank five, the low-energy gauge group is SU(5) and the Higgs multiplets appear as vector-like pairs of **5** and $\bar{\mathbf{5}}$. From Table 1, the number of **5** and $\bar{\mathbf{5}}$ is counted by the cohomologies $H^1(X, \wedge^2 V^\star)$ and $H^1(X, \wedge^2 V^)$, respectively. However, applying the index theorem to $\wedge^2 V$ one can also derive a formula for the chiral asymmetry of these multiplets and one finds [5]

$$\mathrm{ind}(V) = h^1(X, \wedge^2 V^\star) - h^1(X, \wedge^2 V)\ . \tag{25}$$

Our result (23) then implies that

$$n_{\bar{5}} = h^1(X, \wedge^2 V) = -\mathrm{ind}(V)\ , \qquad n_5 = 0\ . \tag{26}$$

Combining this with Eq. (22) shows that we have $-\mathrm{ind}(V)$ complete SU(5) families, each consisting of a **10** and $\bar{\mathbf{5}}$ multiplet, but no vector-like Higgs pairs in **5** and $\bar{\mathbf{5}}$ multiplets.

The absence of Higgs multiplets for both the SO(10) and the SU(5) case is, of course, a concern from a phenomenological point of view. However, it has to be stressed that this result is only valid provided the map f in the monad sequence (15) is indeed generic. It has been shown in Ref. [16], and previously found in the context of other heterotic bundle constructions [14] that a non-generic choice for f can lead to a non-vanishing number of Higgs multiplets. In fact, imposing discrete symmetries on our models as a preparation for the introduction of Wilson lines will require restrictions on the polynomials defining the CICY as well as on f. The number of Higgs multiplets then has to be re-calculated in this context.

Finally, we should compute the number of singlets which correspond to the cohomology $H^1(X, V \otimes V^\star)$. A general formula can be obtained provided the conditions

$$H^1(X, C^\star \otimes C) = H^2(X, C^\star \otimes B) = 0\ . \tag{27}$$

on B and C are satisfied. These conditions can be checked explicitly given our knowledge of line bundle cohomology on CICYs and are indeed satsfied for many of our models. Under this assumption, the number of singlets is given by [17]

$$\begin{aligned} n_1 = \; = \; & h^0(X, B^\star \otimes C) - h^0(X, B^\star \otimes B) - h^0(X, C^\star \otimes C) \\ & + h^0(X, C^\star \otimes B) - h^1(X, C^\star \otimes B) + h^1(X, B^\star \otimes B) + 1 \; , \end{aligned} \tag{28}$$

an expression which can again be evaluated from the known line bundle cohomology. For models which do not satisfy the vanishing conditions (27) more sophisticated methods have to be applied [17] but the number of singlets can still be explicitly calculated even in such cases.

7 Semi-positive monads

One obvious disadvantage of the models considered so far are the relatively large family numbers, evident from the peak at $-\text{ind}(V) \simeq 60$ in Fig. 1 at. These large numbers make it difficult to obtain a model with precisely three families after dividing by a discrete symmetry, given that the order of Calabi-Yau symmetries tends to be relatively small. Clearly, these large family numbers are related to the positivity of the monads. One might, therefore, ask how the situation changes if one considers the class of *semi-positive* monads with entries satisfying $b_i^r \geq 0$ and $c_a^r \geq 0$ (as opposed to $b_i^r > 0$ and $c_a^r > 0$ for positive monads) but are otherwise subject to the same constraints.

The first difficulty one encounters is that this class is not obviously finite. In other words, the conditions $c_1^r(V) = 0$ and $c_{2r}(V) \leq c_{2r}(TX)$ do not lead to a finite number of matrices (b_i^r) and (c_a^r) as was the case for positive monads. Indeed, infinite classes of semi-positive monads satisfying all relevant constraints can be found relatively easily. One example is provided by

$$B = \mathcal{O}_X(1,0)^{\oplus 3} \oplus \mathcal{O}_X(t,1) \; , \quad C = \mathcal{O}_X(t+3,1) \tag{29}$$

for all positive integer t on the CICY with configuration matrix $X = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array} \right]$. For this example, it turns out that all the relevant topological data is independent of t. In fact, so far, we have not been able to find a class where we can show the topological inequivalence of an infinite number of bundles. It is, therefore, not inconceivable that the class of semi-positive monads becomes finite after more sophisticated bundle equivalences are taken into account. This problem is currently under inverstigation [40]

For now, we will simply proceed pragmatically and, as a first attempt, perform a scan of all rank three semi-positive monads with $b_i^r \leq 20$ and $c_a^r \leq 20$ on all 32 CICYs with $h^{1,1}(X) = 2$ (monads with zero entries on cyclic CICYs can be shown to be unstable). We find a list of about 100000 bundles. The distribution of these bundles as a function of the chiral asymmetry is shown in Fig. 3. By comparing with Fig. 1 it is evident that the distribution is peaked at significantly lower values of $-\text{ind}(V)$, as one would expect. As before, we can impose the three-family constraints (7) and (8) on these models. It turns out that about 17000 bundles are consistent with the constraints and, of those, about 7000 satisfy $-\text{ind}(V) \leq 20$. This means we are dealing with a vastly larger class of models and many more pass our rudimentary physical tests. It also turns out that stability can be proven for at least some of these semi-positive monads [34]. However, their particle spectrum can be qualitatively different from that of positive monads in that the number of anti-generations might be non-zero and Higgs multiplets might exist even in the generic case. All these issues require further investigation [40].

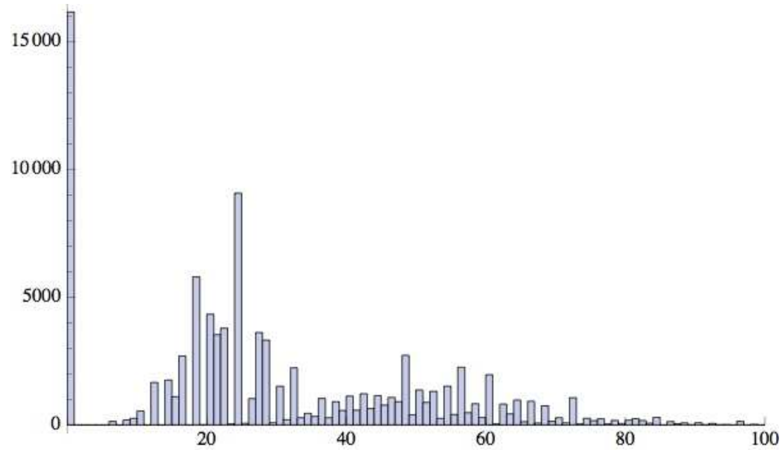

Figure 3: *Number of semi-positive monads with* $b_i^r \leq 20$ *and* $c_a^r \leq 20$ *on CICYs with* $h^{1,1}(X) = 2$ *(vertical axis) as a function of the (negative) index* $\text{ind}(V)$ *(horizontal axis).*

8 Conclusion

In this note, we have reported on progress in constructing heterotic monad bundles on complete intersection Calabi-Yau manifolds (CICYs). We have shown that a certain sub-set of these bundles, the positive monads, is finite and can be completely classified, leading to about 7000 models. Stability can be shown for many of these models and we have explained the basic ideas of the stability proof. Based on these results we conjecture that all positive $\text{SU}(n)$ bundles (where $n = 3, 4, 5$) on (favourable) CICYs are stable. It has also been shown that the complete particle spectrum for these models can be calculated from bundle cohomology. In particular, the number of anti-families vanishes for all models. A phenomenologically problematic feature of positive monads is their relatively large number of families which makes it difficult to obtain three generations after dividing by a discrete symmetry. This has motivated us to consider semi-positive monads. We have shown that they constitute a vastly larger class and that their chiral asymmetry is significantly lower. However, further work is needed to decide if this class of monads is finite and to determine all their relevant properties.

In summary, we have shown that monad bundles on CICYs provide a framework in which large numbers of models can be studied systematically and all physically relevant properties can be extracted in an algorithmic way. We believe that investigating a large class of models in this way is a necessary pre-requisite to extracting a physically successful model from string theory and further work in this direction is under way.

Acknowledgments

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Nonlocal Cosmology

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

A string field theory (SFT) nonlocal model of the cosmological dark energy providing $w < -1$ is briefly surveyed. We summarize recent developments and open problems, as well as point out some theoretical issues related with others applications of the SFT nonlocal models in cosmology, in particular, in inflation and cosmological singularity.

1 Introduction

The origin of the dark energy (DE) is still a fascinating puzzle. Present cosmological observations do not exclude an evolving DE state parameter w. Recent results of WMAP [2] together with the data [1] on Ia supernovae give the following bounds for the DE state parameter $w_{\rm DE} = -1^{+0.14}_{-0.11}$ or without an a priori assumption that the Universe is flat and together with the data on large-scale structure and supernovae $w_{\rm DE} = -1.06^{+0.13}_{-0.08}$.

There are two questions to experimental data:

- Can we rule out a dynamical DE?
- Can we rule out $w < -1$?

Recent data are not enough to answer these two questions and moreover, within the next few years answers on these two questions will not accessible as it has previously expected. However one might wonder whether there is a room for $w < -1$ in a theory.

Dark energy models with the state parameter $w < -1$ violate the null energy condition (NEC). All local models realizing the NEC violation are unstable and violate usual physical requirements. To provide $w < -1$ a string field theory (SFT) nonlocal DE model has been proposed [3].

In this SFT nonlocal model our Universe is considered as a D3 non-BPS brane embedded in the 10 dimensional space-time. The role of the dark energy plays the Neveu-Schwarz (NS) string tachyon leaving in GSO$-$ sector. The tachyon action is dictated by the cubic fermionic SFT [4, 5] and it is nonlocal due to string effects [6].

We postulate a minimal form of the tachyon interaction with gravity. In the spatially flat FRW metric the model is described by a system of two nonlinear nonlocal equations for the tachyon field and the Hubble parameter. The corresponding potential has perturbative and nonperturbative minima. A transition from a perturbative vacuum to a non-perturbative one is interpreted as D-brane decay. It happens that this model under some conditions displays a phantom behaviour [7]. Note that unlike phenomenological phantom models here phantom appears in an effective theory. Since SFT is a consistent theory this approach does not suffer from usual problems which are inevitable for phenomenological phantom models. The UV completion is supposed to be solved by extending the one mode (tachyon) approximation.

SFT in the flat background dictates a particular value of the D-brane tension. It can be found from the requirement that the total energy of the system in the true non-perturbative vacuum is zero. In cosmology this total energy of the system in the true non-perturbative vacuum can be interpreted as the cosmological constant. It has been conjectured that an existence of a rolling solution describing a smooth transition to the true vacuum does define the value of the cosmological constant [3]. We cannot prove this conjecture but arguments to it favor can be given using the local approximation [10]. A recent breakthrough in solving numerically the full nonlinear and nonlocal system of equations [9] also supports this expectation.

2 Model

Our model is given by the following action [3]

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2}+\frac{1}{\lambda_4^2}\left(-\frac{\xi^2\alpha'}{2}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x)+\frac{1}{2}\phi^2(x)-\frac{1}{4}\Phi^4(x)-T'\right)\right)$$

Here $g_{\mu\nu}$, κ and λ_4 are the four-dimensional metric, gravitational coupling constant and scalar field coupling constant, respectively; $\frac{1}{\lambda_4^2}=\frac{v_6M_s^4}{g_o}(\frac{M_s}{M_c})^6$, g_o is the open string dimensionless coupling constant, M_s is the string scale $M_s=1/\sqrt{\alpha'}$ and M_c is a scale of the compactification, v_6 is a number related with a volume of the 6-dimensional compact space. $T'=1/4+\Lambda'$, where Λ' is a dimensionless cosmological constant. ϕ is a tachyon field and Φ is related with ϕ by the following relation $\Phi=e^{\frac{\alpha'}{8}\Box_g}\phi$, where $\Box_g=\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$. This form of the nonlocal interaction is defined by the cubic fermionic SFT (CFSFT) [5]. More precisely, the CFSFT brings a more complicated form of the interaction, but by an analogy with the flat case [8, 7, 9] we believe that this approximation catches essential physical properties of the model and $\xi^2\approx 0.9556$ is a constant defined by the CFSFT.

In the spatially flat FRW metric with a scale factor $a(t)$ the equation for a space homogenous tachyon field Φ and the Friedmann equations have the form [3]

$$\left(\xi^2\mathcal{D}+1\right)e^{-\frac{1}{4}\mathcal{D}}\Phi = \Phi^3,\qquad \mathcal{D}=-\partial_t^2-3H(t)\partial_t,\quad H=\partial_t a/a \tag{1}$$

$$3H^2 = \frac{\kappa^2}{\lambda_4^2}\left(\frac{\xi^2}{2}\partial_t\phi^2-\frac{1}{2}\phi^2+\frac{1}{4}\Phi^4+\mathcal{E}_1+\mathcal{E}_2+T'\right), \tag{2}$$

$$\mathcal{E}_1 = -\frac{1}{8}\int_0^1 ds\left((\xi^2\mathcal{D}+1)\ e^{\frac{s-2}{8}\mathcal{D}}\Phi\right)\cdot\left(\mathcal{D}\ e^{-\frac{1}{8}s\mathcal{D}}\Phi\right), \tag{3}$$

$$\mathcal{E}_2 = -\frac{1}{8}\int_0^1 ds\left(\partial_t(\xi^2\mathcal{D}+1)\,e^{\frac{s-2}{8}\mathcal{D}}\Phi\right)\cdot\left(\partial_t e^{-\frac{1}{8}s\mathcal{D}}\Phi\right). \tag{4}$$

The non-local energy $\mathcal{E}_1$ plays the role of an extra potential term and $\mathcal{E}_2$ the role of the kinetic term. Note that here we use a dimensionless time $t\to t\sqrt{\alpha'}$.

3 How we study our model and what we get

Equations (1) and (2) form a rather complicated system of nonlinear nonlocal equations for functions Φ and $H(t)$ because of the presence of an infinite number of derivatives and a non-flat metric. Before to discuss the methods of study this model let us mention the known methods of study equation (1) in the flat background, $H = 0$,

$$\left(-\xi^2\partial_t^2 + 1\right) e^{\frac{1}{4}\partial_t^2}\Phi(t) = \Phi(t)^3. \tag{5}$$

Equation (2) in the flat case describes the energy conservation [7].

A boundary problem $\Phi(\pm\infty) = \pm 1$ for (5) has been studied using:

- a numerical method [8] based on an integral representation of (5); it is related with a diffusion equation method [11] which uses an auxiliary function of two variables $\Psi(r,t)$ that is the subject of a linear equation and $\Psi(\frac{1}{4},t) = \Phi(t)$;
- a decomposition on local fields [12, 14, 15, 13]; this method works well for linear equations and has been used to study solutions to (5) near vacuum ± 1;
- existence theorems [17, 18, 19];
- almost exact solutions methods [20, 21]; the approach [20] uses a diffusion equation method.

The following two characteristic properties of (5) have been obtained

- an existence of a critical point $\xi^2_{\mathrm{cr}} \approx 1.38$ such that for $\xi^2 < \xi^2_{\mathrm{cr}}$ eq. (5) has a rolling solution [8] interpolating between ± 1;
- an existence of a dominance of an extra non-local kinetic term $\mathcal{E}_2$ over the local kinetic one [7] and as a result, an appearance of a phantom behavior providing $w < -1$.

These result have been obtained using numerical calculations. It is very interesting to study the problem analytically and also try to find approximate models admitting explicit solutions and having above mentioned properties. They could be two or more components local models.

An investigation of non-flat eqs. (1) and (2) is essentially more complicated. The following methods are used:

- A decomposition on local fields and a modification of the potential have been used in [10, 21, 23, 15]. A simplest one phantom mode approximation with an explicit form of the solution $\phi(t) = \tanh(t)$ is realized for a six-order potential [10] and gives $H_0 = 1/3m_p^2$. Assuming that $M_c \sim M_p$ and $M_s \sim 10^{-6.6}M_p$ we get
$$H_0 \sim 10^{-60}M_p.$$
- an analytic approach that is closely related with the diffusion equation method [24].
- A numerical study has been performed in [9], where the diffusion equation method has been used to define $\exp\mathcal{D}$ and a double-step iteration procedure has been proposed.

The following physical effects are found in [9]

- For $\xi^2 < \xi^2_{\mathrm{cr}} \approx 1.18$ and $\Lambda = \Lambda(\xi)$ the system (1), (2) has a rolling solution.

- For $\xi^2 < \xi^2_{\text{shape}}$ and $t > 0$ the Hubble function $H(t)$ is a function which has small fluctuations about a monotonic function $H_l(t)$ with an asymptotic H_0;
  for $\xi^2_{\text{shape}} < \xi^2 < \xi^2_{\text{cr}}$ $H(t)$ describes fluctuations about a function $H_l(t)$ that has two maximum. To realize this approximated shape $H_l(t)$ by local fields one needs at least two fields [23].

As in the flat case it would be very interesting to find approximate analytical solutions which exhibit these properties. Note that two maximum shape regime for $H(t)$ is interesting in a context of building an unified cosmological evolution. Let us also note that there are applications of non-local SFT models to inflation [25, 26, 27, 28] and cosmological singularity ([15, 16] and refs therein).

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BRST charge for the classical $N = 2$ W_3 superalgebra

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

We explicitly demonstrate that the classical $N = 2$ W_3 algebra with zero central charge admits two bases in which it becomes a quadratic superalgebra. As a consequence, for this superalgebra one may construct two different BRST charges Q_1 and Q_2. These charges obey the same boundary conditions and do not anticommute $\{Q_1, Q_2\} = H, [H, Q_{1,2}] = 0$, while $Q_1^2 = Q_2^2 = 0$.

Preface

It is a pleasure for us to send our contribution to the volume dedicated to the 60-th birthday of Professor Joseph L. Buchbinder. Joseph, being extremely brilliant scientist and very attractive person, has done so much to establish an intensive collaboration between Tomsk and Dubna. We all enjoy this collaboration and hope it will continue... forever and a day! In this respect we consider our contribution as a small step towards join activity with Joseph in this area where he is expert.

1 Introduction

Nowadays, the construction of the BRST charges for some, even nonlinear algebras, seems to be a rather algorithmic procedure. Indeed, in the most cases one may just write down the most general ansatz for the charge and than easily fix all the coefficients. Moreover, for some type of algebras (and even for the quantum Lie algebras) there are quite general results concerning explicit structure of the BRST charges (see e.g. [1]). Nevertheless, among the well known nonlinear superalgebras there is at least one special case - the $N = 2$ W_3 superalgebra

[2, 3] - for which the standard procedure of construction of the BRST charge fails. The problem is that in this case one may construct bosonic currents $\mathcal{B}$ from the bosonic ghost-anti-ghosts currents which accompany the fermionic currents of this superalgebra, such that $\mathcal{B}$ possesses zero conformal weight and zero ghost number. Therefore, the general ansatz for the BRST charge for this superalgebra might contain arbitrary functions depending on $\mathcal{B}$. Clearly, in view of this feature the construction of the BRST charge for $N = 2$ W_3 superalgebra can not be done in the standard fashion.

Of course, the reasonable question is what we expect to understand, providing we know the BRST charge for $N = 2$ W_3 superalgebra? Besides the clarification of the situation with infinite tails in the general ansatz for the BRST charge, having it at hand can help to solve a long standing problem of construction of $N = 4$ W_3 algebra. Indeed, in [4] it was shown that explicit expression for BRST current for Virasoro algebra can be used for indirect construction of $N = 2$ super-Virasoro algebra. Later on these results were extended in [5] to the case of famous W_3 algebra [6]. In this paper the authors showed that the BRST current for W_3 algebra [7] can be combined with the proper combinations of the ghost-anti-ghost currents and the currents of W_3 algebra to close on $N = 2$ W_3 algebra. Clearly, to make the next step one needs to know the BRST current for $N = 2$ W_3 algebra. Then repeating all steps of [5] one may hope to reconstruct in this way $N = 4$ W_3 algebra. It is worth noting that to the best of our knowledge all previous attempts to construct $N = 4$ W_3 algebra failed.

In this Letter we present our first results about explicit structure of the classical BRST charge for the classical $N = 2$ W_3 algebra. The full quantum $N = 2$ W_3 algebra is rather complicated. That is why we decided firstly to attack its simpler classical version. Even in this simplest case the general ansatz for BRST current still contains infinite tails in the currents $\mathcal{B}$. So, our next simplification is just to cut all these terms at all. With all these assumptions the ansatz for BRST charge contains 60 terms and the calculations become manageable. We also demonstrate (Section 2) that the classical $N = 2$ W_3 algebra with zero central charge admits two bases in which it becomes a quadratic superalgebra, which slightly simplifies the life. Our main result (Section 3) is that for the classical $N = 2$ W_3 algebra there are two different BRST charges $\tilde{Q}_1$ and $\tilde{Q}_2$, which do not anticommute ($\left\{\tilde{Q}_1, \tilde{Q}_2\right\} = \tilde{H}$, $\left[\tilde{H}, \tilde{Q}_{1,2}\right] = 0, \tilde{Q}_1^2 = 0, \tilde{Q}_2^2 = 0$). We conclude our Letter with a short discussion of this situation.

2 Classical $N = 2$ W_3 algebra with zero central charge

The basic relations defining the classical $N = 2$ W_3 algebra are obtained in [2] in terms of components and in [3] in terms of $N = 2$ superfields. As usually, to construct the classical BRST charge we need a version of the algebra with zero central charge. In the case of nonlinear algebras one can not immediately put the central charge equal to zero due to some features of the nonliner algebras which can be summarized as follows:

- In all known nonlinear algebras with non-zero central chargs c the structure of the terms in the r.h.s. of OPE's schematically reads

$$c + (linear\ terms) + \frac{1}{c}(quadratic\ terms) + \frac{1}{c^2}(qubic\ terms) + \text{etc}$$

- To reach the limit $c = 0$ one has to re-scale some of the currents in the algebra
- After a proper rescaling of the currents in OPE's which contain nonlinear terms only such terms survive. All linear terms in the such OPE's will disappear after rescaling.

In view of these features the structure of $N=2$ W_3 algebra with $c=0$ is much simpler than that of $N=2$ W_3 with an arbitrary central charge. The explicit relations describing $N=2$ W_3 algebra with $c=0$ have been obtained in [3]. Here we will reconstruct these relations pointing reader's attention to some specific peculiarities of this superalgebra which are crucial for the construction of the BRST charge.

To be an $N=2$ supersymmetric extension of W_3, superalgebra should contain two $N=2$ supercurrents $J(Z)$ and $T(Z)$ with conformal spins 1 and 2, respectively.[1] The most general variant of the Superfield Operator Product Expansions (SOPE's) for these currents reads

$$\begin{aligned}
J(Z_1)J(Z_2) &= \frac{\theta_{12}\bar\theta_{12}J}{Z_{12}^2} + \frac{\theta_{12}\bar\theta_{12}\partial J + \bar\theta_{12}\overline{D}J - \theta_{12}DJ}{Z_{12}}, \\
J(Z_1)T(Z_2) &= 2\frac{\theta_{12}\bar\theta_{12}T}{Z_{12}^2} + \frac{\theta_{12}\bar\theta_{12}\partial T + \bar\theta_{12}\overline{D}T - \theta_{12}DT}{Z_{12}}, \\
T(Z_1)T(Z_2) &= \frac{\theta_{12}\bar\theta_{12}U}{Z_{12}^2} + \frac{\frac{1}{2}\theta_{12}\bar\theta_{12}\left(\partial U + \overline{D}\Psi + D\overline{\Psi}\right) + \theta_{12}\Psi - \bar\theta_{12}\overline{\Psi}}{Z_{12}},
\end{aligned} \tag{1}$$

where

$$\theta_{12} = \theta_1 - \theta_2,\ \bar\theta_{12} = \bar\theta_1 - \bar\theta_2, \quad Z_{12} = z_1 - z_2 + \frac{1}{2}\left(\theta_1\bar\theta_2 - \theta_2\bar\theta_1\right), \tag{2}$$

$D, \overline{D}$ are spinor covariant derivatives obeying

$$\left\{D, \overline{D}\right\} = -\partial, \tag{3}$$

and the composite supercurrents are defined as

$$\begin{aligned}
&U = a_1 JT + a_2 J^3, \\
&\Psi = a_3 JDT + a_4 TDJ - a_5 J^2 DJ, \quad \overline{\Psi} = a_3 J\overline{D}T + a_4\overline{D}J - a_5 J^2\overline{D}J.
\end{aligned} \tag{4}$$

Here, $a_1, \ldots, a_5$ are arbitrary, for the time being, coefficients. As one may see from (1), the supercurrent J forms $N=2$ superconformal algebra, while T transforms as a primary, spin-2 supercurrent under this superconformal algebra. The Jacobi identities further restrict the coefficients $a_1, \ldots, a_5$ as follows

$$a_2 = a_5 = \frac{a_4^2 - a_1^2}{16}, \qquad a_3 = -\frac{a_1 + a_4}{2}. \tag{5}$$

One might decide that we are dealing with the two-parameter set of superalgebras. This conclusion is wrong due to the following arguments.

First of all, one may redefine the spin 2 supercurrent T as follows

$$T_N = T + \alpha J^2. \tag{6}$$

The SOPE of the newly defined current T_N with the spin 1 current J remains the same as in (1), while the SOPE of T_N with itself reads as

$$T_N(Z_1)T_N(Z_2) = \frac{\theta_{12}\bar\theta_{12}}{Z_{12}^2}\left[\left(a_1 + 8\alpha\right)JT_N + \frac{a_4^2 - (a_1 + 8\alpha)^2}{16}J^3\right] + \text{ First order poles} \tag{7}$$

Clearly, one can fix this freedom to choose $\alpha = -\frac{a_1}{8}$ to remove the term JT_N from the second pole in (7). This choice is equivalent to putting $a_1 = 0$ in (4),(5) because in r.h.s. of (7) the coefficient a_1 appears only in the combination $(a_1 + 8\alpha)$.

[1]By Z we denote the coordinates of $N=2, d=1$ superspace, $Z = (z, \theta, \bar\theta)$.

Finally, one may rescale the supercurrent $T \to \beta T$ in (1) to completely fix the last coefficient a_4. So, this coefficient in (1) is unessential if it is not equal to zero. In what follows we will choose $a_4 = 8$ to have the simplest SOPE's.

Thus, we conclude that there is only one nonlinear $N = 2$ W_3 algebra with zero central charge which has the following SOPE's

$$
\begin{aligned}
J(Z_1)J(Z_2) &= \frac{\theta_{12}\bar\theta_{12}J}{Z_{12}^2} + \frac{\theta_{12}\bar\theta_{12}\partial J + \bar\theta_{12}\overline{D}J - \theta_{12}DJ}{Z_{12}}, \\
J(Z_1)T(Z_2) &= 2\frac{\theta_{12}\bar\theta_{12}T}{Z_{12}^2} + \frac{\theta_{12}\bar\theta_{12}\partial T + \bar\theta_{12}\overline{D}T - \theta_{12}DT}{Z_{12}}, \\
T(Z_1)T(Z_2) &= 4\frac{\theta_{12}\bar\theta_{12}}{Z_{12}^2}J^3 + \frac{\theta_{12}}{Z_{12}}\left[8DJ\,T - 4J\,DT - 4J^2\,DJ\right] \\
&\quad -\frac{\bar\theta_{12}}{Z_{12}}\left[8\overline{D}J\,T - 4J\,\overline{D}T - 4J^2\,\overline{D}J\right] \\
&\quad +\frac{\theta_{12}\bar\theta_{12}}{Z_{12}}\left[2J\partial T - 4\partial JT - 6\overline{D}JDT - 6DJ\overline{D}T + 8\partial JJ^2\right].
\end{aligned} \tag{8}
$$

With our choice of the parameters a_1, a_4 the algebra (8) contains cubic terms in the SOPE's. It is important to note that there is another possibility to fix these coefficients: $a_1 = \pm 8, a_4 = 8$. With these choice the cubic terms in $(U, \Psi, \overline{\Psi})$ disappear and the superalgebra becomes quadratic one. One may check that these two cases with $a_1 = a_4 = 8$ and $a_1 = -a_4 = -8$ are related with the currents in (8) by a redefinition of the supercurrent T as follows $T_1 = T - J^2$ and $T_2 = T + J^2$. Indeed, with $a_1 = 0, a_4 = 8$ the r.h.s. in (7) does not contain the cubic terms if $\alpha = \pm 1$. Thus our cubic superalgebra (8) has two different "faces" where it shows up as the quadratic algebra.

The existence of two "quadratic faces" of $N = 2$ W_3 superalgebra immediately gives rise to the following problem. It is well known that the BRST charges for quadratically nonlinear algebras contain all currents only linearly. Therefore, in a first "face" the BRST charge should look like $Q_1 = (\text{ ghosts })T_1 + \ldots$ while in the other "face" one may find the BRST charge which reads as $Q_2 = (\text{ ghosts })T_2 + \ldots$. Clearly, these charges being rewritten in the "central" basis (8) where superalgebra is cubic in the supercurrents, are realized as *two different* BRST charges. Thus we conclude that either BRST charge for the classical $N = 2$ W_3 superalgebra with zero central charge does not exist, either we have two of such charges. In the next Section we will demonstrate that the classical $N = 2$ W_3 superalgebra indeed has two BRST charges.

3 Classical BRST charge for $N = 2$ W_3 algebra

To construct the BRST charge for the $N = 2$ W_3 superalgebra (8) one has to introduce the corresponding ghost-anti-ghost $N = 2$ fermionic supercurrents $\{C_j, B_j, C_t, B_t\}$ which obey the standard SOPE's

$$
B_j(Z_1)C_j(Z_2) = \frac{\theta_{12}\bar\theta_{12}}{Z_{12}}, \quad B_t(Z_1)C_t(Z_2) = \frac{\theta_{12}\bar\theta_{12}}{Z_{12}}. \tag{9}
$$

These supercurrents contain all ghost-anti-ghost currents for four bosonic and four fermionic currents in the superalgebra $N = 2$ W_3 among their components.

The improved stress tensor $\tilde J$ has the following form

$$
\tilde J = J + \overline{D}B_jDC_j + DB_j\overline{D}C_j - \partial\left(B_jC_j\right) + \overline{D}B_tDC_t + DB_t\overline{D}C_t - 2\partial\left(B_tC_t\right). \tag{10}
$$

One may check that the supercurrent $\tilde{J}$ has the same OPE with itself as the supercurrent J in (1) and therefore it spans the $N=2$ Virasoro algebra.

With respect to $\tilde{J}$ the ghost supercurrents $\mathcal{A}=\{C_j, B_j, C_t, B_t\}$ transform as primary $N=2$ supercurrents

$$\tilde{J}(Z_1)\mathcal{A}(Z_2) = h_{\mathcal{A}}\frac{\theta_{12}\bar{\theta}_{12}\mathcal{A}}{Z_{12}^2} + \frac{\theta_{12}\bar{\theta}_{12}\partial\mathcal{A} + \bar{\theta}_{12}\overline{D}\mathcal{A} - \theta_{12}D\mathcal{A}}{Z_{12}} \tag{11}$$

with the conformal weights $h_{\mathcal{A}}=\{-1,1,-2,2\}$, respectively.

One of the unpleasant features of the ghost supercurrents for $N=2$ W_3 superalgebra is the possibility to construct the following two combinations

$$X_1 = DC_t\overline{D}B_j, \qquad X_2 = \overline{D}C_tDB_j \ . \tag{12}$$

Both X_1 and X_2 are bosonic supercurrents with zero conformal weights and zero ghost numbers. Therefore, the ansatz for the BRST charge may contain arbitrary functions depending on these combinations of the supercurrents. This is the main problem in the algorithmic construction of BRST charge for $N=2$ W_3 superalgebra.

To overcome this problem we decide to choose an ansatz for the BRST charge which does not contain terms with the supercurrents C_t and B_j simultaneously. So, our ansatz for the BRST current schematically reads

$$\begin{aligned} Q \;=\; & C_jJ + C_tT + JB_tC_jC_t + C_tJJ + (\partial, D\overline{D})\cdot(JB_tC_tC_t) + (\partial, D\overline{D})\cdot(B_jC_jC_j) + \\ & (\partial, D\overline{D})\cdot(B_tC_tC_j) + (\partial, D\overline{D})^2\cdot(B_tC_tC_t) + \\ & (\partial, D\overline{D})^2\cdot(B_tB_tC_tC_tC_t) + (\partial, D\overline{D})\cdot(B_tB_tC_jC_tC_t), \end{aligned} \tag{13}$$

where the derivatives $(\partial, D\overline{D})$ are freely distributed in all possible ways among the currents in the brackets. Moreover, we try to find the BRST currents instead of BRST charges. In other words, we look for a such set of the coefficients in (13) to have the regular SOPE of the Q with itself.

After tedious, but straightforward calculations[2] we find two different solutions for the BRST currents

$$\begin{aligned} Q_1 \;=\; & C_jJ + C_tT - C_tJ^2 + 4JB_tC_t'C_t - 4\overline{D}JB_tC_tDC_t - 4DJB_tC_t\overline{D}C_t \\ & -B_j\overline{D}C_jDC_j - C_jB_tC_t' + C_j\overline{D}B_tDC_t + C_jDB_t\overline{D}C_t - C_jB_t'C_t + \overline{D}B_jC_jDC_j \\ & +DB_jC_j\overline{D}C_j - \overline{D}C_jB_tDC_t + \overline{D}C_jDB_tC_t - DC_jB_t\overline{D}C_t + DC_j\overline{D}B_tC_t \\ & -2B_t\overline{D}B_tC_t[D,\overline{D}]C_tDC_t - 2B_t\overline{D}B_tC_t'C_tDC_t \\ & +2B_tDB_tC_t\overline{D}C_t[D,\overline{D}]C_t - 2B_tDB_tC_t'C_t\overline{D}C_t \\ & +2\overline{D}B_t\overline{D}B_tC_tDC_tDC_t - 4\overline{D}B_tDB_tC_t\overline{D}C_tDC_t \\ & +2DB_tDB_tC_t\overline{D}C_t\overline{D}C_t - 4B_t'B_tC_t\overline{D}C_tDC_t \end{aligned} \tag{14}$$

[2]To find the BRST charge we heavily used the *Mathematica*^{TM} SOPEN2 package [8].

and

$$
\begin{aligned}
Q_2 \;=\;& C_j J + C_t T + C_t J^2 - 8JB_t\overline{D}C_t DC_t + 8J\overline{D}B_t C_t DC_t + 8JDB_t C_t \overline{D}C_t \\
& - B_j\overline{D}C_j DC_j - C_j B_t C_t' + DB_j C_j \overline{D}C_j + \overline{D}B_j C_j DC_j + C_j\overline{D}B_t DC_t \\
& + C_j DB_t \overline{D}C_t - C_j B_t' C_t - \overline{D}C_j B_t DC_t + \overline{D}C_j DB_t C_t - DC_j B_t \overline{D}C_t + DC_j \overline{D}B_t C_t \\
& + 8\overline{D}B_t\overline{D}B_t C_t DC_t DC_t + 8\overline{D}B_t DB_t C_t \overline{D}C_t DC_t + 8DB_t DB_t C_t \overline{D}C_t \overline{D}C_t \\
& - 8B_t' B_t C_t \overline{D}C_t DC_t - 4B_t\overline{D}B_t C_t [D,\overline{D}]C_t DC_t - 4B_t\overline{D}B_t C_t' C_t DC_t \\
& + 4B_t DB_t C_t \overline{D}C_t [D,\overline{D}]C_t - 4B_t DB_t C_t' C_t \overline{D}C_t - 4B_t\overline{D}B_t C_t' C_t DC_t \\
& - 8B_t\overline{D}B_t\overline{D}C_t DC_t DC_t - 8B_t DB_t \overline{D}C_t \overline{D}C_t DC_t \,. \qquad (15)
\end{aligned}
$$

As one may see, in the basics where the superalgebra is quadratic, one of these BRST currents becomes linear in the supercurrents $(J, T_1 = T - J^2)$ and $(J, T_2 = T + J^2)$, respectively as it should be.

It is quite unexpected that neither BRST currents Q_1, Q_2 nor BRST charges $\tilde{Q}_{1,2} = \frac{1}{2\pi i}\int dz d^2\theta\, Q_{1,2}$ do not anticommute. The resulting charge

$$
\tilde{H} = \frac{1}{2\pi i}\int dz d^2\theta\, H = \left\{\tilde{Q}_1, \tilde{Q}_2\right\}
$$

is rather complicated. Schematically it has the following form

$$
\begin{aligned}
H \;=\;& \frac{16}{3}J^2\left(J + 3(\overline{D}B_t DC_t + DB_t\overline{D}C_t)\right)\left(2C_t' C_t - \overline{D}C_t DC_t\right) + 32J^2(B_t C_t)'\overline{D}C_t DC_t + \\
& + (\text{linear in } J \text{ terms}) + (\text{ ghosts terms }). \qquad (16)
\end{aligned}
$$

We also checked that

$$
\left[\tilde{H}, \tilde{Q}_1\right] = 0, \qquad \left[\tilde{H}, \tilde{Q}_2\right] = 0. \qquad (17)
$$

Thus, the charges $(\tilde{H}, \tilde{Q}_1, \tilde{Q}_2)$ form the algebra of $N = 2$ supersymmetric mechanics but the meaning of this fact is unclear for us.

4 Conclusion

In this Letter we constructed two different BRST charges for the classical $N = 2$ W_3 superalgebra. These charges do not anticommute, but together with their anticommutator form the algebra of $N = 2$ supersymmetric mechanics. The main reason for the existence of two different BRST charges is that $N = 2$ W_3 superalgebra possesses two different "aces" where it acquires the quadratic form.

As the immediate applications of our results one may try to construct (at least classical) $N = 4$ W_3 superalgebra along the line suggesting in [5]. Being an extremely interesting (for us), this task is not so important as the full analysis of the gift with two BRST charges. Of course, the $N = 2$ W_3 superalgebra is not a very suitable case for such an analysis due to its rather complicated structure. In this respect it seems reasonable to find a simpler example of an algebra, maybe even finite-dimensional one, having two BRST charges. The main feature this algebra has to possess is an existence of two "faces" in which it becomes a quadratic nonlinear one. Any case, the simultaneous existence of two different BRST charges raises too many questions (the simplest evident question is how to define the physical states in this case?) which have yet to be fully clarified.

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Even and odd geometries on supermanifolds

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Abstract

We analyze from a general perspective all possible supersymmetric generalizations of symplectic and metric structures on smooth manifolds. There are two different types of structures according to the even/odd character of the corresponding quadratic tensors. In general we can have even/odd symplectic supermanifolds, Fedosov supermanifolds and Riemannian supermanifolds. The geometry of even Fedosov supermanifolds is strongly constrained and has to be flat. In the odd case, the scalar curvature is only constrained by Bianchi identities. However, we show that odd Riemannian supermanifolds can only have constant scalar curvature. We also point out that the supersymmetric generalizations of AdS space do not exist in the odd case.

1 Introduction

The two main quadratic geometrical structures of smooth manifolds which play a significant role in classical and quantum physics are Riemannian metrics and symplectic forms. Riemannian geometry is not only basic for the formulation of general relativity but also for the very formulation of gauge field theories. The symplectic structure provides the geometrical framework for classical mechanics (see, e.g. [1]) and field theories [2]. The Fedosov method of quantization by deformation [3] is also formulated in terms of symplectic structures and symplectic connections (the so-called Fedosov manifolds [4]). The introduction of the concept of supermanifold by Berezin [5] (see also [6, 7]) opened new perspectives for geometrical approaches of supergravity and quantization of gauge theories [8, 9, 10]. In summary, the geometry of manifolds and supermanifolds percolates all fundamental physical theories.

In this note we address the classification of possible extensions of symplectic and metric structures to supermanifolds in terms of graded symmetric and antisymmetric second-order

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tensor fields. The cases of even and odd symplectic and Riemannian supermanifolds are analyzed in some detail. Graded non-degenerate Poisson supermanifolds are described by symplectic supermanifolds that if equipped with a symmetric symplectic connection become graded Fedosov supermanifolds. The even case corresponds to a straightforward generalization of Fedosov manifold [4] where the scalar curvature vanishes as for standard Fedosov manifolds. Graded metric supermanifolds equipped with the unique compatible symmetric connection also correspond to graded Riemannian supermanifold. The scalar curvature is non trivial, in general, for odd Riemannian and Fedosov supermanifolds, but in the first case it must always be constant. There is a supersymmetric generalization of AdS space but it is trivial in the odd case.

The paper is organized as follows. In Sect. 2, we consider scalar structures which can be used for the construction of symplectic and metric supermanifolds. The properties of symmetric affine connections on supermanifolds and their curvature tensors are analyzed in Sect. 3. In Sect. 4, we introduce the concepts of even and odd Fedosov supermanifolds and even and odd Riemannian supermanifolds are analyzed in Sect. 5. Finally, we convey the main results in Sect. 6. We use the condensed notation suggested by DeWitt [11] and definitions and notations adopted in [12].

2 Scalar Fields

Let $\mathcal{M}$ be a supermanifold with a dimension $dim\mathcal{M} = N$ and $\{x^i\}, \epsilon(x^i) = \epsilon_i$ a local system of coordinates on in the vicinity of a point $p \in \mathcal{M}$. Let us consider now the most general scalar structures on supermanifolds which can be defined in terms of graded second-rank symmetric and antisymmetric tensor fields.

In general, there exist eight types of second rank tensor fields with the required symmetry properties

$$\omega^{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega^{ji}, \quad \epsilon(\omega^{ij}) = \epsilon(\omega) + \epsilon_i + \epsilon_j, \tag{1}$$
$$\Omega^{ij} = (-1)^{\epsilon_i\epsilon_j}\Omega^{ji}, \quad \epsilon(\Omega^{ij}) = \epsilon(\Omega) + \epsilon_i + \epsilon_j, \tag{2}$$
$$E_{ij} = -(-1)^{\epsilon_i\epsilon_j}E_{ji}, \quad \epsilon(E_{ij}) = \epsilon(E) + \epsilon_i + \epsilon_j, \tag{3}$$
$$g_{ij} = (-1)^{\epsilon_i\epsilon_j}g_{ji}, \quad \epsilon(g_{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j. \tag{4}$$

Using these tensor fields (1)-(4) it is not difficult to built eight scalar structures on a supermanifold:

$$\{A, B\} = \frac{\partial_r A}{\partial x^i}(-1)^{\epsilon_i\epsilon(\omega)}\omega^{ij}\frac{\partial B}{\partial x^j}, \quad \epsilon(\{A, B\}) = \epsilon(\omega) + \epsilon(A) + \epsilon(B), \tag{5}$$
$$(A, B) = \frac{\partial_r A}{\partial x^i}(-1)^{\epsilon_i\epsilon(\Omega)}\Omega^{ij}\frac{\partial B}{\partial x^j}, \quad \epsilon((A, B)) = \epsilon(\Omega) + \epsilon(A) + \epsilon(B), \tag{6}$$
$$E = E_{ij}dx^j \wedge dx^i, \quad \epsilon(E_{ij}dx^j \wedge dx^i) = \epsilon(E), \tag{7}$$
$$g = g_{ij}dx^j\, dx^i, \quad \epsilon(g_{ij}dx^j\, dx^i) = \epsilon(g), \tag{8}$$

where A and B are arbitrary superfunctions.

The bilinear operation $\{A, B\}$ (5) obeys the following symmetry property

$$\{A, B\} = -(-1)^{\epsilon(\omega)+(\epsilon(A)+\epsilon(\omega))(\epsilon(B)+\epsilon(\omega))}\{B, A\} \tag{9}$$

which in the even case ($\epsilon(\omega) = 0$) reduces to

$$\{A, B\} = -(-1)^{(\epsilon(A)\epsilon(B)}\{B, A\} \tag{10}$$

and in the odd case ($\epsilon(\omega)=1$) to

$$\{A,B\}=(-1)^{(\epsilon(A)+1))(\epsilon(B)+1)}\{B,A\}. \tag{11}$$

On the other hand, the bilinear operation (A,B) (6) has the symmetry property

$$(A,B)=(-1)^{\epsilon(\omega)+(\epsilon(A)+\epsilon(\omega))(\epsilon(B)+\epsilon(\omega))}(B,A) \tag{12}$$

which in the even case ($\epsilon(\omega)=0$) reduces to

$$(A,B)=(-1)^{\epsilon(A)\epsilon(B)}(B,A) \tag{13}$$

and in the odd case ($\epsilon(\omega)=1$) to

$$(A,B)=-(-1)^{(\epsilon(A)+1))(\epsilon(B)+1)}(B,A). \tag{14}$$

One can easily check that in the even case ($\epsilon(\omega)=0$) the bilinear operation $\{A,B\}$ satisfies the Jacobi identity

$$\{A,\{B,C\}\}(-1)^{\epsilon(A)(\epsilon(C)}+\{C,\{A,B\}\}(-1)^{\epsilon(C)(\epsilon(B)}+\{B,\{C,A\}\}(-1)^{\epsilon(B)(\epsilon(A)}\equiv 0 \tag{15}$$

if and only if ω satisfies

$$\omega^{ij}\frac{\partial\omega^{kl}}{\partial x^j}(-1)^{\epsilon_i\epsilon_l}+\omega^{lj}\frac{\partial\omega^{ik}}{\partial x^j}(-1)^{\epsilon_l\epsilon_k}+\omega^{kj}\frac{\partial\omega^{li}}{\partial x^j}(-1)^{\epsilon_k\epsilon_i}\equiv 0. \tag{16}$$

In the odd case there is no possibility of satisfying the Jacobi identity for the operation $\{A,B\}$.

On the contrary, the Jacobi's identity for (A,B) can be satisfied

$$\begin{aligned}(A,(B,C))(-1)^{(\epsilon(A)+1)(\epsilon(C)+1)} &+ (C,(A,B))(-1)^{(\epsilon(C)+1)(\epsilon(B)+1)}+\\ &+ (B,(C,A))(-1)^{(\epsilon(B)+1)(\epsilon(A)+1)}\equiv 0\end{aligned} \tag{17}$$

if and only if Ω is odd, $\epsilon(\Omega)=1$, and satisfies

$$\Omega^{ij}\frac{\partial\Omega^{kl}}{\partial x^j}(-1)^{\epsilon_i(\epsilon_l+1)}+\Omega^{lj}\frac{\partial\Omega^{ik}}{\partial x^j}(-1)^{\epsilon_l(\epsilon_k+1)}+\Omega^{kj}\frac{\partial\Omega^{li}}{\partial x^j}(-1)^{\epsilon_k(\epsilon_i+1)}\equiv 0. \tag{18}$$

Therefore, because of the identities (16) and (18), one can identify $\{A,B\}$ ($\epsilon(\{A,B\})=\epsilon(A)+\epsilon(B)$) and (A,B) ($\epsilon((A,B))=\epsilon(A)+\epsilon(B)+1$) with the Poisson bracket and the antibracket respectively.

It is also possible to combine the Poisson bracket associated to ω and the antibracket into the so-called graded Poisson bracket (see, for example, [13, 14, 15, 16]) in the following bilinear operation

$$\begin{aligned}\{A,B\}_g &= \frac{\partial_r A}{\partial x^i}(-1)^{\epsilon_i\epsilon(\omega_g)}\omega_g^{ij}\frac{\partial B}{\partial x^j},\quad \omega_g^{ij}=-(-1)^{\epsilon(\omega_g+\epsilon_i\epsilon_j)}\omega_g^{ji},\\ &\epsilon(\{A,B\}_g)=\epsilon(\omega_g)+\epsilon(A)+\epsilon(B).\end{aligned} \tag{19}$$

From (20) it follows the symmetry property

$$\{A,B\}_g=-(-1)^{(\epsilon(A)+\epsilon(\omega_g))(\epsilon(B)+\epsilon(\omega_g))}\{B,A\}_g. \tag{20}$$

If the tensor fields ω^{ij} satisfy the identities

$$\omega_g^{ij}\frac{\partial\omega_g^{kl}}{\partial x^j}(-1)^{\epsilon_i(\epsilon_l+\epsilon(\omega_g))}+\omega_g^{lj}\frac{\partial\omega_g^{ik}}{\partial x^j}(-1)^{\epsilon_l(\epsilon_k+\epsilon(\omega_g))}+\omega_g^{kj}\frac{\partial\omega_g^{li}}{\partial x^j}(-1)^{\epsilon_k(\epsilon_i+\epsilon(\omega_g))}\equiv 0, \qquad (21)$$

then $\{A,B\}_g$ satisfies the Jacobi identity

$$\{A,\{B,C\}_g\}_g(-1)^{\epsilon_g(A,B.C)} + \{C,\{A,B\}_g\}_g(-1)^{\epsilon_g(B,C,A)} + \qquad (22)$$

$$+ \{B,\{C,A\}_g\}_g(-1)^{\epsilon_g(C,A,B)}\equiv 0 \qquad (23)$$

with $\epsilon_g(A,B,C) = (\epsilon(A)+\epsilon(\omega_g))(\epsilon(C)+\epsilon(\omega_g))$ and plays the role of a graded Poisson bracket.

A supermanifold $\mathcal{M}$ equipped with a Poisson bracket is called a Poisson supermanifold, $(\mathcal{M},\{,\})$. Usually a manifold $\mathcal{M}$ equipped with an non-degenerate antibracket is called an antisymplectic supermanifold $(\mathcal{M},(,))$ or, sometimes, an odd Poisson supermanifold (see, for example, [15, 16]).

In Eq. (3) E denotes a generic graded differential 2-form. If E is closed

$$dE = E_{ij,k}dx^k\wedge dx^j\wedge dx^i = 0 \qquad (24)$$

and non-degenerate, then it defines a graded (even or odd) symplectic supermanifold $(\mathcal{M},E)$ [6]. In terms of tensor fields E_{ij} the condition (24) can be expressed as

$$E_{ij,k}(-1)^{\epsilon_i\epsilon_k}+E_{jk,i}(-1)^{\epsilon_j\epsilon_i}+E_{ki,j}(-1)^{\epsilon_k\epsilon_j}=0,\quad E_{ij}=-(-1)^{\epsilon_i\epsilon_j}E_{ji} \qquad (25)$$

and in terms of inverse tensor fields E^{ij} Eqs. (25) can be rewritten in the form

$$E^{il}\frac{\partial E^{jk}}{\partial x^l}(-1)^{\epsilon_i(\epsilon_k+\epsilon(E))}+E^{kl}\frac{\partial E^{ij}}{\partial x^l}(-1)^{\epsilon_k(\epsilon_j+\epsilon(E))}+E^{jl}\frac{\partial E^{ki}}{\partial x^l}(-1)^{\epsilon_j(\epsilon_i+\epsilon(E))}=0, \qquad (26)$$

where $E^{ij} = -(-1)^{\epsilon(E)+\epsilon_i\epsilon_j}E^{ji}$. Identifying E^{ij} with the tensor field ω^{ij} in (5), one gets in the even case ($\epsilon(E)=0$) the Poisson bracket for which the Jacobi identity (15) follows from (26). Therefore, in the even case there is one-to-one correspondence between non-degenerate Poisson supermanifolds and an even symplectic supermanifolds. In the odd case ($\epsilon(E)=1$), if we assume $E^{ij}=\Omega^{ij}$ in (6) then E^{ij} defines an antibracket for which the Jacobi identity (18) follows from (26). Therefore antisymplectic supermanifolds can be identified with odd symplectic manifolds.

If the tensor field g_{ij} in (8) is non-degenerate, one has a graded metric that can provide a supermanifold $\mathcal{M}$ with a graded (even or odd) metric structure, giving rise to a Riemannian supermanifold $(\mathcal{M},g)$. On the other hand, the inverse tensor field g^{ij} also defines a bilinear operation with symmetry properties (11) or (13) but it does not satisfy the Jacobi identity.

3 Connections in Supermanifolds

Let us consider a covariant derivative ∇ (or an affine connection Γ) on a supermanifold $\mathcal{M}$. In each local coordinate system $\{x\}$ the covariant derivative ∇ is described by its components ∇_i $(\epsilon(\nabla_i)=\epsilon_i)$, which are related to the components the affine connection Γ $\Gamma^i{}_{jk}$, $(\epsilon(\Gamma^i{}_{jk}) = \epsilon_i+\epsilon_j+\epsilon_k)$ by

$$e^i\nabla_j = e^k\Gamma^i{}_{kj}(-1)^{\epsilon_k(\epsilon_i+1)},\quad e_i\nabla_j = -e_k\Gamma^k{}_{ij} \qquad (27)$$

where $\{e_i\}$ and $\{e^i\}$ are the associated bases of the tangent $T\mathcal{M}$ and cotangent $T^*\mathcal{M}$ spaces respectively. The action of the covariant derivative on a tensor field of any rank and type is given in terms of the tensor components, the ordinary derivatives and the connection components (for details see [12]). From here on, we shall consider only symmetric connections

$$\Gamma^i{}_{jk} = (-1)^{\epsilon_j\epsilon_k}\Gamma^i{}_{kj}. \tag{28}$$

The curvature tensor field $R^i{}_{mjk}$ is defined in terms of the commutator of covariant derivatives, $[\nabla_i, \nabla_j] = \nabla_i\nabla_j - (-1)^{\epsilon_i\epsilon_j}\nabla_j\nabla_i$, whose action on a vector field T^i is

$$T^i[\nabla_j, \nabla_k] = -(-1)^{\epsilon_m(\epsilon_i+1)}T^m R^i{}_{mjk}. \tag{29}$$

The choice of factor in r.h.s (29) is dictated by the requirement that the contraction of tensor fields of types $(1,0)$ and $(1,3)$ yield a tensor field of type $(1,2)$. A straightforward calculation yields

$$R^i{}_{mjk} = -\Gamma^i{}_{mj,k} + \Gamma^i{}_{mk,j}(-1)^{\epsilon_j\epsilon_k} + \Gamma^i{}_{jn}\Gamma^n{}_{mk}(-1)^{\epsilon_j\epsilon_m} - \Gamma^i{}_{kn}\Gamma^n{}_{mj}(-1)^{\epsilon_k(\epsilon_m+\epsilon_j)}. \tag{30}$$

The curvature tensor field has a generalized antisymmetry,

$$R^i{}_{mjk} = -(-1)^{\epsilon_j\epsilon_k}R^i{}_{mkj}\,; \tag{31}$$

and satisfies the Jacobi identity,

$$(-1)^{\epsilon_m\epsilon_k}R^i{}_{mjk} + (-1)^{\epsilon_j\epsilon_m}R^i{}_{jkm} + (-1)^{\epsilon_k\epsilon_j}R^i{}_{kmj} \equiv 0\,. \tag{32}$$

Using the Jacobi identity for the covariant derivatives,

$$[\nabla_i,[\nabla_j,\nabla_k]](-1)^{\epsilon_i\epsilon_k} + [\nabla_k,[\nabla_i,\nabla_j]](-1)^{\epsilon_k\epsilon_j} + [\nabla_j,[\nabla_k,\nabla_i]](-1)^{\epsilon_i\epsilon_j} \equiv 0\,, \tag{33}$$

one obtains the Bianchi identity,

$$(-1)^{\epsilon_i\epsilon_j}R^n{}_{mjk;i} + (-1)^{\epsilon_i\epsilon_k}R^n{}_{mij;k} + (-1)^{\epsilon_k\epsilon_j}R^n{}_{mki;j} \equiv 0\,, \tag{34}$$

with the notation $R^n{}_{mjk;i} := R^n{}_{mjk}\nabla_i$.

4 Symplectic supermanifolds

Let us consider a symplectic supermanifold $(\mathcal{M}, \omega)$, i.e. a supermanifold $\mathcal{M}$ with a closed non-degenerate graded differential 2-form ω

$$\omega = \omega_{ij}dx^j \wedge dx^i, \quad \omega_{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega_{ji}. \tag{35}$$

The closure condition of ω, $d\omega = 0$, can be rewritten as

$$\omega_{ij,k}(-1)^{\epsilon_i\epsilon_k} + \omega_{jk,i}(-1)^{\epsilon_i\epsilon_j} + \omega_{ki,j}(-1)^{\epsilon_j\epsilon_k} = 0 \tag{36}$$

in terms of the inverse tensor field ω^{ij}

$$\omega^{ij} = -(-1)^{\epsilon(\omega)+\epsilon_i\epsilon_j}\omega^{ji} \tag{37}$$

and do coincide with identities (21). It means that in the even case ($\epsilon(\omega) = 0$) ω^{ij} defines a nondegenerate Poisson bracket while in the odd case ($\epsilon(\omega) = 1$) it defines an antibracket.

Therefore in the even case there is a one-to-one correspondence between even symplectic supermanifolds and nondegenerate Poisson supermanifold. In the odd case any antisymplectic supermanifold is nothing but an odd symplectic supermanifold.

Let Γ be a symmetric connection of a symplectic supermanifold $(\mathcal{M}, \omega)$. The corresponding covariant derivative ∇ has to verify the compatibility condition $\omega\nabla = 0$ with the symplectic structure ω. In each local coordinate system $\{x^i\}$ the compatibility condition can be expressed as

$$\omega_{ij}\nabla_k = \omega_{ij,k} - \Gamma_{ijk} + \Gamma_{jik}(-1)^{\epsilon_i\epsilon_j} = 0, \quad \omega_{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega_{ji} \tag{38}$$

in terms of the components $\Gamma^i_{\ jk}$ (∇_i) of the symplectic connection ∇, where we use the notation

$$\Gamma_{ijk} = \omega_{in}\Gamma^n_{\ jk}, \quad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k\,. \tag{39}$$

A symplectic supermanifold $(\mathcal{M}, \omega)$ equipped with a symmetric symplectic connection Γ is called a Fedosov supermanifold $(\mathcal{M}, \omega, \Gamma)$.

Let us consider now curvature tensor R_{ijkl} of a symplectic connection

$$R_{ijkl} = \omega_{in}R^n_{\ jkl}, \quad \epsilon(R_{ijkl}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \tag{40}$$

where $R^n_{\ jkl}$ is defined in (30). This tensor has the following symmetry properties

$$R_{ijkl} = -(-1)^{\epsilon_k\epsilon_l}R_{ijlk}, \quad R_{ijkl} = (-1)^{\epsilon_i\epsilon_j}R_{jikl} \tag{41}$$

and satisfies the identity

$$R_{ijkl} + (-1)^{\epsilon_l(\epsilon_i+\epsilon_k+\epsilon_j)}R_{lijk} + (-1)^{(\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_j)}R_{klij} + (-1)^{\epsilon_i(\epsilon_j+\epsilon_l+\epsilon_k)}R_{jkli} = 0. \tag{42}$$

The last statement can be derived from the Jacobi identity

$$(-1)^{\epsilon_j\epsilon_l}R_{ijkl} + (-1)^{\epsilon_l\epsilon_k}R_{iljk} + (-1)^{\epsilon_k\epsilon_j}R_{iklj} = 0\,. \tag{43}$$

together with a cyclic change of indices [17]. The identity (42) involves different components of the curvature tensor with cyclic permutation of all indices, but the sign factors depend on the Grassmann parities of the indices and do not follow a cyclic permutation rule, similar to that of Jacobi identity, but are defined by the permutation of the indices that maps a given set into the original one.

¿From the curvature tensor, R_{ijkl}, and the inverse tensor field ω^{ij} of the symplectic structure ω_{ij}, one can construct the only canonical tensor field of type $(0, 2)$,

$$K_{ij} = \omega^{kn}R_{nikj}(-1)^{\epsilon_i\epsilon_k+(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)} \;=\; R^k_{\ ikj}\,(-1)^{\epsilon_k(\epsilon_i+1)}, \quad \epsilon(K_{ij}) = \epsilon_i + \epsilon_j. \tag{44}$$

This tensor K_{ij} is the Ricci tensor and satisfies the relations [18]

$$[1 + (-1)^{\epsilon(\omega)}](K_{ij} - (-1)^{\epsilon_i\epsilon_j}K_{ji}) = 0. \tag{45}$$

In the even case K_{ij} is symmetric whereas in the odd case there are not restrictions on its (generalized) symmetry properties.

Now, one can define the scalar curvature tensor K by the formula

$$K = \omega^{ji}K_{ij}(-1)^{\epsilon_i+\epsilon_j} = \omega^{ji}\omega^{kn}R_{nikj}(-1)^{\epsilon_i+\epsilon_j+\epsilon_i\epsilon_k+(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)}. \tag{46}$$

¿From the symmetry properties of R_{ijkl}, it follows that

$$[1+(-1)^{\epsilon(\omega)}]K=0, \tag{47}$$

which proves that as in the case of Fedosov manifolds [4] the scalar curvature K vanishes.

However, for odd Fedosov supermanifolds K is, in general, non-vanishing. This fact was quite recently used in Ref. [13] to generalize the BV formalism [8].

Let us consider the Bianchi identity (34) in the form

$$R^n{}_{mij;k} - R^n{}_{mik;j}(-1)^{\epsilon_k\epsilon_j} + R^n{}_{mjk;i}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k)} \equiv 0\,. \tag{48}$$

Contracting indices i and n with the help of (44) we obtain

$$K_{mj;k} - K_{mk;j}(-1)^{\epsilon_k\epsilon_j} + R^n{}_{mjk;n}(-1)^{\epsilon_n(\epsilon_m+\epsilon_j+\epsilon_k+1)} \equiv 0\,. \tag{49}$$

Now using the relations

$$K^i{}_j = \omega^{ik}K_{kj}(-1)^{\epsilon_k},\, K^i{}_{j;m} = \omega^{ik}K_{kj;m}(-1)^{\epsilon_k} \tag{50}$$

$$K^i{}_{j;i}(-1)^{\epsilon_i(\epsilon_j+1)} = \omega^{ik}K_{kj;i}(-1)^{\epsilon_k+\epsilon_i(\epsilon_j+1)}, \tag{51}$$

it follows that

$$K_{,i} = [1-(-1)^{\epsilon(\omega)}]K^j{}_{i;j}(-1)^{\epsilon_j(\epsilon_i+1)}. \tag{52}$$

In the odd case this implies that

$$K_{,i} = 2K^j{}_{i;j}(-1)^{\epsilon_j(\epsilon_i+1)}. \tag{53}$$

In the even case $K_{,i}=0$ but in that case the relation (52) does not provides any new information because in this case $K=0$.

5 Riemannian supermanifolds

Let $\mathcal{M}$ be a supermanifold equipped both with a metric structure g

$$g = g_{ij}\,dx^j dx^i, \quad g_{ij} = (-1)^{\epsilon_i\epsilon_j}g_{ji}, \quad \epsilon(g_{ij}) = \epsilon(g)+\epsilon_i+\epsilon_j\ , \tag{54}$$

and a symmetric connection Δ with a covariant derivative ∇ compatible with the super-Riemannian metric g

$$g_{ij}\nabla_k = g_{ij,k} - g_{im}\Delta^m{}_{jk} - g_{jm}\Delta^m{}_{ik}(-1)^{\epsilon_i\epsilon_j} = 0. \tag{55}$$

It is easy to show that as in the case of Riemannian geometry there exists the unique symmetric connection $\Delta^i{}_{jk}$ which is compatible with a given metric structure. Indeed, proceeding in the same way as in the usual Riemannian geometry one obtains the generalization of celebrated Christoffel formula for the connection in supersymmetric case [12]

$$\Delta^l{}_{ki} = \frac{1}{2}g^{lj}\Big(g_{ij,k}(-1)^{\epsilon_k\epsilon_i} + g_{jk,i}(-1)^{\epsilon_i\epsilon_j} - g_{ki,j}(-1)^{\epsilon_k\epsilon_j}\Big)(-1)^{\epsilon_j\epsilon_i+\epsilon_j+\epsilon(g)(\epsilon_j+\epsilon_l)}. \tag{56}$$

It is straightforward to show that the symbols $\Delta^l{}_{ki}$ in (56) are transformed according with transformation laws for connections. A metric supermanifold $(\mathcal{M}, g)$ equipped with a (even or odd) symmetric connection Δ compatible with a given metric structure g is called a (even or odd) Riemannian supermanifold $(\mathcal{M}, g, \Delta)$.

The curvature tensor of the connection Δ is (56)

$$\mathcal{R}_{ijkl} = g_{in}\mathcal{R}^n{}_{jkl}, \quad \epsilon(\mathcal{R}_{ijkl}) = \epsilon(g) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \tag{57}$$

where $\mathcal{R}^n{}_{jkl}$ is given by (30) by replacing $\Gamma^i{}_{jk}$ for $\Delta^i{}_{jk}$. The curvature tensor has the following symmetry properties [12]

$$\mathcal{R}_{ijkl} = -(-1)^{\epsilon_k\epsilon_l}\mathcal{R}_{ijlk}, \quad \mathcal{R}_{ijkl} = -(-1)^{\epsilon_i\epsilon_j}\mathcal{R}_{jikl}, \quad \mathcal{R}_{ijkl} = \mathcal{R}_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \tag{58}$$

¿From the curvature tensor $\mathcal{R}_{ijkl}$ and the inverse tensor field g^{ij} of the metric g_{ij} defined by

$$g^{ij} = (-1)^{\epsilon(g)+\epsilon_i\epsilon_j} g^{ji}, \quad \epsilon(g^{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j, \tag{59}$$

one can define the only independent tensor field of type $(0,2)$:

$$\begin{aligned} &\mathcal{R}_{ij} = \mathcal{R}^k{}_{ikj}(-1)^{\epsilon_k(\epsilon_i+1)} = g^{kn}\mathcal{R}_{nikj}(-1)^{(\epsilon_k+\epsilon_n)(\epsilon(g)+1)+\epsilon_i\epsilon_k}, \\ &\epsilon(\mathcal{R}_{ij}) = \epsilon_i + \epsilon_j. \end{aligned} \tag{60}$$

It is the generalized Ricci tensor which obeys the symmetry

$$\mathcal{R}_{ij} = (-1)^{\epsilon(g)+\epsilon_i\epsilon_j}\mathcal{R}_{ji}. \tag{61}$$

A further contraction between the metric and Ricci tensors defines the scalar curvature

$$\mathcal{R} = g^{ji}\mathcal{R}_{ij}\ (-1)^{\epsilon_i+\epsilon_j}, \quad \epsilon(\mathcal{R}) = \epsilon(g) \tag{62}$$

which, in general, is non vanishing. Notice that for an odd metric structure the scalar curvature tensor squared is identically equal to zero, $\mathcal{R}^2 = 0$.

Let us consider now relations which follow from the Bianchi identity (34). Repeating all arguments given in the end of previous Section one can derive the following relation between the scalar curvature and the Ricci tensor

$$\mathcal{R}_{,i} = [1 + (-1)^{\epsilon(g)}]\mathcal{R}^j{}_{i;j}(-1)^{\epsilon_j(\epsilon_i+1)}. \tag{63}$$

In the even case we have

$$\mathcal{R}_{,i} = 2\mathcal{R}^j{}_{i;j}(-1)^{\epsilon_j(\epsilon_i+1)}, \tag{64}$$

which is a supersymmetric generalization of the well known relation of Riemannian geometry [19]. In the odd case $\mathcal{R}_{,i} = 0$ and the relation (63) implies that $\mathcal{R}$ =const.

Therefore, odd Riemann supermanifolds can only have constant scalar curvature $\mathcal{R}$ = const.

It is well known that special types of Riemannian manifolds play an important role in modern quantum field theory. In particular, a consistent formulation of higher spin field theories is possible on AdS space (see, for example [20]). In this case the curvature, Ricci and scalar curvature tensors have the form

$$\mathcal{R}_{ijkl} = R(g_{ik}g_{jl} - g_{il}g_{jk}), \quad \mathcal{R}_{ij} = (N-1)Rg_{ij}, \quad \mathcal{R} = N(N-1)R, \tag{65}$$

where N is the dimension of the Riemannian manifold $\mathcal{M}$ with a metric tensor g_{ij} and R is constant. Let us analyze the structure of supersymmetric extensions of AdS spaces (65). If

g_{ij} is the graded metric tensor (54) of the AdS space one can define the following combination of metric tensors

$$T_{ijkl} = g_{ik}g_{jl}(-1)^{\epsilon(g)(\epsilon_i+\epsilon_k)+\epsilon_k\epsilon_j} \tag{66}$$

which transforms as a tensor field. Therefore a natural generalization of (65) satisfies that

$$\begin{aligned}\mathcal{R}_{ijkl} &= R(g_{ik}g_{jl}(-1)^{\epsilon(g)(\epsilon_i+\epsilon_k)+\epsilon_k\epsilon_j} - g_{il}g_{jk}(-1)^{\epsilon(g)(\epsilon_i+\epsilon_l)+\epsilon_l\epsilon_j+\epsilon_l\epsilon_k}) = \\ &= (g_{ik}\, R\, g_{jl}(-1)^{\epsilon_k\epsilon_j} - g_{il}\, R\, g_{jk}(-1)^{\epsilon_l\epsilon_j+\epsilon_l\epsilon_k})(-1)^{\epsilon(g)},\end{aligned} \tag{67}$$

where R ($\epsilon(R) = \epsilon(g)$) is a constant. The Ricci tensor satisfies

$$\mathcal{R}_{ij} = g^{kl}\mathcal{R}_{likj}(-1)^{(\epsilon(g)+1)(\epsilon_k+\epsilon_l)+\epsilon_i\epsilon_k} = R(\mathcal{N}-1)g_{ij}(-1)^{\epsilon(g)} \tag{68}$$

and the scalar curvature tensor verifies that

$$\mathcal{R} = R\mathcal{N}(\mathcal{N}-1), \tag{69}$$

where we denote

$$\mathcal{N} = \delta^i_i(-1)^{\epsilon_i} \tag{70}$$

and $\mathcal{N}$ is nothing but the difference between the number of bosonic and fermionic dimensions of the supermanifold.

The above Riemannian tensors obey the following symmetry properties

$$\begin{aligned}&\mathcal{R}_{ijkl} = -(-1)^{\epsilon_k\epsilon_l}\mathcal{R}_{ijlk}, \quad \mathcal{R}_{ijkl} = -(-1)^{\epsilon(g)+\epsilon_i\epsilon_j}\mathcal{R}_{ijkl}, \\ &\mathcal{R}_{ijkl} = (-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}\mathcal{R}_{klij} + [1-(-1)^{\epsilon(g)}]g_{il}\, R\, g_{jk}(-1)^{\epsilon(g)+\epsilon_l(\epsilon_j+\epsilon_k)}, \\ &\mathcal{R}_{ij} = (-1)^{\epsilon_i\epsilon_j}\mathcal{R}_{ji}.\end{aligned}$$

It is easy to show that in the even case ($\epsilon(g) = 0$) all required symmetry properties for $\mathcal{R}_{ijkl}$ and $\mathcal{R}_{ij}$ are satisfied. Therefore the supersymmetric generalization of (65) has the form

$$\mathcal{R}_{ijkl} = R(g_{ik}g_{jl}(-1)^{\epsilon_k\epsilon_j} - g_{il}g_{jk}(-1)^{\epsilon_l\epsilon_j+\epsilon_l\epsilon_k}),\ \mathcal{R}_{ij} = R(\mathcal{N}-1)g_{ij},\ \mathcal{R} = R\mathcal{N}(\mathcal{N}-1). \tag{71}$$

In the odd case ($\epsilon(g) = 1$) there exists only one possibility to satisfy the symmetry requirements: the vanishing of all curvature tensors

$$R = 0 \longrightarrow \mathcal{R}_{ijkl} = 0, \quad \mathcal{R}_{ij} = 0, \quad \mathcal{R} = 0. \tag{72}$$

6 Conclusions

There are two natural geometric structures of supermanifolds defined by symmetric and antisymmetric graded tensor fields of the second rank: the Poisson bracket defined by an antisymmetric even tensor field of type $(2,0)$ and the antibracket given by an symmetrical odd tensor field of type $(2,0)$. We have have shown that the geometric structures of even and odd symplectic supermanifolds equipped with a symmetric connection compatible with a given symplectic structure are very similar, although only in the even case the scalar curvature has to vanish. In similar way, the structures of even and odd Riemannian supermanifolds equipped with the unique symmetric connection compatible with a given metric structure are also very similar. However, odd Riemannian supermanifolds are strongly constrained by the fact that their scalar curvature has to be constant whereas in the even case the curvature can have any value. It is quite remarkable that the strongest restrictions on the curvatures arise only for even symplectic and odd Riemannian manifolds. In the case of even Riemannian or odd symplectic manifolds, the curvature tensors can be non null and non-constant, respectively. There are several practical implications of the above formal results. The antisymplectic supermanifold underlying the Batalin-Vilkovisky quantization method is just an odd Fedosov supermanifold which as we have shown can have an arbitrary non-vanishing curvature. On the other hand, even Riemannian supermanifolds admit even AdS superspaces as special case, but there is no analogue for odd Riemannian supermanifolds, i.e. there are not odd supersymmetric AdS spaces.

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Coherent states of spinless particle in large magnetic – solenoid field

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

The Aharomov-Bohm effect that arises from a quantum-mechanical description of electron motion in a magnetic field in the presence of an infinitely thin and long solenoid is examined. In the stationary states this effect is manifest in the appearance of an additional series of energy levels shifted with respect to the Landau levels by a magnitude dependent on the magnetic field flux in the solenoid. Quantum states similar to coherent states are formulated that describe quasi-classical electron motion. In these states the existence of the solenoid causes additional electron oscillations with respect to a classical orbit and changes the relations between the parameters of classical motion. This makes it possible to provide a simple interpretation of the Aharonov-Bohm effect.

1 Introduction

By the middle of the 80's the Aharonov-Bohm (AB) effect [1] in low energy physics (see [2] for a general review) was becoming a good instrument for investigating new physical phenomena, principally in condensed matter physics, where the AB ring has been the mainstay of mesoscopic physics research since its inception. It is present in the theory of various

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problems, for example, the study of anyons in the theory of high -T_c superconductivity [3], electronic excitations in a graphitic monolayer (graphene) with topological defects [4], theory of unparticles [5]. The AB field is a field of an infinitely long and infinitesimally thin solenoid. Nonrelativistic effects arising from the combination of the solenoid of a tiny radius and a collinear uniform magnetic field (such a field configuration is named as a magnetic-solenoid field) have first time attracted attention in [6]. This case is interesting in that it represents a sample exactly solvable problem of the quantum states of an electron in a nonuniform magnetic field obtained from a sudden increase in the intensity of one of the lines of force of the uniform field. Recently the interest in the such a superposition has been renewed in connection with planar physics problems, quantum Hall effect, and the AB effect in cyclotron and synchrotron radiations [7, 8, 9]. The problem of taking spin into account in the such a field configuration is an aspect to which much attention has been devoted; for a review see [10].

In some cases, it is more convenient, from both the technical and the fundamental viewpoints, to use coherent states [11], i.e., near-classical states of a quantum system. However, for a particle in the AB potential a construction of such near-classical states turns out to be a nontrivial problem. The point is that the states traditionally used to describe the Aharonov–Bohm effect turn out to be stationary ones. A description on the basis of such states is naturally quite different from the regular classical picture. In order to take advantage of the picture being closest to the classical one, it is necessary to construct coherent states for a particle in the AB potential. In this case, the presence of a uniform magnetic field, which ensures the finiteness of motion, and a discrete spectrum, make it possible to avoid some mathematical difficulties caused by the weak decay of the vector potential of a solenoid at large distances. The coherent states of a charged particle are well known in the case of a uniform magnetic field [12]. In these states, the average values of the coordinates of an electron follow the classical trajectories in a magnetic field. These trajectories are circles whose radii and center coordinates serve as parameters (quantum numbers) of these states in the projection onto the plane perpendicular to the field. It is rather difficult to formulate the coherent states for electron motion in a uniform magnetic field in the presence of an infinitely thin and long solenoid. The analog of such states, that can be named as quasi-coherent states, is formulated for a nonrelativistic electron in article [13]. In the present study we generalize such a representation and carry out a detailed examination of the Aharonov-Bohm effect occurring from spinless relativistic particle motion in a magnetic-solenoid field. The results of the article make it possible to describe the Aharonov–Bohm effect in the classical language and to provide a clear interpretation of the influence of a magnetic-solenoid field on the quantum state of an electron with various “classical” parameters.

We would like to stress that many years ago Professor I. Buchbinder, to whom this Collection is devoted, collaborated with two of the authors (V.B and D.G) of the present article in constructing coherent states of relativistic particles, see [14].

1.1 Classical description

We will consider the motion of an electron with charge $-e$ in a magnetic field that is the superposition of a constant uniform magnetic field of field strength B and the magnetic field of an infinitely thin and long solenoid with a finite internal flux Φ parallel to the uniform field. Taking the coordinate system with the z axis on the solenoid, we will write the magnetic field strength as

$$B_x = B_y = 0\,, \quad B_z = B + \Phi\delta\left(x\right)\delta\left(y\right)\,,\ B > 0. \tag{1}$$

The following vector potential may be assigned to magnetic field (1)

$$A_x = -y\left(\frac{\Phi}{2\pi r^2}+\frac{B}{2}\right) , \quad A_y = x\left(\frac{\Phi}{2\pi r^2}+\frac{B}{2}\right) , \; A_z = 0, \; r^2 = x^2+y^2, \tag{2}$$

whose divergence is zero. Since the electron freely propagates on the z axis, only motion in the perpendicular plane $z = 0$ is nontrivial; this will be examined below.

Nonrelativistic case

The classical equations of motion of the electron derived from the Hamiltonian

$$\mathcal{H} = \frac{\mathcal{P}^2}{2M} , \quad \mathcal{P} = \mathbf{p} + \frac{e}{c}\mathbf{A} , \tag{3}$$

where M is mass of the electron and c is the velocity of light in vacuum., while $\mathbf{p}$ and $\mathcal{P}$ are the canonical and kinetic moments. Solving these equations we find the trajectories that do not intersect the solenoid:

$$x = x_0 + R\cos(\omega t + \varphi_0) , \quad y = y_0 + R\sin(\omega t + \varphi_0) , \tag{4}$$

where $\omega = eB/Mc$, R is the orbital radius,

$$x_0 = x - \frac{c}{eB}\mathcal{P}_y = R_0\cos\alpha , \quad y_0 = y + \frac{c}{eB}\mathcal{P}_x = R_0\sin\alpha \tag{5}$$

are the coordinates of the orbit center; R_0 is its distance from the z axis, where the kinetic momentum is equal to

$$\mathcal{P}_x = M\dot{x} = -\frac{eB}{c}R\sin(\omega t + \varphi_0) , \quad \mathcal{P}_y = M\dot{y} = \frac{eB}{c}R\cos(\omega t + \varphi_0) . \tag{6}$$

The quantities $R, \; \varphi_0, \; R_0$, a are integration constants and characterize the initial conditions. It is also convenient to introduce the conserved quantity

$$\nu = \frac{1}{\hbar}L_z + l_0 + \mu = \frac{\gamma}{2}\left(R^2 - R_0^2\right) , \tag{7}$$

where $l_0 + \mu = \Phi/\Phi_0$, $\Phi_0 = 2\pi\hbar c/e$, and l_0 is integer and $0 \le \mu < 1$, while $L_z = xp_y - yp_x$ is the projection of the angular momentum onto the z axis, and $\gamma = eB/c\hbar$. The latter formula shows that when $\nu > 0$ the orbit encompasses the solenoid, and when $\nu < 0$ the solenoid is outside the orbit. The orbital radius is related to the electron energy by the simple relation

$$\frac{2\mathcal{H}}{M} = \omega^2 R^2 ; \tag{8}$$

and it follows that the dimensionless quantity

$$\gamma R^2 = \frac{v^2}{c^2}\frac{B_0}{B} , \quad B_0 = \frac{M^2c^3}{e\hbar} \tag{9}$$

is quite large for any magnetic field strengths accessible in laboratory even at very small velocities v.

Using the integration constants we may draft conserved dimensionless complex quantities for the classical trajectories that do not intersect the solenoid

$$a_1 = \frac{1}{\sqrt{2\hbar eB/c}}\left(\mathcal{P}_x - i\mathcal{P}_y\right)e^{i\omega t} = \sqrt{\frac{\gamma}{2}}R\exp\left[-i\left(\varphi_0 + \frac{\pi}{2}\right)\right] ,$$
$$a_2 = \sqrt{\frac{\gamma}{2}}\left(\left[x + iy - \frac{c}{eB}\left(\mathcal{P}_y - i\mathcal{P}_x\right)\right]\right) = \sqrt{\frac{\gamma}{2}}\left(x_0 + iy_0\right) = \sqrt{\frac{\gamma}{2}}R_0 e^{i\alpha} . \qquad (10)$$

Then the classical trajectories can be represented as

$$x^2 + y^2 = \frac{2}{\gamma}\left[|a_1|^2 + |a_2|^2 - 2\,|a_1 a_2|\cos\left(\omega t + \chi\right)\right] , \qquad (11)$$

where $\chi = \pi/2 - \arg a_1 - \arg a_2$.

Relativistic case

It is well know, in relativistic quantum mechanics, that the formalism of the light cone variable is a powerful tool in the construction of the coherent states (see ref. [16]). Therefore, we will introduce the following curvilinear coordinate system

$$u_0 = ct - z \; , \quad u_3 = ct + z \; , \quad x = r\cos\varphi \; , \quad y = r\sin\varphi \; . \qquad (12)$$

One can solve the classic relativistic Lorentz equation in the potential (1) to obtain the electron trajectory in the parametric form (with parameter u_0), it is important to note that these trajectories do not pass through the solenoid. This solution can be written in the form

$$ct = \left(1 + \frac{p_z}{p_-}\right)u_0 \; , \quad z = \frac{p_z u_0}{p_-} \; ,$$
$$x = x_0 + R\cos\left(\kappa u_0 + \varphi_0\right) \; , \quad y = y_0 + R\sin\left(\kappa u_0 + \varphi_0\right) \; , \qquad (13)$$

where $p_- = E/c - p_z$ is a integral of motion and

$$x_0 = R_0\cos\alpha \; , \quad y_0 = R_0\sin\alpha \qquad (14)$$

are the coordinates of the center of the orbit in the plane z, R_0 the distance from the center of the orbit to the z axes,

$$\mathcal{P}_x = -\frac{eB}{c}R\sin\left(\kappa u_0 + \varphi_0\right) \; , \quad \mathcal{P}_y = \frac{eB}{c}R\cos\left(\kappa u_0 + \varphi_0\right) \; , \qquad (15)$$

are the kinetic momenta, $\kappa = eB/p_-$ and R is the orbital radius. The quantities R_0, α, R and φ_0 are integral of motion. The relativistic analogous of the Hamiltonian (3), $\mathcal{H} = \mathcal{P}_\perp^2/2p_-$ is related to the radius of the electron orbit by the simple relation

$$2p_-\mathcal{H} = \left(\frac{eB}{c}\right)^2 R^2 \; , \qquad (16)$$

and it follows that the quantity

$$\gamma R^2 = \beta_\perp^2\frac{B_0}{B}\left(\frac{E}{Mc^2}\right)^2 \; , \qquad \beta_\perp^2 = \frac{v_x^2 + v_y^2}{c^2} \; , \qquad (17)$$

where v_x (v_y) are the projection of the electron velocity in the x (y) axes, is quite large for any magnetic field strengths accessible in laboratory, even at very small velocities v. However, note that γR^2 is not large for a strong magnetic field $B \gtrsim B_0$, $B_0 = M^2c^3/e\hbar \simeq 4,4 \times 10^{13}G$. Such fields are observed now for several kinds of astronomical compact objects (pulsars, powerful X-ray sources, soft gamma-ray repeaters, etc.). For instance, at the surface of radio pulsars identified with rotation-powered neutron stars the field strength is up to $10^{14}G$ [18]. Much more intense magnetic fields have been expected to be present near superconductive cosmic strings and at the beginning of the inflation.

Using the integration constants), we may draft conserved dimensionless complex quantities for the classical trajectories that do not intersect the solenoid, relativistic analogous of the (10),

$$
\begin{aligned}
a_1 &= \frac{1}{\sqrt{2\hbar eB/c}} \left(\mathcal{P}_x - i\mathcal{P}_y\right) e^{i\kappa u_0} = \sqrt{\frac{\gamma}{2}} R \exp\left[-i\left(\varphi_0 + \frac{\pi}{2}\right)\right] \ , \\
a_2 &= \sqrt{\frac{\gamma}{2}} \left[x + iy - \frac{c}{eB}\left(\mathcal{P}_y - i\mathcal{P}_x\right)\right] = \sqrt{\frac{\gamma}{2}} \left(x_0 + iy_0\right) = \sqrt{\frac{\gamma}{2}} R_0 e^{i\alpha} \ , \qquad (18)
\end{aligned}
$$

where the substitution $\omega t \to \kappa u_0$ is used.

1.2 Stationary Quantum States

As usual, we will analyze the quantum behavior of the electron in the field (1) by assign operator status to the canonical variables x, y, $p_x = -i\hbar\partial/\partial x$ and $p_y = -i\hbar\partial/\partial y$ and found wave function with determined characteristics. In these article we will concern our self with solution in the form of stationary states and states whose average values draw near classical solutions, i.e., the quasi-coherent states.

Nonrelativistic case

Then expressions (3), (5), and (10) determine the quantum Hamiltonian, the kinetic momentum $\mathcal{P}$ operators, the coordinates of the orbital center x_0 and y_0 and the operators a_1 and a_2 that we will label with the same letters. Thus, the following operator relations are satisfied:

$$
\begin{aligned}
\mathcal{H} &= \frac{\hbar\omega}{2}\left(a_1^+ a_1 + a_1 a_1^+\right) \ , \\
R^2 &= (x - x_0)^2 + (y - y_0)^2 = \frac{2\mathcal{H}}{M\omega^2} = \frac{1}{\gamma}\left(a_1^+ a_1 + a_1 a_1^+\right) \ , \\
R_0^2 &= x_0^2 + y_0^2 = \frac{1}{\gamma}\left(a_2^+ a_2 + a_2 a_2^+\right) \ , \\
\nu &= \frac{1}{\hbar}L_z + l_0 + \mu = \frac{\gamma}{2}\left(R^2 - R_0^2\right) = \frac{1}{2}\left(a_1^+ a_1 + a_1 a_1^+ - a_2^+ a_2 - a_2 a_2^+\right) \ , \\
x_0 &= x - \frac{c}{eB}\mathcal{P}_y = \sqrt{\frac{1}{2\gamma}}\left(a_2^+ + a_2\right) \ , \\
y_0 &= y + \frac{c}{eB}\mathcal{P}_x = i\sqrt{\frac{1}{2\gamma}}\left(a_2^+ - a_2\right) \ , \\
\mathcal{P}_x &= \sqrt{\frac{\hbar eB}{2c}}\left(a_1 e^{-i\omega t} + a_1^+ e^{i\omega t}\right) \ , \quad \mathcal{P}_y = i\sqrt{\frac{\hbar eB}{2c}}\left(a_1 e^{-i\omega t} - a_1^+ e^{i\omega t}\right) \ ,
\end{aligned}
$$

$$x = \sqrt{\frac{1}{2\gamma}} \left(a_2^+ + a_2 + ia_1 e^{-i\omega t} - ia_1^+ e^{i\omega t}\right) ,$$
$$y = \sqrt{\frac{1}{2\gamma}} \left(ia_2^+ - ia_2 - a_1 e^{-i\omega t} - a_1^+ e^{i\omega t}\right) , \quad (19)$$

The operators a_i satisfy the commutation relations

$$\left[a_1, a_1^+\right] = 1 + f , \quad \left[a_2, a_2^+\right] = 1 - f$$
$$\left[a_1, a_2\right] = ife^{i\omega t} , \quad \left[a_1, a_2^+\right] = 0 , \quad (20)$$

where $f = (\Phi/B)\, \delta(x)\, \delta(y)$ is a singular dimensionless function.

Determining the complete time derivatives, we obtain

$$\dot{a}_1 = -i\frac{\omega}{2}\left(fa_1 + a_1 f\right) , \quad \dot{a}_2 = -\frac{\omega}{2}\left(a_1^+ f + fa_1^+\right) e^{i\omega t} , \quad (21)$$

and it is clear that when $\Phi = 0$ the operators a_i are integrals of notion. Moreover, in this case they are also annihilation operators, as indicated by relations (20). In the presence of a solenoid they no longer have these properties and only the combinations $a_i^+ a_i + a_i a_i^+$ remain integrals of motion and commute.

The stationary states of a nonrelativistic electron in magnetic field (1)

$$\Psi\left(t, r, \varphi\right) = \exp\left(-\frac{i}{\hbar}\mathcal{E}t\right) \phi\left(r, \varphi\right) \quad (22)$$

were found and investigated in detail in study [6]. Since Hamiltonian (3) and the operators L_z and R_0^2 commute and are related by the relation

$$L_z + \hbar\left(l_0 + \mu\right) = \frac{1}{\omega}\mathcal{H} - \frac{M\omega}{2}R_0^2 , \quad (23)$$

their eigenfunctions form a complete orthonormalized set of stationary states labeled by two integers $m,\ l\ (m \geq 0)$:

$$\Psi_{m,l}\left(t, r, \varphi\right) = \exp\left(-\frac{i}{\hbar}\mathcal{E}_{m,l} t\right) \phi_{m,l}\left(r, \varphi\right) ,$$
$$\phi_{m,l}\left(r, \varphi\right) = \sqrt{\frac{\gamma}{2\pi}} e^{i(l-l_0)\varphi} \phi_{m,l}\left(r\right) ,$$
$$\phi_{m,l}\left(r\right) = I_{m+|\nu|,m}\left(\xi\right) . \quad (24)$$

Here $\xi = \gamma r^2/2,\ \nu = l + \mu$, while the radial function $I_{n,m}(\xi)$ is the Laguerre function which is related to the Laguerre polynomials $L_m^\alpha(\xi)$ ([17], 8.970, 8.972.1) by

$$I_{m+\alpha,m}(\xi) = \sqrt{\frac{\Gamma\left(m+1\right)}{\Gamma\left(m+\alpha+1\right)}} e^{-\xi/2} \xi^{\alpha/2} L_m^\alpha\left(\xi\right) ,$$
$$L_m^\alpha(\xi) = \frac{1}{m!} e^x \xi^{-\alpha} \frac{d^m}{d\xi^m} e^{-x} \xi^{m+\alpha} . \quad (25)$$

The functions $\phi_{m,l}\left(r\right)$ are selected to be regular [4] at $r = 0$. As it is known for a spinless (non)relativistic electron in magnetic field (1) [8], such a boundary condition specifies one

[4] Here, we use the terms "regular" and "irregular" at $r = 0$ in the following sense. We call a function to be regular if it behaves as r^b at $r = 0$ with $b \geq 0$, and irregular if $b < 0$.

of the four-parameter family of admissible boundary conditions, that is, the concrete self-adjoint extension of the Hamiltonian is selected. It was shown (see more general verification in [10]) that one come to this self-adjoint extension studying the eigenvalue problem with a finite radius solenoid and then shrinking this radius to zero. The eigenvalues of the operators $\mathcal{H}$, L_z, and R_0^2 are equal to

$$\begin{aligned}\mathcal{E}_{m,l} &= \frac{1}{2}\hbar\omega\left(2m+|v|+v+1\right)\ , \quad \left(L_z\right)_{n,m} = \hbar\left(l-l_0\right)\ , \\ \left(R_0^2\right)_{m,l} &= \frac{1}{\gamma}\left(2m+|v|-v+1\right)\ . \end{aligned} \tag{26}$$

respectively.

We see from expression (26) that the parameter l_0 is not significant, since it may be eliminated by overdetermination of the orbital quantum number; in this case wave functions (24) acquire a nonsignificant gauge multiplier. We note, however, that the use of such singular (on the z axis) gauge transforms requires some care and must be accompanied by correct redetermination of the unbounded operators on the z axis.

These formulae reveal that the energy spectrum of an electron in a uniform magnetic field consisting of a series of equidistant levels (Landau levels) decompose into two analogous series separated by $\mu\hbar\omega$ in the presence of the solenoid. Here levels with $\nu > 0$ are shifted only; while the infinitely degenerate levels with $\nu < 0$ remain unperturbed. On the other hand the eigenvalues of the operator R_0^2 experience a shift for values of $\nu < 0$. These physical differences are attributed to the fact that the stationary states with $\nu > 0$ and with $\nu < 0$ experience different types of orbits that have different positions with respect to the solenoid: as indicated by expression (19) the orbit encompasses the solenoid for ν in states with $\nu > 0$, while when $\nu < 0$ the solenoid lies outside the orbit.

It can be stated that the presence of the solenoid in the first case causes an increase in the orbital radius (energy), while in the second case it reduces the distance from its center to the solenoid. It is, however, necessary to remember that in stationary states the concept of orbits is quite arbitrary: due to the uncertainty relations only the radial variables are exactly fixed quantities, while the corresponding angular variables remain completely undetermined.

A convenient technique for a quantum mechanical description of electron behavior in a uniform magnetic field is the operator technique based on the operators a_i and a_i^+ (10). In the presence of the solenoid these operators take the following form in the variables ξ and φ

$$\begin{aligned}a_1 &= -\frac{i}{2\sqrt{\xi}}e^{i\omega t-i\varphi}\left(2\xi\frac{\partial}{\partial\xi}-i\frac{\partial}{\partial\varphi}+l_0+\mu+\xi\right)\ , \\ a_1^+ &= \frac{i}{2\sqrt{\xi}}e^{-i\omega t+i\varphi}\left(-2\xi\frac{\partial}{\partial\xi}-i\frac{\partial}{\partial\varphi}+l_0+\mu+\xi\right)\ , \\ a_2 &= \frac{1}{2\sqrt{\xi}}e^{i\varphi}\left(2\xi\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\varphi}-l_0-\mu+\xi\right)\ , \\ a_2^+ &= \frac{1}{2\sqrt{\xi}}e^{-i\varphi}\left(-2\xi\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\varphi}-l_0-\mu+\xi\right) \end{aligned} \tag{27}$$

and satisfy commutation relations (20) containing the singular function f. The commutators of these operators with Hamiltonian $\mathcal{H}$ and the angular momentum operator L_z,

$$\begin{aligned}
[a_1, \mathcal{H}] &= \frac{\hbar\omega}{2}\left(2a_1 + fa_1 + a_1 f\right) , \\
[a_2, \mathcal{H}] &= -i\frac{\hbar\omega}{2}e^{i\omega t}\left(fa_1^+ + a_1^+ f\right) , \\
[a_1, L_z] &= \frac{\hbar}{2}\left[2a_1 + fa_1 + a_1 f - ie^{i\omega t}\left(a_2^+ f + fa_2^+\right)\right] , \\
[a_2, L_z] &= \frac{\hbar}{2}\left[-2a_2 + fa_2 + a_2 f - ie^{i\omega t}\left(a_1^+ f + fa_1^+\right)\right] ,
\end{aligned}$$

also contain singular terms that makes a number of complications in using the operator method in this case. However under the action of the operators a_i and a_i^+ on the $\Psi_{n,m}$ states that decay sufficiently rapidly when $r \to 0$, these singular terms vanish, which wakes it possible to use these operators as creation and destruction operators for formulating different quantum states of electron notion in magnetic field (1). As a result when the solenoid is present the formulae characteristic of the case of a uniform magnetic field remain valid,

$$\begin{aligned}
a_1\Psi_{m,l} &= \begin{cases} -i\sqrt{m+v}\Psi_{m,l-1} & l > 0 , \\ i\sqrt{m}\Psi_{m-1,l-1} & l < 0 ; \end{cases} \\
a_1^+\Psi_{m,l} &= \begin{cases} i\sqrt{m+v+1}\Psi_{m,l+1} & l \geq 0 , \\ -i\sqrt{m+1}\Psi_{m+1,l+1} & l < -1 ; \end{cases} \\
a_2\Psi_{m,l} &= \begin{cases} -\sqrt{m}\Psi_{m-1,l+1} & l \geq 0 , \\ \sqrt{m+|v|}\Psi_{m,l+1} & l < -1 ; \end{cases} \\
a_2^+\Psi_{m,l} &= \begin{cases} -\sqrt{m+1}\Psi_{m+1,l-1} & l > 0 , \\ \sqrt{m+|v|+1}\Psi_{m,l-1} & l < 0 ; \end{cases}
\end{aligned} \quad (28)$$

with the following exceptions:

$$\begin{aligned}
a_1\Psi_{m,0}(t,r,\varphi) &= -i\sqrt{m+\mu}\phi^{ir}_{m,1}(r,\varphi)\exp\left[-i\omega t\left(n+\mu-\frac{1}{2}\right)\right] , \\
a_1^+\Psi_{m,-1}(t,r,\varphi) &= -i\sqrt{m+1}\phi^{ir}_{m+1,-1}(r,\varphi)\exp\left[-i\omega t\left(m+\frac{3}{2}\right)\right] , \\
a_2\Psi_{m,-1}(t,r,\varphi) &= \sqrt{m+1-\mu}\phi^{ir}_{m,-1}(r,\varphi)\exp\left[-i\omega t\left(m+\frac{1}{2}\right)\right] , \\
a_2^+\Psi_{m,0}(t,r,\varphi) &= -\sqrt{m+1}\phi^{ir}_{m+1,1}(r,\varphi)\exp\left[-i\omega t\left(m+\mu+\frac{1}{2}\right)\right] ,
\end{aligned} \quad (29)$$

where

$$\begin{aligned}
\phi^{ir}_{m,1}(r,\varphi) &= \sqrt{\frac{M\omega}{2\pi\hbar}}e^{-i(1+l_0)\varphi}\phi^{ir}_{m,1}(r) , \\
\phi^{ir}_{m,-1}(r,\varphi) &= \sqrt{\frac{M\omega}{2\pi\hbar}}e^{-il_0\varphi}\phi^{ir}_{m,-1}(r) , \\
\phi^{ir}_{m,1}(r) &= I_{m+\mu-1,m}(\xi) , \ \phi^{ir}_{m,-1}(r) = I_{m-\mu,m}(\xi) .
\end{aligned} \quad (30)$$

The functions $\phi^{ir}_{m,\pm 1}(r)$ are irregular at $r = 0$, unlike the functions $\phi_{m,l}(r)$ in (24), and that do not belong to the selected self-adjoint extension of the Hamiltonian. It means that the operators a_i and a_i^+ are in themselves ill-defined in the presence of AB potential. One can use such operators only as elements of decomposition for well-defined self-adjoint operators. We note that specifically the states $\Psi_{m,0}$ and $\Psi_{m,-1}$ are characterized by the fact that they describe the orbits that come closest to the solenoid and weakly decay when $r \to 0$ are most sensitive to uniform magnetic field distortions caused by the singular magnetic field of the solenoid.

In the case of a uniform magnetic field ($\mu = 0$) the wave function $\Psi_{m,l}$ may be written as

$$\Psi_{m,l} = \begin{cases} \frac{i^{m-l}\left(a_2^+\right)^m\left(a_1^+\right)^{m+l}}{\sqrt{m!(m+l)!}}\Psi_{0,0}\ , & l \geq 0\ , \\ \frac{i^{m}\left(a_1^+\right)^m\left(a_2^+\right)^{m-l}}{\sqrt{m!(m-l)!}}\Psi_{0,0}\ , & l < 0\ , \end{cases} \tag{31}$$

where the single vacuum state, $\Psi_{0,0}$, is used. Due to (29) it is impossible in the presence of a solenoid. However, the nature of energy spectrum (26) makes it possible to carry out such a formulation using the two vacuum states, $\Psi_{0,0}$ and $\Psi_{0,-1}$; see [6]:

$$\Psi_{m,l} = \begin{cases} \frac{i^{m-l}\sqrt{\Gamma(1+\mu)}\left(a_2^+\right)^m\left(a_1^+\right)^{m+l}}{\sqrt{n!\Gamma(n+m+\mu+1)}}\Psi_{0,0}\ , & l \geq 0\ , \\ \frac{i^{m}\sqrt{\Gamma(2-\mu)}\left(a_1^+\right)^m\left(a_2^+\right)^{m-l}}{\sqrt{m!\Gamma(m-l-\mu+1)}}\Psi_{0,-1}\ , & l < -1\ . \end{cases} \tag{32}$$

In the absence of the solenoid ($\mu = 0$) $\Psi_{0,-1} = a_2^+\Psi_{0,0}$ and expression (32) becomes formula (31).

Relativistic spinless case

In this subsection the relativistic electron motion will be described by the Klein-Gordon equation, i.e., we will not consider any effect related to the spin. Our problem is to analyze the influence of the solenoid magnetic flux in the average values of the quantum description of the electron motion. With these goal, we are interesting in solutions of the problem in the form of quasi-coherent states, these states have a behavior very close to the know coherent states, the sense of this similarity will be clarify below.

Using the variables (12) and selecting the u_3 variable, we can find solutions ϕ of the Klein-Gordon equation in the form

$$\phi_{p_-} = L^{-1/2}\exp\left[-\frac{i}{2\hbar}\left(u_3 p_- + \frac{M^2c^2u_0}{p_-}\right)\right]\Psi_{p_-}\left(u_0, r, \varphi\right)\ , \tag{33}$$

where L is a "normalize length" (the range of the u_3 variable), $p_- = E/c - p_z$ is an integral of motion, E the energy and p_z the momentum projection in the z direction. The function Ψ obey a Schrödinger-like equation

$$i\hbar\frac{\partial}{\partial u_0}\Psi = \mathcal{H}\Psi\ , \quad \mathcal{H} = \frac{\vec{\mathcal{P}}_\perp^2}{2p_-}\ ,$$
$$\vec{\mathcal{P}}_\perp = \left(\mathcal{P}_x, \mathcal{P}_y, 0\right)\ . \tag{34}$$

The advantage in search solutions in the form (33) is that this functions can be construct by the known solutions of the two-dimensional non-relativistic problem if we perform the replacements

$$M \to p_-,\ t \to u_0\ ,\ \omega \to \kappa\ . \tag{35}$$

The expressions (34), (13), (14) and (18) determine the quantum Hamiltonian, the kinetic momentum $\mathcal{P}$ operators, the coordinates of the orbital center x_0 and y_0 and the operators a_1 and a_2 that we will label with the same letters. Thus, the following operator relations are satisfied:

$$
\begin{aligned}
\mathcal{H} &= \frac{\hbar\kappa}{2}\left(a_1^+ a_1 + a_1 a_1^+\right) , \\
R_0^2 &= x_0^2 + y_0^2 = \frac{1}{\gamma}\left(a_2^+ a_2 + a_2 a_2^+\right) , \\
2v &= a_1^+ a_1 + a_1 a_1^+ - a_2^+ a_2 - a_2 a_2^+ , \\
\mathcal{P}_x &= \sqrt{\frac{\hbar eB}{2c}}\left(a_1 e^{-i\kappa u_0} + a_1^+ e^{i\kappa u_0}\right) , \quad \mathcal{P}_y = i\sqrt{\frac{\hbar eB}{2c}}\left(a_1 e^{-i\kappa u_0} - a_1^+ e^{i\kappa u_0}\right) , \\
\sqrt{2\gamma}x &= a_2^+ + a_2 + i a_1 e^{-i\kappa u_0} - i a_1^+ e^{i\kappa u_0} , \\
\sqrt{2\gamma}y &= i\left(a_2^+ - a_2\right) - a_1 e^{-i\kappa u_0} - a_1^+ e^{i\kappa u_0} ,
\end{aligned} \tag{36}
$$

The relations above are the same one founded in (20) if we perform the replacement (35).

With the same replacement we obtain the analogous of the equations (20) and (21)

$$
\begin{aligned}
\left[a_1, a_1^+\right] &= 1 + f , \quad \left[a_2, a_2^+\right] = 1 - f , \\
\left[a_1, a_2\right] &= i f e^{i\kappa u_0} , \quad \left[a_1, a_2^+\right] = 0 ,
\end{aligned} \tag{37}
$$

$$
2\dot{a}_1 = -i\kappa\left(f a_1 + a_1 f\right) , \quad 2\dot{a}_2 = -\kappa\left(a_1^+ f + f a_1^+\right) e^{i\kappa u_0} , \tag{38}
$$

where $f = (\Phi/B)\,\delta(x)\,\delta(y)$ and a_i the operators related with (18). Again, when $\Phi = 0$ the operators a_i are integrals of motion and destruction operators and in the presence of a solenoid they no longer have these properties and only the combinations $a_i^+ a_i + a_i a_i^+$ remain integrals of motion and commute.

The Hamiltonian $\mathcal{H}$ in (34) does not depend on u_0 then the "stationary" states of an electron in magnetic field (1) can be written as

$$
\Psi_{p_-}(u_0, r, \varphi) = \exp\left(-\frac{i}{\hbar}\mathcal{E}u_0\right)\phi(r, \varphi) . \tag{39}
$$

Thus, we can can construct a complete and orthonormal set of stationary states in the same manner as for non-relativistic case. Using (35) we find that

$$
\begin{aligned}
\Psi_{m,l}(u_0, r, \varphi) &= \exp\left(-\frac{i}{\hbar}\mathcal{E}_{m,l}u_0\right)\phi_{m,l}(r, \varphi) , \\
\mathcal{E}_{n,m} &= \frac{1}{2}\hbar\varkappa\left(2m + |v| + v + 1\right) ,
\end{aligned} \tag{40}
$$

where the two-dimensional functions $\phi_{m,l}(r, \varphi)$ are given by (24). Then relations (31) and (32) are valid for relativistic case.

In order, to explain the physical difference between states with positive and negative values of l, we will consider the quantum character of the motion of the electron in such states. Although the concept of trajectory do not made sense for the stationary quantum states, we can still define the orbital square operator R^2, using (16), and the operator related with the square distance to the center of the orbit R_0^2. These operators commute with it other. In this case, obviously, the angular variable α becomes completely undefined. In the stationary states we have

$$\gamma R^2 = 2m + |\nu| + \nu + 1 \,, \quad \gamma R_0^2 = 2m + |\nu| - \nu + 1 \tag{41}$$

and, consequently,

$$r^2 = R^2 + R_0^2 = \frac{2}{\gamma}\left(2m + |\nu| + 1\right) \,. \tag{42}$$

From relations (41), follows

$$2\nu = \gamma\left(R - R_0^2\right) \quad . \tag{43}$$

From this result, we can see that states with $\nu > 0$ ($l \geq 0$), even in the quantum theory, can be interpreted as the states that encompass the solenoid, while in the states with $\nu < 0$ ($l \leq -1$) the solenoid remain out side of the orbit. Besides, the states with $l = 0$ and $l = -1$ describe the orbits that come closer to the solenoid.

2 Coherent States

We will now consider the states describing the quasi-classical motion of an electron in magnetic field (1). A familiar example of such states are coherent states determined as the eigenstates of the destruction operators/integrals of motion [11]:

$$a_i \Psi_{z_1,z_2} = z_i \Psi_{z_1,z_2} \,. \tag{44}$$

The operators a_i (10) have the necessary properties due to relations (20) and (21) for the case of a constant uniform magnetic field ($\mu = 0$). The corresponding coherent states may be represented as a series in stationary states (24) or (40) [12]

$$\Psi^{(0)}_{z_1,z_2} = N_0 \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} c^{(0)}_{m,l} \Psi^{(0)}_{m,l} \,, \tag{45}$$

where the coefficients $c^{(0)}_{m,l}$ are equal to

$$c^{(0)}_{m,l} = \frac{(-1)^m \left(iz_1\right)^{m+(|l|+l)/2} \left(z_2\right)^{m+(|l|-l)/2}}{\sqrt{\Gamma\left(m+1\right)\Gamma\left(m+|\nu|+1\right)}} \,, \tag{46}$$

while the index "0" infers $\mu = 0$. Wave functions (45) form a complete, yet nonorthogonal system of functions parametrized by the two complex numbers z_1 and z_2. These functions will be normalized to unity, if we set the normalization constant equal to

$$N_0 = \exp\left(-\frac{1}{2}\left|z_1\right|^2 - \frac{1}{2}\left|z_2\right|^2\right) \,. \tag{47}$$

Non-relativistic case

We can write non-relativistic wave function (45) in explicit form by carrying out summation, using functions (24), or solving system (44) and the Schrödinger equation:

$$\Psi_{z_1,z_2} = N_0 \exp\left\{-\frac{1}{2}\xi - \frac{i}{2}\omega t - iz_1 z_2 e^{-i\omega t} + \sqrt{\xi}\left[z_2 e^{-i\varphi} + iz_1 e^{i\varphi - i\omega t}\right]\right\} \,. \tag{48}$$

Calculating the average values of the quantum mechanical operators $x,\ y,\ \mathcal{P}_x,\ \mathcal{P}_y$ over coherent states (45) we obtain classical trajectories (4)-(6) where the parameters of the trajectories $R,\ \varphi_0,\ R_0$, and α are related to the quantum numbers z_i of the coherent states by the relations

$$z_1 = \sqrt{\frac{\gamma}{2}} R e^{-i(\varphi_0+\pi/2)} \ , \quad z_2 = \sqrt{\frac{\gamma}{2}} R e^{i\alpha} \ . \tag{49}$$

The average value of the Hamiltonian $\mathcal{H}$ over these states is

$$\overline{\mathcal{H}} = \frac{\hbar\omega}{2} + \frac{M\omega^2 R^2}{2} \ , \tag{50}$$

which in fact corresponds to classical expression (8) differing only by the additive quantum correction $1/2\hbar\omega$ which appears due to the quantum variance ($\overline{a^2} \neq \overline{a}^2$). Analogously the average value of the operator $r^2 = x^2 + y^2$ is

$$\overline{r^2} = \frac{1}{\gamma}\left[1 + |z_1|^2 + |z_2|^2 + 2\,|z_1 z_2| \cos\left(\omega t + \varphi_0 - \alpha\right)\right] \ . \tag{51}$$

Therefore in the case of a uniform magnetic field ($\mu = 0$) the average values of the electron coordinate over the coherent states follow classical trajectories (4).

Proceeding to the general case of electron motion in a uniform magnetic field in the presence of the solenoid ($\mu > 0$) we emphasize that formulae (19) for the operators $x,\ y,\ \mathcal{P}_x,\ \mathcal{P}_y$ indicates that as before the average quantum mechanical trajectory still coincide with the classical trajectory, if the average values $\overline{a}_i$ over the corresponding states are time-independent. Here the trajectory parameters $R,\ \varphi_0,\ R_0$ and α will be related to the average values $\overline{a}_i$ by relations formally analogous to relations (49):

$$\overline{a_1} = \sqrt{\frac{\gamma}{2}} R e^{-i(\varphi_0+\pi/2)} \ , \quad \overline{a_2} = \sqrt{\frac{\gamma}{2}} R e^{i\alpha} \tag{52}$$

If the average values $\overline{a}_i$ depend on t, the average quantum mechanical trajectory will be different from the classical trajectory.

In the presence of the solenoid ($\mu > 0$) the operators a_i are no longer integrals of motion, do not commute and lose their status as annihilation operators. Therefore they cannot be used to formulate coherent state system (44). On the other hand it may be demonstrated that there are no linear first order differential operators having properties necessary for this. In the general case $\mu > 0$ the annihilation operators/integrals of motion for the Schrödinger equation with Hamiltonian (3) are integral operators that, although they may be formally drafted based on known exact solutions of the problem they are nonetheless cumbersome in form which wakes them unsuitable for practical application.

In addition in order to describe the possible influence of the magnetic field flux in the solenoid on the average electron trajectory in a uniform magnetic field we will draft other, quasi-coherent, states by determining them through the wave functions of stationary states (24) as the series

$$\Psi_{z_1,z_2} = N \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \Psi_{m,l} \ , \tag{53}$$

where

$$c_{m,l} = \frac{(-1)^m \left(i z_1\right)^{m+(|l|+l)/2} \left(z_2\right)^{m+(|l|-l)/2}}{\sqrt{\Gamma\left(m+1\right)\Gamma\left(m+|v|+1\right)}} \ , \tag{54}$$

generalizing formulae (45), (46) to the case $\mu > 0$. From the unity normalization of wave function (53) we obtain the expression for the normalization constant N (see Appendix):

$$N^{-2} = q^{-\mu} I_\mu(2q) + q^{-\mu} R_\mu\left(q, |z_1|^2\right) + q^\mu R_{-\mu}\left(q, |z_2|^2\right) ,$$

$$R_\alpha(q, p) = 2pe^p \int_0^1 x^{\alpha+1} e^{-px^2} I_\alpha(2qx)\, dx , \tag{55}$$

where $q = |z_1 z_2|$, while $I_\alpha(x)$ is the Bessel function of the imaginary argument. when $\mu = 0$ known expression (47) follows. In the particular case $z_2 = 0$ formula (55) becomes

$$N^{-2} = \frac{1}{\Gamma(1+\mu)} \Phi\left(1, 1+\mu; |z_1|^2\right) , \tag{56}$$

where $\Phi(\alpha, \beta;)$ is a degenerate hypergeometric function.

In order to find the average quantum mechanical trajectory over states (53), i.e., in order to determine the time dependence of the operators $x,\ y,\ \mathcal{P}_x,\ \mathcal{P}_y,\ x_0,\ y_0$ (19), it is sufficient to find the average values of the operators a_i. Using expressions (53), (54) and accounting for relations (28) and (29), we obtain (see Appendix)

$$\overline{a_1} = z_1 - N^2 \left[z_1 |z_2|^2 q^{\mu-1} I_{1-\mu}(2q) + i z_2^* F(1-\mu, \omega t)\right] ,$$

$$\overline{a_2} = z_2 - N^2 z_2 \left[q^{-\mu} I_\mu(2q) - F(\mu, \omega t)\right] , \tag{57}$$

where we introduce the convention

$$F(\beta, x) = \frac{\sin \pi\beta}{\pi} e^{i\beta x} \sum_{k=-\infty}^{\infty} I_k(2q) \frac{e^{ik(x+\chi)}}{k+\beta} , \tag{58}$$

while $\chi = \pi/2 - \arg z_1 - \arg z_2$.

It is clear that the solenoid ($\mu > 0$) causes a change in the average values $\overline{a}_i$ that are equal to the parameters z_i of the coherent state when $\mu = 0$. This change is due to the violation of relations (28) when $\mu > 0$ for transition between the states in $\Psi_{m,l}$ with $l = 0$ and $l = -1$ belonging to different ranges of levels (26). It is these states in which the probability of finding the electron near the solenoid is the greatest and, as discussed above, they are the most sensitive to distortions in the uniform magnetic field caused by the solenoid. Specifically, the existence of such states in the expansion of quasi-coherent state (53) causes the average values $\overline{a}_i$ (57) to become time-dependent. In turn this causes the average quantum mechanical trajectory of the electron to differ from the classical trajectory. The fundamental qualitative result is that oscillations of frequency $\mu\omega$ caused by the magnetic field flux in the solenoid arise near the classical orbit.

In order to examine in greater detail the nature of electron notion in quasi-coherent state (53) we will calculate the average values of the Hamiltonian $\mathcal{H}$ and the operator r^2. From a simple calculation (see the Appendix), using relations (28) and (29) we obtain

$$\overline{\mathcal{H}} = \hbar\omega \left\{\frac{1}{2} + |z_1|^2 + N^2 \left[q^{1-\mu} I_{\mu-1}(2q) - q^{1+\mu} I_{1-\mu}(2q)\right]\right\} \tag{59}$$

and

$$r^2 = \frac{1}{\gamma} \left\{1 + |z_1|^2 + |z_2|^2 - 2|z_1 z_2| \cos(\omega t + \chi) + \right.$$

$$\left. + N^2 q^{-\mu} \left[q I_{\mu-1}(2q) - |z_2|^2 I_\mu(2q)\right] - N^2 q^\mu \left[q I_{1-\mu}(2q) - |z_2|^2 I_{-\mu}(2q)\right]\right\} . \tag{60}$$

These averages do not contain oscillations of frequency $\mu\omega$ where $\overline{\mathcal{H}}$ and, consequently, due to relation (19) $\overline{R^2}$, are generally time-dependent. On the other hand average value $\overline{R}$ consistent with (52) and (57) is in the general case independent of t and experiences oscillations with frequency $\mu\omega$. Consequently the additional oscillations to the orbit are also caused by the quantum variance.

In the particular case $z_2 = 0$ consistent with expressions (57) we have $\overline{a}_1 = z_1$ and $\overline{a}_2 = 0$ and therefore in this state the average quantum mechanical trajectory agrees with the classical trajectory whose center is at the coordinate origin ($R_0 = 0$). However the dependence of the average electron energy (59) on the parameter z_1 (the average value of R),

$$\overline{\mathcal{H}} = \hbar\omega \left\{ \frac{1}{2} + |z_1|^2 + \mu \left[\Phi\left(1, 1+\mu; |z_1|^2\right)\right]^{-1} \right\} , \tag{61}$$

is different from expression (51) which is valid when $\mu = 0$. Therefore even in this particular case when the average quantum mechanical trajectory agrees with the classical trajectory the relations between the orbital parameters are different from the classical parameters and have an explicit dependence on the magnetic field flux in the solenoid. By measuring these parameters with sufficient accuracy we may establish the magnitude of this flux.

We note that the difference between relations (61) and (51) caused by the presence of the solenoid ($\mu > 0$) is quite small in practical realizable conditions. Indeed, as discussed above, the dimensionless parameter $|z_1|^2 = \gamma R^2/2$ (9) can be taken as large. Therefore by using the asymptotics of the degenerate hypergeometric function, we obtain from formula (61) the approximate expression

$$\overline{\mathcal{H}} \approx \hbar\omega \left\{ \frac{1}{2} + \frac{\gamma}{2}R^2 + \mu \left(\frac{\gamma}{2}R^2\right)^{\mu} \exp\left(-\frac{\gamma}{2}R^2\right) \right\} . \tag{62}$$

In order to again emphasize the important role of transitions between states with $l = 0$ and $l = -1$ in analyzing the influence of the magnetic field flux in the solenoid on the characteristic features of the average quantum mechanical trajectory of the electron, we will consider quasi-coherent states of the type

$$\Psi^{(+)} = N^{(+)} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} c_{m,l} \Psi_{m,l} , \quad \Psi^{(-)} = N^{(-)} \sum_{m=0}^{\infty} \sum_{l=-\infty}^{-1} c_{m,l} \Psi_{m,l} , \tag{63}$$

where, as before, the coefficients $c_{m,l}$ are determined by formulae (54). The normalization constants are equal to

$$N^{(\pm)} = \left\{ \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{q^{2m} |z_1 z_2|^{2l}}{\Gamma(m+1)\Gamma(m+l+1\pm\mu)} \right\}^{-1/2} . \tag{64}$$

Calculating the averages over states (63) we find

$$\overline{a_1^{(\pm)}} = z_1 , \quad a_2^{(\pm)} = z_2 \left[1 \mp \left(N^{(\pm)}\right)^2 q^{\mp\mu} I_{\pm}(2q) \right] , \tag{65}$$

$$\overline{\mathcal{H}^{(\pm)}} = \hbar\omega \left[\frac{1}{2} + |z_1|^2 \pm \left(N^{(\pm)}\right)^2 q^{1\mp\mu} I_{\pm(\mu-1)}(2q) \right] . \tag{66}$$

Since by definition no transitions occur between states with $l = 0$ and $l = -1$ in calculating any average values over states (63), the average values $\overline{a}_i$ are independent of time and the average quantum mechanical orbit always coincides with the classical orbit. However the relations between the orbital parameters are different from the classical relations. Indeed, relation (65) indicates that $|z_1|^2 = \gamma R^2/2$, while relation (66) establishes a relation between the average energy and the orbital radius. This relation is explicitly dependent on the magnetic field flux a in the solenoid. Therefore we way determine μ by measuring the energy $\mathcal{H}$ and the radii R and R_0.

In conclusion we will demonstrate that these states indeed describe quasi-classical electron motion. For this purpose we will calculate, for example, the Heisenberg uncertainty relation for states $\Psi^{(+)}$. The simple yet cumbersome calculation yields

$$\overline{\Delta\mathcal{P}_x^2\Delta x^2} = \frac{1}{4}\left[1+2\left(N^{(+)}\right)^2 q^{1-\mu} I_{\mu-1}(2q)\right]\left\{2+\left(N^{(+)}\right)^2 q^{-\mu}\times\right.$$
$$\left[(z_2+z_2^*)^2 I_\mu(2q)\left(1-\left(N^{(+)}\right)^2 q^{-\mu} I_\mu(2q)\right)+2\mu I_\mu(2q)\right.$$
$$\left.\left.-(z_2-z_2^*)^2|z_2|^{-2} qI_{1+\mu}(2q)-4qI_\mu(2q)\cos(\omega t+\chi)\right]\right\} \quad (67)$$

where the parameter χ was the same as in formula (58). It is clear that in states $\Psi^{(+)}$, as would be expected, the minimum of the uncertainty relations equal to $1/4$ is not realized, although the right half of (67) remains bounded for all t. For the particular case $z_2 = 0$ the dependence on t vanishes, although in this case the minimum of the right half is not achieved.

Therefore it is possible to formulate different types of states in the problem of the quantum motion of an electron in a uniform magnetic field in the presence of a solenoid for which the average quantum mechanical trajectory coincides with the classical trajectory or is very close to it. It is also possible on the average to satisfy select classical relations between the physical quantities. However it is always possible to identify other relations between the measured quantities that will differ from the corresponding classical relations and will have an explicit dependence on the magnetic flux in the solenoid. This represents the manifestation of the Aharonov-Bohm effect in the quasi-classical states of an electron in a magnetic field.

Relativistic case

As discussed after the equation (36), for the relativistic cases all the relations of the preceding section remain valid if we perform the substitution (35). In this formulation, the quantity u_0 make the role of "time".

Once more, we can construct the quasi-coherent states by a combination of solutions (40) in the form (53) with coefficients (54) and the average values of the quantum operator over these states give results very close (in the sense of previous section) to the classical behavior of the electron in the field (1). The previous discussion of the influence of the solenoid field over the electron trajectories also remains valid.

3 Appendix

3.1 Calculation of the Normalization Constant N

From the normalization of wave function (53) to unity we have

$$
\begin{aligned}
N^{-2} &= \sum_{m=0}^{\infty}\sum_{l=-\infty}^{\infty}|c_{m,l}|^2 \\
&= \sum_{m=0}^{\infty}\left[\sum_{l=0}^{\infty}\frac{|z_1|^{2(m+l)}|z_2|^{2m}}{\Gamma(m+1)\Gamma(m+l+\mu+1)}+\sum_{l=1}^{\infty}\frac{|z_1|^{2m}|z_2|^{(2m+l)}}{\Gamma(m+1)\Gamma(m+l-\mu+1)}\right] \\
&= \sum_{l=0}^{\infty}|z_1|^{2l}q^{-l-\mu}I_{l+\mu}(2q)+\sum_{l=1}^{\infty}|z_2|^{2l}q^{-l+\mu}I_{l-\mu}(2q)
\end{aligned}
$$

where $q=|z_1z_2|$, since (see formula 5.2.10.7 in part 1 [15])

$$
\sum_{m=0}^{\infty}\frac{q^{2m}}{\Gamma(m+1)\Gamma(m+1+\alpha)}=q^{-\alpha}I_{\alpha}(2q)\ .
$$

Then using formula 5.8.3.1 in part 2 [15],

$$
\begin{aligned}
\sum_{l=0}^{\infty}p^lI_{l+\alpha}(2q) &= 2qe^{pq}\int_0^1 x^{\alpha}e^{-pqx^2}I_{\alpha-1}(2qx)\,dx \\
&= I_{\alpha}(2q)+2pqe^{pq}\int_0^1 x^{\alpha+1}e^{-pqx^2}I_{\alpha}(2qx)\,dx,
\end{aligned}
$$

we finally obtain

$$
N^{-2}=q^{-\mu}I_{\mu}(2q)+q^{-\mu}R_{\mu}\left(q,|z_1|^2\right)+q^{\mu}R_{-\mu}\left(q,|z_2|^2\right)\ ,
$$

where

$$
R_{\alpha}(q,p)=2pe^{p}\int_0^1 x^{\alpha+1}e^{-px^2}I_{\alpha}(2qx)\,dx\ .
$$

When $\mu=0$ we have

$$
N_0^{-2}=\exp\left(|z_1|^2+|z_2|^2\right)\ .
$$

In the asymptotic domain $|z_1|\to\infty,\ |z_2|\to\infty$ we obtain

$$
N^{-2}\approx|z_2|^{2\mu}\exp\left(|z_1|^2+|z_2|^2\right)\ ,
$$

since (see formula 2.15.5.4 in part 2 [15])

$$
\int_0^{\infty}x^{\alpha+1}e^{-\beta x^2}I_{\alpha}(x)\,dx=(2\beta)^{-\alpha-1}\exp\left(\frac{1}{4\beta}\right)\ .
$$

In the particular case $z_2=0$ we have

$$
N^{-2}=\frac{1}{\Gamma(1+\mu)}\Phi\left(1,1+\mu;|z_1|^2\right)\ .
$$

3.2 Calculation of the Averages a_i

Accounting for relation (28), (29) and using formula 2.19.14.18 in part 2 [15],

$$\int_0^\infty e^{-x} L_{m'}^{\alpha}(x) L_m^{-\alpha}(x)\, dx = \frac{(-1)^{m+m'}\Gamma(m'+\alpha+1)\Gamma(m-\alpha+1)}{\Gamma(m'+1)\Gamma(m+1)\Gamma(m'-m+\alpha+1)\Gamma(m-m'-\alpha+1)}$$
$$= \frac{\sin(\pi\alpha)}{\pi(m'-m+\alpha)} \frac{\Gamma(m'+\alpha+1)\Gamma(m-\alpha+1)}{\Gamma(m'+1)\Gamma(m+1)} ,$$

we will write out the matrix elements of the operator a_1 over the standard states (24)

$$\langle a_1\rangle_{m',l';m,l} = \delta_{l',l-1} \begin{cases} -i\sqrt{m+v}\delta_{m,m'} , & l>0 \\ i\sqrt{m}\delta_{m'+1,m} & l<0 \\ -\frac{\sin(\pi\mu)e^{i\omega t(m'-m-\mu+1)}}{\pi(m'-m-\mu+1)} \sqrt{\frac{\Gamma(m'-\mu+2)\Gamma(m+\mu+1)}{\Gamma(m'+1)\Gamma(m+1)}} & l=0 \end{cases}$$

$$\langle a_2\rangle_{m',l';m,l} = \delta_{l',l+1} \begin{cases} -\sqrt{m}\delta_{m',m-1} , & l\geq 0 \\ \sqrt{m+|v|}\delta_{m',m} & l<-1 \\ \frac{\sin(\pi\mu)e^{i\omega t(m'-m+\mu)}}{\pi(m'-m+\mu)} \sqrt{\frac{\Gamma(m'-\mu+1)\Gamma(m-\mu+2)}{\Gamma(m'+1)\Gamma(m+1)}} & l=-1 \end{cases} .$$

The average values of the a_i operators over states (53) are equal to

$$\overline{a_i} = N^2 \sum_{m,m'=0}^{\infty} \sum_{l,l'=-\infty}^{\infty} c^*_{m',l'} c_{m,l} \langle a_i\rangle_{m',l';m,l} .$$

Thus,

$$\overline{a_1} = N^2 \sum_{m,m'=0}^{\infty} \sum_{l=-\infty}^{\infty} c^*_{m',l-1} c_{m,l} \langle a_1\rangle_{m',l-1;m,l}$$
$$= N^2 \sum_{m=0}^{\infty} \left[\sum_{l=1}^{\infty} (-i)\sqrt{m+v} c^*_{m,l-1} c_{m,l} + \sum_{l=-\infty}^{-1} i\sqrt{m} c^*_{m-1,l-1} c_{m,l} + \right.$$
$$\left. + \sum_{m'=0}^{\infty} c^*_{m',-1} c_{m,0} \langle a_1\rangle_{m',-1;m,0} \right]$$
$$= N^2 \sum_{m=0}^{\infty} \left[z_1 \sum_{l=0}^{\infty} c^*_{m,l} c_{m,l} + z_1 \sum_{l=-\infty}^{-2} c^*_{m,l} c_{m,l} + z_1 c^*_{m,-1} c_{m,-1} + \right.$$
$$\left. + \sum_{m'=0}^{\infty} c^*_{m',-1} c_{m,0} \langle a_1\rangle_{m',-1;m,0} - z_1 c^*_{m,-1} c_{m,-1} \right]$$
$$= z_1 + \delta z_1 ,$$

where

$$\delta z_1 = N^2 \sum_{m=0}^{\infty} \left[-z_1 c^*_{m,-1} c_{m,-1} + \sum_{m'=0}^{\infty} c^*_{m',-1} c_{m,0} \langle a_1\rangle_{m',-1;m,0} \right]$$
$$= -N^2 \left\{ i z_2^* \frac{\sin(\pi\mu)}{\pi} \sum_{m,m'=0}^{\infty} \frac{(i z_1^* z_2^*)^{m'} (-i z_1 z_2)^m}{\Gamma(m'+1)\Gamma(m+1)} \frac{e^{i\omega t\left(m'-m-\mu+1\right)}}{m'-m-\mu+1} + \right.$$
$$\left. + z_1 |z_2|^2 \sum_{m=0}^{\infty} \frac{|z_1 z_2|^2}{\Gamma(m+1)\Gamma(m-\mu+2)} \right\} .$$

Then we obtain

$$\overline{a_1} = z_1 - N^2 \left[z_1 |z_2|^2 q^{\mu-1} I_{1-\mu}(2q) + i z_2^* F(1-\mu, \omega t) \right] ,$$

where we use the convention

$$F(\beta, x) = \frac{\sin \pi\beta}{\pi} \sum_{m,m'=0}^{\infty} \frac{(i z_1^* z_2^*)^{m'} (-i z_1 z_2)^m}{\Gamma(m'+1)\Gamma(m+1)(m'-m+\beta)} e^{ix(m'-m+\beta)} .$$

Analogously

$$\overline{a_2} = z_2 - N^2 z_2 \left[q^{-\mu} I_\mu(2q) - F(\mu, \omega t) \right] .$$

When $\mu = 0$, by virtue of the relation

$$\lim_{\beta\to 0} \frac{\sin(\pi\beta)}{\pi(m'-m+\beta)} = \delta_{m',m} ,$$

we obtain $\overline{a_i} = z_i$.

We will simplify the expression for $F(\beta, x)$. Since

$$\frac{e^{ix(m'-m+\beta)}}{m'-m+\beta} = -i \lim_{\varepsilon\to 0} \int_x^\infty e^{-\varepsilon y + iy(m'-m+\beta)} dy$$

we have

$$\begin{aligned} F(\beta, x) &= -i \frac{\sin \pi\beta}{\pi} \lim_{\varepsilon\to 0} \int_x^\infty dy\, e^{-\varepsilon y + iy\beta} \sum_{m,m'=0}^{\infty} \frac{\left(i z_1^* z_2^* e^{iy}\right)^{m'} \left(-i z_1 z_2 e^{-iy}\right)^m}{\Gamma(m'+1)\Gamma(m+1)} \\ &= -i \frac{\sin \pi\beta}{\pi} \lim_{\varepsilon\to 0} \int_x^\infty dy\, \exp\left(-\varepsilon y + i\beta y + i z_1^* z_2^* e^{iy} - i z_1 z_2 e^{-iy}\right) \\ &= -i \frac{\sin \pi\beta}{\pi} \lim_{\varepsilon\to 0} \int_x^\infty dy\, \exp\left[-\varepsilon y + i\beta y + 2q\cos(y+\chi)\right] , \end{aligned}$$

where $\chi = \pi/2 - \arg z_1 - \arg z_2$.

Accounting for

$$e^{2q\cos(y+\chi)} = \sum_{k=-\infty}^{\infty} e^{ik(y+\chi)} I_k(2q) ,$$

we finally obtain

$$F(\beta, x) = \frac{\sin \pi\beta}{\pi} e^{i\beta x} \sum_{k=-\infty}^{\infty} I_k(2q) \frac{e^{ik(y+\chi)}}{k+\beta} .$$

3.3 Calculation of the Average Energy

The average value of $\overline{\mathcal{H}}$ over states (53) is

$$\begin{aligned}
\overline{\mathcal{H}} &= N^2 \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \mathcal{E}_{m,l} \left|c_{m,l}\right|^2 \\
&= \hbar\omega \left\{ \frac{1}{2} + N^2 \sum_{m=0}^{\infty} \left[\sum_{l=0}^{\infty} (m+l+\mu) \left|c_{m,l}\right|^2 + \sum_{l=-\infty}^{-1} m \left|c_{m,l}\right|^2 \right] \right\} \\
&= \hbar\omega \left\{ \frac{1}{2} + N^2 \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left|z_1\right|^{2(m+l)} \left|z_2\right|^{2m}}{\Gamma(m+1)\,\Gamma(m+l+\mu)} + N^2 \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{\left|z_1\right|^{2m} \left|z_2\right|^{2(m+l)}}{\Gamma(m)\,\Gamma(m+l-\mu+1)} \right\} \\
&= \hbar\omega \left\{ \frac{1}{2} + \left|z_1\right|^2 + N^2 \sum_{m=0}^{\infty} \frac{\left|z_1 z_2\right|^{2m}}{\Gamma(m+1)\,\Gamma(m+\mu)} - N^2 \sum_{m=1}^{\infty} \frac{\left|z_1 z_2\right|^{2m}}{\Gamma(m)\,\Gamma(m-\mu+1)} \right\} .
\end{aligned}$$

Thus

$$\overline{\mathcal{H}} = \hbar\omega \left\{ \frac{1}{2} + \left|z_1\right|^2 + N^2 \left[q^{1-\mu} I_{\mu-1}(2q) - q^{1+\mu} I_{1-\mu}(2q) \right] \right\} .$$

In particular, when $\mu = 0$, we obtain the familiar result

$$\overline{\mathcal{H}} = \hbar\omega \left(\frac{1}{2} + \left|z_1\right|^2 \right) .$$

For the case $z_2 = 0$ we have

$$\overline{\mathcal{H}} = \hbar\omega \left\{ \frac{1}{2} + \left|z_1\right|^2 + \mu \left[\Phi\left(1, 1+\mu; \left|z_1\right|^2\right) \right]^{-1} \right\} .$$

We now calculate the average values of the operator R^2, R_0^2 and r^2. It is obvious that

$$\overline{R^2} = \frac{2\overline{\mathcal{H}}}{M\omega^2} .$$

Then accounting for

$$\left\langle R_0^2 \right\rangle_{m',l';m,l} = \frac{2}{\gamma} \delta_{m',m} \delta_{l'l} \left(m + \frac{|v| - v}{2} + \frac{1}{2} \right) ,$$

after analogous calculation we obtain

$$\overline{R^2} = \frac{2}{\gamma} \left\{ \frac{1}{2} + \left|z_2\right|^2 + \left|z_2\right|^2 N^2 \left[q^{\mu} I_{-\mu}(2q) - q^{-\mu} I_{\mu}(2q) \right] \right\} .$$

The matrix elements of r^2 over the standard states are equal to

$$\begin{aligned}
\left\langle r^2 \right\rangle_{m',l';m,l} = \frac{2}{\gamma} \delta_{l',l} \{ & (2m + |v| + 1)\, \delta_{m'm} - \\
& - e^{i\omega t} \sqrt{(m+1)(m+|v|+1)} \delta_{m',m+1} - e^{-i\omega t} \sqrt{m(m+|v|)} \delta_{m'+1,m} \} .
\end{aligned}$$

Hence we have for the average over the stationary states

$$\left\langle r^2 \right\rangle_{m,l} = \left\langle R_0^2 \right\rangle_{m,l} + \left\langle R^2 \right\rangle_{m,l} .$$

Calculating the average over states (53) we obtain

$$\overline{r^2} = \frac{2}{\gamma} N^2 \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{|z_1|^{2m+|l|+l} |z_2|^{2m+|l|-l}}{\Gamma(m+1)\Gamma(m+|v|+1)} \times$$
$$\times \left\{ (2m+|v|+1) - i z_1^* z_2^* e^{i\omega t} + i z_1 z_2 e^{-i\omega t} \right\} .$$

Simple calculation yields the final result

$$\overline{r^2} = \frac{2}{\gamma} \left\{ 1 + |z_1|^2 + |z_2|^2 - 2q\cos(\omega t + \chi) + \right.$$
$$\left. + N^2 q^{-\mu} \left[q I_{\mu-1}(2q) - |z_2|^2 I_\mu(2q) \right] - N^2 q^\mu \left[q L_{1-\mu}(2q) - |z_2|^2 I_{-\mu}(2q) \right] \right\} .$$

When $\mu = 0$ we have

$$\overline{r^2} = \frac{2}{\gamma} \left[1 + |z_1|^2 + |z_2|^2 + 2q\cos(\omega t + \varphi_0 - \alpha) \right] ,$$

which corresponds to the classical expression with the correction for quantum variance.

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Density matrix of the Universe: origin of inflation and cosmological acceleration stages

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

We present an overview of a recently suggested new model of quantum initial conditions for the Universe in the form of the cosmological density matrix. This density matrix originally suggested in the Euclidean quantum gravity framework turns out to describe the microcanonical ensemble in the Lorentzian quantum gravity of spatially closed cosmological models. This ensemble represents an equipartition in the physical phase space of the theory (sum over everything), but in terms of the observable spacetime geometry it is peaked about the set of cosmologies limited to a bounded range of the cosmological constant. This suggests the mechanism of constraining the landscape of string vacua and a possible solution to the dark energy problem in the form of the quasi-equilibrium decay of the microcanonical state of the Universe. The effective Friedmann equation governing this decay incorporates the effect of the conformal anomaly of quantum fields and features a new mechanism for a cosmological acceleration stage – big boost scenario. We also briefly discuss the relation between our model, the AdS/CFT correspondence and RS and DGP braneworlds.

1 Introduction

It is widely recognized that Euclidean quantum gravity (EQG) is a lame duck in modern particle physics and cosmology. After its summit in early and late eighties (in the form of the cosmological wavefunction proposals [1, 2] and baby universes boom [3]) the interest in this theory gradually declined, especially, in cosmological context, where the problem of quantum initial conditions was superseded by the concept of stochastic inflation [4]. EQG could not stand the burden of indefiniteness of the Euclidean gravitational action [5] and the cosmology debate of the tunneling vs no-boundary proposals [6].

Thus, a recently suggested EQG density matrix of the Universe [7] is hardly believed to be a viable candidate for the initial state of the Universe, even though it avoids the infrared catastrophe of small cosmological constant Λ, generates an ensemble of quasi-thermal universes in the limited range of Λ, and suggests a strong selection mechanism for the landscape of string vacua [7, 8]. Here we want to give a brief overview of these results and also justify them by deriving from first principles of Lorentzian quantum gravity applied to a microcanonical ensemble of closed cosmological models.

This work is dedicated to Iosif Buchbinder on the occasion of his sixtieth birthday, and he might be interested to know how his wide scope of scientific results in renormalization theory, conformal invariance, supersymmetry and quantum gravity finds applications in quantum cosmology underlying the origin of our Universe and its phenomenology. In particular, he might be fascinated with a possibly paradigmatic role of thermodynamics and statistical physics – the field in which he started his scientific career – in the solution of the cosmological constant and dark energy problems.

In Sect.2 we begin with the construction of the EQG density matrix of spatially closed cosmology and its application to the anomaly driven cosmological model originally considered in [9, 10] – the works which represented perhaps first examples of self-consistent inflationary models. These earlier works disregarded the formulation of initial conditions which later were considered in the form of the no-boundary proposal [1] and, more recently, using the AdS/CFT correspondence [11]. In contrast to [1] we allow the possibility that the initial state of the Universe is a mixed state and compute its statistical sum [7, 8]. The latter can be calculated within the $1/N_{\rm cdf}$-expansion in the number, $N_{\rm cdf}$, of conformally invariant fields under the assumption that they outnumber all other degrees of freedom. This statistical sum is dominated by the set of the quasi-thermal instantons with the effective cosmological constant of the early Universe belonging to a finite range, strictly bounded from above and from below [7, 8]. It also shows that the vacuum Hartle-Hawking instantons are excluded from the initial conditions, having *infinite positive* Euclidean gravitational effective action [7] due to the contribution of the conformal anomaly.

In Sect.3 we justify this construction by proving that the suggested EQG density matrix in fact represents the microcanonical ensemble in cosmology [12]. In view of the peculiarities of spatially closed cosmology this ensemble describes ultimate (unit weight) equipartition in the physical phase space of the theory. This can be interpreted as a sum over Everything, thus emphasizing a distinguished role of this candidate for the initial state of the Universe.

In Sect.4 we analyze the cosmological evolution in this model with the initial conditions set by the instantons of [7, 8]. In particular, we derive the modified Friedmann equation incorporating the effect of the conformal anomaly at late radiation and matter domination stages [13]. This equation shows that the vacuum (Casimir) part of the energy density is "degravitated" via the effect of the conformal anomaly – the Casimir energy does not weigh. Moreover, together with the recovery of the general relativistic behavior, this equation can feature a stage of cosmological acceleration followed by what we call a *big boost* singularity. At this singularity the scale factor acceleration grows in finite cosmic time up to infinity with a finite limiting value of the Hubble factor, when the Universe again enters a quantum phase demanding for its description a UV completion of the low-energy semiclassical theory.

In Sect.5 we discuss the possibility of realizing this scenario within the AdS/CFT and braneworld setups, in particular, when the conformal anomaly and its effective action are induced on 4D boundary/brane from the type IIB supergravity theory in the 5D bulk. We also comment on the relation between our model and the DGP setup.

2 EQG density matrix in the model of anomaly driven cosmology

A density matrix $\rho(q, q')$ in Euclidean quantum gravity [14] is related to a spacetime having two disjoint boundaries Σ and Σ' associated with its two entries q and q' (collecting both gravity and matter observables), see Fig.1. The metric and matter configuration on this spacetime $[g, \phi]$ interpolates between q and q', thus establishing mixing correlations.

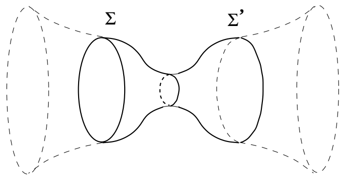

Figure 1: The picture of Euclidean spacetime underlying the EQG density matrix, whose two arguments are associated with the surfaces Σ and Σ'. Dashed lines depict the Lorentzian signature spacetime nucleating at Σ and Σ'.

This obviously differs from the pure Hartle-Hawking state $|\Psi_{HH}\rangle$ which can also be formulated in terms of a special density matrix $\hat{\rho}_{HH}$. For the latter the spacetime bridge between Σ and Σ' is broken, so that the spacetime is a union of two disjoint hemispheres which smoothly close up at their poles (Fig.2) — a picture illustrating the factorization of $\hat{\rho}_{HH} = |\Psi_{HH}\rangle\langle\Psi_{HH}|$.

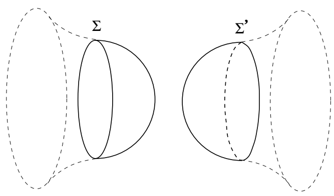

Figure 2: Density matrix of the pure Hartle-Hawking state represented by the union of two instantons of vacuum nature.

Analogously to the prescription for the Hartle-Hawking state [1], the EQG density matrix can be defined by the path integral [7, 8] over gravitational g and matter ϕ fields on the spacetime of the above type interpolating between the observables q and q' respectively at Σ and Σ',

$$\rho(q, q') = e^{\Gamma} \int D[g, \phi] \exp\big(-S_E[g, \phi]\big), \tag{1}$$

where $S_E[g, \phi]$ is the classical Euclidean action of the system. In view of the density matrix normalization $\mathrm{tr}\hat{\rho} = 1$ the corresponding statistical sum $\exp(-\Gamma)$ is given by a similar path integral,

$$e^{-\Gamma} = \int_{\rm periodic} D[g, \phi] \exp\big(-S_E[g, \phi]\big), \tag{2}$$

over periodic fields on the toroidal spacetime with identified boundaries Σ and Σ', see Fig.3.

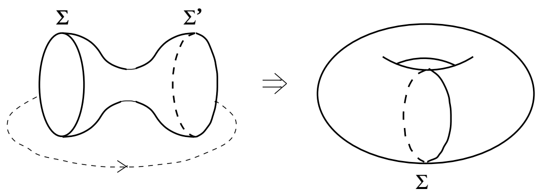

Figure 3: Calculation of the statistical sum represented by a toroidal compactification with periodically identified Euclidean time.

For a closed cosmology with the S^3-topology of spatial sections this statistical sum can be represented by the path integral over the periodic scale factor $a(\tau)$ and lapse function $N(\tau)$ of the minisuperspace metric

$$ds^2 = N^2(\tau)\, d\tau^2 + a^2(\tau)\, d^2\Omega^{(3)} \tag{3}$$

on the toroidal $S^1 \times S^3$ spacetime [7, 8]

$$e^{-\Gamma} = \int\limits_{\rm periodic} D[\,a, N\,]\; e^{-\Gamma_E[\,a,\,N\,]}, \tag{4}$$

$$e^{-\Gamma_E[\,a,\,N]} = \int\limits_{\rm periodic} D\Phi(x)\, e^{-S_E[\,a,\,N;\,\Phi(x)\,]}. \tag{5}$$

Here $\Gamma_E[\,a,\,N]$ is the Euclidean effective action of all inhomogeneous "matter" fields which include also metric perturbations on minisuperspace background $\Phi(x) = (\phi(x), \psi(x), A_\mu(x), h_{\mu\nu}(x), ...)$, $S_E[a, N; \Phi(x)] \equiv S_E[\,g, \phi\,]$ is the original classical action of the theory under the decomposition of the full configuration space into the minisuperspace and perturbations sectors,

$$[\,g, \phi\,] = [\,a(\tau), N(\tau);\, \Phi(x)\,], \tag{6}$$

and the integration also runs over periodic fields $\Phi(x)$.

Under the assumption that the system is dominated by free matter fields conformally coupled to gravity this action is exactly calculable by the conformal transformation taking the metric (3) into the static Einstein metric with $a = \text{const}$ [7]. In units of the Planck mass $m_P = (3\pi/4G)^{1/2}$ the action reads

$$\Gamma_E[\,a, N\,] = m_P^2 \int d\tau\, N \left\{ -aa'^2 - a + \frac{\Lambda}{3}a^3 + B\left(\frac{a'^2}{a} - \frac{a'^4}{6a}\right) + \frac{B}{2a} \right\} + F(\eta), \tag{7}$$

where

$$F(\eta) = \pm \sum_\omega \ln\left(1 \mp e^{-\omega\eta}\right), \quad \eta = \int d\tau N/a, \tag{8}$$

and $a' \equiv da/Nd\tau$. The first three terms in curly brackets represent the classical Einstein action with a primordial cosmological constant Λ, the B-terms correspond to the contribution of the conformal anomaly and the contribution of the vacuum (Casimir) energy ($B/2a$) of conformal fields on a static Einstein spacetime. $F(\eta)$ is the free energy of these fields – a typical boson or fermion sum over field oscillators with energies ω on a unit 3-sphere, η playing the role of the inverse temperature — an overall circumference of the toroidal instanton measured in units of the conformal time. The constant B,

$$B = \frac{3\beta}{4m_P^2} = \frac{\beta G}{\pi}, \tag{9}$$

is determined by the coefficient β of the topological Gauss-Bonnet invariant $E = R_{\mu\nu\alpha\gamma}^2 - 4R_{\mu\nu}^2 + R^2$ in the overall conformal anomaly of quantum fields

$$g_{\mu\nu}\frac{\delta\Gamma_E}{\delta g_{\mu\nu}} = \frac{1}{4(4\pi)^2} g^{1/2}\left(\alpha\Box R + \beta E + \gamma C_{\mu\nu\alpha\beta}^2\right) \tag{10}$$

($C_{\mu\nu\alpha\beta}^2$ is the Weyl tensor squared term). For a model with N_0 scalars, $N_{1/2}$ Weyl spinors and N_1 gauge vector fields it reads [16]

$$\beta = \frac{1}{360}\left(2N_0 + 11N_{1/2} + 124N_1\right). \tag{11}$$

The coefficient γ does not contribute to (7) because the Weyl tensor vanishes for any FRW metric. What concerns the coefficient α is more complicated. A nonvanishing α induces higher derivative terms $\sim \alpha(a'')^2$ in the action and, therefore, adds one extra degree of freedom to the minisuperspace sector of a and N and results in instabilities[1]. But α can be renormalized to zero by adding a finite *local* counterterm $\sim R^2$ admissible by the renormalization theory. We assume this *number of degrees of freedom preserving* renormalization to keep theory consistent both at the classical and quantum levels [7]. It is interesting that this finite renormalization changes the value of the Casimir energy of conformal fields in closed Einstein cosmology in such a way that for all spins this energy is universally expressed in terms of the same conformal anomaly coefficient B (corresponding to the $B/2a$ term in (7)) [7]. As we will see, this leads to the gravitational screening of the Casimir energy, mediated by the conformal anomaly of quantum fields.

Ultimately, the effective action (7) contains only two dimensional constants – the Planck mass squared (or the gravitational constant) $m_P^2 = 3\pi/4G$ and the cosmological constant Λ. They have to be considered as renormalized quantities. Indeed, the effective action of conformal fields contains divergences, the quartic and quadratic ones being absorbed by the renormalization of the initially singular bare cosmological and gravitational constants to yield finite renormalized m_P^2 and Λ [15]. Logarithmically divergent counterterms have the same structure as curvature invariants in the anomaly (10). When integrated over the spacetime closed toroidal FRW instantons they identically vanish because the $\Box R$ term is a total derivative, the Euler number E of $S^3 \times S^1$ is zero, $\int d^4x g^{1/2}E = 0$, and $C_{\mu\nu\alpha\beta} = 0$. There is however a finite tail of these vanishing logarithmic divergences in the form of the conformal anomaly action which incorporates the coefficient β of E in (10) and constitutes a major contribution to Γ_E — the first two B-dependent terms of (8)[2]. Thus, in fact, this model when considered in the leading order of the $1/N$-expansion (therefore disregarding loop

[1] In Einstein theory this sector does not contain physical degrees of freedom at all, which solves the problem of the formal ghost nature of a in the Einstein Lagrangian. Addition of higher derivative term for a does not formally lead to a ghost – the additional degree of freedom has a good sign of the kinetic term as it happens in $f(R)$-theories, but still leads to the instabilities discovered in [10].

[2] These terms can be derived from the metric-dependent Riegert action [17] or the action in terms of the conformal factor relating two metrics [18, 19, 20] and generalize the action of [9] to the case of a spatially closed cosmology with $\alpha = 0$.

effects of the graviton and other non-conformal fields) is renormalizable in the minisuperspace sector of the theory.

The path integral (4) is dominated by the saddle points — solutions of the equation $\delta\Gamma_E/\delta N(\tau) = 0$ which reads as

$$-\frac{a'^2}{a^2} + \frac{1}{a^2} - B\left(\frac{1}{2}\frac{a'^4}{a^4} - \frac{a'^2}{a^4}\right) = \frac{\Lambda}{3} + \frac{C}{a^4}, \tag{12}$$

with C given by

$$C = \frac{B}{2} + \frac{dF(\eta)}{d\eta}, \quad \eta = 2k\int_{\tau_-}^{\tau_+}\frac{d\tau}{a}. \tag{13}$$

Note that the usual (Euclidean) Friedmann equation is modified by the anomalous B-term and the radiation term C/a^4. The constant C sets the amount of radiation and satisfies the bootstrap equation (13), where $B/2$ is the contribution of the Casimir energy, and

$$\frac{dF(\eta)}{d\eta} = \sum_\omega \frac{\omega}{e^{\omega\eta} \mp 1} \tag{14}$$

is the energy of the gas of thermally excited particles with the inverse temperature η. The latter is given in (13) by the k-fold integral between the turning points of the scale factor history $a(\tau)$, $\dot{a}(\tau_\pm) = 0$. This k-fold nature implies that in the periodic solution the scale factor oscillates k times between its maximum and minimum values $a_\pm = a(\tau_\pm)$, see Fig.4

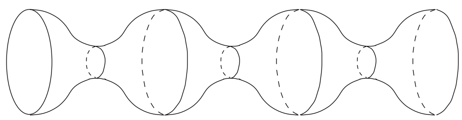

Figure 4: The garland segment consisting of three folds of a simple instanton glued at surfaces of a maximal scale factor.

As shown in [7], such solutions represent garland-type instantons which exist only in the limited range of values of the cosmological constant

$$0 < \Lambda_{\rm min} < \Lambda < \frac{3\pi}{2\beta G}, \tag{15}$$

and eliminate the vacuum Hartle-Hawking instantons corresponding to $a_- = 0$. Hartle-Hawking instantons are ruled out in the statistical sum by their infinite positive effective action which is due to the contribution of the conformal anomaly. Hence the tree-level predictions of the theory are drastically changed.

The upper bound of the range (15) is entirely caused by the quantum anomaly – it represents a new quantum gravity scale which tends to infinity when one switches the quantum effects off, $\beta \to 0$. The lower bound $\Lambda_{\rm min}$ can be attributed to both radiation and anomaly, and can be obtained numerically for any field content of the model. For a large number of conformal fields, and therefore a large β, the lower bound is of the order $\Lambda_{\rm min} \sim 1/\beta G$. Thus the restriction (15) can be regarded as a solution of the cosmological constant problem in the early Universe, because specifying a sufficiently high number of conformal fields one can achieve a primordial value of Λ well below the Planck scale where the effective theory applies, but high enough to generate a sufficiently long inflationary stage. Also this restriction can be potentially considered as a selection criterion for the landscape of string vacua [7, 12].

3 Microcanonical density matrix of the Universe: sum over Everything

The period of the quasi-thermal instantons is not a freely specifiable parameter and can be obtained as a function of G and Λ from Eqs. (12)-(13). Therefore this model clearly does not describe a canonical ensemble, but rather a microcanonical ensemble [12] with only two freely specifiable dimensional parameters — the renormalized gravitational and renormalized cosmological constants as discussed above.

To show this, contrary to the EQG construction of the above type, consider the density matrix as the canonical path integral in *Lorentzian* quantum gravity. Its kernel in the representation of 3-metrics and matter fields denoted below as q reads

$$\rho(q_+, q_-) = e^{\Gamma} \int\limits_{q(t_\pm)=\,q_\pm} D[\,q, p, N\,]\; e^{\,i\int_{t_-}^{t_+} dt\,(p\,\dot{q} - N^\mu H_\mu)}, \tag{16}$$

where the integration runs over histories of phase-space variables $(q(t), p(t))$ interpolating between $q_\pm$ at $t_\pm$ and the Lagrange multipliers of the gravitational constraints $H_\mu = H_\mu(q, p)$ — lapse and shift functions $N(t) = N^\mu(t)$. The measure $D[\,q, p, N\,]$ includes the gauge-fixing factor of the delta function $\delta\,[\,\chi\,] = \prod_t \prod_\mu \delta(\chi^\mu)$ of gauge conditions χ^μ and the relevant ghost factor [21, 22] (condensed index μ includes also continuous spatial labels). It is important that the integration range of N^μ,

$$-\infty < N < +\infty, \tag{17}$$

generates in the integrand the delta-functions of the constraints $\delta(H) = \prod_\mu \delta(H_\mu)$. As a consequence the kernel (16) satisfies the set of Wheeler-DeWitt equations

$$\hat{H}_\mu\big(q, \partial/i\partial q\big)\,\rho(\,q, q') = 0, \tag{18}$$

and the density matrix (16) can be regarded as an operator delta-function of these constraints

$$\hat\rho \sim \text{“}\prod_\mu \delta(\hat{H}_\mu)\text{”}. \tag{19}$$

This expression should not be understood literally because the multiple delta-function here is not uniquely defined, for the operators $\hat{H}_\mu$ do not commute and form an open algebra. Moreover, exact operator realization $\hat{H}_\mu$ is not known except the first two orders of a semiclassical $\hbar$-expansion [23]. Fortunately, we do not need a precise form of these constraints, because we will proceed with their path-integral solutions adjusted to the semiclassical perturbation theory.

The very essence of our proposal is the interpretation of (16) and (19) as the density matrix of a *microcanonical* ensemble in spatially closed quantum cosmology. A simplest analogy is the density matrix of an unconstrained system having a conserved Hamiltonian $\hat{H}$ in the microcanonical state with a fixed energy E, $\hat\rho \sim \delta(\hat{H} - E)$. A major distinction of (19) from this case is that spatially closed cosmology does not have freely specifiable constants of motion like the energy or other global charges. Rather it has as constants of motion the Hamiltonian and momentum constraints H_μ, all having a particular value — zero. Therefore, the expression (19) can be considered as a most general and natural candidate for the quantum state of the *closed* Universe. Below we confirm this fact by showing that in the physical sector the corresponding statistical sum is a uniformly distributed (with a unit weight) integral over entire phase space of true physical degrees of freedom. Thus, this is the sum over Everything. However, in terms of the observable quantities, like spacetime geometry, this distribution turns out to be nontrivially peaked around a particular set of universes. Semiclassically this distribution is given by the EQG density matrix and the saddle-point instantons of the above type [7].

From the normalization of the density matrix in the physical Hilbert space we have

$$1 = \mathrm{Tr}_{\rm phys}\,\hat\rho = \int dq\,\mu\big(q,\partial/i\partial q\big)\,\rho(q,q')\Big|_{q'=q} \\ = e^{\varGamma} \int\limits_{\rm periodic} D[\,q,p,N\,]\,e^{\,i\int dt(p\,\dot q - N^\mu H_\mu)}. \tag{20}$$

Here in view of the coincidence limit $q' = q$ the integration runs over periodic histories $q(t)$, and $\mu\big(q,\partial/i\partial q\big) = \hat\mu$ is the measure which distinguishes the physical inner product in the space of solutions of the Wheeler-DeWitt equations $(\psi_1|\psi_2) = \langle\psi_1|\hat\mu|\psi_2\rangle$ from that of the space of square-integrable functions, $\langle\psi_1|\psi_2\rangle = \int dq\,\psi_1^*\psi_2$. This measure includes the delta-function of unitary gauge conditions $\chi^\mu = \chi^\mu(q,p)$ and an operator factor incorporating the relevant ghost determinant [23].

On the other hand, in terms of the physical phase space variables the Faddeev-Popov path integral equals [21, 22]

$$\int D[\,q,p,N\,]\,e^{\,i\int dt\,(p\,\dot q - N^\mu H_\mu)} \\ = \int Dq_{\rm phys}\,Dp_{\rm phys}\,e^{i\int dt\,(p_{\rm phys}\,\dot q_{\rm phys} - H_{\rm phys}(t))} \\ = \mathrm{Tr}_{\rm phys}\,\Big(\mathrm{T}\,e^{-i\int dt\,\hat H_{\rm phys}(t)}\Big)\,, \tag{21}$$

where T denotes the chronological ordering. The physical Hamiltonian and its operator realization $\hat H_{\rm phys}(t)$ are nonvanishing here only in unitary gauges explicitly depending on time [23], $\chi^\mu(q,p,t)$. In static gauges, $\partial_t\chi^\mu = 0$, they vanish, because the full Hamiltonian in closed cosmology is a combination of constraints.

The path integral (21) is gauge-independent on-shell [21, 22] and coincides with that in the static gauge. Therefore, from Eqs.(20)-(21) with $\hat H_{\rm phys} = 0$, the statistical sum of our microcanonical ensemble equals

$$e^{-\varGamma} = \mathrm{Tr}_{\rm phys}\,\mathbf{I}_{\rm phys} = \int dq_{\rm phys}\,dp_{\rm phys} \\ = \text{sum over Everything}. \tag{22}$$

Here $\mathbf{I}_{\rm phys} = \delta(q_{\rm phys} - q'_{\rm phys})$ is a unit operator in the physical Hilbert space, whose kernel when represented as a Fourier integral yields extra momentum integration (2π-factor included into $dp_{\rm phys}$). This sum over Everything (as a counterpart to the concept of creation from “anything” in [24]), not weighted by any nontrivial density of states, is a result of general covariance and closed nature of the Universe lacking any freely specifiable constants of motion. The Liouville integral over entire *physical* phase space, whose structure and topology is not known, is very nontrivial. However, below we show that semiclassically it is determined by EQG methods and supported by instantons of [7] spanning a bounded range of the cosmological constant.

Integration over momenta in (20) yields a Lagrangian path integral with a relevant measure and action

$$e^{-\varGamma} = \int D[\,q, N\,]\; e^{iS_L[\,q,\,N\,]}. \qquad (23)$$

As in (20) integration runs over periodic fields (not indicated explicitly but assumed everywhere below) even despite the Lorentzian signature of the underlying spacetime. Similarly to the decomposition (6) of [7, 8] leading to (4)-(5), we decompose $[\,q, N\,]$ into a minisuperspace $[\,a_L(t), N_L(t)\,]$ and the "matter" $\varPhi_L(x)$ variables, the subscript L indicating their Lorentzian nature. With a relevant decomposition of the measure $D[\,q, N\,] = D[\,a_L, N_L\,] \times D\varPhi_L(x)$, the microcanonical sum reads

$$e^{-\varGamma} = \int D[\,a_L, N_L\,]\; e^{i\varGamma_L[\,a_L,\,N_L\,]}, \qquad (24)$$

$$e^{i\varGamma_L[\,a_L,\,N_L\,]} = \int D\varPhi_L(x)\; e^{iS_L[\,a_L,\,N_L;\,\varPhi_L(x)]}, \qquad (25)$$

where $\varGamma_L[\,a_L,\, N_L\,]$ is a Lorentzian effective action. The stationary point of (24) is a solution of the effective equation $\delta\varGamma_L/\delta N_L(t) = 0$. In the gauge $N_L = 1$ it reads as a Lorentzian version of the Euclidean equation (12) and the bootstrap equation for the radiation constant C with the Wick rotated $\tau = it$, $a(\tau) = a_L(t)$, $\eta = i\int dt/a_L(t)$. However, with these identifications C turns out to be purely imaginary (in view of the complex nature of the free energy $F(i\int dt/a_L))$. Therefore, no periodic solutions exist in spacetime with a *real* Lorentzian metric.

On the contrary, such solutions exist in the Euclidean spacetime. Alternatively, the latter can be obtained with the time variable unchanged $t = \tau$, $a_L(t) = a(\tau)$, but with the Wick rotated lapse function

$$N_L = -iN, \quad iS_L[\,a_L, N_L; \phi_L] = -S_E[\,a, N; \varPhi\,]. \qquad (26)$$

In the gauge $N = 1$ ($N_L = -i$) these solutions exactly coincide with the instantons of [7]. The corresponding saddle points of (24) can be attained by deforming the integration contour (17) of N_L into the complex plane to pass through the point $N_L = -i$ and relabeling the real Lorentzian t with the Euclidean τ. In terms of the Euclidean $N(\tau)$, $a(\tau)$ and $\varPhi(x)$ the integrals (24) and (25) take the form of the path integrals (4) and (5) in EQG,

$$i\varGamma_L[\,a_L,\, N_L] = -\varGamma_E[\,a,\, N\,]. \qquad (27)$$

However, the integration contour for the Euclidean $N(\tau)$ runs from $-i\infty$ to $+i\infty$ through the saddle point $N = 1$. This is the source of the conformal rotation in Euclidean quantum gravity, which is called to resolve the problem of unboundedness of the gravitational action and effectively renders the instantons a thermal nature, even though they originate from the microcanonical ensemble. This mechanism implements the justification of EQG from the canonical quantization of gravity [25] (see also [26] for the black hole context).

4 Cosmological evolution and Big Boost scenario

The gravitational instantons of Sect.2 can be regarded as setting initial conditions for the cosmological evolution in the physical spacetime with the Lorentzian signature. Indeed, those initial conditions can be viewed as those at the nucleation of the Lorentzian spacetime from the Euclidean spacetime at the maximum value of the scale factor $a_+ = a(\tau_+)$ at the turning point of the Euclidean solution τ_+ — the minimal (zero extrinsic curvature) surface of the instanton. For the contribution of the one-fold instanton to the density matrix of the Universe this nucleation process is depicted in Fig. 1.

The Lorentzian evolution can be obtained by analytically continuing the Euclidean time into the complex plane by the rule $\tau = \tau_+ + it$. Correspondingly the Lorentzian effective equation follows from the Euclidean one (12) as

$$\frac{\dot a^2}{a^2} + \frac{1}{a^2} - \frac{B}{2}\left(\frac{\dot a^2}{a^2} + \frac{1}{a^2}\right)^2 = \frac{\Lambda}{3} + \frac{C - B/2}{a^4}, \tag{28}$$

where the dot, from now on, denotes the derivative with respect to the Lorentzian time t. This can be solved for the Hubble factor as

$$\frac{\dot a^2}{a^2} + \frac{1}{a^2} = \frac{1}{B}\left\{1 - \sqrt{1 - 2B\left(\frac{\Lambda}{3} + \frac{\mathcal{C}}{a^4}\right)}\right\}, \tag{29}$$

$$\mathcal{C} \equiv C - \frac{B}{2}. \tag{30}$$

We have thus obtained a modified Friedmann equation in which the overall energy density, including both the cosmological constant and radiation, very nonlinearly contributes to the square of the Hubble factor.

An interesting property of this equation is that the Casimir energy does not weigh. Indeed the term $B/2a^4$ is completely subtracted from the full radiation density C/a^4 in the right hand side of (28) and under the square root of (29). Only "real" thermally excited quanta contribute to the right-hand side of (29). Indeed, using (13), the radiation contribution $\mathcal{C}/a^4$ is seen to read simply as

$$\frac{\mathcal{C}}{a^4} = \frac{1}{a^4}\sum_\omega \frac{\omega}{e^{\omega\eta} \mp 1}. \tag{31}$$

This is an example of the gravitational screening which is now being intensively searched for the cosmological constant [27, 28]. As we see, in our case, this mechanism is mediated by the conformal anomaly action, but it applies not to the cosmological constant, but rather to the Casimir energy which has the equation of state of radiation $p = \varepsilon/3$. This gravitational screening is essentially based on the above mentioned renormalization that eradicates higher derivatives from the effective action and thus preserves the minisuperspace sector free from dynamical degrees of freedom.

After nucleation from the Euclidean instanton at the turning point with $a = a_+$ and $\dot a_+ = 0$ the Lorentzian Universe starts expanding, because $\ddot a_+ > 0$. Therefore, the radiation quickly dilutes, so that the primordial cosmological constant starts dominating and can generate an inflationary stage. It is natural to assume that the primordial Λ is not fundamental, but is due to some inflaton field. This effective Λ is nearly constant during the Euclidean stage and the inflation stage, and subsequently leads to a conventional exit from inflation by the slow roll mechanism[3].

During a sufficiently long inflationary stage, particle production of conformally non-invariant matter takes over the polarization effects of conformal fields. After being thermalized

[3]In the Euclidean regime this field also stays in the slow roll approximation, but in view of the oscillating nature of a scale factor it does not monotonically decay. Rather it follows these oscillations with much lower amplitude and remains nearly constant during all Euclidean evolution, whatever long this evolution is (as it happens for garland instantons with the number of folds $k \to \infty$).

at the exit from inflation this matter gives rise to an energy density $\varepsilon(a)$ which should replace the energy density of the primordial cosmological constant and radiation. Therefore, at the end of inflation the combination $\Lambda/3 + \mathcal{C}/a^4$ should be replaced according to

$$\frac{\Lambda}{3} + \frac{\mathcal{C}}{a^4} \to \frac{8\pi G}{3}\,\varepsilon(a) \equiv \frac{8\pi G}{3}\,\rho(a) + \frac{\mathcal{C}}{a^4}. \tag{32}$$

Here $\varepsilon(a)$ denotes the full energy density including the component $\rho(a)$ resulting from the decay of Λ and the radiation density of the primordial conformal matter $\mathcal{C}/a^4$. The dependence of $\varepsilon(a)$ on a is of course determined by the equation of state via the stress tensor conservation, and $\rho(a)$ also includes its own radiation component emitted by and staying in (quasi)equilibrium with the baryonic part of the full $\varepsilon(a)$.

Thus the modified Friedmann equation finally takes the form

$$\frac{\dot a^2}{a^2} + \frac{1}{a^2} = \frac{\pi}{\beta G}\left\{1 - \sqrt{1 - \frac{16G^2}{3}\,\beta\varepsilon}\right\}, \tag{33}$$

where we expressed B according to (9).

In the limit of small subplanckian energy density $\beta G^2\varepsilon \equiv \beta\varepsilon/\varepsilon_P \ll 1$ the modified equation goes over into the ordinary Friedmann equation in which the parameter β completely drops out

$$\frac{\dot a^2}{a^2} + \frac{1}{a^2} = \frac{8\pi G}{3}\,\varepsilon. \tag{34}$$

Therefore within this energy range the standard cosmology is recovered. Depending on the effective equation of state, a wide set of the standard scenarios of late cosmological evolution can be obtained, including those showing a cosmic acceleration, provided some kind of dark energy component is present [29, 30].

The range of applicability of the GR limit (34) depends however on β. This makes possible a very interesting mechanism to happen for a very large β. Indeed, the value of the argument of the square root in (33) can be sufficiently far from 1 even for small ε provided $\beta \sim N_{\rm cdf} \gg 1$. Moreover, one can imagine a model with a variable number of conformal fields $N_{\rm cdf}(t)$ inducing a time-dependent, and implicitly a scale factor-dependent β, $\beta = \beta(a)$. If $\beta(a)$ grows with a faster than the rate of decrease of $\varepsilon(a)$, then the solution of (33) can reach a singular point, labeled below by ∞, at which the square root argument vanishes and the cosmological acceleration becomes infinite. This follows from the expression

$$\frac{\ddot a}{a} \sim \frac{4\pi}{3\beta G}\,\frac{a(G^2\beta\varepsilon)'}{\sqrt{1 - 16G^2\beta\varepsilon/3}}, \tag{35}$$

where prime denotes the derivative with respect to a. This expression becomes singular at $t = t_\infty$ even though the Hubble factor $H^2 \equiv (\dot a^2/a^2 + 1/a^2)$ remains finite when $(G^2\beta\varepsilon)_\infty = 3/16$, $H^2_\infty = 16\pi(G\varepsilon)_\infty/3$.

Assuming for simplicity that the matter density has a dust-like behavior and β grows as a power law in a

$$G\varepsilon \sim \frac{1}{a^3}, \quad G\beta \sim a^n, \quad n > 3, \tag{36}$$

one easily finds an inflection point $t = t_*$ when the cosmological acceleration starts after the deceleration stage, $(G^2\beta\varepsilon)_* = 3(n-2)/4(n-1)^2$,

$$H^2_* = \frac{8\pi}{3}\,\frac{n-1}{n-2}\,(G\varepsilon)_*. \tag{37}$$

Also it is useful to comment on the duration of the acceleration stage before reaching the singularity. If we identify our epoch with some instant t_0 soon after t_*, this duration until the singularity can be estimated by disregarding the spatial curvature term. It reads $t_\infty - t_* \sim \sqrt{B_0} \sim H_0^{-1}$, which is comparable to the age of the Universe. Thus, although the acceleration stage does not pass the eternity test of [31], its duration is very large.

Nevertheless, the evolution ends in this model with the curvature singularity, $\ddot{a} \to \infty$, reachable in a finite proper time. Unlike the situation with a big brake singularity of [32] it cannot be extended beyond this singularity analytically even by smearing it out or taking into account its weak integrable nature. In contrast to [32] the acceleration at the singularity is positive. Hence, we called this type of singularity a *big boost*. The effect of the conformal anomaly drives the expansion of the Universe to the maximum value of the Hubble constant, after which the solution becomes complex. This, of course, does not make the model a priori inconsistent, because for $t \to t_\infty$ an infinitely growing curvature invalidates the semiclassical and $1/N$ approximations. This is a new essentially quantum stage which requires a UV completion of the effective low-energy theory.

5 AdS/CFT correspondence and braneworld settings

What can be the mechanism of a variable and indefinitely growing β? One such mechanism is well known – phase transitions in cosmology between different vacua can give a mass m to an initially massless particle. This results in the loss of conformal invariance of the corresponding particle, which instead of contributing to the vacuum polarization by its own β factor starts generating the Coleman-Weinberg type potential $\sim m^4 \ln(m^2/\mu^2)$. However this effect is weak and decreases the effective value of β, which is opposite to what we need.

Another mechanism was suggested in [12]. It relies on the possible existence, motivated by string theory, of extra dimensions whose size is evolving in time. Indeed, theories with extra dimensions provide a qualitative mechanism to promote β to the level of a modulus variable which can grow with the evolving size L of those dimensions, as we now explain. The parameter β basically counts the number $N_{\rm cdf}$ of conformal degrees of freedom, $\beta \sim N_{\rm cdf}$ (see Eq.(11)). However, if one considers a string theory in a space time with more than four dimensions, the extra-dimension being compact with typical size L, the effective 4-dimensional fields arise as Kaluza-Klein (KK) and winding modes with masses (see e.g. [33])

$$m_{n,w}^2 = \frac{n^2}{L^2} + \frac{w^2}{\alpha'^2}\, L^2 \tag{38}$$

(where n and w are respectively the KK and winding numbers), which break their conformal symmetry. These modes remain approximately conformally invariant as long as their masses are much smaller than the spacetime curvature, $m_{n,w}^2 \ll H_0^2 \sim m_P^2/N_{\rm cdf}$. Therefore the number of conformally invariant modes changes with L. Simple estimates show that the number of pure KK modes ($w = 0$, $n \leq N_{\rm cdf}$) grows with L as $N_{\rm cdf} \sim (m_P L)^{2/3}$, whereas the number of pure winding modes ($n = 0$, $w \leq N_{\rm cdf}$) grows as L decreases as $N_{\rm cdf} \sim (m_P \alpha'/L)^{2/3}$. Thus, it is possible to find a growing β in both cases with expanding or contracting extra dimensions. In the first case it is the growing tower of superhorizon KK modes which *makes* the horizon scale $H_0 \sim m_P/\sqrt{N_{\rm cdf}} \sim m_P/(m_P L)^{1/3}$ decrease as L increases to infinity. In the second case it is the tower of superhorizon winding modes which makes this scale decrease with the decreasing L as $H_0 \sim m_P(L/m_P\alpha')^{1/3}$. At the qualitative level of this discussion so far, such a scenario is flexible enough to accommodate the present day acceleration scale (though, at the price of fine-tuning an enormous coefficient governing the expansion or contraction of L).

However, string (or rather string-inspired) models can offer a more explicit construction of these ideas. In particular, some guidance can be obtained from the AdS/CFT picture. Indeed, in this picture [34] a higher dimensional theory of gravity, namely type IIB supergravity compactified on $AdS_5 \times S^5$, is seen to be equivalent to a four dimensional conformal theory, namely $\mathcal{N} = 4$ $SU(N)$ SYM, thought to live on the boundary of AdS_5 space-time. An interesting arena for a slight generalization of these ideas is the Randall-Sundrum model [35] where a 3-brane is put in the inside of AdS_5 space-time resulting in a large distance recovery of 4D gravity without the need for compactification. This model has a dual description. On the one hand it can just be considered from a 5D gravity perspective, on the other hand it can also be described, thanks to the AdS/CFT picture, by a 4D conformal field theory coupled to gravity.

Indeed, in this picture, the 5D SUGRA — a field-theoretic limit of compactified type IIB string theory — induces on the brane of the underlying AdS background the quantum effective action of the conformally invariant 4D $\mathcal{N} = 4$ $SU(N)$ SYM theory coupled to the 4D geometry of the boundary. The multiplets of this CFT contributing according to (11) to the total conformal anomaly coefficient β are given by $(N_0, N_{1/2}, N_1) = (6N^2, 4N^2, N^2)$ [37], so that

$$\beta = \frac{1}{2}\,N^2. \tag{39}$$

The parameters of the two theories are related by the equation [34, 36, 11]

$$\frac{L^3}{2G_5} = \frac{N^2}{\pi}, \tag{40}$$

where L is the radius of the 5D AdS space-time with the negative cosmological constant $\Lambda_5 = -6/L^2$ and G_5 is the 5D gravitational constant. The radius L is also related to the 't Hooft parameter of the SYM coupling $\lambda = g^2_{SYM}N$ and the string length scale $l_s = \sqrt{\alpha'}$, $L = \lambda^{1/4} l_s$. The generation of the 4D CFT from the local 5D supergravity holds in the limit when both N and λ are large. This guarantees the smallness of string corrections and establishes the relation between the weakly coupled tree-level gravity theory in the bulk ($G_5 \to 0$, $L \to \infty$) and the strongly coupled 4D CFT ($g^2_{SYM} \gg 1$). Moreover, as said above, the AdS/CFT correspondence explains the mechanism of recovering general relativity theory on the 4D brane of the Randall-Sundrum model [36, 11]. The 4D gravity theory is induced on the brane from the 5D theory with the negative cosmological constant $\Lambda_5 = -6/L^2$. In the one-sided version of this model the brane has a tension $\sigma = 3/8\pi G_5 L$ (the 4D cosmological constant is given by $\Lambda_4 = 8\pi G_4 \sigma$), and the 4D gravitational constant $G_4 \equiv G$ turns out to be

$$G = \frac{2G_5}{L}. \tag{41}$$

One recovers 4D General Relativity at low energies and for distances larger than the radius of the AdS bulk, L. Thus, the CFT dual description of the 5D Randall-Sundrum model is very similar to the model considered above. Moreover, even though the CFT effective action is not exactly calculable for $g^2_{SYM} \gg 1$ it is generally believed that its conformal anomaly is protected by extended SUSY [39] and is exactly given by the one-loop result (10). Therefore it generates the exact effective action of the anomalous (conformal) degree of freedom given by (8), which guarantees a good $1/N_{\rm cdf}$-approximation for the gravitational dynamics.

Applying further the above relations it follows a relation between our β coefficient and the radius L of the AdS space-time, given by $\beta G = \pi L^2/2$. Introducing this in the modified

Friedmann equation (33), the latter becomes explicitly depending on the size of the 5D AdS spacetime as given by

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{2}{L^2}\left\{1 - \sqrt{1 - L^2\left(\frac{8\pi G}{3}\,\rho + \frac{\mathcal{C}}{a^4}\right)}\right\}, \tag{42}$$

where we have reintroduced the decomposition (32) of the full matter density into the decay product of the inflationary and matter domination stages, with energy density ρ, and the thermal excitations of the primordial CFT (31).

For low energy density, $GL^2\rho \ll 1$ and $L^2\mathcal{C}/a^4 \ll 1$, in the approximation beyond the leading order, cf. Eq.(34), the modified Friedmann equation coincides with the modified Friedmann equation in the Randall-Sundrum model [38]

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{8\pi G}{3}\,\rho\left(1 + \frac{\rho}{2\sigma}\right) + \frac{\mathcal{C}}{a^4}, \tag{43}$$

here $\sigma = 3/8\pi G_5 L = 3/4\pi G L^2$ is the Randall-Sundrum brane tension and $\mathcal{C}$ is the braneworld constant of motion [38, 41].[4] Note that the thermal radiation on the brane (of non-Casimir energy nature) is equivalent to the mass of the bulk black hole associated with this constant. This fact can be regarded as another manifestation of the AdS/CFT correspondence in view of the known duality between the bulk black hole and the thermal CFT on the brane [41].

Interestingly, this comparison between our model and the Randall-Sundrum framework also allows one to have some insight on the phenomenologically allowed physical scales. Indeed, it is well known that the presence of an extra-dimension in the Randall-Sundrum model, or in the dual language, that of the CFT, manifests itself typically at distances lower than the *AdS* radius L. Hence, it is perfectly possible to have a large number of conformal fields in the Universe, *à la* Randall-Sundrum, without noticing their presence in the everyday experiments, provided L is small enough. Moreover, if one uses the scenario of [7] to set the initial conditions for inflation, it provides an interesting connection between the Hubble radius of inflation, given by eq. (15), and the distance at which the presence of the CFT would manifest itself in gravity experiments, both being given by L. Last, it seems natural in a string theory setting, to imagine that the *AdS* radius L can depend on time, and hence on the scale factor.

In this case, assuming that the AdS/CFT picture still holds when L is adiabatically evolving, one can consider the possibility that $GL^2\varepsilon$ is large, and that $L^2(t)$ grows faster than $G\varepsilon(t)$ decreases during the cosmological expansion. One would then get the cosmological acceleration scenario of the above type followed by the big boost singularity.

In this case, however, should this acceleration scenario correspond to the present day accelerated expansion, L should be of the order of the present size of the Universe, i.e. $L^{-2} \sim H_0^2$. Since the Randall-Sundrum mechanism recovers 4D GR only at distances beyond the curvature radius of the AdS bulk, $r \gg L$, it means that local gravitational physics of our model (42) at the acceleration stage is very different from the 4D general relativity. Thus this mechanism can hardly be a good candidate for generating dark energy in real cosmology.

It is interesting that there exists an even more striking example of a braneworld setup dual to our anomaly driven model. This is the generalized DGP model [40] including together with the 4D and 5D Einstein-Hilbert terms also the 5D cosmological constant, Λ_5, in the special case of the *vacuum* state on the brane with a vanishing matter density $\rho = 0$. In contrast to the Randall-Sundrum model, for which this duality holds only in the low energy limit —

[4] We assume that the dark radiation term is redshifted, as a grows, faster than the matter term and expand to the second order in ρ, but the first order in $\mathcal{C}$.

small ρ and small $\mathcal{C}/a^4$, vacuum DGP cosmology *exactly* corresponds to the model of [7] with the 4D cosmological constant Λ simulated by the 5D cosmological constant Λ_5.

Indeed, in this model (provided one neglects the bulk curvature), gravity interpolates between a 4D behaviour at small distances and a 5D behaviour at large distances, with the crossover scale between the two regimes being given by r_c,

$$\frac{G_5}{2G} = r_c, \tag{44}$$

and in the absence of stress-energy exchange between the brane and the bulk, the modified Friedmann equation takes the form [42]

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} - r_c^2 \left(\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} - \frac{8\pi G}{3}\,\rho \right)^2 = \frac{\Lambda_5}{6} + \frac{\mathcal{C}}{a^4}. \tag{45}$$

Here $\mathcal{C}$ is the same as above constant of integration of the bulk Einstein's equation, which corresponds to a nonvanishing Weyl tensor in the bulk (or a mass for a Schwarzschild geometry in the bulk) [38, 41]. It is remarkable that this equation with $\rho = 0$ exactly coincides with the modified Friedmann equation of the anomaly driven cosmology (28) under the identifications

$$B \equiv \frac{\beta G}{\pi} = 2 r_c^2, \tag{46}$$

$$\Lambda = \frac{\Lambda_5}{2}. \tag{47}$$

These identifications imply that in the DGP limit $G \ll r_c^2$, the anomaly coefficient β is much larger than 1.

This looks very much like the generation of the vacuum DGP model for any value of the dark radiation $\mathcal{C}/a^4$ from the anomaly driven cosmology with a very large $\beta \sim m_P^2 r_c^2 \gg 1$. However, there are several differences. A first important difference between the conventional DGP model and the anomaly driven DGP is that the former does not incorporate the self-accelerating branch [42, 43] of the latter. This corresponds to the fact that only one sign of the square root is admissible in Eq.(29) — a property dictated by the instanton initial conditions at the nucleation of the Lorentzian spacetime from the Euclidean one. So, one does not have to worry about possible instabilities associated with the self-accelerating branch [44].

Another important difference concerns the way the matter energy density manifests itself in the Friedmann equation for the non-vacuum case. In our 4D anomaly driven model it enters the right hand side of the equation as a result of the decay (32) of the effective 4D cosmological constant Λ, while in the DGP model it appears inside the parenthesis of the left hand side of equation (45). Therefore, the DGP Hubble factor reads as

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{8\pi G}{3}\,\rho + \frac{1}{2r_c^2}\left\{ 1 - \sqrt{1 - 4r_c^2\left(\frac{\Lambda_5}{6} + \frac{\mathcal{C}}{a^4} - \frac{8\pi G}{3}\,\rho\right)} \right\} \tag{48}$$

(note the negative sign of ρ under the square root and the extra first term on the right hand side). In the limit of small ρ, $\mathcal{C}/a^4$ and Λ_5, the above equation yields a very different behavior from the GR limit of the anomaly driven model (34),

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \simeq \frac{\Lambda_5}{6} + \frac{\mathcal{C}}{a^4} + r_c^2 \left(\frac{\Lambda_5}{6} + \frac{\mathcal{C}}{a^4} - \frac{8\pi G}{3}\,\rho \right)^2. \tag{49}$$

For vanishing Λ_5 and $\mathcal{C}/a^4$ this behavior corresponds to the 5D dynamical phase [42, 43] which is realized in the DGP model for a very small matter energy density on the brane $\rho \ll 3/32\pi G r_c^2 \sim m_P^2/r_c^2$.

Of course, in this range the DGP braneworld reduces to a vacuum brane, but one can also imagine that the 5D cosmological constant decays into matter constituents similar to (32) and thus simulates the effect of ρ in Eq.(33). This can perhaps provide us with a closer correspondence between the anomaly driven cosmology and the non-vacuum DGP case. But here we would prefer to postpone discussions of such scenarios to future analyses and, instead, focus on the generalized *single-branch* DGP model to show that it also admits the cosmological acceleration epoch followed by the big boost singularity.

Indeed, for positive Λ_5 satisfying a very weak bound

$$\Lambda_5 > \frac{3}{2r_c^2} \tag{50}$$

Eq.(48) has a solution for which, during the cosmological expansion with $\rho \to 0$, the argument of the square root vanishes and the acceleration tends to $\pm\infty$. For the effective a-dependence of r_c^2 and $G\rho$ analogous to (36), $r_c^2(a) \sim a^n$ and $G\rho(a) \sim 1/a^3$, the acceleration becomes positive at least for $n \geq 0$,

$$\frac{\ddot{a}}{a} \simeq \frac{n + 32\pi G\, r_c^2 \rho}{4r_c^2 \sqrt{1 + 4r_c^2 \Big(\frac{8\pi G}{3}\,\rho - \frac{\Lambda_5}{6} - \frac{\mathcal{C}}{a^4}\Big)}} \to +\infty. \tag{51}$$

This is the big boost singularity labeled by ∞ and having a finite Hubble factor $(\dot{a}^2/a^2 + 1/a^2)_\infty = \Lambda_5/6 + 1/4r_c^2$.

Thus, the *single-branch* DGP cosmology can also lead to a big boost version of acceleration. For that to happen, one does not actually need a growing r_c (which can be achieved at the price of having a time dependent G_5 — itself some kind of a modulus, in a string inspired picture). The DGP crossover scale r_c can be constant, $n = 0$, and the big boost singularity will still occur provided the lower bound (50) is satisfied [5]. When Λ_5 violates this bound, the acceleration stage is eternal with an asymptotic value of the Hubble factor squared $H^2 = \dot{a}^2/a^2$ given by $\big(1 - \sqrt{1 - 2r_c^2\Lambda_5/3}\big)/2r_c^2$.

6 Conclusions

To summarize, within a minimum set of assumptions (the equipartition in the physical phase space (22)), we have the mechanism of generating a limited range of the positive cosmological constant which is likely to constrain the landscape of string vacua and get the full evolution of the Universe as a quasi-equilibrium decay of its initial microcanonical state[6].

We have obtained the modified Friedmann equation for this evolution in the anomaly dominated cosmology. This equation exhibits a gravitational screening of the quantum Casimir energy of conformal fields — this part of the total energy density does not weigh, being degravitated due to the contribution of the conformal anomaly. Also, in the low-density limit this equation does not only show a recovery of the standard general relativistic behavior, but

[5] More precisely, one should also take into account here the modification due the dark radiation contribution $\mathcal{C}/a^4$. However, the latter is very small at late stages of expansion.

[6] Thus, contrary to anticipations of Sidney Coleman that "there is nothing rather than something" regarding the actual value of the cosmological constant [3], one can say that something (rather than nothing) comes from everything.

also coincides with the dynamics of the Randall-Sundrum cosmology within the AdS/CFT duality relations. Moreover, for a very large and rapidly growing value of the Gauss-Bonnet coefficient β in the conformal anomaly this equation features a regime of cosmological acceleration followed by a big boost singularity. At this singularity the acceleration factor grows in finite proper time up to infinity with a finite limiting value of the Hubble factor. A proper description of the late phase of this evolution, when the Universe enters again a quantum phase, would require a UV completion of the low-energy semiclassical theory.

A natural mechanism for a growing β can be based on the idea of an adiabatically evolving scale associated with extra dimensions [12] and realized within the picture of AdS/CFT duality, according to which a conformal field theory is induced on the 4D brane from the 5D non-conformal theory in the bulk. As is well known, this duality sheds light on the 4D general relativistic limit in the Randall-Sundrum model [36, 11]. Here we observed an extended status of this duality from the cosmological perspective — the generalized Randall Sundrum model with the Schwarzschild-AdS bulk is equivalent to the anomaly driven cosmology for small energy density. In particular, the radiation content of the latter is equivalent to the dark radiation term $\mathcal{C}/a^4$ pertinent to the Randall-Sundrum braneworld with a bulk black hole of mass $\mathcal{C}$.

Another intriguing observation concerns the *exact* correspondence between the anomaly driven cosmology and the vacuum DGP model generalized to the case of a nonvanishing bulk cosmological constant Λ_5. In this case a large β is responsible for the large crossover scale r_c, (44). For positive Λ_5 satisfying the lower bound (50) this model also features a big boost scenario even for stabilized β. Below this bound (but still for positive $\Lambda_5 > 0$, because a negative Λ_5 would imply a time of maximal expansion from which the Universe would start recollapsing) the cosmological evolution eventually enters an eternal acceleration phase. However, the DGP model with matter on the brane can hardly be equivalent to the 4D anomaly driven cosmology, unless one has some mechanism for Λ_5 to decay and to build up matter density on the brane.

Unfortunately, our scenario put in the framework of the AdS/CFT correspondence with adiabatically evolving scale of extra dimension cannot agree with the observed dark energy, because, for the required values of the parameters, the local gravitational physics of this model would becomes very different from the 4D general relativity.

In general, the idea of a very large central charge of CFT algebra, underlying the solution of the hierarchy problem in the dark energy sector and particle phenomenology, seems hovering in current literature [45, 46]. Our idea of a big growing β belongs to the same scope, but its realization seems missing a phenomenologically satisfactory framework. In essence, it can be considered as an attempt to cross a canyon in two endeavors — the leap of 32 decimal orders of magnitude in $N_{\rm cdf}$ of [46], separating the electroweak and Planckian scales, versus our 120 orders of magnitude needed to transcend separation between the Hubble and Planckian scales. Both look equally speculative from the viewpoint of local phenomenology.

Probably some other modification of this idea can be more productive. In particular, an alternative mechanism of running β could be based on the winding modes. These modes do not seem to play essential role in the AdS/CFT picture with a big scale of extra dimensions L, because they are heavy in this limit. On the contrary, this mechanism should work in the opposite case of contracting extra dimensions, for which the restrictions from local gravitational physics do not apply (as long as for $L \to 0$ the short-distance correction go deeper and deeper into UV domain).

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Complete and Consistent Non-Minimal String Corrections to Supergravity

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

We give a complete and consistent solution to string corrected (deformed), $D = 10$, $N = 1$ supergravity as the non-minimal low energy limit of string theory. We solve the Bianchi identities with suitable constraints to second order in the string slope parameter. In so doing we pave the way for continuing the study of the many applications of these results. We also modify, reaffirm and correct a previously given incomplete solution, and we introduce an important adjustment to the known first order results.

1 Introduction

This work is inspired by the celebration of the activity of Prof. Buchbinder. One of us (S.B.) has been a steady collaborator of the Tomsk group in the past decade, with 16 original papers published in collaboration. Most of them involved Prof. Anton Galajinsky and two were directly related to the work of Prof. Buchbinder. The latter ones dealt with noncommutative field theory, however many discussions and interactions with Prof. Buchbinder over the years, concerned issues related to supersymmetry and supergravity theories, an area where Prof. Buchbinder obtained some of his numerous prestigious achievements. Hence it is quite suitable to present in this dedicated volume an investigation that connects to the stimulating and seemingly almost everlasting issue of string-corrected ten-dimensional supergravity theories.

The route to finding a manifestly supersymmetric theory of D= 10, N=1 supergravity at second order in the string slope parameter has encountered many difficulties over the years. Some years ago a solution to D=10, N=1 Supergravity as the low energy limit of String Theory was given at first order in the string slope parameter and was recently re-calculated [1]. In a sense this was a minimal solution. This approach was founded on what is nowadays refered to, as the scenario of Gates and collaborators; (see [1], [2], and references therein). Other varied approaches are also pursued, however the power of this older approach is now being vindicated. A partial second order solution was recently given in [3] and [4]. It was incomplete and therefore in doubt due do an unsatisfactory assumption in the curvature sector, as well as a computational error. Here we reaffirm that that solution is correct up to a curvature term, and in particular that the proposed X tensor is valid. We then show that the results obtained can be used to solve the curvature Bianchi identity, equation (3). We achieve this by introducing ${R^{(1)}}_{ab\gamma}{}^{\delta}$, and then imposing a condition on it which also modifies the old first order results. The difficulties that prevented the complete closing of the Bianchi identities at second order are overcome. We present the full set of equations that consistently satisfy all required Bianchi identities. As the work in itself is lengthy we leave finding the equations of motion and other applications for another letter. We also do not list results explicitly solved by Bianchi identies. For this approach it is required that we solve the Bianchi identities for D=10 N=1 supergravity in superspace at second order in the string slope parameter, and in the presence of the Lorentz Chern-Simons Form, using the so called Beta Function Favored Constraints [5].[1] This approach has been detailed to first order in [1], and to second order in [3], so we will not recount it here. We show that all results fall neatly into place in a very elegant way, therefore further vindicating the whole original scenario. We note here that it appears also to work consistently at third order, as we have proceeded to that order, and that is for yet another work.

2 Review of Solution and Notation

The Bianchi identities in Superspace are as follows:

$$[[\nabla_{[A}, \nabla_B\}, \nabla_{C)}\} = 0 \tag{1}$$

Here we have switched off Yang-Mills fields and the commutator is given by

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C \nabla_C + \frac{1}{2} R_{ABd}{}^e M_e{}^d \tag{2}$$

A solution must be found in such a way that all if the identities are simultaneously satisfied. A small alteration in one sector will change the whole picture. Most of the resulting identities are listed in [1] and [4], so we will not list them here. The second order solution given in part in [3] and [4] to some extent was based upon an Ansatz for the so called X tensor, as well as extensive algebraic manipulations. The necessity for introducing the X tensor was predicted by Gates et. al., [1]. In [3], and [4], the following Bianchi identity was not properly solved:

$$T_{(\alpha\beta|}{}^{\lambda} R_{|\gamma)\lambda de} \; - \; T_{(\alpha\beta|}{}^{g} R_{|\gamma)gde} \; - \; \nabla_{(\alpha|} R_{\beta\gamma)de} \; = \; 0 \tag{3}$$

It is crucial to show that all of the second order torsions and curvatures satisfy this identity. Also ${R^{(2)}}_{\gamma gde}$ is required, in order to complete the set. Various ideas, such as finding a new

[1] In earlier works, this made the determination of a D=10 globally supersymmetric and Lorentz covariant higher derivative Yang-Mills action possible, to order γ^3 (see e.g. [6]), an important result for topologically nontrivial gauge vector field configurations, as in the case of compactified string theories on manifolds with topologically nontrivial properties.

X tensor, imposing constraints on the spinor derivative $\nabla_\alpha \chi_\beta$ at second order or adjusting the super current A_{abc} at second order were considered. We have found that including these adjustments and constraints is unnecessary, and might in fact be wrong.

In this paper we find a complete and consistent solution. We also point out that equation (58) in reference [3] (or equation (115) in reference [4]) is wrong.

In order to avoid a proliferation of terms we maintain the same notation and conventions as in [1], but to avoid recasting the first order results, we denote all quantities by the order in the slope parameter

$$R_{ABde} = R^{(0)}{}_{ABde} + R^{(1)}{}_{ABde} + R^{(2)}{}_{ABde} + ...$$

$$T_{AD}{}^G = T^{(0)}{}_{AD}{}^G + T^{(1)}{}_{AD}{}^G + T^{(2)}{}_{AD}{}^G ...$$

In this work we make some improvements to the notation of references [3]. For example an apparently fundamental object is the following:

$$\Omega^{(1)}{}_{gef} = L^{(1)}{}_{gef} - \frac{1}{4} A^{(1)}{}_{gef} \tag{4}$$

and its spinor derivative which we denote simply as

$$\Omega^{(1)}{}_{\alpha gef} = \nabla_\gamma \{ L^{(1)}{}_{gef} - \frac{1}{4} A^{(1)}{}_{gef} \} \tag{5}$$

We leave it like this for brevity of notation. The numerical superscript refers to the order of the quantity. A crucial input at first order is that for the super-current $A^{(1)}{}_{gef}$. The choice made for on-shell conditions in [1] and hence also [3], is as follows:

$$A^{(1)}{}_{gef} = i\gamma \sigma_{gef\epsilon\tau} T^{mn\epsilon} T_{mn}{}^\tau \tag{6}$$

In [3], we proposed the form of the X tensor to read

$$T^{(2)}{}_{\alpha\beta}{}^d \ = \sigma^{pqref}{}_{\alpha\beta} X_{pqrefd} = -\frac{i\gamma}{6} \sigma^{pqref}{}_{\alpha\beta} H^{(0)d}{}_{ef} A^{(1)}{}_{pqr}. \tag{7}$$

A fundamental result which was used in every Bianchi identity and which is very lengthy to derive is the following:

$$\begin{aligned} T^{(0)}{}_{(\alpha\beta|}{}^\lambda \sigma^{pqref}{}_{|\gamma)\lambda} A^{(1)}{}_{pqr} H^{(0)}{}_{def} - \sigma^{pqref}{}_{(\alpha\beta|} H^{(0)}{}_{def} \nabla_{|\gamma)} A^{(1)}{}_{pqr} \\ = -24 \sigma^g{}_{(\alpha\beta|} H^{(0)}{}_d{}^{ef} [\Omega^{(1)}{}_{\gamma gef}] \end{aligned} \tag{8}$$

We note however in this paper that this result can be achieved indirectly by using the first order results found in [1], in conjunction with the Bianchi identity (3), listed in this paper. We found that the following dimension one half torsion is given uniquely by:

$$T^{(2)}{}_{\alpha\beta}{}^{\lambda} \;=\; -\,\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}T_{ef}{}^{\lambda}. \tag{9}$$

It was then shown that together with the proposed X tensor Ansatz as well as equation (8) and other observations and results, the H sector Bianchi identities as listed in [1], [2] could be solved simultaneously with the torsions (10) and (11) as listed below

$$T_{(\alpha\beta|}{}^{\lambda}T_{|\gamma)\lambda}{}^{d} \;-\; T_{(\alpha\beta|}{}^{g}T_{|\gamma)g}{}^{d} \;-\; \nabla_{(\alpha|}T_{\beta\gamma)}{}^{d} \;=\; 0 \tag{10}$$

and

$$T_{(\alpha\beta|}{}^{\lambda}T_{|\gamma)\lambda}{}^{\delta} \;-\; T_{(\alpha\beta|}{}^{g}T_{|\gamma)g}{}^{\delta} \;-\; \nabla_{(\alpha|}T_{|\beta\gamma)}{}^{\delta} - \;\frac{1}{4}R_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \;=\; 0. \tag{11}$$

We find the second order solutions to (10) to be given by (7) and the following

$$\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} = 4\gamma\sigma^{g}{}_{(\alpha\beta|}\Omega^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} - \frac{i\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi}, \tag{12}$$

$$\begin{aligned}
T^{(2)}{}_{\gamma ab} = +2\gamma[\Omega^{(1)}{}_{\gamma[a|ef]}H^{(0)}{}_{|b]}{}^{ef} + \sigma_{ab}{}_{\gamma}{}^{\phi}[\frac{\gamma}{3}\Omega^{(1)}{}_{\phi gef}H^{(0)gef}] \\
-\frac{\gamma}{6}\sigma_{[a|}{}^{g}{}_{\gamma}{}^{\phi}\{\Omega^{(1)}{}_{\phi|b]ef}H^{(0)}{}_{g}{}^{ef} + \Omega^{(1)}{}_{\phi gef}H^{(0)}{}_{|b]}{}^{ef}\} \\
-\frac{i\gamma}{12}A^{(1)}{}_{pqr}\sigma^{pqrg}{}_{[a|\phi\lambda}T^{(0)}{}_{|b]g}{}^{\lambda} \\
-\frac{i\gamma}{72}\sigma_{ab}{}_{\gamma}{}^{\phi}\sigma^{pqreg}{}_{\phi\lambda}A^{(1)}{}_{pqr}T^{(0)}{}_{eg}{}^{\lambda} \\
\frac{i\gamma}{144}A^{(1)}{}_{pqr}\sigma_{[a|}{}^{g}{}_{\gamma}{}^{\phi}[\sigma^{pqre}{}_{|b]\phi\lambda}T^{(0)}{}_{eg}{}^{\lambda} + \sigma^{pqre}{}_{g\phi\lambda}T^{(0)}{}_{e|b]}{}^{\lambda}]
\end{aligned} \tag{13}$$

In equation (11), we notice the occurrence of the term

$$-\nabla_{(\alpha|}T^{(0)}{}_{|\beta\gamma)}{}^{\delta(Order2)} = [2\delta_{(\alpha|}{}^{\delta}\delta_{|\beta)}{}^{\lambda} + \sigma^{g}{}_{(\alpha\beta|}\sigma_{g}{}^{\delta\lambda}]\nabla_{|\gamma)}\chi_{\lambda}{}^{(2)}. \tag{14}$$

This was not properly considered in references [3]. In this work we find that there is no need to modify the spinor derivative of χ_α at second order so that an additional constraint on this derivative is unnecessary. For the solution of (11) we extract after some algebra, and neat cancelations, the candidates

$$T^{(2)}{}_{\gamma g}{}^{\delta} = 2\gamma\; T^{(0)ef\,\delta}\Omega^{(1)}{}_{\gamma gef} \tag{15}$$

And

$$R^{(2)}{}_{\alpha\beta de} = -\;\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}R^{(0)}{}_{efde}. \tag{16}$$

We now must show that all of the above found results satisfy (3).

3 New Solution for $R^{(2)}{}_{\lambda gde}$

We must show that we can close equation (3) using the results (7), (9), (15), and (16). As mentioned, various approaches such as implementing the previously suggested constraints did not work, nor was there any way to manipulate the terms using the sigma matrix algebra. Eventually the following procedure provides a solution. At second order the Bianchi identity (3) becomes

$$T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}R^{(2)}{}_{|\gamma)\lambda de} + T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de} - T^{(0)}{}_{(\alpha\beta|}{}^{g}R^{(2)}{}_{|\gamma)gde} - T^{(2)}{}_{(\alpha\beta|}{}^{g}R^{(0)}{}_{|\gamma)gde} - \nabla_{(\alpha|}[R^{(0)}{}_{|\beta\gamma)de}{}^{Order(2)} + R^{(1)}{}_{|\beta\gamma)de}{}^{Order(2)} + R^{(2)}{}_{|\beta\gamma)de}{}^{Order(2)}] = 0. \tag{17}$$

Using the results listed above we arrive at

$$-i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} + T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[-\frac{i\gamma}{12}\sigma^{pqrab}{}_{|\gamma)\lambda}A^{(1)}{}_{pqr}R^{(0)}{}_{abde}] - \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de} + \frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}H^{(0)g}{}_{ab}A^{(1)}{}_{pqr}R^{(0)}{}_{|\gamma)gde} - \nabla_{(\gamma|}\{-2i\sigma^{g}{}_{|\alpha\beta)}\Pi^{(0)+(1)}{}_{gde} + \frac{i}{24}\sigma^{pqr}{}_{de|\alpha\beta)}A^{(1)}{}_{pqr} - \frac{i}{12}\sigma^{pqrab}{}_{|\alpha\beta)}A^{(1)}{}_{pqr}R^{(0)}{}_{abde}\} = 0. \tag{18}$$

Here we encounter second order contributions from zero order terms but in solvable form. (That is where we can extract a quantity symmetrized with a sigma matrix)We define

$$\Pi_{g}{}^{ef} = L_{g}{}^{ef} - \frac{1}{8}A_{g}{}^{ef}. \tag{19}$$

Now again using out key relation (8) we obtain

$$-i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} + 2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(0)}{}_{abde}[\Omega^{(1)}{}_{|\gamma)gab}] - \nabla_{(\gamma|}\{-2i\sigma^{g}{}_{|\alpha\beta)}\Pi^{(0)+(1)}{}_{gde}\} - \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de} + \frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}H^{(0)g}{}_{ab}A^{(1)}{}_{pqr}R^{(0)}{}_{|\gamma)gde} + \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}R^{(0)}{}_{abde}] - \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)(Order(2))}_{pqr}] = 0 \tag{20}$$

Of particular concern and interest is the last term in (20). It was thought that a possible modification of $A^{(1)}{}_{pqr}$, or a contribution from $A^{(2)}{}_{pqr}$ would be necessary. Here we may avoid such a modification. In advance we anticipate that the solution will be as follows:

$$+i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} = +2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(0)}{}_{abde}[\Omega^{(1)}{}_{|\gamma)g}{}^{ab}] + \nabla_{(\gamma|}\{2i\sigma^{g}{}_{|\alpha\beta)}\Pi^{(0)+(1)}{}_{gde}\}^{Order(2)} \tag{21}$$

And

$$
-\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de}+\frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}H^{(0)g}{}_{ab}A^{(1)}{}_{pqr}R^{(0)}{}_{|\gamma)gde}
+\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}R^{(0)}{}_{abde}]-\frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}^{(Order(2)}_{pqr}]=0 \quad (22)
$$

We need to show that (22) does in fact vanish. We must begin with the Bianchi identity that gives the spinor derivative of $T_{kl}{}^{\tau}$.

$$
\nabla_{\gamma}T_{kl}{}^{\tau}=T_{\gamma[k|}{}^{\lambda}T_{\lambda|l]}{}^{\tau}+T_{\gamma[k}{}^{g}T_{g|l]}{}^{\tau}+T_{kl}{}^{\lambda}T_{\lambda\gamma}{}^{\tau}+T_{kl}{}^{g}T_{g\gamma}{}^{\tau}-\nabla_{[k|}T_{|l]\gamma}{}^{\tau}-R_{kl\gamma}{}^{\tau}. \quad (23)
$$

At first order this simplifies to

$$
\nabla_{\gamma}T_{kl}{}^{\tau\, Order(1)}=-R^{(1)}{}_{kl\gamma}{}^{\tau}-\frac{1}{48}[2H^{(0)}{}_{klg}\sigma^{g}{}_{\gamma\lambda}\sigma^{pqr\lambda\tau}A^{(1)}{}_{pqr}-\sigma_{[k|\gamma\lambda}\sigma^{pqr\lambda\tau}(\nabla_{|l]}A^{(1)}{}_{pqr})]. \quad (24)
$$

We now write the last term in (22), using the ten dimensional metric so that the unsolved part becomes

$$
-\frac{i}{12}\sigma^{pqrab}{}_{(\alpha\beta|}\{\gamma A^{(1)}{}_{pqr}[T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de}+T^{(0)g}_{ab}R^{(0)}{}_{|\gamma)gde}-\nabla_{|\gamma)}R^{(0)}{}_{abde}]
+\frac{1}{2}\eta_{ad}\,\eta_{be}\nabla_{|\gamma)}A^{(1)}{}^{Order(2)}_{pqr}\}=0 \quad (25)
$$

Using the definition of $A^{(1)}{}_{pqr}$ (6), yields

$$
+\frac{\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}\sigma_{pqr\epsilon\tau}T^{kl\epsilon}\{\gamma T_{kl}{}^{\tau}[T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de}+T^{(0)g}_{ab}R^{(0)}{}_{|\gamma)gde}-\nabla_{|\gamma)}R^{(0)}{}_{abde}]
+\eta_{ad}\eta_{be}\nabla_{|\gamma)}T_{kl}{}^{\tau}\}=0 \quad (26)
$$

We now use equation (24) and the properties of the sigma matrices. After some algebra we choose to impose a condition on $R^{(1)}{}_{kl\gamma}{}^{\tau}$. We require

$$
R^{(1)}{}_{kl\gamma}{}^{\tau}=+\frac{\gamma}{100}T_{kl}{}^{\tau}[T_{mn}{}^{\lambda}R^{(0)}{}_{\gamma\lambda}{}^{mn}+T^{(0)}{}_{mn}{}^{g}R^{(0)}{}_{\gamma g}{}^{mn}-\nabla_{\gamma}R^{(0)}{}_{mn}{}^{mn}]
-\frac{1}{48}[2H^{(0)}{}_{klg}\sigma^{g}{}_{\gamma\lambda}\sigma^{rst\lambda\tau}A^{(1)}{}_{rst}-2\sigma_{[k|\gamma\lambda}\sigma^{rst\lambda\tau}\nabla_{|l]}A^{(1)}{}_{rst}] \quad (27)
$$

This can now be added to the list of first order results quoted in [1]. $R^{(1)}{}_{kl\gamma}{}^{\tau}$ was not defined in [1]. We obtain as we required,

$$
R^{(2)}{}_{\gamma gde}=2\gamma R^{(0)}{}_{abde}[\Omega^{(1)}{}_{\gamma g}{}^{ab}]+2\nabla_{\gamma}\{\Pi^{(0)+(1)}{}_{gde}\}^{Order(2)} \quad (28)
$$

As a check we can also examine another Bianchi identity. The following Bianchi identity also includes $R^{(2)}{}_{\alpha bde}$:

$$
\frac{1}{4}R_{(\alpha|amn}\sigma^{mn}{}_{|\beta)}{}^{\gamma}+T_{\alpha\beta}{}^{g}T_{ga}{}^{\gamma}+T_{\alpha\beta}{}^{\lambda}T_{\lambda a}{}^{\gamma}+T_{a(\alpha|}{}^{\lambda}T_{|\beta)\lambda}{}^{\gamma}-T_{a(\alpha|}{}^{g}T_{|\beta)g}{}^{\gamma}
-\nabla_{(\alpha|}T_{|\beta)a}{}^{\gamma}-\nabla_{a}T_{\alpha\beta}{}^{\gamma}=0 \quad (29)
$$

This Bianchi identity after some cancelations results in the following expression:

$$
\begin{aligned}
\frac{1}{4}R^{(2)}{}_{(\alpha|amn}\sigma^{mn}{}_{|\beta)}{}^{\gamma} + 2\gamma\{\nabla_{(\alpha|}\Omega^{(1)}{}_{aef}\}[-\frac{1}{4}R^{(0)}{}_{ef}{}^{mn}\sigma^{\gamma}_{mn|\beta)}] \\
+i\sigma^{g}{}_{\alpha\beta}T^{(2)}{}_{ga}{}^{\gamma} - \frac{i\gamma}{6}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}H^{(0)g}{}_{ef}T^{(0)}{}_{ga}{}^{\gamma} + T^{(0)}{}_{\alpha\beta}{}^{\lambda}T^{(2)}{}_{\lambda a}{}^{\gamma} \\
-2\gamma T^{(0)}{}_{ef}{}^{\gamma}[\nabla_{(\alpha|}\nabla_{|\beta)}\{\Omega^{(1)}{}_{aef}\}] \\
+\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}\{\nabla_{a}A^{(1)}{}_{pqr}\}T^{(0)}{}_{ef}{}^{\gamma} + \frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}[\nabla_{a}T^{(0)}{}_{ef}{}^{\gamma}] \\
+[\delta_{(\alpha|}{}^{\lambda}\delta_{|\beta)}{}^{\phi} + \sigma^{g}{}_{\alpha\beta}\sigma_{g}{}^{\lambda\phi}]\nabla_{a}\chi_{\phi}{}^{Order2} \\
+\frac{1}{4}\sigma^{nm}{}_{(\alpha|}{}^{\gamma}\nabla_{|\beta)}\Pi_{amn}{}^{Order(2)} = 0
\end{aligned}
\tag{30}
$$

This identity also predicts the same form for $R^{(2)}{}_{\alpha amn}$. However it also includes a great deal of other information which we plan to include in another letter.

4 Conclusions

We have found a consistent and manifestly supersymmetric solution to the Bianchi identities for D=10, N=1 supergravity, with string corrections to second order in the slope parameter. We have reaffirmed the results and the proposed X tensor of [3] and [4], and we have solved the remaining previously intractable curvature. We have used the first order result for $A^{(1))}{}_{pqr}$ as first given in [1]. We have not modified it at second order. We have not imposed the constraint

$$
T_{\alpha b}{}^{\delta} = -\frac{1}{48}\sigma_{b\alpha\lambda}\sigma^{pqr\lambda\delta}A_{pqr} \tag{31}
$$

at second order. This is a conventional constraint and so could have been be imposed to all orders. However we dropped it in favor of requiring an adjustment to $T_{\alpha b}{}^{\delta(2)}$ as given by equation (15).

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6 Appendix I

Here for convenience we list the torsions curvatures and H sector results to second order, simply by including the results found at first order in [1].

$$
H_{\alpha\beta\gamma} = 0 + Order(\gamma^{3}) \tag{32}
$$

$$H_{\alpha\beta d} = +\frac{i}{2}\sigma_{d\alpha\beta} + 4i\gamma\sigma^{g}{}_{\alpha\beta}H_{\gamma}{}^{ef}H_{d}{}^{ef}$$
$$\sigma_{\alpha\beta}{}^{g}[8i\gamma H^{(0)}{}_{def}L^{(1)}{}_{g}{}^{ef} \;-\; i\gamma H^{(0)}{}_{def}A^{(1)}{}_{g}{}^{ef}]$$
$$+\;\sigma^{pqref}{}_{\alpha\beta}[\frac{i\gamma}{12}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] + Order(\gamma^{3}) \tag{33}$$

$$H_{\alpha ab} = +2i\gamma[-\sigma_{[a|\alpha\beta}T_{ef}{}^{\beta}G_{|b]}{}^{ef} - 2\sigma_{e\alpha\beta}T_{f[a|}{}^{\beta}G_{|b]}{}^{ef}]$$
$$2\gamma[\nabla_{\alpha}(H^{(0)}_{[a|ef}H^{(0)}{}_{|b]}{}^{ef} \;-\; \sigma_{ab\alpha}{}^{\phi}\nabla_{\phi}(H^{(0)}{}_{gef}H^{gef})]$$
$$+2i\gamma\sigma_{[a|\alpha\phi}T_{ef}{}^{\phi}\Pi^{(1)}{}_{|b]}{}^{ef} -\; 2i\gamma\sigma_{ab\alpha}{}^{\lambda}\sigma_{g\lambda\phi}T_{ef}{}^{\phi}\Pi^{(1)gef}$$
$$-\;\frac{\gamma}{6}\sigma^{g}{}_{[a|\alpha}{}^{\phi}\sigma_{|b]\lambda\phi}T_{ef}{}^{\lambda}\Pi^{(1)}{}_{g}{}^{ef} -\; \frac{\gamma}{6}\sigma^{g}{}_{[a|\alpha}{}^{\phi}\sigma_{g\lambda\phi}T_{ef}{}^{\lambda}\Pi^{(1)}{}_{|b]}{}^{ef}$$
$$-\;4\gamma R^{(1)}{}_{\alpha[a|}{}^{ef}H^{(0)}{}_{|b]ef} + T^{(2)}{}_{\alpha ab} + Order(\gamma^{3}) \tag{34}$$

$$T_{\alpha\beta}{}^{g} = i\sigma_{\alpha\beta}{}^{g} - \frac{i\gamma}{6}\sigma^{pqref}{}_{\alpha\beta}H^{(0)d}{}_{ef}A^{(1)}{}_{pqr} + Order(\gamma^{3}) \tag{35}$$

$$T_{abc} = -2L_{abc} \tag{36}$$

$$T_{\alpha\beta}{}^{\gamma} = -[\delta_{(\alpha|}{}^{\gamma}\delta_{|\beta)}{}^{\delta} + \sigma^{g}{}_{\alpha\beta}\sigma_{g}{}^{\gamma\delta}]\chi_{\delta} \;-\; \frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}T_{ef}{}^{\gamma} + Order(\gamma^{3}) \tag{37}$$

$$T_{\alpha g}{}^{\delta} = -\frac{1}{48}\sigma_{g\alpha\phi}\sigma^{pqr\phi\delta}A^{(1)}_{pqr} + 2\gamma\; T^{(0)ef\delta}\Omega^{(1)}{}_{\alpha gef} + Order(\gamma^{3}) \tag{38}$$

$$\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} = 4\gamma\sigma^{g}{}_{(\alpha\beta|}\Omega_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} - \frac{i\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi} \tag{39}$$

Or symmetrized,

$$T_{\gamma ab} = +2\gamma[\Omega^{(1)}{}_{\gamma[a|ef]}H^{(0)}{}_{|b]}{}^{ef} + \sigma_{ab\;\gamma}{}^{\phi}[\frac{\gamma}{3}\Omega^{(1)}{}_{\phi gef}H^{(0)gef}]$$
$$-\frac{\gamma}{6}\sigma_{[a|}{}^{g}{}_{\gamma}{}^{\phi}\{\Omega^{(1)}{}_{\phi|b]ef}H^{(0)}{}_{g}{}^{ef} + \Omega^{(1)}{}_{\phi gef}H^{(0)}{}_{|b]}{}^{ef}\}$$
$$-\frac{i\gamma}{12}A^{(1)}{}_{pqr}\sigma^{pqrg}{}_{[a|\phi\lambda}T^{(0)}{}_{|b]g}{}^{\lambda}$$
$$-\frac{i\gamma}{72}\sigma_{ab\;\gamma}{}^{\phi}\sigma^{pqreg}{}_{\phi\lambda}A^{(1)}{}_{pqr}T^{(0)}{}_{eg}{}^{\lambda}$$
$$\frac{i\gamma}{144}A^{(1)}{}_{pqr}\sigma_{[a|}{}^{g}{}_{\gamma}{}^{\phi}[\sigma^{pqre}{}_{|b]\phi\lambda}T^{(0)}{}_{eg}{}^{\lambda} + \sigma^{pqre}{}_{g\phi\lambda}T^{(0)}{}_{e|b]}{}^{\lambda}] + Order(\gamma^{3}) \tag{40}$$

$$R_{\alpha\beta de} = -2i\sigma^{g}{}_{\alpha\beta}\Pi_{gde}{}^{(1)} + \frac{i}{24}\sigma^{pqref}{}_{\alpha\beta}A_{pqr}{}^{(1)} - \frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}R_{efde} + Order(\gamma^3) \quad (41)$$

Where

$$\Pi^{(1)}{}_{g}{}^{ef} = L^{(1)}{}_{g}{}^{ef} - \frac{1}{8}A^{(1)}{}_{g}{}^{ef} \quad (42)$$

$$R_{\alpha gde} = -i\sigma_{[d|\alpha\phi}T_{g|e]}{}^{f} + i\gamma\sigma_{[g|\alpha\ \phi}T_{kl}{}^{\phi}R^{kl}{}_{|de]} +2\gamma R^{(0)}{}_{abde}[\Omega^{(1)}{}_{\alpha g}{}^{ab}] + 2\nabla_{\alpha}\{\Pi^{(0)+(1)}{}_{gde}\}^{Order(2)} + Order(\gamma^3) \quad (43)$$

$$A_{abc} = i\gamma\sigma_{gef\gamma\lambda}T^{mn\gamma}T_{mn}{}^{\lambda} \quad (44)$$

$$R^{(1)}{}_{kl\gamma}{}^{\tau} = +\frac{\gamma}{100}T_{kl}{}^{\tau}[T_{mn}{}^{\lambda}R^{(0)}{}_{\gamma\lambda}{}^{mn} + T^{(0)}{}_{mn}{}^{g}R^{(0)}{}_{\gamma g}{}^{mn} - \nabla_{\gamma}R^{(0)}{}_{mn}{}^{mn}] -\frac{1}{48}[2H^{(0)}{}_{klg}\sigma^{g}{}_{\gamma\lambda}\sigma^{rst\lambda\tau}A^{(1)}{}_{rst} - 2\sigma_{[k|\gamma\lambda}\sigma^{rst\lambda\tau}\nabla_{|l]}A^{(1)}{}_{rst}] \quad (45)$$

The spinor derivative of L_{abc} is solved and available from a Bianchi identity. We will list it in a later paper. $R^{(2)}{}_{kl\gamma}{}^{\tau}$ if it exits will likely show up from third order calculations of (3).

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Conformal Invariance in Deformed N=4 SYM Theory

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Abstract

We discuss a class of deformations of $\mathcal{N} = 4$ SYM theory and look for conditions under which the theory would be conformally invariant and finite. Applying the algorithm of perturbative adjustments of the couplings we construct the family of theories which are conformal up to 3 loops in the non-planar case and up to 4 loops in the planar one. We found particular solutions in the planar case when the conformal condition seems to be exhausted in the one loop order and present the arguments that these solutions might be valid in any loop order.

1 Introduction

During the last decade much attention has been paid to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) and its deformations obtained by the orbifold [1] or orientifold [2] projection, or by adding the marginal deformations [3] to the Lagrangian. Such deformations lead to theories with less supersymmetry but possibly inheriting some attractive features of the original $\mathcal{N} = 4$ SYM theory, namely the conformal invariance, integrability [4] in the planar limit, and, especially, its connection with dual string theory via the AdS/CFT correspondence.

Since the original version of the AdS/CFT correspondence [5] there appeared several modifications [6]. However, at present time it is not clear how to build gravity dual to an arbitrary superconformal supersymmetric gauge theory and which properties of the gauge theories are necessary for existence of this correspondence. In this context the marginally deformed SYM theories are of special interest.

In case of deformed $\mathcal{N}=4$ SYM theory [3] the initial symmetry is broken down to $\mathcal{N}=1$ supersymmetry, and $SU(4)_R$ global group down to $U(1)_R$. One of such examples is the so-called β-deformation of the original $\mathcal{N}=4$ SYM theory. Its gravity dual was constructed by Lunin and Maldacena [7] and significant role in this duality plays the $U(1)\times U(1)$ global symmetry of the β-deformed theory which was associated with isometries of the deformed $AdS_5\times\tilde{S}_5$ background. There were also attempts to construct the gravity dual to the full Leigh-Strassler deformation [8].

¿From the field theory side the investigation of the β-deformed case was made in [9, 10, 11]. The case of the full Leigh-Strassler deformation was less investigated. In this paper we are looking for the conformal invariance of the full Leigh-Strassler deformation. Using the dimensional regularization (reduction) we found conditions of conformal invariance up to four loops in the planar limit and up to three loops in the non-planar one.

There are special cases when the conformal conditions are exhausted in the one-loop order. In case of the beta-deformed theory in the planar limit this corresponds to real values of β. We also found such solutions for the full Leigh-Strassler deformation. We present them below and conjecture that they might be valid in any loop order.

2 The Leigh-Strassler Deformation of the $\mathcal{N}=4$ SYM Theory

The so called Leigh-Strassler deformation can be obtained by modification of the superpotential in the original $\mathcal{N}=4$ SYM theory written in terms of $\mathcal{N}=1$ superfields

$$S = \int d^8z Tr\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \left(\frac{1}{2g^2}\int d^6z Tr(W^\alpha W_\alpha) + \int d^6z\, \mathcal{W} + h.c.\right) \quad (1)$$

in such a way that

$$\begin{aligned} \mathcal{W}_{N=4\ SYM} &= ig(Tr(\Phi_1\Phi_2\Phi_3) - Tr(\Phi_1\Phi_3\Phi_2)) \rightarrow \\ \mathcal{W}_{LS\ SYM} &= i[h_1 Tr(\Phi_1\Phi_2\Phi_3) - h_2 Tr(\Phi_1\Phi_3\Phi_2) + \frac{h_3}{3}\sum_{i=1}^{3} Tr(\Phi_i^3)], \end{aligned} \quad (2)$$

where Φ_i with $i=1,2,3$ are the three chiral superfields of the original $\mathcal{N}=4$ SYM theory in adjoint representation of the gauge group $SU(N)$ and the couplings h_1, h_2, h_3 are in general complex. The beta-deformed case in the same notation corresponds to

$$h_1 = hq,\ h_2 = h/q,\ q = e^{i\pi\beta} \text{ and } h_3 = 0.$$

The superpotential (2) brakes the $SU(4)_R$ symmetry of the original $\mathcal{N}=4$ theory down to $U(1)_R$. In addition it is invariant under cyclic permutations of $(\Phi_1,\ \Phi_2,\ \Phi_3)$ and exchange: $\beta \leftrightarrow 1-\beta$ or in our notation $h_1 \leftrightarrow -h_2$.

Now one has to examine the UV divergences. In case of interest as in any $\mathcal{N}=1$ SYM theory formulated in terms of $\mathcal{N}=1$ superfields one has two types of divergent diagrams, those of the chiral field propagator and of the gauge field one. Moreover, the gauge propagator is not independent: its divergences are related to the chiral propagators [12]. Since in the Leigh-Strassler deformed $\mathcal{N}=4$ SYM case one has the same field content as in $\mathcal{N}=4$ SYM, so $\sum T(R) = 3C(G)$ and everything is defined by the chiral field anomalous dimension γ. Hence, the conformal invariance being understood as the vanishing of the beta function is valid on the sub-manifold in the coupling constant space which is defined by condition:

$$\gamma(g,\ \{h_i\}) = 0, \quad (3)$$

where $\{h_i\} = (h_1,\ h_2,\ h_3)$. One can solve this condition choosing the Yukawa couplings in the form of perturbation series over g [13]:

$$h_i = \alpha_{0i} g + \alpha_{1i} g^3 + \alpha_{2i} g^5 + ...\ , i = 1...3. \tag{4}$$

If the anomalous dimensions of the chiral fields vanish, so do the gauge and Yukawa beta functions and the theory is conformally invariant.

Conformal invariance also means that the theory is finite, i.e. all UV divergencies cancel (or in some gauges the sum of divergencies) and the renormalization factors Z (or their products) are equal to 1 or finite. In the context of dimensional regularization this can be achieved by adding to expansion over g (4) a similar expansion over the parameter of dimensional regularization $\varepsilon = 4 - D$, i.e. one has the two fold expansion instead of one fold one [14]

$$\begin{aligned} h_i &= g\left(a_i + \alpha_{0i}^{(1)}\varepsilon + \alpha_{0i}^{(2)}\varepsilon^2 + ... + \alpha_{0i}^{(n-2)}\varepsilon^{n-2} + \alpha_{0i}^{(n-1)}\varepsilon^{n-1} + \alpha_{0i}^{(n)}\varepsilon^n + ...\right) \\ &+ g^3\left(\alpha_{1i}^{(0)} + \alpha_{1i}^{(1)}\varepsilon + \alpha_{1i}^{(2)}\varepsilon^2 + ... + \alpha_{1i}^{(n-2)}\varepsilon^{n-2} + \alpha_{1i}^{(n-1)}\varepsilon^{n-1} + ...\right) \\ &+ g^5\left(\alpha_{2i}^{(0)} + \alpha_{2i}^{(1)}\varepsilon + \alpha_{2i}^{(2)}\varepsilon^2 + ... + \alpha_{2i}^{(n-2)}\varepsilon^{n-2} + ...\right) \\ &+ .... \\ &+ g^{2n-1}\left(\alpha_{n-2i}^{(0)} + \alpha_{n-2i}^{(1)}\varepsilon + ...\right) \\ &+ g^{2n+1}\left(\alpha_{n-1i}^{(0)} + ...\right). \end{aligned} \tag{5}$$

In a given order of PT equal n one needs all terms of the double expansion with a total power of $g^2 \cdot \varepsilon$ equal n. The existing freedom of choice of the coefficients $\alpha_{ki}^{(m)}$ is enough to get *simultaneously* the vanishing of the anomalous dimensions (read *conformal invariance*) and of the pole terms in Z factors (read *finiteness*). The coefficients from $\alpha_{ni}^{(0)}$ to $\alpha_{0i}^{(n)}$ calculated in n-th order of PT are related. One can not put either of them to zero in an arbitrary way. For more complete discussion and some examples of how these procedure works see our paper [15].

Our goal now is to calculate several terms of the double expansion (5) and to look for particular solutions when expansion breaks down at the first terms. In dimensional regularization (reduction) and $\overline{MS}$ renormalization scheme the anomalous dimension of a chiral superfield has the following form in n-th order of PT:

$$\gamma(g,\ \{h_i\}) = \sum_{k=1}^{n} k\ c_{1k}(g,\ \{h_i\}), \tag{6}$$

where c_{1k} are the coefficients at the lowest order pole in Z_2^{-1}. In the 1-loop order one has for the chiral field renormalization constant

$$Z_2^{-1} = 1 - \frac{N}{(4\pi)^2}\left(f(\{h_i\}, N) - 2g^2\right)\frac{1}{\varepsilon}. \tag{7}$$

where

$$f(\{h_i\}, N) = \sum_{i,k=1}^{3} f_{ik} h_i \bar{h}_k = (1 - \frac{2}{N^2})(|h_1|^2 + |h_2|^2) + \frac{2}{N^2}(h_1\bar{h}_2 + h_2\bar{h}_1) + (1 - \frac{4}{N^2})|h_3|^2, \tag{8}$$

Thus the one-loop conformal condition takes the form

$$f(\{h_i\},N) - 2g^2 = 0. \tag{9}$$

To fulfil it, the coefficients $\{a_i\}$ in (4) must satisfy the requirement

$$\sum_{i,k=1}^{3} f_{ik} a_i \bar{a}_k = 2. \tag{10}$$

In higher loops one has the following situation: up to three loops in the planar case (or up to two loops in the non-planar case) the coefficients c_{ik} have the form

$$c_{nk} = (f(\{h_i\},N) - 2g^2) P_{nk}(h_i, g^2, N), \; n = 1,..,3, \; k = 1,...,n \tag{11}$$

and vanish provided (10) is satisfied. This means that the one-loop conformal condition (9) is valid up to 3 loops in the planar case and up to two loops in the non-planar case. In even higher orders new contributions appear and eq.(11) is modified.

2.1 Three-Loop Conformal Condition in Non-Planar Case

Starting from three loops in the non-planar case one has the new contribution coming from the set of supergraphs with the "cross" topology shown in Fig.1. Eq.(11) then takes the form:

$$c_{nk} = (f(\{h_i\},N) - 2g^2) P_{nk}(\{h_i\}, g^2, N) + G_{nk}(\{h_i\},N), \; n \geq 3, \; k = 1,...,n \;, \tag{12}$$

where

$$G_{nk}(\{h_i\},N) = \sum_{i,p=1}^{3} (G_{nk})_{ip} (h_i \bar{h}_p)^n, \tag{13}$$

is a homogeneous polynomial, and

$$G_{nk}(\{a_i g\},N) \neq 0, \tag{14}$$

i.e. G_{nk} do not vanish when applying the one loop conformal condition (10) and to achieve conformal invariance one has to take more terms of the double expansion. At this order of PT to get simultaneously conformal and finite theory one needs the following terms of expansion (5):

$$h_i \;\; = \;\; g\left(a_1 + \alpha_{0i}^{(2)} \varepsilon^2 + g^2 \alpha_{2i}^{(1)} \varepsilon^1 + g^4 \alpha_{4i}^{(0)}\right), \; i = 1,2,3. \tag{15}$$

The explicit form of G_{31} is

$$\begin{aligned} G_{31}(\{h_i\},N) \;\; &= \;\; -\frac{1}{128} \frac{6\zeta(3)}{(4\pi)^6} \frac{N^2-4}{N^3} \times \qquad (16) \\ &\Big\{ |h_1-h_2|^2 \left(N^2 |h_1^2+h_2^2+h_1 h_2|^2 - 9N^2 |h_1|^2 |h_2|^2 + 5|h_1-h_2|^4 \right) \\ &-18|h_3|^2 \left((N^2-5)|h_1^2+h_2^2|^2 - (N^2-10)\left(\bar{h}_1 \bar{h}_2 (h_2^2+h_1^2) + c.c. \right) - 20|h_1|^2|h_2|^2 \right) \\ &+ \left(\bar{h}_3^3 (h_1-h_2)((N^2+20)(h_1^2+h_2^2) + 10(N^2-4)h_1 h_2) + c.c. \right) - 8(N^2-10)(|h_3|^2)^3 \Big\} \end{aligned}$$

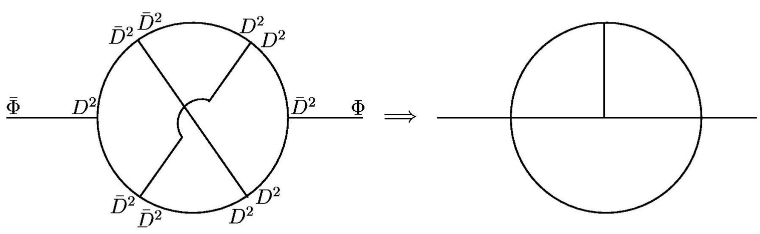

Figure 1: The topology of the relevant divergent non-planar supergraphs and their scalar counterpart at 3 loops

Now we follow the standard procedure [15] and find that up to 3 loops the couplings $\{h_i\}$ must satisfy the following condition:

$$\sum_{i,k=1}^{3} f_{ik} h_i \bar{h}_k = (1 - \frac{2}{N^2})(|h_1|^2 + |h_2|^2) + \frac{2}{N^2}(h_1\bar{h}_2 + h_2\bar{h}_1) + (1 - \frac{4}{N^2})|h_3|^2$$
$$= g^2 \left\{ 2 - \frac{\zeta_3}{128} G_{31}^{\Sigma} \varepsilon^2 - \frac{2\zeta_3}{128} G_{31}^{\Sigma} \left(\frac{g^2 N}{16\pi^2} \right) \varepsilon + \frac{18\zeta_3}{128} G_{31}^{\Sigma} \left(\frac{g^2 N}{16\pi^2} \right)^2 \right\} \quad (17)$$

where we have defined:

$$G_{31}(\{a_i g\}, N) = -\frac{N^3}{128} \frac{6\zeta(3)}{(4\pi)^6} G_{31}^{\Sigma} g^6. \quad (18)$$

For the bare couplings one has:

$$\sum_{i,k=1}^{3} f_{ik}(h_i \bar{h}_k)|_B = g_B^2 \left\{ 2 - \frac{\zeta_3}{128} G_{31}^{\Sigma} \varepsilon^2 + ... \right\} \quad (19)$$

2.2 Four-Loop Conformal Condition in the Planar Limit

The situation is simplified in the planar (large N of the $SU(N)$ gauge group) limit. In this case in the one loop conformal condition (8) only the diagonal terms $f_{ik},\ i = k$ survive

$$f(\{h_i\}, N) = \sum_{i,k=1}^{3} f_{ik} h_i \bar{h}_k = |h_1|^2 + |h_2|^2 + |h_3|^2, \quad (20)$$

so from (9) one has:

$$|h_1|^2 + |h_2|^2 + |h_3|^2 - 2g^2 = 0. \quad (21)$$

At four loops the only non vanishing contribution to G_{41} comes from the set of planar supergraphs with new "ladder" topology (see Fig.2). Contribution of this set of chiral supergraphs to the chiral propagator renormalization constant in the planar limit is:

$$
\begin{aligned}
&c_{41}(\{h_i\}, g^2, N) = \qquad (22)\\
&= \frac{5}{2}\zeta(5)\frac{N^4}{(4\pi)^8}\{(|h_1|^2+|h_2|^2+|h_3|^2)^4-(2g^2)^4+(|h_1|^2-|h_2|^2)^4+(|h_3|^2)^4\\
&+6(|h_3|^2)^2(|h_1|^2+|h_2|^2)^2+24|h_3|^2|h_1|^2|h_2|^2(|h_1|^2+|h_2|^2)+\\
&+8h_3^3(|h_2|^2\bar h_1^3-|h_1|^2\bar h_2^3)+8\bar h_3^3(|h_2|^2h_1^3-|h_1|^2h_2^3)\\
&-8|h_3|^2(h_2^3\bar h_1^3+h_1^3\bar h_2^3)-4|h_3|^2(|h_1|^2+|h_2|^2)^3-4(|h_3|^2)^3(|h_1|^2+|h_2|^2)\}.
\end{aligned}
$$

Hereafter the chiral-gauge $\bar\Phi V\Phi$ contributions proportional to $|h_1|^2+|h_2|^2+|h_3|^2-2g^2$ are omitted. Note that $G_{41}=c_{41}$ in this case, and does not vanish at the one-loop conformal condition.

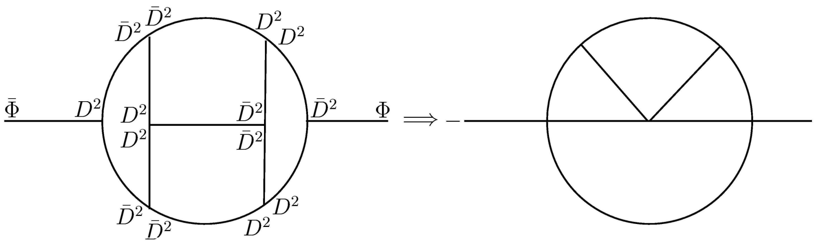

Figure 2: The topology of the relevant divergent planar supergraphs and their scalar counterpart at 4 loops

With account of non-vanishing contribution to G_{41} one needs the following terms of expansion (5)

$$
h_i \;=\; g\left(a_i+\alpha_{0i}^{(3)}\varepsilon^3+g^2\alpha_{2i}^{(2)}\varepsilon^2+g^4\alpha_{4i}^{(1)}\varepsilon+g^6\alpha_{6i}^{(0)}\right),\ i=1,2,3. \qquad (23)
$$

Again as the previous case one gets a finite and conformal theory up to four loops if the renormalized Yukawa couplings are chosen to satisfy the condition

$$
\sum_{i,k=1}^{3} f_{ik}h_i\bar h_k = |h_1|^2+|h_2|^2+|h_3|^2 = g^2\left\{2+\frac{5}{18}\zeta_5 G_{41}^{\Sigma}\varepsilon^3+\frac{5}{3}\zeta_5 G_{41}^{\Sigma}(\frac{g^2N}{16\pi^2})\varepsilon^2\right.
$$
$$
\left. +\quad 5\zeta_5 G_{41}^{\Sigma}(\frac{g^2N}{16\pi^2})^2\varepsilon+10\zeta_5 G_{41}^{\Sigma}(\frac{g^2N}{16\pi^2})^3+...\right\}.
$$

For the bare couplings one has

$$
|h_1|_B^2+|h_2|_B^2+|h_3|_B^2 = g_B^2\left\{2+\frac{5}{18}\zeta_5 G_{41}^{\Sigma}\varepsilon^3+...\right\}. \qquad (24)
$$

The explicit form of G_{41}^{Σ} is:

$$
\begin{aligned}
G_{41}^{\Sigma} \;=\;& \big\{(a_3\overline{a}_3)^4+(a_1\overline{a}_1-a_2\overline{a}_2)^4+6(a_3\overline{a}_3)^2(a_1\overline{a}_1+a_2\overline{a}_2)^2\\
+\;& 24a_1\overline{a}_1a_2\overline{a}_2a_3\overline{a}_3(a_1\overline{a}_1+a_2\overline{a}_2)+8a_3^3(a_2\overline{a}_2\overline{a}_1^3-a_1\overline{a}_1\overline{a}_2^3)\\
+\;& 8\overline{a}_3^3(a_2\overline{a}_2a_1^3-a_1\overline{a}_1a_2^3)-8a_3\overline{a}_3(\overline{a}_1^3a_2^3+\overline{a}_2^3a_1^3)\\
-\;& 4ca_3\overline{a}_3(a_1\overline{a}_1+a_2\overline{a}_2)^3-4(a_3\overline{a}_3)^3(a_1\overline{a}_1+a_2\overline{a}_2)\big\}. \qquad (25)
\end{aligned}
$$

3 Exploring the Conformal Conditions

Consider now if one can find such values of (h_1, h_2, h_3) that G_{31} in the non-planar case and G_{41} in the planar case vanish meaning that the one-loop conformal condition is valid up to three or four loops, correspondingly.

In non-planar case, similar to the beta-deformed theory, we have not found any solution for vanishing of G_{31} which has a simple form and might be valid in any order of PT. In the planar case, on the contrary, we found two families of simple solutions of equation $G_{41} = 0$:

$$\text{Solution \# 1}: \quad \begin{cases} h_1 = ge^{i\alpha}(A-B), \\ h_2 = ge^{i\alpha}(A+B), \\ h_3 = 2ge^{i\alpha}B, \end{cases} \tag{26}$$

where A, B, α are arbitrary *real* numbers. The one-loop conformal condition brings us to the following relation between A and B: $B^2 = \frac{1-A^2}{3}$. If this condition is satisfied, then $G_{41} = 0$ for arbitrary α and $-1 \le A \le 1$.

$$\text{Solution \# 2}: \quad \begin{cases} h_1 = -ge^{i\alpha}, \\ h_2 = 0, \\ h_3 = ge^{i\beta}, \end{cases} \quad \text{or} \quad \begin{cases} h_1 = 0, \\ h_2 = ge^{i\alpha}0, \\ h_3 = ge^{i\beta}, \end{cases} \tag{27}$$

where α and β are arbitrary real phases. However, not all of these solutions are genuine. Some of them happen to be unitary equivalent to the β-deformed case.

Indeed, as was first noticed in [16] considering the full Leigh-Strassler deformation one can find the special points in the parameter space of $\{h_1, h_2, h_3\}$ at which the theory is unitary equivalent to the beta-deformed $\mathcal{N} = 4$ SYM theory. To see this, consider a general unitary matrix $U(3)$ $(UU^+ = 1)$.

$$U = \begin{pmatrix} c_1 & c_3 s_1 & s_1 s_3 \\ -c_2 s_1 & c_1 c_3 - e^{iy} s_2 s_3 & e^{iy} c_3 s_2 + c_1 c_2 s_3 \\ s_1 s_2 & -c_1 c_3 s_2 - e^{iy} c_2 s_3 & e^{iy} c_2 c_3 - c_1 s_2 s_3 \end{pmatrix}$$

where $s_i = \sin(x_i)$ and $c_i = \cos(x_i)$.

Taking now the beta-deformed theory and making an arbitrary unitary transformation of the fields

$$\Phi_i = U_{ij} \Psi_j, \tag{28}$$

we demand the new theory to be of the Leigh-Strassler type. It leads to the following allowed values for the transformation parameters

$$\begin{cases} x_1 = \pm \arccos(\frac{1}{\sqrt{3}}) + \pi k, \\ x_2 = \frac{\pi}{4} + \frac{\pi l}{2}, \\ x_3 = \frac{\pi}{4} + \frac{\pi m}{2}, \\ y = \frac{\pi}{2} + \pi n. \end{cases} \tag{29}$$

As the result the superpotential which is obtained from the beta-deformed SYM theory by unitary transformation (28) with parameters fixed by (29) has the form

$$W = iTr\left(\tilde{h_1}\Psi_1\Psi_2\Psi_3 - \tilde{h_2}\Psi_1\Psi_3\Psi_2\right) + i\frac{\tilde{h_3}}{3}\sum_{i=1}^{3} Tr(\Psi_i^3), \tag{30}$$

where

$$\begin{cases} \tilde{h_1} = i(a-b) \\ \tilde{h_2} = i(a+b) \\ \tilde{h_3} = 2ib \end{cases} \quad \text{or} \quad \begin{cases} \tilde{h_1} = e^{\pm\frac{\pi}{3}}(a-b) \\ \tilde{h_2} = e^{\pm\frac{\pi}{3}}(a+b) \\ \tilde{h_3} = -2ib \end{cases} \tag{31}$$

The parameters a and b are linked with the original couplings h_1 and h_2 by

$$\begin{cases} a = \pm\frac{1}{2}(h_1+h_2), \\ b = \pm\frac{1}{i2\sqrt{3}}(h_1-h_2). \end{cases} \tag{32}$$

The chiral propagators calculated in the full Leigh-Strassler deformed theory (30) with the coupling chosen as (31,32) will be the same as calculated in the beta-deformed theory.

Looking back to our solutions we find that the solution # 1 coincides with the left part of (31). This means that the obtained theory is unitary equivalent to the beta-deformed case. For the solution # 2 if the parameters α and β satisfy $\alpha-\beta=\frac{2\pi m}{3}$ one again has a theory which is unitary equivalent to the real beta-deformed one.

Thus, the only non-trivial solution that exists in the planar limit and leads to conformal theory (up to 4 loops at least) corresponds to the superpotential which can be written in the form

$$\mathcal{W} = ih\int d^6z(qTr\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\sum_{i=1}^{3}\frac{Tr(\Phi_i^3)}{3}). \tag{33}$$

where $|h|^2=g^2$ and $|q|=1$, but $q \neq e^{i\frac{\pi n}{3}}$.

4 Exact conformal invariance?

One may wonder if the theory defined by the superpotential (33) is exactly conformal in the planar limit like the real beta-deformed one. Due to the unitary equivalence to real beta-deformed theory for particular values of the phase the absence of phase-dependent terms would mean the exact conformal invariance of the theory.

¿From the superpotential (33) one can notice that only phase-dependent structures that can emerge are of the form

$$(|h_3|^2)^n(|h_1|^2)^l[(h_3\bar{h_1})^{3k}+(\bar{h}_3h_1)^{3k}], k=0,1,...$$

Hence, if $h_1=hq,\ h_3=\frac{h}{q},\ q=e^{i\gamma}$ the only phase-dependent contribution looks like

$$const \times \cos(6k\gamma).$$

Since we know that when $q=e^{i\frac{\pi n}{3}}$ the theory is unitary equivalent to the real beta-deformed one, it should be exactly conformal for $\gamma=\pi n/3$. This corresponds to $\cos(6k\gamma)=\cos(2\pi kn)=1$ for arbitrary k and n.

So the question is whether it is possible to construct a diagram which is phase-dependent in the planar limit. This happened to be not a simple task for the following reasons:

1. All possible phase-dependent "boxes" are suppressed in the planar limit. Thus, the possible phase-dependent diagram should contain more complicated structures.

2. The diagram containing a polygon higher than the "box" has many external legs. Being inserted into the chiral propagator it again produces "boxes" thus containing no phases.

As the result, at least up to twenty loops, one can not construct a potentially phase-dependent diagram in the planar limit. We assume, though we have no rigorous proof yet, that in the planar limit such a phase-dependent structure does not emerge in any order of PT.

Thus our conjecture is that the theory defined by the superpotential (33) with $|q| = 1$ is exactly conformal in the planar limit.

5 Conclusion

We have investigated here the conformal conditions for the full Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory both in the planar and non-planar cases. The conformal condition was found up to four loops in the planar limit and up to three loops in non-planar case. We would like to emphasize the obtained theory *simultaneously* conformal invariant and finite since these two requirements are *identical*. This can be achieved properly adjusting the Yukawa couplings order by order in PT. In the framework of dimensional regularization this requires the double series over the gauge coupling g and the parameter of dimensional regularization ε.

Since in the full Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory there is an extra coupling constant we have more freedom in our theory. So looked for the solutions where the one-loop conformal condition is exact and at the same time which are not obtainable from the real beta deformation of the $\mathcal{N} = 4$ SYM theory by unitary transformation. We did not find such solutions in the non-planar case, but in the planar limit we found one potentially interesting solution. We made certain that in the planar limit the one-loop conformal condition in this case is valid up to twenty loops and p it might be also valid in all orders of PT.

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The Origin of Horizons in 2+1-dimensional Black Holes

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Abstract

Gravitational collapse of matter to a black hole proceeds by the formation of a horizon, which typically expands from a point, assumes a varying shape in the case of asymmetrical collapse, and finally settles down to a constant, simple shape. The exact asymmetrical collapse solutions of the Einstein equations in 2+1 dimensional spacetimes are used to investigate this development of the horizon.

Of the many unusual properties of black holes the horizon is the most characteristic and special. It can be considered an attribute of a black hole here and now, even though it cannot be measured or experienced locally and is not determined until the late-time evolution of the black hole is known. Most of the physical properties that can be attributed to horizons (as in the membrane paradigm) refer to stationary black holes or changes between stationary states; much less is known about stages when there is rapid change in the horizon. It is therefore interesting to investigate horizons at their time of formation, when they are not spherical and not even smooth. This is done here for the case of black holes in 2+1 dimensional spacetime with a negative cosmological constant $\Lambda = 1/\ell^2$. There the horizon at at a given time is a (generally not round) circle. The deviations of this closed curve from smoothness will be of particular interest.

The defining property of a horizon is that it divides spacetime into an interior and an exterior. Points in the exterior can be causally (by timelike or null curves) connected to an asymptotic region, whereas no causal curve from points in the interior end at infinity; instead they typically reach in a finite time a region that is in some sense singular.

As a global property the horizon is not easily found, except in highly symmetrical cases. Otherwise one usually does not have exact solutions and the horizon has to be found numerically. In 2+1 dimensions all solutions of the Einstein equations are locally anti de Sitter and differ only in the way these constant negative curvature regions are connected together. These connections are well understood geometrically and one therefore in effect has a large number of exact solutions of the Einstein equations, even if a single metric is not written down for these spacetimes. This simplicity is attributed to the absence of gravitational waves in three dimensions, so such waves cannot be used to perturb the horizon of the most symmetric "BTZ" black hole. But non-uniform matter distributions can perturb this symmetry. In 2+1 dimensions there exist point particles (conical singularities). This simplest form of matter represents extreme concentrations and causes interesting dynamics of the horizon (see [1] for background of classical 2+1 dimensional black hole theory).

By a somewhat loose language we call "the horizon" both the two-dimensional subspace of spacetime, and its intersection with a spacelike time slice, similar to the way a "particle" can mean both the worldline and its position on a spacelike surface. The horizon is a null surface, hence penetrable by timelike curves in one direction only. It is generated by the last light rays that can get out to infinity, and this extremal property implies that these rays are null geodesics. Because each point on a null surface has a unique null tangent, the generators cannot cross on the horizon. Wherever they cross is a boundary point of the horizon, the point where those generators enter the horizon. The curve on which generators cross is the boundary curve, which must be spacelike because there are curves arbitrarily close to the boundary on the horizon, and those cannot be timelike or null [1] . Thus the (connected) network of spacelike boundary curves define the development of the horizon. The network has finite length because the horizon expands from it and only increases in length, until it reaches the finite constant horizon length that measures the finite total mass of the black hole. (Of course it may happen – as in the Schwarzschild or BTZ black hole – that the generators do not cross anywhere. In that case the horizon has no starting point, the black hole is eternal.)

The network will always lie on a spacelike surface, which could be chosen as one of the time slices. In that notion of time all parts of the horizon start simultaneously on this network. In order to show the development of the horizon as a sequence in time, other coordinates are used, which unlike Schwarzschild coordinates involve a change in time coordinate on the horizon itself. A familiar example is the collapse of a spherical shell in Eddington-Finkelstein coordinates [4]. Successive collapse of two shells is studied in [5]. The horizon starts at a point at the center of the shell and expands as a light cone until it reaches the shell. After the shell has crossed the horizon, the latter no longer changes its radius, it is a cylinder joined to this light cone.

Because the horizon is a global object, it does not follow ordinary notions of causality; for example, in the spherically symmetrical case it is determined only by the matter distribution, but not in a causal way: it "anticipates" future changes in this distribution. If the collapsing shell is followed by another collapsing shell, the horizon radius continues to expand after the first shell has crossed, and becomes constant only after passing the second shell. If there are non-spherical perturbations in the shell's matter one expects perturbations in the horizon *before* it crosses the shell, since the black hole uniqueness does not allow perturbations after the black hole has fully formed.[2] It is therefore appropriate to follow the horizon backwards in time as it crosses matter, to find out how it changes.

In Figure 1 a section of horizon (position 4) is incident as time decreases on a local concentration of matter. The top of the figure is the outside of the horizon, and we consider the horizon moving in a downward direction toward earlier times. The matter retards the parts of the horizon that have passed through the matter (position 3) and acts as a gravitational lens that bends the horizon generators to an imperfect focus F (position 2). If we now continue along all the generators further backward in time we reach self-intersecting curves as in position 1 which enclose a triangular region R, whose points are in the future (above) of

[1] The general properties of black hole horizons are discussed in [2]. Numerical study of horizon in 3+1 dimensions is given in [3].

[2] In four-dimensional spacetime this is only true after any gravitational radiation emitted during the collapse has dispersed. In three-dimensional spacetimes there is no gravitational radiation, so the horizon has its final shape immediately after it has crossed all matter.

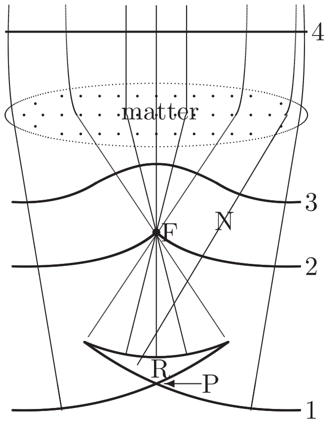

Figure 1: Focusing of horizon generators due to matter.

some part of this curve, as well as in the past (below) of another part. To determine whether R is outside or inside the horizon we recall that all points causally connected to infinity lie outside the horizon. The region R can be connected to infinity, for example by rays like the null line N. Thus the inside of the horizon at that early time is only the bottom region, not including R. As we have gone backwards in time the horizon has lost generators, and it is about to lose the two generators going through the point P. In the forward time direction new generators enter the horizon at P, and later along the curve joining P to F. The horizon is not smooth (both on a spacelike surface and as a subspace of space-time) at these points where the generators enter, and because the slope changes, two generators enter at such points. The entry point does not move along either generator, and therefore must exceed the speed of light: it moves along a spacelike curve, as noted above. This curve is usually called the horizon "caustic". Its motion ends at the paraxial focus F, after which the horizon becomes smooth.

If the mass concentration is a point-like particle – that is, a conical singularity on spacelike surfaces – the deflection of null rays is independent of impact parameter from the particle. Therefore the focus is *at* the particle, and the horizon caustic ends at the particle itself. In a vacuum spacetime with cosmological constant and point particles, due to analyticity, a caustic must end either on a point particle or at a vertex with other caustics. Since generically it takes three null surfaces to intersect at a point, three caustics typically come together at a vertex. In cases of special symmetry more caustics can join together.

As a simple example, consider the case when the collapsing shell is replaced by n equidistant point masses starting from rest on a circle of radius R. The initial geometry of any arrangement of point masses defines a total (ADM) mass-energy through the behavior of its geometry in the asymptotic region. There the metric can always be brought into the BTZ form of single black hole,

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)dt^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}} + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2$$

Here we consider the non-rotating (time symmetric) case, where $J = 0$. If the parameter m is negative, no black hole is formed; the particles simply move on radial geodesics until they collide at the center. The negative m measures the total particle mass, that is, the total

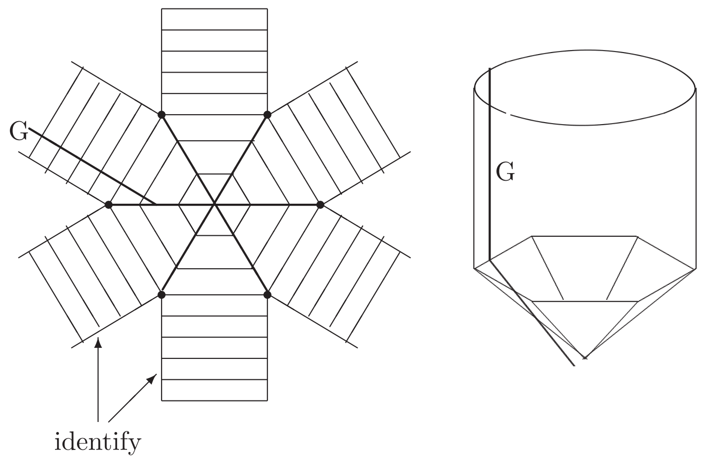

Figure 2: The horizon at various times as it passes six identical particles (left) and a representation of the two-dimensional horizon surface in a three-dimensional spacetime (right).

conical angle deficit of the particles. When $m > 0$ a black hole will definitely be formed. The value of m can be computed by hyperbolic geometry on the initial spacelike surface from the conical angle deficits of the point particles and their distances. Because of this difference between particle and black hole mass it is always possible to determine from the initial date, and without having to integrate the time development, whether an arrangement of particles will collapse to a black hole or not.

The more complicated hyperbolic geometry necessary for analyzing a general arrangement of particles can be replaced by the simpler Euclidean geometry in the limit of small cosmological constant, or equivalently, of small system size. The general features of the horizon formation will not be affected by this approximation (but details may vary; for example, curves of constant curvature will be replaced by straight lines). During the interesting time when the horizon is forming and rapidly passing the masses we can regard the particles as static in flat spacetime, with a cylindrical exterior that approximates the geometry near the throat of a black hole. (A conical exterior, on the other hand, would mean that the total mass of the system is that of a particle.) The condition that the exterior is cylindrical and the system represents a black hole is then simply that the sum of the particles' angle deficits should be 2π. Thus the angle deficit of each particle must be $2\pi/n$.

When a black hole is formed from n symmetrically arranged particles it is clear from symmetry that the horizon's caustics will be radial geodesics between the center and each particle[3]. The horizon starts at the center and expands into a polygon with n vertices that lie on the horizon caustics. After it passes the particles it is a smooth circle whose circumference remains constant. Since this spacetime is static and flat except at the particles, its spacelike geometry can be represented on flat paper by making cuts along the cylinder's generator to the particles and opening up the angle deficits, with the understanding that corresponding points along the cuts are to be identified. The result for the case of 6 masses is shown in the left part of Figure 2. Each of the six dots that represent the particles is the vertex of a wedge with the $60°$ deficit angle that is to be cut out and edges identified. The thicker lines are the horizon's caustics, and the horizon itself is shown at various times by the thinner

[3]Collapse of two point masses on a BTZ background is considered in [6].

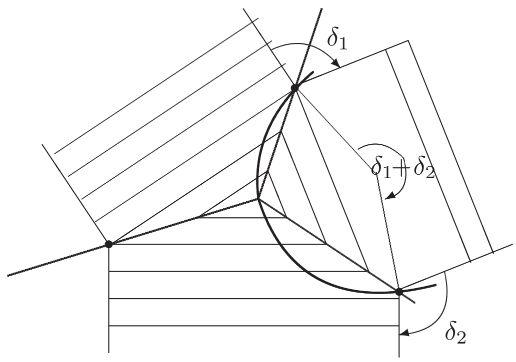

Figure 3: Construction of horizon for three particles. The particles of angle deficit $\delta_1, \delta_2, \delta_3$ are located at the three dots. The center of a circle passing through particles 1 and 2 is located by the requirement that radii to the particles subtend an angle $\delta_1 + \delta_2$. Similar circles (not shown) are drawn for the two other particle pairs. Their intersection is the origin of the caustics (thicker lines) that connect this origin to the particles. The horizon at various times is then drawn as triangles whose vertex angles are bisected by the caustics.

lines. This figure can also be regarded as a kind of level-line diagram of the horizon in the three dimensional spacetime. The right part of Figure 2 is an attempt at a three dimensional drawing of the horizon surface in a kind of Eddington-Finkelstein coordinate system. The bottom part is standard Minkowski space in which the six triangular null surfaces are inclined at 45°, but at the top the light cones are tilted so that the exterior horizon can be shown as a cylinder of constant circumference, as in the Eddington-Finkelstein picture. The line G is a typical horizon generator that enters the horizon at a caustic.

When we are given a less symmetrical arrangement of particles, even at the time-symmetric moment, it is rather more complicated to find the horizon. Consider two generators that intersect and enter the horizon at a caustic that ends at one of the particles. The angle between the generators at entry must be such that when they pass the particle on opposite sides they are deflected to become parallel, because the exterior part of the horizon has parallel generators, like that of an eternal black hole. Their angle at entry is therefore equal to the particle's deficit angle. Since the horizon wave front is normal to the generator, the angle by which the horizon changes at a caustic must also equal the corresponding particle's deficit angle. So we know the exterior angles of the polygon that is the horizon before it has passed the particles. In the case of three particles these three angles determine the shape of the triangular horizon, and we confine attention to that case.

With three particles the three horizon caustics come together at a point that can be regarded as the origin of the horizon. It is equidistant from the sides of any (later) horizon triangle, and therefore the caustics are bisectors of that triangle's edges. The angle between two caustics is then the average of the angle deficit of the corresponding particles (Figure 3). It remains only to place this tree of three caustics on the given triangle so that each vertex lies on its caustic. This can be done constructing the intersection of three circles, as explained in the caption. With the caustics located within the triangle of the three particles, the horizon is then found as the triangles whose angle bisectors are the caustics. The horizon will generally cross the particles one at a time, and accordingly lose its discontinuities not simultaneously but one after the other.

The case of more particles can be analyzed in a similar way (see the details in [7]). Figure

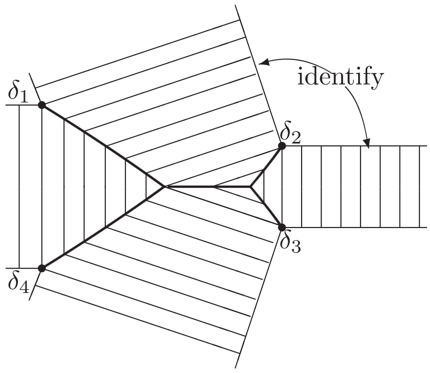

Figure 4: Four unequal masses with less symmetry, their caustics and horizon sequence.

4 gives an example. The generic tree of caustics has three caustics coming together at a vertex, therefore there will be internal caustics without end points on a particle. A vertex that is connected to two particles and not the origin of the horizon can be considered as a single particle that replaces those two particles in the early development of the horizon. If the particles have motion and angular momentum the qualitative horizon development remains unchanged, but there is the additional complication that the generators cannot be assumed to be perpendicular to the momentary horizon. Our analysis also has qualitative implications for less concentrated matter distributions than point particles. In that case the focus where the caustics end is some distance away from the mass, as in Figure 1, and the horizon can become smooth before it crosses the matter.

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Ekpyrotic Cosmology and Non-Gaussianity

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Abstract

New Ekpyrotic Cosmology is an alternative scenario of the early universe which relies on a phase of slow contraction before the Big Bang. During this epoch, we show that nearly scale-invariant density perturbations are generated with the observed amplitude and red spectral tilt. We calculate the 3-point and 4-point correlation functions of these primordial density perturbations and find a generically large non-Gaussian signal. This is in contrast with slow-roll inflation, which predicts negligible non-Gaussianity. The model is also distinguishable from alternative inflationary scenarios with large non-Gaussianity but differing shape dependence of the correlation functions. The Ekpyrotic phase is followed by an epoch in which the Null Energy Condition is violated, allowing a "bounce" from contraction to an expanding universe. This is accomplished via "ghost condensation".

1 Introduction

Over the past decade observations of the microwave background temperature anisotropy have revealed that the large scale structure in our universe originates from primordial perturbations that are nearly scale-invariant and adiabatic. Since these coincide with the predictions of the simplest inflationary models, this is widely regarded as evidence for inflation. However this does not constitutes a proof, and it is prudent to keep in mind that the seeds for structure formation could originate from a different mechanism. Ultimately our faith in inflation must rely on the absence of a compelling alternative paradigm.

The ekpyrotic scenario is an alternative candidate theory of early-universe cosmology. Instead of invoking a short burst of accelerated expansion from a hot initial state, as in inflation, the ekpyrotic scenario relies on a cold beginning followed by a phase of very slow contraction. Despite such diametrically opposite dynamics, both models predict a flat, homogeneous and isotropic universe, endowed with a nearly scale invariant spectrum of density perturbations, and are, therefore, equally successful at accounting for all current cosmological observations.

An important drawback of the original ekpyrotic theory [1] is how to avoid the big crunch singularity without introducing ghosts or other pathologies. Moreover, the fate of perturbations through the bounce is ambiguous — whereas the scalar field fluctuations are scale invariant during the contracting phase, the curvature perturbation on uniform-density hypersurfaces, ζ, is not. And since the latter remains constant on super-horizon scales, one generally expects its (unacceptably blue) spectrum to be preserved irrespective of the bounce physics. Despite considerable work indicating that stringy effects at the bounce may positively alter this conclusion [2], the issue of matching conditions remains controversial.

Both of these issues have been resolved in the recently proposed New Ekpyrotic scenario [3]. In [3], we derived a fully *non-singular* bounce within a controlled and ghost-free four-dimensional effective theory using the ghost condensation mechanism [4]. Moreover, ζ acquires a scale invariant spectrum well before the bounce, thanks to an entropy perturbation generated by a second scalar field [3, 5, 6]. Thus New Ekpyrotic Cosmology appears to be a consistent alternative to the inflationary scenario.

A distinguishing prediction lies in the tensor spectrum [1]: inflation predicts scale invariant primordial gravity waves, whereas ekpyrosis does not. Detecting tensor modes from CMB B-mode polarization could rule out the ekpyrotic scenario, whereas an absence of detection would not discriminate between the two models.

Here, we focus on another key observable: the non-Gaussianity of primordial density perturbations. We show that New Ekpyrotic Cosmology generically predicts a large level of non-Gaussianity, potentially just below current sensitivity levels and detectable by near-future CMB experiments.

We calculate the 3-point and 4-point functions. For typical parameter values, the amplitude of the 3-point function is generically large, with $f_{\rm NL}$ around the current WMAP bound [7]: $-36 < f_{\rm NL} < 100$. That is, assuming all parameters are $\mathcal{O}(1)$, $f_{\rm NL}$ approaches the limits of this bound, depending on the sign of a parameter. These values are well above the expected sensitivity of the Planck experiment: $|f_{\rm NL}| \lesssim 20$. The amplitude of the 4-point function is also generically large: $\tau_{\rm NL} \sim 10^4$, which is again near the estimated WMAP bound and within the reach of Planck: $\tau_{\rm NL} \lesssim 600$ [8].

This is in stark contrast with the highly Gaussian spectrum predicted by slow-roll inflation. Comparably large non-Gaussianity does arise in non-slow roll models, such as DBI inflation [9], and whenever the precursor of density fluctuations is a light spectator field, such as in the curvaton [10, 11] or modulon scenarios [12, 13]. However, as we will see, New Ekpyrosis predicts a different shape dependence in momentum space for the 3- and/or 4-point spectrum than the simplest such models.

Non-Gaussianity therefore offers a distinguishing prediction of New Ekpyrotic Cosmology, potentially testable in CMB experiments within the next few years.

2 New Ekpyrotic Cosmology

As with inflation, ekpyrosis relies on a scalar field ϕ rolling down a potential $\mathcal{V}(\phi)$. Instead of being flat and positive, however, here $\mathcal{V}(\phi)$ must be steep, negative and nearly exponential in form. For concreteness, we take

$$\mathcal{V}(\phi) = -V_0 e^{-\phi/\Lambda} \,, \tag{1}$$

where $\Lambda \equiv \sqrt{\epsilon}M_{\rm Pl}$ and $\epsilon \ll 1$. The Friedmann and scalar field equations then yield a background scaling solution,

$$a(t) \sim (-t)^{2\epsilon}\,;\;\; \bar{\phi}(t) = \Lambda \log\left(\frac{V_0}{2\Lambda^2(1-6\epsilon)}\,t^2\right)\,, \tag{2}$$

with Hubble parameter $H = 2\epsilon/t$. Since $\epsilon \ll 1$, this describes a slowly-contracting universe with rapidly increasing H, again in contrast with the rapid expansion and nearly constant H in inflation.

In single-field ekpyrosis, fluctuations in ϕ acquire a scale invariant spectrum. As we review shortly, this traces back to the fact that the above solution satisfies $\bar{\mathcal{V}}_{,\phi\phi} = -2/t^2$. However this contribution exactly projects out of ζ, leaving the latter with an unacceptably blue spectrum. Since ζ is conserved on super-horizon scales barring entropy perturbations, it is generally expected to match continuously through the bounce, although stringy effects could alter this picture [2].

New Ekpyrotic Cosmology introduces a second field, χ, as the progenitor of the scale-invariant perturbation spectrum [3, 5]. This field has no dynamics during the ekpyrotic phase and remains approximately fixed at $\bar{\chi} = 0$. However, as we describe below, its fluctuations generate a scale-invariant spectrum of entropy perturbations, which gets imprinted onto ζ at the end of the ekpyrotic phase.

An essential condition in obtaining a scale-invariant spectrum is that at $\bar{\chi} = 0$ the curvature of the potential be nearly the same along the χ and ϕ directions: $\bar{V}_{,\chi\chi} \approx \bar{V}_{,\phi\phi}$. An example of such a potential is

$$V(\phi,\chi) = \mathcal{V}(\phi)\left(1 + \frac{\chi^2}{2\Lambda^2} + \frac{\alpha_3}{3!}\frac{\chi^3}{\Lambda^3} + \frac{\alpha_4}{4!}\frac{\chi^4}{\Lambda^4} + \ldots\right)\,. \tag{3}$$

The higher-order χ terms are naturally expected to be suppressed by the same scale Λ as the quadratic term, hence the form (3). For simplicity we take $\alpha_3, \alpha_4, \ldots$ to be constants. While potential (3) yields a slightly blue spectral tilt, a more general potential is presented in [3] which allows for the observed red tilt without altering the conclusions for non-Gaussianity arrived at in this paper. Note that the required field trajectory lies along an unstable point. However, a pre-ekpyrotic, stabilizing phase can easily create initial conditions so that this trajectory is arbitrarily close to the tachyonic ridge [3].

Power spectrum for χ: Since our space-time background is nearly static, we ignore gravity in studying χ perturbations. To linear order, the Fourier modes $\delta\chi_k^{(0)}$ around $\bar{\chi} = 0$ satisfy a free field equation with time-dependent mass $\bar{V}_{,\chi\chi} = \bar{V}_{,\phi\phi} = -2/t^2$:

$$\delta\ddot{\chi}_k^{(0)} + \left(k^2 - \frac{2}{t^2}\right)\delta\chi_k^{(0)} = 0\,. \tag{4}$$

Assuming the usual adiabatic vacuum, we find

$$\delta\chi_k^{(0)} = \frac{e^{-ikt}}{\sqrt{2k}}\left(1 - \frac{i}{kt}\right)\,. \tag{5}$$

On super-Hubble scales, $k(-t) \ll 1$, the power spectrum, defined by $\langle\delta\chi_k^{(0)}\delta\chi_{k'}^{(0)}\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')P_\chi(k)$, is

$$k^3 P_\chi(k) = \frac{1}{2t^2}\,, \tag{6}$$

which is scale invariant. Including gravity and departing from the pure exponential form (1) results in small deviations from scale invariance. This can yield a small red tilt, consistent with current CMB observations [3].

Evolution of ζ: We focus for simplicity on the regime where all relevant modes are well-outside the horizon, $k \ll aH$. In the small-gradient approximation, the metric can be written as $ds^2 = -\mathcal{N}^2 dt^2 + e^{2\zeta(\vec{x},t)} a^2(t) d\vec{x}^2$ [14], where $\mathcal{N}$ is the lapse function, and ζ is the curvature perturbation. The evolution of ζ on uniform-density hypersurfaces is governed by

$$\dot{\zeta} = 2H \frac{\delta V}{\dot{\bar{\phi}}^2 - 2\delta V} , \tag{7}$$

where $\delta V \equiv V(\phi, \chi) - V(\bar{\phi}, \bar{\chi})$. A key simplification is that $\delta\phi$ has a steep blue spectrum at long wavelengths and, hence, can be neglected. Thus, for the potential (3), we have $\delta V \approx \mathcal{V}(\bar{\phi})\delta\chi^2/2\Lambda^2 + \ldots$

To proceed further, one needs an expression for $\delta\chi$ to higher-order than the "free" part $\delta\chi^{(0)}$. To do this, we solve $\ddot{\delta\chi} + \bar{V}_{\chi\chi}\delta\chi = 0$, valid at long wavelengths, perturbatively: $\delta\chi = \delta\chi^{(0)} + \delta\chi^{(1)} + \ldots$ To lowest order, this equation reduces to (4) in the limit $k \to 0$. The next order, $\delta\chi^{(1)}$, satisfies $\ddot{\delta\chi}^{(1)} + \bar{V}_{\chi\chi}\delta\chi^{(1)} + \bar{V}_{,\chi\chi\chi}(\delta\chi^{(0)})^2/2 = 0$. Using (2), (3) and $\delta\chi^{(0)} \sim 1/t$, we find

$$\delta\chi = \delta\chi^{(0)} + \frac{\alpha_3}{4\Lambda}\left(\delta\chi^{(0)}\right)^2 + \ldots \tag{8}$$

Substituting into (7), one can integrate to obtain

$$\zeta_{\rm ek} = \frac{1}{2}\left(\frac{\delta\chi^{(0)}}{M_{\rm Pl}}\right)^2 + \frac{5\alpha_3}{18\sqrt{\epsilon}}\left(\frac{\delta\chi^{(0)}}{M_{\rm Pl}}\right)^3 + \ldots \tag{9}$$

The ekpyrotic phase must eventually end if the universe is to undergo a smooth bounce and reheat into a hot big bang phase. This is achieved by adding a feature to the potential (3) which eventually pushes χ away from the tachyonic ridge [3]. Denote the time at which ekpyrosis stops as $t_{\rm end}$. For simplicity, we model this with $V_{,\chi}$ suddenly becoming non-zero and nearly constant at $\chi = 0$. Denote this constant by $V_{,\chi}|$. The exit phase is assumed to last for a time interval Δt which is short compared to a Hubble time: $|H_{\rm end}|\Delta t \ll 1$. This will be the case provided the potential satisfies

$$\epsilon_\chi \equiv \frac{H_{\rm end}^4 M_{\rm Pl}^2}{V_{,\chi}|^2} \lesssim 1 . \tag{10}$$

The exit phase generates an additional contribution to ζ. To compute this in the rapid-exit approximation, we can treat the right-hand side of (7) as approximately constant. In evaluating this constant, note that, to leading order in $\delta\chi$, we have $\delta V \approx V_{,\chi}|\delta\chi = \pm H_{\rm end}^2 M_{\rm Pl}\delta\chi/\sqrt{\epsilon_\chi}$. (Higher-order terms in $\delta\chi$ yield small corrections to (9) and are therefore negligible.) Thus ζ changes from $\zeta_{\rm ek}$ by an amount ζ_c during the exit, given by

$$\zeta_c = \mp 2\sqrt{\epsilon}\beta \frac{\delta\chi(t_{\rm end})}{M_{\rm Pl}} , \tag{11}$$

where $\beta \equiv |H_{\rm end}|\Delta t\sqrt{\epsilon/\epsilon_\chi}$. Noting that to lowest order $\delta\chi \approx \delta\chi^{(0)}$ and substituting (6) evaluated at $t_{\rm end}$, it follows from (11) that the ζ power spectrum is

$$k^3 P_\zeta(k) = \frac{4\epsilon\beta^2}{M_{\rm Pl}^2} k^3 P_\chi(k) = \beta^2 \frac{H_{\rm end}^2}{2\epsilon M_{\rm Pl}^2} . \tag{12}$$

Up to the prefactor β^2, this is identical to the inflationary result, with ϵ playing the role of the usual slow-roll parameter. In the exit mechanism of [3], β denotes the overall change in angle in the field trajectory: $\beta = \Delta\theta$.

Let us pause to discuss the parameter values that satisfy the CMB constraint $k^3 P_\zeta(k) \approx 10^{-10}$. Although H passes through zero at the bounce, as argued in [3] its magnitude is essentially the same at the beginning of the hot big bang phase as it was at $t_{\rm end}$, the end of the ekpyrotic phase. In other words, $H_{\rm end}$ sets the reheat temperature in the expanding phase. For GUT-scale reheat temperature, we have $H_{\rm end}/M_{\rm Pl} \approx 10^{-6}$. Meanwhile, β is a free parameter whose value depends on the exit dynamics. For the explicit exit mechanism of [3], however, the natural value is $\beta \sim \mathcal{O}(1)$. In this case, setting $k^3 P_\zeta(k) = 10^{-10}$ implies $\epsilon \approx 10^{-2}$. We will henceforth take $\beta = 1$ and $\epsilon = 10^{-2}$ as fiducial parameter values.

Combining (11) with (8) and (9) yields

$$\zeta(x) = \zeta_c(x) + \frac{1}{8\epsilon\beta^2}\zeta_c^2(x) \mp \frac{5\alpha_3}{144\epsilon^2\beta^3}\zeta_c^3(x) + \dots . \tag{13}$$

The exit from the ekpyrotic phase is followed by a ghost condensate phase which leads to a *non-singular* bounce and reheating. Meanwhile, χ gets stabilized and further evolution is governed by the single scalar ϕ. It follows that ζ is conserved through the bounce and emerges unscathed in the hot big bang phase.

3 Non-Gaussianity

3-point function: The 3-point ζ correlation function in New Ekpyrotic Cosmology is given by [3]

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(k_1, k_2, k_3)\,, \tag{14}$$

where the shape function $B(k_1, k_2, k_3)$ is

$$B(k_1, k_2, k_3) = \frac{6}{5} f_{\rm NL}\left\{P_\zeta(k_1)P_\zeta(k_2) + {\rm perm.}\right\} . \tag{15}$$

This is of the so-called *local* form [15]. Equations (14) and (15) are consistent with $\zeta(x)$ of the form $\zeta(x) = \zeta_g(x) + \frac{3}{5} f_{NL}\zeta_g^2(x)$, where ζ_g is Gaussian. The correlation function is evaluated at $t_{\rm end}$ ignoring gravity.

Thus the 3-point function is fully specified by $f_{\rm NL}$ [16]. This parameter receives two contributions. To begin with, the non-Gaussianity of $\delta\chi$, due to its cubic interaction in (3), is inherited by ζ through (11). Following Maldacena [18], the $\delta\chi$ 3-point function is given by

$$\begin{aligned} \langle \delta\chi_1\delta\chi_2\delta\chi_3\rangle &= -i\int_{-\infty}^{t_{\rm end}} ds \langle 0|[\delta\chi_1\delta\chi_2\delta\chi_3, \mathcal{H}_{\rm int}(s)]|0\rangle \\ &\quad + {\rm c.\,c.}\,, \end{aligned} \tag{16}$$

where $\delta\chi_i \equiv \delta\chi(x_i)$, and $\mathcal{H}_{\rm int} = \mathcal{V}(\bar{\phi})\alpha_3\chi^3/3!\Lambda^3$ is the cubic interaction Hamiltonian from (3). An explicit calculation yields the *intrinsic* contribution

$$f_{\rm NL}^{\rm int} = \mp\frac{5}{24}\frac{\alpha_3}{\beta\epsilon}\,. \tag{17}$$

The $\mp$ sign corresponds to choosing $V_{,\chi}|$ to be $\pm$.

The second contribution comes from the non-linear relation between $\delta\chi$ and ζ embodied in (11) and (13). Even if $\delta\chi$ were Gaussian, this non-linearity would make ζ non-Gaussian. This *conversion* contribution to $f_{\rm NL}$ is:

$$f_{\rm NL}^{\rm conv} = \frac{5}{24}\frac{1}{\beta^2\epsilon}\,. \tag{18}$$

Summing (17) and (18) yields a combined $f_{\rm NL}$:

$$f_{\rm NL} \equiv f_{\rm NL}^{\rm int} + f_{\rm NL}^{\rm conv} = \frac{5}{24\beta^2\epsilon}\left(1 \mp \alpha_3\beta\right)\,. \tag{19}$$

Since this is inversely proportional to $\epsilon \ll 1$, non-Gaussianity tends to be large in New Ekpyrotic Cosmology. Related ekpyrotic models [6, 17] also give $f_{\rm NL} \sim \epsilon^{-1}$. (A ghost condensate bounce and second scalar field are also invoked in [6], albeit without an explicit conversion mechanism; and while the two-field ekpyrotic phase of [17] is similar to ours, the bounce physics remains unspecified.) This is in sharp contrast with slow-roll inflation, where $f_{\rm NL}$ is *proportional* to the slow-roll parameters and therefore unobservably small. For concreteness, consider our fiducial model with GUT-scale reheating, $\beta = 1$ and $\epsilon = 10^{-2}$. Taking, for example, the $-$ sign in (17) and choosing $2.728 > \alpha_3 > -3.8$ yields $f_{\rm NL}$ within the present WMAP 2σ range: $-36 < f_{\rm NL} < 100$. Thus $\alpha_3 \sim \mathcal{O}(1)$ yields a non-Gaussian signal near the WMAP bound. Lower reheating temperatures correspond to smaller ϵ and, therefore, larger non-Gaussian signal. Of course, $|f_{\rm NL}|$ can always be made smaller by taking β, ϵ to be larger and/or by suitably choosing α_3.

4-point function: The connected 4-point function,

$$\begin{aligned}\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\\ \cdot[T(k_1,k_2,k_3,k_4)+T'(k_1,k_2,k_3,k_4)],\end{aligned} \tag{20}$$

involves two different shape functions, evaluated at $t_{\rm end}$:

$$\begin{aligned} T &= \frac{1}{2}\tau_{\rm NL}\left\{P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_{14}) + 23\ \text{perm.}\right\}\,;\\ T' &= \kappa_{\rm NL}\left\{P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3) + 3\ \text{perm.}\right\}, \end{aligned} \tag{21}$$

where $\vec{k}_{ij} \equiv \vec{k}_i + \vec{k}_j$. Thus T and T' are specified respectively by the $\tau_{\rm NL}$ and $\kappa_{\rm NL}$ parameters. (Note that $\kappa_{\rm NL}$ is proportional to the f_2 parameter of [8].) Equations (20) and (21) are consistent with $\zeta(x)$ of the form $\zeta(x) = \zeta_g(x) + \frac{\sqrt{\tau_{\rm NL}}}{2}\zeta_g^2(x) + \frac{\kappa_{\rm NL}}{6}\zeta_g^3(x)$, where ζ_g is Gaussian. Note that we can obtain $\tau_{\rm NL}$ immediately by simply comparing its definition with that of $f_{\rm NL}$:

$$\tau_{\rm NL} = \frac{36}{25}f_{\rm NL}^2 = \frac{1}{16\beta^4\epsilon^2}\left(1 \mp \alpha_3\beta\right)^2\,. \tag{22}$$

This was also checked by explicitly computing the three and four-point functions.

Let us now consider $\kappa_{\rm NL}$. It receives two contributions: i) an intrinsic piece due to cubic and quartic $\delta\chi$ interactions; ii) a conversion piece from the non-linear relation between $\delta\chi$ and ζ. The first contribution arises from cubic and quartic terms in χ in the potential (3). An explicit calculation gives

$$\kappa_{\rm NL}^{\rm int} = \frac{2\alpha_4 + 3\alpha_3^2}{40\beta^2\epsilon^2}\,. \tag{23}$$

The second contribution is encoded in the ζ_c^2 and ζ_c^3 terms in (13). Comparing with (20), we obtain

$$\kappa_{\rm NL}^{\rm conv} = \mp\frac{5\alpha_3}{24\beta^3\epsilon^2}\,. \tag{24}$$

Combining the above results, we find

$$\kappa_{\rm NL} \equiv \kappa_{\rm NL}^{\rm int} + \kappa_{\rm NL}^{\rm conv} = \frac{\alpha_3(9\alpha_3\beta \mp 25) + 6\alpha_4\beta}{120\beta^3\epsilon^2}\,. \quad (25)$$

Both $\tau_{\rm NL}$ and $\kappa_{\rm NL}$ are inversely proportional to ϵ^2 and therefore also tend to be relatively large. Note that $\tau_{\rm NL}$ is always positive, whereas $\kappa_{\rm NL}$ can be positive, zero or negative depending on the choices of α_3 and α_4. For instance, our fiducial parameter values for GUT-scale reheating with $\alpha_3, \alpha_4 \sim \mathcal{O}(1)$ yield $\tau_{\rm NL} \sim 10^4$, which is around the estimated bound for the WMAP experiment [8]. Lower non-Gaussianity can again be achieved by taking larger β, ϵ and/or by a suitable choice of α_3 and α_4.

4 Discussion

The simplest inflationary models, consisting of one or more slowly-rolling scalar fields, all predict negligible 3-point and higher-order correlation functions. Non-Gaussianity therefore offers a robust test to distinguish New Ekpyrotic Cosmology from slow-roll inflation.

Significant inflationary non-Gaussianity can be obtained in non-slow-roll models, such as DBI inflation, albeit with a distinguishable shape dependence. Our 3-point function is "local", characterized by a momentum dependence that peaks for squeezed triangles, whereas the DBI amplitude peaks for equilateral triangles [15].

Large non-Gaussianity may also be achieved in the curvaton scenario. The curvaton 3-point function is also of the local form and hence cannot be used to distinguish curvatons from New Ekpyrotic Cosmology. There is, however, an essential difference at the 4-point level. In the simplest curvaton model, the progenitor of density perturbations is a free field. Thus, $\kappa_{\rm NL} \sim f_{\rm NL}$ [11]. In contrast, in New Ekpyrosis, $\tau_{\rm NL}$ and $\kappa_{\rm NL}$ are generically of comparable magnitude ($\sim \epsilon^{-2}$) and are expected to exhibit a distinguishable shape dependence. More intricate curvaton models with self-interactions can also yield large $\kappa_{\rm NL}$. Similarly for general modulon scenarios [13].

Near-future non-Gaussianity observations will, therefore, test the new ekpyrotic paradigm and can potentially distinguish it from its inflationary alternatives.

In this paper we have used a simplifying approximation so as to obtain an analytic expression for non-Gaussianity. Our results give the exact parametric dependence while the details of the potential, roll-off time and so on are encoded in model dependent parameters, such as β. We have checked that different quasi-analytic approximations continue to give the same parametric dependence as presented here, although with model-dependent coefficients that can at most differ from those in this paper by factors of order unity.

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On Some Effects of Space-Time Noncommutativity on Gravitational Theory and Black Hole Physics

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Professor I. L. Buchbinder has been active in various fields of theoretical physics, among them gravity, e.g. [1, 2, 3], and noncommutative field theories, e.g. [4, 5, 6]. For this volume honouring his contribution to theoretical physics, I have chosen a work representative of our common interests.

Abstract

Based on a gauge theory of gravitation in noncommutative space-time, the deformed Reissner-Nordström solution is found. Interesting features related to the black hole physics and the cosmological constant are discussed.

1 Introduction

Questioning the nature of space-time at infinitely small scales has been a fundamental issue for physics. It is generally believed that the visionary Riemann hinted to a possible breakdown of space-time as a manifold already in 1854, in his famous inaugural lecture [7]. The quantum nature of space-time, expressed as noncommutativity of space-time coordinates, has been lately a subject of active research, especially in connection with string theory [8].

Naturally, various effects of space-time noncommutativity in cosmology have been studied, principally motivated by the fact that, since noncommutativity is believed to be significant at the Planck scale - the same scale where quantum gravity effects become important - it is most sensible to search for signatures of noncommutativity in the cosmological observations (for a

[1]Work done in collaboration with M. R. Setare, A. Tureanu and G. Zet.

review, see [9] and references therein). One of the most compelling reasons for the study of noncommutative inflation is the fact that in an inflationary model the physical wavelengths observed today in cosmological experiments emerged from the Planckian region in the early stages of inflation, and thus carry the effects of the Planck scale physics [10]. Among other things, the observed anisotropies of the cosmic microwave background (CMB) may be caused by the noncommutativity of space-time [11]. The noncommutativity has been taken into account either through space-space uncertainty relations or space-time uncertainty relations [12], as well as noncommutative description of the inflaton (with gravity as background which is not affected by noncommutativity) [11]. On the side of noncommutative black-hole physics, the studied effect of noncommutativity was the smearing of the mass-density of a static, spherically-symmetric, particle-like gravitational source [13].

These very interesting ideas have been developed lacking a noncommutative theory of gravity. Although various proposals have been made (see, for a list of references, [14]), an ultimate noncommutative theory of gravity is still elusive. We believe that the most natural way towards this goal is the gauging of the twisted Poincaré symmetry [15]. Although the formulation of twisted internal gauge theories has not yet been achieved [16], the possibility of gauging the (space-time) twisted Poincaré algebra has not been ruled out and the issue is under investigation.

At the moment, one of the most coherent approaches to noncommutative gravity is the one proposed by Chamseddine [17], consisting in gauging the noncommutative $SO(4,1)$ de Sitter group and using the Seiberg-Witten map with subsequent contraction to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$. Although this formulation is not a final theory of noncommutative gravity, it still can serve as a concrete model to be studied, whose main features shall illustrate at least qualitatively the influence of quantum space-time on gravitational effects. The study of specific examples as such can cast light upon the reasonable and unreasonable assumptions proposed so far in the field. Besides, up to now, there have been no calculations presented in the literature (except [14]) to obtain the metric by solving a NC version of gravitational theory, dues to the technical difficulty of the task.

In a recent paper [14] a deformed Schwarzschild solution in noncommutative gauge theory of gravitation was obtained based on [17]. The gravitational gauge potentials (tetrad fields) were calculated for the Schwarzschild solution and the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$ was defined. According to the result of [14] corrections appear only in the second order of the expansion in Θ, i.e. there are no first order correction terms.

In this paper we attempt to extend the results of [14] to include as well the Reissner-Nordström solution. Having these two classical solutions known in the noncommutative setup, we can embark upon a more rigorous study of noncommutative black-hole physics. Black hole thermodynamical quantities depend on the Hawking temperature via the usual thermodynamical relations. The Hawking temperature undergoes corrections from many sources: the quantum corrections [18], the self-gravitational corrections [19], and the corrections due to the generalized uncertainty principle [20]. In this paper we focus on the corrections due to the space-space noncommutativity.

The results of this paper have been obtained using a program devised for GRTensor II application of Maple. For the self-consistence of the paper we shall present, in the commutative case, results of the de Sitter gauge theory with spherical symmetry, obtained with a similar type of program [21] and recall the derivation of the metric tensor components in the noncommutative case, as obtained in [14].

2 de Sitter gauge theory with spherical symmetry

2.1 Commutative case

In the following we shall sketch the principal aspects of a model of gauge theory for gravitation having the de Sitter group (dS) as local symmetry and gravitational field created by a point-like source of mass m and carrying also the electric charge Q. The detailed treatment, including the analytical GRTensor II program used for the calculations, can be found in Ref. [21].

The base manifold is a four-dimensional Minkowski space-time M_4, in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{1}$$

The corresponding metric $g_{\mu\nu}$ has the following non-zero components:

$$g_{00} = -1, \qquad g_{11} = 1, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2\theta. \tag{2}$$

The infinitesimal generators of the 10-dimensional de Sitter group will be denoted by Π_a and $M_{ab} = -M_{ba}$, $a, b = 1, 2, 3, 0$, where Π_a generate the de Sitter "translations" and M_{ab} - the Lorentz transformations. In order to give a general formulation of the gauge theory for the de Sitter group dS, we will denote the generators Π_a and M_{ab} by X_A, $A = 1, 2, \ldots, 10$. The corresponding 10 gravitational gauge fields will be the tetrads $e^a_\mu(x)$, $a = 0, 1, 2, 3$, and the spin connections $\omega^{ab}_\mu(x) = -\omega^{ba}_\mu(x)$, $[ab] = [01], [02], [03], [12], [13], [23]$. Then, the corresponding components of the strength tensor can be written in the standard form, as the torsion tensor:

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \left(\omega^{ab}_\mu e^c_\nu - \omega^{ab}_\nu e^c_\mu\right)\eta_{bc}\,, \tag{3}$$

with η_{ab} the flat space metric, and the curvature tensor:

$$F^{ab}_{\mu\nu} \equiv R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \left(\omega^{ac}_\mu\omega^{db}_\nu - \omega^{ac}_\nu\omega^{db}_\mu\right)\eta_{cd} + 4\lambda^2\left(\delta^b_c\delta^a_d - \delta^a_c\delta^b_d\right)e^c_\mu e^d_\nu, \tag{4}$$

where λ is a real parameter. The integral of action associated to the gravitational gauge fields $e^a_\mu(x)$ and $\omega^{ab}_\mu(x)$ will be chosen as [22]:

$$S_g = \frac{1}{16\pi G}\int d^4x\, e\, F, \tag{5}$$

where $e = \det(e^a_\mu)$ and

$$F = F^{ab}_{\mu\nu}\, e^\mu_a\, e^\nu_b. \tag{6}$$

Here, $e^\mu_a(x)$ denotes the inverse of $e^a_\mu(x)$ satisfying the usual properties:

$$e^a_\mu e^\mu_b = \delta^a_b, \qquad e^a_\mu e^\nu_a = \delta^\nu_\mu. \tag{7}$$

We assume that the source of the gravitation creates also an electromagnetic field $A_\mu(x)$, with the standard action [23]:

$$S_{em} = -\frac{1}{4Kg^2}\int d^4x\, e\, A^a_\mu\, A^\mu_a, \tag{8}$$

where $A^a_\mu = A^\nu_\mu e^a_\nu$, $A^\nu_\mu = e^\nu_a e^\rho_b \eta^{ab} A_{\mu\rho}$ and respectively $A^\mu_a = A^\nu_\mu e^\nu_a$, with $A_{\mu\rho}$ being the electromagnetic field tensor, $A_{\mu\rho} = \partial_\mu A_\rho - \partial_\rho A_\mu$. Here K is a constant that will be chosen in a convenient form to simplify the solutions of the field equations and g is the gauge coupling constant [23].

Then, the total integral of action associated to the system composed of the two fields is given by the sum of the expressions (5) and (8):

$$S = \int d^4x \left[\frac{1}{16\pi G} F - \frac{1}{4Kg^2} A^a_\mu \, A^\mu_a \right] e. \tag{9}$$

The field equations for the gravitational potentials $e^a_\mu(x)$ are obtained by imposing the variational principle $\delta_e S = 0$ with respect to $e^a_\mu(x)$. They are [24]:

$$F^a_\mu - \frac{1}{2} F \, e^a_\mu = 8\pi G T^a_\mu, \tag{10}$$

where F^a_μ is defined by:

$$F^a_\mu = F^{ab}_{\mu\nu} \, e^\nu_b, \tag{11}$$

and T^a_μ is the energy-momentum tensor of the electromagnetic field [25]:

$$T^a_\mu = \frac{1}{Kg^2} \left(A^b_\mu \, A^a_\nu \, e^\nu_b - \frac{1}{4} A^b_\nu \, A^\nu_b \, e^a_\mu \right). \tag{12}$$

The field equations for the other gravitational gauge potentials $\omega^{ab}_\mu(x)$ are equivalent with:

$$F^a_{\mu\nu} = 0. \tag{13}$$

The solutions of the field equations (10) and (13) were obtained in [21], under the assumption that the gravitational field has spherical symmetry and it is created by a point-like source of mass m, which also produces, due to its constant electric charge Q, the electromagnetic field $A_\mu(x)$. The particular form of the spherically symmetric gravitational gauge field adopted in [21] is given by the following Ansatz:

$$e^0_\mu = (A, 0, 0, 0)\,, \qquad e^1_\mu = \left(0, \frac{1}{A}, 0, 0 \right), \qquad e^2_\mu = (0, 0, r, 0)\,, \qquad e^3_\mu = (0, 0, 0, r\sin\theta)\,, \tag{14}$$

and

$$\begin{gathered} \omega^{01}_\mu = (U, 0, 0, 0)\,, \qquad \omega^{02}_\mu = \omega^{03}_\mu = 0, \qquad \omega^{12}_\mu = (0, 0, A, 0)\,, \\ \omega^{13}_\mu = (0, 0, 0, A\sin\theta)\,, \qquad \omega^{23}_\mu = (0, 0, 0, \cos\theta)\,, \end{gathered} \tag{15}$$

where A and U are functions only of the 3D radius r. With the above expressions the components of the tensors $F^a_{\mu\nu}$ and $F^{ab}_{\mu\nu}$ defined by the Eqs. (3) and (4) were computed. Here we give only the expressions of $F^{ab}_{\mu\nu}$ components, which we need to use further, in the derivation of the expressions of the deformed tetrads:

$$\begin{aligned} F^{01}_{10} &= U' + 4\lambda^2, \qquad F^{02}_{20} = A\left(U + 4\lambda^2 r\right), \qquad F^{03}_{30} = A\sin\theta\left(U + 4\lambda^2 r\right), \\ F^{12}_{21} &= \frac{-AA' + 4\lambda^2 r}{A}, \qquad F^{13}_{31} = \frac{\left(-AA' + 4\lambda^2 r\right)\sin\theta}{A}, \\ F^{23}_{32} &= \left(1 - A^2 + 4\lambda^2 r^2\right)\sin\theta, \end{aligned} \tag{16}$$

where A' and U' denote the derivatives with respect to the variable r.

Using the field equations, the solution is obtained [21] as:

$$U = -AA',$$

$$A^2 \quad = \quad 1 + \frac{\alpha}{r} + \frac{Q^2}{r^2} + \beta r^2, \tag{17}$$

where α and β are constants of integration. It is well-known [25] that the constant α is determined by the mass m of the point-like source that creates the gravitational field, by comparison with the Newtonian limit at very large distances:

$$\alpha = -2m. \tag{18}$$

The other constant β was determined in [21] (see also [26]) as $\beta = 4\lambda^2 = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, such that the solution finally reads:

$$A^2 = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2, \qquad U = -\frac{m}{r^2} + \frac{Q^2}{r^3} + \frac{\Lambda}{3} r. \tag{19}$$

If we consider the contraction $\Lambda \to 0$, then the de Sitter group becomes the Poincaré group, and the solution (19) reduces to the Reissner-Nordström one.

2.2 Noncommutative case using the Seiberg-Witten map

The noncommutative corrections to the metric of a space-time with spherically symmetric gravitational field have been obtained in [14], based on the general outline developed by Chamseddine [17].

The noncommutative structure of the space-time is determined by the commutation relation

$$[x^\mu, x^\nu] = i\,\Theta^{\mu\nu}, \tag{20}$$

where $\Theta^{\mu\nu} = -\,\Theta^{\nu\,\mu}$ are constant parameters. It is well known that noncommutative field theory on such a space-time requires is defined by introducing the star product "*" between the functions f and g defined over this space-time:

$$(f{*}g)\,(x) = f\,(x)\; e^{\frac{i}{2}\,\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\;\overrightarrow{\partial_\nu}} g\,(x)\,. \tag{21}$$

The gauge fields corresponding to the de Sitter gauge symmetry for the noncommutative case are denoted by $\hat{e}^a_\mu\,(x,\,\Theta)$ and $\hat{\omega}^{ab}_\mu\,(x,\,\Theta)$, generically denoted by $\hat{\omega}^{AB}_\mu\,(x,\,\Theta)$, with the obvious meaning for the indices A, B. The main idea of the Seiberg-Witten map is to expand the noncommutative gauge fields, transforming according to the noncommutative gauge algebra, in terms of commutative gauge fields, transforming under the corresponding commutative gauge algebra, in such a way that the noncommutative and commutative gauge transformations are compatible, i.e.

$$\hat{\omega}^{AB}_\mu(\omega) + \delta_{\hat{\lambda}}\hat{\omega}^{AB}_\mu(\omega) = \hat{\omega}^{AB}_\mu(\omega + \delta_\lambda \omega). \tag{22}$$

where $\delta_{\hat{\lambda}}$ are the infinitesimal variations under the noncommutative gauge transformations and δ_λ are the infinitesimal variations under the commutative gauge transformations.

Using the Seiberg-Witten map [8], one obtains the following noncommutative corrections up to the second order [17]:

$$\omega^{AB}_{\mu\nu\rho}\,(x) \quad = \quad \frac{1}{4}\;\{\omega_\nu,\,\partial_\rho\omega_\mu + F_{\rho\mu}\}^{AB}\,, \tag{23}$$

$$\begin{aligned} \omega^{AB}_{\mu\nu\rho\lambda\tau}\,(x) \quad &= \quad \frac{1}{32}\;(-\{\omega_\lambda, \partial_\tau\{\omega_\nu, \partial_\rho\omega_\mu + F_{\rho\mu}\}\} + 2\{\omega_\lambda, \{F_{\tau\nu}, F_{\mu\rho}\}\} \\ &- \quad \{\omega_\lambda, \{\omega_\nu, D_\rho F_{\tau\mu} + \partial_\rho F_{\tau\mu}\}\} - \{\{\omega_\nu, \partial_\rho\omega_\lambda + F_{\rho\lambda}\}\,, (\partial_\tau\omega_\mu + F_{\tau\mu})\} \end{aligned} \tag{24}$$

$$+ \quad 2\left[\partial_\nu \omega_\lambda, \partial_\rho \left(\partial_\tau \omega_\mu + F_{\tau\mu}\right)\right]\Big)^{AB},$$

where

$$\{\alpha, \beta\}^{AB} = \alpha^{AC}\,\beta_C^B + \beta^{AC}\,\alpha_C^B, \quad [\alpha, \beta]^{AB} = \alpha^{AC}\,\beta_C^B - \beta^{AC}\,\alpha_C^B \tag{25}$$

and

$$D_\mu F_{\rho\sigma}^{AB} = \partial_\mu F_{\rho\sigma}^{AB} + \left(\omega_\mu^{AC}\,F_{\rho\sigma}^{D\,B} + \omega_\mu^{BC}\,F_{\rho\sigma}^{D\,A}\right)\eta_{CD}. \tag{26}$$

The noncommutative tetrad fields were obtained in [17] up to the second order in Θ in the limit $\Lambda \to 0$ as:

$$\hat{e}_\mu^a(x, \Theta) = e_\mu^a(x) - i\,\Theta^{\nu\rho}\,e_{\mu\nu\rho}^a(x) + \Theta^{\nu\rho}\,\Theta^{\lambda\tau}\,e_{\mu\nu\rho\lambda\tau}^a(x) + O\left(\Theta^3\right), \tag{27}$$

where

$$e_{\mu\nu\rho}^a = \frac{1}{4}\left[\omega_\nu^{a\,c}\partial_\rho e_\mu^d + \left(\partial_\rho\omega_\mu^{a\,c} + F_{\rho\mu}^{a\,c}\right)e_\nu^d\right]\eta_{c\,d}, \tag{28}$$

$$\begin{aligned}
e_{\mu\nu\rho\lambda\tau}^a &= \frac{1}{32}\Big[2\left\{F_{\tau\nu}, F_{\mu\rho}\right\}^{a\,b}\,e_\lambda^c - \omega_\lambda^{a\,b}\left(D_\rho\,F_{\tau\mu}^{c\,d} + \partial_\rho\,F_{\tau\mu}^{c\,d}\right)e_\nu^m\,\eta_{d\,m} \\
&- \left\{\omega_\nu, \left(D_\rho F_{\tau\mu} + \partial_\rho F_{\tau\mu}\right)\right\}^{a\,b}\,e_\lambda^c - \partial_\tau\left\{\omega_\nu, \left(\partial_\rho\,\omega_\mu + F_{\rho\mu}\right)\right\}^{a\,b}\,e_\lambda^c \\
&- \omega_\lambda^{a\,b}\,\partial_\tau\left(\omega_\nu^{c\,d}\,\partial_\rho e_\mu^m + \left(\partial_\rho\,\omega_\mu^{c\,d} + F_{\rho\mu}^{c\,d}\right)e_\nu^m\right)\eta_{dm} + 2\,\partial_\nu\omega_\lambda^{a\,b}\partial_\rho\partial_\tau\,e_\mu^c \\
&- 2\,\partial_\rho\left(\partial_\tau\,\omega_\mu^{a\,b} + F_{\tau\mu}^{a\,b}\right)\partial_\nu\,e_\lambda^c - \left\{\omega_\nu, \left(\partial_\rho\omega_\lambda + F_{\rho\lambda}\right)\right\}^{a\,b}\,\partial_\tau\,e_\mu^c \\
&- \left(\partial_\tau\,\omega_\mu^{a\,b} + F_{\tau\mu}^{a\,b}\right)\left(\omega_\nu^{c\,d}\partial_\rho e_\lambda^m + \left(\partial_\rho\,\omega_\lambda^{c\,d} + F_{\rho\lambda}^{c\,d}\right)e_\nu^m\,\eta_{d\,m}\right)\Big]\,\eta_{b\,c}.
\end{aligned} \tag{29}$$

Using the hermitian conjugate $\hat{e}_\mu^{a\dagger}(x, \Theta)$ of the deformed tetrad fields given in (27),

$$\hat{e}_\mu^{a\,\dagger}(x, \Theta) = e_\mu^a(x) + i\,\Theta^{\nu\rho}\,e_{\mu\nu\rho}^a(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau}e_{\mu\nu\rho\lambda\tau}^a(x) + O\left(\Theta^3\right). \tag{30}$$

the real deformed metric was introduced in [14] by the formula:

$$\hat{g}_{\mu\nu}(x, \Theta) = \frac{1}{2}\,\eta_{a\,b}\,\left(\hat{e}_\mu^a * \hat{e}_\nu^{b\,\dagger} + \hat{e}_\mu^b * \hat{e}_\nu^{a\,\dagger}\right). \tag{31}$$

2.3 Second order corrections to Reissner-Nordström de Sitter solution

Using the Ansatz (14)-(15), we can determine the deformed Reissner-Nordström de Sitter metric by the same method as the Schwarzschild metric was obtained in [14]. To this end, we have to obtain first the corresponding components of the tetrad fields $\hat{e}_\mu^a(x, \Theta)$ and their complex conjugated $\hat{e}_\mu^{a+}(x, \Theta)$ given by the Eqs. (27) and (30). With the definition (31) it is possible then to obtain the components of the deformed metric $\hat{g}_{\mu\nu}(x, \Theta)$.

Taking only space-space noncommutativity, $\Theta_{0i} = 0$ (due to the known problem with unitarity), we choose the coordinate system so that the parameters $\Theta^{\mu\nu}$ are given as:

$$\Theta^{\mu\nu} = \begin{pmatrix} 0 & \Theta & 0 & 0 \\ -\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu, \nu = 1, 2, 3, 0. \tag{32}$$

The non-zero components of the tetrad fields $\hat{e}_\mu^a(x, \Theta)$ are:

$$\begin{aligned}
\hat{e}_1^1 &= \frac{1}{A} + \frac{A''}{8}\,\Theta^2 + O(\Theta^3), \\
\hat{e}_2^1 &= -\frac{i}{4}\left(A + 2\,r\,A'\right)\,\Theta + O(\Theta^3),
\end{aligned} \tag{33}$$

$$\begin{aligned}
\hat{e}_2^2 &= r + \frac{1}{32}\left(7A\,A' + 12\,r\,A'^2 + 12\,r\,A\,A''\right)\Theta^2 + O(\Theta^3),\\
\hat{e}_3^3 &= r\sin\theta - \frac{i}{4}(\cos\theta)\,\Theta + \frac{1}{8}\left(2r\,A'^2 + rAA'' + 2AA' - \frac{A'}{A}\right)(\sin\theta)\,\Theta^2 + O(\Theta^3),\\
\hat{e}_0^0 &= A + \frac{1}{8}\left(2\,r\,A'^3 + 5\,r\,A\,A'\,A'' + r\,A^2\,A''' + 2\,A\,A'^2 + A^2\,A''\right)\Theta^2 + O(\Theta^3)\,,
\end{aligned}$$

where A', A'', A''' are first, second and third derivatives of $A(r)$, respectively, with A^2 given in (19).

Then, using the definition (31), we obtain the following non-zero components of the deformed metric $\hat{g}_{\mu\nu}\,(x,\Theta)$ up to the second order:

$$\begin{aligned}
\hat{g}_{1\,1}\,(x,\Theta) &= \frac{1}{A^2} + \frac{1}{4}\,\frac{A''}{A}\,\Theta^2 + O(\Theta^4), \qquad (34)\\
\hat{g}_{22}\,(x,\Theta) &= r^2 + \frac{1}{16}\left(A^2 + 11\,r\,A\,A' + 16\,r^2\,A'^2 + 12\,r^2 A\,A''\right)\Theta^2 + O(\Theta^4),\\
\hat{g}_{33}\,(x,\Theta) &= r^2\,\sin^2\,\theta\\
&+ \frac{1}{16}\left[4\left(2\,r\,A\,A' - r\frac{A'}{A} + r^2\,A\,A'' + 2\,r^2\,A'^2\right)\sin^2\,\theta + \cos^2\theta\right]\Theta^2 + O(\Theta^4)\\
\hat{g}_{00}\,(x,\Theta) &= -A^2 - \frac{1}{4}\left(2\,r\,A\,A'^3 + r\,A^3\,A''' + A^3\,A'' + 2\,A^2\,A'^2 + 5\,r\,A^2\,A'\,A''\right)\Theta^2\\
&+ O(\Theta^4).
\end{aligned}$$

For $\Theta \to 0$ we obtain the commutative Reissner-Nordström de Sitter solution with $A^2 = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2$.

We should mention that the expressions for the noncommutative corrections to the deformed tetrad fields and noncommutative metric elements are the same as the ones obtained in [14], although here we have also the electromagnetic field involved. The reason is that the first order noncommutative corrections to the electromagnetic field in the Seiberg-Witten map approach, i.e. $A_\mu^{(1)}$ is a pure gauge and thus can be gauged away [27]. As a result, in this order noncommutative corrections involving the electromagnetic field do not appear.

Now, if we insert A into (34), then we obtain the deformed Reissner-Nordström-de Sitter metric with corrections up to the second order in Θ. Its non-zero components are:

$$\begin{aligned}
\hat{g}_{11} &= \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right)^{-1} + \frac{(-2mr^3 + 3m^2r^2 + 3Q^2r^2 - 6mQ^2r + 2Q^4)}{16r^2(r^2 - 2mr + Q^2)}\Theta^2, \qquad (35)\\
\hat{g}_{22} &= r^2 + \frac{r^4 - 17mr^3 + 34m^2r^2 + 27Q^2r^2 - 75mQ^2r + 30Q^4}{16r^2(r^2 - 2mr + Q^2)}\Theta^2\,,\\
\hat{g}_{33} &= r^2\sin^2\,\theta + \frac{\cos^2\theta(r^4 + 2mr^3 - 7Q^2r^2 - 4m^2r^2 + 16mQ^2r - 8Q^4)}{16r^2(r^2 - 2mr + Q^2)}\Theta^2\\
&+ \frac{(-4mr^3 + 4m^2r^2 + 8Q^2r^2 - 16mQ^2r + 8Q^4)}{16r^2(r^2 - 2mr + Q^2)}\Theta^2\,,\\
\hat{g}_{00} &= -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right) + \frac{4mr^3 - 9Q^2r^2 - 11m^2r^2 + 30mQ^2r - 14Q^4}{4r^6}\Theta^2\,.
\end{aligned}$$

In the expressions above, we have neglected the terms containing $\Lambda\Theta^2$ and $\Lambda^2\Theta^2$, which are extremely small by comparison to the other corrections. In the further calculations, we retained the Λ corrections coming only from the 0th order in Θ expression of the metric.

2.4 Noncommutative scalar curvature and cosmological constant

It is well known that, in the commutative case, the scalar curvature of the vacuum solutions (like the Schwarzschild and Reissner-Nordström, when $\Lambda = 0$) vanishes, i.e. the corresponding space-time is Ricci-flat. It is interesting to study whether this property holds also in the noncommutative case. Moreover, this study is motivated also by the fact that in the commutative case the addition of a cosmological term leads to nonvanishing scalar curvatures even in the space devoid of any gravitational source. Should the scalar curvature not vanish for the deformed vacuum solutions, the noncommutative behaviour, in some sense or another, may naturally imitate the commutative solution with cosmological term.

The noncommutative Riemann tensor is expanded in powers of Θ as [17]:

$$\hat{F}^{ab}_{\mu\nu} = F^{ab}_{\mu\nu} + i\Theta^{\rho\tau}F^{ab}_{\mu\nu\rho\tau} + \Theta^{\rho\tau}\Theta^{\kappa\sigma}F^{ab}_{\mu\nu\rho\tau\kappa\sigma} + O(\Theta^3)\,, \tag{36}$$

where

$$F^{ab}_{\mu\nu\rho\tau} = \partial_\mu\omega^{ab}_{\nu\rho\tau} + (\omega^{ac}_\mu\omega^{db}_{\nu\rho\tau} + \omega^{ac}_{\mu\rho\tau} + \omega^{db}_\nu - \frac{1}{2}\partial_\rho\omega^{ac}_\mu\partial_\tau\omega^{db}_\nu)\eta_{cd} - (\mu \leftrightarrow \nu) \tag{37}$$

and

$$F^{ab}_{\mu\nu\rho\tau\kappa\sigma} = \partial_\mu\omega^{ab}_{\nu\rho\tau\kappa\sigma} + (\omega^{ac}_\mu\omega^{db}_{\nu\rho\tau\kappa\sigma} + \omega^{ac}_{\mu\rho\tau\kappa\sigma} + \omega^{db}_\nu - \omega^{ac}_{\mu\rho\tau}\omega^{db}_{\nu\kappa\sigma} - \frac{1}{4}\partial_\rho\partial_\kappa\omega^{ac}_\mu\partial_\tau\partial_\sigma\omega^{db}_\nu)\eta_{cd} - (\mu \leftrightarrow \nu)\,, \tag{38}$$

where $\omega^{ab}_{\mu\nu\rho}$ and $\omega^{ab}_{\mu\nu\rho\lambda\tau}$ are given by Eqs. (23) and (24), respectively, with $A = a$ and $B = b$. After we calculate $\hat{F}^{ab}_{\mu\nu}$ and have also $\hat{e}^a_\mu$ we can obtain the noncommutative scalar curvature:

$$\hat{F} = \hat{e}^\mu_a * \hat{F}^{ab}_{\mu\nu} * \hat{e}^\nu_b \tag{39}$$

where $\hat{e}^\mu_a$ is the inverse of $\hat{e}^a_\mu$ with respect to the star product, i.e. $\hat{e}^\mu_a * \hat{e}^b_\mu = \delta^b_a$. The general expression of the scalar curvature, expanded in powers of Θ, is:

$$\begin{aligned}\hat{F} = F \quad &+ \quad \Theta^{\rho\tau}\Theta^{\kappa\sigma}(e^\mu_a F^{ab}_{\mu\nu\rho\tau\kappa\sigma}e^\nu_b + e^\mu_{a\rho\tau\kappa\sigma}F^{ab}_{\mu\nu}e^\nu_b + e^\mu_a F^{ab}_{\mu\nu}e^\nu_{b\rho\tau\kappa\sigma} - e^\mu_{a\rho\tau}F^{ab}_{\mu\nu}e^\nu_{b\kappa\sigma} \\ &- \quad e^\mu_{a\rho\tau}F^{ab}_{\mu\nu\kappa\sigma}e^\nu_b - e^\mu_a F^{ab}_{\mu\nu\rho\tau}e^\nu_{b\kappa\sigma}) + O(\Theta^4)\,.\end{aligned} \tag{40}$$

Here, we have to calculate also $e^\mu_{a\rho\tau}$ and $e^\mu_{a\rho\tau\kappa\sigma}$, using:

$$\hat{e}^\mu_a = e^\mu_a - i\Theta^{\nu\rho}e^\mu_{a\nu\rho} + \Theta^{\nu\rho}\Theta^{\kappa\sigma}e^\mu_{a\nu\rho\kappa\sigma} + O(\Theta^3)\,. \tag{41}$$

The noncommutative scalar curvature for the Reissner-Nordström de Sitter solution is then obtained in the form:

$$\begin{aligned}\hat{F} \quad = \quad & 4\Lambda + \frac{96mr^5 - 552m^2r^4 - 72Q^2r^4 + 896m^3r^3 + 1174mQ^2r^3 - 2740m^2Q^2r^2}{32r^8(r^2 - 2mr + Q^2)}\Theta^2 \\ + \quad & \frac{-550Q^4r^2 + 2362mQ^4r - 614Q^6}{32r^8(r^2 - 2mr + Q^2)}\Theta^2 \\ + \quad & \frac{(-16m^2r^4 + 28Q^2r^4 - 24mQ^2r^3 + 12Q^4r^2)}{32r^8(r^2 - 2mr + Q^2)}(\cot^2\theta)\ \Theta^2\end{aligned} \tag{42}$$

For the charge $Q = 0$ one obtains the scalar curvature for the deformed Schwarzschild de Sitter solution, which is also non-zero.

The very interesting feature of these scalar curvatures is that for finite values of the radius r they are non-zero for the pure Schwarzschild and Reissner-Nordström vacuum solutions (when $\Lambda = 0$), although asymptotically they do vanish. In a away, this situation locally mimics the presence of a nonvanishing cosmological term in the Einstein equations, where it is well known that the space-time curvature does not vanish even in the absence of any matter.

Another point of interest is the apparent singularity of the correction terms given by the vanishing of the denominator in (42) at $r_0 = m \pm \sqrt{m^2 - Q^2}$. Inspired by the removal of divergencies from the usual Schwarzschild metric by a change of coordinates (see, e.g., [25]), we could be tempted to use the same procedure here. However, since $\hat{F}$ is a scalar, the singularity can not be removed by a change of coordinates. This aspect could jeopardize this approach to noncommutative gravity. As we shall comment further, after deriving the corrected horizon radii, this singularity is immaterial for the domain of validity of (42).

3 Noncommutativity corrections to the thermodynamical quantities of black holes

In this section we derive the corrections to the thermodynamical quantities due to the space-space noncommutativity.

For the Reissner-Nordstrom-de Sitter metric, there are four roots of $A^2(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2$, denoted r_i, with $i = 1, ..., 4$. In the Lorentzian section, $0 \leq r < \infty$, the first root is negative and has no physical significance. The second root r_2 is the inner (Cauchy) black-hole horizon, $r = r_3 = r_+$ is the outer (Killing) horizon, and $r = r_4 = r_c$ is the cosmological (acceleration) horizon. The solution of

$$A^2(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 = 0 \tag{43}$$

is found as a series expansion in the cosmological constant:

$$r = r_0 + a\Lambda + b\Lambda^2 + ... , \tag{44}$$

where r_0 is the Reissner-Nordström horizon radius because for $\Lambda = 0$ we obtain the Reissner-Nordström solution. We know that:

$$r_0 = m \pm \sqrt{m^2 - Q^2} \, . \tag{45}$$

Therefore, we obtain the following cosmological and black-hole horizon radius solutions with cosmological constant, respectively:

$$r_c = m + \sqrt{m^2 - Q^2} + \frac{(m + \sqrt{m^2 - Q^2})^5}{6(m(m + \sqrt{m^2 - Q^2}) - Q^2)}\Lambda \, , \tag{46}$$

$$r_+ = m - \sqrt{m^2 - Q^2} + \frac{(m + \sqrt{m^2 - Q^2})^5}{6(m(m + \sqrt{m^2 - Q^2}) - Q^2)}\Lambda \, . \tag{47}$$

On the other hand, for the Schwarzschild-de Sitter metric, with $A^2(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2$, the number of positive roots (and thus the number of event horizons) depends on the ratio of m and $\frac{8\pi}{\Lambda}$. There are no event horizons if $m > \frac{1}{3\sqrt{\Lambda}}$. There is only one event horizon, $r_1 = \frac{1}{\sqrt{\Lambda}}$, if $m = \frac{1}{3\sqrt{\Lambda}}$. In the case $m < \frac{1}{3\sqrt{\Lambda}}$, there are two distinct event horizons:

$$r_2 = \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{\pi}{3} + \frac{1}{3}\arctan\sqrt{\frac{1}{9m^2\Lambda} - 1}\right) , \tag{48}$$

$$r_3 = \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{\pi}{3} - \frac{1}{3}\arctan\sqrt{\frac{1}{9m^2\Lambda} - 1}\right). \tag{49}$$

It can be shown that $r_2 < r_1 < r_3$.

In the noncommutative case, we consider the corrected event horizon radius up to the second order as

$$\hat{r}_{1,2} = A_{1,2} + B_{1,2}\Theta + C_{1,2}\Theta^2 . \tag{50}$$

Substituting the above $\hat{r}_{1,2}$ into the equation $\hat{g}_{00} = 0$, we obtain the corrected cosmological and black hole (Killing) event horizon radii respectively as solutions of this equation:

$$\begin{aligned} \hat{r}_1 &= m + \sqrt{m^2 - Q^2} + \frac{(m + \sqrt{m^2 - Q^2})^5}{6(m(m + \sqrt{m^2 - Q^2}) - Q^2)}\Lambda \\ &+ \frac{(6m^4 + \sqrt{m^2 - Q^2}(6m^3 - 8mQ^2) - 11Q^2m^2 + 5Q^4)}{8(8m^5 + \sqrt{m^2 - Q^2}(8m^4 - 8m^2Q^2 + Q^4) - 12m^3Q^2 + 4mQ^4)}\Theta^2 \end{aligned} \tag{51}$$

and

$$\begin{aligned} \hat{r}_2 &= m - \sqrt{m^2 - Q^2} + \frac{(m + \sqrt{m^2 - Q^2})^5}{6(m(m + \sqrt{m^2 - Q^2}) - Q^2)}\Lambda \\ &+ \frac{(6m^4 - \sqrt{m^2 - Q^2}(6m^3 - 8mQ^2) - 11Q^2m^2 + 5Q^4)}{8(8m^5 - \sqrt{m^2 - Q^2}(8m^4 - 8m^2Q^2 + Q^4) - 12m^3Q^2 + 4mQ^4)}\Theta^2 \end{aligned} \tag{52}$$

The distance between the corrected event horizon radii is given by following relation in an example case, when $m = 2Q$

$$\hat{d} = \hat{r}_1 - \hat{r}_2 = d - \Delta d = 2\sqrt{3}Q + \frac{51\sqrt{3}}{4Q}\Theta^2 \tag{53}$$

Therefore in the noncommtative space-time the distance between horizons is more than in the commutative case. Then we obtain from (53):

$$\frac{\Delta d}{d} = \frac{51\Theta^2}{8Q^2} \tag{54}$$

The ratio of this change due to the noncommutativity correction to the distance has a value which is much too small.

Returning to the singularity of the scalar curvature mentioned in the end of the previous section, one can now see that at the corrected horizons (51) and (52) the noncommutative scalar curvature (42) is well behaved, since the denominator does not vanish. In fact, the noncommutative corrections to the horizon radii are such, that the points $r_0 = m \pm \sqrt{m^2 - Q^2}$ are inside the black hole, while the metric and the scalar curvature $\hat{F}$ have been obtained outside of the black hole. As a result, the scalar curvature (42) is not singular throughout its domain of validity.

The modified Hawking-Bekenstein temperature and the horizon area of the Reissner-Nordström de Sitter black hole in noncommutative space-time to the second order in Θ are as follows, respectively:

$$\begin{aligned} \hat{T}_+ &= \frac{1}{4\pi}\frac{d\hat{g}_{00}(\hat{r}_1)}{dr} = \frac{m^2 - m\sqrt{m^2 - Q^2} - Q^2}{2\pi(m - \sqrt{m^2 - Q^2})} \\ &+ \frac{Q^2(-4m^2Q^2\sqrt{m^2 - Q^2} - 48m^5 + 68m^3Q^2 - 21mQ^4 + \sqrt{m^2 - Q^2}Q^4)}{12\pi(m - \sqrt{m^2 - Q^2})^4(-m^2 - m\sqrt{m^2 - Q^2} + Q^2)}\Lambda \end{aligned} \tag{55}$$

$$
\begin{aligned}
&+ \quad \Big(\frac{(448m^9-1648Q^2m^7+2112Q^4m^5-1091Q^6m^3+179Q^8m)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \quad \frac{\sqrt{m^2-Q^2}(-2240Q^2m^6+2597Q^4m^4-1053Q^6m^2+612m^8+84Q^8)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \quad \frac{(m^2-Q^2)^{3/2}(264Q^2m^4-473Q^4m^2-152m^6+51Q^6)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \quad \frac{(m^2-Q^2)^{5/2}(16Q^2m^2-12m^4)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\Big)\frac{\Theta^2}{16\pi},
\end{aligned}
$$

$$
\begin{aligned}
\hat{A}_+ &= 4\pi\hat{r}_1^2 = 4\pi((m-\sqrt{m^2-Q^2})^2+\frac{(m-\sqrt{m^2-Q^2})(m+\sqrt{m^2-Q^2})^5}{3(m(m+\sqrt{m^2-Q^2})-Q^2)}\Lambda) \qquad (56)\\
&+ \frac{\pi(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)}\Theta^2 .
\end{aligned}
$$

The corresponding quantities for the cosmological horizon are as follows, respectively:

$$
\begin{aligned}
\hat{T}_c &= \frac{-1}{4\pi}\frac{d\hat{g}_{00}(\hat{r}_2)}{dr} = \frac{-m^2-m\sqrt{m^2-Q^2}+Q^2}{2\pi(m+\sqrt{m^2-Q^2})} \qquad (57)\\
&+ \frac{(-4m^2-4m)\sqrt{m^2-Q^2}+5Q^2)(m+\sqrt{m^2-Q^2})}{12\pi(-m^2-m\sqrt{m^2-Q^2}+Q^2)}\Lambda\\
&+ \Big(\frac{(448m^9-1648Q^2m^7+2112Q^4m^5-1091Q^6m^3+179Q^8m)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \frac{\sqrt{m^2-Q^2}(-2240Q^2m^6+2597Q^4m^4-1053Q^6m^2+612m^8+84Q^8)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \frac{(m^2-Q^2)^{3/2}(264Q^2m^4-473Q^4m^2-152m^6+51Q^6)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\\
&+ \frac{(m^2-Q^2)^{5/2}(16Q^2m^2-12m^4)}{(m+\sqrt{m^2-Q^2})^7[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}\Big)\frac{\Theta^2}{16\pi},
\end{aligned}
$$

$$
\begin{aligned}
\hat{A}_c &= 4\pi\hat{r}_2^2 = 4\pi((m+\sqrt{m^2-Q^2})^2+\frac{(m+\sqrt{m^2-Q^2})^6}{3(m(m+\sqrt{m^2-Q^2})-Q^2)}\Lambda) \qquad (58)\\
&+ \frac{\pi(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)}\Theta^2 .
\end{aligned}
$$

According to the Bekenstein-Hawking formula the thermodynamic entropy of a black hole is proportional to the area of the event horizon $S = A/4$, where A is the area of the horizon. The corrected entropy due to noncommutativity for the black-hole horizon and the cosmological horizon are:

$$
\begin{aligned}
\hat{S}_+ = \frac{\hat{A}_+}{4} &= \pi^2((m-\sqrt{m^2-Q^2})^2+\frac{(m-\sqrt{m^2-Q^2})(m+\sqrt{m^2-Q^2})^5}{3(m(m+\sqrt{m^2-Q^2})-Q^2)}\Lambda) \qquad (59)\\
&+ \frac{\pi\Theta^2(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)}
\end{aligned}
$$

$$+ \quad \frac{\pi\Theta^2(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{4[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]}$$

$$\begin{aligned} \hat{S}_c = \frac{\hat{A}_c}{4} &= \pi^2((m+\sqrt{m^2-Q^2})^2 + \frac{(m+\sqrt{m^2-Q^2})^6}{3(m(m+\sqrt{m^2-Q^2})-Q^2)}\Lambda) \\ &+ \frac{\pi\Theta^2(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)} \\ &+ \frac{\pi\Theta^2(m+\sqrt{m^2-Q^2})[6m^4-11Q^2m^2+5Q^4+\sqrt{m^2-Q^2}(6m^3-8Q^2m)]}{4[8m^5-12Q^2m^3+4Q^4m+\sqrt{m^2-Q^2}(8m^4-8Q^2m^2+Q^4)]} \end{aligned} \tag{60}$$

If we consider the $Q = 0$ case, we obtain the corresponding quantities for the Schwarzschild de Sitter black holes.

4 Conclusions and discussions

Following Ref. [21], in the present paper we constructed a gauge theory for gravitation using the de Sitter group as the local symmetry. The gravitational field has been described by gauge potentials. The solutions of the gauge field equations were studied considering a spherically symmetric case. Assuming that the source of the gravitational field is a point-like mass electrically charged, we obtained the Reissner-Nordström solution. Then, a deformation of the gravitational field has been performed along the lines of Ref. [17] by gauging the noncommutative de Sitter $SO(4,1)$ group and using the Seiberg-Witten map. The corresponding space-time is also of Minkowski type, but endowed now with spherical noncommutative coordinates. We determined the deformed gauge fields up to the second order in the noncommutativity parameters $\Theta^{\mu\nu}$. The deformed gravitational gauge potentials (tetrad fields) $\hat{e}^a_\mu(x,\Theta)$ have been obtained by contracting the noncommutative gauge group $SO(4,1)$ to the Poincaré (inhomogeneous Lorentz) group $ISO(3,1)$. Then, we have calculated these potentials for the case of the Reissner-Nordström solution and defined the corresponding deformed metric $\hat{g}_{\mu\nu}(x,\Theta)$. By finding the Reissner-Nordström solution, as well as the Schwarzschild solution in [14], for a noncommutative theory of gravity we came closer to plausible black-hole physics on noncommutative space-time. The event horizon of the black hole undergoes corrections from the noncommutativity of space as in Eq. (51). Since the noncommutativity parameter is small in comparison with the length scales of the system, one can consider the noncommutative effect as perturbations of the commutative counterpart. Then we have obtained the corrections to the temperature and entropy given in Eqs. (55) and (56).

The noncommutativity of space-time drastically changes the topology of the space-time in the vicinity of the source in the presence of gravitational fields, in the sense that the curvature is not zero, locally, while asymptotically is does vanish. This situation is, in a limited sense, similar to the effect of a nonvanishing cosmological term in usual Einstein's equations. It could not be a priori ruled out that in a fully consistent treatment of a noncommutative theory of gravity, without expansion in Θ, the effects of the cosmological constant could be less locally imitated by the noncommutativity. In any case, one can say that the NC corrections are of the same form as those arising from the quantum gravity effects [28].

The use of the Seiberg-Witten map for constructing the noncommutative gauge theory of gravity leads inevitably to some loss of information, at least concerning the "big picture", i.e. the global features of the space-time. The reason is that the compatibility of the commutative and noncommutative gauge transformation is required at algebraic level, for infinitesimal transformations. As a result, the noncommutative fields and, consequently, the observables, will always be expressed as power series in Θ, starting from the 0th order, which is inevitably the corresponding field or observable of the commutative theory. Thus, the Seiberg-Witten map approach is useful for calculating corrections, but some phenomena which may be peculiar to the entire noncommutative setting will be concealed. The phenomenon of UV/IR mixing [29] is a show-case for this. If we did perturbation in Θ, the nonplanar diagram (which is finite when using the whole star-product) would no more be finite and the planar diagram would remain UV-divergent.

This features of the Seiberg-Witten map may hide interesting aspects when it comes to singularities. In this paper we have obtained the same singularity structure for the Schwarzschild and Reissner-Nordström metrics: if the 0th order in Θ is singular, then higher order corrections could never cancel this singularity.[2]. This is valid for the deformed Ricci scalar curvature, as well as the Kretschmann invariant, $\hat{F}^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu\rho\sigma}$, where $\hat{F}^{\mu\nu}_{\rho\sigma}$ is the deformed Riemann tensor. This is in sharp contradiction with the conclusions of Ref. [13], where a nonsingular de Sitter geometry was found in the origin. In Ref. [13] the noncommutativity is taken into account by one of its major effects, the infinite nonlocality which it produces - the source of gravitational field is not point-like, but it has a Gaussian extension, while the noncommutativity effects of the gravitational field have not been taken into account. A clear-cut conclusion can be provided only by a full treatment of the noncommutative theory of gravity.

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Some Lagrangians with Zeta Function Nonlocality

Dedicated to Professor I.L. Buchbinder on the occasion of his 60th anniversary

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Abstract

Some nonlocal and nonpolynomial scalar field models originated from p-adic string theory are considered. Infinite number of spacetime derivatives is governed by the Riemann zeta function through d'Alembertian $\Box$ in its argument. Construction of the corresponding Lagrangians begins with the exact Lagrangian for effective field of p-adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of over all n. Some basic classical field properties of these scalar fields are obtained. In particular, some trivial solutions of the equations of motion and their tachyon spectra are presented. Field theory with Riemann zeta function nonlocality is also interesting in its own right.

1 Introduction

The first paper on a p-adic string is published in 1987 [1]. After that various p-adic structures have been observed not only in string theory but also in many other models of modern mathematical physics (for a review of the early days developments, see e.g. [2, 3]).

One of the remarkable achievements in p-adic string theory is construction of a field model for open scalar p-adic string [4, 5]. The effective tachyon Lagrangian is very simple and exact. It describes four-point scattering amplitudes as well as all higher ones at the tree-level.

This field theory approach to p-adic string theory has been significantly pushed forward when was shown [6] that it may describes tachyon condensation and brane descent relations. After this success, many aspects of p-adic string dynamics have been investigated and compared with dynamics of ordinary strings (see, e.g. [7, 8, 9, 10] and references therein). Noncommutative deformation of p-adic string world-sheet with a constant B-field was investigated in [11] (on p-adic noncommutativity see also [12]). A systematic mathematical study of spatially homogeneous solutions of the relevant nonlinear differential equations of motion has

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been of considerable interest (see [9, 13, 14, 15] and references therein). Some possible cosmological implications of p-adic string theory have been also investigated [16, 17, 18, 19, 20]. It was proposed [21] that p-adic string theories provide lattice discretization to the world-sheet of ordinary strings. As a result of these developments, some nontrivial features of ordinary string theory have been reproduced from the p-adic effective action. Moreover, there have been established many similarities and analogies between p-adic and ordinary strings.

Adelic approach to the string scattering amplitudes enables to connect p-adic and ordinary counterparts ([2, 3] as a review, and see also [22]). Moreover, it eliminates unwanted prime number parameter p contained in p-adic amplitudes and also cures the problem of p-adic causality violation. Adelic generalization of quantum mechanics was also successfully formulated, and it was found a connection between adelic vacuum state of the harmonic oscillator and the Riemann zeta function [23]. Recently, an interesting approach toward foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [24]. Note that p-adic and ordinary sectors of the four point adelic string amplitudes separately contain the Riemann zeta function (see, e.g. [2], [3] and [25]).

The present paper is mainly motivated by our intention to obtain the corresponding effective Lagrangian for adelic scalar string. Hence, as a first step we investigate possibilities to derive Lagrangian related to the p-adic sector of adelic string. Starting with the exact Lagrangian for the effective field of p-adic tachyon string, extending prime number p to arbitrary natural number n and undertaking various summations of such Lagrangians over all n, we obtain some scalar field models with the operator valued Riemann zeta function. Emergence of the Riemann zeta function at the classical level can be regarded as its analog of quantum scattering amplitude. This zeta function controls spacetime nonlocality. In the sequel we shall construct and explore some classical field models which should help in investigation of some properties of adelic scalar strings.

2 Construction of zeta nonlocal Lagrangians

The exact tree-level Lagrangian of effective scalar field φ for open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]\,, \tag{1}$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian.

The equation of motion for (1) is

$$p^{-\frac{\Box}{2m_p^2}}\,\varphi = \varphi^p\,, \tag{2}$$

and its properties have been studied by many authors (see e.g. [9, 13, 14, 15] and references therein).

Prime number p in (1) and (2) can be replaced by any natural number $n \geq 2$ and such expressions also make sense. Moreover, if $p = 1+\varepsilon \to 1$ there is the limit of (1)

$$\mathcal{L} = \frac{m^D}{g^2}\,\Big[\frac{1}{2}\,\varphi\,\frac{\Box}{m^2}\,\varphi + \frac{\varphi^2}{2}\,(\ln\varphi^2 - 1)\Big] \tag{3}$$

which corresponds to the ordinary bosonic string in the boundary string field theory [26].

Now we want to introduce a model which incorporates all the above string Lagrangians (1) with p replaced by $n \in \mathbb{N}$. To this end, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \Big[-\frac{1}{2} \phi\, n^{-\frac{\Box}{2m_n^2}} \phi + \frac{1}{n+1} \phi^{n+1} \Big] , \tag{4}$$

whose explicit realization depends on particular choice of coefficients C_n, string masses m_n and coupling constants g_n. To avoid a divergence problem in $1/(n-1)$ when $n = 1$ one has to take that $C_n\, m_n^D/g_n^2$ is proportional to $n-1$. In this paper we shall consider a case when coefficients C_n are proportional to $n-1$, while masses m_n as well as coupling constants g_n do not depend on n, i.e. $m_n = m$, $g_n = g$. Since this is an approach towards effective Lagrangian of an adelic string it seems natural to take mass and coupling constant independent on particular p or n. To emphasize that Lagrangian (4) describes a new field, which is different from a particular p-adic one, we introduced notation ϕ instead of φ. The two terms in (4) with $n = 1$ are equal up to the sign, but we remain them because they provide the suitable form of total Lagrangian L.

2.1 Case $C_n = \frac{n-1}{n^{2+h}}$

Let us first consider the case

$$C_n = \frac{n-1}{n^{2+h}} , \tag{5}$$

where h is a real number. The corresponding Lagrangian is

$$L_h = \frac{m^D}{g^2} \Big[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2} - h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \Big] \tag{6}$$

and it depends on parameter h.

According to the famous Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2\,m^2} - h} = \prod_p \frac{1}{1 - p^{-\frac{\Box}{2\,m^2} - h}} .$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} , \quad s = \sigma + i\tau , \quad \sigma > 1 , \tag{7}$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (7) we can rewrite (6) in the form

$$L_h = \frac{m^D}{g^2} \Big[-\frac{1}{2} \phi\, \zeta\Big(\frac{\Box}{2\,m^2} + h\Big) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \Big] . \tag{8}$$

Here $\zeta\Big(\frac{\Box}{2\,m^2} + h\Big)$ acts as a pseudodifferential operator

$$\zeta\Big(\frac{\Box}{2\,m^2} + h\Big) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk}\, \zeta\Big(-\frac{k^2}{2\,m^2} + h\Big) \tilde{\phi}(k)\, dk , \tag{9}$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\,\phi(x)\,dx$ is the Fourier transform of $\phi(x)$. Lagrangian L_0, with the restriction on momenta $-k^2 = k_0^2 - \overrightarrow{k}^2 > (2-2h)\,m^2$ and field $|\phi| < 1$, is analyzed in [27]. In the sequel we shall consider Lagrangian (8) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1}\,\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\Big[-\frac{1}{2}\,\phi\,\zeta\Big(\frac{\Box}{2\,m^2} + h\Big)\,\phi + \mathcal{AC}\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\,\phi^{n+1}\Big]\,, \tag{10}$$

where $\mathcal{AC}$ denotes analytic continuation.

Nonlocal dynamics of this field ϕ is encoded in the pseudodifferential form of the Riemann zeta function. When the d'Alembertian is in the argument of the Riemann zeta function we say that we have zeta nonlocality. Accordingly, this ϕ is a zeta nonlocal scalar field.

Potential of the above zeta scalar field (10) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\,\Big(\frac{\phi^2}{2}\,\zeta(h) - \mathcal{AC}\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\,\phi^{n+1}\Big)\,, \tag{11}$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ-function vanishes at $h = -2, -4, -6, \cdots$.

The equation of motion in differential and integral form is

$$\zeta\Big(\frac{\Box}{2\,m^2} + h\Big)\,\phi = \mathcal{AC}\sum_{n=1}^{+\infty} n^{-h}\,\phi^n\,, \tag{12}$$

$$\frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} e^{ixk}\,\zeta\Big(-\frac{k^2}{2\,m^2} + h\Big)\,\tilde{\phi}(k)\,dk = \mathcal{AC}\sum_{n=1}^{+\infty} n^{-h}\,\phi^n\,, \tag{13}$$

respectively. It is clear that $\phi = 0$ is a trivial solution for any real h. Existence of other trivial solutions depends on parameter h. When $h > 1$ we have another constant trivial solution $\phi = 1$.

In the weak field approximation ($|\phi(x)| \ll 1$) the above expression (13) becomes

$$\int_{\mathbb{R}^D} e^{ikx}\,\Big[\zeta\Big(-\frac{k^2}{2\,m^2} + h\Big) - 1\Big]\,\tilde{\phi}(k)\,dk = 0\,, \tag{14}$$

which has a solution $\tilde{\phi}(k) \neq 0$ if equation

$$\zeta\Big(\frac{-k^2}{2\,m^2} + h\Big) = 1 \tag{15}$$

is satisfied. According to the usual relativistic kinematic relation $k^2 = -k_0^2 + \overrightarrow{k}^2 = -M^2$, equation (15) in the form

$$\zeta\Big(\frac{M^2}{2\,m^2} + h\Big) = 1\,, \tag{16}$$

determines mass spectrum $M^2 = \mu_h\,m^2$, where set of values of spectral function μ_h depends on h.

Equation (16) gives infinitely many tachyon mass solutions. Namely, function $\zeta(s)$ is continuous for real $s \neq 1$ and changes sign crossing its zeros $s = -2n,\, n \in \mathbb{N}$. According to relation $\zeta(1-2n) = -B_{2n}/(2n)$ and values of the Bernoulli numbers ($B_0 = 1,\, B_1 = -1/2,\, B_2 = 1/6,\, B_4 = -1/30,\, B_6 = 1/42,\, B_8 = -1/30,\, B_{10} = 5/66,\, B_{12} =$

$-691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $B_{18} = 43867/798$, $\cdots$) it follows that $|\zeta(1-2n)| = |B_{2n}/(2n)| > 1$ if and only if $n \geq 9$. Taking into account also regions where $\zeta(1-2n) > 0$ we conclude that $\zeta(s) = 1$ has two solutions when $-20-4j < s < -18-4j$ for every $j = 0, 1, 2, \cdots$. Consequently, for any $h \in \mathbb{Z}$, we obtain infinitely many tachyon masses M^2:

$$M^2 = -(40 + 8j + 2h - a_j)\, m^2 \quad \text{and} \quad M^2 = -(36 + 8j + 2h + b_j)\, m^2, \tag{17}$$

where $a_j \ll 1$, $b_j \ll 1$ and $j = 0, 1, 2, \cdots$.

An elaboration of the above Lagrangian for $h = 0, \pm 1, \pm 2$ is presented in [28].

2.2 Case $C_n = \frac{n^2-1}{n^2}$

In this case Lagrangian (4) becomes

$$L = \frac{m^D}{g^2}\Big[-\frac{1}{2}\, \phi \sum_{n=1}^{+\infty} \Big(n^{-\frac{\Box}{2m^2}+1} + n^{-\frac{\Box}{2m^2}} \Big)\, \phi + \sum_{n=1}^{+\infty} \phi^{n+1} \Big] \tag{18}$$

and it yields

$$L = \frac{m^D}{g^2}\Big[-\frac{1}{2}\, \phi \Big\{ \zeta\Big(\frac{\Box}{2\, m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\, m^2}\Big) \Big\}\, \phi + \frac{\phi^2}{1-\phi} \Big]. \tag{19}$$

The corresponding potential is

$$V(\phi) = -\frac{m^D}{g^2}\, \frac{31 - 7\phi}{24\,(1-\phi)}\, \phi^2\,, \tag{20}$$

which has the following properties: $V(0) = V(31/7) = 0\,,\ V(1 \pm 0) = \pm\infty\,,\ V(\pm\infty) = -\infty$. At $\phi = 0$ potential has local maximum.

The equation of motion is

$$\Big[\zeta\Big(\frac{\Box}{2\, m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\, m^2}\Big) \Big]\, \phi = \frac{\phi((\phi-1)^2 + 1)}{(\phi-1)^2}\,, \tag{21}$$

which has only $\phi = 0$ as a constant real solution. Its weak field approximation is

$$\Big[\zeta\Big(\frac{\Box}{2\, m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\, m^2}\Big) - 2 \Big]\, \phi = 0\,, \tag{22}$$

which implies condition on the mass spectrum

$$\zeta\Big(\frac{M^2}{2\, m^2} - 1\Big) + \zeta\Big(\frac{M^2}{2\, m^2}\Big) = 2\,. \tag{23}$$

From (23) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79\, m^2$ and many tachyon solutions when $M^2 < -38\, m^2$.

3 Extension by ordinary Lagrangian

Let us now add ordinary bosonic Lagrangian (3) to the above constructed ones, i.e. $\mathbf{L}_h = L_h + \mathcal{L}$ and $\mathbf{L} = L + \mathcal{L}$.

Respectively, one has Lagrangian, potential, equation of motion and mass spectrum condition:

$$\mathbf{L}_h = \frac{m^D}{g^2}\Big[\frac{\phi}{2}\Big\{\frac{\Box}{m^2} - \zeta\Big(\frac{\Box}{2\,m^2} + h\Big)\Big\}\phi + \frac{\phi^2}{2}\,(\ln\phi^2 - 1) + \mathcal{AC}\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\Big]\,, \tag{24}$$

$$\mathbf{V}_h(\phi) = \frac{m^D}{g^2}\Big[\frac{\phi^2}{2}\,\Big(\zeta(h) + 1 - \ln\phi^2\Big) - \mathcal{AC}\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\,\phi^{n+1}\Big]\,, \tag{25}$$

$$\Big[\zeta\Big(\frac{\Box}{2\,m^2} + h\Big) - \frac{\Box}{m^2}\Big]\,\phi = \phi\,\ln\phi^2 + \mathcal{AC}\sum_{n=1}^{+\infty}\frac{\phi^n}{n^h}\,, \tag{26}$$

$$\zeta\Big(\frac{M^2}{2\,m^2} + h\Big) - \frac{M^2}{m^2} = -1\,. \tag{27}$$

An analysis of these expressions depending on parameter h will be presented elsewhere. When $C_n = \frac{n^2-1}{n^2}$ one respectively obtains:

$$\mathbf{L} = \frac{m^D}{g^2}\Big[\frac{\phi}{2}\Big\{\frac{\Box}{m^2} - \zeta\Big(\frac{\Box}{2\,m^2} - 1\Big) - \zeta\Big(\frac{\Box}{2\,m^2}\Big) - 1\Big\}\phi + \frac{\phi^2}{2}\,\ln\phi^2 + \frac{\phi^2}{1-\phi}\Big]\,, \tag{28}$$

$$\mathbf{V}(\phi) = \frac{m^D}{g^2}\,\frac{\phi^2}{2}\,\Big[\zeta(-1) + \zeta(0) + 1 - \ln\phi^2 - \frac{1}{1-\phi}\Big]\,, \tag{29}$$

$$\Big[\zeta\Big(\frac{\Box}{2\,m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\,m^2}\Big) - \frac{\Box}{m^2} + 1\Big]\,\phi = \phi\,\ln\phi^2 + \phi + \frac{2\phi - \phi^2}{(1-\phi)^2}\,, \tag{30}$$

$$\zeta\Big(\frac{M^2}{2\,m^2} - 1\Big) + \zeta\Big(\frac{M^2}{2\,m^2}\Big) = \frac{M^2}{m^2}\,. \tag{31}$$

Potential (29) has one local minimum $\mathbf{V}(0) = 0$ and two local maxima, which are approximately: $\mathbf{V}(-0.6) \approx 0.15\,\frac{m^D}{g^2}$ and $\mathbf{V}(0.3) \approx 0.06\,\frac{m^D}{g^2}$. It has also the following properties: $\mathbf{V}(1 \pm 0) = \pm\infty$ and $\mathbf{V}(\pm\infty) = -\infty$.

In addition to many tachyon solutions, equation (31) has two solutions with positive mass: $M^2 \approx 2.67\,m^2$ and $M^2 \approx 4.66\,m^2$.

4 Concluding remarks

As a first step towards construction of an effective field theory for adelic open scalar string, we have found a few Lagrangians which contain all corresponding n-adic Lagrangians ($n \in \mathbb{N}$). As a result one obtains that an infinite number of spacetime derivatives and related nonlocality are governed by the Riemann zeta function. Potentials are nonpolynomial. Tachyon mass spectra are determined by definite equations and they are contained in all the above cases. p-Adic Lagrangians can be easily restored from a zeta Lagrangian using just an inverse procedure for its construction.

This paper contains some basic classical properties of the introduced scalar field with zeta function nonlocality. There are rather many classical aspects which should be investigated. One of them is a systematic study of the equations of motion and nontrivial solutions. In the quantum sector it is desirable to investigate scattering amplitudes and make comparison with adelic string.

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Interacting massless higher spins in the BRST approach

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Abstract

We give a short review of the construction of gauge invariant Lagrangians for interacting bosonic higher spin fields.

1 Introduction

Higher spin gauge theory has been a subject of intensive research (see [1] for recent reviews) during the last decades. Other than being a fascinating topic by itself this theory has attracted a significant amount of attention due to its close relation with the string and M-theory.

The problem of building consistent interactions between massless higher spin fields is highly nontrivial. An important landmark was reached in [2]–[3] with the understanding that the AdS background can accommodate consistent self interactions of massless higher spin fields. This picture has two crucial features: the presence of an infinite tower of massless higher spin fields and nonlocality. It seems to be of extreme importance to understand the results of [2]–[3] in terms of the so called "metric– like" formulation of the higher spin fields where basic objects are tensor fields of an arbitrary rank and symmetry. For this reasons the method of BRST constructions [4]–[13] (see also [14]–[15] for the gauge invariant approach and [16] for the light cone approach) seems to be the most appropriate one. The method of BRST constructions is based upon the principle of gauge invariance. Namely the free Lagrangians must possess enough gauge invariance to remove nonphysical states-ghosts- from the spectrum, while interactions are constructed via consistent deformations of the "free" abelian gauge transformations.

Below we describe a general method for such constructions [10] and give some explicit examples [11], which are exact up to the first order in the coupling constant g. Finally, we derive a vertex [12] which is invariant in all orders in g and is derived from the exact vertex of the open bosonic string field theory [17].

2 Free Massless Fields

A way to construct the nilpotent BRST charge whose cohomologies describe arbitrary massless reducible representations of the Poincare group is to start with the BRST charge for the open bosonic string

$$\mathcal{Q} = \sum_{k,l=-\infty}^{+\infty} (C_{-k}L_k - \frac{1}{2}(k-l) : C_{-k}C_{-l}B_{k+l} :) - C_0, \tag{1}$$

perform the rescaling of oscillator variables

$$c_k = \sqrt{2\alpha'}C_k, \qquad b_k = \frac{1}{\sqrt{2\alpha'}}B_k, \qquad c_0 = \alpha' C_0, \qquad b_0 = \frac{1}{\alpha'}B_0, \tag{2}$$

$$\alpha_k^\mu \to \sqrt{k}\alpha_k^\mu$$

and then take the high energy limit $\alpha' \to \infty$. In this way one obtains a BRST charge

$$Q = c_0 l_0 + \tilde{Q} - b_0\mathcal{M} \tag{3}$$

$$\tilde{Q} = \sum_{k=1}^{\infty}(c_k l_k^+ + c_k^+ l_k), \quad \mathcal{M} = \sum_{k=1}^{\infty} c_k^+ c_k, \quad l_0 = p^\mu p_\mu, \qquad l_k^+ = p^\mu \alpha_{k\mu}^+ \tag{4}$$

which is nilpotent in any space-time dimension. The oscillator variables obey the usual (anti)commutator relations

$$[\alpha_\mu^k, \alpha_\nu^{l,+}] = \delta^{kl}\eta_{\mu\nu}, \quad \{c^{k,+}, b^l\} = \{c^k, b^{l,+}\} = \{c_0^k, b_0^l\} = \delta^{kl}\,, \tag{5}$$

and the vacuum in the Hilbert space is defined as

$$\alpha_k^\mu|0\rangle = 0, \quad c_k|0\rangle = 0 \quad k > 0, \qquad b_k|0\rangle \ = \ 0 \qquad k \geq 0. \tag{6}$$

Let us note that one can take the value of k to be any fixed number without affecting the nilpotency of the BRST charge (3). Fixing the value $k = 1$ one obtains the description of totally symmetric massless higher spin fields, with spins $s, s-2, ..1/0$. The string functional (named "triplet" [18]) in this simplest case has the form

$$|\Phi\rangle = |\phi_1\rangle + c_0|\phi_2\rangle = |\varphi\rangle + c^+ \ b^+ \ |d\rangle + c_0 \ b^+ \ |c\rangle$$

whereas for an arbitrary value of k one has the so called "generalised triplet"

$$|\Phi\rangle = \frac{c_{k_1}^+ \dots c_{k_p}^+ b_{l_1}^+ \dots b_{l_p}^+}{(p!)^2}|D_{k_1,\dots l_p}^{l_1,\dots l_p}\rangle + \frac{c_0 c_{k_1}^+ \dots c_{k_{p-1}}^+ b_{l_1}^+ \dots b_{l_p}^+}{(p-1)!p!}|C_{k_1,\dots k_{p-1}}^{l_1,\dots l_p}\rangle,$$

where the vectors $|D_{l_1,\dots l_p}^{k_1,\dots k_p}\rangle$ and $|C_{l_1,\dots l_p}^{k_1,\dots k_p}\rangle$ are expanded only in terms of oscillators $\alpha_k^{\mu+}$, and the first term in the ghost expansion of (7) with $p = 0$ corresponds to the state $|\varphi\rangle$ in (7). One can show that the whole spectrum of the open bosonic string decomposes into an infinite number of generalised triplets, each of them describing a finite number of fields with mixed symmetries [6].

In order to describe the cubic interactions one introduces three copies ($i = 1, 2, 3$) of the Hilbert space defined above, as in bosonic OSFT [17]. Then the Lagrangian has the form

$$L \ = \ \sum_{i=1}^{3} \int dc_0^i \langle\Phi_i|\, Q_i \,|\Phi_i\rangle \ \ + g(\int dc_0^1 dc_0^2 dc_0^3 \langle\Phi_1|\langle\Phi_2|\langle\Phi_3||V\rangle + h.c)\,, \tag{7}$$

where $|V\rangle$ is the cubic vertex and g is a string coupling constant. The Lagrangian (7) is completely invariant with respect to the nonabelian gauge transformations

$$\delta|\Phi_i\rangle = Q_i|\Lambda_i\rangle - g\int dc_0^{i+1}dc_0^{i+2}[(\langle\Phi_{i+1}|\langle\Lambda_{i+2}| + \langle\Phi_{i+2}|\langle\Lambda_{i+1}|)|V\rangle]\,, \tag{8}$$

provided that the vertex $|V\rangle$ satisfies the BRST invariance condition

$$Q|V\rangle = \sum_i Q_i|V\rangle = \sum_i Q_i V c_0^1 c_0^2 c_0^3|0\rangle\,. \tag{9}$$

The function V is a polynomial of the combinations

$$\gamma^{+,ij}_{(kp)} = c^{+,i}_{(k)}b^{+,j}_{(p)}\,,\quad \beta^{+,ij}_{(kp)} = c^{+,i}_{(k)}b^{j}_{0,(p)}\quad M^{+,ij}_{(kp)} = \frac{1}{2}\alpha^{+,\mu,i}_{(k)}\alpha^{+,j}_{\mu(p)}\,,\quad l^{+,ij}_{(k)} = \alpha^{+,i,\mu}_{(k)}p^j_\mu \tag{10}$$

with the coefficients to be determined from the BRST invariance conditions. The ghost number of the function V is zero as it is for the combinations (10). The additional constraints imposed by the closure of the algebra of gauge transformations will be discussed in the last section. The gauge parameter $|\Lambda\rangle$ in each individual Hilbert space has the ghost structure

$$|\Lambda\rangle = b^+|\lambda\rangle \tag{11}$$

for the totally symmetric case, while the gauge parameters for the generalised triplets take the form

$$|\Lambda\rangle = \frac{c^+_{k_1}\dots c^+_{k_p}b^+_{l_1}\dots b^+_{l_{p+1}}}{(p!)(p+1)!}|\Lambda^{l_1,\dots l_{p+1}}_{k_1,\dots k_p}\rangle + \frac{c_0c^+_{k_1}\dots c^+_{k_{p-1}}b^+_{l_1}\dots c^+_{l_{p+1}}}{(p-1)!(p+1)!}|\hat\Lambda^{l_1,\dots l_{p+1}}_{k_1,\dots k_{p-1}}\rangle.$$

Further, the interaction vertex must belong to the nontrivial cohomologies of the BRST charge $Q=\sum_i Q_i$. And finally since the number operator

$$\tilde N = \alpha^{+,i,\mu}_{(k)}\alpha^i_{\mu,(k)} + b^{+,i}_{(k)}c^i_{(k)} + c^{+,i}_{(k)}b^i_{(k)} \tag{12}$$

commutes with the BRST charge Q one can solve (9) for each eigenvalue of $\tilde N$ separately.

In order to extend the discussion to the case of an arbitrary dimensional AdS space let us recall some relevant facts about the triplet formulation on anti –de Sitter space [6]– [7] we restrict ourselves to the case of totally symmetric fields on $\mathcal{D}$-dimensional AdS space-time i.e., to the case of the "triplet". The formulas given for the case of a flat space-time background apply to the case of AdS space as well (see [10] for details) , but now the ordinary partial derivative is replaced by the operator

$$p_\mu \;=\; -\,i\,\left(\nabla_\mu + \omega_\mu^{ab}\,\alpha^+_{\;a}\,\alpha_{\;b}\right)\;, \tag{13}$$

where ω_μ^{ab} is the spin connection of AdS and ∇_μ is the AdS covariant derivative. The AdS counterpart of the BRST charge (3) has the form

$$\begin{aligned} Q \;&=\; c_0(l_0 + \frac{1}{L^2}(N^2 - 6N + 6 + \mathcal{D} - \frac{\mathcal{D}^2}{4} - 4M^+M + c^+b(4N-6) \\ &+\; b^+c(4N-6) + 12c^+bb^+c - 8c^+b^+M + 8M^+bc)) + c^+l + cl^+ - c^+cb_0 \end{aligned} \tag{14}$$

where l_0 is the AdS covariant d'Alembertian, L is the radius of the AdS space and

$$N \;=\; \alpha^{\mu+}\alpha_\mu \;+\; \frac{\mathcal{D}}{2},\quad M = \frac{1}{2}\alpha^\mu\alpha_\mu\,. \tag{15}$$

3 Some explicit examples

Spin-2 with two scalars We assign the field with spin two to the third Fock space, put the scalars in the first and the second Fock spaces respectively, and the vector field in the third Fock space. Since the oscillators $\alpha_\mu^{i,+}, c^{i,+}$ and $b^{i,+}$ occur only in the third Fock space we omit the index i for them in what follows. Therefore we have [11]

$$|\Phi_3\rangle = (\frac{1}{2!}h_{\mu\nu}(x)\alpha^{\mu+}\alpha^{\nu+} + D(x)c^+b^+ - iC_\mu(x)\alpha^{\mu+}c_0^3b^+)|0\rangle\,, \tag{16}$$

$$|\Lambda\rangle = i\lambda_\mu(x)\alpha^{\mu+}b^+|0\rangle\,. \tag{17}$$

According to our discussion in the previous section since the operator $\tilde{N}$ acting on the product $|\Phi_1\rangle|\Phi_2\rangle|\Phi_3\rangle$ gives 2, then the operator $\tilde{N}$ acting on the function V should give 2 as well in order to "saturate" the last term in (7). Solving the BRST invariance condition (9) one obtains the Lagrangian

$$L = L_{free} + L_{int}\,, \tag{18}$$

$$\begin{aligned} L_{free} &= (\partial_\mu\phi_1)(\partial^\mu\phi_1) + (\partial_\mu\phi_2)(\partial^\mu\phi_2) + m^2(\phi_1^2+\phi_2^2) + (\partial_\rho h_{\mu\nu})(\partial^\rho h^{\mu\nu}) \\ &\quad -4(\partial_\mu h^{\mu\nu})C_\nu - 4(\partial_\mu C^\mu)D - 2(\partial_\mu D)(\partial^\mu D) + 2C_\mu C^\mu\,, \end{aligned} \tag{19}$$

$$\begin{aligned} L_{int} &= C_{2,0}\ (h^{\mu\nu}(\partial_\mu\partial_\nu\phi_1)\phi_2 + h^{\mu\nu}(\partial_\mu\partial_\nu\phi_2)\phi_1 - 2h^{\mu\nu}(\partial_\mu\phi_1)(\partial_\nu\phi_2)) \\ &\quad -C_{2,1}\ \phi_1\phi_2(h_\mu^\mu - 2D)\,. \end{aligned} \tag{20}$$

and the relevant gauge transformations

$$\delta\phi_1 = C_{2,0}\ (2\lambda^\mu\partial_\mu\phi_2 + \phi_2\partial_\mu\lambda^\mu)\,, \tag{21}$$

$$\delta\phi_2 = C_{2,0}\ (2\lambda^\mu\partial_\mu\phi_1 + \phi_1\partial_\mu\lambda^\mu)\,, \tag{22}$$

$$\delta h_{\mu\nu} = \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu, \quad \delta C_\mu = \Box\lambda_\mu, \quad \delta D = \partial_\mu\lambda^\mu \tag{23}$$

where $C_{2,0}$ and $C_{2,1}$ are arbitrary real constants. Note that we have added a mass - term for the scalars in the Lagrangian. Curiously enough the Lagrangian describing the interaction of two massless scalars with a spin two triplet is still gauge invariant after the addition of the mass terms for the scalar. This opens the interesting possibility to start with the Lagrangian for the free massive scalars and gauge its symmetries. In this way one recovers the Lagrangian given above after gauging the symmetries generated by the parameter λ_μ. A similar result holds for the case of two scalars interacting with a spin 3 gauge field. In this case one gauges the symmetries of the free Lagrangian generated by the parameter $\lambda_{\mu\nu}$ [11].

According to our general construction we have obtained the cubic vertex which involves two different scalars and the triplet with higher spin 2. To obtain the interaction of a single scalar with the spin-2 field we need to set $\phi_1 = \phi_2$ *. It should also be noted that for $\phi_1 = \phi_2$ (20) is equivalent to the linearised interaction of a scalar field with gravity. The generalisation for the coupling of a spin-2 triplet with an arbitrary number of scalar fields n goes in an analogous manner.

In $AdS_{\mathcal{D}}$ we replace ordinary derivatives with covariant ones. There will be no other changes for the gauge transformation rules (i.e., for all fields $\delta_{AdS} = \delta$) (23) except for

$$\delta_{AdS}C_\mu = \delta C_\mu + \frac{1-\mathcal{D}}{L^2}\lambda_\mu\,, \tag{24}$$

*Note that setting i.e., $\phi_2 = 0$ is meaningless since in our formalism that would mean to consider two Fock spaces, hence no cubic interaction vertex.

The free Lagrangian is modified to include the standard AdS "mass -terms" of order $1/L^2$

$$\Delta L_{free} = -\frac{1}{L^2}(2h^\mu_\mu h^\nu_\nu - 16h^\mu_\mu D + 2h_{\mu\nu}h^{\mu\nu} + (4\mathcal{D}+12)D^2 + (2\mathcal{D}-6)\ (\phi_1^2+\phi_2^2))\,. \quad (25)$$

The interaction part also changes and gains an additional piece

$$\Delta L_{int.} = C_{2,0}\ \frac{\mathcal{D}-1}{L^2} D\phi_1\phi_2\,. \quad (26)$$

This is an additional interaction of the D scalar with a "spin-0" current.

Spin-3 with two scalars

The spin-3 triplet is described by the field [11]

$$|\Phi_3\rangle = (\frac{1}{3!}h_{\mu\nu\rho}(x)\alpha^{\mu+}\alpha^{\nu+}\alpha^{\rho+} + D_\mu(x)\alpha^{\mu+}c^+b^+ - \frac{i}{2}C_{\mu\nu}(x)\alpha^{\mu+}\alpha^{\nu+}c_0^3b^+)|0\rangle, \quad (27)$$

$$|\Lambda\rangle = \frac{i}{2}\lambda_{\mu\nu}(x)\alpha^{\mu+}\alpha^{\nu+}b^+|0\rangle\,. \quad (28)$$

Now one has to solve the BRST invariance condition (9) for the polynomial V, such that $\tilde{N}V = 3V$. As a result one gets the gauge transformations

$$\delta\phi_1 = 3i\ C_{3,0}\ (4\lambda^{\mu\nu}\partial_\mu\partial_\nu\phi_2 + \phi_2\partial_\mu\partial_\nu\lambda^{\mu\nu} + 4(\partial_\mu\phi_2)(\partial_\nu\lambda^{\mu\nu})) + i\ C_{3,1}\ \phi_2\lambda^\mu_\mu\,, \quad (29)$$

$$\delta\phi_2 = -3i\ C_{3,0}\ (4\lambda^{\mu\nu}\partial_\mu\partial_\nu\phi_1 + \phi_1\partial_\mu\partial_\nu\lambda^{\mu\nu} + 4(\partial_\mu\phi_1)(\partial_\nu\lambda^{\mu\nu})) - i\ C_{3,1}\ \phi_1\lambda^\mu_\mu\,, \quad (30)$$

$$\delta h_{\mu\nu\rho} = \partial_\mu\lambda_{\nu\rho} + \partial_\nu\lambda_{\mu\rho} + \partial_\rho\lambda_{\mu\nu}, \quad \delta C_{\mu\nu} = \Box\lambda_{\mu\nu}, \quad \delta D_\mu = \partial_\nu\lambda^\nu_\mu\,. \quad (31)$$

and the free and interacting parts of the Lagrangian

$$\begin{aligned} L_{free} \ &= \ (\partial_\mu\phi_1)(\partial^\mu\phi_1) + (\partial_\mu\phi_2)(\partial^\mu\phi_2) + m^2(\phi_1^2+\phi_2^2) + (\partial_\tau h_{\mu\nu\rho})(\partial^\tau h^{\mu\nu\rho}) \quad (32) \\ &\quad -6(\partial_\rho h^{\mu\nu\rho})C_{\mu\rho} - 12(\partial_\mu C^{\mu\nu})D_\nu - 6(\partial_\mu D_\nu)(\partial^\mu D^\nu) + 3C_\mu C^\mu\,, \end{aligned}$$

$$\begin{aligned} L_{int.} \ &= \ i\ C_{3,0}\ (h^{\mu\nu\rho}\phi_1\partial_\mu\partial_\nu\partial_\rho\phi_2 - h^{\mu\nu\rho}\phi_2\partial_\mu\partial_\nu\partial_\rho\phi_1 - 3h^{\mu\nu\rho}(\partial_\mu\partial_\nu\phi_2)(\partial_\rho\phi_1) \\ &\quad +3h^{\mu\nu\rho}(\partial_\mu\partial_\nu\phi_1)(\partial_\rho\phi_2)) \\ &\quad +i\ C_{3,1}\ (h^{\mu\nu}_\nu - 2D^\mu)(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1) + h.c. \quad (33) \end{aligned}$$

where $C_{3,0}$ and $C_{3,1}$ are arbitrary real constants. Note that in this case, had we set $\phi_1 = \phi_2$ the interaction would vanish. Unlike the previous example for the case of an interacting triplet with higher spin 3 with two scalars one cannot make the scalars ϕ_1 and ϕ_2 equal to each other so one needs a complex scalar in analogy with scalar electrodynamics. There is one more difference with respect to the previous example, namely when doing the deformation to the $AdS_\mathcal{D}$ case, apart from changing ordinary derivatives to covariant ones, both the Lagrangian and gauge transformation rules for scalars get deformed

$$\begin{aligned} \Delta L_{free} &= -\frac{1}{L^2}(6h^{\mu\rho}_\mu h^\nu_{\nu\rho} - 48h^{\mu\nu}_\mu D_\nu - (\mathcal{D}-3)h_{\mu\nu\rho}h^{\mu\nu\rho} + \\ &+18(\mathcal{D}+3)D^\mu D_\mu + (2\mathcal{D}-6)\ (\phi_1^2+\phi_2^2)) \quad (34) \end{aligned}$$

$$\Delta L_{int} = i\ C_{3,0}\ \frac{6\mathcal{D}}{L^2}\ D^\mu\ (\phi_1\nabla_\mu\phi_2 - \phi_2\nabla_\mu\phi_1) + h.c. \quad (35)$$

$$\delta_{AdS}\phi_1 = \delta_0\phi_1 - i\ C_{3,0}\ \frac{6}{L^2}\lambda^\mu_\mu\phi_2, \quad \delta_{AdS}\phi_2 = \delta\phi_2 + i\ C_{3,0}\ \frac{6}{L^2}\lambda^\mu_\mu\phi_1, \quad (36)$$

$$\delta_{AdS}C_{\mu\nu} = \delta C_{\mu\nu} + \frac{2(1-\mathcal{D})}{L^2}\lambda_{\mu\nu} + \frac{2}{L^2}g_{\mu\nu}\lambda^\rho_\rho. \quad (37)$$

4 An exact vertex

In this subsection we will give a solution to the cubic vertex which is exact to all orders in the constant g [12]. We begin first with the simple case of a vertex for totally symmetric fields. This means we consider only one set of oscillators as in (5). The form of the vertex can be deduced from the high energy limit of the corresponding vertex of OSFT. In bosonic OSFT the cubic vertex has the form

$$|V_3\rangle = \int dp_1\ dp_2\ dp_3\ (2\pi)^d\ \delta^d(p_1+p_2+p_3) \tag{38}$$

$$\times exp\left(\frac{1}{2}\sum_{i,j=1}^{3}\sum_{n,m=0}^{\infty}\alpha^{+,i}_{n,\mu}\ N^{ij}_{nm}\alpha^{+,j}_{m,\nu}\ \eta^{\mu\nu}+\sum_{i,j=1}^{3}\sum_{n\geq 1,m\geq 0}c^{+,i}_{n}X^{ij}_{nm}b^{+,j}_{m}\right)\ |-\rangle_{123},$$

$$|-\rangle_{123} = c_0^1 c_0^2 c_0^3|0\rangle$$

where the solution is given in terms of the Neumann coefficients.

Let us note first that since the BRST charge takes the form (3), it can be truncated to contain any finite number of oscillator variables [6]. For this reason it is possible to look for the BRST invariant vertex that describes the interaction among only totally symmetric tensor fields of arbitrary rank, without the inclusion of modes with mixed symmetries. One possibility is to start from the OSFT vertex (38) and keep only terms proportional to at least one momentum p^r_μ in the exponential, therefore dropping all trace operators $(\alpha^r_\mu\eta^{\mu\nu}\alpha^s_\nu)$, as one does when obtaining the BRST charge (3) from (1) since they are leading in the $\alpha'\to\infty$ limit. However, since these terms are exponentiated and the term $\alpha^{+,r}_{n,\mu}N^{rs}_{n0}p^s_\mu$ is of the same order as $\alpha^{+,r}_{n,\mu}N^{rs}_{n0}p^s_\mu\ (\alpha^{+,r}_{n,\mu}\ N^{rs}_{nm}\alpha^{+,s}_{m,\nu})^p,\ m,n\geq 1$, a priori one can keep them both . The same is true regarding the ghost part where although the term $c^{+,r}_n b^s_0$ is leading compared to the term $c^{+,r}_n X^{rs}_{nm}b^s_m,\ n,m\geq 1$ one cannot neglect the later one in the exponential. Let us stress that all these terms will be essential to maintain the off shell closure of the algebra of gauge transformations and complete gauge invariance of the action.

Based on the discussion above one can make the following ansatz for the vertex which describes interactions between massless totally symmetric fields with an arbitrary spin

$$|V\rangle = V^1\times V^{mod}|-\rangle_{123} \tag{39}$$

where the vertex contains two parts: a part considered in [19] (see also [20])

$$V^1 = exp\ (\ Y_{ij}l^{+,ij}+Z_{ij}\beta^{+,ij}\ ). \tag{40}$$

and the part which ensures the closure of the nonabelian algebra as well as the gauge invariance at all orders in g

$$V^{mod} = exp\ (\ S_{ij}\gamma^{+,ij}+P_{ij}M^{+,ij}\ ), \tag{41}$$

where $P_{ij}=P_{ji}$. Putting this ansatz into the BRST invariance condition and using momentum conservation $p^1_\mu+p^2_\mu+p^3_\mu=0$ one can obtain a solution for Y^{rs} and Z^{rs}

$$Z_{i,i+1}+Z_{i,i+2}=0 \tag{42}$$

$$Y_{i,i+1} = Y_{ii}-Z_{ii}-1/2(Z_{i,i+1}-Z_{i,i+2})$$

$$Y_{i,i+2} = Y_{ii}-Z_{ii}+1/2(Z_{i,i+1}-Z_{i,i+2}).$$

$$S_{ij} = P_{ij} = 0 \qquad i \neq j \qquad (43)$$
$$P_{ii} - S_{ii} = 0 \qquad i = 1, 2, 3$$

In what follows we will assume cyclic symmetry in the three Fock spaces which implies along with (42)

$$Z_{12} = Z_{23} = Z_{31} = Z_a, \quad Z_{21} = Z_{13} = Z_{32} = Z_b = -Z_a, \qquad (44)$$
$$Y_{12} = Y_{23} = Y_{31} = Y_a, \quad Y_{21} = Y_{13} = Y_{32} = Y_b,$$
$$Y_{ii} = Y, \quad Z_{ii} = Z, \quad P_{ii} = -S_{ii} = -S = P.$$

By having determined the form of the vertex from (42) and (43) we will proceed in computing the commutator of two gauge transformations with gauge parameters $|\Xi\rangle$ and $|\Lambda\rangle$. One can check via direct computations that the Lagrangian is invariant in all orders in g provided

$$|S|^2 = 1 \qquad (45)$$

In a similar manner one can prove the closure of the algebra at order g^2. In general, closure of the algebra to order $O(g)$ implies

$$[\delta_\Lambda,\ \delta_\Xi]|\Phi_1\rangle = \delta_{\tilde{\Lambda}}|\Phi_1\rangle = Q_1|\tilde{\Lambda}_1\rangle - g[(\langle\Phi_2|\langle\tilde{\Lambda}_3| + \langle\Phi_3|\langle\tilde{\Lambda}_2|)|V\rangle] +\ O(g^2) \qquad (46)$$

where

$$|\tilde{\Lambda}_1\rangle = g(\langle\Lambda_2|\langle\Xi_3| + \langle\Lambda_3|\langle\Xi_2|)|V\rangle +\ O(g^2). \qquad (47)$$

and therefore for a commutator of two gauge transformations one gets

$$[\delta_\Lambda, \delta_\Xi]|\Phi_1\rangle = Q_1|\tilde{\Lambda_1}\rangle \qquad (48)$$
$$+g^2[\langle V|\,(|\Phi_1\rangle|\Lambda_3\rangle + |\Lambda_1\rangle|\Phi_3\rangle\ )\,\langle\Xi_3||V\rangle + \langle V|\,(|\Phi_1\rangle|\Lambda_2\rangle + |\Lambda_1\rangle|\Phi_2\rangle)\,\langle\Xi_2||V\rangle$$
$$-\langle V|\,(|\Phi_1\rangle|\Xi_3\rangle + |\Xi_1\rangle|\Phi_3\rangle)\,\langle\Lambda_3||V\rangle - \langle V|\,(|\Phi_1\rangle|\Xi_2\rangle + |\Xi_1\rangle|\Phi_2\rangle)\,\langle\Lambda_2||V\rangle$$

where we have suppressed the integrations over the ghost fields of (8). One can show that the condition (45) leads to the closure of the algebra at order g^2.

The case of arbitrary mixed symmetry fields is completely analogous to the construction for totally symmetric fields. As in (39) we make the ansatz

$$V = exp\ (\sum_{n=1}^{\infty}\ Y_{ij}^{(n)} l_{ij}^{+,(n)} +\ Z_{ij}^{(n)} \beta_{ij}^{+,(n)}) \times exp\ (\sum_{n,m=1}^{\infty}\ S_{ij}^{(nm)} \gamma_{ij}^{+,(nm)} + P_{ij}^{(nm)} M_{ij}^{+,(nm)}) \qquad (49)$$

where in this case we are summing over n, m as well. We put the oscillator level indices in parentheses in order to distinguish them from the Fock space ones. The oscillator algebra takes the form

$$[\alpha_\mu^{(m),i}, \alpha_\nu^{+,(n),j}] = \delta^{mn}\delta^{ij} g_{\mu\nu}, \quad \{c^{+,(m),i}, b^{(n),j}\} = \{c^{(m),i}, b^{+,(n),j}\} = \delta^{mn}\delta^{ij}. \qquad (50)$$

The BRST invariance with respect to (3) implies

$$Z_{i,i+1}^{(r)} + Z_{i,i+2}^{(r)} = 0 \qquad (51)$$

$$Y_{i,i+1}^{(r)} = Y_{ii}^{(r)} - Z_{ii}^{(r)} - \frac{1}{2}(Z_{i,i+1}^{(r)} - Z_{i,i+2}^{(r)}) \qquad (52)$$

$$Y_{i,i+2}^{(r)} = Y_{ii}^{(r)} - Z_{ii}^{(r)} + \frac{1}{2}(Z_{i,i+1}^{(r)} - Z_{i,i+2}^{(r)}). \quad (53)$$

and

$$\begin{aligned} S_{ij}^{(ps)} = P_{ij}^{(ps)} = 0 \qquad & i \neq j \ or \ p \neq s \\ P_{ii}^{(ss)} - S_{ii}^{(ss)} = 0 \qquad & i = 1, 2, 3 \end{aligned} \quad (54)$$

We can once more choose a cyclic solution in the three Fock spaces as in (44) and in this way get an obvious generalisation of (51). Finally, as in the case of only one oscillator, the complete invariance of the vertex requires

$$|S^{(r)}|^{\ 2} = 1, \quad (55)$$

for all r being integer numbers.

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Massive charged particle on $AdS_2 \times S^2$

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

Different aspects of particle dynamics on $AdS_2 \times S^2$ background with two–form flux are discussed. These include explicit solution to the equations of motion and a canonical transformation to conformal mechanics.

1 Introduction

It is a great pleasure to contribute to the volume celebrating the 60th birthday of Professor I.L. Buchbinder. I was lucky to attend the lecture course on Quantum Field Theory which Professor Buchbinder gave in 1991-1992 for graduate students at Tomsk State University. To a large extend those beautiful lectures led me to choose the research career. I would like to heartily congratulate Professor Buchbinder and wish him every success.

Recently particle dynamics on $AdS_2 \times S^2$ background with two–form flux was extensively investigated [1]–[13]. This line of research was mostly motivated by the study of different aspects of AdS_2/CFT_1 correspondence (for a review see [14]). As was demonstrated in [1], an interesting example of this correspondence is provided by a massive charged particle propagating near the horizon of an extreme Reissner-Nordström black hole. The geometry characterizing this case is $AdS_2 \times S^2$ and in the limit of large black hole mass one recovers the conventional $d = 1$ conformal mechanics [15]. In particular, an infinite number of quantum states of a particle probe localized near the horizon of the black hole was related to the absence of a ground state in the conformal mechanics and the necessity to redefine the Hamiltonian [15].

It is worth mentioning also that the conformal group $SO(2, d-1)$ is the isometry group of anti de Sitter space AdS_d. As a result, particle models on such a background automatically exhibit conformal symmetry. Since anti de Sitter space describes the near horizon geometry of a wide class of extreme black holes (for a review see e.g. [16]), it was conjectured [1, 17] that the study of conformal invariant models in $d = 1$ and their AdS duals might provide

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new insight into quantum mechanics of black holes. This idea was pushed further in [18]–[20], where a conformal mechanics on the moduli space of a system of static black holes in d=5 was constructed and investigated.

A relation between a particle probe propagating near the horizon of an extreme Reissner-Nordström black hole and the conformal mechanics was established in a specific limit when the black hole mass M is large, the difference between the particle mass and the absolute value of its charge $(m-|e|)$ tends to zero and $M^2(m-|e|)$ is kept fixed [1]. It should be noted, however, that in this limit angular variables effectively decouple and show up only in an indirect way via the effective coupling constant. In [9] a simple conformal mechanics was constructed which is canonically equivalent to a particle moving on $AdS_2 \times S^2$ background, with both the radial and angular variables retained. The clue to finding the canonical transformation was to require the conserved charges to coincide in both theories.

In [1, 9] the magnetic charge on the extreme Reissner-Nordström black hole was set to zero. The purpose of this note is to generalize the results of [9] to the case when a particle probe also couples to the magnetic charge of the black hole. The corresponding conformal mechanics is shown to coincide with the bosonic limit of a new variant of $N = 4$ superconformal mechanics constructed recently in [11]. We also discuss in some detail the aspects of classical dynamics of a particle propagating on $AdS_2 \times S^2$ background with two–form flux.

In the next section we briefly review the near horizon geometry of the extreme Reissner-Nordström black hole solution of Einstein–Maxwell equations. In sect. 3 we use global symmetry of the model in order to construct solutions of equations of motion. Peculiar features of the orbits are discussed. In Sect. 4 a simple conformal mechanics in three dimensions is considered. It can be viewed as the bosonic limit of the $N = 4$ superconformal particle [11]. A canonical transformation relating the conformal mechanics to the particle moving near the horizon of the extreme Reissner-Nordström black hole (with a non–vanishing magnetic charge) is constructed.

2 Geometry of background fields

Our starting point is the extreme Reissner-Nordström black hole solution of Einstein–Maxwell equations (for a review see e.g. [16])

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 , \quad A = -\frac{q}{r} dt + p \cos\theta d\varphi . \qquad (1)$$

Here M, q, p are the mass, the electric and magnetic charges, respectively, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the standard metric on a sphere. For the extreme solution one has $M = \sqrt{q^2 + p^2}$. Throughout the paper we use units for which $G = 1$.

The near horizon limit is most easily accessible in isotropic coordinates ($r \rightarrow r - M$) which cover the region outside the horizon only

$$ds^2 = -\left(1 + \frac{M}{r}\right)^{-2} dt^2 + \left(1 + \frac{M}{r}\right)^2 \left(dr^2 + r^2 d\Omega^2\right) . \qquad (2)$$

When $r \rightarrow 0$ the metric takes the form

$$ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 + M^2 d\Omega^2 , \qquad (3)$$

while implementing the limit in the two–form field strength, one finds the background vector field

$$A = \frac{q}{M^2} r dt + p \cos\theta d\varphi \ . \tag{4}$$

The last two lines give the Bertotti-Robinson solution of the Einstein–Maxwell equations.

From (3) it follows that in the near horizon limit the space–time geometry is the product of a two-dimensional sphere of radius M and a two-dimensional pseudo Riemannian space–time with the metric

$$ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 \ . \tag{5}$$

The latter proves to be the metric of AdS_2. In order to see this, consider the hyperboloid in $\mathbf{R}^{2,1}$

$$-\eta_{AB} x^A x^B = M^2 \ , \qquad \eta_{AB} = \mathrm{diag}(-,+,-) \ , \tag{6}$$

parameterized by the Poincaré coordinates (t, r)

$$x^0 = \frac{1}{2r}(1 + r^2(M^2 - t^2)), \quad x^1 = \frac{1}{2r}(1 - r^2(M^2 + t^2)), \quad x^2 = Mrt \ . \tag{7}$$

Since $x^0 - x^1 > 0$, the local coordinates cover only half of the hyperboloid[1]. Calculating the metric $ds^2 = \eta_{AB} dx^A dx^B$ induced on the surface (7) and making the shift $r \to M^2 r$, one gets precisely (5). Notice that in this picture the black hole mass M is equal to the radius of S^2 (AdS_2). It is worth mentioning also that, by construction, the isometry group of the metric (3) is $SO(2,1) \times SO(3)$.

To summarize, the background geometry is that of the $AdS_2 \times S^2$ space–time with the 2–form flux.

3. Particle dynamics on $AdS_2 \times S^2$

Having fixed the geometry of background fields, we now consider the action of a relativistic particle moving on such a background, i.e. near the horizon of the extreme Reissner-Nordström black hole

$$S = -\int dt \left(m\sqrt{(r/M)^2 - (M/r)^2 \dot{r}^2 - M^2(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2)} + eqr/M^2 + ep\cos\theta \dot{\varphi} \right) \ , \tag{8}$$

where m, e are the mass and the electric charge of a particle, respectively.

The dynamics is most easily analyzed within the Hamiltonian formalism. Introducing the momenta $(p_r, p_\theta, p_\varphi)$ canonically conjugate to (r, θ, φ), one finds the Hamiltonian

$$H = (r/M)\left(\sqrt{m^2 + (r/M)^2 p_r^2 + (1/M)^2(p_\theta^2 + \sin^{-2}\theta(p_\varphi + ep\cos\theta)^2)} + eq/M\right) \ , \tag{9}$$

which generates time translations. In agreement with the isometries of the background metric one also finds the conserved quantities

$$K = M^3/r\left(\sqrt{m^2 + (r/M)^2 p_r^2 + (1/M)^2(p_\theta^2 + \sin^{-2}\theta(p_\varphi + ep\cos\theta)^2)} - eq/M\right) + \\ + t^2 H + 2trp_r \ , \qquad D = tH + rp_r \ , \tag{10}$$

[1]In order to avoid closed time–like curves, one considers the universal covering of the hyperboloid with $-\infty < t < \infty$, $0 < r$.

which generate special conformal transformations and dilatations, respectively. Together with the Hamiltonian they form $so(2,1)$ algebra

$$\{H, D\} = H \ , \quad \{H, K\} = 2D \ , \quad \{D, K\} = K \ . \tag{11}$$

The generators of rotations

$$\begin{aligned} J_1 &= -p_\varphi \cot\theta \cos\varphi - p_\theta \sin\varphi - ep\cos\varphi \sin^{-1}\theta \ , \\ J_2 &= -p_\varphi \cot\theta \sin\varphi + p_\theta \cos\varphi - ep\sin\varphi \sin^{-1}\theta \ , \\ J_3 &= p_\varphi \ , \qquad \{J_i, J_j\} = \epsilon_{ijk} J_k \ , \qquad \epsilon_{123} = 1 \ . \end{aligned} \tag{12}$$

enter the Hamiltonian via the Casimir element

$$J_i J_i = p_\theta^2 + \sin^{-2}\theta (p_\varphi + ep\cos\theta)^2 + (ep)^2 \ , \tag{13}$$

and, hence, are conserved due to the $su(2)$ algebra they form.

When analyzing solutions of equations of motion, two distinct cases should be examined. First consider the situation when the magnetic charge of the black hole vanishes

$$p = 0 \ , \quad M = |q| \ . \tag{14}$$

In this case the particle moves on a plane orthogonal to the angular momentum vector J_i. Making use of the rotation invariance one can choose the reference frame where J_i is along x^3-axis, i.e.

$$\theta = \pi/2 \ , \quad p_\theta = 0 \quad \rightarrow \quad J_1 = 0, \quad J_2 = 0, \quad J_3 = p_\varphi = L \ , \tag{15}$$

with L a constant[2]. Then from the conservation laws (9) and (10) one can fix the dynamics of the radial coordinate

$$r(t) = \frac{EM^2}{\sqrt{a^2(t) + b^2} + c} \ , \qquad p_r(t) = \frac{a(t)(\sqrt{a^2(t) + b^2} + c)}{EM^2} \ , \tag{16}$$

where and $E = H$ is the energy and we abbreviated

$$a(t) = D - tE \ , \qquad b^2 = m^2 M^2 + L^2 \ , \qquad c = eq \ . \tag{17}$$

Evolution of the angular variable is found by straightforward integration

$$\varphi(t) = -\frac{L}{\sqrt{b^2 - c^2}} \left(\arctan\frac{a(t)}{\sqrt{b^2 - c^2}} - \arctan\frac{ca(t)}{\sqrt{a^2(t) + b^2}\sqrt{b^2 - c^2}} \right) + \varphi_0 \ , \quad p_\varphi(t) = L \ . \tag{18}$$

It is important to notice that the conserved charges (9), (10) also specify the value of the Casimir element of $so(2,1)$ algebra realized in the model in terms of parameters of the particle and those of the background

$$EK - D^2 = b^2 - c^2 = M^2(m^2 - e^2) + L^2 \ . \tag{19}$$

This should correlate with the bound $b^2 - c^2 > 0$ revealed by the explicit solution given above which also assures that the energy of the particle $E = (r/M)(\sqrt{a^2(t) + b^2} + c)$ is positive

[2]We assume that $L \neq 0$. When $L = 0$ the particle travels towards the horizon at $r = 0$ along a straight line.

even if c is negative. Indeed, if $c < 0$ then from the condition $b^2 - c^2 > 0$ one immediately gets

$$(\sqrt{a^2(t) + b^2} + c)(\sqrt{a^2(t) + b^2} - c) > 0 \ , \tag{20}$$

which means that the first factor entering the last line is positive.

As $\dot{r}$ is proportional to $a(t)$ with a positive coefficient, depending on the initial data, the particle either goes directly towards the black hole horizon located at $r = 0$, or it moves away for some time, slows down with the turning point at $t = D/E$, and then travels back towards $r = 0$. The orbit looks particularly simple when the particle is electrically neutral, i.e. $c = 0$

$$r(\varphi) = \frac{EM^2}{b} |\cos(b(\varphi - \varphi_0)/L)| \ . \tag{21}$$

The corresponding plot indeed reproduces the qualitative behavior mentioned above.

Now consider the case when the magnetic charge p of a black hole does not vanish. In this case the particle moves on the cone (turning to Cartesian coordinates)

$$\frac{x^i J_i}{\sqrt{x^2}} = -ep \ . \tag{22}$$

As before, one can use the rotation invariance so as to pass to the reference frame where J_i is along x^3-axis. This specifies the canonical pair (θ, p_θ)

$$J_1 = 0, \quad J_2 = 0, \quad J_3 = p_\varphi = L \quad \rightarrow \quad \cos\theta = -ep/L \ , \quad p_\theta = 0 \ , \tag{23}$$

and imposes the natural bound $|\frac{ep}{L}| \leq 1$. The solution of equations of motion for $(r(t), p_r(t))$ and $(\varphi(t), p_\varphi(t))$ proves to maintain the previous form (16), (18) with $a(t)$ and c unchanged, but b^2 modified

$$b^2 = m^2 M^2 + L^2 - (ep)^2 \ . \tag{24}$$

Thus, for $p \neq 0$ the qualitative behavior of a particle is similar to the previous case but this time it is confined to move on the cone (22).

4. A relation to conformal mechanics

The conventional conformal mechanics in one dimension is governed by the action functional [15]

$$S = \frac{1}{2} \int dt \left(\dot{x}^2 - \frac{g}{x^2} \right) \ , \tag{25}$$

where g is the coupling constant. Passing to the Hamiltonian formalism one finds the conserved charges

$$H' = \frac{p^2}{2} + \frac{g}{2x^2} \ , \quad D' = tH' - \frac{1}{2}xp, \quad K' = t^2 H' - t(xp) + \frac{1}{2}x^2 \ , \tag{26}$$

which altogether form the $so(2,1)$ algebra (11). Guided by this observation, the authors of [1] argued that the quantum mechanics of a test particle moving near the horizon of the extreme Reissner-Nordström black hole[3] reproduces the old conformal mechanics (25) in the limit of large black hole mass

$$M \rightarrow \infty \ , \qquad (m - e) \rightarrow 0 \ , \tag{27}$$

[3] In [1] only the case of the vanishing magnetic charge $p = 0$ was discussed.

with $M^2(m-e)$ fixed. The relation between the two models was recognized to be a manifestation of the AdS_2/CFT_1 correspondence.

In [9] the conformal mechanics (25) was extended by a couple of angular variables in such a way that the resulting model is related to a particle moving near the horizon of the extreme Reissner-Nordström black hole by a canonical transformation (for a related work see [10, 12]). The construction in [9] does not appeal to any specific limit and is valid for any finite value of the black hole mass.

In this section we generalize the results of [9] to the case when a test particle couples to the magnetic charge of the black hole. In constrast to the calculation in [9], the use of the rotation invariance allows us to simplify the analysis notably.

Consider a specific extension of the model (25) by two angular degrees of freedom (Θ, Φ)

$$S = \frac{1}{2} \int dt \left(\dot{x}^2 + \frac{1}{4} x^2 (\dot{\Theta}^2 + \sin^2 \Theta \dot{\Phi}^2) - \frac{g}{x^2} - 2\nu \cos \Theta \dot{\Phi} \right) , \tag{28}$$

where ν is a new coupling constants and x is now treated as a radial coordinate in the enlarged configuration space. This theory arises, in particular, in the bosonic limit of the $N=4$ superconformal mechanics associated with the supergroup $D(2,1;\alpha)$ [11] for $\alpha=-1$. Notice, however, that in [11] the coupling g was identified with ν^2. That the new degrees of freedom do not destroy the conformal symmetry of the original model is most easily verified within the Hamiltonian formalism. Indeed, given the Hamiltonian

$$H' = \frac{p^2}{2} + \frac{g}{2x^2} + \frac{2}{x^2} (p_\Theta^2 + \sin^{-2} \Theta (p_\Phi + \nu \cos \Theta)^2) , \tag{29}$$

where (p, p_Θ, p_Φ) designate momenta canonically conjugate to (x, Θ, Φ), the generators of dilatations D' and special conformal transformations K' are constructed following the prescription in (26) and the full algebra proves to be $so(2,1)$.

As might be anticipated from the form of the action (28), the new variables accommodate rotation invariance. The corresponding generators are derived from (12) by the obvious change of the canonical variables and the coupling constants $(\varphi, p_\varphi) \to (\Phi, p_\Phi)$, $(\theta, p_\theta) \to (\Theta, p_\Theta)$, $ep \to \nu$. They are trivially conserved because the angular variables enter the Hamiltonian via the Casimir element of $so(3)$ algebra realized in the model.

Now let us demonstrate that the system (29) and a particle on $AdS_2 \times S^2$ background with the 2–form flux are related by a canonical transformation. In order to simplify the analysis, first we use the rotation invariance intrinsic to both the models and pass for each system to the reference frame where the conserved angular momentum is along x^3–axis[4]. This allows one to disregard the pairs (θ, p_θ), and (Θ, p_Θ)

$$\cos \theta = -ep/L , \quad p_\theta = 0 , \quad \cos \Theta = -\nu/L , \quad p_\Theta = 0 . \tag{30}$$

Following [9], we then search for a canonical transformation which brings the symmetry generators characterizing the model (8) (including the Hamiltonian) precisely to those of the system (28). Comparing the conserved charges in both the pictures, one immediately finds the desired transformation

$$\begin{aligned} x &= \left[\frac{2M^2}{r} \left(\sqrt{m^2M^2 + (rp_r)^2 + \left(Lp_\varphi - (ep)^2\right)^2 / (L^2 - (ep)^2)} - eq \right) \right]^{1/2} , \\ p &= -2rp_r \left[\frac{2M^2}{r} \left(\sqrt{m^2M^2 + (rp_r)^2 + \left(Lp_\varphi - (ep)^2\right)^2 / (L^2 - (ep)^2)} - eq \right) \right]^{-1/2} , \\ p_\Phi &= p_\varphi . \end{aligned} \tag{31}$$

[4] As explained below, our construction implies $p_\Phi = p_\varphi = L$ on–shell.

The corresponding Poisson brackets are indeed canonical. When establishing the correspondence above, one has to specify the couplings of the conformal mechanics in terms of the parameters characterizing a particle on $AdS_2 \times S^2$

$$\nu = ep \ , \qquad g = 4(m^2M^2 - (eq)^2) \ . \tag{32}$$

A transformation law of the last missing variable Φ is then found with the help of (31). Imposing the Poisson bracket relations

$$\{\Phi, x\} = 0 \ , \qquad \{\Phi, p\} = 0 \ , \qquad \{\Phi, p_\Phi\} = 1 \ , \tag{33}$$

which are to be calculated with respect to the variables (r, p_r), (φ, p_φ), and making the ansatz

$$\Phi = \varphi + A(s, p_\varphi) \ , \tag{34}$$

where $s = (rp_r)$ and $A(s, p_\varphi)$ is an arbitrary function to be fixed below, one then reduces (33) to a single ordinary differential equation for $A(s, p_\varphi)$ as a function of the variable s. This yields the solution

$$A(s, p_\varphi) = -\frac{\alpha}{\sqrt{k^2 - c^2}} \left(\arctan \frac{s}{\sqrt{k^2 - c^2}} + \arctan \frac{cs}{\sqrt{k^2 - c^2}\sqrt{k^2 + s^2}} \right) \ , \tag{35}$$

where we denoted

$$\alpha = \frac{L(Lp_\varphi - (ep)^2)}{(L^2 - (ep)^2)} \ , \quad k^2 = m^2M^2 + \frac{(Lp_\varphi - (ep)^2)^2}{(L^2 - (ep)^2)} \ , \quad c = eq \ . \tag{36}$$

Thus, the canonical transformation exposed above establishes the equivalence relation between the charged massive particle moving near the horizon of the extreme Reissner-Nordström black hole, which has a non–vanishing magnetic charge, and the conformal mechanics (28). Different aspects of dynamics in one model can be studied in terms of the other and vice versa. It is noteworthy that the equivalence holds for any fixed value of the black hole mass and does not refer to any specific limit. This is to be contrasted with the consideration in [1].

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Hidden symmetries of non-minimal 5D supergravity

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Abstract

We discuss hidden symmetries of five-dimensional supergravity coupled to vector multiplets inherited from diagonal dimensional reduction of the eleven-dimensional supergravity on T^6. This model containing three vector and three constrained scalars, recently attracted attention in connection with supertubes and black rings. We perform dimensional reduction along the lines of the Julia and Pope formalism which directly reveals the symmetries of the scalar cosets in terms of the so called dilaton vectors. By hidden symmetries we understand classical U-duality groups arising in dimensional reduction of 5D theory to four and three dimensions. In four dimensions the U-duality group is $SL(2,\mathbb{R})^3$, in three dimensions it is $SO(4,4)$. Identification of the vectors and freezing out the scalar moduli leads to contraction of $SO(4,4)$ to $G_{2(2)}$ corresponding to the hidden symmetry of minimal 5D supergravity. This contraction provides a simple physical interpretation of the embedding of $G_{2(2)}$ into $SO(4,4)$. We obtain a new representation of the coset $G_{2(2)}/SL(2,\mathbb{R})^2$ of the 3D sigma model of the reduced minimal 5D supergravity in terms of this embedding.

1 Introduction

This paper is devoted to hidden symmetries in five-dimensional supergravity coupled to vector fields. It is a pleasure for us to participate in the volume dedicated to professor I.L. Buchbinder whose contribution to the development of supersymmetric field theories is widely recognized [1].

Current interest to five-dimensional supergravity is partly related to the discovery of new soliton solutions which were called black rings [2]. These solutions violate the no-hair theorems of the four-dimensional gravity/supergravity and give an interesting example of a soliton with the topology of a closed string. Supersymmetric black rings can be found solving the BPS equations of five-dimensional supergravity. More general (non-supersymmetric) black

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rings are supposed to exists and serve as their thermal counterparts. Black rings can carry conserved (electric) charges associated with the gauge fields, as well as (non-conserved) dipole charges of the magnetic type. Some particular solutions for thermal black rings were found by an educated guess, but the most general solution still remains unknown. Since we are interested in solutions effectively depending only on two spatial coordinates, one can hope that dimensional reduction machinery, which reveals the so called hidden symmetries of the theory, may help to solve this problem.

Dimensional reduction of minimal five-dimensional supergravity [3, 4] to four and three dimensions was considered in number of papers [5, 6, 7, 8, 9, 10, 11]. The three-dimensional classical U-duality group of the reduced theory is of particular importance an the above respect, since it opens a way to create various solution generating techniques. This U-duality is given by the lowest rank exceptional group G_2, more precisely,(for the Euclidean signature of the three-space) by its non-compact version $G_{2(2)}$. The corresponding generating technique opening a way to construct classical solution depending on three coordinates was recently developed in [12, 13]. It amounts to using the 7×7 matrix representation of the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ and classifying the $G_{2(2)}$ isometries acting in the space of potentials as a way to endow known solution with additional parameters such as charges or angular momenta. In principle the problem to construct the most general five-parametric black ring solution of the minimal five-dimensional supergravity can be solved using this technique. However, the algebraic structure of G_2 and, correspondingly, the geometry of the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ is rather complicated, and stimulate us to search for alternative techniques.

It turns out, that suitable extension of minimal five-dimensional supergravity leads to a more clear U-duality group $SO(4,4)$, whose matrix realization is less sophisticated. Consequently, truncation of this model leads again to the $G_{2(2)}$ supergravity, now in terms of an embedding of this group into $SO(4,4)$. So finally we get a somewhat simpler matrix representation of the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$. The non-minimal theory is also interesting by itself as providing wider classes of solitons known as supertubes. This extension is $U(1)^3$ five-dimensional supergravity with three vector fields which can be regarded as a truncated toroidal compactification of the $D = 11$ supergravity

$$I_{11} = \frac{1}{16\pi G_{11}} \int \left(R_{11} \star_{11} 1 - \frac{1}{2} F_{[4]} \wedge \star_{11} F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right), \tag{1}$$

where $F_{[4]} = dA_{[3]}$, according to the ansatz

$$\begin{aligned} ds_{11}^2 &= ds_5^2 + X^1 \left(dz_1^2 + dz_2^2 \right) + X^2 \left(dz_3^2 + dz_4^2 \right) + X^3 \left(dz_5^2 + dz_6^2 \right), \\ A_{[3]} &= A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 + A^3 \wedge dz_5 \wedge dz_6 \,. \end{aligned} \tag{2}$$

Here z^i, $i = 1, \ldots, 6$ are coordinates parameterizing the torus T^6. The three scalars $X^I, (I = 1, 2, 3)$ and the one-forms A^I depend only on the five-dimensional coordinates. The scalars X^I satisfy the constraint $X^1 X^2 X^3 = 1$, this implies the five-dimensional metric ds_5^2 to be the Einstein-frame metric. The reduced five-dimensional action reads

$$\begin{aligned} I_5 &= \frac{1}{16\pi G_5} \int \left(R_5 \star_5 1 - \frac{1}{2} G_{IJ} dX^I \wedge \star_5 dX^J - \frac{1}{2} G_{IJ} F^I \wedge \star_5 F^J \right. \\ &- \left. \frac{1}{6} \delta_{IJK} F^I \wedge F^J \wedge A^K \right), \\ G_{IJ} &= \text{diag}\left((X^1)^{-2},\ (X^2)^{-2},\ (X^3)^{-2} \right), \quad F_I = dA_I, \quad I, J, K = 1, 2, 3, \end{aligned} \tag{3}$$

where the Chern-Simons coefficients $\delta_{IJK} = 1$ if the indices I, J, K is a permutation of 1,2,3, and zero otherwise. Supersymmetric solutions to this theory were studied in a number of papers[14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] The most general ring solution to this theory constructed so far is a family of non-supersymmetric black rings depending on three conserved charges Q_I, three dipole charges q_I, and a radius of S^1 [25], with the mass M and two angular momenta J_ψ, J_ϕ being some function of these seven free parameters. An existence of a larger family of non-supersymmetric black rings with nine-parameters $(M, J_\psi, J_\phi, Q_I, q_I)$ is expected reducing to the solutions of [21, 22, 23, 24] in the supersymmetric limit. The generating technique developed in the present paper introduces a sufficient number of parameters to construct the nine-parametric solution.

It is worth noting that the ansatz (2) is far from being the general toroidal compactification of the $D = 11$ supergravity. The general toroidal reduction leads to the five-dimensional theory with 27 vector fields and 42 scalar fields parameterizing a coset $E_{6(6)}/USp(8)$. Correspondingly, the general black ring must contain 27 conserved charges and 27 dipole charges. More accurate analysis [26] shows that 24 conserved charges can be generated from the above three by duality transformations, while the number of independent dipole charges is 15 (the number of the independent four-cycles of T^6).

Contraction to minimal 5D supergravity is effected via an identification of the vector fields

$$A^1 = A^2 = A^3 = \frac{1}{\sqrt{3}} A$$

and elimination of the scalars $X^1 = X^2 = X^3 = 1$. This leads to the Lagrangian

$$\mathcal{L}_5 = R_5 \star_5 1 - \frac{1}{2} F \wedge \star_5 F - \frac{1}{3\sqrt{3}} F \wedge F \wedge A.$$

In this case our results go back to those of the Refs. [12, 13]. However, our matrix representation of the coset $SO(4,4)/(SO(4) \times SO(4))$ leads upon contraction to a new representation of the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ which is different from that given in [12, 13].

2 D=4 reduction

In this section we investigate the Kaluza-Klein reduction of the D=5 action (3) to four dimensions. We assume that the 5D space-time has the structure $\mathcal{M}_5 = \mathcal{M}_4 \times T^1$, where T^1 is a circle, and can be parameterized by the coordinates $\{x^\mu, z\}$, $\mu = 1, \ldots, 4$ with z relating to the circle. Following to the standard procedure we decompose the 5D metric as

$$ds_5^2 = e^{\frac{\phi}{\sqrt{3}}} ds_4^2 + e^{-\frac{2\phi}{\sqrt{3}}} (dz + a)^2, \tag{4}$$

where the is $ds_4^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$, the Kaluza-Klein one-form is $a = a_\mu dx^\mu$ and ϕ is the dilaton. In a similar way the 5D vector fields $A^I(x^\mu, z)$ are decomposed as

$$A^I(x^\mu, z) = A^I(x^\mu) + u^I dz, \tag{5}$$

where u^I are the axions. All the above fields do not depend on z. Inserting these decompositions into the 5D action we get the 4D lagrangian

$$\begin{aligned} \mathcal{L}_4 \;=\;& R_4 \star 1 - \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} G_{IJ} \star dX^I \wedge dX^J - \frac{1}{2} e^{\frac{2\phi}{\sqrt{3}}} G_{IJ} \star du^I \wedge du^J \\ &- \frac{1}{2} e^{-\sqrt{3}\phi} \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{2} e^{-\frac{\phi}{\sqrt{3}}} G_{IJ} \star F^I \wedge F^J - \frac{1}{2} \delta_{IJK} dA^I \wedge dA^J u^K, \end{aligned} \tag{6}$$

where $\mathcal{F} = da$ and $F^I = dA^I - du^I \wedge a$ are the field strength two-forms. Our purpose is to rewrite this lagrangian in the form exhibiting the classical duality symmetry. First of all we consider the scalar part of (6) written in the following form

$$\begin{aligned} e_4^{-1}\mathcal{L}_{scal} &= \frac{1}{2}\Big((\partial\phi)^2 + G_{IJ}\partial X^I\partial X^J + e^{\frac{2\phi}{\sqrt{3}}}G_{IJ}\partial u^I\partial u^J\Big) \\ &= \hat{\mathcal{G}}_{AB}(\hat\Phi)\partial\hat\Phi^A\partial\hat\Phi^B, \quad A,B = 1,\dots,6, \end{aligned} \tag{7}$$

where $\partial \equiv \partial/\partial x^\mu$, e_4 is the Hodge dual to unity: $e_4 \equiv \star 1 = \sqrt{-g}\, d^4x$ and all index operations refer to the metric $g_{\mu\nu}$. The potentials $\hat\Phi^A$ combine the six variables $\{X^1, X^2, \phi, u^I\}$ and realize the map $\hat\Phi^A:\ x^\mu \in \mathcal{M}_4 \ \rightarrow\ \hat\Phi^A(x^\mu) \in \mathcal{M}_{scal}$ between the 4D Minkowskian space-time and the target space with the metric $\hat{\mathcal{G}}_{AB}(\hat\Phi)$. Replacing the dilaton ϕ and the moduli X^I by new variables α^I: $\alpha^I = \phi/\sqrt{3} - \ln X^I$ enable us to simplify $\mathcal{L}_{scal}$ as follows

$$e_4^{-1}\mathcal{L}_{scal} = \frac{1}{2}\sum_I\Big((\partial\alpha^I)^2 + e^{2\alpha^I}(\partial u^I)^2\Big).$$

The structure of the scalar manifold $\mathcal{M}_{scal}$ becomes more clear if we introduce three complex potentials $z^I = u^I + ie^{-\alpha^I}$:

$$e_4^{-1}\mathcal{L}_{scal} = \frac{1}{2}\sum|\partial z^I|^2/(\mathrm{Im}\ z^I)^2.$$

It is well know that lagrangian $\mathcal{L} = \frac{1}{2}|\partial z|^2/(\mathrm{Im}\ z)^2$ invariant under the group $SL(2,\mathbb{R})$ and the corresponding target space metric is the Kähler space $SL(2,\mathbb{R})/SO(2)$. So in our case the isometry group of $\mathcal{M}_{scal}$ is $\hat G = (SL(2,\mathbb{R}))^3$ and the corresponding target space is $\hat G/\hat H = \mathcal{M}_{scal} = (SL(2,\mathbb{R})/SO(2))^3$ with metric

$$\hat{\mathcal{G}}_{AB}(\hat\Phi)d\hat\Phi^A d\hat\Phi^B = \frac{1}{2}\Big((d\phi)^2 + G_{IJ}dX^I dX^J + e^{\frac{2\phi}{\sqrt{3}}}G_{IJ}du^I du^J\Big) = \frac{1}{2}\sum|dz^I|^2/(\mathrm{Im}\ z^I)^2.$$

As the second step, we reformulate the tensor part of the lagrangian (6) according with the structure of the bosonic lagrangian of $N=2$ supergravity coupled to vector multiplets (for a review see the Ref.[27]). We express it in terms of the field two-forms $\widetilde F^I$ and $\mathcal{F}$ obeying to the Bianchi identities $d\widetilde F^I = 0$ and $d\mathcal{F} = 0$ respectively. To determine the two-forms $\widetilde F^I$ one needs to extract the exterior derivative $d(u^I a)$ in $F^I = dA^I - du^I \wedge a$. As result we have $F^I = \widetilde F^I + u^I\mathcal{F}$, where $\widetilde F^I = d\widetilde A \equiv d(A^I - u^I a)$. Inserting the two-forms F^I and dA^I expressed via $\widetilde F$ and $\mathcal{F}$ into (6) and integrating by parts the terms like $\delta_{IJK}\widetilde F^I \wedge du^J u^K \wedge a$ and $\delta_{IJK}du^I u^J u^K \wedge a \wedge \mathcal{F}$ we will obtain for the tensor part of the 4D lagrangian

$$\begin{aligned} \mathcal{L}_{tens} &= \frac{1}{2}\Big[e^{-\sqrt{3}\phi}\star\mathcal{F}\wedge\mathcal{F} + e^{-\frac{\phi}{\sqrt{3}}}G_{IJ}\Big(\star\widetilde F^I\wedge\widetilde F^J + 2\star\mathcal{F}u^{[I}\wedge\widetilde F^{J]} + u^I u^J\star\mathcal{F}\wedge\mathcal{F}\Big) \\ &+ \delta_{IJK}\Big(\widetilde F^I\wedge\widetilde F^J u^K + \widetilde F^I u^J u^K\wedge\mathcal{F} + \frac{1}{3}u^I u^J u^K\mathcal{F}\wedge\mathcal{F}\Big)\Big]. \end{aligned} \tag{8}$$

Denote the field strength tensor and its Hodge dual as $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu\wedge dx^\nu$ and $\star\mathcal{F}_{\mu\nu} = \frac{1}{2}\mathcal{F}^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}$, where $\epsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric Levi-Civita tensor related with the Levi-Civita tensor density $\varepsilon_{\alpha\beta\mu\nu}$ as $\epsilon_{\alpha\beta\mu\nu} = (-g)^{1/2}\varepsilon_{\alpha\beta\mu\nu}$. Assuming that $dx^\mu\wedge dx^\nu\wedge dx^\alpha\wedge dx^\beta = -\epsilon^{\mu\nu\alpha\beta}e_4$, we find

$$\star\mathcal{F}\wedge\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}e_4 = \frac{1}{2}\mathcal{F}^2 e_4, \quad \mathcal{F}\wedge\mathcal{F} = -\frac{1}{2}\mathcal{F}_{\mu\nu}\star\mathcal{F}^{\mu\nu}e_4 = -\frac{1}{2}\mathcal{F}\star\mathcal{F}e_4.$$

Note that in the 4D space-time with Lorentzian signature the double Hodge dual is $\star\star = -1$. We also combine field tensors $\widetilde{F}^I_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$ into the 4-column $\mathcal{B}_{\mu\nu} = \begin{pmatrix} \widetilde{F}^I_{\mu\nu} \\ \mathcal{F}_{\mu\nu} \end{pmatrix}$. With these definition one can rewrite (8) in the matrix form [28, 29]

$$e_4^{-1}\mathcal{L}_{tens} = \frac{1}{4}\mathcal{B}^T_{\alpha\beta}(\hat{\mu}\mathcal{B}^{\alpha\beta} - \frac{1}{\sqrt{2}}\hat{\nu}\star\mathcal{B}^{\alpha\beta}),$$

where the symmetric 4×4 matrices $\hat{\mu}$ and $\hat{\nu}$ are given by

$$\hat{\mu} = \begin{pmatrix} e^{-\frac{\phi}{\sqrt{3}}}G_{IJ} & e^{-\frac{\phi}{\sqrt{3}}}G_{IJ}u^J \\ e^{-\frac{\phi}{\sqrt{3}}}G_{IJ}u^J & G_{IJ}u^I u^J + e^{-\sqrt{3}\phi} \end{pmatrix}, \quad \hat{\nu} = \sqrt{2}\begin{pmatrix} \delta_{IJK}u^K & \frac{1}{2}\delta_{IJK}u^J u^K \\ \frac{1}{2}\delta_{IJK}u^J u^K & 2u^1u^2u^3 \end{pmatrix}.$$

This lagrangian yields the field equations for $\mathcal{B}^T_{\alpha\beta}$: $\nabla_\alpha(\hat{\mu}\mathcal{B}^{\alpha\beta} - \frac{1}{\sqrt{2}}\hat{\nu}\star\mathcal{B}^{\alpha\beta}) = 0$. Introducing the dual field strength $\mathcal{H}_{\alpha\beta}$ as $\star\mathcal{H}^{\alpha\beta} = \hat{\mu}\mathcal{B}^{\alpha\beta} - \frac{1}{\sqrt{2}}\hat{\nu}\star\mathcal{B}^{\alpha\beta}$ we see that the above equations are the Bianchi identities for $\mathcal{H}_{\alpha\beta}$. Therefore the lagrangian $\mathcal{L}_{tens}$ take the form which manifestly has the duality symmetry

$$e_4^{-1}\mathcal{L}_{tens} = \frac{1}{4}\mathcal{B}^T_{\alpha\beta}\star\mathcal{H}^{\alpha\beta} = \frac{1}{8}\Im^T\Sigma_1\star\Im, \quad \Im = \begin{pmatrix} \mathcal{B}_{\alpha\beta} \\ \mathcal{H}_{\alpha\beta} \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It can be checked that relation between $\Im$ and $\star\Im$ is given by

$$\Im = \Omega\,\hat{P}\star\Im,$$

where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the 8×8 symplectic invariant and $\hat{P}$ is the 8×8 matrix depending on the potentials of the scalar manifolds $\mathcal{M}_{scal}$

$$\hat{P} = \begin{pmatrix} \hat{\mu} + \hat{\nu}\hat{\mu}^{-1}\hat{\nu} & \hat{\nu}\hat{\mu}^{-1} \\ \hat{\mu}^{-1}\hat{\nu} & \hat{\mu}^{-1} \end{pmatrix}.$$

The matrix $\hat{P}$ provides the matrix representation γ of the coset element $\pi(\hat{\Phi}^A)$, namely $\gamma : \pi \in \mathcal{M}_{scal} \rightarrow \gamma(\pi) = \hat{P}$. It is evident from the following expression $\hat{\mathcal{G}}_{AB}d\hat{\Phi}^A d\hat{\Phi}^B = -\frac{1}{16}\mathrm{Tr}(d\hat{P}\,d\hat{P}^{-1}) = -\frac{1}{8}\mathrm{Tr}(d\mu d\mu^{-1} - d\nu\mu^{-1}d\nu\mu^{-1})$.

Let the fields $\hat{\Phi}$ be subject to the diffeomorphism $\hat{\Phi}^A \rightarrow \hat{\Phi}^{A\prime}$, which leave invariant the target space metric. This diffeomorphism corresponds to the action of some element $\hat{g}$ belonging to the isometry group of the target space $\hat{g} \in \hat{G}$. In terms of the matrix representation γ this means that the coset matrix $\hat{R} \equiv \Sigma_1\hat{P}$ transforms as $\hat{R} \rightarrow \hat{R}' = \gamma(\hat{g})\hat{R}\gamma(\hat{g}^{-1})$. Inserting the expression $\star\Im = -\Omega\hat{P}\Im$ into the $\mathcal{L}_{tens}$ and keeping in mind that $\Sigma_1\Omega = -\Omega\Sigma_1$ we will obtain for the tensor part of the lagrangian:

$$e_4^{-1}\mathcal{L}_{tens} = \frac{1}{8}\Im^T\Omega\hat{R}\Im.$$

If we now demand this lagrangian to be invariant under the action of $\gamma(\hat{g})$, we get the restrictions for the element $\widetilde{g} \in \hat{G}$ acting on the column as $\Im \rightarrow \gamma(\widetilde{g})\Im$. Performing the transformation we have

$$\mathcal{L}_{tens} \rightarrow \mathcal{L}'_{tens} = \frac{1}{8}\Im^T\gamma(\widetilde{g})^T\Omega\gamma(\hat{g})\hat{R}\gamma(\hat{g}^{-1})\gamma(\widetilde{g})\Im.$$

Thus the conditions for $\gamma(\widetilde{g})$ are $\gamma(\widetilde{g}) = \gamma(\hat{g})$ and $\gamma(\hat{g})^T\Omega\gamma(\hat{g}) = \Omega$. The former relation means that there is the symplectic embedding of the isometry group into the symplectic group $\hat{G} \to Sp(8,\mathbb{R})$ [30]. In other words, $\gamma(\hat{g})$ provides the symplectic representation of $\hat{g}$ which rotates the fields $\Im$. Note the full 4D lagrangian can be written in the following form

$$e_4^{-1}\mathcal{L} = R_4 + \frac{1}{16}\mathrm{Tr}(\partial\hat{R}\,\partial\hat{R}^{-1}) - \frac{1}{8}\Im^T\Omega\hat{R}\Im.$$

3 D=3 reduction

Dimensional reduction of $U(1)^3$ five-dimensional supergravity (3) on a two-torus was discussed recently in [31] and we reproduce some of the results here assuming that the 5D Minkowskian space-time admits two independent Killing symmetries. Thus the initial space-time can be represented as $\mathcal{M}_5 = \mathcal{M}_3 \times \Sigma$, where Σ is T^2 if both Killing vectors are asymptotically space-like or $T^1 \times \mathbb{R}$ if one of their is asymptotically time-like. The full set of 5D coordinates then splits on $x^i \in \mathcal{M}_3,\ i = 1,\dots,3$ and $z^a \in \Sigma,\ a = 4,5$. The decomposition of the 5D metric on the fields independent on z^a is given by

$$ds_5^2 = \mathrm{e}^{-\frac{2}{\sqrt{3}}\varphi_1}(dz^4 + \hat{\mathcal{A}}^4)^2 + \kappa\mathrm{e}^{\frac{1}{\sqrt{3}}\varphi_1-\varphi_2}(dz^5 + \mathcal{A}^5)^2 + \mathrm{e}^{\frac{1}{\sqrt{3}}\varphi_1+\varphi_2}ds_3^2,$$

where $\mathcal{A}^5$ is the 3D KK one-form, $\hat{\mathcal{A}}^4 = \mathcal{A}^4 + \chi dz^5$ is the 4D KK one-form reducing to the 3D KK one-form $\mathcal{A}^4$ and the axion χ and finally φ_1, φ_2 is a set of the dilaton. The factor κ is responsible for the signature: $\kappa = 1$ for z^5 space-like, and $\kappa = -1$ for z^5 time-like. This ansatz can be represented in the 2D-covariant form

$$ds_5^2 = \lambda_{ab}(dz^a + a^a)(dz^b + a^b) - \kappa\tau^{-1}h_{ij}dx^i dx^j,$$

where the 2×2 matrix is introduced

$$\lambda = \mathrm{e}^{-\frac{2}{\sqrt{3}}\varphi_1}\begin{pmatrix} 1 & \chi \\ \chi & \chi^2 + \kappa\mathrm{e}^{\sqrt{3}\varphi_1-\varphi_2}\end{pmatrix}, \qquad \det\lambda \equiv -\tau = \kappa\mathrm{e}^{-\frac{1}{\sqrt{3}}\varphi_1-\varphi_2}, \tag{9}$$

and a^a are the KK one-forms: $a^4 = \mathcal{A}^4 - \chi\mathcal{A}^5,\ a^5 = \mathcal{A}^5$. The 5D $U(1)$ gauge fields A^I reduce to the 3D one-forms $A^I(x^i)$ and the six axions denoted as the 2D-covariant doublet $\psi_a^I = (u^I, v^I)$ with the index a relative to the metric λ_{ab}

$$A^I(x^i, z^4, z^5) = A^I(x^i) + \psi_a^I dz^a = A^I(x^i) + u^I dz^4 + v^I dz^5.$$

Therefore the lagrangian (3) in the new variables will read [32]

$$\begin{aligned} e_3^{-1}\mathcal{L}_3 \;&=\; R_3 - \frac{1}{2}(\partial\vec{\varphi})^2 - \frac{1}{2}G_{IJ}\partial X^I\partial X^J - \frac{1}{2}\mathrm{e}^{\frac{2}{\sqrt{3}}\varphi_1}G_{IJ}\partial u^I\partial u^J & (10)\\ &-\; \frac{1}{2}\mathrm{e}^{\varphi_2-\frac{1}{\sqrt{3}}\varphi_1}G_{IJ}\partial v^I\partial v^J - \frac{1}{2}\mathrm{e}^{\varphi_2-\frac{3}{\sqrt{3}}\varphi_1}(\partial\chi)^2 - \frac{1}{4}\kappa\tau G_{IJ}F^IF^J & (11)\\ &-\; \frac{1}{4}\mathrm{e}^{-\frac{3}{\sqrt{3}}\varphi_1-\varphi_2}(\mathcal{F}^4)^2 - \frac{1}{4}\kappa\mathrm{e}^{-2\varphi_2}(\mathcal{F}^5)^2 - \frac{1}{2}e_3^{-1}\delta_{IJK}\varepsilon^{ab}d\psi_a^I\wedge d\psi_b^J\wedge A^K, \end{aligned}$$

with $\vec{\varphi} = (\varphi_1, \varphi_2)$ and $\varepsilon^{ab} = -\varepsilon^{ba}$, $\varepsilon^{45} = 1$. The field strength tensors entering (10) are defined as

$$F^I = dA^I - d\psi_a^I\wedge a^a, \quad \mathcal{F}^4 = da^4 + \chi da^5, \quad \mathcal{F}^5 = da^5. \tag{12}$$

An approach suggested in the Ref. [33] has an advantage to provide the roots of the hidden symmetry group directly in terms of the so called *dilaton vectors* (coefficients in the dilaton exponentials entering the reduced action). Following this we replace the scalar fields $\vec{\phi}$, X^I by the new set of scalars $\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ related to the old variables as

$$\phi_1 = \frac{1}{\sqrt{2}}\left(-\ln(X^3) + \frac{1}{\sqrt{3}}\varphi_1 + \varphi_2\right), \quad \phi_2 = \frac{1}{\sqrt{2}}\left(\ln(X^3) - \frac{1}{\sqrt{3}}\varphi_1 + \varphi_2\right), \tag{13}$$

$$\phi_3 = \frac{1}{\sqrt{2}}\left(\ln(X^3) + \frac{2}{\sqrt{3}}\varphi_1\right), \qquad \phi_4 = \frac{1}{\sqrt{2}} \ln\frac{X^1}{X^2} \tag{14}$$

and rewrite the lagrangian (10) with dilaton exponentials manifest (we take for simplicity $\kappa = 1$)

$$\begin{aligned} e_3^{-1}\mathcal{L}_3 &= R_3 - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{I,a} \mathrm{e}^{\vec{e}_{I+3(a-3)}\cdot\vec{\phi}}\,(\partial\psi^I_a)^2 - \frac{1}{4}\sum_I \mathrm{e}^{-\vec{e}_I\cdot\vec{\phi}}(F^I)^2 \\ &- \frac{1}{4}\sum_a \mathrm{e}^{-\vec{e}_{a+6}\cdot\vec{\phi}}\,(\mathcal{F}^a)^2 - \frac{1}{2}\mathrm{e}^{\vec{e}_{12}\cdot\vec{\phi}}\,(\partial\chi)^2 + e_3^{-1}\mathcal{L}_{CS}, \end{aligned} \tag{15}$$

where $\mathcal{L}_{CS}$ is the Chern-Simons term. The set of the four-dimensional vectors $\vec{e}_k$, $k = 1, \ldots, 12$ given by

$$\begin{aligned} \vec{e}_1 &= \sqrt{2}(1,0,0,1), & \vec{e}_2 &= \sqrt{2}(1,0,0,-1), & \vec{e}_3 &= \sqrt{2}(0,1,1,0), \\ \vec{e}_4 &= \sqrt{2}(0,0,1,-1), & \vec{e}_5 &= \sqrt{2}(0,0,1,1), & \vec{e}_6 &= \sqrt{2}(1,-1,0,0), \\ \vec{e}_7 &= \sqrt{2}(0,1,0,-1), & \vec{e}_8 &= \sqrt{2}(0,1,0,1), & \vec{e}_9 &= \sqrt{2}(1,0,-1,0), \\ \vec{e}_{10} &= \sqrt{2}(1,0,1,0), & \vec{e}_{11} &= \sqrt{2}(1,1,0,0), & \vec{e}_{12} &= \sqrt{2}(0,1,-1,0). \end{aligned}$$

To obtain a purely scalar $3D$ lagrangian we have to perform *dualisation* of the 2-forms F^I and $\mathcal{F}^a$. Following the lines of [33] we introduce into the lagrangian (10) three Lagrange multipliers μ_I ensuring the Bianchi identities for the two-forms $F^I - \psi^I_a da^a = dA^I - d(\psi^I_a a^a)$ and two Lagrange multipliers ω_a ensuring the Bianchi identities for the two-forms $da^a = \gamma^a{}_b\mathcal{F}^b$, where $\gamma^4{}_4 = \gamma^5{}_5 = 1$, $\gamma^4{}_5 = -\chi$. We also rewrite the Chern-Simons term as (see Eq. (3.29) in [33]):

$$\mathcal{L}_{CS} = \frac{1}{2}\delta_{IJK}\varepsilon^{ab}(d\psi^I_a\psi^J_b \wedge F^K + \frac{1}{3}d\psi^I_a\psi^J_b\psi^K_c\gamma^c{}_d \wedge \mathcal{F}^d)$$

Integrating by parts the terms with the Lagrangian multipliers we can represent (10) as

$$\begin{aligned} \mathcal{L}_3 &= R_3 \star \mathbf{1} - \frac{1}{2}\star d\vec{\phi}\wedge d\vec{\phi} - \frac{1}{2}\sum_{I,a} \mathrm{e}^{\vec{e}_{I+3(a-3)}\cdot\vec{\phi}} \star d\psi^I_a \wedge d\psi^I_a - \frac{\tau\kappa}{2}G_{IJ}\star F^I\wedge F^J + G_I\wedge F^I \\ &- \frac{1}{2}\mathrm{e}^{-\vec{e}_{10}\cdot\vec{\phi}}\star\mathcal{F}^4\wedge\mathcal{F}^4 - \frac{1}{2}\kappa\,\mathrm{e}^{-\vec{e}_{11}\cdot\vec{\phi}}\star\mathcal{F}^5\wedge\mathcal{F}^5 + G_a\wedge\mathcal{F}^a, \end{aligned}$$

where the one-forms G_I, G_a are related to the scalars μ_I, ω_a as follows:

$$\begin{aligned} &G_I = d\mu_I + \frac{1}{2}\delta_{IJK}d\psi^J_a\psi^K_b\varepsilon^{ab}, \\ &G_7 = V_7, \quad G_8 = V_8 - \chi V_7, \\ &V_a = d\omega_a - \psi^I_a\Big(d\mu_I + \frac{1}{6}\delta_{IJK}d\psi^J_b\psi^K_c\varepsilon^{bc}\Big). \end{aligned}$$

Then, eliminating the initial two-forms F^I, $\mathcal{F}^a$ via the equations of motion

$$F^I = \tau^{-1}G^{IJ}\star G_J, \quad \mathcal{F}^4 = -\kappa\mathrm{e}^{\vec{e}_{10}\cdot\vec{\phi}}\star G_4, \quad \mathcal{F}^5 = -e^{\vec{e}_{11}\cdot\vec{\phi}}\star G_5, \tag{16}$$

we obtain the lagrangian in the dual terms:

$$\begin{aligned}\mathcal{L}_3 &= R_3 \star \mathbf{1} - \frac{1}{2} \star d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_{I,a} \mathrm{e}^{\vec{e}_{I+3(a-3)} \cdot \vec{\phi}} \star d\psi_a^I \wedge d\psi_a^I \\ &+ \frac{1}{2} \tau^{-1} G^{IJ} \star G_I \wedge G_J - \kappa \frac{1}{2} \mathrm{e}^{\vec{e}_{10} \cdot \vec{\phi}} \star G_4 \wedge G_4 - \frac{1}{2} e^{\vec{e}_{11} \cdot \vec{\phi}} \star G_5 \wedge G_5,\end{aligned}$$

where G^{IJ} is the inverse moduli metric G_{IJ}. Note that the signs in the dilaton exponents were inverted under dualisation. The Eqs. (12) together with the relations $\mathrm{e}^{-\vec{e}_{10} \cdot \vec{\phi}} = -\kappa \tau \lambda_{44}$, $\mathrm{e}^{-\vec{e}_{11} \cdot \vec{\phi}} = \tau(\chi \lambda_{45} - \lambda_{55})$, which follow from the definitions (9) and (13)-(14), enable us to rewrite the Eqs.(16) as the dualisation equations covariant with respect to all indices:

$$\begin{aligned}\tau \lambda_{ab} da^b &= \star V_a, \\ dA^I &= d\psi_b^I \wedge a^b + \tau^{-1} G^{IJ} \star G_J,\end{aligned}$$

or, explicitly:

$$\begin{aligned}\lambda_{ab} \partial^{[i} a^{j]b} &= \frac{1}{2\tau\sqrt{h}} \varepsilon^{ijk} \left[\partial_k \omega_a - \psi_a^I \left(\partial_k \mu_I + \frac{1}{6} \delta_{IJK} \partial_k \psi_c^J \psi_d^K \varepsilon^{cd} \right) \right], \\ \partial^{[i} A^{j]I} &= a^{b[j} \partial^{i]} \psi_b^I + \frac{1}{2\tau\sqrt{h}} \varepsilon^{ijk} G^{IJ} \left(\partial_k \mu_J + \frac{1}{2} \delta_{JKL} \partial_k \psi_a^K \psi_b^L \varepsilon^{ab} \right),\end{aligned}$$

where the antisymmetrization is assumed with 1/2.

Combining all the above formulas we can present the dualized action as that of a $3D$ gravity coupled sigma model:

$$I_3 = \frac{1}{16\pi G_3} \int \sqrt{|h|} \left(R_3 - \mathcal{G}_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right) d^3x,$$

where h^{ij} is the inverse metric of the three-space, R_3 is the corresponding Ricci scalar and $\mathcal{G}_{AB}(\Phi^A)$, $A, B = 1, \ldots, 16$ is the metric of the target space parameterized by sixteens scalar variables $\Phi^A = (\vec{\phi}, \psi^I, \mu_I, \chi, \omega_p)$, which can be read off from the following line element:

$$\begin{aligned}dl^2 &= \mathcal{G}_{AB} d\Phi^A d\Phi^B \qquad (17) \\ &= \frac{1}{2} \Big((d\vec{\phi})^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_1+\phi_4)} (G_1)^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_1-\phi_4)} (G_2)^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_3+\phi_2)} (G_3)^2 \\ &+ \mathrm{e}^{\sqrt{2}(\phi_3-\phi_4)} (du^1)^2 + e^{\sqrt{2}(\phi_4+\phi_3)} (du^2)^2 + e^{\sqrt{2}(\phi_1-\phi_2)} (du^3)^2 \\ &+ \kappa \mathrm{e}^{\sqrt{2}(\phi_2-\phi_4)} \left(dv^1 - \chi du^1 \right)^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_4+\phi_2)} \left(dv^2 - \chi du^2 \right)^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_1-\phi_3)} \left(dv^3 - \chi du^3 \right)^2 \\ &+ \kappa \mathrm{e}^{\sqrt{2}(\phi_1+\phi_3)} (G_7)^2 + \mathrm{e}^{\sqrt{2}(\phi_1+\phi_2)} (G_8)^2 + \kappa \mathrm{e}^{\sqrt{2}(\phi_2-\phi_3)} d\chi^2 \Big).\end{aligned}$$

This line element can be more concisely rewritten in the 2D-covariant form:

$$\begin{aligned}dl^2 &= \frac{1}{2} G_{IJ} (dX^I dX^J + d\psi^{I^T} \lambda^{-1} d\psi^J) - \frac{1}{2} \tau^{-1} G^{IJ} G_I G_J \\ &+ \frac{1}{4} \mathrm{Tr} \left(\lambda^{-1} d\lambda \lambda^{-1} d\lambda \right) + \frac{1}{4} \tau^{-2} d\tau^2 - \frac{1}{2} \tau^{-1} V^T \lambda^{-1} V.\end{aligned}$$

The set of the dilaton vectors $\vec{e}_k$, $k = 1, \ldots, 12$ is directly related to the root system of the isometry algebra of the target space [33]. One can check that the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_7, \vec{e}_8, \vec{e}_9, \vec{e}_{10}$,

$\vec{e}_{11}$ are expressed in terms of $\vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_{12}$ as follows

$$\begin{aligned} \vec{e}_I &= \sum_{K\neq I} \vec{e}_{K+3} + \vec{e}_{12}, \quad \vec{e}_{I+6} = \vec{e}_{I+3} + \vec{e}_{12}, \\ \vec{e}_{10} &= \sum_{K} \vec{e}_{K+3} + \vec{e}_{12}, \quad \vec{e}_{11} = \sum_{K} \vec{e}_{K+3} + 2\vec{e}_{12}. \end{aligned}$$

It is clear that the vectors $\vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_{12}$ are the simple roots forming the Dynkin diagram of $so(8)$ [34].

The signature of the target space is $+16$ for $\kappa = 1$ (dimensional reduction in all space-like directions) and $(+8, -8)$ $\kappa = -1$ (one of the reduced dimensions is time-like). Then it is easy to recognize that the isometry group G is actually the non-compact form $SO(4,4)$ of the $SO(8)$, whose Killing metric has the signature $(-12, +16)$, while the target space is the coset $SO(4,4)/(SO(4) \times SO(4))$ for $\kappa = 1$ and $SO(4,4)/(SO(2,2) \times SO(2,2))$ for $\kappa = -1$. For these both symmetric spaces the scalar curvature $\mathcal{R}$ is constant and negative

$$\mathcal{R} = -96.$$

Denoting the four-dimensional Cartan subalgebra of $so(4,4)$ as $\vec{H}$, and the generators corresponding to the non-zero roots $\pm\vec{e}_k$, $k = 1, \ldots, 12$ as $P^{\pm I}$, $W_{\pm I}$, $Z_{\pm I}$, $\Omega^{\pm a}$, $X^{\pm}$, with $I = 1, 2, 3$, $a = 4, 5$, we will have the relations

$$P^{\pm I} \leftrightarrow \pm\vec{e}_I, \quad W_{\pm I} \leftrightarrow \pm\vec{e}_{I+3}, \quad Z_{\pm I} \leftrightarrow \pm\vec{e}_{I+6}, \quad \Omega^{\pm a} \leftrightarrow \pm\vec{e}_{a+6}, \quad X^{\pm} \leftrightarrow \pm\vec{e}_{12}, \quad X \equiv X^{+}.$$

The commutators of these generators with the Cartan subalgebra $\vec{H}$ read:

$$\begin{aligned} [\vec{H}, X^{\pm}] &= \pm\vec{e}_{12} X^{\pm}, \\ [\vec{H}, \Sigma^a_{\pm I}] &= \pm\vec{e}_{I+3(a-3)} \Sigma^a_{\pm I}, \\ [\vec{H}, \Omega^{\pm a}] &= \pm\vec{e}_{a+6} \Omega^{\pm a}, \\ [\vec{H}, P^{\pm I}] &= \pm\vec{e}_I P^{\pm I}, \end{aligned} \tag{18}$$

where we have arranged W_I, Z_I into a column vector $\Sigma_I = \begin{pmatrix} W_I \\ Z_I \end{pmatrix}$. The remaining non-zero commutators are obtained from the relations between the root vectors

$$\vec{e}_{I+3} + \vec{e}_{J+6} = \vec{e}_K, \quad \vec{e}_{I+3} + \vec{e}_{12} = \vec{e}_{I+6}, \quad \vec{e}_{I+3(l+1)} + \vec{e}_I = \vec{e}_{l+10} \ (l = 0, 1), \quad \vec{e}_{12} + \vec{e}_{10} = \vec{e}_{11}.$$

where in the first equations I, J, K are all different. One finds

$$\begin{aligned} [\Sigma^a_{\pm I}, \Sigma^b_{\pm J}] &= \mp\varepsilon^{ab} \delta_{IJK} P^{\pm K}, \quad [\Sigma^a_{\mp I}, \Sigma^b_{\pm J}] = \pm\varepsilon^{ab} \delta_{IJ} X^{\pm}, \\ [X^{\pm}, W_{\pm I}] &= \mp Z_{\pm I}, \quad [X^{\mp}, Z_{\pm I}] = \mp W_{\pm I}, \\ [\Sigma^a_{\pm I}, P^{\pm J}] &= \mp\delta^J_I \Omega^{\pm a}, \quad [\Sigma^a_{\pm I}, P^{\mp J}] = \pm\varepsilon^a_{\ .b} \delta_{IK} \delta^{KJL} \Sigma^b_{\mp L}, \\ [X^{\pm}, \Omega^{\pm 4}] &= \mp\Omega^{\pm 5}. \end{aligned} \tag{19}$$

4 Matrix representation

As a convenient representative of the 3D sigma-model coset $\pi(\Phi^A)$ one can choose the matrix representation $\gamma : \pi \to \gamma(\pi) \equiv \mathcal{V}$, where $\mathcal{V}$ is the upper triangular matrix. We assume that $\mathcal{V}$ transforms under the global action of the symmetry group G by the right multiplication and under the local action of the isotropy group H by the left multiplication: $\mathcal{V} \to \mathcal{V}' = h(\Phi)\mathcal{V}g$, where g and h belong to the matrix representation γ of G and H respectively. Given this representative, one can construct the $H-$invariant matrix

$$\mathcal{M} = \mathcal{V}^T K \mathcal{V}, \tag{20}$$

where K is an involution matrix invariant under H:

$$h(\Phi)^T K h(\Phi) = K, \tag{21}$$

(dependent on the coset signature parameter κ). Then the transformation of $\mathcal{M}$ under G will be $\mathcal{M} \to \mathcal{M}' = g^T \mathcal{M} g$. The target space metric (17) in terms of the matrix $\mathcal{M}$ will read

$$dl^2 = -\frac{1}{8}\mathrm{Tr}(d\mathcal{M} d\mathcal{M}^{-1}). \tag{22}$$

The desired upper-triangular matrix $\mathcal{V}$ can be constructed by an exponentiation of the Borel subalgebra of the Lie algebra of G consisting of the Cartan H and the positive-root E_+ generators (in what follows we omit the sign + in the indices):

$$\mathcal{V} = \mathcal{V}_H \mathcal{V}_{E_+} = \mathcal{V}_H \mathcal{V}_X \mathcal{V}_\Psi \mathcal{V}_\Omega \mathcal{V}_P,$$

where the matrices $\mathcal{V}_H,\ \mathcal{V}_X,\ \mathcal{V}_\Psi,\ \mathcal{V}_\Omega,\ \mathcal{V}_P$ are the exponentials:

$$\begin{aligned} \mathcal{V}_H &= \mathrm{e}^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}, \\ \mathcal{V}_X &= \mathrm{e}^{\chi X}, \\ \mathcal{V}_\Psi &= \mathrm{e}^{\psi^I \Sigma_I}, \\ \mathcal{V}_\Omega &= \mathrm{e}^{\omega_a \Omega^a}, \\ \mathcal{V}_P &= \mathrm{e}^{\mu_I P^I}, \end{aligned} \tag{23}$$

where we assume that generators $\vec{H}, X$ and so on belong to the matrix representation γ of the Lie algebra of G. Using (22), one can rewrite the target space metric in terms of the matrix current $\mathcal{J} = d\mathcal{V}\mathcal{V}^{-1}$ as follows:

$$dl^2 = \frac{1}{4}\mathrm{Tr}(\mathcal{J}^2) + \frac{1}{4}\mathrm{Tr}(\mathcal{J}^{\mathrm{T}} K \mathcal{J} K^{-1}).$$

Using the Eqs.(23) and the commutators (18) and (19) for the positive-root generators, one can show that the matrix current one-form $\mathcal{J}$ is spanned by the Borel subalgebra generators as follows:

$$\begin{aligned} \mathcal{J} = d\mathcal{V}\,\mathcal{V}^{-1} \;&=\; \frac{1}{2}d\vec{\phi}\cdot\vec{H} + \mathrm{e}^{\frac{1}{2}\vec{e}_{12}\cdot\vec{\phi}}\, d\chi\, X + \sum_I \mathrm{e}^{\frac{1}{2}\vec{e}_{I+3}\cdot\vec{\phi}}\, du^I\, W_I + \sum_I \mathrm{e}^{\frac{1}{2}\vec{e}_{I+6}\cdot\vec{\phi}} \big(dv^I \\ &- \chi du^I\big)\, Z_I + \sum_a \mathrm{e}^{\frac{1}{2}\vec{e}_{a+6}\cdot\vec{\phi}}\, G_a\, \Omega^a + \sum_I \mathrm{e}^{\frac{1}{2}\vec{e}_I\cdot\vec{\phi}}\, G_I\, P^I. \end{aligned}$$

We choose the following 8×8 matrix representation γ of the so(4,4) algebra

$$E = \begin{pmatrix} A & B \\ C & -A^{\hat{T}} \end{pmatrix}, \tag{24}$$

where $A,\ B,\ C$ are the 4×4 matrices, $A,\ B$ being antisymmetric, $B=-B^T,\ C=-C^T$, and the symbol $\widehat{T}$ in $A^{\widehat{T}}$ means the the transposition with respect to the minor diagonal. The diagonal matrices $\vec{H}$ are given by the following $A-$type matrices (with $B=0=C$):

$$A_{H_1} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{H_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{H_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{H_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Twelve generators corresponding to the positive roots are given by the upper-triangular matrices $E_k,\ k=1,\ldots,12,$. From these the generators labeled by $k=2,4,6,7,9,12$ are of pure A-type (with $B=0=C$):

$$A_{E_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_6} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{E_7} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{E_9} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_{12}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

while the other six are of pure B type (with $A=0=C$)

$$B_{E_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_{E_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{E_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{E_8} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_{E_{10}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{E_{11}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The correspondence with the previously introduced generators is as follows:

$$P^I \leftrightarrow E_I, \quad W_I \leftrightarrow E_{I+3}, \quad Z_I \leftrightarrow E_{I+6}, \quad \Omega^a \leftrightarrow E_{a+6}, \quad X \leftrightarrow E_{12}.$$

In this representation, the matrices corresponding to the negative roots,

$$P^{-I} \leftrightarrow E_{-I}, \quad W_{-I} \leftrightarrow E_{-(I+3)}, \quad Z_{-I} \leftrightarrow E_{-(I+6)}, \quad \Omega^{-a} \leftrightarrow E_{-(a+6)}, \quad X^- \leftrightarrow E_{-12},$$

are the transposed ones with respect to the positive roots matrices: $E_{-k}=(E_k)^T$. The following normalization conditions are assumed: $\mathrm{Tr}(H_i,H_j)=4\delta_{ij},\ i,j=1,\ldots,4,\quad \mathrm{Tr}(E_k,E_{-k})=2$, and the involution matrix K is chosen as

$$K=\mathrm{diag}(\kappa,\kappa,1,1,1,1,\kappa,\kappa).$$

The generators of the isotropy subgroup are selected by the Eq.(21). They are given by the following linear combinations of the above generators:

$$P^I - \kappa P^{-I}, \quad Z_I - \kappa Z_{-I}, \quad W_I - W_{-I}, \quad X - \kappa X^-, \quad \Omega^4 - \kappa\Omega^{-4}, \quad \Omega^5 - \Omega^{-5}.$$

Therefore given matrix representation enable us after the exponentiation of the Borel subalgebra (23) to get the coset matrix in the following block form (see in detail [31]):

$$\mathcal{V} = \begin{pmatrix} S & R \\ 0 & \widetilde{S} \end{pmatrix}, \quad \widetilde{S} = (S^{-1})^{\widehat{T}},$$

where S and R are 4×4 matrices. Finally, using the definition (20) one can finds the gauge-invariant representative of the coset:

$$\mathcal{M} = \begin{pmatrix} \mathcal{P} & \mathcal{P}\mathcal{Q} \\ \mathcal{Q}^T\mathcal{P} & \widetilde{\mathcal{P}} + \mathcal{Q}^T\mathcal{P}\mathcal{Q} \end{pmatrix}, \quad \mathcal{Q} = S^{-1}R, \quad \mathcal{P} = S^T\mathcal{E}S, \quad \widetilde{\mathcal{P}} = \widetilde{S}^T\widetilde{\mathcal{E}}\widetilde{S},$$

with $\mathcal{E} = \mathrm{diag}(\kappa,\kappa,1,1)$, $\widetilde{\mathcal{E}} = \kappa\mathcal{E}$.

5 Contraction to $G_{2(2)}$

The present model reduces to minimal five-dimensional supergravity under the following indentifications

$$\psi^1 = \psi^2 = \psi^3 = \psi, \quad \mu_1 = \mu_2 = \mu_3 = \mu, \quad X^1 = X^2 = X^3 = 1, \tag{25}$$

leading to

$$\phi_1 = \frac{1}{\sqrt{2}}(\varphi_2 + \frac{1}{\sqrt{3}}\varphi_1), \quad \phi_2 = \frac{1}{\sqrt{2}}(\varphi_2 - \frac{1}{\sqrt{3}}\varphi_1), \quad \phi_3 = \sqrt{\frac{2}{3}}\varphi_1, \quad \phi_4 = 0.$$

In this case the target space metric of the three-dimensional sigma-model will read

$$\begin{aligned} dl^2 &= \frac{1}{2}\big(d\varphi_1^2 + d\varphi_2^2 + 3\kappa e^{\varphi_2 + \frac{1}{\sqrt{3}}\varphi_1} G^2 + 3e^{\frac{2}{\sqrt{3}}\varphi_1} du^2 + 3\kappa e^{\varphi_2 - \frac{1}{\sqrt{3}}\varphi_1}(dv - \chi du)^2 + \kappa e^{\varphi_2 + \sqrt{3}\varphi_1} G_4^2 \\ &+ e^{2\varphi_2} G_5^2 + \kappa e^{\varphi_2 - \sqrt{3}\varphi_1} d\chi^2\big), \end{aligned} \tag{26}$$

where the one-forms are

$$\begin{aligned} G &= d\mu + v du - u dv, \\ G_4 &= V_4, \quad G_5 = V_5 - \chi V_4, \\ V_a &= d\omega_a - \psi_a\left(3d\mu + \varepsilon^{bc} d\psi_b \psi_c\right) \end{aligned}$$

This manifold is invariant under the $G_{2(2)}$ subgroup of the $SO(4,4)$. Dimensional reduction of the $D = 5$ minimal supergravity to three dimensions was recently studied in [12, 13]. In the notation of [12, 13] the indices a, b, c take the values $0, 1$, the coordinate z^4 is time-like and the matrix λ is related to the present one by transposition with respect to the minor diagonal. In matrix terms the target-space metric (26) reads

$$dl^2 = \frac{1}{4}\mathrm{Tr}\left(\lambda^{-1}d\lambda\lambda^{-1}d\lambda\right) + \frac{1}{4}\tau^{-2}d\tau^2 + \frac{3}{2}d\psi^T\lambda^{-1}d\psi - \frac{1}{2}\tau^{-1}V^T\lambda^{-1}V - \frac{3}{2}\tau^{-1}G^2.$$

This coincides with the result of [12, 13] for the Euclidean signature of the three-space ($\kappa = -1$). The above contraction to $G_{2(2)}$ can be described in terms of the root space as follows. Consider the root vectors of the $so(4,4)$ algebra in the following basis:

$$\begin{aligned}
\vec{e}_1 &= \left(s-\frac{1}{2}, s+\frac{1}{2}, \frac{1}{2}, s\right), \qquad \vec{e}_2 = \left(1,-1,\frac{1}{2},s\right), \qquad \vec{e}_3 = \left(-s-\frac{1}{2}, \frac{1}{2}-s, \frac{1}{2}, s\right), \\
\vec{e}_4 &= \left(\frac{1}{2}-s, -s-\frac{1}{2}, 1, 0\right), \qquad \vec{e}_5 = (-1,1,1,0), \qquad \vec{e}_6 = \left(s+\frac{1}{2}, s-\frac{1}{2}, 1, 0\right), \\
\vec{e}_7 &= \left(\frac{1}{2}-s, -s-\frac{1}{2}, -\frac{1}{2}, s\right), \quad \vec{e}_8 = \left(-1,1,-\frac{1}{2},s\right), \quad \vec{e}_9 = \left(s+\frac{1}{2}, s-\frac{1}{2}, -\frac{1}{2}, s\right), \\
\vec{e}_{10} &= \left(0,0,\frac{3}{2},s\right), \qquad \vec{e}_{11} = (0,0,0,2s), \qquad \vec{e}_{12} = \left(0,0,-\frac{3}{2},s\right), \quad s = \frac{\sqrt{3}}{2}.
\end{aligned}$$

Examination of this pattern shows that the following combinations of the triplets of the $so(4,4)$ root vectors

$$\vec{\alpha}_{\pm 4} = \frac{1}{3}\sum_I \vec{e}_{\pm I}, \quad \vec{\alpha}_{\pm 1} = \frac{1}{3}\sum_I \vec{e}_{\pm (I+3)}, \quad \vec{\alpha}_{\pm 3} = \frac{1}{3}\sum_I \vec{e}_{\pm (I+6)},$$

together with
$$\vec{\alpha}_{\pm 5} = \vec{e}_{\pm 10}, \quad \vec{\alpha}_{\pm 6} = \vec{e}_{\pm 11}, \quad \vec{\alpha}_{\pm 2} = \vec{e}_{\pm 12},$$
form the standard set of the G_2 roots satisfying the relations:

$$\vec{\alpha}_{\pm 3} = \pm(\vec{\alpha}_1 + \vec{\alpha}_2), \quad \vec{\alpha}_{\pm 4} = \pm(2\vec{\alpha}_1 + \vec{\alpha}_2), \quad \vec{\alpha}_{\pm 5} = \pm(3\vec{\alpha}_1 + \vec{\alpha}_2), \quad \vec{\alpha}_{\pm 6} = \pm(3\vec{\alpha}_1 + 2\vec{\alpha}_2).$$

The corresponding generators read:

$$\begin{aligned}
&M_1 = \frac{\sqrt{2}}{3}(H_1 - H_2 + 2H_3), \quad M_2 = \sqrt{\frac{2}{3}}(H_1 + H_2), \\
&P^{\pm} = \frac{1}{\sqrt{3}}\sum P^{\pm I}, \quad Z_{\pm} = \frac{1}{\sqrt{3}}\sum Z_{\pm I}, \quad W_{\pm} = \frac{1}{\sqrt{3}}\sum W_{\pm I}, \quad \Omega^{\pm a}, \quad X^{\pm}.
\end{aligned}$$

They obey the following commutation relations in the Cartan-Weyl form:

$$\begin{aligned}
&[P^+, P^-] = \frac{1}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\
&[W_+, W_-] = M_1, \\
&[Z_+, Z_-] = -\frac{1}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\
&[\Omega^4, \Omega^{-4}] = \frac{3}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \quad [\Omega^5, \Omega^{-5}] = \frac{\sqrt{3}}{2}M_2, \\
&[X^+, X^-] = -\frac{3}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\
&[W_\pm, P^\pm] = \mp\Omega^{\pm 4}, \quad [Z_\pm, P^\pm] = \mp\Omega^{\pm 5}, \\
&[W_\pm, Z_\pm] = \mp P^\pm, \\
&[X^\pm, W_\pm] = \mp Z^\pm, \\
&[X^\pm, \Omega^{\pm 4}] = \mp\Omega^{\pm 5},
\end{aligned}$$

and so on.

Contracting the set of the potentials Φ^A according to the conditions (25), we obtain the following representation for the coset blocks $\mathcal{P}$ and $\mathcal{Q}$:

$$\mathcal{Q} = \begin{pmatrix} \mu, & \omega - \mu\psi^T, & 0 \\ \widetilde{\psi}^T, & \mu\sigma_3, & \widetilde{\omega} - \mu\widetilde{\psi}^T \\ 0, & \psi^T, & -\mu \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\mathcal{P} = -\tau^{-1} \begin{pmatrix} 1, & \eta^T, & \mu \\ \eta, & \eta\eta^T - \tau\widetilde{\lambda}, & \eta\mu - \tau\widetilde{\lambda}\widetilde{\psi}^T \\ \mu, & \eta^T\mu - \tau\widetilde{\psi}\widetilde{\lambda}, & \mu^2 - \tau - \tau\widetilde{\psi}\widetilde{\lambda}\widetilde{\psi}^T \end{pmatrix}, \quad \eta = \sigma_3\psi.$$

This gives a 8×8 representation of the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ of the minimal five-dimensional supergravity reduced to three dimensions, alternative to the 7×7 one given in [12, 13].

6 Conclusions

In this paper we have considered dimensional reduction of the $U(1)^3$ $5D$ supergravity compactified on a circle and and a two-torus. This particular non-minimal five-dimensional supergravity model was widely discussed recently in connection with the supertubes and the black rings. Its dimensional reduction reveal hidden duality symmetries which may serve as tools for generating new classical solutions for this theory. In the four-dimensional theory the duality group is $SL(2,\mathbb{R})^3$, and we gave a convenient matrix realization of it. This is a subgroup of the U-duality group of the three-dimensional theory resulting from the compactification on a two-torus. The latter theory after dualisation of the vector fields to scalars can be presented as the the three-dimensional gravity coupled sigma model on symmetric spaces $SO(4,4)/(SO(4) \times SO(4))$ or $SO(4,4)/(SO(2,2) \times SO(2,2))$ depending on the signature of the three-space. The classical U-duality group of the three-dimensional theory is the 28-parametric non-compact group $SO(4,4)$ which acts transitively on the target space. An identification of the three vector fields and freezing out the two scalar moduli reduce the present theory to minimal five-dimensional supergravity with the three-dimensional U-duality group $G_{2(2)}$, which was extensively studied recently along the same lines [12, 13]. For this limiting case we have presented a new matrix representation for the coset $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ in terms of the 8×8 matrices. Hopefully this new representation will be useful in the search of new soliton solution of five-dimensional supergravity.

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Using Exact Foldy-Wouthuysen Transformation for a Dirac Fermion in Torsion and Magnetic Fields Background

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Abstract

The discussion of experimental manifestations of torsion at low energies are mainly related to the torsion-spin interaction. In this respect the behavior of Dirac field and the spinning particle in an external torsion field deserves and received very special attention. In the present paper we consider the combined action of torsion and strong magnetic field on the massive spinor field and on the corresponding particle. Despite in this case the Hamiltonian doesn't admit the exact Foldy-Wouthuysen transformation, one can perform the single transformation which is, simultaneously, exact in magnetic field and perturbative in the torsion field backgrounds. It is remarkable that this new method enables one to reproduce some known perturbative results and also derive the non-relativistic equations of motion for a spin$-\frac{1}{2}$ particle. Our results confirm and generalize the ones obtained by Buchbinder et al in 1992.

1 Introduction

One of the most natural extensions of General Relativity is related to the inclusion of torsion which is supposed to describe, along with the metric, the space-time geometry and physical properties. The background of gravity with torsion are well known on both classical and quantum levels (see, e.g., the reviews [1]). The physical aspects of torsion gravity has a long story of study (see, e.g., [1, 4, 2, 3, 5, 6] for extensive reviews and references). The issue which always called special attention was interaction of torsion with the spinor field and with the spinning particle [7, 8, 9, 10]. In particular, the papers [11, 12, 13] were devoted

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to the nonrelativistic approximation of Dirac equation and in [12, 13], correspondingly, the Pauli equation and Foldy-Wouthuysen transformation have been obtained for the fermion field on combined electromagnetic and torsion background. One can use these results for exploring the possible manifestations of torsion in the domain of atomic physics [12, 14]. Let us notice that the completely antisymmetric background torsion has been also investigated as an independent parameter measuring the Lorentz and CPT symmetries violations (see, e.g., [15, 16] for the reviews).

The Foldy-Wouthuysen transformation provides, in general, more detailed information about the nonrelativistic approximation, specially if the exact version of this transformation is employed [18, 19, 20]. It is, in principle, safer to perform the exact transformation, for otherwise there is a certain risk to miss some important terms. Recently it has been shown that this is the case for the spinor field in the weak gravitational field [21]. Therefore it is worthwhile to construct the exact Foldy-Wouthuysen transformation for the case of torsion and electromagnetic background. Besides the possible technical advantages, this calculation may provide the opportunity to explore the impact of a strong magnetic field on the possible effect of torsion. One can imagine, for instance, the situation when the magnetic field could enforce the effect of torsion and thus make the upper bound for torsion more precise. Recently, we have used this approach for the case of fermion on the combined background of the gravitational wave and magnetic field and found that in fact there are potentially interesting nonlinear effects like the ones described above. In the present paper we consider the case of torsion. Indeed, the same approach can be used, also, for other Lorentz and CPT violating terms [15, 16].

The usual perturbative Foldy-Wouthuysen transformation can be constructed for the Dirac field interacting with great variety of external fields, including torsion [13]. However, the possibility to have the exact Foldy-Wouthuysen transformation depends on whether some special condition (the existence of the involution operator) is satisfied for a given choice of external fields. Indeed, the exact transformation is more complicated and more interesting from the mathematical point of view [18, 19]. As it was also mentioned above, in this paper we are interested in the set of two external fields - one is torsion and another one is constant and uniform magnetic field. One can safely assume that torsion is very week, for otherwise it would be easy to detect [6], while the magnetic field of our interest showld be very strong, because we expect that the presence of this field may enforce the effect of torsion. Therefore, our goal should be the Foldy-Wouthuysen transformation which is exact in magnetic field but may be just linear in torsion. Actually, the exact Foldy-Wouthuysen transformation with torsion is not possible because the corresponding Hamiltonian does not admit the involution operator. However, even in this situation the technique of exact Foldy-Wouthuysen transformation turns out to be useful and efficient. One can make some *as hoc* modification of the torsion-dependent term in the Hamiltonian, such that the modified expression admits the involution operator. Then one can use the known technique developed for the exact Foldy-Wouthuysen transformation. The main point is that, in the linear approximation, the mentioned modification can be easily removed from the final result. In this way we can reproduce the known perturbative result [13] in a technically much more economic way and also get the Foldy-Wouthuysen Hamiltonian which the terms which show explicitly the mixture between the torsion and magnetic field. In other words, we have derived a Hamiltonian which is exact in magnetic field and linear in torsion.

After performing the Foldy-Wouthuysen transformation we derive the non-relativistic equations of motion for the particle with spin $\frac{1}{2}$. The result manifests the same field mixing due to the exact nature of the Foldy-Wouthuysen transformation.

The paper is organized as follows. In the next section we present a brief information about gravity with torsion and rederive the Hamiltonian of the Dirac field. Section 3 is devoted to the exact Foldy-Wouthuysen transformation and in section 4 the semiclassical nonrelativistic equation of motion for the spinning particle is obtained. In section 5 we discuss the linear expansion in the torsion field and finally in last section we draw our conclusions.

2 Hamiltonian of Dirac field interacting with torsion

Let us start with some details about gravity with torsion. We shall use the notations of [6]. In the space - time with torsion $T^{\alpha}_{\beta\gamma}$ the connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ is non symmetric, and we write $\tilde{\Gamma}^{\alpha}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}_{\gamma\beta} = T^{\alpha}_{\beta\gamma}$. In order to find the explicit expression for $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, one can use the metricity condition $\tilde{\nabla}_{\mu} g_{\alpha\beta} = 0$. The solution for the connection can be easily found in the form

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + K^{\alpha}_{\beta\gamma}\,, \tag{1}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is the Christoffel symbol and $K^{\alpha}_{\beta\gamma}$ is the contorsion tensor

$$K^{\alpha}_{\beta\gamma} = \frac{1}{2}\left(T^{\alpha}_{\beta\gamma} - T_{\beta\ \gamma}^{\ \alpha} - T_{\gamma\ \beta}^{\ \alpha}\right). \tag{2}$$

It proves useful to divide torsion into following irreducible components: the trace $T_{\beta} = T^{\alpha}_{\beta\alpha}$, the pseudotrace $S^{\nu} = \varepsilon^{\alpha\beta\mu\nu} T_{\alpha\beta\mu}$ and the tensor $q^{\alpha}_{\beta\gamma}$, satisfying the conditions $q^{\alpha}_{\beta\alpha} = \varepsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0$. Then torsion can be written in the form

$$T_{\alpha\beta\mu} = \frac{1}{3}\left(T_{\beta} g_{\alpha\mu} - T_{\mu} g_{\alpha\beta}\right) - \frac{1}{6}\,\varepsilon_{\alpha\beta\mu\nu} S^{\nu} + q_{\alpha\beta\mu}\,. \tag{3}$$

For the Dirac field is in an external gravitational field with torsion we can perform the minimal covariant generalization

$$\eta_{\mu\nu} \to g_{\mu\nu} \qquad , \qquad \partial_{\mu} \to \nabla_{\mu} \qquad , \qquad \int d^4x \to \int d^4x\sqrt{-g}\,.$$

Let us notice that the covariant derivative of the spinor field ψ is defined as

$$\begin{aligned} \tilde{\nabla}_{\mu}\psi &= \partial_{\mu}\psi + \frac{i}{2}\tilde{w}_{\mu}^{\ ab}\sigma_{ab}\psi\,, \\ \tilde{\nabla}_{\mu}\bar{\psi} &= \partial_{\mu}\bar{\psi} - \frac{i}{2}\tilde{w}_{\mu}^{\ ab}\bar{\psi}\sigma_{ab}\,, \end{aligned} \tag{4}$$

where $\tilde{w}_{\mu}^{\ ab}$ are the components of spinor connection. We use the standard representation for the Dirac matrices (see, for example, [17])

$$\begin{aligned} \beta &= \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_i = \beta\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \\ \gamma_5 &= \gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \sigma_{a\,b} = \frac{i}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)\,. \end{aligned} \tag{5}$$

The expression for spinor connection which agrees with (2) has the form

$$\tilde{w}_{\mu}^{\ ab} = \frac{1}{4}(e^b_{\nu}\partial_{\mu}e^{\nu a} - e^a_{\nu}\partial_{\mu}e^{\nu b}) + \bar{\Gamma}^{\alpha}_{\nu\mu}(e^{\nu a}e^b_{\alpha} - e^{\nu b}e^a_{\alpha})\,. \tag{6}$$

As far as metric and torsion are independent fields, and the physical effects of both in the laboratory conditions are supposed to be very weak, it is a good idea to explore the torsion effects separately and assume that the metric is the flat Minkowski one. Then the equation (6) boils down to

$$\tilde{w}_\mu^{\ ab} = K^\alpha_{\ \nu\mu}(e^{\nu a}e^b_\alpha - e^{\nu b}e^a_\alpha)\,. \tag{7}$$

The action of spinor field minimally coupled with torsion has the form

$$S = \frac{i}{2}\int d^4x\sqrt{-g}\,\left(\bar{\psi}\gamma^\mu\tilde{\nabla}_\mu\psi - \tilde{\nabla}_\mu\bar{\psi}\gamma^\mu\psi\,-\,2im\bar{\psi}\psi\right), \tag{8}$$

where m is the mass of the Dirac field. In what follows we shall consider only the torsion effects and therefore restrict ourselves by the only special case of flat metric. So we put $g_{\mu\nu} = \eta_{\mu\nu}$ and keep $T^\alpha_{\ \beta\gamma}$ arbitrary. After certain algebra the expression (6) can be rewritten in the form

$$S = \int d^4x\,\left\{i\bar{\psi}\gamma^\mu(\partial_\mu + i\eta_1\gamma_5 S_\mu)\psi + m\bar{\psi}\psi\right\} \tag{9}$$

with $\eta_1 = 1/8$. One can see that the spinor field minimally interacts only with the pseudovector S_μ part of the torsion tensor. The nonminimal interaction is more complicated. According to [22, 6] (see also further references therein) the consistent quantum theory can be constructed only for the nonminimal interaction of Dirac field with torsion. Therefore in what follows we shall keep the parameter η_1 arbitrary.

3 Exact Foldy-Wouthuysen transformation

Consider the spin-1/2 particle in an external torsion and electromagnetic fields. We are going to consider the magnetitic and torsion fields which can only vary with time, but do not depend on the space coordinates. The equation of motion which follows from the action (9) has the form

$$i\hbar\,\frac{\partial\Psi}{\partial t}\; = \;H\Psi\; = \;\left(c\overrightarrow{\alpha}\cdot\overrightarrow{p} - e\overrightarrow{\alpha}\cdot\overrightarrow{A} - \eta_1\overrightarrow{\alpha}\cdot\overrightarrow{S}\gamma_5 + e\Phi + \eta_1\gamma_5 S_0 + mc^2\beta\right)\Psi\,. \tag{10}$$

Here we used notations $A_\mu = (\Phi\,,\,\overrightarrow{A})$, $S_\mu = (S_0\,,\,\overrightarrow{S})$. In case of constant magnetic field one can set $\Phi = 0$. Before start making the exact Foldy-Wouthuysen transformation, one has to consider the conditions

$$JH + HJ = 0\;, \qquad \text{where} \qquad J = i\gamma_5\beta \tag{11}$$

is the involution operator. Only those theories where the Hamiltonian admits the involution operator, enable one to perform the exact Foldy-Wouthuysen transformation [18, 19, 20, 21]. The case of potential corresponding to the constant magnetic field satisfies this criterion [18]. However, direct inspection shows that the term $\eta_1\overrightarrow{\alpha}\,\overrightarrow{S}\gamma_5$ in (10) does not satisfy the condition (11). Thus, the Hamiltonian (10) does not enable one to perform the exact Foldy-Wouthuysen transformation. However, due to the weakness of the torsion field we are really interested only in the linear order in torsion while the magnetic field should be treated exactly.

Let us make an *ad hoc* modification of the the term $\eta_1\overrightarrow{\alpha}\,\overrightarrow{S}\gamma_5$, that is multiply it by the β-matrix. The modified term satisfies the condition (11) and now the the exact Foldy-Wouthuysen transformation is perfectly possible. The main point is that, in the linear order in the torsion field, an extra β has no effect. The reason is that, after deriving the final Hamiltonian operator, it will have the block diagonal structure. We are interested only in the upper block of Hamiltonian which is even (after transformation) to perform physical analysis. At least in the first order in $1/m$ it does not matter if this term is multiplied by β or not, because beta has the form (5) and its upper block is just the unity matrix. As a result we arrive at what one can call semi-exact Foldy-Wouthuysen transformation, because it is exact in only part of external fields and linear in other external fields.

After all, the Hamiltonian we are going to deal with has the form

$$H \,=\, c\vec{\alpha}\cdot\vec{p} - e\vec{\alpha}\cdot\vec{A} - \,\eta_1\vec{\alpha}\cdot\vec{S}\gamma_5\beta + \eta_1\gamma_5 S_0 + mc^2\beta\,. \tag{12}$$

According to the standard prescription [18], the next step is to obtain H^2. Direct calculations gives the result

$$\begin{aligned} H^2 \;&=\; (c\vec{p} - e\vec{A} - \eta_1\vec{\Sigma}S_0)^2 + m^2c^4 + 2\eta_1 mc^2\vec{\Sigma}\cdot\vec{S} \\ &+\; (\eta_1)^2(\vec{S})^2 + \hbar ce\vec{\Sigma}\cdot\vec{B} - 2(\eta_1)^2(S_0)^2 + i\eta_1\gamma_5\beta\vec{\Sigma}\cdot\big[\vec{S}\times(c\vec{p} - e\vec{A})\big]\,. \end{aligned} \tag{13}$$

The last term in (13) is odd. After computing $\sqrt{H^2}$ we should multiply this by J (see equation (17)). If we do this we get a term that transforms, under parity, in a different way of the other terms in the Hamiltonian. To avoid this situation we suppose that $\vec{S}$, $\vec{\Sigma}$ and $\vec{A}$ satisfy the relation

$$\vec{\Sigma}\cdot[\vec{S}\times(c\vec{p} - e\vec{A})] = 0\,. \tag{14}$$

In this way we obtain the operator H^2 of the form

$$\begin{aligned} H^2 \;&=\; (c\vec{p} \,-\, e\vec{A} \,-\, \eta_1\vec{\Sigma}S_0)^2 \,+\, m^2c^4 + 2\eta_1 mc^2\vec{\Sigma}\cdot\vec{S} \\ &+\; \hbar ce\vec{\Sigma}\cdot\vec{B} \,+\, (\eta_1)^2(\vec{S})^2 \,-\, 2(\eta_1)^2\,(S_0)^2\,. \end{aligned} \tag{15}$$

The complete form for H^2 is additionally discussed in appendix.

In order to get the transformed Hamiltonian H^{tr} we rewrite H^2 as $H^2 = A^2 + B$ with A being m-dependent terms in H^2 and B the ones that do not depend on mass. In this case we present

$$A = mc^2 + \eta_1\vec{\Sigma}\cdot\vec{S}\,.$$

Than, we search for an operator K in the form

$$K = A + \frac{1}{A}K_1 + K_1\frac{1}{A} + \vartheta(\frac{1}{A^2})\,, \tag{16}$$

such that $K^2 = A^2$. Finally, using (15) and the fact that

$$H^{tr} \,=\, UHU^* \,=\, \beta[\sqrt{H^2}]^{EVEN} + J[\sqrt{H^2}]^{ODD}\,, \tag{17}$$

where the even (odd) terms in (17) are the ones that commutate (anticommutate) with the matrix β, we get

$$\begin{aligned} H^{tr} \;&=\; \beta mc^2 + \frac{\beta}{2mc^2}(c\vec{p} - e\vec{A} - \eta_1\vec{\Sigma}S_0)^2 + \beta\eta_1\vec{\Sigma}\cdot\vec{S} \\ &+\; \beta\frac{\hbar e}{2mc}\vec{\Sigma}\cdot\vec{B} - \beta\frac{(\eta_1)^2}{mc^2}(S_0)^2\,. \end{aligned} \tag{18}$$

The next step is to present the Dirac fermion ψ in the form

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{\frac{-imc^2 t}{\hbar}} \quad , \tag{19}$$

and use the equation

$$i\hbar \partial_t \psi = H\psi \tag{20}$$

to derive the Hamiltonian for the two-spinor φ. Inserting (19) into (20), we obtain the two-component equation

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \left(-mc^2 + H\right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} . \tag{21}$$

Using the fact that the transformed Hamiltonian is an even function, we obtain, in the φ sector, the nonrelativistic Hamiltonian

$$H_\varphi^{tr} = \frac{1}{2m} (\overrightarrow{\Pi})^2 + B_0 + \overrightarrow{\sigma} \cdot \overrightarrow{Q} \,,$$

$$\overrightarrow{\Pi} = \overrightarrow{p} - \frac{e}{c}\overrightarrow{A} - \frac{\eta_1}{c} S_0 \overrightarrow{\sigma} \quad , \quad B_0 = -\frac{(\eta_1)^2}{mc^2} (S_0)^2 \quad , \quad \overrightarrow{Q} = \eta_1 \overrightarrow{S} + \frac{\hbar e}{2mc} \overrightarrow{B} \,. \tag{22}$$

The epressions above are exactly the same as derived in [12] and in [13] through the usual perturbative Foldy-Wouthuysen transformation.

One can also perform the canonical quantization of the theory in a way similar to [12]. In order to make this we introduce the operators of coordinate $\hat{x}_i$, momenta $\hat{p}_i$ and spin $\hat{\sigma}_i$ and implement the equal-time commutation relations of the following form:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad , \quad [\hat{x}_i, \hat{\sigma}_j] = [\hat{p}_i, \hat{\sigma}_j] = 0 \quad , \quad [\hat{\sigma}_i, \hat{\sigma}_j] = 2i\varepsilon_{ijk}\hat{\sigma}_k \,. \tag{23}$$

The Hamiltonian operator $\hat{H}$ which corresponds to the energy (22) is easily constructed in terms of the operators $\hat{x}_i, \hat{p}_i, \hat{\sigma}_i$ and then these operators yield the equations of motion

$$i\hbar \frac{d\hat{x}_i}{dt} = [\hat{x}_i, H] \quad , \quad i\hbar \frac{d\hat{p}_i}{dt} = [\hat{p}_i, H] \quad , \quad i\hbar \frac{d\hat{\sigma}_i}{dt} = [\hat{\sigma}_i, H] \,. \tag{24}$$

After the computation of the commutators in (24) we arrive at the explicit form of the operator equations of motion. Now we can omit all the terms which vanish when $\hbar \to 0$. Thus we obtain the classical equations which can be interpreted as the (quasi)classical equations of motion for the particle in an external torsion and electromagnetic fields. Note that the operator arrangement problem is irrelevant because of the use of $\hbar \to 0$ limit. The straightforward calculations lead to the equations

$$\frac{dx_i}{dt} = \frac{1}{m}\left(p_i - \frac{e}{c}A_i - \frac{\eta_1}{c}\sigma_i S_0\right) = v_i \,,$$

$$\frac{dp_i}{dt} = \frac{1}{m}\left(p^j - \frac{e}{c}A^j - \frac{\eta_1}{c}\sigma^j S_0\right) \frac{e}{c}\frac{\partial A_j}{\partial x^i} \,,$$

$$\frac{d\sigma_i}{dt} = \left[\overrightarrow{R} \times \overrightarrow{\sigma}\right]_i \quad , \quad \overrightarrow{R} = \frac{2\eta_1}{\hbar}\left[\overrightarrow{S} - \frac{1}{c}\overrightarrow{v} S_0\right] + \frac{e}{mc}\overrightarrow{B} \,. \tag{25}$$

Using the first and second of equations (25) it is possible to obtain

$$m\frac{dv_i}{dt} = -\frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}\left[\overrightarrow{v} \times \overrightarrow{B}\right]_i - \frac{\eta_1}{c}\sigma_i \frac{\partial S_0}{\partial t} - \frac{\eta_1}{c} S_0 \frac{d\sigma_i}{dt} \,. \tag{26}$$

4 Linear expansion in S_μ

Now let us take again the equation (15) and rewrite it using the linear approximation in S_μ. From now on all the terms that have power greater than two in S_μ will be considered neglectable. We get

$$H^2 = H_0^2 + 2\eta_1 mc^2 \overrightarrow{\Sigma} \cdot \overrightarrow{S} + 2\eta_1\gamma_5 S_0 \overrightarrow{\alpha} \cdot (c\overrightarrow{p} - e\overrightarrow{A}) \,, \tag{27}$$

where

$$H_0^2 = (c\overrightarrow{p} - e\overrightarrow{A})^2 + \hbar ce \overrightarrow{\Sigma} \cdot \overrightarrow{B} + m^2c^4 \,. \tag{28}$$

The idea now is to consider the expansion of $\sqrt{H^2}$ not only in terms of the parameter m, but also in terms of S_μ. To perform this let us presented equation (27) in the form

$$H^2 = H_0^2 \{ 1 + \frac{2\eta_1 mc^2 \overrightarrow{\Sigma} \cdot \overrightarrow{S}}{H_0^2} + \frac{2\eta_1\gamma_5 S_0 \overrightarrow{\alpha} \cdot (c\overrightarrow{p} - e\overrightarrow{A})}{H_0^2} \} \,. \tag{29}$$

The next step is to extract the square root of (29). The term H_0^2 we expand in power series in $1/m$ (going to the second order in $1/m$) and we get the same result of [18] which we call H_0^{EK}

$$H_o^{EK} = \sqrt{H_0^2} = mc^2 + \frac{(c\overrightarrow{p} - e\overrightarrow{A})^2}{2mc^2} + \frac{\hbar e}{2mc} \overrightarrow{\Sigma} \cdot \overrightarrow{B} \,. \tag{30}$$

The term between brackets in (29) we expand em power series in S_μ,

$$\sqrt{H^2} = H_0^{EK} \{ 1 + \frac{\eta_1}{mc^2} \overrightarrow{\Sigma} \cdot \overrightarrow{S} + \frac{\eta_1\gamma_5 S_0}{m^2c^4} \overrightarrow{\alpha} \cdot (c\overrightarrow{p} - e\overrightarrow{A}) -$$

$$- \frac{\eta_1}{m^3c^6} \overrightarrow{\Sigma} \cdot \overrightarrow{S} (c\overrightarrow{p} - e\overrightarrow{A})^2 - \frac{\eta_1 \hbar ce}{m^3c^6} \overrightarrow{\Sigma} \cdot \overrightarrow{S} \overrightarrow{\Sigma} \cdot \overrightarrow{B} \} \,. \tag{31}$$

In this equation we impose the condition (see appendix)

$$\overrightarrow{\Sigma} \cdot (\overrightarrow{S} \times \overrightarrow{B}) = 0 \tag{32}$$

and get the final Hamiltonian for this case, using (17)

$$H'^{\,tr} = \beta mc^2 + \beta \frac{(c\overrightarrow{p} - e\overrightarrow{A} - \eta_1 S_0 \overrightarrow{\Sigma})^2}{2mc^2} \big(1 - \frac{\eta_1}{2mc^2} \overrightarrow{\Sigma} \cdot \overrightarrow{S} \big) +$$

$$+ \beta \frac{\hbar ce}{2mc^2} \overrightarrow{\Sigma} \cdot \overrightarrow{B} + \beta\eta_1 \overrightarrow{\Sigma} \cdot \overrightarrow{B} - \beta \frac{\hbar ce \eta_1}{2m^2c^4} \overrightarrow{S} \cdot \overrightarrow{B} . \tag{33}$$

Here, we used prime in H, just to make difference between Hamiltonians (22) and (33). In Hamiltonian (33) we apply the same algorithm used between equations (19) and (22) and finally get the Hamiltonian for the two-spinor φ

$$H_\varphi'^{\,tr} = \frac{1}{2m} (\overrightarrow{\Pi})^2 \big(1 - \frac{\eta_1}{2mc^2} \overrightarrow{\sigma} \cdot \overrightarrow{S} \big) + B_1 + \overrightarrow{\sigma} \cdot \overrightarrow{Q} \,,$$

$$\vec{\Pi} = \vec{p} - \frac{e}{c}\vec{A} - \frac{\eta_1}{c} S_0 \vec{\sigma} \quad , \quad B_1 = -\frac{\eta_1 \hbar c e}{2m^2c^4}\vec{S}\cdot\vec{B} \quad , \quad \vec{Q} = \eta_1 \vec{S} + \frac{\hbar e}{2mc}\vec{B}\,. \tag{34}$$

The next step is to derive the equations of motion using the same procedure as the one applied in [12]. In our case the equations of motion are

$$v_i = \frac{dx_i}{dt} = \big(1 - \frac{\eta_1}{2mc^2}\vec{\sigma}\cdot\vec{S}\big)\,\frac{1}{m}\,(P_i - \frac{e}{c}A_i - \frac{\eta_1}{c}S_0\sigma_i)\,,$$

$$\frac{dp_i}{dt} = \big(1 - \frac{\eta_1}{2mc^2}\vec{\sigma}\cdot\vec{S}\big)\,\frac{1}{m}\,\Big(p^j - \frac{e}{c}A^j - \frac{\eta_1}{c}\sigma^j S_0\Big)\,\frac{e}{c}\frac{\partial A_j}{\partial x^i}\,,$$

$$\frac{d\sigma_i}{dt} = [\vec{r}\times\vec{\sigma}]_i \quad , \quad \vec{r} = \frac{2\eta_1}{\hbar}\left[(1 - \frac{v^2}{4c^2})\vec{S} - \frac{1}{c}\vec{v}S_0\right] + \frac{e}{mc}\vec{B}\,. \tag{35}$$

Using the first two equations of (35), we write

$$m\frac{dv_i}{dt} = -\frac{e}{c}\frac{\partial A_i}{\partial t}(1 - \frac{\eta_1}{2mc^2}\vec{\sigma}\cdot\vec{S}) + \frac{e}{c}\left[\vec{v}\times\vec{B}\right]_i(1 - \frac{\eta_1}{2mc^2}\vec{\sigma}\cdot\vec{S}) -$$

$$-\frac{\eta_1}{c}\,\sigma_i\,\frac{\partial S_0}{\partial t} - \frac{\eta_1}{c}\,S_0\,\frac{d\sigma_i}{dt} - \frac{\eta_1}{2c^2}\,v_i\,\frac{d}{dt}\,(\vec{\sigma}\cdot\vec{S})\,. \tag{36}$$

All the new terms in these equations, in comparison with (25) and (26), are in order $1/m^2$, as it should be. The second term in equation (36) shows an interesting effect. This equation is the analogous to Lorentz force acting on a particle with interacts with external electromagnetic field. The term where S_μ appear can be seen as corrections for this case. Thinking like that, this term shows an explicity mixture between torsion and magnetic field. One can suppose a situation when the magnetic field is strong enough to compensate the fact that S_μ is weak and this term begin to deflect the particle motion in some notable way.

5 Discussion and conclusion

In this paper, we have derived Foldy-Wouthuysen transformation for the Dirac spinor field on the combined background of torsion and constant uniform magnetic fields. We have constructed this for fermion in a new manner, using technique developed for exact Foldy-Wouthuysen transformation. Despite the torsion case doesn‘t admit the exact transformation, the method of [21, 18, 19, 20] proves efficient and in particular we were able to reproduce known results [12, 13] in a much more economic way. The main practical output of our work is expression for the Hamiltonian (34) with is linear in torsion and non-perturbative in external (constant) magnetic field. The same structure was obtained for the non-relativistic equations of motion for a spinning particle. The motivation for the presence of the magnetic field is to check the possibility of the amplification of the influence of the torsion on a Dirac particle. According to our calculations (35) and (36) such an effect is possible. The sufficient strong magnetic field may enforce the effect of a weak torsion. In principle, this result can meet some application in astrophysics with torsion.

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6 Appendix

If we write the complete form for H^2 equation (15), without supposing the condition (14), we get

$$\begin{aligned} H^2 &= (c\vec{p} - e\vec{A} - \eta_1 \vec{\Sigma} S_0)^2 + m^2c^4 + 2\eta_1 mc^2 \vec{\Sigma}\cdot\vec{S} \\ &+ (\eta_1)^2(\vec{S})^2 + \hbar ce\vec{\Sigma}\cdot\vec{B} - 2(\eta_1)^2(S_0)^2 + i\eta_1\gamma_5\beta\vec{\Sigma}\cdot\left[\vec{S}\times(c\vec{p} - e\vec{A})\right] . \end{aligned} \quad (37)$$

The last term in (37) is odd. After computing $\sqrt{H^2}$ we should multiply this by J (see equation (17)). If we do this we get a term that transforms, under parity, in a different way of the other terms in the Hamiltonian. To avoid this situation we suppose (14) and we showed in (25) and (26) and that if (14) is satisfied the equations of motion we get is the same of [12].

The second point to note in this appendix is the condition (32), that we imposed in (31). If we didn‘t do this we should find instead of equation (32), the result

$$\sqrt{H^2} = H_0^{EK} + \eta_1\vec{\Sigma}\cdot\vec{S} + \frac{\eta_1\gamma_5 S_0}{mc^2}\vec{\alpha}\cdot(c\vec{p} - e\vec{A}) -$$

$$-\frac{\eta_1}{2m^2c^4}\vec{\Sigma}\cdot\vec{S}(c\vec{p} - e\vec{A})^2 - \frac{\eta_1\hbar ce}{2m^2c^4}\vec{S}\cdot\vec{B} - \frac{i\hbar ce\eta_1}{2m^2c^4}\vec{\Sigma}\cdot(\vec{S}\times\vec{B}) . \quad (38)$$

The last term in this equation is even. Therefore if we apply equation (17) in (38) we find an imaginary term in final Hamiltonian. In analogy to the situation described above, for the condition (14), in order to avoid an imaginary term, we supposed the condition (32).

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Gauged supersymmetric mechanics

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

On a few simple examples we explain salient features of the superfield gauging of isometries in the models of extended supersymmetric mechanics. The gauging procedure provides a manifestly supersymmetric realization of $d = 1$ automorphic dualities which relate to each other various irreducible off- shell multiplets of $d = 1$ extended supersymmetry with the same number of physical fermions but different splittings of bosonic fields into the physical and auxiliary subsets.

1 Introduction

The study of $d = 1$ supersymmetric models (supersymmetric mechanics) is of importance from many points of view. One of the basic motivations is that the $d = 1$ models provide a useful laboratory for exploring characteristic features of higher- dimensional supersymmetric theories. In particular, various versions of superconformal mechanics are related to the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence and so give an opportunity to improve the understanding of the general "gravity/gauge" paradigm. The superconformal $d = 1$ models also describe the physics of supersymmetric black holes. Another source of interest in supersymmetric mechanics models is the search for new superextensions of one-dimensional integrable systems such as Calogero or Calogero-Moser systems.

The one-dimensional supersymmetry possesses some peculiar features which are not shared by its higher-dimensional counterparts. This concerns, e.g. the irreducible off-shell multiplets of the $d = 1$ extended supersymmetry. They are characterized by three integer numbers $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ where $\mathbf{n}_1$ is the number of physical bosons, $\mathbf{n}_2$ is the number of physical fermions and $\mathbf{n}_3 = \mathbf{n}_2 - \mathbf{n}_1$ stands for that of auxiliary bosons. Some of these off-shell multiplets can be obtained as a reduction of those of the $d > 1$ supersymmetries, while others have as their $d > 1$ counterparts essentially on-shell multiplets (leaving aside a possibility of adding an infinite number of auxiliary fields in the framework of the harmonic superspace approach). Also, many off-shell linear multiplets of extended $d = 1$ supersymmetry have nonlinear analogs which transform under the relevant $d = 1$ supersymmetry in intrinsically nonlinear way.

One more specific feature of the $d = 1$ models is the so called *automorphic duality* [1, 2, 3, 4] which relates to each other the supermultiplets with different divisions of the set of bosonic fields into the physical and auxiliary subsets. This kind of relationships was established in [1]-[4] at the linear level of free actions. Generalizations to the case of interacting multiplets were discussed in [5, 6, 7, 8]. In particular, in [7], basically using the component approach, many results related to the linear and nonlinear automorphic dualities in $\mathcal{N} = 4, d = 1$ supersymmetry were summarized and the exceptional role of the "root" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ as the generating one for other $\mathcal{N} = 4$ multiplets was pointed out.

Recently, in refs. [9, 10, 11], we proposed a systematic superfield way of relating various multiplets of $d = 1$ supersymmetry by using the procedure of gauging isometries in superspace. In the present contribution we explain this method and its possible implications on a few simple examples of the $\mathcal{N} = 1, \mathcal{N} = 2$ and $\mathcal{N} = 4, d = 1$ supersymmetric mechanics models.

2 $\mathcal{N}$=1 and $\mathcal{N}$=2 examples

The basic principles of our gauging construction can be explained already on the simple $\mathcal{N}$=1, d=1 example. Let the coordinate set (t, θ) parametrize $\mathcal{N}$=1, d=1 superspace and $\Phi(t, \theta) = \phi(t) + \theta\chi(t)$ be a scalar $\mathcal{N}$=1 superfield comprising the $\mathcal{N}$=1 supermultiplet $(\mathbf{1}, \mathbf{1}, \mathbf{0})$. The invariant free action of Φ is

$$S_{N=1} = -i \int dt d\theta\, \partial_t \Phi D\Phi = \int dt \left[(\partial_t \phi)^2 + i\chi\partial_t\chi\right], \tag{1}$$

where

$$D = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}, \quad D^2 = i\partial_t, \quad \int d\theta\theta = 1. \tag{2}$$

The action (1) is invariant under constant shifts

$$\Phi' = \Phi + \lambda. \tag{3}$$

Let us now gauge this shifting symmetry by replacing $\lambda \rightarrow \Lambda(t, \theta)$ in (3). To gauge-covariantize the action (1), we are led to introduce the fermionic "gauge superfield" $\Psi(t, \theta) = \psi(t) + i\theta A(t)$ transforming as

$$\Psi\,' = \Psi + D\Lambda, \tag{4}$$

and to substitute the "flat" derivatives in (1) by the gauge-covariant ones

$$S^{gauge}_{N=1} = -i \int dt d\theta\, \nabla_t \Phi \mathcal{D}\Phi, \tag{5}$$

with

$$\nabla_t \Phi = \partial_t \Phi + iD\Psi, \quad \mathcal{D}\Phi = D\Phi - \Psi. \tag{6}$$

Taking into account that $\Lambda(t, \theta)$ is an arbitrary superfunction, one can choose the "unitary gauge" in (5)

$$\Phi = 0, \tag{7}$$

in which

$$S^{gauge}_{N=1} = -\int dt d\theta\, D\Psi\Psi = \int dt \left(i\psi\partial_t\psi + A^2\right). \tag{8}$$

This is just the free action of the $\mathcal{N}$=1 multiplet $(\mathbf{0}, \mathbf{1}, \mathbf{1})$, with $A(t)$ being the auxiliary bosonic field. Thus we observe the phenomenon of transmutation of the physical bosonic field

$\phi(t)$ of the $\mathcal{N}$=1 multiplet $(\mathbf{1},\mathbf{1},\mathbf{0})$ into an auxiliary bosonic field $A(t)$ of another off-shell $\mathcal{N}$=1 multiplet, the $(\mathbf{0},\mathbf{1},\mathbf{1})$ one. This comes about as a result of gauging a shift isometry of the action (1) of the former multiplet. In other words, the $(\mathbf{0},\mathbf{1},\mathbf{1})$ action (8) is a particular gauge of the covariantized $(\mathbf{1},\mathbf{1},\mathbf{0})$ action (5).

One can come to the same final result by choosing a Wess-Zumino gauge, in which $\Psi(t,\theta)$ takes the form

$$\Psi_{WZ}(t,\theta) = i\theta\, A(t)\,, \quad \delta A(t) = \partial_t \lambda(t)\,, \quad \lambda(t) = \Lambda(t,\theta)|_{\theta=0}\,. \tag{9}$$

In this gauge (5) becomes

$$S^{gauge}_{N=1} = \int dt \left[(\partial_t\phi - A)^2 + i\chi\partial_t\chi\right] \tag{10}$$

and the residual gauge freedom acts as an arbitrary shift of $\phi(t)$, $\phi'(t) = \phi(t) + \lambda(t)$. Fixing this freedom by the gauge condition $\phi(t) = 0$, we once again come to the $(\mathbf{0},\mathbf{1},\mathbf{1})$ free action.

On this simplest example we see that the phenomenon of "duality" between the physical and auxiliary degrees of freedom in d=1 supermultiplets can be given a clear interpretation in terms of gauging appropriate isometries of the relevant superfield actions. After gauging, some physical (Goldstone) bosons become pure gauge and can be eliminated, while the relevant d=1 "gauge fields" acquire status of auxiliary fields. This treatment can be extended to higher $\mathcal{N}$ d=1 supersymmetries. In the next Sections we shall discuss how it works in $\mathcal{N}$=4 mechanics.

In the $\mathcal{N}$=2 case an analog of the gauge superfield $\Psi(t,\theta)$ is the real superfield $\mathcal{V}(t,\theta,\bar\theta)$ with the transformation law

$$\mathcal{V}\,'(t,\theta,\bar\theta) = \mathcal{V}(t,\theta,\bar\theta) + \frac{i}{2}\left[\Lambda(t_L,\theta) - \bar\Lambda(t_R,\bar\theta)\right], \quad t_L = t + i\theta\bar\theta\,,\; t_R = \overline{(t_L)} \tag{11}$$

Though this transformation law mimics that of $\mathcal{N}=1, 4D$ gauge superfield, in the WZ gauge only one bosonic field survives, as in (9):

$$\mathcal{V}_{WZ} = \theta\bar\theta A(t)\,. \tag{12}$$

By making use of $\mathcal{V}$, one can study various gaugings of $\mathcal{N}$=2 supersymmetric mechanics models and establish, in this way, the relations between off-shell $\mathcal{N}$=2 multiplets $(\mathbf{2},\mathbf{2},\mathbf{0})$, $(\mathbf{1},\mathbf{2},\mathbf{1})$ and $(\mathbf{0},\mathbf{2},\mathbf{2})$. As an example, let us consider the gauging of the $U(1)$ phase invariance of the free action of the $(\mathbf{2},\mathbf{2},\mathbf{0})$ multiplet.

This multiplet is described by the chiral $\mathcal{N}$=2 superfield $\varphi(t_L,\theta) = \phi(t_L) + \theta\psi(t_L)$, with the bilinear action

$$S^{free}_{N=2} = -\int dt d^2\theta \left[D\varphi(t_L,\theta)\bar D\bar\varphi(t_R,\bar\theta) + 4c\,\varphi(t_L,\theta)\bar\varphi(t_R,\bar\theta)\right], \tag{13}$$

where

$$D = \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}\,,\; \bar D = -\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}\,.$$

In (13) the first term yields the standard free action of the involved fields, $\sim \partial_t\phi\partial_t\bar\phi + \ldots$, while the second piece is an $\mathcal{N}$=2 superextension of the WZ-type Lagrangian $\sim i\left(\partial_t\phi\bar\phi - \partial_t\bar\phi\phi\right)$.

The action (13) possesses the evident $U(1)$ invariance $\varphi' = e^{-i\lambda}\varphi$, $\bar\varphi' = e^{i\lambda}\bar\varphi$ with a constant parameter λ. Under the local version of these transformations

$$\varphi\,' = e^{-i\Lambda}\varphi\,, \quad \bar\varphi\,' = e^{i\bar\Lambda}\bar\varphi\,, \quad \Lambda = \Lambda(t_L,\theta)\,,\; \bar\Lambda = \bar\Lambda(t_R,\bar\theta) \tag{14}$$

the action ceases to be invariant and should be covariantized with the help of the gauge superfield $\mathcal{V}$ with the transformation law (11):

$$S^{gauge}_{N=2} = -\int dt d^2\theta \left[\mathcal{D}\varphi(t_L,\theta)\bar{\mathcal{D}}\bar{\varphi}(t_R,\bar\theta)\, e^{2\mathcal{V}} + 4c\,\varphi(t_L,\theta)\bar{\varphi}(t_R,\bar\theta)\, e^{2\mathcal{V}} + 2\xi\,\mathcal{V}\right]. \quad (15)$$

Here

$$\mathcal{D} = D + 2D\,\mathcal{V}\,, \quad \bar{\mathcal{D}} = \bar{D} + 2\bar{D}\,\mathcal{V}\,, \quad (16)$$

and we also added a Fayet-Iliopoulos (FI) term for $\mathcal{V}$. Using the gauge freedom (14), one can choose the manifestly supersymmetric "unitary" gauge

$$\varphi = 1\,. \quad (17)$$

Now the gauge freedom (14) has been fully "compensated" and $\mathcal{V}$ becomes a general real $\mathcal{N}$=2, d=1 superfield with the off-shell content $(\mathbf{1},\mathbf{2},\mathbf{1})$. The action (15) in this particular gauge becomes the specific action of the latter multiplet:

$$S^{W}_{N=2} = -\int dt d^2\theta \left(DW\bar{D}W + c\,W^2 + 2\xi\,\ln W\right), \quad (18)$$

where we redefined $W = 2e^{\mathcal{V}}$. Thus we started from the bilinear action of the multiplet $(\mathbf{2},\mathbf{2},\mathbf{0})$, gauged its $U(1)$ symmetry and came to the action of the multiplet $(\mathbf{1},\mathbf{2},\mathbf{1})$ with a non-trivial superpotential as a result of the special gauge-fixing in the covariantized $(\mathbf{2},\mathbf{2},\mathbf{0})$ action.

The superpotential in (18) is generated by the WZ and FI terms in the gauge-covariantized action, and it is interesting to see what kind of scalar component potential they produce. Expanding W as

$$W(t,\theta,\bar\theta) = \rho(t) + \theta\chi(t) - \bar\theta\bar\chi(t) + \theta\bar\theta\omega(t)\,, \quad (19)$$

and neglecting fermions, we find

$$S^{W(bos)}_{N=2} = \int dt \left[(\partial_t\rho)^2 + \omega^2 - 2c\,\rho\omega - 2\xi\,\omega\rho^{-1}\right], \quad (20)$$

which, after eliminating the auxiliary field $\omega(t)$, is reduced to the simple expression

$$S^{W(bos)}_{N=2} = \int dt \left[(\partial_t\rho)^2 - c^2\rho^2 - \frac{\xi^2}{\rho^2} - 2c\xi\right]. \quad (21)$$

Thus, gauging the free action of the $(\mathbf{2},\mathbf{2},\mathbf{0})$ multiplet, we finally arrived at the action of the multiplet $(\mathbf{1},\mathbf{2},\mathbf{1})$ with a non-trivial scalar potential.

Note that the action (13) possesses also a shift isometry $\varphi' = \varphi + \omega$, ω being a complex constant parameter. One can alternatively gauge this isometry, and in the corresponding "unitary" gauge $\varphi = 0$ also recover a superfield action of $(\mathbf{1},\mathbf{2},\mathbf{1})$ multiplet. It looks like (18), but with the term $\sim W$ instead of $\ln W$ (this linear term can in fact be removed by a shift of W).

3 $\mathcal{N} = 4$ examples

3.1 General setting

Because the $\mathcal{N}$=4, d=1 harmonic superspace (HSS) plays the central role in our construction for the $\mathcal{N}$=4 case, we start by recollecting the basics of this approach [12, 13, 5, 9].

The $\mathcal{N}$=4, d=1 superspace is defined as the following coordinate set

$$z = (t, \theta_i \;\bar\theta^i)\,, \quad \bar\theta^i = \overline{(\theta_i)}\,. \tag{22}$$

The covariant spinor derivatives are

$$\begin{aligned} &D^i = \frac{\partial}{\partial\theta_i} + i\bar\theta^i\partial_t\,, \quad \bar D_i = \frac{\partial}{\partial\bar\theta^i} + i\theta_i\partial_t = -\overline{(D^i)}\,, \\ &\{D^i, \bar D_j\} = 2i\delta^i_j\partial_t\,,\; \{D^i, D^j\} = \{\bar D_i, \bar D_j\} = 0\,. \end{aligned} \tag{23}$$

The $\mathcal{N}$=4, d=1 harmonic superspace (HSS) in the *central* basis is the following set

$$(z, u) = (t,\; \theta_i\,,\; \bar\theta^i\,,\; u^\pm_i)\,. \tag{24}$$

Here $u^\pm_i \in SU(2)_A/U(1)$ are the $SU(2)_A$ harmonic variables:

$$u^-_i = \overline{(u^{+i})}\,, \quad u^{+i}u^-_i = 1 \;\Leftrightarrow\; u^+_i u^-_k - u^+_k u^-_i = \varepsilon_{ik}\,. \tag{25}$$

The coordinates of $\mathcal{N}$=4, d=1 HSS in the *analytic* basis are

$$\big(t_A = t - i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)\,,\; \theta^\pm = \theta^i u^\pm_i\,,\; \bar\theta^\pm = \bar\theta^i u^\pm_i\,,\; u^\pm_k\big)\,. \tag{26}$$

The analytic subspace of HSS is defined as

$$(t_A, \theta^+, \bar\theta^+, u^\pm_i) \equiv (\zeta, u). \tag{27}$$

It is closed under the $\mathcal{N}$=4 supersymmetry

$$\delta t_A = -2i\left(\epsilon^i u^-_i\bar\theta^+ - \bar\epsilon^i u^-_i\theta^+\right), \quad \delta\theta^+ = \epsilon^i u^+_i\,,\; \delta\bar\theta^+ = \bar\epsilon^i u^+_i\,,\; \delta u^\pm_i = 0\,, \tag{28}$$

and is real with respect to the generalized conjugation $\widetilde{\ }$ [12].

In the analytic basis, the spinor and harmonic derivatives read

$$\begin{aligned} &D^+ = \frac{\partial}{\partial\theta^-}\,,\; \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}\,,\; D^- = -\frac{\partial}{\partial\theta^+} + 2i\bar\theta^-\partial_{t_A}\,,\; \bar D^- = \frac{\partial}{\partial\bar\theta^+} + 2i\theta^-\partial_{t_A}\,, \\ &D^{++} = \partial^{++} - 2i\theta^+\bar\theta^+\partial_{t_A} + \theta^+\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^-}\,, \\ &D^{--} = \partial^{--} - 2i\theta^-\bar\theta^-\partial_{t_A} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+}\,, \quad \partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\,. \end{aligned} \tag{29}$$

The derivatives D^+ , $\bar D^+$ are short in the analytic basis, whence it follows that one can define analytic $\mathcal{N}$=4 superfields $\Phi^{(q)}(\zeta, u)$

$$D^+\Phi^{(q)} = \bar D^+\Phi^{(q)} = 0 \quad\Rightarrow\quad \Phi^{(q)} = \Phi^{(q)}(\zeta, u)\,, \tag{30}$$

where q is the external harmonic $U(1)$ charge. This Grassmann harmonic analyticity is preserved by the harmonic derivative D^{++} : when applied to $\Phi^{(q)}(\zeta, u)$, this derivative yields an analytic $\mathcal{N}$=4, d=1 superfield of charge $(q+2)$.

We shall deal with the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ which is described by a doublet analytic superfield $q^{+a}(\zeta, u)$ of charge 1 satisfying the harmonic constraint [1]

$$D^{++}q^{+a} = 0 \quad\Rightarrow\quad q^{+a}(\zeta, u) = f^{ia}(t)u^+_i + \theta^+\chi^a(t) + \bar\theta^+\bar\chi^a(t) + 2i\theta^+\bar\theta^+\partial_t f^{ia}(t)u^-_i\,. \tag{31}$$

[1]For brevity, in what follows we frequently omit the index "A" of t_A.

It satisfies the pseudoreality condition

$$\widetilde{q^{+a}} = -q^{+a} \;\Rightarrow\; \overline{(f^{ia})} = \epsilon_{ab}\epsilon_{ik} f^{kb}\,,\;\; \overline{(\chi^a)} = \bar{\chi}_a\,. \tag{32}$$

A general off-shell action for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet is

$$S_q = \int dudtd^4\theta\, \mathcal{L}(q^{+a}, q^{-b}, u^{\pm}), \quad q^{-a} \equiv D^{--}q^{+a}. \tag{33}$$

The free action reads

$$S_q^{\rm free} = -\frac{1}{4}\int dudtd^4\theta\,(q^{+a}D^{--}q^{+}_a) = \frac{i}{2}\int dud\zeta^{(-2)}\,(q^{+a}\partial_t q^{+}_a). \tag{34}$$

The free action (34) and constraint (31) exhibit a 7-parameter group of symmetries [10]. Some of them can be extended to the interaction case, leading to certain restrictions on the form of the general action (33). In terms of the component fields, these symmetries become isometries of the target bosonic metric. We list here some symmetries of this sort preserving the constraint (31). We are interested only in those symmetries which commute with $\mathcal{N}$=4 supersymmetry and so can be gauged without passing to the local supersymmetry [9].

The list of the relevant isometries is

1. *Shift*:
$$\delta_1 q^{+a} = \lambda_1 m^a{}_b u^{+b}\,. \tag{35}$$

2. $SU(2)_{PG}$ *rotations*:
$$\delta_{su(2)} q^{+a} = \lambda^a{}_b\, q^{+b}\,,\;\; \lambda^a_a = 0\,. \tag{36}$$

3. $U(1) \subset SU(2)_{PG}$ *rotation*:
$$\delta_2 q^{+a} = \lambda_2 c^a{}_b q^{+b}\,, \quad c^a{}_a = 0\,. \tag{37}$$

4. *Scale transformation*:
$$\delta_3 q^{+a} = \lambda_3 q^{+a}\,. \tag{38}$$

The transformations 1, 2 and 3 are invariances of the free action of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet. Requiring invariance under the rescalings 4 picks up a more complicated action, with a non-trivial bosonic target space metric.

The $\mathcal{N}$=4, d=1 "gauge multiplet" is described by a charge 2 unconstrained analytic superfield $V^{++}(\zeta,u)$ the gauge transformation of which in the abelian case reads

$$\delta V^{++} = D^{++}\Lambda\,, \tag{39}$$

with $\Lambda(\zeta,u)$ being a charge zero unconstrained analytic superfield parameter. Using this gauge freedom, one can choose the Wess-Zumino gauge, in which the gauge superfield becomes

$$V^{++}(\zeta,u) = 2i(\theta^+\bar{\theta}^+)A(t), \quad \delta A(t) = -\partial_t\Lambda_0(t),\;\; \Lambda_0 = \Lambda(\zeta,u)|_{\theta=0}\,. \tag{40}$$

As in the $\mathcal{N}$=2, d=4 HSS [12, 13], V^{++} gauge- covariantizes the analyticity-preserving harmonic derivative D^{++}. Assume that the analytic superfield $\Phi^{(q)}$ is transformed under some abelian gauge isometry as

$$\delta_\Lambda \Phi^{(q)} = \Lambda\,\mathcal{I}\,\Phi^{(q)}\,, \tag{41}$$

where $\mathcal{I}$ is the corresponding generator. Then the harmonic derivative D^{++} is covariantized as

$$D^{++}\Phi^{(q)} \quad\Longrightarrow\quad \mathcal{D}^{++}\Phi^{(q)} = (D^{++} - V^{++}\mathcal{I})\Phi^{(q)}\,. \tag{42}$$

One can also define the second, non-analytic harmonic connection V^{--}

$$\mathcal{D}^{--} = D^{--} - V^{--}\mathcal{I}\,, \quad \delta V^{--} = D^{--}\Lambda\,. \tag{43}$$

From the requirement of preserving the algebra of flat harmonic derivatives,

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = D^0\,, \quad [D^0, \mathcal{D}^{\pm\pm}] = \pm 2\mathcal{D}^{\pm\pm}\,, \tag{44}$$

the well-known harmonic zero-curvature equation follows

$$D^{++}V^{--} - D^{--}V^{++} = 0\,. \tag{45}$$

It specifies V^{--} in terms of V^{++}. One can also define the covariant spinor derivatives

$$\mathcal{D}^- = [\mathcal{D}^{--}, D^+] = D^- + (D^+V^{--})\,\mathcal{I}\,, \quad \bar{\mathcal{D}}^- = [\mathcal{D}^{--}, \bar{D}^+] = \bar{D}^- + (\bar{D}^+V^{--})\,\mathcal{I}\,, \tag{46}$$

as well as the covariant time derivative $\mathcal{D}_t$:

$$\{D^+, \bar{\mathcal{D}}^-\} = 2i\mathcal{D}_t\,, \quad \mathcal{D}_t = \partial_t - \frac{i}{2}(D^+\bar{D}^+V^{--})\,\mathcal{I}\,. \tag{47}$$

The vector gauge connection

$$V \equiv D^+\bar{D}^+V^{--}, \quad \delta V = -2i\partial_{t_A}\Lambda\,, \tag{48}$$

is an analytic superfield, so $\mathcal{D}_t$ preserves the analyticity. In the gauge (40)

$$V \quad\Longrightarrow\quad 2i\,A(t)\,. \tag{49}$$

We will exploit these relations in the sample examples below.

3.2 Gauging a shift symmetry

We shall specialize to the case of a shift symmetry (35) with $m^a{}_b = \delta^a{}_b$. The gauging procedure as its first step involves replacing the global parameter λ_1 by a superfield $\Lambda_1(t, \theta, \bar{\theta}, u)$ which depends on the coordinates of harmonic superspace. We require the local transformations to respect the analyticity, and thus Λ_1 is an analytic superfield

$$\delta_1 q^{+a} = \Lambda_1 u^{+a}\,, \qquad D^+\Lambda_1 = \bar{D}^+\Lambda_1 = 0 \;\Leftrightarrow\; \Lambda_1 = \Lambda_1(t_A, \theta^+, \bar{\theta}^+, u^\pm)\,. \tag{50}$$

The harmonic constraint needs to be covariantized. This can be done by introducing an analytic gauge superfield $V^{++}(t_A, \theta^+, \bar{\theta}^+, u^\pm)$ with the gauge transformation law (39). Then, the covariantized harmonic constraint reads

$$\nabla^{++}q^{+a} = D^{++}q^{+a} - V^{++}u^{+a} = 0\,.$$

The gauge-covariantization of $D^{--}q^{+a}$ is

$$\nabla^{--}q^{+a} = D^{--}q^{+a} - V^{--}u^{+a}\,.$$

where V^{--} was defined in (43), (45). The covariantization of the free action (34) is

$$S_g = \int dt d^4\theta du\, q^{+a}\nabla^{--} q^+_a\,. \tag{51}$$

The gauge transformation (50) implies

$$\delta_1\left(q^{+a}u^-_a\right) = \Lambda_1\,.$$

Thus we may choose a supersymmetric unitary gauge such that

$$q^{+a}u^-_a = 0\,. \tag{52}$$

Then, what remains from the superfield q^{+a} is the projection $W^{++} = q^{+a}u^+_a$. The harmonic constraint expresses V^{++} in terms of W^{++} , and also properly constrains W^{++}

$$\nabla^{++}q^{+a} = 0 \;\Rightarrow\; \begin{cases} V^{++} = W^{++} \\ D^{++}W^{++} = 0 \end{cases}. \tag{53}$$

We recognize W^{++} as the superfield providing the description of the $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet:

$$\begin{aligned} D^{++}W^{++} = 0 \quad \Rightarrow \quad W^{++}(\zeta,u) \;\; &= \;\; w^{(ik)}(t)u^+_i u^+_k + \theta^+\psi^i(t)u^+_i + \bar\theta^+\bar\psi^i(t)u^+_i \\ &\quad + i\theta^+\bar\theta^+[F(t) + 2\partial_t w^{(ik)}(t)u^+_i u^-_k]. \end{aligned} \tag{54}$$

The gauge invariant action S_g reduces to the action $S_g \propto \int dtd^4\theta W^{++}(D^{--})^2W^{++}$ – the free $(\mathbf{3},\mathbf{4},\mathbf{1})$ action.

Instead of a supersymmetric gauge, we might equally choose the Wess-Zumino (WZ) gauge

$$V^{++} = 2i\theta^+\bar\theta^+A(t)\,, \tag{55}$$

the only surviving component in V^{++} being the gauge field $A(t)$. in the supermultiplet in the WZ gauge. in [9]. The residual gauge freedom of the component fields is given by

$$\delta_1 A(t) = -\partial_t\lambda_1(t)\,, \quad \delta_1 f^{ia}(t) = \lambda_1\epsilon^{ai}\,, \quad \delta_1\chi^a(t) = 0\,, \quad \lambda_1 = \Lambda_1|_{\theta=\bar\theta=0}\,. \tag{56}$$

In WZ gauge, the gauge invariant action S_g becomes, in terms of components,

$$S_g \sim \int dt\left[(\dot f^{ia} - A\epsilon^{ia})(\dot f_{ia} + A\epsilon_{ia}) + i\dot{\bar\chi}^a\chi_a\right]. \tag{57}$$

The essential degrees of freedom are revealed by imposing the further (unitary) gauge

$$\delta_1(f^{ia}\epsilon_{ia}) = 2\lambda_1(t)\,, \;\Rightarrow\; \textit{unitary gauge}: f^{ia}\epsilon_{ia} = 0\,. \tag{58}$$

The action then becomes

$$S_g \sim \int dt\left[\dot f^{(ia)}\dot f_{(ia)} + i\bar\chi^a\dot\chi_a + 2A^2\right]. \tag{59}$$

The remaining fields are a triplet of physical bosons $f^{(ia)}$, a complex doublet of fermions χ^a, $\bar\chi_a$ and an auxiliary field A. This is just the component content of the $(\mathbf{3},\mathbf{4},\mathbf{1})$ supermultiplet.

In order to reproduce the most general sigma-model type superfield action of the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1}) \leftrightarrow W^{++}$, one should start from the general superfield q^+ action invariant under the shifts (50) and pass to the gauged action by the same rules as above.

3.3 An example of non-abelian gauging

As our final example we consider non-abelian gauging of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet action. The covariantized action in a fixed gauge yields an action of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet. Like in the previous example, we shall limit our consideration to the gauging of the free action of the q^{+a} multiplet (34). The gauging of the general $SU(2)_{PG}$ invariant actions is considered in [10].

The action (34) exhibits a manifest invariance under the global $SU(2)_{PG}$ transformations (36). Let us gauge this symmetry by changing $\lambda^a_b \rightarrow \Lambda^a_b(\zeta, u)$:

$$\delta q^{+a} = \Lambda^a_b q^{+b}\,, \quad \Lambda^a_a = 0\,. \tag{60}$$

The constraint (31) is covariantized to

$$D^{++}q^{+a} - V^{++a}{}_b q^{+b} = 0\,, \tag{61}$$

where the traceless analytic gauge connection $V^{++a}{}_b$ is transformed as

$$\delta V^{++a}{}_b = D^{++}\Lambda^a_b + \Lambda^a_c V^{++c}{}_b - V^{++a}{}_c \Lambda^c_b\,. \tag{62}$$

Using this freedom, one can pass to the WZ gauge as in the abelian case (40)

$$V^{++a}{}_b = 2i\theta^+\bar\theta^+ A^a{}_b(t)\,,\ \delta_r A^a{}_b = -\partial_t \Lambda_{(0)}{}^a{}_b + \Lambda_{(0)}{}^a{}_c A^c_b - A^a_c \Lambda_{(0)}{}^c{}_b\,. \tag{63}$$

The action (34) written in the analytic superspace is covariantized by changing

$$\partial_t q^{+a} \quad \Rightarrow \quad \nabla_t q^{+a} = \partial_t q^{+a} - \frac{i}{2} V^a_b q^{+b}\,, \tag{64}$$

where

$$V^a_b = D^+\bar D^+ V^{--a}{}_b\,, \quad D^{++}V^{--a}{}_b - D^{--}V^{++a}{}_b - V^{++a}{}_c V^{--c}{}_b + V^{--a}{}_c V^{++c}{}_b = 0\,. \tag{65}$$

In the WZ gauge (63):

$$V^{--a}{}_b = 2i\theta^-\bar\theta^-\, A^a{}_b(t)\,, \quad V^a_b = D^+\bar D^+ V^{--a}{}_b = 2iA^a{}_b(t)\,. \tag{66}$$

In this gauge, the solution of the covariantized constraint (61) is obtained from the solution (31) just by the replacement

$$\partial_t f^{ia} \;\Rightarrow\; \nabla_t f^{ia} = \partial_t f^{ia} + A^a{}_b f^{ib}\,. \tag{67}$$

After substituting this covariantized solution for q^{+a} into the covariantization of the action (34) and performing there the Grassmann and harmonic integration, we arrive at the following component action

$$S^{bos}_{na} = \int dt\, \left(\nabla_t f^{ia}\nabla_t f_{ia} - i\chi^a \nabla_t \bar\chi_a\right). \tag{68}$$

Splitting f^{ia} as

$$f^{ia} = \varepsilon^{ia}\frac{1}{\sqrt2}\, f + f^{(ia)}\,, \tag{69}$$

and assuming that f has a non-vanishing constant vacuum part, $f =< f > + \ldots, < f >\neq 0$, we observe that the symmetric part in (69) can be fully gauged away by the residual $SU(2)$ gauge freedom

$$f^{ia} \quad \Rightarrow \quad \varepsilon^{ia}\frac{1}{\sqrt2}\, f\,. \tag{70}$$

In this gauge, the action (68) becomes

$$S_{na}^{bos} = \int dt \left[(\partial_t f)^2 - i\chi^a \partial_t \bar{\chi}_a + f^2 \frac{1}{2} A^{(ab)} A_{(ab)} - i\chi_{(a} \bar{\chi}_{b)} A^{(ab)} \right] \tag{71}$$

where the former gauge field $A^{(ab)}$ becomes a triplet of auxiliary fields. So, gauging the "Pauli-Gürsey" $SU(2)$ symmetry of the *free* action of $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet and choosing the appropriate gauge in the resulting covariantized action, we arrived at the action (71) which describes an *interacting* system of 1 physical bosonic field $f(t)$, the fermionic doublet $\chi^a(t)$ and the triplet of auxiliary fields $A^{(ab)}(t)$, that is just the field content of off-shell $\mathcal{N}$=4, d=1 multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$. The $\mathcal{N}$=4 supersymmetry of the action (71) is guaranteed, since we started from the manifestly supersymmetric action and just fixed some gauges in it. Note that after eliminating the auxiliary field from (71), the latter takes the following on-shell form

$$S_{na}^{bos} = \int dt \left[(\partial_t f)^2 - i\chi^a \partial_t \bar{\chi}_a + \frac{3}{8f^2} (\chi^a \chi_a)(\bar{\chi}_a \bar{\chi}^a) \right]. \tag{72}$$

Like in the previous examples, the passing from the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ to the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ by gauging $SU(2)_{PG}$ can be performed at the level of superfield actions, without resorting to WZ gauge at the intermediate steps [10].

4 Conclusions

The gauging procedure in supersymmetric mechanics exemplified above has in fact a wide range of applicability. It was shown in [9]-[11] that all general superfield actions of the linear $\mathcal{N}$=4 multiplets with four fermions, i.e. $(\mathbf{3},\mathbf{4},\mathbf{1})$, $(\mathbf{2},\mathbf{4},\mathbf{2})$, $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$, as well as of nonlinear versions of these multiplets, can be recovered by gauging the subclasses of the general superfield q^{+a} action which enjoy invariance under one or another symmetry implementable on q^{+a} and giving rise to the proper isometries of the bosonic cores of these particular q^{+a} actions. The "root" q^{+a} superfield can be taken either in its standard linear form or in various nonlinear forms [9, 14]. Thus the full variety of all possible models of $\mathcal{N}$=4 mechanics is embodied by the most general gauged q^+ mechanics and can be alternatively described in terms of the analytic $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplets q^{+a} (both linear and nonlinear) and analytic gauge superfields V^{++} (both abelian and non-abelian). This fact suggests, in particular, that the analytic $\mathcal{N}$=4, $d = 1$ superspace is the fundamental underlying superspace of $\mathcal{N}$=4, $d = 1$ supersymmetry.

As one of the possible directions of the further development of this gauging approach, it would be tempting to generalize it to higher $\mathcal{N}$ supermechanics, e.g. $\mathcal{N}$=8 mechanics [15].

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Акустические фононы в гидродинамике и метрика Шварцшильда

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Аннотация

Показано, что акустические фононы в жидкости описываются волновым уравнением с эффективной четырехмерной метрикой, описывающей нетривиальную геометрию пространства-времени. Эта метрика определяется классическими нерелятивистскими уравнениями гидродинамики. Дан эвристический “вывод” решения Шварцшильда в координатах Пенлеве–Гулстранда. В таком виде метрика Шварцшильда конформно эквивалентна эффективной метрике для акустических фононов.

Acoustic phonons in liquid are shown to be governed by the wave equation with the effective four-dimensional metric describing nontrivial geometry of the space-time. This metric is defined by the classical nonrelativistic equations of hydrodynamics. The heuristic “derivation” of the Schwarzschild metric in Panlevé–Gullstrand coordinates is presented. The Schwarzschild metric in this form is conformally equivalent to the metric for acoustic phonons.

1 Введение

В физике всегда большой интерес вызывали аналогии между явлениями из разных областей. В настоящей статье мы покажем, что некоторые явления в гидродинамике и общей теории относительности описываются уравнениями, которые во многих отношениях похожи и обладают рядом одинаковых свойств.

В последние годы акустические фононы в жидкости [1, 2] привлекают все возрастающий интерес, как физически наглядная модель черных дыр. Уравнения движения для акустических фононов следуют из классических нерелятивистских уравнений гидродинамики следующим образом. Рассмотрим некоторое точное решение уравнений гидродинамики. Тогда уравнение для малых возбуждений (фононов) вблизи этого решения сводится к уравнению Даламбера с нетривиальной эффективной четырехмерной метрикой лоренцевой сигнатуры, в которой роль скорости света играет скорость звука в жидкости.

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Отличие от общей теории относительности сводится к тому, что эффективная метрика для фононов определяется уравнениями гидродинамики, а не уравнениями Эйнштейна. Тем не менее уравнение для фононов задается нетривиальной четырехмерной метрикой, для которой тензор кривизны отличен от нуля. Другими словами, фононы двигаются на многообразии с нетривиальной геометрией. При этом возможно возникновение горизонтов, когда скорость течения жидкости превышает скорость звука, и, следовательно, образование акустических аналогов черных дыр.

Мы приведем также эвристический "вывод" решения Шварцшильда, исходя из уравнений нерелятивистской механики и преобразования Галилея [3]. Небольшая цепочка (нестрогих!) преобразований переводит метрику Лоренца в метрику Шварцшильда, записанную в координатах Пенлеве–Гулстранда [4, 5, 6]. Показано, что эта метрика конформно эквивалентна эффективной метрике для акустических фононов.

2 Акустические фононы в гидродинамике

В настоящем разделе мы следуем выводу уравнения для фононов, предложенному в [1, 2] (см. также [7]). Рассмотрим 4-мерное галилеево пространство-время с декартовой системой координат $\{x^\alpha\}$, $\alpha = 0, 1, 2, 3$, которые мы будем обозначать индексами из начала греческого алфавита $\alpha, \beta, \dots$. Координату $x^0 \in \mathbb{R}$ мы отождествляем с временем, $x^0 = t$. Пространственные координаты $\{x^\mu\} \in \mathbb{R}^3$ мы будем обозначать индексами из середины греческого алфавита $\mu, \nu, \dots$. Жидкость без вязкости называется идеальной и описывается плотностью $\rho(x)$, давлением $p(x)$ и вектором скорости $\vec{v} = \{v^\mu(x)\}$, где $x = \{x^\alpha\}$. Движение идеальной жидкости или идеального газа в пространстве определяется следующей замкнутой системой из пяти нелинейных уравнений для пяти переменных (см., например, [8])

$$\rho\dot{\vec{v}} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p + \vec{f}, \tag{1}$$

$$\dot{\rho} + \operatorname{div}(\rho\vec{v}) = 0, \tag{2}$$

$$p = p(\rho), \tag{3}$$

где точка обозначает дифференцирование по времени. Уравнение (1) называется уравнением Эйлера и представляет собой второй закон Ньютона для элемента объема жидкости. Здесь $\vec{f}(x)$ – плотность внешних сил. Например, в гравитационном поле $\vec{f} = -\rho\nabla\varphi$, где $\varphi(x)$ – потенциал гравитационного поля. В дальнейшем мы ограничимся только этим случаем. Уравнение (2) является уравнением непрерывности. Последнее уравнение (3) является уравнением состояния жидкости, которое характеризует саму жидкость и считается заданным. Здесь мы предполагаем, что давление жидкости зависит только от ее плотности. Такая жидкость представляет собой один из наиболее распространенных видов жидкости и называется баротропной. Для идеального газа уравнение состояния имеет вид

$$p = \frac{\rho}{\mu}RT, \tag{4}$$

где μ, R и T есть молекулярный вес, универсальная газовая постоянная и абсолютная температура. При постоянной температуре $T = \mathsf{const}$ давление идеального газа прямо пропорционально плотности. Для несжимаемой жидкости $\rho = \mathsf{const}$ и уравнение непрерывности принимает вид $\operatorname{div}\vec{v} = 0$. Уравнения (1)–(3) записаны в стандартном для гидродинамики виде, где не делается различие между верхними и нижними индексами.

Если уравнение состояния задано, то давление p (или плотность ρ) можно исключить из системы уравнений движения. Для этого заметим, что

$$\nabla p = \frac{dp}{d\rho}\nabla\rho = c^2\nabla\rho,$$

где введена скорость звука $c(\rho)$

$$c^2 = \frac{dp}{d\rho}. \tag{5}$$

Тогда уравнение Эйлера можно переписать в виде

$$\dot{\vec{v}} + (\vec{v}\nabla)\vec{v} = -c^2\frac{\nabla\rho}{\rho} - \nabla\varphi, \tag{6}$$

где скорость звука $c = c(\rho)$ рассматривается, как заданная функция от плотности жидкости. Уравнение Эйлера (6) вместе с уравнением непрерывности (2) однозначно описывают движение такой жидкости при заданных начальных и граничных условиях. Воспользовавшись тождеством

$$\frac{1}{2}\nabla v^2 = [\vec{v}, \operatorname{rot}\vec{v}] + (\vec{v}\nabla)\vec{v}, \qquad v^2 = (\vec{v})^2,$$

уравнение Эйлера (1) после деления на ρ можно переписать в виде

$$\dot{\vec{v}} - [\vec{v}, \operatorname{rot}\vec{v}] = -\nabla\left(h + \frac{v^2}{2} + \varphi\right),$$

где введена *энтальпия* жидкости

$$h(p) = \int_0^p \frac{dp'}{\rho(p')}.$$

Предположим, что движение жидкости является безвихревым:

$$\operatorname{rot}\vec{v} = 0 \quad \Leftrightarrow \quad \partial_\mu v_\nu - \partial_\nu v_\mu = 0. \tag{7}$$

На геометрическом языке это означает, что 1-форма $dx^\mu v_\mu$ является замкнутой на пространственном сечении $x^0 = \mathsf{const}$. Тогда локально существует потенциальное поле $\psi(x)$ (потенциал) такое, что

$$v_\mu = -\partial_\mu\psi. \tag{8}$$

Для безвихревой жидкости уравнение Эйлера эквивалентно уравнению Бернулли

$$-\dot{\psi} + h + \frac{(\nabla\psi)^2}{2} + \varphi = F(t), \tag{9}$$

где $F(t)$ – произвольная функция времени. Поскольку потенциал ψ определен с точностью до добавления произвольной функции времени, то, без ограничения общности, положим $F(t) = 0$. Допустим, что мы имеем точное решение уравнений гидродинамики $\rho_0(x)$, $p_0(x)$ и $\vec{v}_0(x) = -\nabla\psi_0$. Получим уравнение, описывающее распространение акустических возбуждений (фононов) вблизи этого решения. Пусть

$$\begin{aligned} \rho &\approx \rho_0 + \epsilon\rho_1, \\ p &\approx p_0 + \epsilon p_1, \\ v^\mu &\approx v_0^\mu + \epsilon v_1^\mu, \\ \psi &\approx \psi_0 + \epsilon\psi_1. \end{aligned} \tag{10}$$

где $\epsilon \ll 1$ – малый параметр разложения. При этом мы считаем внешние силы заданными $\varphi = \varphi_0$.

В дальнейшем мы будем использовать обозначения, принятые в дифференциальной геометрии, и различать верхние и нижние индексы. Пространственные индексы в декартовой системе координат поднимаются и опускаются с помощью метрики $\eta_{\mu\nu} = -\delta_{\mu\nu}$ и ее обратной, которая отличается от евклидовой метрики знаком. В наших обозначениях $v_\mu = \partial_\mu \psi$, $v^\mu = \eta^{\mu\nu}\partial_\nu\psi = -\partial_\mu\psi$.

Уравнение Бернулли (9) в нулевом и первом порядке по ϵ имеет вид

$$\epsilon^0: \quad -\partial_0\psi_0 + h_0 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\psi_0\partial_\nu\psi_0 + \varphi_0 = 0, \tag{11}$$

$$\epsilon^1: \quad -\partial_0\psi_1 + \frac{p_1}{\rho_0} - v_0^\mu\partial_\mu\psi_1 = 0, \tag{12}$$

где учтено разложение для энтальпии

$$h(p) \approx h(p_0) + \epsilon\frac{p_1}{\rho_0}.$$

Учтем, что

$$\rho_1 = \frac{d\rho}{dp}p_1 = \frac{p_1}{c^2},$$

и найдем поправку к плотности ρ_1 из уравнения (12)

$$\rho_1 = \frac{\rho_0}{c^2}(\partial_0\psi_1 + v_0^\mu\partial_\mu\psi_1). \tag{13}$$

Уравнение непрерывности в нулевом и первом порядке по ϵ имеет вид

$$\epsilon^0: \quad \partial_0\rho_0 + \partial_\mu(\rho_0 v_0^\mu) = 0, \tag{14}$$

$$\epsilon^1: \quad \partial_1\rho_1 + \partial_\mu(\rho_0 v_1^\mu + \rho_1 v_0^\mu) = 0. \tag{15}$$

Подставим во второе уравнение решение для поправки к плотности (13). В результате получим уравнение для поправки к потенциалу скорости:

$$\partial_0\left[\frac{\rho_0}{c^2}(\partial_0\psi_1 + v_0^\mu\partial_\mu\psi_1)\right] + \partial_\mu\left[\rho_0\partial^\mu\psi_1 + \frac{\rho_0}{c^2}v_0^\mu(\partial_0\psi_1 + v_0^\nu\partial_\nu\psi_1)\right] = 0. \tag{16}$$

Это волновое уравнение для $\psi_1(x)$ полностью определяет распространение акустических колебаний в движущейся жидкости, описываемой плотностью $\rho_0(x)$ и полем скоростей $v_0^\mu(x)$, с заданным уравнением состояния $c = c(\rho)$. Если решение для $\psi_1(x)$ известно, то поправка к плотности ρ_1 однозначно определяется формулой (13).

Уравнение (16), как легко проверить, можно записать в матричных обозначениях

$$\partial_\alpha(f^{\alpha\beta}\partial_\beta\psi_1) = 0,$$

где

$$f^{\alpha\beta} = \frac{\rho_0}{c^2}\begin{pmatrix} 1 & v_0^\nu \\ v_0^\mu & c^2\eta^{\mu\nu} + v_0^\mu v_0^\nu. \end{pmatrix}$$

Введем метрику $g_{\alpha\beta}$ и ее обратную $g^{\alpha\beta}$

$$g_{\alpha\beta} = \frac{\rho_0}{c}\begin{pmatrix} c^2 + v_0^\mu v_{0\mu} & -v_{0\nu} \\ -v_{0\mu} & \eta_{\mu\nu} \end{pmatrix}, \quad g^{\alpha\beta} = \frac{1}{\rho_0 c}\begin{pmatrix} 1 & v_0^\nu \\ v_0^\mu & c^2\eta^{\mu\nu} + v_0^\mu v_0^\nu \end{pmatrix}. \tag{17}$$

Эта метрика имеет лоренцеву сигнатуру $(+---)$, и ее определитель равен

$$g = \det g_{\alpha\beta} = -\frac{\rho_0^4}{c^2}.$$

Обратная метрика $g^{\alpha\beta}$ отличается от матрицы $f^{\alpha\beta}$ простым множителем

$$g^{\alpha\beta} = \frac{c}{\rho_0^2} f^{\alpha\beta}.$$

Теперь уравнение для акустических фононов можно переписать в инвариантном (относительно общих преобразований координат) виде

$$\frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta\psi) = 0, \tag{18}$$

где мы, для простоты обозначений, отбросили индекс у поправки к потенциалу скорости.

Таким образом распространение фононов в движущейся жидкости описывается инвариантным волновым уравнением в четырехмерном пространстве-времени с нетривиальной метрикой лоренцевой сигнатуры (17). Эта метрика определяется плотностью ρ_0, скоростью звука c и полем скоростей v_0^μ, которые удовлетворяют исходным уравнениям (11), (15). Подчеркнем, что движение самой жидкости происходит в плоском галилеевом пространстве-времени, а распространение акустических возбуждений в этой движущейся жидкости описывается волновым уравнением на псевдоримановом пространстве-времени с нетривиальной "эффективной" метрикой.

Эффективная метрика определяется четырьмя функциями $\rho_0(x)$ и $v_0(x)$, которые удовлетворяют уравнениям гидродинамики. При постановке задачи Коши для однозначного определения этих функций необходимо задать четыре произвольные функции в качестве начальных условий. В общей теории относительности метрика удовлетворяет уравнениям Эйнштейна и имеет две распространяющихся степени свободы. При постановке задачи Коши для уравнений Эйнштейна также необходимо задать четыре произвольные функции на пространственноподобном сечении: по две на каждую степень свободы, так как уравнения движения второго порядка.

Нулевая компонента метрики g_{00} в (17) меняет знак в тех точках пространства-времени, где течение жидкости становится сверхзвуковым: $c^2 = \vec{v}^2 = -v^\mu v_\mu$. Эти поверхности в пространстве соответствуют горизонтам черных дыр. Действительно, поскольку скорость фононов ограничена скоростью звука в жидкости, то они не могут покинуть область сверхзвукового течения. Следовательно, в быстро текущей жидкости для фононов могут образовываться аналоги черных дыр в общей теории относительности.

Интервал, соответствующий метрике (17), имеет вид

$$ds^2 = \frac{\rho_0}{c}\left[c^2dt^2 + \eta_{\mu\nu}(dx^\mu - v_0^\mu dt)(dx^\nu - v_0^\nu dt)\right] \tag{19}$$

В следующем разделе мы покажем, что метрику Шварцшильда можно записать в виде, конформно эквивалентном (19).

3 Эвристический "вывод" решения Шварцшильда

Эвристический подход к метрике Шварцшильда описан в [3]. Рассмотрим плоское пространство-время Минковского $\mathbb{R}^{1,3}$ со следующей декартовой системой координат

$\{x^0 = t, x^1, x^2, x^3\} = \{x^0, x^\mu\}$, $\mu = 1, 2, 3$. Согласно закону всемирного тяготения тело массы M, находящееся в начале координат, создает в точке $x \in \mathbb{R}^{1,3}$ гравитационный потенциал

$$\varphi(x) = -\frac{GM}{r}, \tag{20}$$

где G – гравитационная постоянная, а $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ – расстояние от начала координат. Рассмотрим систему координат, которая связана с наблюдателями, свободно падающими на центр вдоль радиусов из бесконечно удаленной точки, где они имели нулевую скорость. Тогда из закона сохранения энергии следует, что на расстоянии r от начала координат у них будет скорость

$$v^\mu = -\sqrt{\frac{2GM}{r}}\frac{x^\mu}{r}, \tag{21}$$

где x^μ/r – компоненты единичного радиального вектора. Поскольку свободно падающие наблюдатели не чувствуют гравитационного поля, то в связанной с ними системе отсчета метрика будет совпадать с метрикой Лоренца:

$$ds^2 = c^2(dt_{\text{F}})^2 - (dx^1_{\text{F}}) - (dx^2_{\text{F}})^2 - (dx^3_{\text{F}})^2, \tag{22}$$

где c – скорость света, и индекс F обозначает, что система координат связана со свободно падающими наблюдателями.

Предположим, что свободно падающая система координат связана с декартовой системой координат в пространстве-времени, которую можно интерпретировать, как систему координат бесконечно удаленного наблюдателя, преобразованием Галилея:

$$dt_{\text{F}} = dt, \quad dx^\mu_{\text{F}} = dx^\mu + v^\mu dt. \tag{23}$$

Тогда простые вычисления приводят к следующей метрике в системе координат бесконечно удаленного наблюдателя:

$$\begin{aligned} ds^2 &= \left(c^2 - \frac{2GM}{r}\right)dt^2 + 2\sqrt{\frac{2GM}{r}}\frac{1}{r}x^\mu dx_\mu dt + dx^\mu dx_\mu = \\ &= \left(c^2 - \frac{2GM}{r}\right)dt^2 - 2\sqrt{\frac{2GM}{r}}dtdr - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) = \\ &= c^2dt^2 - \left(dr + \sqrt{\frac{2GM}{r}}dt\right)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{24}$$

где мы перешли в сферическую систему координат t, r, θ, φ. Напомним, что в наших обозначениях $x^\mu x_\mu = -(x^1)^2 - (x^2)^2 - (x^3)^2 = -r^2$.

Метрика (24) есть метрика Шварцшильда в координатах Пенлеве–Гулстранда [4, 5, 6], которые используются не столь часто, как координаты Шварцшильда. Отличительной особенностью координат Пенлеве–Гулстранда является то, что в каждый момент времени $t = \mathsf{const}$ метрика является локально евклидовой. За эту наглядность пришлось заплатить наличием в метрике недиагонального слагаемого $dtdr$.

При "выводе" метрики Шварцшильда (24) было использовано четыре предположения: закон всемирного тяготения (20); закон сохранения энергии (21); предположение о том, что в инерциальной системе координат метрика плоская (22), и "преобразование Галилея" (23). Все четыре постулата настолько привычны в нерелятивистской механике, что "вывод" решения Шварцшильда кажется безукоризненным. Однако, это не так. Мы не случайно поставили кавычки у слова "вывод", поскольку приведенные рассуждения некорректны по следующей причине. "Преобразование Галилея" (23) записано для дифференциалов, а не для самих координат и является неинтегрируемым. А именно, второе уравнение в (23) эквивалентно системе уравнений в частных производных

$$\frac{dx^\mu_{\text{F}}}{dx^\nu} = \delta^\mu_\nu,$$
$$\frac{dx^\mu_{\text{F}}}{dt} = -\sqrt{\frac{2GM}{r}}\frac{x^\mu}{r}.$$

Эта система уравнений нетривиальна, т.к. скорость v^μ нетривиально зависит от точки пространства Минковского. Легко проверить, что условия интегрируемости для этой системы уравнений не выполнены, и, следовательно, не существует таких функций $x^\mu_{\text{F}}(x,t)$, что для дифференциалов выполнены равенства (23) даже локально. В остальном "вывод" решения Шварцшильда безупречен. Отсутствие преобразования координат такого, которое переводит метрику Лоренца (22) в метрику Шварцшильда, следует также из общих соображений. Метрика Лоренца является плоской и соответствующий ей тензор кривизны равен нулю. Для метрики Шварцшильда тензор Риччи, как следствие уравнений Эйнштейна, равен нулю всюду, за исключением начала координат, где он неопределен. Полный же тензор кривизны, как хорошо известно, отличен от нуля. Следовательно, метрика Лоренца не может быть связана с метрикой Шварцшильда никаким преобразованием координат. С физической точки зрения, в "оправдание" эвристического "вывода" можно привести следующий аргумент. Вдали от притягивающего тела скорости малы, и свободно падающий наблюдатель в небольшой окрестности и в течении небольшого промежутка времени может приближенно считать, что находится в инерциальной системе координат, которая движется равномерно и прямолинейно. Тогда равенства (23) можно считать выполненными приближенно. Это показывает, что в данном случае правдоподобные рассуждения приводят к неправильному ответу.

Запишем метрику (24) в координатах Шварцшильда. Для этого достаточно ввести новую временну́ю координату:

$$T = t + \frac{2\sqrt{2GM}}{c^2}\left[\frac{\sqrt{2GM}}{c}\,\mathsf{arcth}\sqrt{\frac{2GM}{rc^2}} - \sqrt{r}\right].$$

Отсюда следует связь дифференциалов:

$$dT = dt - \frac{1}{c}\sqrt{\frac{2GM}{rc^2}}\frac{dr}{1-\frac{2GM}{rc^2}}.$$

Теперь нетрудно проверить, что в новых координатах метрика (24) совпадает с обычной метрикой Шварцшильда:

$$ds^2 = c^2\left(1-\frac{2GM}{rc^2}\right)dT^2 - \frac{dr^2}{1-\frac{2GM}{rc^2}} - r^2(d\theta^2 + \mathsf{sin}^2\theta d\varphi^2).$$

Таким образом, доказано, что метрика (24) является метрикой Шварцшильда.

Метрику Шварцшильда в координатах Пенлеве–Гулстранда (24) можно переписать в эквивалентном виде

$$ds^2 = c^2dt^2 + \eta_{\mu\nu}(dx^\mu + v^\mu dt)(x^\nu + v^\nu dt), \tag{25}$$

где компоненты скорости v^μ были определены ранее (21). В таком виде метрика Шварцшильда конформно эквивалентна эффективной метрике для распространения фононов в жидкости (19), которая была получена в предыдущем разделе. Разница заключается в том, что для эффективной метрики, скорость v_0 является решением уравнений гидродинамики (1)–(3), в то время как в решении Шварцшильда выражение для скорости v следует из уравнений Эйнштейна.

4 Заключение

В настоящей статье показано, что уравнение, описывающее распространение акустических возбуждений (фононов) в идеальной баротропной безвихревой жидкости, может быть записано в инвариантном относительно общих преобразований координат виде. Это – волновое уравнение для скалярного поля на многообразии с метрикой лоренцевой сигнатуры. Компоненты соответствующей метрики определяются полем скоростей и скоростью звука в жидкости, которые являются решением классических нерелятивистских уравнений гидродинамики. Компоненты метрики имеют особенности в тех точках, где скорость звука равна скорости течения жидкости. Эти особенности соответствуют горизонтам и, следовательно, в жидкости возможно образование аналогов черных дыр. Представляет большой интерес исследование геометрических характеристик и глобальной структуры пространства-времени для эффективной метрики, возникающей в гидродинамике.

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Quartic curvature terms in inflationary cosmology [1]

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

We consider the four-dimensional gravity with the most general fourth-order (in the spacetime curvature) terms originating from superstrings/M-theory in the leading order with respect to the Regge slope parameter, and study their impact on the evolution of the Hubble scale in the context of the FRW cosmology. We propose the generalized Friedmann equations, find the conditions on the quartic terms that allow the existence of exact solutions describing an inflationary (de Sitter) universe, and solve the constraints imposed by the scale factor duality. In particular, we rule out the on-shell-superstrings-induced gravitational effective action given by square of the Bel-Robinson tensor alone.

1 Introduction

I know Professor Joseph L'vovich Buchbinder for about 30 years since I was a student at the Physics Department of Tomsk State University in Russia. He was teaching us a course of General Relativity in 1981, and I was very impressed by both the subject and the way of his presentation. Professor I. Buchbinder was not yet a professor at that time, and we, the students, were rather afraid of Einstein theory. I remember the joke, invented by a student at that time: 'the genius is a student who was able to understand General Relativity'. Nevertheless, I do remember that all students in my group did their best to understand the lectures, also in part because it were very unusual lectures. First, they were about Physics of course, but like a course in Mathematics, with iron logics and a lot of calculations. Second, Dr. Buchbinder had his lecture notes, but he never used them during his lectures, he always started with the last equation of the previous lecture that he always remembered, and then continued his calculations in front of us, by explaining every step. It was the best lesson I had, as regards doing calculations in theoretical physics. Now I am teaching General Relativity to

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Japanese students in Tokyo, like I did it earlier in Germany, by following the same style of presentation I learned from Professor I.L. Buchbinder.

It is my pleasure to contribute a research paper to the Volume in Honor of Professor I.L. Buchbinder. I chose the title closely related to Cosmology, String Theory and General Relativity. It is fair to say that this paper grew out of the first course in General Relativity I got from Professor I.L. Buchbinder in 1981 in Tomsk.

2 Beyond Einstein theory

The General Relativity is a classical field theory of gravitation. Being a classical theory, it has its limits, and is valid only at energy scales much less than the Planck scale where quantum gravity effects become strong. Quantum gravity is needed, for example, in order to understand physics of black holes (especially near the end-point of their evaporation), and the cosmological (big bang) singularity.

At present, there is only one serious candidate for a quantum gravity theory, namely, the theory of strings and branes (M-theory) [1]. It is worthy to mention that M-theory is the theory under construction, its non-perturbative definition is lacking. Nevertheless, M-theory pretends to describe both gravity and all other gauge interactions altogether, at all energy scales, while it requires extra dimensions and supersymmetry for consistency. The predictive power of String Theory is low at present, since any testable predictions (by near future accelerator experiments) can only be based on the low-energy effective action of superstrings, which can only be derived perturbatively, while its precise form is unknown. However, this discouraging conclusion may be avoided, in principle, if the effective string (energy) scale is well below the Planck scale.

In this paper, I would like to briefly discuss an early Universe, an inflation and a cosmological singularity, by using the toy (pure) gravity model, induced by the leading quantum gravity corrections coming from M-theory, in the context of the four-dimensional FRW cosmology with a metric

$$ds^2_{\rm FRW} = -dt^2 + a^2(t)(dx_i)^2 \tag{2.1}$$

where $a(t)$ is the (FRW) scale factor, and $i = 1, 2, 3$.

To begin with, I would like to recall some well known problems in cosmology with the usual (standard) Einstein equations.

First, let me recall the famous Hawking-Penrose theorem about the impossibility of indefinite continuation of geodesics [2]. Their statement is usually interpreted as the proof of existence of a singularity in any exact solution to the Einstein equations. For instance, the standard (FRW) cosmology has an initial cosmological singularity (a big bang), which also implies that the (classical) Einstein theory is incomplete towards the initial singularity.

Second, there is a problem in describing inflation, when using the standard Einstein equations. The homogeneity and isotropy of the Universe, as well as the observed spectrum of density perturbations, are well explained by the inflationary cosmology [3], which means

$$\ddot{a}(t) > 0 \ . \tag{2.2}$$

However, the Einstein equations (in Raychaudhuri form),

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \ , \tag{2.3}$$

with the (perfect fluid matter) energy density $\rho(t)$ and the pressure $P(t)$, then imply

$$\rho + 3P = \frac{1}{4\pi G} R_{00} < 0 \ , \tag{2.4}$$

i.e. the obvious violation of the Strong Energy Condition, $R_{00} > 0$, as well as a negative pressure, $P < -\frac{1}{3}\rho < 0$. Those conditions are not necessarily fatal, because one may imagine the existence of such exotic matter in our Universe, but one would have to assume its existence and adjust all the exotic matter parameters by hand.

Inflation is usually arranged by introducing a scalar field (inflaton) and choosing its (everywhere non-negative) scalar potential [3]. Despite of the simplicity of many those inflationary scenarios, the origin of their key ingredients, such as (unobserved) inflaton and its scalar potential, remain obscure. As is well known, the Standard Model of elementary particles has no inflaton.

In recent years, many brane inflation scenarios were proposed (see e.g. ref. [4] for a review), including their embeddings into the (warped) compactified superstring models, in a good package with the phenomenological constraints coming from particle physics (see e.g. ref. [5]). However, it also did not contribute to revealing the orgin of the key ingredients of inflation. At the same time, it greatly increased the number of possibilities up to 10^{100} (known as the String Landscape) too.

The inflation driven by a scalar potential, and its engineering by strings and branes, are by no means required in string theory. There are other scenarios, such as the pre-big bang superstring cosmology [6] based on the scale factor duality (see Sec. 4). In that scenario the cosmological singularity can be avoided, while the big bang is no longer regarded as the beginning of Everything, but only as a smooth 'phase transition' between two different cosmological regimes.

Yet another possibility could be a modification of the gravitational part of Einstein equations by terms of the higher order in the spacetime curvature [7]. It does not need an inflaton or an exotic matter, while the higher-curvature terms already appear in the effective action of superstrings [1]. We would like to consider the last opportunity in this paper, in the contxet of pure gravity with quartic curvature terms.

3 Setup

The perturbative strings are merely defined on-shell (in the form of quantum amplitudes), while they give rise to the infinitely many higher-curvature corrections to the Einstein equations, to all orders in the Regge slope parameter α' and the string coupling g_s, whose finite form is unknown and is beyond our control. However, it still makes sense to consider the leading corrections to the Einstein equations, coming from strings and branes. Being valid for limited energy scales, the results to be obtained from them cannot be conclusive, but they may offer both qualitative and technical insights into cosmology, within the well defined and very restrictive framework. Our approach to inflation is based on the Einstein equations modified by the leading superstring-generated gravitational terms to be considered *on equal footing* with the Einstein terms, i.e. non-perturbatively, in four space-time dimensions. We assume that the quantum g_s-corrections can be suppressed against the leading α'-corrections, whereas all the moduli, including a dilaton and an axion, are somehow stabilized by fluxes, in a warped compactification to four dimensions (after spontaneous supersymmetry breaking).

There are five perturbatively consistent superstring models in ten spacetime dimensions (see e.g. the book [1]). All those models are related by duality transformations, while in our discussion in this paper we will only consider the gravitational sector of type-II strings. In addition, there exists a parent theory behind all those superstring models, it is called M-theory and is eleven-dimensional [1]. Though not so much is known about the non-perturbative M-theory, there are nevertheless the well-established facts that (i) the M-theory low-energy effective action is given by the 11-dimensional supergravity, and (ii) the leading quantum corrections to the 11-dimensional supergravity from M-theory in the bosonic sector are also known [1].

To match the constraints imposed by particle physics, M-theory is supposed to be compactified to one of the superstring models in ten dimensions, and then down to four spacetime dimensions e.g., on a Calabi-Yau complex three-fold [1]. Alternatively, M-theory may be directly compactified down to four real dimensions on a 7-dimensional special (G_2) holonomy manifold [8].

As regards the gravitational sector of the compactified four-dimensional type-II superstrings, it suffices to perform a warped compactification from eleven to four dimensions, with a metric

$$ds^2_{11} = e^{2A(y)} ds^2_{\rm FRW} + e^{-2A(y)} (dy_a)^2 \ , \tag{3.0}$$

where $ds^2_{\rm FRW}$ is the FRW metric (2.1) in 4-dimensional spacetime, whereas y_a, with $a = 4,5,6,7,8,9,10$, are the coordinates of the compactified 7-manifold, and $A(y)$ is the warp factor.

We put all the four-dimensional scalars (like a dilaton, an axion and moduli) into the matter stress-energy tensor (in Einstein frame), and assume that they are somehow stabilized to certain fixed values. Also, we do not consider any M-theory/superstrings solitons such as M- or D-branes. After dimensional reduction from eleven dimesnions, the only gravitational terms coming from type-II superstrings in four dimensions, are given by (see e.g. ref. [9] for details)

$$S_4 = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, (R + \beta J_R) \tag{1}$$

where we have introduced two dimensional coupling constants, κ and β, and the quartic curvature scalar J_R in four spacetime dimensions,

$$J_R = R^{mijn} R_{pijq} R_m{}^{rsp} R^q{}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R_m{}^{rsp} R^q{}_{rsn} + O(R_{mn}) \tag{2}$$

with all vector indices now belonging to four-dimensions, $i,j = 0,1,2,3$. The full curvature terms are given by the square of the Bel-Robinson tensor (see Appendix), whereas any quartic terms with at least one Ricci tensor can be added to eq. (2), just because the on-shell superstring theory does not fix those terms at all. There are about a hundred of the Ricci-dependent terms in the most general off-shell gravitational effective action quartic in the curvature. It implies a hundred of new coefficients, which makes fixing the off-shell quartic superstring effective action to be very difficult.

The gravitational action is to be added to a matter action, which lead to the *modified* Einstein equations of motion,

$$R_{ij} - \tfrac{1}{2} g_{ij} R + \beta \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ij}} \left(\sqrt{-g} J_R \right) = \kappa^2 T_{ij} \tag{3}$$

where T_{ij} stands for the energy-momentum tensor of all the matter fields (including dilaton and axion).

When all the Ricci tensor-dependent terms in eq. (2) are ignored, the modified gravitational equations of motion are given by

$$\begin{aligned}\kappa^2 T_{ij} = R_{ij} - \frac{1}{2}g_{ij}R + \beta \Big[&-\frac{1}{2}g_{ij}J_R - R_{mhk(i}R_{j)rt}{}^{m}\left(R^{kqsr}R^{t}{}_{qs}{}^{h} + R^{ksqt}R^{hr}{}_{qs}\right) \\ &- R_{kqs(i}R_{j)rmt}\left(R^{hsqt}R^{krm}{}_{k} - R^{thsq}R_{h}{}^{rmk}\right) + \left(R_{itrj}R^{ksqt}R^{h}{}_{sq}{}^{r}\right)_{(;k;h)} \\ &+ \left(R_{isqt}R^{rktm}R_{j}{}^{sq}{}_{k}\right)_{(;r;m)} - \left(R^{hrs}{}_{(i}R_{j)mnr}R_{h}{}^{mnk} + R^{sht}{}_{(i}R_{j)mnl}R^{kmn}{}_{h}\right)_{(;k;s)}\Big]\end{aligned} \tag{4}$$

4 Exact solutions, and scale factor duality

Our motivation is based on the observation that the Standard Model (SM) of elementary particles does not have an inflaton. M-theory/superstrings have plenty of inflaton candidates but any inflationary mechanism based on a scalar field is highly model-dependent. When one wants the universal geometrical mechanism of inflation based on gravity only, it should occur due to some Planck scale physics to be described by the higher curvature terms (cf. ref. [7]).

On the experimental side, it is known that the vacuum energy density $\rho_{\rm inf}$ during inflation is bounded from above by a (non)observation of tensor fluctuations of the Cosmic Microwave Background (CMB) radiation [10],

$$\rho_{\rm inf} \leq \left(10^{-3}M_{\rm Pl}\right)^4 \tag{5}$$

It severly constrains but does not exclude the possibility of the geometrical inflation originating from the purely gravitational sector of string theory, because the factor of 10^{-3} above may be just due to some numerical coefficients.

Due to a single arbitrary function $a(t)$ in the FRW metric, it is enough to take only one gravitational equation of motion in eq. (3) without matter, namely, its mixed 00-component. As is well known [3], the spatial (3-dimensional) curvature can be ignored in a very early universe, so we choose the manifestly conformally-flat FRW metric (2.1). It leads to a purely gravitational equation of motion having the form

$$3H^2 \equiv 3\left(\frac{\dot{a}}{a}\right)^2 = \beta P_8\left(\frac{\dot{a}}{a},\frac{\ddot{a}}{a},\frac{\dddot{a}}{a},\frac{\ddddot{a}}{a}\right) , \tag{6}$$

where P_8 is a *polynomial* with respect to its arguments,

$$P_8 = \sum_{\substack{n_1+2n_2+3n_3+4n_4=8,\\ n_1,n_2,n_3,n_4\geq 0}} c_{n_1n_2n_3n_4}\left(\frac{\dot{a}}{a}\right)^{n_1}\left(\frac{\ddot{a}}{a}\right)^{n_2}\left(\frac{\dddot{a}}{a}\right)^{n_3}\left(\frac{\ddddot{a}}{a}\right)^{n_4} \tag{7}$$

Here the sum goes over the *integer* partitions $(n_1, 2n_2, 3n_3, 4n_4)$ of 8, the dots stand for the derivatives with respect to time t, and $c_{n_1n_2n_3n_4}$ are some real coefficients. The highest derivative can enter only linearly, $n_4 = 0, 1$.

Equation (6) is our generalized Friedmann equation that we are going to apply for describing a very early universe.

The FRW Ansatz yields the non-vanishing curvatures [11]

$$R^0{}_{i0j} = \delta_{ij}\ \ddot{a}\ a, \quad R^i{}_{jkl} = \left(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk}\right)(\dot{a})^2, \quad R^i_j = -\delta^i_j\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right] \tag{8}$$

where $i, j = 1, 2, 3$. For example, in the case of the $(BR)^2$ gravity (3.4), after a straightforward (though quite tedious) calculation of the mixed 00-equation with the curvatures (8) and no matter, we find

$$3H^2 + \beta \left[9 \left(\frac{\ddot{a}}{a} \right)^4 - 36H^2 \left(\frac{\ddot{a}}{a} \right)^3 + 84H^4 \left(\frac{\ddot{a}}{a} \right)^2 - 36H \left(\frac{\ddot{a}}{a} \right)^2 \left(\frac{\dddot{a}}{a} \right) \right.$$
$$\left. +63H^8 - 72H^3 \left(\frac{\ddot{a}}{a} \right) \left(\frac{\dddot{a}}{a} \right) + 48H^6 \left(\frac{\ddot{a}}{a} \right) - 24H^5 \left(\frac{\dddot{a}}{a} \right) \right] = 0 \tag{9}$$

It is remarkable that the 4th order time derivatives (present in various terms of eq. (4)) cancel, whereas the square of the 3rd order time derivative of the scale factor, $\dddot{a}^{\,2}$, does not appear at all in this equation. [3]

Our generalized Friedmann equation (6) applies to *any* combination of the quartic curvature terms in the action, including the Ricci-dependent terms. The coefficients $c_{n_1 n_2 n_3 n_4}$ in eq. (7) can be thought of as linear combinations of the coefficients in the most general quartic curvature action. The polynomial (7) has just about ten undetermined coefficients, which is much less than a hundred coefficients in the most general quartic curvature action.

Moreover, eqs. (6) and (7) have the particular structure that allows the existence of generic exact inflationary solutions without a spacetime singularity. Indeed, when using the most naive (de Sitter) Ansatz for the scale factor,

$$a(t) = a_0 e^{ht} \quad , \tag{10}$$

with some real positive constants a_0 and h, and substituting eq. (10) into eq. (6) we get $3h^2 = (\#)\beta h^8$, whose coefficient $(\#)$ is just a sum of all c-coefficients in eq. (7). Assuming the $(\#)$ to be positive, we find a solution for the effective Hubble constant h,

$$h = \left(\frac{3}{\#\beta} \right)^{1/6} \tag{11}$$

This solution in non-perturbative in β, i.e. it is impossible to get it when considering the quartic curvature terms as a perturbation. Of course, the assumption that we are dealing with the leading correction, implies $ht \ll 1$. It leads to the natural hierarchy

$$\kappa M_{\rm KK} \ll 1 \quad \text{or} \quad l_{\rm Pl} \ll l_{\rm KK} \tag{12}$$

where we have introduced the four-dimensional Planck scale $l_{\rm Pl} = \kappa$ and the compactification scale $l_{\rm KK} = M_{\rm KK}^{-1}$.

The exact solution (10) is non-singular, while (when $h > 0$) it describes an inflationary isotropic and homogeneous early universe. [4] Once the universe expands, the curvatures decrease, so that the higher curvature terms cease to be the dominant contributions against the matter terms we ignored in the equations of motion. The matter terms may provide a mechanism for ending the geometrical inflation and reheating (i.e. a Graceful Exit to the standard cosmology).

It remains to be investigated how string theory affects the coefficients in our generalized Friedmann equation. Here we apply the scale factor duality argument [13] by requiring our

[3] Taking Weyl tensors instead of Riemann curvatures leads to all vanishing coefficients.
[4] Similar exact de Sitter solutions were also found in ref. [12].

equation (6) to be invariant under the duality transformation

$$a(t) \leftrightarrow \frac{1}{a(t)} \equiv b(t) \tag{13}$$

It is straightforward to calculate how the right-hand-side of eq. (6) transforms under the duality by differentiating eq. (13) first. We find

$$\begin{aligned} \frac{\dot{a}}{a} &= -\frac{\dot{b}}{b} \ , \quad \frac{\ddot{a}}{a} = -\frac{\ddot{b}}{b} + 2\left(\frac{\dot{b}}{b}\right)^2 \ , \\ \frac{\dddot{a}}{a} &= -\frac{\dddot{b}}{b} + 6\left(\frac{\dot{b}}{b}\right)\left(\frac{\ddot{b}}{b}\right) - 6\left(\frac{\dot{b}}{b}\right)^3 \ , \\ \frac{\ddddot{a}}{a} &= -\frac{\ddddot{b}}{b} + 6\left(\frac{\ddot{b}}{b}\right)^2 + 8\left(\frac{\dot{b}}{b}\right)\left(\frac{\dddot{b}}{b}\right) - 36\left(\frac{\dot{b}}{b}\right)^2\left(\frac{\ddot{b}}{b}\right) + 24\left(\frac{\dot{b}}{b}\right)^4 \end{aligned} \tag{14}$$

In order to see how the scale factor duality affects the polynomial P_8, we consider the case with the 3rd order time derivatives, motivated by eq. (9). We introduce the notation

$$\frac{\dot{a}}{a} = x \ , \quad \frac{\ddot{a}}{a} = y \ , \quad \frac{\dddot{a}}{a} = z \tag{15}$$

so that the duality invariance condition reads

$$P_8(-x, 2x^2 - y, 6xy - 6x^3 - z) = P_8(x, y, z) \tag{16}$$

The structure of the polynomial P_8 in eq. (7), as the sum over partitions of 8, restricts a solution to eq. (16) to be most quadratic in z,

$$P_8(x, y, z) = a_2(x, y)z^2 + b_5(x, y)z + c_8(x, y) \tag{17}$$

whose coefficients are polynomials in (x, y), of the order being given by their subscripts, i.e.

$$\begin{aligned} a_2(x, y) &= a_0x^2 + a_1y \ , \\ b_5(x, y) &= b_0x^5 + b_1x^3y + b_2xy^2 \ , \\ c_8(x, y) &= c_4y^4 + c_3y^3x^2 + c_2y^2x^4 + c_1yx^6 + c_0x^8 \end{aligned} \tag{18}$$

After a substitution of eqs. (17) and (6.14) into eq. (16), we get an *over*determined system of linear equations on the coefficients. Nevertheless, we find that there is a consistent general solution,

$$\begin{aligned} P_8(x, y, z) = {} & a_0x^2z^2 + (b_0x^5 - 3a_0xy^2)z \\ & + c_4y^4 + (9a_0 - 4c_4)y^3x^2 + c_2y^23x^4 \\ & + (8c_4 - 18a_0 - 3b_0 - 2c_2)yx^6 + c_0x^8 \end{aligned} \tag{19}$$

parameterized by merely five real coefficients $(a_0, b_0, c_4, c_2, c_0)$. Requiring the existence of the exact solution (10), i.e. the positivity of (#) in eq. (11), yields

$$5c_4 + c_0 > 11a_0 + 2b_0 + c_2 \tag{20}$$

Further constraints are needed in order to restrict the values of the undetermined coefficients. As regards the 'minimal' $(BR)^2$, we found that neither the duality condition nor the inequality (20) are satisfied by the coefficients present in eq. (9). We interpret it as the clear indications that some additional Ricci-dependent terms *have to be added* to the $(BR)^2$ terms or, equivalently, the $(BR)^2$ gravity is ruled out as the off-shell effective action for superstrings.

To the end of this section, we would like to mention about some possible simplifications and generalizations.

The last equation (8) apparently implies that the Ricci-dependent terms in P_8 should have the factor of $(y + 2x^2)$. Hence, it may be possible to completely eliminate both the 4th and 3rd order time derivatives in our generalized Friedmann equations, though we are not sure that this choice is fully consistent. However, if so, instead of eq. (16) we would get another duality condition,

$$P_8(-x, 2x^2 - y) = P_8(x, y) \tag{21}$$

whose most general solution is simpler,

$$P_8(x, y) = c_0 x^8 + c_5 y(y - 2x^2)\left[y(y - 2x^2) - 4x^6\right] + c_6 x^4 y(y - 2x^2) \tag{22}$$

with merely three, yet to be determined coefficients (c_0, c_5, c_6).

We would like to emphasize that our results above can be generalized to any finite order with respect to the spacetime curvatures in the off-shell superstring effective action, because it amounts to increasing the order of the polynomial P. We may even speculate about the form of the generalized Friedmann equation to *all* orders in the curvature. It depends upon whether (i) there will be some finite maximal order of the time derivatives there, or (ii) the time derivatives of arbitrarily high order appear (we do not know about it). Given the case (i), we just drop the requirement that the right-hand-side of our cosmological equation (6) is a polynomial, and take a duality-invariant *function* P instead. In the case (ii), we should replace the function by a *functional*, thus getting a non-local equation having the form

$$H^2 = \frac{\dot{a}^2}{a^2} = \beta P[a(t)] \tag{23}$$

whose functional P is subject to the non-trivial duality constraint

$$P[a(t)] = P[1/a(t)] \ . \tag{24}$$

5 Conclusion

The higher curvature terms in the gravitational action defy the Hawking-Penrose theorem about the existence of a spacetime singularity in any exact solution to the Einstein equations. As we demonstrated in this paper, the initial cosmological singularity can be easily avoided by condsidering the superstring-motivated higher curvature terms on equal footing (i.e. non-perturbatively) with the Einstein-Hilbert term.

As regards inflation, though we showed the natural existence of inflationary (de Sitter) exact solutions without a spacetime singularity under rather generic conditions on the coefficients in the higher-derivative terms, it is by no means sufficient, because our geometrical inflation has no end. In fact, we assumed the dominance of the higher curvature gravitational terms over all matter contributions in a very early universe at the Planck scale. However, given the expansion of the universe under the geometrical inflation, the spacetime curvatures would decrease, so that the matter terms could no longer be ignored. The latter may effectively replace the geometrical inflation by another matter-dominated mechanism, allowing the inflation to continue substantially below the Planck scale. Needless to say, more research is needed to submit a specific mechanism for that.

The higher curvature terms are also relevant for the alternative (to inflation) Brandenberger-Vafa scenario of string gas cosmology [14] — see e.g. ref. [15] for a recent investigation of the higher curvature corrections there. [5]

Appendix: Bel-Robinson tensor

The detailed structure and physical meaning of the *full* curvature terms in eq. (2) are easily revealed via their connection to the four-dimensional *Bel-Robinson* (BR) tensor [16]. The latter is well known in general relativity [17]. We review here the main on-shell properties of the BR tensor. The BR tensor is defined by

$$T_R^{iklm} = R^{ipql}R^k{}_{pq}{}^m + {}^*R^{ipql*}R^k{}_{pq}{}^m \tag{25}$$

whose structure is quite similar to that of the Maxwell stress-energy tensor,

$$T_{ij}^{\rm Maxwell} = F_{ik}F_j{}^k + {}^*F_{ik}{}^*F_j{}^k \quad , \quad F_{ij} = \partial_i A_j - \partial_j A_i \tag{26}$$

First, we quote some identities [9] valid on-shell, i.e modulo Ricci tensor dependent terms,

$$T_{ijkl}^2 = 8J_R = \tfrac{1}{4}(R_{ijkl}^2)^2 + \tfrac{1}{4}({}^*R_{ijkl}R^{ijkl})^2 \tag{27}$$

in a slightly abused notation (all contractions of indices are covariant). We also find

$$\begin{aligned} T_{ijkl}^2 = 8J_R &= -\tfrac{1}{4}({}^*R_{ijkl}^2)^2 + \tfrac{1}{4}({}^*R_{ijkl}R^{ijkl})^2 \\ &= \tfrac{1}{4}(P_4^2 - E_4^2) = \tfrac{1}{4}(P_4 + E_4)(P_4 - E_4) \end{aligned} \tag{28}$$

where we have introduced the Euler and Pontryagin topological densities in four dimensions.

In addition [16, 17], the on-shell BR tensor is *fully symmetric* with respect to its vector indices, and is *traceless*,

$$T_{ijkl} = T_{(ijkl)} \quad , \qquad T^i_{ikl} = 0 \quad , \tag{29}$$

(ii) it is covariantly *conserved* (though the BR tensor is not a physical current!),

$$\nabla^i T_{ijkl} = 0 \quad , \tag{30}$$

and it has *positive* 'energy' density,

$$T_{0000} > 0 \quad . \tag{31}$$

The BR tensor is related to the gravitational energy-momentum *pseudo*-tensors [17]. It can be most clearly seen in *Riemann Normal Coordinates* (RNC) at any *given* point in spacetime. The RNC are defined by the relations

$$g_{ij} = \eta_{ij} \ , \quad g_{ij,k} = 0 \ , \quad g_{ij,mn} = -\tfrac{1}{3}(R_{imjn} + R_{injm}) \tag{32}$$

so that the derivatives of Christoffel symbols read as follows:

$$\Gamma^i_{jk,l} = -\tfrac{1}{3}(R^i{}_{jkl} + R^i{}_{kjl}) \tag{33}$$

[5] The higher curvature terms were considered only perturbatively in ref. [15].

Raising and lowering of vector indices in RNC are performed with Minkowski metric η_{ij} and its inverse η^{ij}, whereas all traces in the last two eqs. (32) and (33) vanish,

$$\eta^{ij}g_{ij,mn} = \eta^{ij}\Gamma^k_{ij,l} = \Gamma^i_{ij,k} = \Gamma^i_{jk,i} = 0 \tag{34}$$

Moreover, there exists the remarkable non-covariant relation (valid only in RNC) [17]

$$T_{ijkl} = \partial_k\partial_l \left(t^{LL}_{ij} + \tfrac{1}{2} t^E_{ij} \right) \tag{35}$$

where the symmetric *Landau-Lifshitz* (LL) gravitational pseudo-tensor [18]

$$\begin{aligned}(t_{LL})^{ij} = &- \eta^{ip}\eta^{jq}\Gamma^k_{pm}\Gamma^m_{qk} + \Gamma^i_{mn}\Gamma^j_{pq}\eta^{mp}\eta^{nq} - \left(\Gamma^m_{np}\Gamma^j_{mq}\eta^{in}\eta^{pq} + \Gamma^m_{np}\Gamma^i_{mq}\eta^{jn}\eta^{pq}\right) \\ &+ h^{ij}\Gamma^m_{np}\Gamma^n_{mq}\eta^{pq}\end{aligned} \tag{36}$$

and the non-symmetric *Einstein* (E) (or canonical) gravitational pseudo-tensor [19]

$$(t^E)^i_j = \left(-2\Gamma^i_{mp}\Gamma^m_{jq} + \delta^i_j\Gamma^n_{pm}\Gamma^m_{qn}\right)\eta^{pq} \tag{37}$$

have been introduced in RNC, in terms of the Christoffel symbols.

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Electromagnetic radiation in even-dimensional spacetimes

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

The basic concepts and mathematical constructions of the Maxwell–Lorentz electrodynamics in flat spacetime of an arbitrary even dimension $d = 2n$ are briefly reviewed. We show that the retarded field strength $\mathcal{F}_{\mu\nu}^{(2n)}$ due to a point charge living in a $2n$-dimensional world can be algebraically expressed in terms of the retarded vector potentials $\mathcal{A}_{\mu}^{(2m)}$ generated by this charge as if it were accommodated in $2m$-dimensional worlds nearby, $2 \le m \le n+1$. With this finding, the rate of radiated energy-momentum of the electromagnetic field takes a compact form.

1 Introduction

This paper is dedicated to Professor Iosif Buchbinder in celebration of his sixtieth birthday. A marvellous feature of my friend Iosif is his ability to grasp the essence of a challenging problem in theoretical physics and interpret it quite plainly. He makes a major effort to attain the greatest possible clarity in a complex subject. The analysis of the Maxwell–Lorentz electrodynamics in even-dimensional spacetimes presented in this paper will hopefully be found to be made in the same vein.

The physics in higher spacetime dimensions is of basic current interest. String-inspired large extra-dimensional models [1], [2], [3] and braneworld scenarios [4] [5], [6], [7], [8] (for a review see [9], [10]) offer promise for a better understanding of a rich variety of high-energy phenomena which is expected to be discovered at the Large Hadron Collider at CERN, and other coming into service colliders. Our main concern in this paper is with the concept of radiation in higher-dimensional classical electrodynamics. A further refinement of this concept is needed if we are to gain a more penetrating insight into the self-interaction problem [11], [12], [13]. Recently, this problem was addressed in Refs. [14], [15], [16], [17] [18], [19]. It should be stressed that spacetime manifolds in these papers were assumed to be flat. (Conceivably it might be prematurely to embark on a study of the radiation in curved manifolds until the energy-momentum problem in general relativity is completely solved.) We also note that the idea of radiation in odd-dimensional worlds falls far short of being clear-cut because Huygens's principle fails in odd spacetime dimensions [20], and the same is true for massive vector fields. Consequently, we will restrict our consideration to classical electrodynamics in Minkowski spacetime of even dimension $d = 2n$.

The paper is organized as follows. Section 2 outlines the state of the art of the $2n$-dimensional Maxwell–Lorentz theory, notably the methods for solving Maxwell's equations with the source composed of a single point charge. A central result of this section is given by equations (40)–(44) suggesting that the retarded field strength $\mathcal{F}^{(2n)}_{\mu\nu}$ due to a point charge living in a $2n$-dimensional world can be algebraically expressed in terms of the retarded vector potentials $\mathcal{A}^{(2m)}_{\mu}$ generated by this charge as if it were accommodated in $2m$-dimensional worlds nearby, with m being within the limits $2 \le m \le n+1$. It is then shown in Sec. 3 that the rate of radiated energy-momentum of the electromagnetic field in $2n$-dimensional spacetime takes a compact form, Eqs. (72) and (73). Some implications of these results are discussed in Sec. 4.

We adopt the metric of the form $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, \dots, -1)$, and follow the conventions of Ref. [12] throughout.

2 Vector potentials, prepotentials, and field strengths

Consider a single charged point particle moving along a timelike world line in flat spacetime of an arbitrary even dimension $d = 2n$, $n = 1, 2, \dots$ With reference to the aforementioned string-inspired models and braneworld scenarios, our prime interest is with d in the range from $d = 2$ to $d = 10$. The world line $z^\mu(s)$ is regarded as a smooth function of the proper time s. Suppose that the Maxwell–Lorentz electrodynamics is still valid. This is tantamount to stating that the field sector is given by

$$\mathcal{L} = -\frac{1}{4\Omega_{d-2}}\, F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu, \tag{1}$$

$$j^\mu(x) = e\int_{-\infty}^{\infty} ds\, v^\mu(s)\, \delta^d\left[x - z(s)\right], \tag{2}$$

and the retarded boundary condition is imposed on the vector potential A_μ. Here, Ω_{d-2} is the area of the unit $(d-2)$-sphere, $v^\mu = \dot z^\mu = dz^\mu/ds$ is the d-velocity, and $\delta^d(R)$ is the d-dimensional Dirac delta-function. In what follows the value of the charge will be taken to be unit, $e = 1$.

The field equation resulting from (1) reads

$$\mathcal{E}^\mu = \partial_\nu F^{\mu\nu} + \Omega_{d-2} j^\mu = 0. \tag{3}$$

This is accompanied by the Bianchi identity

$$\mathcal{E}^{\lambda\mu\nu} = \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0. \tag{4}$$

We take the general solution to (4), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and choose the Lorenz gauge condition $\partial_\mu A^\mu = 0$ to put (3) into the form

$$\Box\, A^\mu = \Omega_{d-2}\, j^\mu. \tag{5}$$

There are two alternative procedures for integrating the wave equation (5). The Green's function approach holds much favor (for a review see [20], [12]). The retarded Green's function satisfying

$$\Box\, G_{\rm ret}(x) = \delta^d(x) \tag{6}$$

is given by

$$G_{\rm ret}(x) = \tfrac{1}{2\pi^n}\,\theta(x_0)\,\delta^{(n-1)}(x^2), \quad d = 2n. \tag{7}$$

Here, $\delta^{(n-1)}(x^2)$ is the delta-function differentiated $n-1$ times with respect to its argument. With the Green's function (7) at our disposal it becomes possible to obtain the retarded vector potential

$$A^\mu(x) = \Omega_{d-2}\int_{-\infty}^{\infty} ds\, G_{\rm ret}\,(R)\, v^\mu(s), \tag{8}$$

where

$$R^\mu = x^\mu - z^\mu(s) \tag{9}$$

is the null four-vector drawn from the retarded point $z^\mu(s)$ on the world line, where the signal is emitted, to the point x^μ, where the signal is received.

The other procedure consists of using the ansatz of a particular form [14], [12]. To illustrate, the pertinent ansätze for $d = 2, 4, 6$ are given, respectively, by

$$A^{(2)}_\mu = \alpha(\rho)\, R_\mu, \tag{10}$$

$$A^{(4)}_\mu = f(\rho)\, R_\mu + g(\rho)\, v_\mu, \tag{11}$$

$$A^{(4)}_\mu = \Omega(\rho,\lambda)\, R_\mu + \Phi(\rho,\lambda)\, v_\mu + \Psi(\rho,\lambda)\, a_\mu. \tag{12}$$

Here, $a_\mu = \dot v_\mu$ is the d-acceleration, and α, f, g, Ω, Φ, Ψ are unknown scalar functions. The functions α, f, g are assumed to depend on the retarded invariant distance

$$\rho = R\cdot v, \tag{13}$$

while Ω, Φ, Ψ are taken to depend on ρ and the retarded invariant variable

$$\lambda = R\cdot a - 1. \tag{14}$$

Let us introduce a further null vector c_μ aligned with R_μ,

$$R_\mu = \rho\, c_\mu. \tag{15}$$

We insert any one of the ansätze (10), (11), (12) in (3), perform differentiations of the retarded variables using the rules

$$\partial_\mu s = c_\mu, \tag{16}$$

$$\partial_\mu \rho = v_\mu + \lambda c_\mu, \tag{17}$$

$$\partial_\mu R^\lambda = \delta^\lambda{}_\mu - v^\lambda\, c_\mu, \tag{18}$$

and solve the resulting ordinary differential equations (for detail see [14], [12]) to obtain

$$A^{(2)}_\mu = -R_\mu, \tag{19}$$

$$F^{(2)}_{\mu\nu} = c_\mu v_\nu - c_\nu v_\mu, \tag{20}$$

$$A^{(4)}_\mu = \tfrac{v_\mu}{\rho}\,, \tag{21}$$

$$F^{(4)}_{\mu\nu} = c_\mu U^{(4)}_\nu - c_\nu U^{(4)}_\mu, \quad U^{(4)}_\mu = -\lambda \,\tfrac{v_\mu}{\rho^2} + \tfrac{a_\mu}{\rho}\,, \tag{22}$$

$$A^{(6)}_\mu = \tfrac{1}{3}\left(-\lambda \,\tfrac{v_\mu}{\rho^3} + \tfrac{a_\mu}{\rho^2}\right)\,, \tag{23}$$

$$F^{(6)}_{\mu\nu} = \tfrac{1}{3}\left(c_\mu U^{(6)}_\nu - c_\nu U^{(6)}_\mu + \tfrac{a_\mu v_\nu - a_\nu v_\mu}{\rho^3}\right), \quad U^{(6)}_\mu = \left[3\lambda^2 - \rho^2\,(\dot a\cdot c)\right]\tfrac{v_\mu}{\rho^4} - 3\lambda\,\tfrac{a_\mu}{\rho^3} + \tfrac{\dot a_\mu}{\rho^2}. \tag{24}$$

Note the overall factor $\frac{1}{3}$ in (23). The origin of this numerical factor is most easily understood if we apply Gauss' law to the case that $a_\mu = 0$ and $\dot a_\mu = 0$. To simplify our notations as much as possible, we introduce the *net* vector potentials and field strengths, $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ (as opposed to the ordinary vector potentials and field strengths, A_μ and $F_{\mu\nu}$, whose normalization is consistent with Gauss' law):

$$A^{(2p)}_\mu = N_p^{-1}\mathcal{A}^{(2p)}_\mu, \quad F^{(2p)}_{\mu\nu} = N_p^{-1}\mathcal{F}^{(2p)}_{\mu\nu}, \tag{25}$$

where

$$N_p = (p-1)!!\,. \tag{26}$$

It is an easy matter to extend the sequence of the ansätze shown in (10), (11), (12) to any $d = 2n$ with integer $n \ge 1$. Based on the anzatz for $d = 2n$, we come to the anzatz for $d = 2n+2$ by appending a term proportional to the $(n-1)$th derivative of v_μ with respect to s, and assuming that the unknown functions depend on ρ, together with scalar products of R_μ and derivatives of v_μ up to the $(n-1)$th derivative inclusive.

Proceeding in these lines, we get

$$\mathcal{A}^{(2)}_\mu = -R_\mu, \tag{27}$$

$$\mathcal{A}^{(4)}_\mu = \tfrac{v_\mu}{\rho}\,, \tag{28}$$

$$\mathcal{A}^{(6)}_\mu = -\lambda\,\tfrac{v_\mu}{\rho^3} + \tfrac{a_\mu}{\rho^2}\,, \tag{29}$$

$$\mathcal{A}^{(8)}_\mu = \left[3\lambda^2 - \rho^2\,(\dot a\cdot c)\right]\tfrac{v_\mu}{\rho^5} - 3\lambda\,\tfrac{a_\mu}{\rho^4} + \tfrac{\dot a_\mu}{\rho^3}\,, \tag{30}$$

$$\begin{aligned}
\mathcal{A}^{(10)}_\mu &= \left[-15\lambda^3 + 10\lambda\rho^2\,(\dot a\cdot c) - \rho^2 a^2 - \rho^3\,(\ddot a\cdot c)\right]\tfrac{v_\mu}{\rho^7} + \left[15\lambda^2 - 4\rho^2\,(\dot a\cdot c)\right]\tfrac{a_\mu}{\rho^6}\\
&- 6\lambda\,\tfrac{\dot a_\mu}{\rho^5} + \tfrac{\ddot a_\mu}{\rho^4}\,,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\mathcal{A}^{(12)}_\mu &= \left\{105\lambda^2\left[\lambda^2 - \rho^2(\dot a\cdot c)\right] + 15\lambda\rho^2[\rho(\ddot a\cdot c) + a^2] - \tfrac{5}{2}\,\rho^3(a^2)^{\cdot} - \rho^4(\dddot a\cdot c)\right.\\
&+ \left.10\rho^4(\dot a\cdot c)^2\right\}\tfrac{v_\mu}{\rho^9} + 5\left\{3\lambda\left[-7\lambda^2 + 4\rho^2(\dot a\cdot c)\right] - \rho^2[\rho(\ddot a\cdot c) + a^2]\right\}\tfrac{a_\mu}{\rho^8}\\
&+ 5\left[9\lambda^2 - 2\rho^2(\dot a\cdot c)\right]\tfrac{\dot a_\mu}{\rho^7} - 10\lambda\,\tfrac{\ddot a_\mu}{\rho^6} + \tfrac{\dddot a_\mu}{\rho^5}\,.
\end{aligned} \tag{32}$$

Another way of looking at $\mathcal{A}^{(2p)}_\mu$ is to invoke the notion of prepotential. The prepotential H_μ of the vector potential A_μ is defined as

$$A_\mu = \Box\, H_\mu. \tag{33}$$

One can check that

$$N_{p+1}\,\Box\, A^{(2p)}_\mu = (d-2p)\,N_p\,A^{(2p+2)}_\mu, \quad p \ge 1. \tag{34}$$

In other words, any $2n$-dimensional retarded vector potential $A^{(2n)}_\mu$ (up to a normalization factor) is the prepotential of the $(2n+2)$-dimensional retarded vector potential $A^{(2n+2)}_\mu$. Furthermore, $\mathcal{A}^{(2p)}_\mu$ can be produced by acting on $\mathcal{A}^{(2)}_\mu$ $p-1$ times with the wave operator:

$$\mathcal{A}^{(2p)}_\mu = Z^{-1}_{d,p}\,\Box^{p-1}R_\mu = -Z^{-1}_{d,p}\,\Box^{p-1}\mathcal{A}^{(2)}_\mu, \tag{35}$$

where

$$Z_{d,p} = (d-2)\,(d-4)\cdots(d-2p)\,N_p = \tfrac{2^p(n-1)!}{(n-p-1)!}\,N_p. \tag{36}$$

All the resulting vector potentials $\mathcal{A}^{(2p)}_\mu$, beginning with $p=2$, obey the Lorenz gauge condition. To see this, we note that $\partial^\mu R_\mu = d-1$, and so $\Box\,\partial^\mu\mathcal{A}^{(2)}_\mu = 0$.

This technique provides a further significant advantage if we observe that the action of the wave operator amounts to the action of the first-order differential operator

$$\tfrac{1}{\rho}\,\tfrac{d}{ds}\,. \tag{37}$$

We thus have

$$\mathcal{A}^{(2p)}_\mu = -\left(\tfrac{1}{\rho}\,\tfrac{d}{ds}\right)^{p-1}\mathcal{A}^{(2)}_\mu. \tag{38}$$

Indeed, (38) derives from (7) and (8) by noting that $dR^2/ds = -2\rho$, $dR_\mu/ds = -v_\mu$, and so

$$-\tfrac{1}{\rho}\,\tfrac{d}{ds}\,\mathcal{A}^{(2)}_\mu = \tfrac{1}{\rho}\,\tfrac{d}{ds}\,R_\mu = \tfrac{v_\mu}{\rho} = \mathcal{A}^{(4)}_\mu. \tag{39}$$

We now take a closer look at the field strengths $F^{(2)}_{\mu\nu}$, $F^{(4)}_{\mu\nu}$, and $F^{(6)}_{\mu\nu}$ shown, respectively, in (20), (22), and (24). When their structure is compared with that of the vector potentials $\mathcal{A}^{(2)}_\mu$, $\mathcal{A}^{(4)}_\mu$, $\mathcal{A}^{(6)}_\mu$, $\mathcal{A}^{(8)}_\mu$ displayed in (27)–(30), it is apparent that

$$\mathcal{F}^{(2)} = -\mathcal{A}^{(2)}\wedge\mathcal{A}^{(4)}, \tag{40}$$

$$\mathcal{F}^{(4)} = -\mathcal{A}^{(2)}\wedge\mathcal{A}^{(6)}, \tag{41}$$

$$\mathcal{F}^{(6)} = -\mathcal{A}^{(2)}\wedge\mathcal{A}^{(8)} - \mathcal{A}^{(4)}\wedge\mathcal{A}^{(6)}. \tag{42}$$

In addition, one can verify that

$$\mathcal{F}^{(8)} = -\mathcal{A}^{(2)}\wedge\mathcal{A}^{(10)} - 2\mathcal{A}^{(4)}\wedge\mathcal{A}^{(8)}, \tag{43}$$

$$\mathcal{F}^{(10)} = -\mathcal{A}^{(2)}\wedge\mathcal{A}^{(12)} - 3\mathcal{A}^{(4)}\wedge\mathcal{A}^{(10)} - 2\mathcal{A}^{(6)}\wedge\mathcal{A}^{(8)}. \tag{44}$$

We come to recognize that the retarded field strength $\mathcal{F}^{(2p)}_{\mu\nu}$ can be expressed in a very compact and elegant form in terms of retarded vector potentials $\mathcal{A}^{(2m)}_\mu$, $2\le m\le p+1$. Recall, the canonical representation of a general 2-form $\omega^{(2n)}$ in spacetime of dimension $d=2n$ is the sum of n exterior products of 1-forms:

$$\omega^{(2n)} = f_1\wedge f_2 + \cdots + f_{2n-1}\wedge f_{2n}. \tag{45}$$

In particular, by (45), $\omega^{(10)}$ is decomposed into the sum involving five terms. However, (44) shows that the retarded field strength contains only three exterior products, two less than the canonical representation.

The validity of relations (40)–(44) can be seen by inspection. To derive them in a regular way, we take (40) as the starting point. If we apply $Z^{-1}_{d,p}\,\Box^{p-1}$ to the left-hand side of this equation, then, in view of (35), we obtain $\mathcal{F}^{(2p)}$. Applying $p-1$ times the first-order differential operator (37) to the right-hand side of (40) and taking into account Leibnitz's rule for differentiation of the product of two functions, in view of (38), we come to the desired result.

This explains the puzzling fact that the gauge-independent quantity $\mathcal{F}^{(2p)}$ is an algebraic function of gauge-dependent quantities $\mathcal{A}^{(2m)}$. By the construction, the vector potentials $\mathcal{A}^{(2m)}_\mu$, $m\ge 1$, are subject to the Lorenz gauge condition. Therefore, such $\mathcal{A}^{(2m)}_\mu$ leave room for gauge modes $\partial_\mu\chi$ with χ being solutions to the wave equation, $\Box\chi=0$. In our derivation of

(41)–(44), we are entitled to apply the wave operator $\Box$, rather than the first-order differential operator (37), to the right-hand side of (40). All feasible gauge modes are then killed by the action of $\Box$.

We close this section with a remark about the behavior of the retarded electromagnetic field at spatial infinity. In general, $\mathcal{F}^{(2n)}$ can be represented as the sum of exterior products of retarded vector potentials $\mathcal{A}^{(2p)} \wedge \mathcal{A}^{(2n-2p+4)}$. It is easy to understand that the infrared properties of $\mathcal{F}^{(2n)}$ are controlled by the term $\mathcal{A}^{(2)} \wedge \mathcal{A}^{(2n+2)}$. (In fact, a comparison of the long-distance behavior of $\mathcal{A}^{(2)} \wedge \mathcal{A}^{(2n+2)}$ and $\mathcal{A}^{(4)} \wedge \mathcal{A}^{(2n)}$ will suffice for the present purposes. Since the least falling terms of $\mathcal{A}^{(2n+2)}$ and $\mathcal{A}^{(2n)}$ scale, respectively, as ρ^{-n} and ρ^{1-n}, the leading long-distance asymptotics of $\mathcal{A}^{(2)} \wedge \mathcal{A}^{(2n+2)}$ is given by ρ^{1-n} while that of $\mathcal{A}^{(4)} \wedge \mathcal{A}^{(2n)}$ is given by ρ^{-n}.) We segregate in $\mathcal{A}^{(2n+2)}$ the term scaling as ρ^{-n} by introducing the vectors

$$\mathfrak{b}_\mu^{(2n+2)} = \lim_{\rho\to\infty} \rho^n \mathcal{A}_\mu^{(2n+2)} \tag{46}$$

and

$$\bar{\mathcal{A}}_\mu^{(2n+2)} = \tfrac{1}{\rho^n}\, \mathfrak{b}_\mu^{(2n+2)}. \tag{47}$$

All infrared irrelevant terms are erased by this limiting procedure, so that

$$\mathcal{A}^{(2)} \wedge \bar{\mathcal{A}}^{(2n+2)} \tag{48}$$

represents the infrared part of $\mathcal{F}^{(2n)}$.

We write explicitly $\mathfrak{b}_\mu^{(2n+2)}$ for $n = 1, 2, 3, 4, 5$:

$$\mathfrak{b}_\mu^{(4)} = v_\mu, \tag{49}$$

$$\mathfrak{b}_\mu^{(6)} = -\,(a\cdot c)\, v_\mu + a_\mu, \tag{50}$$

$$\mathfrak{b}_\mu^{(8)} = \Big[3\,(a\cdot c)^2 - (\dot a\cdot c)\Big]\, v_\mu - 3\,(a\cdot c)\, a_\mu + \dot a_\mu, \tag{51}$$

$$\begin{aligned}\mathfrak{b}_\mu^{(10)} = &-\Big[15\,(a\cdot c)^3 - 10\,(a\cdot c)\,(\dot a\cdot c) + (\ddot a\cdot c)\Big]\, v_\mu\\ &+\Big[15\,(a\cdot c)^2 - 4\,(\dot a\cdot c)\Big]\, a_\mu - 6\,(a\cdot c)\,\dot a_\mu + \ddot a_\mu,\end{aligned} \tag{52}$$

$$\begin{aligned}\mathfrak{b}_\mu^{(12)} &= \Big\{5\Big[3\cdot 7\,(a\cdot c)^2 \Big((a\cdot c)^2 - (\dot a\cdot c)\Big) + 2\,(\dot a\cdot c)^2 + 3\,(a\cdot c)\,(\ddot a\cdot c)\Big] - (\dddot a\cdot c)\Big\}\, v_\mu\\ &- 5\Big\{3\,(a\cdot c)\,[7\,(a\cdot c)^2 - 4(\dot a\cdot c)] + (\ddot a\cdot c)\Big\}\, a_\mu + 5\Big[9\,(a\cdot c)^2 - 2(\dot a\cdot c)\Big]\,\dot a_\mu\\ &- 10\,(a\cdot c)\,\ddot a_\mu + \dddot a_\mu.\end{aligned} \tag{53}$$

It follows from (50)–(53) that $\mathfrak{b}^{(6)}, \ldots, \mathfrak{b}^{(12)}$ are subject to the constraint

$$R\cdot \mathfrak{b}^{(2n+2)} = 0, \tag{54}$$

while $\mathfrak{b}^{(4)}$ is not. To derive (54), we note that, far apart from the the world line, the field appears (locally) as a plane wave moving along a null ray that points toward the propagation vector k_μ,

$$\mathcal{A}_\mu \sim \epsilon_\mu \phi(k\cdot x), \tag{55}$$

$$\mathcal{F}_{\mu\nu} \sim (k_\mu\epsilon_\nu - k_\nu\epsilon_\mu)\,\phi'. \tag{56}$$

Here, ϵ_μ is the polarization vector, ϕ is an arbitrary smooth function of the phase $k\cdot x$, and the prime stands for the derivative with respect to the phase. Recall that $\partial^\mu\mathcal{A}^{(2n)}_\mu=0$ for $n\ge 2$. In view of (55), this equation becomes

$$(k\cdot\epsilon)\,\phi'=0, \tag{57}$$

which implies that the polarization vector is orthogonal to the propagation vector. On the other hand, $\mathcal{A}^{(2n+2)}_\mu$ approaches $\bar{\mathcal{A}}^{(2n+2)}_\mu$ as $\rho\to\infty$. Now the null vector R_μ acts as the propagation vector k_μ. A comparison between (48) and (56) shows that ϵ_μ should be identified with $\mathfrak{b}^{(2n+2)}_\mu$.

Since $\partial^\mu R_\mu=d-1$, the vector potential $\mathcal{A}^{(2)}_\mu$ does not obey the Lorenz gauge condition, and hence (54) is not the case for $\mathfrak{b}^{(4)}_\mu$.

To sum up, the polarization of the retarded electromagnetic field is an imprint of the next even dimension $d+2$, excluding $d=2$ which is immune from the effect of $d=4$.

3 Radiation

Apart from the overall numerical factor, the metric stress-energy tensor of the electromagnetic field takes the same form in any dimension,

$$\Theta_{\mu\nu}=\tfrac{1}{\Omega_{d-2}}\left(F_\mu{}^\alpha F_{\alpha\nu}+\tfrac{\eta_{\mu\nu}}{4}\,F^{\alpha\beta}F_{\alpha\beta}\right). \tag{58}$$

Let us substitute (8) into (58). Since the result is to be integrated over $(d-1)$-dimensional spacelike surfaces, $\Theta^{\mu\nu}$ is conveniently split into two parts, nonintegrable and integrable,

$$\Theta^{\mu\nu}=\Theta^{\mu\nu}_{\rm I}+\Theta^{\mu\nu}_{\rm II}. \tag{59}$$

Here, our concern is only with the integrable part $\Theta^{\mu\nu}_{\rm II}$. To identify this part of the stress-energy tensor as the *radiation*, we check the fulfilment of the following conditions [21], [12]:

(i) $\Theta^{\mu\nu}_{\rm I}$ and $\Theta^{\mu\nu}_{\rm II}$ are dynamically independent off the world line, that is,

$$\partial_\mu\Theta^{\mu\nu}_{\rm I}=0,\quad \partial_\mu\Theta^{\mu\nu}_{\rm II}=0, \tag{60}$$

(ii) $\Theta^{\mu\nu}_{\rm II}$ propagates along the future light cone C_+ drawn from the emission point, and

(iii) the energy-momentum flux of $\Theta^{\mu\nu}_{\rm II}$ goes as ρ^{2-d} implying that the same amount of energy-momentum flows through spheres of different radii.

It has been found in the previous section that the infrared behavior[1] of $\mathcal{F}^{(2n)}$ is controlled by

$$\mathcal{A}^{(2)}\wedge\mathcal{A}^{(2n+2)}. \tag{61}$$

More precisely, the leading long-distance term

$$\mathcal{A}^{(2)}\wedge\bar{\mathcal{A}}^{(2n+2)}, \tag{62}$$

where $\bar{\mathcal{A}}^{(2n+2)}_\mu$ is defined in (47), is responsible for the infrared properties of $\mathcal{F}^{(2n)}$.

With (58), it is apparent that $\Theta^{\mu\nu}_{\rm II}$ is built up solely from the term shown in (62),

$$\Theta^{\mu\nu}_{\rm II}=\frac{-1}{N_n^2\Omega_{2n-2}}\,R^\mu R^\nu\left(\bar{\mathcal{A}}^{(2n+2)}\right)^2=\frac{-1}{N_n^2\Omega_{2n-2}\rho^{2n-2}}\,c^\mu c^\nu\left(\mathfrak{b}^{(2n+2)}\right)^2. \tag{63}$$

[1]The term ‘infrared’ is used here in reference to what can be described by means of quantities which are either regular or having integrable singularities at the world line.

Let us check that $\Theta^{\mu\nu}_{\rm II}$ given by (63) meets every condition (i)–(iii), and hence this quantity is reasonable to call the radiation[2]. In view of (47), the scaling properties of this $\Theta^{\mu\nu}_{\rm II}$ are in agreement with (iii). Furthermore, since the surface element of the future light cone C_+ is

$$d\sigma^\mu = c^\mu \rho^{2n-2} d\rho\, d\Omega_{2n-2}, \tag{64}$$

where c^μ is a null vector on C_+, the flux of $\Theta^{\mu\nu}_{\rm II}$ through C_+ vanishes, $d\sigma_\mu \Theta^{\mu\nu}_{\rm II} = 0$. Therefore, $\Theta^{\mu\nu}_{\rm II}$ propagates along C_+ to suit (ii).

To verify that condition (i) holds, let us note that, for $\Theta^{\mu\nu}$ and j^μ written, respectively, as (58) and (2),

$$\partial_\nu \Theta^{\mu\nu} = -F^{\mu\nu} j_\nu. \tag{65}$$

Off the world line, (65) becomes

$$\partial_\mu \Theta^{\mu\nu} = 0, \tag{66}$$

and hence either of two local conservation laws (60) implies the other one. It is sufficient to verify the conservation law for the $\Theta^{\mu\nu}_{\rm II}$. We have

$$\partial_\mu \Theta^{\mu\nu}_{\rm II} \propto \partial_\mu \left[R^\mu R^\nu \left(\bar{\mathcal{A}}^{(2n+2)} \right)^2 \right] = R^\nu \left[2n \left(\bar{\mathcal{A}}^{(2n+2)} \right)^2 + (R\cdot\partial) \left(\bar{\mathcal{A}}^{(2n+2)} \right)^2 \right]. \tag{67}$$

Here the second equation is obtained using the differentiation rule (18) and the fact that $\delta^\mu{}_\mu = 2n$. Let us take into account that $\mathfrak{b}^{(2n+2)}_\mu$ depends on v^α, a^α, ... and their scalar products with c^α. Since

$$(R\cdot\partial)\{c^\nu,\ v^\nu,\ a^\nu,\ \dot{a}^\nu, \ldots\} = 0, \quad (R\cdot\partial)\,\rho = \rho, \tag{68}$$

we apply $(R\cdot\partial)$ to $\bar{\mathcal{A}}^{(2n+2)}_\mu$ defined in (47) to conclude from (67) that $\partial_\mu \Theta^{\mu\nu}_{\rm II} = 0$. This is just the required result.

By (54), $\mathfrak{b}^{(2n+2)}_\mu$ is orthogonal to a null vector R_μ. This suggests that $\mathfrak{b}^{(2n+2)}_\mu$ is a linear combination of a spacelike vector and the null vector R_μ itself. Referring to (48), $\bar{\mathcal{A}}^{(2n+2)}_\mu$ is defined up to adding $k\, R_\mu$, where k is an arbitrary constant. If we impose the Lorenz gauge condition to this additional term, then $k\,(n-1) = 0$, that is, $k = 0$ for $n \neq 1$. When it is considered that $\mathfrak{b}^{(2n+2)}_\mu$ is spacelike, (63) shows that $\Theta^{00}_{\rm II} \geq 0$. We thus see that $\Theta^{00}_{\rm II}$ represents positive field energy flowing outward from the source.

Let us calculate the radiation rate. The radiation flux through a $(d-2)$-dimensional sphere enclosing the source is constant for any radius of the sphere. Therefore, the terms of $\Theta_{\mu\nu}$ responsible for this flux scale as ρ^{2-d}. The radiated energy-momentum is defined by

$$\mathcal{P}^\mu = \int_\Sigma d\sigma_\nu\, \Theta^{\mu\nu}_{\rm II}, \tag{69}$$

where Σ is a $(d-1)$-dimensional spacelike hypersurface. Since $\Theta^{\mu\nu}_{\rm II}$ involves only integrable singularities, and $\partial_\nu \Theta^{\mu\nu}_{\rm II} = 0$, the surface of integration Σ in (69) may be chosen arbitrarily. It is convenient to deform Σ to a tubular surface T_ϵ of small invariant radius $\rho = \epsilon$ enclosing the world line. The surface element on this tube is

$$d\sigma^\mu = \partial^\mu \rho\, \rho^{d-2}\, d\Omega_{d-2}\, ds = (v^\mu + \lambda c^\mu)\, \epsilon^{d-2}\, d\Omega_{d-2}\, ds. \tag{70}$$

[2] Strictly speaking, the radiation is represented by (63) only when $n \geq 2$. In a world with one temporal and one spatial dimension, the radiation is absent [14], [12].

Equation (69) becomes

$$\mathcal{P}_\mu^{(2n)} = -\frac{1}{N_n^2\Omega_{2n-2}}\int_{-\infty}^{s} ds\int d\Omega_{2n-2}\, c_\mu \left(\mathfrak{b}^{(2n+2)}\right)^2, \tag{71}$$

so that the radiation rate is given by

$$\dot{\mathcal{P}}_\mu^{(2n)} = -\frac{1}{N_n^2\Omega_{2n-2}}\int d\Omega_{2n-2}\, c_\mu \left(\mathfrak{b}^{(2n+2)}\right)^2. \tag{72}$$

This can be recast as

$$\dot{\mathcal{P}}_\mu^{(2n)} = -\frac{1}{Z_{d,n}^2\Omega_{2n-2}}\int d\Omega_{2n-2}\, c_\mu \left(\lim_{\rho\to\infty}\rho^n\Box^n R_\alpha\right)^2. \tag{73}$$

Conceivably this form of the radiation rate might find use in a wider context of gauge theories.

The solid angle integration is greatly simplified if we introduce the spacelike normalized vector u^μ orthogonal to v^μ,

$$c^\mu = v^\mu + u^\mu\,, \tag{74}$$

and observe that the integrands are expressions homogeneous of some degree in u^μ. Consider

$$I_{\mu_1\cdots\mu_p} = \frac{1}{\Omega_{d-2}}\int d\Omega_{d-2}\, u_{\mu_1}\cdots u_{\mu_p}\,. \tag{75}$$

In the case of odd number of multiplying vectors u^μ, the integrals vanish. If the number of multiplying vectors u^μ is even, then the integration are made through the use of the following formulas

$$I_{\mu\nu} = -\left(\tfrac{1}{d-1}\right)\overset{v}{\perp}_{\mu\nu}\,, \tag{76}$$

$$I_{\alpha\beta\mu\nu} = \tfrac{1}{(d-1)(d+1)}\left(\overset{v}{\perp}_{\mu\nu}\overset{v}{\perp}_{\alpha\beta} + \overset{v}{\perp}_{\alpha\mu}\overset{v}{\perp}_{\beta\nu} + \overset{v}{\perp}_{\alpha\nu}\overset{v}{\perp}_{\beta\mu}\right), \tag{77}$$

$$\begin{aligned} I_{\alpha\beta\gamma\lambda\mu\nu} = -\tfrac{1}{(d-1)(d+1)(d+3)}\Big(&\overset{v}{\perp}_{\alpha\beta}\overset{v}{\perp}_{\gamma\lambda}\overset{v}{\perp}_{\mu\nu} + \overset{v}{\perp}_{\alpha\beta}\overset{v}{\perp}_{\gamma\mu}\overset{v}{\perp}_{\lambda\nu} + \overset{v}{\perp}_{\alpha\beta}\overset{v}{\perp}_{\gamma\nu}\overset{v}{\perp}_{\lambda\mu}\\ &+ \overset{v}{\perp}_{\alpha\gamma}\overset{v}{\perp}_{\beta\lambda}\overset{v}{\perp}_{\mu\nu} + \overset{v}{\perp}_{\alpha\gamma}\overset{v}{\perp}_{\beta\mu}\overset{v}{\perp}_{\lambda\nu} + \overset{v}{\perp}_{\alpha\gamma}\overset{v}{\perp}_{\beta\nu}\overset{v}{\perp}_{\lambda\mu} + \overset{v}{\perp}_{\alpha\lambda}\overset{v}{\perp}_{\beta\gamma}\overset{v}{\perp}_{\mu\nu}\\ &+ \overset{v}{\perp}_{\alpha\lambda}\overset{v}{\perp}_{\beta\mu}\overset{v}{\perp}_{\gamma\nu} + \overset{v}{\perp}_{\alpha\lambda}\overset{v}{\perp}_{\beta\nu}\overset{v}{\perp}_{\gamma\mu} + \overset{v}{\perp}_{\alpha\mu}\overset{v}{\perp}_{\beta\nu}\overset{v}{\perp}_{\gamma\lambda} + \overset{v}{\perp}_{\alpha\mu}\overset{v}{\perp}_{\beta\gamma}\overset{v}{\perp}_{\lambda\nu}\\ &+ \overset{v}{\perp}_{\alpha\mu}\overset{v}{\perp}_{\beta\lambda}\overset{v}{\perp}_{\gamma\nu} + \overset{v}{\perp}_{\alpha\nu}\overset{v}{\perp}_{\beta\mu}\overset{v}{\perp}_{\gamma\lambda} + \overset{v}{\perp}_{\alpha\nu}\overset{v}{\perp}_{\beta\lambda}\overset{v}{\perp}_{\gamma\mu} + \overset{v}{\perp}_{\alpha\nu}\overset{v}{\perp}_{\beta\gamma}\overset{v}{\perp}_{\lambda\mu}\Big), \end{aligned} \tag{78}$$

which are readily derived (see, e. g., [12]). Here,

$$\overset{v}{\perp}_{\mu\nu} = \eta_{\mu\nu} - v_\mu v_\nu \tag{79}$$

is the operator that projects vectors onto a hyperplane with normal v^μ, The number of terms in such decompositions of $I_{\mu_1\cdots\mu_k}$ proliferates with k: $I_{\mu_1\cdots\mu_4}$ contains 3 monomials $\overset{v}{\perp}\overset{v}{\perp}$, $I_{\mu_1\cdots\mu_6}$ involves $3\cdot 5$ monomials $\overset{v}{\perp}\overset{v}{\perp}\overset{v}{\perp}$, $I_{\mu_1\cdots\mu_8}$ comprises $3\cdot 5\cdot 7$ monomials $\overset{v}{\perp}\overset{v}{\perp}\overset{v}{\perp}\overset{v}{\perp}$, etc. If $k\ge 6$, then calculations with $I_{\mu_1\cdots\mu_k}$ are rather tedious, so that we restrict our discussion to the dimensions $d=4$ and $d=6$. In these cases we need only handling $I_{\mu\nu}$ and $I_{\alpha\beta\mu\nu}$.

Using the identities

$$v^2 = 1, \quad (v\cdot a) = 0, \quad (v\cdot\dot a) = -a^2, \tag{80}$$

we find from (50) and (51) that

$$\left(\mathfrak{b}^{(6)}\right)^2 = (a\cdot u)^2 + a^2, \tag{81}$$

$$\left(\mathfrak{b}^{(8)}\right)^2 = \left[(\overset{v}{\perp}\dot a)^2 + 9\,(a\cdot u)^2 a^2 + 9\,(a\cdot u)^4 + (\dot a\cdot u)^2\right] - 3\left[(a^2)^{\cdot}\,(a\cdot u) + 2\,(a\cdot u)^2(\dot a\cdot u)\right]. \tag{82}$$

Thus, in four and six dimensions, the radiation rate is given, respectively, by

$$\dot{\mathcal{P}}^{(4)}_\mu = -\tfrac{2}{3}\,a^2 v_\mu \tag{83}$$

and

$$\dot{\mathcal{P}}^{(6)}_\mu = \tfrac{1}{9}\,\tfrac{1}{5\cdot 7}\left\{4\left[16\,(a^2)^2 - 7\,\dot a^2\right]v_\mu - 3\cdot 5\left(a^2\right)^{\cdot}\,a_\mu + 6\,a^2(\overset{v}{\perp}\dot a)_\mu\right\}. \tag{84}$$

4 Discussion and outlook

Let us summarize our discussion of the methods for obtaining retarded field configurations due to a single point charge in $2n$-dimensional Minkowski spacetime. The retarded Green's function technique is presently accepted as the standard approach. Iwanenko and Sokolow [20] pioneered the use of this technique. The approach based on the ansätze of a particular form, such as those defined in (10), (11), and (12), was developed in Ref. [14]. This procedure for solving Maxwell's equations (without resort to Green's functions) is found to be of particular assistance in solving the Yang–Mills equations [12]. It seems likely that the tool of greatest practical utility involves the notion of prepotential, in particular the simplest way for calculating the retarded vector potential $\mathcal{A}^{(2n)}_\mu$ is given by Eq. (38).

Close inspection of exact solutions to d-dimensional Maxwell's equations shows that the retarded field strength $\mathcal{F}^{(2n)}_{\mu\nu}$ generated by a point charge living in a $2n$-dimensional world is expressed in terms of the retarded vector potentials $\mathcal{A}^{(2m)}_\mu$ due to this charge in $2m$-dimensional worlds nearby, Eqs. (40)–(44). The fact that the state of the retarded electromagnetic field in a given even-dimensional manifold is entangled with those of contiguous even-dimensional manifolds may be the subject of far-reaching philosophical speculations. To illustrate, it follows from (41) that, while living in $d = 4$, a charge feels a specific impact from $d = 2$ and $d = 6$. The responsibility for this entanglement may rest with either coexistence on an equal footing of different $2p$-branes in some braneworld scenario or manifestation of contiguous 'parallel' realms.

A notable feature of Eqs. (40)–(44) is that the world line $z^\mu(s)$ of the charge generating these field configurations is described by different numbers of the principal curvatures κ_j for different spacetime dimensions. To be specific, we refer to Eq. (41). The world line appearing in $\mathcal{A}^{(2)}_\mu$ is a planar curve, specified solely by κ_1, while that appearing in $\mathcal{A}^{(6)}_\mu$ is a curve characterized (locally) by five essential parameters κ_1, κ_2, κ_3, κ_4, κ_5. If we regard the world line $z^\mu(s)$ in $\mathbb{M}_{1,2n-1}$ as the basic object, then both projections of this curve onto lower-dimensional spacetimes and its extensions to higher-dimensional spacetimes are rather arbitrary. Nevertheless, Eqs. (40)–(44) are invariant under variations of these mappings of the world line $z^\mu(s)$.

The advanced fields $\mathcal{F}_{\rm adv}$ can be also represented as the sums of exterior products of 1-forms $\mathcal{A}_{\rm adv}$ similar to (40)–(44), whereas combinations $\alpha\,\mathcal{F}_{\rm ret} + \beta\,\mathcal{F}_{\rm adv}$, $\alpha\beta \neq 0$, are not. Therefore, Eqs. (40)–(44) do not hold for field configurations satisfying the Stückelberg–Feynman boundary condition. We thus see that the remarkably simple structures displayed in Eqs. (40)–(44) are inherently classical.

Based on Eqs. (40)–(44), we put the rate of radiated energy-momentum of electromagnetic field in a compact form, Eqs. (72) and (73). Let us recall that there are two alternative concepts of radiation, proposed by Dirac and Teitelboim (for a review see [11]); the latter was entertained in Sec. 3. Although these concepts have some points in common, they are not equivalent. Accordingly, the fact [afforded by (63) and (72)] that the radiation in $2n$-dimensional spacetime is an infrared phenomenon stemming from the next even dimension $d = 2n+2$ cannot be clearly recognized until the Teitelboim's definition of radiation is invoked.

Why is it essential to draw the stress-energy tensor for introducing the concept of radiation? It is still common to see the assertion that the degrees of freedom related to the radiation may be identified directly in $\mathcal{F}^{(2n)}$ if one takes the piece of $\mathcal{F}^{(2n)}$ shown in (62) as the 'radiation field'. However, this assertion is erroneous. First, the construction $\mathcal{A}^{(2)} \wedge \bar{\mathcal{A}}^{(2n+2)}$ which allegedly plays the role of radiation field is in no sense dynamically independent of the rest of $\mathcal{F}^{(2n)}$. Second, the bivector $\varpi = \mathcal{A}^{(2)} \wedge \mathcal{A}^{(2n+2)}$ is deprived of information about the vector $\bar{\mathcal{A}}_\mu^{(2n+2)}$. A pictorial view of ϖ is the parallelogram of the vectors $\mathcal{A}_\mu^{(2)}$ and $\mathcal{A}_\mu^{(2n+2)}$. The bivector ϖ is independent of concrete directions and magnitudes of the constituent vectors $\mathcal{A}_\mu^{(2)}$ and $\mathcal{A}_\mu^{(2n+2)}$; ϖ depends only on the parallelogram's orientation and area $\mathfrak{S} = |\mathcal{A}^{(2)} \cdot \mathcal{A}^{(2n+2)}|$. By virtue of (54), $\bar{\mathcal{A}}_\mu^{(2n+2)}$ makes no contribution to $\mathfrak{S}$. It can be shown (much as was done in [12], p. 181) that the term scaling as ρ^{1-n} can be eliminated by a local $\mathrm{SL}(2,\mathbb{R})$ transformation of the plane spanned by the vectors $\mathcal{A}^{(2)}$ and $\mathcal{A}^{(2n+2)}$ which leaves the bivector ϖ invariant. In other words, there is a reference frame in which the 'radiation field' (62) vanishes over all spacetime (except for the future null infinity).

The implication of this argument is that the radiation is determined not only by the retarded field $\mathcal{F}^{(2n)}$ as such but also by the frame of reference in which $\mathcal{F}^{(2n)}$ is measured. On the other hand, the stress-energy tensor $\Theta^{\mu\nu}$ is not invariant under such $\mathrm{SL}(2,\mathbb{R})$ transformations. $\Theta^{\mu\nu}$ carries information about both the field $\mathcal{F}^{(2n)}$ and the frame which is used to describe $\mathcal{F}^{(2n)}$.

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BRST approach to Lagrangian Construction for Massive Higher Spin Fields

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

We review the recently developed general gauge invariant approach to Lagrangian construction for massive higher spin fields in Minkowski and AdS spaces of arbitrary dimension. Higher spin Lagrangian, describing the dynamics of the fields with any spin, is formulated with help of BRST-BFV operator in auxiliary Fock space. No off-shell constraints on the fields and gauge parameters are imposed. The construction is also applied to tensor higher spin fields with index symmetry corresponding to a multirow Young tableau.

1 Introduction

It gives me great pleasure to contribute a paper to the volume devoted to the 60 year Jubilee of Professor I.L. Buchbinder. I defended my Ph.D. thesis under his supervision and after this all my scientific activity is connected with him. I would like to congratulate him heartily on his birthday and to express sincere gratitude for his invaluable help in my scientific researches. One of the topics of our researches is the Lagrangian construction for massive higher spin fields and the present paper is a review of our joint progress in this direction.

Higher spin field problem attracts much attention during a long time. At present, there exist the various approaches to this problem although the many aspects are still far to be completely clarified (see e.g. [1] for recent reviews of massless higher spin field theory). This paper is a brief survey of recent state of gauge invariant approach to massive higher spin field theory.

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The standard BFV or BRST-BFV construction (see the reviews [2]) arose at operator quantization of dynamical systems with first class constraints. The systems under consideration are characterized by first class constraints in phase space T_a, $[T_a, T_b] = f^c_{ab}T_c$. Then BRST-BFV charge is introduced according to the rule

$$Q = \eta^a T_a + \tfrac{1}{2}\eta^b\eta^a f^c_{ab}\mathcal{P}_c, \qquad Q^2 = 0, \tag{1}$$

where η^a and $\mathcal{P}_a$ are canonically conjugate ghost variables (we consider here the case $gh(T) = 0$, then $gh(\eta^a) = 1$, $gh(\mathcal{P}_a) = -1$) satisfying the relations $\{\eta^a, \mathcal{P}_b\} = \delta^a_b$. After quantization the BRST-BFV charge becomes an Hermitian operator acting in extended space of states including ghost operators, the physical states in the extended space are defined by the equation $Q|\Psi\rangle = 0$. Due to the nilpotency of the BRST-BFV operator, $Q^2 = 0$, the physical states are defined up to transformation $|\Psi'\rangle = |\Psi\rangle + Q|\Lambda\rangle$ which is treated as a gauge transformation.

Application of BRST-BFV construction in the higher spin field theory [3] is inverse to above quantization problem. The initial point are equations, defining the irreducible representations of Poincare or AdS groups with definite spin and mass, the BRST-BFV operator is constructed on the base of these constraints and finally the higher spin Lagrangian is found on the base of BRST-BFV operator. Generic procedure looks as follows. The equations defining the representations are treated as the operators of first class constraints in some auxiliary Fock space. However, in the higher spin field theory a part of these constraints are non-Hermitian operators and in order to construct a Hermitian BRST-BFV operator we have to involve the operators which are Hermitian conjugate to the initial constraints and which are not the constraints. Then for closing the algebra to the complete set of operators we must add some more operators which are not constraints as well. Because of presence of such operators the standard BRST-BFV construction can not be applied it its literal form. However, as we will see, this problem can be solved.

2 Massive bosonic field

We illustrate the method used for Lagrangian construction on the base of massive bosonic field in Minkowski d-dimensional space. It is well known that the totally symmetric tensor field $\Phi_{\mu_1\cdots\mu_s}$, describing the irreducible spin-s massive representation of the Poincare group must satisfy the following constraints

$$(\partial^2 + m^2)\Phi_{\mu_1\cdots\mu_s} = 0, \qquad \partial^{\mu_1}\Phi_{\mu_1\mu_2\cdots\mu_s} = 0, \qquad \eta^{\mu_1\mu_2}\Phi_{\mu_1\cdots\mu_s} = 0. \tag{2}$$

In order to describe all higher integer spin fields simultaneously it is convenient to introduce Fock space generated by creation and annihilation operators a^+_μ, a_μ with vector Lorentz index $\mu = 0, 1, 2, \ldots, d-1$ satisfying the commutation relations

$$\left[a_\mu, a^+_\nu\right] = -\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = (+, -, \ldots, -). \tag{3}$$

Then we define the operators

$$l_0 = -p^2 + m^2, \qquad l_1 = a^\mu p_\mu, \qquad l_2 = \tfrac{1}{2}a^\mu a_\mu, \tag{4}$$

where $p_\mu = -i\frac{\partial}{\partial x^\mu}$. These operators act on states in the Fock space

$$|\Phi\rangle = \sum_{s=0}^{\infty}\Phi_{\mu_1\cdots\mu_s}(x)a^{\mu_1+}\cdots a^{\mu_s+}|0\rangle \tag{5}$$

which describe all integer spin fields simultaneously if the following constraints on the states take place

$$l_0|\Phi\rangle = 0, \qquad l_1|\Phi\rangle = 0, \qquad l_2|\Phi\rangle = 0. \tag{6}$$

If constraints (6) are fulfilled for the general state (5) then constraints (2) are fulfilled for each component $\Phi_{\mu_1\cdots\mu_s}(x)$ in (5) and hence the relations (6) describe all free massive higher spin bosonic fields simultaneously. Our purpose is to describe the Lagrangian construction for the massive higher spin fields on the base of BRST-BFV approach, therefore first what we should find is the Hermitian BRST-BFV operator. It means, we should have a system of Hermitian constraints. In the case under consideration the constraint l_0 is Hermitian, $l_0^+ = l_0$, however the constraints l_1, l_2 are not Hermitian. We extend the set of the constraints l_0, l_1, l_2 adding two new operators $l_1^+ = a^{\mu+}p_\mu$, $l_2^+ = \frac{1}{2}a^{\mu+}a_\mu^+$. As a result, the set of operators l_0, l_1, l_2, l_1^+, l_2^+ is invariant under Hermitian conjugation. We want to point out that operators l_1^+, l_2^+ are not constraints on the space of bra-vectors (5) since they may not annihilate the physical states. Taking Hermitian conjugation of (6) we see that l_1^+, l_2^+ together with l_0 are constraints on the space of bra-vectors

$$\langle\Phi|l_0 = 0, \qquad \langle\Phi|l_1 = 0, \qquad \langle\Phi|l_2 = 0. \tag{7}$$

Algebra of the operators l_0, l_1, l_1^+, l_2, l_2^+ is open in terms of commutators of these operators. We will suggest the following procedure of consideration. We want to use the BRST-BFV construction in the simplest (minimal) form corresponding to closed algebras. To get such an algebra we add to the above set of operators, all operators generated by the commutators of l_0, l_1, l_1^+, l_2, l_2^+. Doing such a way we obtain two new operators

$$m^2 \qquad \text{and} \qquad g_0 = -a_\mu^+ a^\mu + \tfrac{d}{2}. \tag{8}$$

The resulting algebra are written in Table 1. In this table the first arguments of the commutators and explicit expressions for all the operators are listed in the left column and the second argument of commutators are listed in the upper row.

Let us emphasize once again that operators l_1^+, l_2^+ are not constraints on the space of ket-vectors. The constraints in space of ket-vectors are l_0, l_1, l_2 (6) and they are the first class constraints in this space. Analogously, the constraints in space of bra-vectors are l_0, l_1^+, l_2^+ (7) and they also are the first class constraints but only in this space, not in space of ket-vectors. Since the operator m^2 is obtained from the commutator

$$[l_1, l_1^+] = l_0 - m^2, \tag{9}$$

where l_1 is a constraint in the space of ket-vectors (6) and l_1^+ is a constraint in the space of bra-vectors (7), then it can not be regarded as a constraint neither in the ket-vector space nor in the bra-vector space. Analogously the operator g_0 is obtained from the commutator

$$[l_2, l_2^+] = g_0, \tag{10}$$

where l_2 is a constraint in the space of ket-vectors (6) and l_2^+ is a constraint in the space of bra-vectors (7). Therefore g_0 can not also be regarded as a constraint neither in the ket-vector space nor in the bra-vector space.

One can show that a straightforward use of BRST-BFV construction as if all the operators l_0, l_1, l_2, l_1^+, l_2^+, g_0, m^2 are the first class constraints doesn't lead to the proper equations (6) for any spin. This happens because among the above hermitian operators there are operators

	l_0	l_1	l_1^+	l_2	l_2^+	g_0	m^2
l_0	0	0	0	0	0	0	0
l_1	0	0	$l_0 - m^2$	0	$-l_1^+$	l_1	0
l_1^+	0	$-l_0 + m^2$	0	l_1	0	$-l_1^+$	0
l_2	0	0	$-l_1$	0	g_0	$2l_2$	0
l_2^+	0	l_1^+	0	$-g_0$	0	$-2l_2^+$	0
g_0	0	$-l_1$	l_1^+	$-2l_2$	$2l_2^+$	0	0
m^2	0	0	0	0	0	0	0

Table 1: Operator algebra generated by the constraints

which are not constraints (g_0 and m^2 in the case under consideration) and they bring two more equations (in addition to (6)) onto the physical field (5). Thus we must somehow get rid of these supplementary equations.

The method of avoiding the supplementary equations consists in constructing the new enlarged expressions for the operators of the algebra, so that the Hermitian operators which are not constraints will be zero.

Let us act as follows. We enlarge the representation space of the operator algebra by introducing the additional (new) creation and annihilation operators and enlarge expressions for the operators (see [4] for more details)

$$l_i \quad \longrightarrow \quad L_i = l_i + l'_i, \qquad\qquad l_i = \{l_0, l_1, l_1^+, l_2, l_2^+, g_0, m^2\}$$

The enlarged operators must satisfy two conditions:
1) They must form an algebra $[L_i, L_j] \sim L_k$;
2) The operators which can't be regarded as constraints must be zero or contain arbitrary parameters whose values will be defined later from the condition of reproducing the correct equations of motion.

In the case of higher spin fields in Minkowski space the algebra of the operators is a Lie algebra

$$[l_i, l_j] = f_{ij}^k \, l_k. \tag{11}$$

In this case we can construct the additional parts of the operators l'_i which satisfy the same algebra (11) $[l'_i, l'_j] = f_{ij}^k \, l'_k$ using the method described in [10] and since the initial operators l_i commute with the additional parts l'_j we get that the enlarged operators satisfy the same algebra $[L_i, L_j] = f_{ij}^k \, L_k$ (11). After this the BRST-BFV operators Q' can be constructed in the usual way (1).

Now one need to define the arbitrary parameters. As explained in [4] we should assume that the state vectors $|\Psi\rangle$ and the gauge parameters $|\Lambda\rangle$ in the extended Fock space, including the ghost fields, must be independent of the ghosts corresponding to the Hermitian operators which are not constraints. Let us denote these ghost as η_G and η_M corresponding to the extended operators $G_0 = g_0 + g_0'$ and $M^2 = m^2 + m'^2$ respectively.

Let us extract the dependence of the BRST-BFV operator on the ghosts η_G, $\mathcal{P}_G$, η_M, $\mathcal{P}_M$

$$Q' \quad = \quad Q + \eta_G(\sigma + h) + \eta_M(m^2 + m'^2) - \eta_2^+\eta_2\mathcal{P}_G + \eta_1^+\eta_1\mathcal{P}_M, \qquad (12)$$

where $\sigma + h = g_0 + g_0' +$ *ghost fields*, with h and m'^2 being the arbitrary parameters to be defined. After this the equation on the physical states in the BRST-BFV approach $Q'|\Psi\rangle = 0$ yields three equations

$$Q|\Psi\rangle = 0, \qquad gh(|\Psi\rangle) = 0, \qquad (13)$$

$$(\sigma + h)|\Psi\rangle = 0, \qquad (m^2 + m'^2)|\Psi\rangle = 0. \qquad (14)$$

From the two equations in (14) we find the possible values of h and m'^2 whereas equation (13) is equation on the physical state. This equation on the physical state can be obtained from the Lagrangian

$$-\mathcal{L} = \int d\eta_0 \ \langle\Psi|KQ|\Psi\rangle. \qquad (15)$$

In eq. (15) above the standard scalar product in the Fock space is used and K is a specific invertible operator providing the reality of the Lagrangian (see [4] for more details). The latter acts as the unit operator in the entire Fock space, but for the sector controlled by the auxiliary creation and annihilation operators used at constructing the additional parts.

Because of nilpotensy of the BRST-BFV operator Q' (12) equation on the physical state (13) is invariant under the reducible gauge transformations

$$\delta|\Psi\rangle = Q|\Lambda\rangle, \qquad gh(|\Lambda\rangle) = -1, \qquad (16)$$

$$\delta|\Lambda\rangle = Q|\Omega\rangle, \qquad gh(|\Omega\rangle) = -2. \qquad (17)$$

We assume that the arbitrary parameters in eqs. (16), (17) have been fixed by conditions (14). Since all the ghost are fermionic we can not write a gauge parameter with ghost number -3 and therefore the chain of the gauge transformations is finite.

3 Lagrangian construction for the fermionic fields

The Lagrangian construction for the fermionic higher spin theories have two specific differences compared to the bosonic ones and demands some comments.

One of the specific features consists in that we have the fermionic operators in the algebra of constraints and corresponding them the bosonic ghosts. We can write these ghosts in any power in the Fock space states and therefore the gauge parameters can have an arbitrary negative number. As a result the chain of gauge transformations (16), (17) can be continued. But due to the first eq. of (14) the chain of the gauge transformations will be finite for each spin and the order of reducibility grows with the spin of the field (see [5] for further details).

Another specific features is that unlike the bosonic case, in the fermionic theory we must obtain Lagrangian which is linear in derivatives. But if we try to construct Lagrangian similar to the bosonic case (15) we obtain Lagrangian which will be the second order in derivatives. To overcome this problem one first partially fixes the gauge and partially solves some field equations. Then the obtained equations are still Lagrangian and thus we can derive the correct Lagrangian (see [5] for further details).

Using this method, the Lagrangians for the massive fermionic higher spin fields have been obtained [5].

4 Lagrangian construction for the fields in AdS

The main difference of the Lagrangian construction in AdS space is that the algebra generated by the constraints is nonlinear, but it has a special structure. The structure of the algebra looks like [6]

$$[\,l_i, l_j] \quad = \quad f_{ij}^k l_k + f_{ij}^{km} l_k l_m, \tag{18}$$

where f_{ij}^k, f_{ij}^{km} are constants. The constants f_{ij}^{km} are proportional to the scalar curvature and disappear in the flat limit.

We describe the method of finding the enlarged expressions for the operators of the algebra (18) [6], (see also [7]). First, we enlarge the representation space by introducing the additional creation and annihilation operators and construct new operators of the algebra $l_i \to L_i = l_i + l'_i$, where l'_i is the part of the operator which depends on the new creation and annihilation operators only (and constants of the theory like the mass m and the curvature).

Then we demand that the new operators L_i are in involution relations

$$[\,L_i, L_j] \quad \sim \quad L_k. \tag{19}$$

Since $[\,l_i, l'_j] = 0$ we have

$$\begin{aligned}[\,L_i, L_j] = [\,l_i, l_j] + [\,l'_i, l'_j] \quad = \quad & f_{ij}^k L_k - (f_{ij}^{km} + f_{ij}^{mk}) l'_m L_k + f_{ij}^{km} L_k L_m \\ & - f_{ij}^k l'_k + f_{ij}^{km} l'_m l'_k + [\,l'_i, l'_j].\end{aligned}$$

Then in order to provide (19) the last three terms must be canceled. Thus we find the algebra of the additional parts

$$[\,l'_i, l'_j] \quad = \quad f_{ij}^k l'_k - f_{ij}^{km} l'_m l'_k \tag{20}$$

and also we find the deformed algebra for the enlarged operators

$$[\,L_i, L_j] \quad = \quad f_{ij}^k L_k - (f_{ij}^{km} + f_{ij}^{mk}) l'_m L_k + f_{ij}^{km} L_k L_m. \tag{21}$$

We see that the algebra (21) of the enlarged operators L_i is changed in comparison with the algebra (18) of the initial operators l_i.

There exists the method [10] which allows us to construct explicit expressions for the additional parts on the base of their algebra (20). Thus the problem of constructing of the additional parts for the nonlinear algebra (18) can be solved. Let us remind that the additional parts corresponding to operators which are not constraints must linearly contain arbitrary parameters (whose values will be defined later from the condition of reproducing the correct equations of motion) and therefore the trivial solution is not allowed.

Next we discuss the aspects of constructing the BRST-BFV operator caused by the nonlinearity of the operator algebra using the massive bosonic higher spin fields in AdS space [6], [7] as an example. The construction of BRST-BFV operator is based on following general principles:

1. The BRST-BFV operator Q' is Hermitian, $Q'^{+} = Q'$, and nilpotent, $Q'^2 = 0$.

2. The BRST-BFV operator Q' is built using a set of first class constraints. In the case under consideration the operators $\tilde{L}_0$, L_1, L_1^+, L_2, L_2^+, G_0 are used as a set of such constraints.

3. The BRST-BFV operator Q' satisfies the special initial condition

$$Q'\Big|_{\mathcal{P}=0} \quad = \quad \eta_0 \tilde{L}_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_G G_0.$$

Straightforward calculation of the commutators allows us to find the algebra of the enlarged operators. In particular for the bosonic fields in AdS space we get the following commutation relations [6]

$$\begin{aligned}
[L_1, \tilde{L}_0] &= (\gamma-\beta)rL_1 + 4\beta r L_1^+ L_2 - 4\beta r l_1'^+ L_2 - 4\beta r l_2' L_1^+ \\
&\quad + 2\beta r G_0 L_1 - 2\beta r l_1' G_0 - 2\beta r g_0' L_1, \qquad (22)\\
[\tilde{L}_0, L_1^+] &= (\gamma-\beta)rL_1^+ + 4\beta r L_2^+ L_1 - 4\beta r l_2'^+ L_1 - 4\beta r l_1' L_2^+ \\
&\quad + 2\beta r L_1^+ G_0 - 2\beta r l_1'^+ G_0 - 2\beta r g_0' L_1^+, \qquad (23)\\
[L_1, L_1^+] &= \tilde{L}_0 - \gamma r G_0 + 4(2-\beta)r(l_2'^+ L_2 + l_2' L_2^+) \\
&\quad - 2(2-\beta)r g_0' G_0 + (2-\beta)r(G_0^2 - 2G_0 - 4L_2^+ L_2). \qquad (24)
\end{aligned}$$

All possible ways to order the operators in the right hand sides of (22)–(24) can be described in terms of arbitrary real parameters ξ_1, ξ_2, ξ_3, ξ_4, ξ_5. The arbitrariness in the BRST-BFV operator caused by the parameter ξ_i is resulted in arbitrariness of introducing the auxiliary fields in the Lagrangians and hence does not affect the dynamics of the basic field (see [6] for the details). After that, the construction of the Lagrangians for the fields in AdS space goes the practically the same way as for fields in Minkowsky space.

Using this method, the Lagrangians for the bosonic [6] and for fermionic [8] massive higher spin fields in AdS space have been constructed.

5 Fields corresponding to an arbitrary Young tableau

Now we consider the Lagrangian construction for the fields corresponding to non sqaure Young tableau using a Young tableau with 2 rows ($s_1 \geq s_2$)

$$\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \cdots & \cdots & \cdots & \mu_{s_1} \\ \hline \nu_1 & \nu_2 & \cdots & \nu_{s_2} & \multicolumn{2}{c}{} \\ \cline{1-4}\end{array}. \qquad (25)$$

The tensor field is symmetric with respect to permutation of each type of the indices[2] $\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = \Phi_{(\mu_1\cdots\mu_{s_1}),\,(\nu_1\cdots\nu_{s_2})}(x)$ and in addition must satisfy the following equations

$$(\partial^2 + m^2)\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = 0, \qquad (26)$$

$$\partial^{\mu_1}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = \partial^{\nu_1}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = 0, \qquad (27)$$

$$\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}} = \eta^{\nu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\nu_2\cdots\nu_{s_2}} = \eta^{\mu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}} = 0, \quad (28)$$

$$\Phi_{(\mu_1\cdots\mu_{s_1},\,\nu_1)\cdots\nu_{s_2}}(x) = 0. \qquad (29)$$

Then we define Fock space generated by creation and annihilation operators

$$[a_i^\mu, a_j^{+\nu}] = -\eta^{\mu\nu}\delta_{ij}, \qquad \eta^{\mu\nu} = diag(+,-,-,\cdots,-) \qquad i,j = 1,2. \qquad (30)$$

The number of pairs of creation and annihilation operators one should introduce is determined by the number of rows in the Young tableau corresponding to the symmetry of the tensor field. Thus we introduce two pairs of such operators. An arbitrary state vector in this Fock space has the form

$$|\Phi\rangle \;=\; \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x)\, a_1^{+\mu_1}\cdots a_1^{+\mu_{s_1}} a_2^{+\nu_1}\cdots a_2^{+\nu_{s_2}}|0\rangle. \qquad (31)$$

[2]The indices inside round brackets are to be symmetrized.

To get equations (26)–(29) on the coefficient functions we introduce the following operators

$$l_0 = -p^{\mu}p_{\mu} + m^2, \qquad l_i = a_i^{\mu}p_{\mu}, \qquad l_{ij} = \tfrac{1}{2}a_i^{\mu}a_{j\mu} \qquad g_{12} = -a_1^{+\mu}a_{2\mu} \tag{32}$$

where $p_{\mu} = -i\partial_{\mu}$. One can check that restrictions (26)–(29) are equivalent to

$$l_0|\Phi\rangle = 0\,, \qquad l_i|\Phi\rangle = 0\,, \qquad l_{ij}|\Phi\rangle = 0\,, \qquad g_{12}|\Phi\rangle = 0 \tag{33}$$

respectively.

Now we can generalize this construction to the fields corresponding to k-row Young tableau. For this purpose one should introduce Fock space generated by k pairs of creation and annihilation operators (30), where $i, j = 1, 2, \ldots, k$, and then introduce operators[3] (32), but now with $i, j = 1, 2, \ldots, k$. After this the Lagrangian construction can be carried out as usual [9]. Using this method Lagrangians for the massive bosonic field corresponding to 2-rows Young tableau was constructed in [9].

6 Summary

In this paper we have briefly considered the basic principles of gauge invariant Lagrangian construction for massive higher spin fields[4]. This method can be applied to any free higher spin field model in Minkowski and AdS spaces. It is interesting to point out that the Lagrangians obtained possess a reducible gauge invariance and for the fermionic fields the order of reducibility grows with value of the spin. Recent applications of BRST-BFV approach to interaction higher spin theories are discussed in [12].

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Wandering in five-dimensional curved superspace

Dedicated to Professor I. L. Buchbinder
On the Occasion of His 60th Birthday

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Abstract

This is a brief review of the superspace formulation for five-dimensional $\mathcal{N} = 1$ matter-coupled supergravity recently developed by the authors.

1 Introduction

Historically, the first attempt to formulate five-dimensional $\mathcal{N} = 1$ (often called $\mathcal{N} = 2$) supergravity in an off-shell superspace setting was made in [1] shortly before its on-shell component formulation was given [2, 3]. Inspired by [2], Howe [4] (see also [5]) proposed a superspace formulation for the minimal multiplet of 5D $\mathcal{N} = 1$ supergravity ("minimal" in the sense of superconformal tensor calculus). After Howe's work [4], 5D $\mathcal{N} = 1$ curved superspace has been abandoned for 25 years. General matter couplings in 5D $\mathcal{N} = 1$ supergravity have been constructed within on-shell components approaches [6, 7, 8] and within the superconformal tensor calculus [9, 10].

In 2007, we began the program of developing a superspace formulation for 5D $\mathcal{N} = 1$ matter-coupled supergravity. We first elaborated supersymmetric field theory in 5D $\mathcal{N} = 1$ anti-de Sitter superspace which is a maximally symmetric curved background [11]. This was followed by a fully-fledged supergravity formalism developed in a series of papers [12, 13, 14]. In these publications, we not only reproduced the main results of the superconformal tensor approach [9, 10], but also proposed new off-shell supermultiplets and more general supergravity-matter systems. The present note is a brief review of our construction.

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Looking back at the 25 year history of 5D $\mathcal{N}=1$ curved superspace, one can notice a striking historical curiosity. In 1982, Howe had the right superspace setting for pure supergravity – the minimal multiplet [4], which was the starting point of our approach [12, 13]. The same multiplet also occurs within the superconformal tensor calculus [9, 10] by coupling the Weyl multiplet to an Abelian vector multiplet and then gauge fixing some local symmetries (the vector multiplet is one of two compensators required to describe Poincaré supergravity). So why didn't Howe make use of his formulation to construct Poincaré supergravity and its matter couplings? A partial answer is quite simple. Even in rigid supersymmetry with eight supercharges in diverse dimensions, adequate approaches to generate off-shell supermultiplets and supersymmetric actions appeared only in 1984. They go by the names *harmonic superspace* [15, 16] and *projective superspace* [17, 18].

This note is organized as follows. In section 2 we review, following [14], the superspace formulation for the Weyl multiplet of conformal supergravity. Covariant projective supermultiplets and the supersymmetric action principle are introduced in section 3. The same section also contains a few examples of interesting dynamical systems.

2 5D conformal supergravity in superspace

We start by describing the superspace formulation for 5D conformal supergravity [14]. Let $z^{\hat M}=(x^{\hat m},\theta^{\hat\mu}_i)$ be local bosonic (x) and fermionic (θ) coordinates parametrizing a curved five-dimensional $\mathcal{N}=1$ superspace $\mathcal{M}^{5|8}$ ($\hat m=0,1,\cdots,4$, $\hat\mu=1,\cdots,4$, and $i=\underline{1},\underline{2}$). The Grassmann variables $\theta^{\hat\mu}_i$ obey the 5D pseudo-Majorana reality condition $\overline{\theta^{\hat\mu}_i}=\theta^i_{\hat\mu}=\varepsilon_{\hat\mu\hat\nu}\varepsilon^{ij}\theta^{\hat\nu}_j$. The tangent-space group is chosen to be $\mathrm{SO}(4,1)\times\mathrm{SU}(2)$, and the superspace covariant derivatives $\mathcal{D}_{\hat A}=(\mathcal{D}_{\hat a},\mathcal{D}^i_{\hat\alpha})$ have the form

$$\mathcal{D}_{\hat A}=E_{\hat A}+\tfrac12\Omega_{\hat A}{}^{\hat b\hat c}(z)M_{\hat b\hat c}+\Phi_{\hat A}{}^{kl}(z)J_{kl}\ . \tag{1}$$

Here $E_{\hat A}=E_{\hat A}{}^{\hat M}(z)\partial_{\hat M}$ is the supervielbein, with $\partial_{\hat M}=\partial/\partial z^{\hat M}$; $M_{\hat b\hat c}$ and $\Omega_{\hat A}{}^{\hat b\hat c}$ are the Lorentz generators and connection respectively (both antisymmetric in $\hat b$, $\hat c$); J_{kl} and $\Phi_{\hat A}{}^{kl}$ are respectively the SU(2) generator and connection (symmetric in k, l). The generators of $\mathrm{SO}(4,1)\times\mathrm{SU}(2)$ act on the covariant derivatives as follows:[1]

$$[M_{\hat\alpha\hat\beta},\mathcal{D}^k_{\hat\gamma}]=\varepsilon_{\hat\gamma(\hat\alpha}\mathcal{D}^k_{\hat\beta)}\ ,\qquad [M_{\hat a\hat b},\mathcal{D}_{\hat c}]=2\eta_{\hat c[\hat a}\mathcal{D}_{\hat b]}\ ,\qquad [J^{kl},\mathcal{D}^i_{\hat\alpha}]=\varepsilon^{i(k}\mathcal{D}^{l)}_{\hat\alpha}\ , \tag{2}$$

where $J^{kl}=\varepsilon^{ki}\varepsilon^{lj}J_{ij}$ and $M_{\hat\alpha\hat\beta}=M_{\hat\beta\hat\alpha}=(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}M_{\hat a\hat b}$ and $(\Sigma^{\hat a\hat b})_{\hat\alpha}{}^{\hat\beta}$ are the spinor Lorentz generators, $\Sigma^{\hat a\hat b}=-\frac14[\Gamma^{\hat a},\Gamma^{\hat b}]$, with $\Gamma^{\hat a}$ the 5D Dirac matrices (see the appendix in [13] for our notation and conventions).

The supergravity gauge group is generated by local transformations of the form

$$\delta_K\mathcal{D}_{\hat A}=[K,\mathcal{D}_{\hat A}]\ ,\quad \delta_K U=K\,U\ ,\qquad K=K^{\hat C}(z)\mathcal{D}_{\hat C}+\tfrac12K^{\hat c\hat d}(z)M_{\hat c\hat d}+K^{kl}(z)J_{kl}\ , \tag{3}$$

with all the gauge parameters obeying natural reality and symmetry conditions, and otherwise arbitrary. In (3) we have also included the transformation rule for a tensor superfield $U(z)$, with its indices suppressed.

The covariant derivatives obey (anti)commutation relations of the general form

$$[\mathcal{D}_{\hat A},\mathcal{D}_{\hat B}\}=T_{\hat A\hat B}{}^{\hat C}\mathcal{D}_{\hat C}+\tfrac12R_{\hat A\hat B}{}^{\hat c\hat d}M_{\hat c\hat d}+R_{\hat A\hat B}{}^{kl}J_{kl}\ , \tag{4}$$

[1] The operation of (anti)symmetrization of n indices is defined to involve a factor $(n!)^{-1}$.

where $T_{\hat{A}\hat{B}}{}^{\hat{C}}$ is the torsion, and $R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}$ and $R_{\hat{A}\hat{B}}{}^{kl}$ are the SO(4,1) and SU(2) curvature tensors, respectively.

To describe the Weyl multiplet of conformal supergravity [9, 10], the torsion has to be constrained as [14]:

$$T^{i\,j\,\hat{c}}_{\hat{\alpha}\hat{\beta}} \;=\; -2\mathrm{i}\varepsilon^{ij}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}, \qquad T^{i\,j\,\hat{\gamma}}_{\hat{\alpha}\hat{\beta}k} \;=\; T^{i}_{\hat{\alpha}\hat{b}}{}^{\hat{c}} \;=\; 0, \qquad T_{\hat{a}\hat{b}}{}^{\hat{c}} \;=\; T_{\hat{a}\hat{\beta}(j}{}^{\hat{\beta}}{}_{k)} \;=\; 0\;. \tag{5}$$

With these constraints, it can be shown that the torsion and curvature tensors are expressed in terms of four dimension-1 tensor superfields S^{ij}, $C_{\hat{a}}{}^{ij}$, $X_{\hat{a}\hat{b}}$, and $N_{\hat{a}\hat{b}}$, and their covariant derivatives. The superfields S^{ij}, $C_{\hat{a}}{}^{ij}$ are symmetric in i, j, while $X_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$ are antisymmetric in $\hat{a}\,,\hat{b}$. All these tensors are real $\overline{S^{ij}} = S_{ij}$, $\overline{C_{\hat{a}}{}^{ij}} = C_{\hat{a}ij}$, $\overline{X_{\hat{a}\hat{b}}} = X_{\hat{a}\hat{b}}$, $\overline{N_{\hat{a}\hat{b}}} = N_{\hat{a}\hat{b}}$.

The covariant derivatives obey the (anti)commutation relations [14]:

$$\begin{aligned}
\{\mathcal{D}^i_{\hat{\alpha}},\mathcal{D}^j_{\hat{\beta}}\} \;=\;& -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat{\alpha}\hat{\beta}} - \mathrm{i}\,\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}X^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} + \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Gamma_{\hat{a}})_{\hat{\alpha}\hat{\beta}}N_{\hat{b}\hat{c}}M_{\hat{d}\hat{e}}\\
&-\frac{\mathrm{i}}{2}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}}C_{\hat{c}}{}^{ij}M_{\hat{d}\hat{e}} + 4\mathrm{i}\,S^{ij}M_{\hat{\alpha}\hat{\beta}} + 3\mathrm{i}\,\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}S^{kl}J_{kl}\\
&-\mathrm{i}\,\varepsilon^{ij}C_{\hat{\alpha}\hat{\beta}}{}^{kl}J_{kl} - 4\mathrm{i}\Big(X_{\hat{\alpha}\hat{\beta}} + N_{\hat{\alpha}\hat{\beta}}\Big)J^{ij}\;,
\end{aligned}\tag{6a}$$

$$\begin{aligned}
[\mathcal{D}_{\hat{a}},\mathcal{D}^j_{\hat{\beta}}] \;=\;& \frac{1}{2}\Big((\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}S^j{}_k - X_{\hat{a}\hat{b}}(\Gamma^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}\delta^j_k - \frac{1}{4}\,\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}N^{\hat{d}\hat{e}}(\Sigma^{\hat{b}\hat{c}})_{\hat{\beta}}{}^{\hat{\gamma}}\delta^j_k + (\Sigma_{\hat{a}}{}^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}C_{\hat{b}}{}^j{}_k\Big)\mathcal{D}^k_{\hat{\gamma}}\\
&+\text{curvature terms}\;.
\end{aligned}\tag{6b}$$

The dimension-1 components of the torsion, S^{ij}, $X_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$ and $C_{\hat{a}}{}^{ij}$, obey some differential constraints implied by the Bianchi identities [14].

The fact that the supergeometry introduced corresponds to 5D conformal supergravity, manifests itself in the invariance of the constraints (5) under infinitesimal super-Weyl transformations of the form[2]

$$\delta_\sigma\mathcal{D}^i_{\hat{\alpha}} \;=\; \sigma\mathcal{D}^i_{\hat{\alpha}} + 4(\mathcal{D}^{\hat{\gamma}i}\sigma)M_{\hat{\gamma}\hat{\alpha}} - 6(\mathcal{D}_{\hat{\alpha}k}\sigma)J^{ki}\;, \tag{7a}$$

$$\delta_\sigma\mathcal{D}_{\hat{a}} \;=\; 2\sigma\mathcal{D}_{\hat{a}} + \mathrm{i}(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}}(\mathcal{D}^k_{\hat{\gamma}}\sigma)\mathcal{D}_{\hat{\delta}k} - 2(\mathcal{D}^{\hat{b}}\sigma)M_{\hat{a}\hat{b}} + \frac{\mathrm{i}}{4}(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}}(\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{D}^{l)}_{\hat{\delta}}\sigma)J_{kl}\;, \tag{7b}$$

where the scalar superfield σ is real and unconstrained. The components of the dimension-1 torsion can be seen to transform as follows:

$$\delta_\sigma S^{ij} \;=\; 2\sigma S^{ij} + \frac{\mathrm{i}}{2}\,\mathcal{D}^{\hat{\alpha}(i}\mathcal{D}^{j)}_{\hat{\alpha}}\sigma\;, \qquad \delta_\sigma C_{\hat{a}}{}^{ij} = 2\sigma C_{\hat{a}}{}^{ij} + \mathrm{i}\,(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(i}_{\hat{\gamma}}\mathcal{D}^{j)}_{\hat{\delta}}\sigma\;, \tag{8a}$$

$$\delta_\sigma X_{\hat{a}\hat{b}} \;=\; 2\sigma X_{\hat{a}\hat{b}} - \frac{\mathrm{i}}{2}\,(\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}}\mathcal{D}^k_{\hat{\alpha}}\mathcal{D}_{\hat{\beta}k}\sigma\;, \qquad \delta_\sigma N_{\hat{a}\hat{b}} = 2\sigma N_{\hat{a}\hat{b}} - \mathrm{i}\,(\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}}\mathcal{D}^k_{\hat{\alpha}}\mathcal{D}_{\hat{\beta}k}\sigma\;. \tag{8b}$$

It follows from here that $W_{\hat{a}\hat{b}} := X_{\hat{a}\hat{b}} - \frac{1}{2}N_{\hat{a}\hat{b}}$ transforms homogeneously,

$$\delta_\sigma W_{\hat{a}\hat{b}} = 2\sigma W_{\hat{a}\hat{b}}\;. \tag{9}$$

Therefore, $W_{\hat{a}\hat{b}}$ is a superspace generalization of the Weyl tensor.

It turns out that the super-Weyl transformations can be used to gauge away the superfield $C_{\hat{a}}{}^{ij}$. Imposing the super-Weyl gauge condition

$$C_{\hat{a}}{}^{ij} = 0\;, \tag{10}$$

[2]The finite form for the super-Weyl transformations has been given in [19].

is equivalent to extending the set of constraints (5) by an additional dimension-1 constraint which is $T_{\hat{a}}{}_{(\hat{\beta}\hat{\gamma})}^{(j\,k)} = 0$ [14]. The resulting superspace geometry provides an alternative description of the Weyl multiplet. Because of (10), the full set of constraints is now invariant under the super-Weyl transformations (7a)–(7b) generated by a constrained parameter σ. The corresponding constraint is

$$\mathcal{D}^{(i}_{\hat{\alpha}}\mathcal{D}^{j)}_{\hat{\beta}}\sigma - \frac{1}{4}\varepsilon_{\hat{\alpha}\hat{\beta}}\mathcal{D}^{\hat{\gamma}(i}\mathcal{D}^{j)}_{\hat{\gamma}}\sigma = 0 \ . \tag{11}$$

Another consequence of (10) in conjunction with the Bianchi identities is that S^{ij} satisfies the equation

$$\mathcal{D}^{(i}_{\hat{\gamma}}S^{jk)} = 0 \ . \tag{12}$$

If not specifically mentioned, eq. (10) will be assumed in what follows.

The Weyl multiplet can naturally be coupled to a non-Abelian vector multiplet. This is achieved by introducing gauge-covariant derivatives $\boldsymbol{\mathcal{D}}_{\hat{A}} = \mathcal{D}_{\hat{A}} + \mathcal{V}_{\hat{A}}(z)$, with $\mathcal{V}_{\hat{A}}$ a gauge connection taking its values in the Lie algebra of the gauge group. Then the algebra (4) turns into

$$[\boldsymbol{\mathcal{D}}_{\hat{A}}, \boldsymbol{\mathcal{D}}_{\hat{B}}\} = T_{\hat{A}\hat{B}}{}^{\hat{C}}\boldsymbol{\mathcal{D}}_{\hat{C}} + \tfrac{1}{2}R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} + R_{\hat{A}\hat{B}}{}^{kl}J_{kl} + \mathcal{F}_{\hat{A}\hat{B}} \ . \tag{13}$$

An irreducible off-shell vector multiplet emerges if $\mathcal{F}_{\hat{A}\hat{B}}$ is constrained as $\mathcal{F}^{i\,j}_{\hat{\alpha}\hat{\beta}} \propto \varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}\mathcal{W}$ (compare with [5]). The field strength $\mathcal{W}$ possesses the super-Weyl transformation $\delta_\sigma\mathcal{W} = 2\sigma\mathcal{W}$ and obeys the following Bianchi identity:

$$\boldsymbol{\mathcal{D}}^{(i}_{\hat{\alpha}}\boldsymbol{\mathcal{D}}^{j)}_{\hat{\beta}}\mathcal{W} - \frac{1}{4}\varepsilon_{\hat{\alpha}\hat{\beta}}\boldsymbol{\mathcal{D}}^{\hat{\gamma}(i}\boldsymbol{\mathcal{D}}^{j)}_{\hat{\gamma}}\mathcal{W} = 0 \ . \tag{14}$$

Associated with the vector multiplet is the composite superfield [14]

$$\mathcal{G}^{ij} := \mathrm{tr}\left\{\mathrm{i}\,\boldsymbol{\mathcal{D}}^{\hat{\alpha}(i}\mathcal{W}\boldsymbol{\mathcal{D}}^{j)}_{\hat{\alpha}}\mathcal{W} + \frac{\mathrm{i}}{2}\mathcal{W}\boldsymbol{\mathcal{D}}^{ij}\mathcal{W} - 2S^{ij}\mathcal{W}^2\right\} , \qquad \boldsymbol{\mathcal{D}}^{ij} := \boldsymbol{\mathcal{D}}^{\hat{\alpha}(i}\boldsymbol{\mathcal{D}}^{j)}_{\hat{\alpha}} \ . \tag{15}$$

It is characterized by the following fundamental properties:

$$\mathcal{D}^{(i}_{\hat{\alpha}}\mathcal{G}^{jk)} = 0 \ , \qquad \delta_\sigma\mathcal{G}^{ij} = 6\sigma\mathcal{G}^{ij} \ . \tag{16}$$

Let $\mathcal{W} = W\,\boldsymbol{Z}$, with $\boldsymbol{Z}$ the generator, be the field strength of an Abelian vector multiplet. Then, eq. (14) coincides in form with the constraint (11) obeyed by the super-Weyl parameter. If the vector multiplet is characterized by $W(z) \neq 0$ everywhere in superspace, super-Weyl transformations can be used to impose the gauge $W = 1$. The resulting geometry (13) describes the minimal multiplet of 5D supergravity [4].

3 Kinematics and dynamics in curved projective superspace

We have reviewed the geometric description of 5D conformal supergravity in superspace. Let us now turn to a brief discussion of a large family of off-shell supermultiplets coupled to conformal supergravity, which can be used to describe supersymmetric matter. They were introduced in [14] under the name *covariant projective supermultiplets*. These supermultiplets are a curved-superspace extension of the 5D superconfomal projective multiplets [20]. The latter are ordinary projective supermultiplets [18] with respect to the super-Poincaré subgroup of the 5D superconformal group.

It is useful to introduce auxiliary isotwistor coordinates $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$ in addition to the superspace coordinates $z^{\hat M} = (x^{\hat m}, \theta_i^{\hat\mu})$. All the coordinates u_i^+ and $z^{\hat M}$ are defined to be inert under the tangent-space group. In particular, the variables u_i^+ do not transform under the local SU(2) group, and hence they are covariantly constant, $\mathcal{D}_{\hat A} u_j^+ = 0$. It follows from (6a) that the operators $\mathcal{D}_{\hat\alpha}^+ := u_i^+ \mathcal{D}_{\hat\alpha}^i$ obey the following algebra (the constraint (10) is not assumed from here until eq. (21) including):

$$\{\mathcal{D}_{\hat\alpha}^+, \mathcal{D}_{\hat\beta}^+\} = -4\mathrm{i}\left(X_{\hat\alpha\hat\beta} + N_{\hat\alpha\hat\beta}\right) J^{++} + 4\mathrm{i}\, S^{++} M_{\hat\alpha\hat\beta} - \frac{\mathrm{i}}{2}\varepsilon^{\hat a\hat b\hat c\hat d\hat e}(\Sigma_{\hat a\hat b})_{\hat\alpha\hat\beta} C_{\hat c}{}^{++} M_{\hat d\hat e} \;, \qquad (17)$$

where $J^{++} := u_i^+ u_j^+ J^{ij}$ and $S^{++} := u_i^+ u_j^+ S^{ij}$.

A covariant projective supermultiplet of weight n, $Q^{(n)}(z, u^+)$, is defined to be a scalar superfield that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:
(i) it obeys the covariant analyticity constraint

$$\mathcal{D}_{\hat\alpha}^+ Q^{(n)} = 0 \;; \qquad (18)$$

(ii) it is a homogeneous function of u^+ of degree n, that is,

$$Q^{(n)}(z, c\,u^+) = c^n\, Q^{(n)}(z, u^+) \;, \qquad c \in \mathbb{C} \setminus \{0\} \;; \qquad (19)$$

(iii) infinitesimal gauge transformations (3) act on $Q^{(n)}$ as follows:

$$\begin{aligned} \delta_K Q^{(n)} &= \left(K^{\hat C} \mathcal{D}_{\hat C} + K^{ij} J_{ij}\right) Q^{(n)} \;, \\ K^{ij} J_{ij} Q^{(n)} &= -\tfrac{1}{(u^+u^-)}\left(K^{++} D^{--} - n\, K^{+-}\right) Q^{(n)} \;, \qquad K^{\pm\pm} = K^{ij}\, u_i^\pm u_j^\pm \;, \end{aligned} \qquad (20)$$

where $D^{--} = u^{-i}\partial/\partial u^{+i}$. The right-hand side in (20) involves an additional isotwistor, u_i^- which is subject to the condition $(u^+u^-) = u^{+i}u_i^- \neq 0$, and is otherwise arbitrary. By construction, $Q^{(n)}$ is independent of u^-, i.e. $\partial Q^{(n)}/\partial u^{-i} = 0$. One can see that $\delta Q^{(n)}$ is also independent of the isotwistor u^-, that is $\partial(\delta Q^{(n)})/\partial u^{-i} = 0$, due to (19). It follows from (20) that $J^{++} Q^{(n)} \equiv 0$ which is the integrability condition for the constraint (18). It is important to note that, because of (ii), the isotwistor u_i^+ plays the role of homogeneous global coordinates for $\mathbb{C}P^1$ and the covariant projective multiplets live in curved projective superspace $\mathcal{M}^{5|8} \times \mathbb{C}P^1$.

In the case of conformal supergravity, we have to address the issue of how covariant projective multiplets may consistently vary under the super-Weyl transformations. If a weight-n projective superfield $Q^{(n)}$ is chosen to transforms homogeneously, $\delta_\sigma Q^{(n)} \propto \sigma Q^{(n)}$, then its transformation law turns out to be uniquely fixed by the constraint (18) to be

$$\delta_\sigma Q^{(n)} = 3n\, \sigma Q^{(n)} \;. \qquad (21)$$

Without the assumption of homogeneity, it is easy to construct examples of covariant projective multiplets which do not respect (21). The superfield S^{++} is a particularly important example. Due to eq. (12) (from here on we only consider the geometry with $C_{\hat a}{}^{ij} = 0$), S^{++} is a projective superfield of weight two, $\mathcal{D}_{\hat\alpha}^+ S^{++} = 0$. In accordance with (8a), its super-Weyl transformation is inhomogeneous

$$\delta_\sigma S^{++} = 2\sigma S^{++} + \frac{\mathrm{i}}{2}(\mathcal{D}^+)^2\sigma \;, \qquad (\mathcal{D}^+)^2 := \mathcal{D}^{+\hat\alpha}\mathcal{D}_{\hat\alpha}^+ \;. \qquad (22)$$

Another important example of weight-two projective multiplet is given by $\mathcal{G}^{++} := \mathcal{G}^{ij} u_i^+ u_j^+$ with $\mathcal{G}^{ij}$ the descendant associated with the Yang-Mills field strength $\mathcal{W}$ defined in (15). It satisfies the constraint $\mathcal{D}^+_{\hat\alpha} \mathcal{G}^{++} = 0$, and possesses the super-Weyl transformation law $\delta_\sigma \mathcal{G}^{++} = 6\sigma \mathcal{G}^{++}$ [14].

If $Q^{(n)}(u^+)$ is a covariant projective multiplet, its complex conjugate $\bar{Q}^{(n)}(\overline{u^+})$ is no longer of the same type. However, one can introduce a generalized *smile*-conjugation, $Q^{(n)} \to \widetilde{Q}^{(n)}$,

$$\widetilde{Q}^{(n)}(u^+) \equiv \bar{Q}^{(n)}\big(\overline{u^+} \to \widetilde{u}^+\big) \ , \qquad \widetilde{u}^+ = \mathrm{i}\, \sigma_2\, u^+ \ , \tag{23}$$

which acts on the space of covariant projective weight-n multiplets, because of relation $\widetilde{\mathcal{D}^+_{\hat\alpha} Q^{(n)}} = (-1)^{\epsilon(Q^{(n)})} \mathcal{D}^{+\hat\alpha} \widetilde{Q}^{(n)}$. One can see that $\widetilde{\widetilde{Q}}^{(n)} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be defined for n even.

To define a locally supersymmetric and super-Weyl invariant action, one needs two prerequisites [14]: (i) a Lagrangian $\mathcal{L}^{++}(z,u^+)$ which is a real projective multiplet of weight two and which possesses the super-Weyl transformation $\delta_\sigma \mathcal{L}^{++} = 6\sigma \mathcal{L}^{++}$; (ii) an Abelian vector multiplet with its field strength $W(z)$ non-vanishing everywhere. The action is:

$$S(\mathcal{L}^{++}) = \tfrac{2}{3\pi} \oint (u^+ \mathrm{d}u^+) \int \mathrm{d}^5x\, \mathrm{d}^8\theta\, E\, \tfrac{\mathcal{L}^{++} W^4}{(G^{++})^2} \ , \qquad E^{-1} = \mathrm{Ber}\,(E_{\hat A}{}^{\hat M}) \ . \tag{24}$$

Here $G^{++} := G^{ij} u_i^+ u_j^+$, where G^{ij} is the descendant (15) associated with W. Note that $S(\mathcal{L}^{++})$ is invariant under arbitrary re-scalings $u_i^+(t) \to c(t)\, u_i^+(t)$, $\forall c(t) \in \mathbb{C} \setminus \{0\}$, where t denotes the evolution parameter along the integration contour. The action can be shown to be invariant under supergravity gauge transformations (3) and (20), see [14, 13]. To see that $S(\mathcal{L}^{++})$ is invariant under super-Weyl transformations, one has only to note that $\delta_\sigma E = -2\sigma E$ and make use of the transformation rules $\delta_\sigma \mathcal{L}^{++} = 6\sigma \mathcal{L}^{++}$, $\delta_\sigma W = 2\sigma W$ and $\delta_\sigma G^{++} = 6\sigma G^{++}$.

The crucial property of $S(\mathcal{L}^{++})$ is that it is independent of the concrete choice of W, provided $\mathcal{L}^{++}$ is independent of such a vector multiplet. Another important feature of the action introduced is that (24) provides a natural extension of the action principle in flat projective superspace [17, 20].

Since the action (24) is super-Weyl invariant, one can choose the super-Weyl gauge $W = 1$. Then, the action functional (24) takes the form given in [13] in the case of the 5D minimal multiplet.

Now we are in a position to give some interesting examples of supergravity-matters systems. Let $\mathbb{V}(z,u^+)$ denote the tropical prepotential[3] for the Abelian vector multiplet W appearing in the action (24). The prepotential is a real weight-zero projective multiplet possessing the gauge invariance

$$\delta \mathbb{V} = \lambda + \tilde{\lambda} \ , \tag{25}$$

with λ a weight-zero arctic multiplet. A hypermultiplet can be described by an arctic weight-one multiplet $\Upsilon^+(z,u^+)$ and its smile-conjugate $\widetilde{\Upsilon}^+$. Consider a gauge invariant Lagrangian of the form (with the gauge transformation of Υ^+ being $\delta\Upsilon^+ = -\xi\lambda\Upsilon^+$)

$$\mathcal{L}^{++} = \tfrac{1}{\kappa^2} \mathbb{V}\, G^{++} - \widetilde{\Upsilon}^+ \mathrm{e}^{\xi \mathbb{V}}\, \Upsilon^+ \ , \tag{26}$$

with κ the gravitational coupling constant, and ξ a cosmological constant. It describes Poincaré supergravity if $\xi = 0$, and pure gauge supergravity with $\xi \neq 0$.

[3] See [12] for the definition of covariant arctic and tropical multiplets.

The dynamics of the Yang-Mills supermultiplet can be described by the Lagrangian $\mathcal{L}^{++}_{\rm YM} = g^{-2}\, \mathbb{V}\, \mathcal{G}^{++}$, with g the coupling constant (compare with the rigid supersymmetric case [21]).

A system of arctic weight-one multiplets $\Upsilon^+(z, u^+)$ and their smile-conjugates $\widetilde{\Upsilon}^+$ can be described by the Lagrangian

$$\mathcal{L}^{++} = {\rm i}\, K(\Upsilon^+, \widetilde{\Upsilon}^+) \ , \tag{27}$$

with $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real analytic function of n complex variables Φ^I, where $I = 1, \dots, n$. For $\mathcal{L}^{++}$ to be a weight-two real projective superfield, it is sufficient to require

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) \ . \tag{28}$$

This is a curved superspace generalization of the general model for superconformal polar multiplets [20] (see also [11]).

Given a system of interacting arctic weight-zero multiplets $\mathbf{\Upsilon}$ and their smile-conjugates $\widetilde{\mathbf{\Upsilon}}$, their coupling to supergravity can be described by the Lagrangian

$$\mathcal{L}^{++} = G^{++}\, \mathbf{K}(\mathbf{\Upsilon}, \widetilde{\mathbf{\Upsilon}}) \ , \tag{29}$$

with $\mathbf{K}(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real function which is not required to obey any homogeneity condition. The corresponding action is invariant under Kähler transformations of the form

$$\mathbf{K}(\mathbf{\Upsilon}, \widetilde{\mathbf{\Upsilon}}) \ \to \ \mathbf{K}(\mathbf{\Upsilon}, \widetilde{\mathbf{\Upsilon}}) + \mathbf{\Lambda}(\mathbf{\Upsilon}) + \bar{\mathbf{\Lambda}}(\widetilde{\mathbf{\Upsilon}}) \ , \tag{30}$$

with $\mathbf{\Lambda}(\Phi^I)$ a holomorphic function.

Happy Birthday, Ioseph L'vovich !!!

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WDVV solutions from orthocentric polytopes and Veselov systems

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract

$\mathcal{N}$=4 superconformal n-particle quantum mechanics on the real line is governed by two prepotentials, U and F, which obey a system of partial nonlinear differential equations generalizing the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation. For U=0 one remains with the WDVV equation which suggests an ansatz for F in terms of a set of covectors to be found. One approach constructs such covectors from suitable polytopes, another method solves Veselov's $\vee$-conditions in terms of deformed Coxeter root systems. I relate the two schemes for the A_n example.

1 Introduction

The issue of constructing $\mathcal{N}$=4 superconformal extensions of Calogero-type multi-particle quantum mechanics in one dimension has been attacked in several works [1]–[4]. In [1, 2] it was discovered that this task leads to the (generalized) Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation known from two-dimensional topological field theory [5, 6]. A physicist's classification of $\mathcal{N}$=4 superconformal mechanics models based on particular WDVV solutions has been advanced in [3, 4], where new models (with a purely quantum potential based on orthocentric simplices) were found. Independently, mathematicians' efforts revealed WDVV solutions derived from Coxeter systems and certain deformations thereof and lead to the notion of Veselov $\vee$-systems [7]–[13]. In the current paper I relate the mathematics approach with the physicist's picture for solving the (generalized) WDVV equation. In particular, the deformed A_n solutions of [9] will be mapped to the orthocentric simplices of [4].

In section 2 I recall the formulation $\mathcal{N}$=4 superconformal n-particle mechanics in terms of $su(1,1|2)$ generators. The closure of the superconformal algebra poses constraints on the interaction, which for an ansatz quartic in the fermionic coordinates lead to the WDVV equation plus a homogeneity condition for a quantum prepotential F and to related differential equations for a classical prepotential U. Section 3 expresses these prepotentials in terms of a system of covectors, thereby turning the differential to nonlinear algebraic equations. Putting U to zero, a family of WDVV solutions is constructed in section 4. Its covectors deform the A_n root system and are parametrized by the shape moduli of orthocentric n-simplices. The different formulations of the WDVV equation are related in section 5, where the geometry of the deformed A_n $\vee$-systems is made explicit.

2 WDVV equations from N=4 superconformal quantum mechanics

Let me consider a quantum mechanical system of n identical particles with unit mass on the real line, described by positions x^i and momenta p_i, and enhanced by fermionic degrees of freedom ψ^i_α and $\bar\psi^{i\alpha} = (\psi^i_\alpha)^\dagger$, where $i = 1, \dots, n$ and $\alpha = 1, 2$. Spinor indices are raised and lowered with the invariant tensor $\epsilon^{\alpha\beta}$ and its inverse $\epsilon_{\alpha\beta}$, where $\epsilon^{12} = 1$. Further, I impose the canonical quantization rules[1]

$$[x^i, p_j] = \mathrm{i}\delta_j{}^i \qquad \text{and} \qquad \{\psi^i_\alpha, \bar\psi^{j\beta}\} = \delta_\alpha{}^\beta \delta^{ij} , \tag{1}$$

with all other (anti)commutators vanishing. At this stage I have introduced a Euclidean metric (δ_{ij}) in the configuration space $\mathbb{R}^{n|4n}/S_n$.

I want the dynamics to be invariant under $\mathcal{N}$=4 superconformal transformations. Their generators $\{H, Q_\alpha, \bar Q^\alpha, D, J_a, S_\alpha, \bar S^\alpha, K\}$, with $a = 1, 2, 3$ and $(Q_\alpha)^\dagger = \bar Q^\alpha$ as well as $(S_\alpha)^\dagger = \bar S^\alpha$, form a (centrally extended) $su(1,1|2)$ algebra defined by the following non-vanishing (anti)commutation relations,

$$\begin{aligned}
&[D,H] = -\mathrm{i}\,H , && [H,K] = 2\mathrm{i}\,D , \\
&[D,K] = +\mathrm{i}\,K , && [J_a,J_b] = \mathrm{i}\,\epsilon_{abc}J_c , \\
&\{Q_\alpha,\bar Q^\beta\} = 2\,H\delta_\alpha{}^\beta , && \{Q_\alpha,\bar S^\beta\} = +2\mathrm{i}\,(\sigma_a)_\alpha{}^\beta J_a - 2\,D\delta_\alpha{}^\beta - \mathrm{i}\,C\delta_\alpha{}^\beta , \\
&\{S_\alpha\,,\,\bar S^\beta\} = 2\,K\delta_\alpha{}^\beta , && \{\bar Q^\alpha,S_\beta\} = -2\mathrm{i}\,(\sigma_a)_\beta{}^\alpha J_a - 2\,D\delta_\beta{}^\alpha + \mathrm{i}\,C\delta_\beta{}^\alpha , \\
&[D,Q_\alpha] = -\tfrac{1}{2}\mathrm{i}\,Q_\alpha , && [D,S_\alpha] = +\tfrac{1}{2}\mathrm{i}\,S_\alpha , \\
&[K,Q_\alpha] = +\mathrm{i}\,S_\alpha , && [H,S_\alpha] = -\mathrm{i}\,Q_\alpha , \\
&[J_a,Q_\alpha] = -\tfrac{1}{2}\,(\sigma_a)_\alpha{}^\beta Q_\beta , && [J_a,S_\alpha] = -\tfrac{1}{2}\,(\sigma_a)_\alpha{}^\beta S_\beta , \\
&[D,\bar Q^\alpha] = -\tfrac{1}{2}\mathrm{i}\,\bar Q^\alpha , && [D,\bar S^\alpha] = +\tfrac{1}{2}\mathrm{i}\,\bar S^\alpha , \\
&[K,\bar Q^\alpha] = +\mathrm{i}\,\bar S^\alpha , && [H,\bar S^\alpha] = -\mathrm{i}\,\bar Q^\alpha , \\
&[J_a,\bar Q^\alpha] = \tfrac{1}{2}\,\bar Q^\beta(\sigma_a)_\beta{}^\alpha , && [J_a,\bar S^\alpha] = \tfrac{1}{2}\,\bar S^\beta(\sigma_a)_\beta{}^\alpha .
\end{aligned} \tag{2}$$

Here, $\epsilon_{123} = 1$, C stands for the central charge, and $\{\sigma_1, \sigma_2, \sigma_3\}$ denote the Pauli matrices.

[1] I suppress $\hbar$ except for illustrative purposes.

For a realization of the generators I try (repeated indices are summed over) [1]–[4]

$$\begin{aligned}
K &= \frac{1}{2}x^i x^i \ , \qquad S_\alpha = x^i \psi^i_\alpha \ , \qquad \bar{S}^\alpha = x^i \bar{\psi}^{i\alpha} \ , \\
D &= -\frac{1}{4}(x^i p_i + p_i x^i) \ , \qquad J_a = \frac{1}{2}\bar{\psi}^{i\alpha}(\sigma_a)_\alpha{}^\beta \psi^i_\beta \ , \\
Q_\alpha &= \big(p_j - \mathrm{i}\, x^i\, U_{ij}(x)\big)\, \psi^j_\alpha \ - \frac{\mathrm{i}}{2}\, x^i\, F_{ijkl}(x)\, \langle \psi^j_\beta\, \psi^{k\beta} \bar{\psi}^l_\alpha \rangle \ , \\
\bar{Q}^\alpha &= \big(p_j + \mathrm{i}\, x^i\, U_{ij}(x)\big)\, \bar{\psi}^{j\alpha} - \frac{\mathrm{i}}{2}\, x^i\, F_{ijkl}(x)\, \langle \psi^{j\alpha} \bar{\psi}^{k\beta} \bar{\psi}^l_\beta \rangle \ , \\
H &= \frac{1}{2}p_i p_i \ + \ V_B(x) \ - \ U_{ij}(x)\langle \psi^i_\alpha \bar{\psi}^{j\alpha} \rangle \ + \ \frac{1}{4}F_{ijkl}(x)\langle \psi^i_\alpha \psi^{j\alpha} \bar{\psi}^{k\beta} \bar{\psi}^l_\beta \rangle \ ,
\end{aligned} \tag{3}$$

with completely symmetric unknown functions V_B, U_{ij} and F_{ijkl} homogeneous of degree -2 in $x \equiv \{x^1, \ldots, x^n\}$. Here, the symbol $\langle \ldots \rangle$ stands for symmetric (or Weyl) ordering. The ordering ambiguity present in the fermionic sector affects the bosonic potential V_B. In contrast to the $\mathcal{N}$=2 superconformal extensions [14, 15], the closure of the algebra demands the quartic term, and a nonzero central charge requires the quadratic term. Hence, there does not exist a free mechanical representation of the algebra (2). A prototypical model is of the Calogero type,

$$V_B \ = \ \sum_{i<j} \frac{g^2}{(x^i - x^j)^2} \ , \qquad U_{ij} \ = ? \ , \qquad F_{ijkl} \ = ? \ . \tag{4}$$

Inserting the representation (3) into the algebra (2), one produces a fairly long list of constraints on V_B, U_{ij} and F_{ijkl}. One of the consequences is that [1, 2, 3]

$$\begin{aligned}
U_{ij} &= \partial_i \partial_j U \qquad \text{and} \qquad F_{ijkl} \ = \ \partial_i \partial_j \partial_k \partial_l F \ , \\
V_B &= \frac{1}{2}\,(\partial_i U)(\partial_i U) \ + \ \frac{\hbar^2}{8}\,(\partial_i \partial_j \partial_k F)(\partial_i \partial_j \partial_k F) \ ,
\end{aligned} \tag{5}$$

which introduces two scalar prepotentials. Note that a quadratic polynomial in F or a constant in U are irrelevant. The constraints then turn into the following system of nonlinear partial differential equations [2, 3] (see also [1]),

$$(\partial_i \partial_k \partial_p F)(\partial_j \partial_l \partial_p F) \ = \ (\partial_j \partial_k \partial_p F)(\partial_i \partial_l \partial_p F) \quad , \qquad x^i \partial_i \partial_j \partial_k F \ = \ -\delta_{jk} \ , \tag{6}$$

$$\partial_i \partial_j U - (\partial_i \partial_j \partial_k F)\, \partial_k U \ = \ 0 \quad , \qquad x^i \partial_i U \ = \ -C \ , \tag{7}$$

which I refer to as the "structure equations". Notice that these equations are quadratic in F but only linear in U. The first of (6) is a kind of zero-curvature condition for a connection $\partial^3 F$. It coincides with the (generalized) WDVV equation known from topological field theory [5, 6]. The first of (7) is a kind of covariant constancy for ∂U in the $\partial^3 F$ background. Since its integrability implies the WDVV equation projected onto ∂U, I call it the "flatness condition".

The right equations in (6) and (7) represent homogeneity conditions for U and F. They are are inhomogeneous with constants δ_{jk} and C (the central charge) on the right-hand side and display an explicit coordinate dependence. Furthermore, the second equation in (6) can be integrated twice, arriving at

$$(x^i \partial_i - 2)F \ = \ -\frac{1}{2}\, x^i x^i \qquad \text{and} \qquad x^i \partial_i U \ = \ -C \ . \tag{8}$$

where I used the freedom in the definition of F to put the integration constants – a linear function on the right-hand side – to zero.

There are some dependencies among the equations (6) and (7). The contraction of two left equations with x^i is a consequence of the two right equations, and therefore only the components orthogonal to x are independent, effectively reducing the dimension to $n{-}1$. This means that only

$\frac{1}{12}n(n-1)^2(n-2)$ WDVV equations need to be solved and only $\frac{1}{2}n(n-1)$ flatness conditions have to be checked. For n=2 in particular, the single WDVV equation follow from the homogeneity condition in (6), and the three flatness conditions are all equivalent. Hence, the nonlinearity of the structure equations becomes only relevant for $n\geq 3$.

3 Covector ansatz for the prepotentials

For a particular solution to (8), I make the ansatz [1, 3, 4]

$$F \;=\; -\frac{1}{2}\sum_\alpha f_\alpha\;\alpha(x)^2\,\ln|\alpha(x)| \qquad\text{and}\qquad U \;=\; -\sum_\alpha g_\alpha\;\ln|\alpha(x)| \tag{9}$$

with real coefficients f_α and g_α, where α runs over a finite set of (unlabelled) noncollinear covectors in $\mathbb{R}^n$, i.e.

$$\alpha(x) \;=\; \alpha_i\,x^i \qquad \text{for each covector } \alpha\;. \tag{10}$$

The center-of-mass degree of freedom corresponds to $\alpha(x)=\rho(x)\equiv\sum_i x^i$, and the relative particle motion is translation invariant only if $\alpha_i\rho_i=0\quad\forall\alpha{\neq}\rho$, meaning that the other covectors span only the hyperplane perpendicular to ρ and $\{\alpha\}$ decomposes orthogonally. Identical particles require the set $\{\alpha\}$ to be invariant (up to sign) under permutations of the components α_i and enforce equality of the f_α (and g_α) coefficients for permutation-related covectors. Relative translation invariance and permutation symmetry are coordinate-dependent properties; they are not preserved by a generic SO(n) coordinate transformation. Therefore, demanding either will severely restrict the coordinate choice. Finally, a rescaling of α may be absorbed into a renormalization of f_α. Therefore, only the rays $\mathbb{R}_+\alpha$ are invariant data. I cannot, however, change the sign of f_α in this manner.

Compatibility of (9) with the conditions (8) directly yields

$$\sum_\alpha f_\alpha\,\alpha_i\alpha_j \;=\; \delta_{ij} \qquad\text{and}\qquad \sum_\alpha g_\alpha \;=\; C\;. \tag{11}$$

The second relation fixes the central charge, and the g_α are independent free couplings if not forced to zero. The first relation amounts to a decomposition of the identity (δ_{ij}) into (usually non-orthogonal) rank-one projectors and imposes $\frac{1}{2}n(n{+}1)$ relations on the coefficients f_α for a given set $\{\alpha\}$.

From (9) one derives

$$\partial_i\partial_j\partial_k F \;=\; -\sum_\alpha f_\alpha\,\tfrac{\alpha_i\alpha_j\alpha_k}{\alpha(x)} \qquad\text{and}\qquad \partial_i U \;=\; -\sum_\alpha g_\alpha\,\tfrac{\alpha_i}{\alpha(x)}\;, \tag{12}$$

and so the bosonic part of the potential takes the form

$$V_B \;=\; \frac{1}{2}\sum_{\alpha,\beta}\tfrac{\alpha\cdot\beta}{\alpha(x)\,\beta(x)}\Big(g_\alpha g_\beta \;+\; \frac{\hbar^2}{4}f_\alpha f_\beta\,(\alpha\cdot\beta)^2\Big) \tag{13}$$

with the covector scalar product

$$\alpha\cdot\beta \;=\; \alpha_i\,\delta^{ij}\beta_j \;=\; \alpha_i\,\beta_i\;. \tag{14}$$

The remaining structure equations in (6) and (7) become

$$\sum_{\alpha,\beta} f_\alpha f_\beta\,\tfrac{\alpha\cdot\beta}{\alpha(x)\,\beta(x)}\,(\alpha\wedge\beta)^{\otimes 2} \;=\; 0 \qquad\qquad \text{and} \tag{15}$$

$$\sum_\beta\Big(g_\beta\,\tfrac{1}{\beta(x)} \;-\; f_\beta\sum_\alpha g_\alpha\,\tfrac{\alpha\cdot\beta}{\alpha(x)}\Big)\,\tfrac{1}{\beta(x)}\;\beta\otimes\beta \;=\; 0 \tag{16}$$

with

$$(\alpha\wedge\beta)^{\otimes 2}_{ijkl} \;=\; (\alpha_i\beta_j-\alpha_j\beta_i)(\alpha_k\beta_l-\alpha_l\beta_k) \qquad \text{and} \qquad (\beta\otimes\beta)_{ij} \;=\; \beta_i\,\beta_j\ . \tag{17}$$

The task is to first solve (15) and (11), i.e. find sets $\{\alpha, f_\alpha\}$, and then to determine $\{g_\alpha\}$ from (16), subject to (11). Many F backgrounds do not admit a $C{\neq}0$ solution, but a homogeneous U can always be found [4]. I close the section with a simplifying observation. If a set of covectors decomposes into mutually orthogonal subsets, (15) and (16) hold for each subset individually, and their prepotentials just add up to the total F or U. Therefore, one may restrict the analysis to indecomposable covector sets.

4 WDVV solutions from orthocentric simplices

For the rest of the paper I put U to zero and investigate solutions to the WDVV equations (15), subject to the homogeneity condition

$$\sum_\alpha f_\alpha\ \alpha\otimes\alpha \;=\; \mathbb{1}\ . \tag{18}$$

Let me look for indecomposable sets of covectors obeying the WDVV equation (15). In one dimension, the equation is trivial. For $n{=}2$, it follows from the homogeneity condition (18), which can actually be satisfied for *any* set $\{\alpha\}$ of coplanar covectors [4]. Nevertheless, it is instructive to outline the simplest examples. For the case of two covectors $\{\alpha,\beta\}$ one is forced to $\alpha{\cdot}\beta=0$. For three coplanar covectors $\{\alpha,\beta,\gamma\}$, the homogeneity condition (18) uniquely fixes the f coefficients to

$$f_\alpha \;=\; -\tfrac{\beta\cdot\gamma}{\alpha\wedge\beta\ \gamma\wedge\alpha} \qquad \text{and cyclic}\ , \tag{19}$$

due to the identity

$$\beta{\wedge}\gamma\ \beta{\cdot}\gamma\ \alpha^i\alpha^j \;+\; \text{cyclic} \;=\; -\alpha{\wedge}\beta\ \beta{\wedge}\gamma\ \gamma{\wedge}\alpha\ \delta^{ij}\ . \tag{20}$$

The traceless part of the homogeneity condition should imply the single WDVV equation (15) in two dimensions. Indeed, the choice (19) turns the latter into

$$\alpha{\wedge}\beta\ \gamma(x) \;+\; \beta{\wedge}\gamma\ \alpha(x) \;+\; \gamma{\wedge}\alpha\ \beta(x) \;=\; 0 \tag{21}$$

which is identically true. Without loss of generality I may assume that $\alpha+\beta+\gamma=0$, i.e. the three covectors form a triangle. In this case I have $\alpha{\wedge}\beta=\beta{\wedge}\gamma=\gamma{\wedge}\alpha=2A$, where the area A of the triangle may still be scaled to $\frac{1}{2}$, and (19) simplifies to

$$f_\alpha \;=\; -\tfrac{\beta\cdot\gamma}{4\,A^2} \qquad \text{and cyclic}\ . \tag{22}$$

In dimension $n{=}3$, the minimal set of three covectors must form an orthogonal basis, with $f_\alpha^{-1}=\alpha{\cdot}\alpha$. Let me skip the cases of four and five covectors and go to the situation of six covectors because the homogeneity condition (18) then precisely determines all f coefficients. However, it is not true that six generic covectors can be scaled to form the edges of a polytope. The space of six rays in $\mathbb{R}^3$ modulo rigid SO(3) is nine dimensional, while the space of tetrahedral shapes (modulo size) has only five dimensions. In order to generalize the $n{=}2$ solution above, let me assume that my six covectors can be scaled to form a tetrahedron, with volume V and edges $\{\alpha,\beta,\gamma,\alpha',\beta',\gamma'\}$ where α' is skew to α and so on. Any such tetrahedron is determined by giving three nonplanar covectors, say $\{\alpha,\beta,\gamma'\}$, which up to rigid rotation are fixed by six parameters, corresponding to the shape and size of the tetrahedron.

The triangle result (22) can be employed to patch together the unique solution to the homogeneity condition (18) for the tetrahedron, but only if the geometric constraints

$$\alpha\cdot\alpha' \;=\; 0\ , \qquad \beta\cdot\beta' \;=\; 0\ , \qquad \gamma\cdot\gamma' \;=\; 0 \tag{23}$$

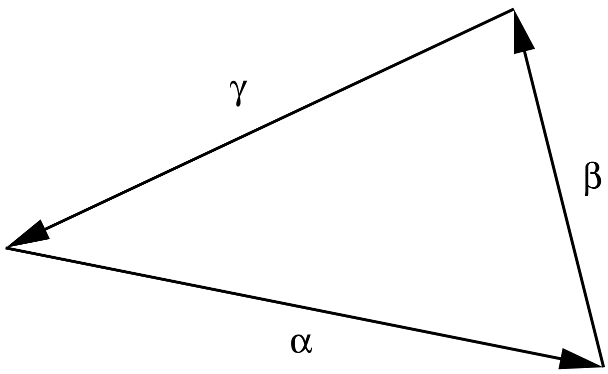

Figure 1: Triangular configuration of covectors

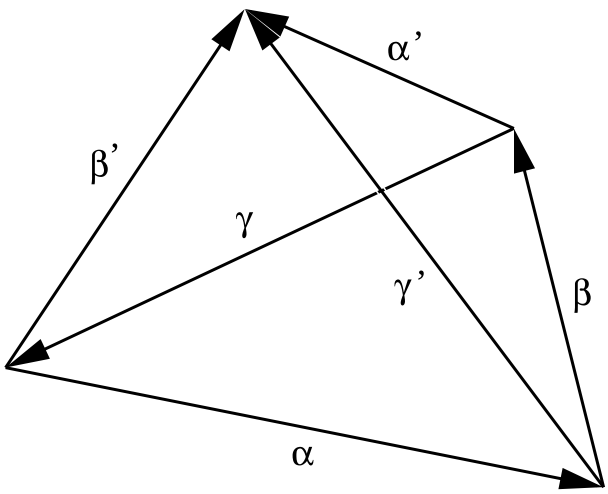

Figure 2: Tetrahedral configuration of covectors

are obeyed for the pairs of skew edges. In this situation, the identity

$$\beta{\cdot}\gamma\ \beta'{\cdot}\gamma'\ \alpha^i\alpha^j\ +\ \beta{\cdot}\gamma'\ \beta'{\cdot}\gamma\ \alpha'^i\alpha'^j\ +\ \text{cyclic}\ =\ -36\,V^2\,\delta^{ij}\ , \tag{24}$$

guarantees the homogeneity condition (18) for

$$f_\alpha\ =\ -\tfrac{\beta\cdot\gamma\ \beta'\cdot\gamma'}{36\,V^2} \qquad \text{and} \qquad f_{\alpha'}\ =\ -\tfrac{\beta\cdot\gamma'\ \beta'\cdot\gamma}{36\,V^2} \tag{25}$$

plus their cyclic images. Tetrahedra subject to (23) are called "orthocentric" [16]. They are characterized by the fact that all four altitudes are concurrent (in the orthocenter) and their feet are the orthocenters of the faces. The space of orthocentric tetrahedra is of codimension two inside the space of all tetrahedra and represents a three-parameter deformation of the A_3 root system (ignoring the overall scale).

What about the WDVV equation in this case? The 15 pairs of edges in the double sum of (15) group into four triples corresponding to the tetrahedron's faces plus the three skew pairs. It is not hard to see that for each face the contributions add to zero, and so the concurrent edge pairs do not contribute to the double sum in (15). This leaves the three skew pairs, but their contribution is killed by the orthocentricity constraint (23), and the WDVV equation is indeed obeyed.

Although I do not know the f coefficients for a general tetrahedron, I can offer the following proof that the WDVV equation already enforces the orthocentricity. Consider the limit $\hat{n}(x) \to \infty$ for some fixed covector $\hat{n}$ of unit length. Decomposing

$$\alpha \;=\; \alpha{\cdot}\hat{n}\,\hat{n} + \alpha_\perp \qquad \longrightarrow \qquad \alpha(x) \;=\; \alpha{\cdot}\hat{n}\,\hat{n}(x) + \alpha_\perp(x) \tag{26}$$

we see that any factor $\frac{1}{\alpha(x)}$ vanishes in this limit unless $\alpha{\cdot}\hat{n} = 0$. Thus, only covectors perpendicular to $\hat{n}$ survive in (15) and (16), reducing the system to the hyperplane orthogonal to $\hat{n}$. On the other hand, any solution to these equations, being an identity in x, must carry over to a solution of the limiting equations, which correspond to the dimensionally reduced system. In a general tetrahedron, take $\hat{n} \propto \alpha\wedge\alpha'$. Then, the limit $\hat{n}(x) \to \infty$ in (15) retains only the covectors α and α', and the WDVV equation reduces to a single term, which vanishes only for $\alpha{\cdot}\alpha' = 0$. Equivalently, the plane spanned by α and α' contains no further covector, and two covectors in two dimensions must be orthogonal. The same argument applies to $\beta{\cdot}\beta'$ and $\gamma{\cdot}\gamma'$, completing the proof.

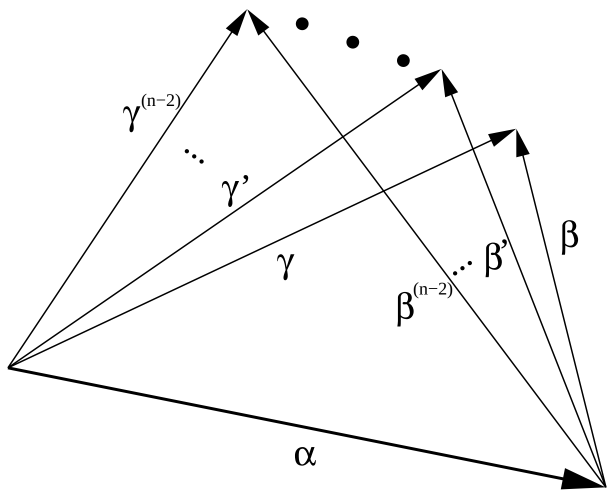

Figure 3: Faces sharing an edge of an n-simplex

This scheme may be taken to any dimension n. A simplicial configuration of $\frac{1}{2}n(n{+}1)$ covectors is already determined by n independent covectors, which modulo SO(n) are given by $\frac{1}{2}n(n{+}1)$ parameters. The homogeneity condition (18) uniquely fixes the f coefficients. Employing an iterated dimensional reduction to any plane spanned by a skew pair of edges and realizing that no other edge lies in such a plane, one sees that the WDVV equation always demands such an edge pair to be orthogonal. This condition renders the n-simplex orthocentric and reduces the number of degrees of freedom to $n{+}1$ (now including the overall scale given by the n-volume V). In this situation I can write down the unique solution to both the homogeneity condition and the WDVV equation,

$$f_\alpha \;=\; \frac{\beta{\cdot}\gamma\;\beta'{\cdot}\gamma'\;\beta''{\cdot}\gamma''\;\ldots\;\beta^{(n-2)}{\cdot}\gamma^{(n-2)}}{(n!\;V)^2}\;, \tag{27}$$

where the edge α is shared by the $n{-}1$ faces $\langle\alpha\beta\gamma\rangle$, $\langle\alpha\beta'\gamma'\rangle$, ..., $\langle\alpha\beta^{(n-2)}\gamma^{(n-2)}\rangle$, and I have oriented all edges as pointing away from α. This formula works because any sub-simplex, in particular any tetrahedral building block, is itself orthocentric. To summarize, the WDVV solutions for simplicial covector configurations in any dimension are exhausted by an n-parameter deformation of the A_n root system. The n moduli are relative angles and do not include the $\frac{1}{2}n(n{+}1)$ trivial covector rescalings, which, apart from the common scale, destroy the tetrahedron.

These findings suggest that covector configurations corresponding to deformations of other roots systems may solve the WDVV equations as well. For verification, I propose to consider the polytopes associated with the weight systems of a given Lie algebra, since their edge sets are built from the root covectors. The idea is then to relax the angles of such polytopes and analyze the constraints from the homogeneity and WDVV equations. The above n-dimensional orthocentric hypertetrahedra emerge simply from the fundamental representations of A_n. Extending this strategy to other representations and Lie algebras could lead to many more solutions.

5 WDVV solutions from Veselov systems

In the mathematical literature, the (generalized) WDVV equation is usually formulated as

$$W_i\,W_k^{-1}W_j \;=\; W_j\,W_k^{-1}W_i \qquad \text{for} \quad i,j,k=1,\dots,n\ , \tag{28}$$

where W_i is an $n{\times}n$ matrix with entries

$$(W_i)_{lm} \;=\; \partial_i\partial_l\partial_m W \qquad \text{for} \quad W=W(y^1,\dots,y^n)\ , \tag{29}$$

and $\partial_i \equiv \dfrac{\partial}{\partial y^i}$. It is easy to show [7] that (28) is equivalent to

$$W_i\,G^{-1}W_j \;=\; W_j\,G^{-1}W_i \qquad \text{with} \quad G=-y^k\,W_k\ , \tag{30}$$

which in components reads

$$(\partial_i\partial_l\partial_p W)\,G^{pq}(\partial_q\partial_m\partial_j W) \;=\; (\partial_j\partial_l\partial_p W)\,G^{pq}(\partial_q\partial_m\partial_i W)\ , \tag{31}$$

where the index position distinguishes between the metric G and its inverse G^{-1}. For the covector ansatz (9)

$$W \;=\; -\frac12\sum_\beta f_\beta\;\beta(y)^2\;\ln|\beta(y)| \tag{32}$$

it follows that

$$W_i \;=\; -\sum_\beta f_\beta\;\tfrac{\beta_i}{\beta(y)}\;\beta\otimes\beta \qquad\longrightarrow\qquad G \;=\; \sum_\beta f_\beta\;\beta\otimes\beta\ . \tag{33}$$

How is this related to the material of the previous sections? Comparing with (18), it seems that one must impose the additional condition of $G=-\mathbb{1}$. However, this is not so, because such a choice may be achieved by a linear coordinate change

$$x^i \;=\; y^j\,M_j{}^i \qquad\longrightarrow\qquad \beta_i \;=\; M_i{}^j\,\alpha_j \tag{34}$$

so that for $F(x)=W(y)$ one gets

$$W_i \;=\; M_i{}^j\,F_j \qquad \text{and} \qquad G_{lm} \;=\; -y^k\,W_{klm} \;=\; -M_l{}^i M_m{}^j\,x^k F_{kij} \;=\; M_l{}^i\,\delta_{ij}\,M^{\top j}{}_m\ , \tag{35}$$

where the right equation in (6) was used in the last step. This converts the metric (G_{ij}) of the y-frame to the Euclidean metric (δ_{ij}) in the x-frame,[2] and changes the covector scalar product accordingly,

$$\beta{\cdot}\beta' \;=\; \beta_i\,G^{ij}\beta'_j \;=\; \alpha_k\,M^{\top k}{}_i\,G^{ij}M_j{}^l\,\alpha'_l \;=\; \alpha_k\,\delta^{kl}\alpha'_l \;=\; \alpha{\cdot}\alpha'\ , \tag{36}$$

in short:

$$G \;=\; M\,\delta\,M^\top \qquad \text{and} \qquad \delta \;=\; M^\top G^{-1}M\ . \tag{37}$$

[2]Note that for the y-frame one must replace δ^{ij} with G^{ij} in the quantization rule (1).

Thus, solutions to (31) of the form (32) can be translated to solutions to (6) of the form (9) by a linear transformation.

For a prominent example, I turn to the n-parameter deformation of the A_n root system first proposed in [9],

$$\{\beta\} \;=\; \bigl\{\sqrt{c_i c_j}\,(e^i - e^j)\;,\;\sqrt{c_i}\,e^i \;\big|\; 1\le i<j\le n\bigr\} \qquad \text{(no sums)}\;, \tag{38}$$

where $e^i(y) = y^i$ and the c_i are arbitrary (positive) parameters. It was shown that this covector set satisfies the so-called $\vee$-conditions, which implies that (with $f_\beta = 1$) it provides an n-parameter family of solutions (32) to the WDVV equation. For this case, the metric and its inverse are quickly evaluated,

$$G_{ij} \;=\; \bigl(1+\textstyle\sum_k c_k\bigr)\,c_i\,\delta_{ij} - c_i c_j \qquad\text{and}\qquad G^{ij} \;=\; \bigl(1+\textstyle\sum_k c_k\bigr)^{-1}\bigl(c_i^{-1}\delta^{ij}+1\bigr)\;, \tag{39}$$

but in order to compute the corresponding transformation matrix M (or its inverse M^{-1}) via (37) one has to diagonalize G (or G^{-1}), which is not an easy task.

However, in order to interpret the solution (38) in the x-frame, it suffices to study its geometric (frame-independent) properties. First, I rescale each β by shifting the square roots into f_β coefficients,

$$\{\gamma\} \;=\; \bigl\{e^i - e^j\;,\;e^i\bigr\} \qquad\text{and}\qquad \{f_\gamma\} \;=\; \bigl\{c_i c_j\;,\;c_i\bigr\} \qquad\text{for}\quad 1\le i<j\le n\;, \tag{40}$$

and observe that the new covectors fulfil the incidence relations of an n-simplex. Second, I must figure out the angles formed by its edges,

$$\cos\angle(\gamma,\gamma') \;=\; \tfrac{\gamma\cdot\gamma'}{\sqrt{\gamma\cdot\gamma\;\gamma'\cdot\gamma'}} \qquad\text{with}\quad \gamma\cdot\gamma' \;=\; \gamma_i\,G^{ij}\,\gamma'_j\;. \tag{41}$$

These angles depend on the deformation parameters c_i, except for

$$e^i\cdot(e^j - e^k) \;=\; 0 \qquad\text{for } i,\,j,\,k \text{ mutually distinct}\;, \tag{42}$$

which means that non-concurrent edges are orthogonal to one another! This is a frame-independent statement and qualifies the polytope based on (38) as an orthocentric one.

Clearly, I have rediscovered the solution family of section 4. As a side result, one obtains an explicit parametrization of orthocentric n-simplices,

$$\{\alpha\}(c) \;=\; \bigl\{M^{-1}(e^i - e^j)\;,\;M^{-1}e^i\bigr\}\;, \tag{43}$$

where the c_i-dependence enters via the matrix M^{-1}. The (physical) geometries corresponding to the other known $\vee$-systems remain to be worked out.

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Dark energy, inflation and dark matter from modified $F(R)$ gravity

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Abstract

We review modified $F(R)$ gravity as realistic candidate to describe the observable universe expansion history. We show that recent cosmic acceleration, radiation/matter-dominated epoch and inflation could be realized in the framework of $F(R)$-gravity in the unified way. For some viable classes of $F(R)$-gravity, the Newton law is respected and there is no so-called matter instability (the very heavy positive mass for additional scalar degree of freedom is generated). The reconstruction program in modified gravity is also reviewed and it is demonstrated that *any* time-evolution of the universe expansion could be realized in $F(R)$-gravity. These models remain to be realistic also in the presence of non-minimal gravitational coupling with usual matter. It is shown that same model which passes local tests and predicts the unification of inflation with cosmic acceleration also describes dark matter thanks to presence of additional scalar degree of freedom and chameleon mechanism.

1 Introduction

Modified gravity suggests very natural answers to resolution of several fundamental cosmological problems. For instance, the observable universe expansion history may be described by modified gravity. Indeed, it gives very beautiful unification of the early-time inflation and late-time acceleration thanks to different role of gravitational terms relevant at small and at large curvature. Moreover, the coincidence problem may be solved in such theory simply by the universe expansion. Some models of modified gravity are predicted by string/M-theory considerations.

From another side, dark matter may be described totally in terms of modified gravity. Moreover, modified gravity may be useful in high energy physics, for instance, to solve the hierarchy or gravity-GUTs unification problems. Finally, modified gravity may pass the local tests and cosmological bounds.

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Usually the evolution of the universe can be described by the FRW equation:

$$\frac{3}{\kappa^2}H^2 = \rho \ . \tag{1}$$

Here the spatial part of the universe is assumed to be flat. We denote the Hubble rate by H, which is defined in terms of the scale factor a by

$$H \equiv \frac{\dot a}{a} \ . \tag{2}$$

In (1), ρ expresses the energy density of the usual matter, dark matter, and dark energy. The dark energy could be cosmological constant and/or a matter with 'equation of state (EoS)' parameter w, which is less than $-1/3$ and is defined by

$$w \equiv \frac{p}{\rho} \ . \tag{3}$$

Instead of including unknown exotic matter or energy, one may consider the modification of gravity, which corresponds to the change of the l.h.s. in (1).

An example of such modified gravity pretending to describe dark energy could be the scalar-Einstein-Gauss-Bonnet gravity [1], whose action is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)\mathcal{G}\right\} \ . \tag{4}$$

Here $\mathcal{G}$ is Gauss-Bonnet invariant:

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ . \tag{5}$$

Another example is so-called $F(R)$-gravity (for a review, see [2]). In $F(R)$-gravity models[2], the scalar curvature R in the Einstein-Hilbert action

$$S_{\rm EH} = \int d^4x\sqrt{-g}R \ , \tag{6}$$

is replaced by a proper function of the scalar curvature:

$$S_{F(R)} = \int d^4x\sqrt{-g}F(R) \ . \tag{7}$$

Recently, an interesting realistic theory has been proposed in [3], where $F(R)$ is given by

$$F(R) = \frac{1}{2\kappa^2}\left(R + f_{HS}(R)\right) \ , \quad f_{HS}(R) = -\frac{m^2 c_1 \left(R/m^2\right)^n}{c_2\left(R/m^2\right)^n + 1} \ . \tag{8}$$

In this model, R is large even in the present universe, and $f_{HS}(R)$ could be expanded by the inverse power series of R:

$$f_{HS}(R) \sim -\frac{m^2 c_1}{c_2} + \frac{m^2 c_1}{c_2^2}\left(\frac{R}{m^2}\right)^{-n} + \cdots \ , \tag{9}$$

Then there appears an effective cosmological constant $\Lambda_{\rm eff}$ as $\Lambda_{\rm eff} = m^2 c_1/c_2$, which generates the accelerating expansion in the present universe

In the HS-model (8), there occurs a flat spacetime solution, where $R = 0$, since the following condition is satisfied:

$$\lim_{R\to 0} f_{HS}(R) = 0 \ . \tag{10}$$

An interesting point in the HS model is that several cosmological conditions could be satisfied.

In the next section, we review on the general properties of $F(R)$-gravity. After some versions of $F(R)$-gravity were proposed as a model of the dark energy, there were indicated several problems/viability criteria, which we review in Section 3. It is shown how the critique of modified gravity may be removed for realistic models. In Section 4, we propose models [4] and [5], which unify the early-time inflation and the recent cosmic acceleration and pass several cosmological constraints. Reconstruction program for $F(R)$-gravity is reviewed in Section 5. The partial reconstruction scenario is proposed. Section six is devoted to the description of dark matter in terms of viable modified gravity where composite scalar particle from $F(R)$ gravity plays the role of dark particle. Non-minimal modified gravity is discussed in section seven. Some summary and outlook is given in the last section.

2 General properties of $F(R)$-gravity

In this section, the general properties of the $F(R)$-gravity are reviewed. For general $F(R)$-gravity, one can define an effective equation of state (EoS) parameter. The FRW equations in Einstein gravity coupled with perfect fluid are:

$$\rho = \frac{3}{\kappa^2} H^2 \ , \quad p = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right) \ . \tag{11}$$

For modified gravities, one may define an effective EoS parameter as follows:

$$w_{\rm eff} = -1 - \frac{2\dot{H}}{3H^2} \ . \tag{12}$$

The equation of motion for modified gravity is given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2} T_{(m)\mu\nu} \ . \tag{13}$$

By assuming spatially flat FRW universe,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \ , \tag{14}$$

the FRW-like equation follows:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right) F'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right) F''(R) + \kappa^2 \rho_{(m)} \tag{15}$$

There may be several (often exact) solutions of (15). Without any matter, assuming that the Ricci tensor could be covariantly constant, that is, $R_{\mu\nu} \propto g_{\mu\nu}$, Eq.(13) reduces to the algebraic equation:

$$0 = F(R) - 2RF(R) \ . \tag{16}$$

If Eq.(16) has a solution, the Schwarzschild (or Kerr) - (anti-)de Sitter space is an exact vacuum solution (see[6] and refs. therein).

When $F(R)$ behaves as $F(R) \propto R^m$ and there is no matter, there appears the following solution:

$$H \sim \frac{-\frac{(m-1)(2m-1)}{m-2}}{t} \ , \tag{17}$$

which gives the following effective EoS parameter:

$$w_{\rm eff} = -\frac{6m^2 - 7m - 1}{3(m-1)(2m-1)} \ . \tag{18}$$

When $F(R) \propto R^m$ again but if the matter with a constant EoS parameter w is included, one may get the following solution:

$$H \sim \frac{\frac{2m}{3(w+1)}}{t} \ , \tag{19}$$

and the effective EoS parameter is given by

$$w_{\rm eff} = -1 + \frac{w+1}{m} \ . \tag{20}$$

This shows that modified gravity may describe early/late-time universe acceleration.

3 Problems with $F(R)$-gravity

Immediately, after the $F(R)$-models were proposed as models of the dark energy, there appeared several works [7, 8] (and more recently in [9, 10]) criticizing such theories.

First of all, we comment on the claim in [7]. Note that one can rewrite $F(R)$-gravity in the scalar-tensor form. By introducing the auxiliary field A, we rewrite the action (7) of the $F(R)$-gravity in the following form:

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right)+F(A)\right\} \ . \tag{21}$$

By the variation over A, one obtains $A = R$. Substituting $A = R$ into the action (21), one can reproduce the action in (7). Furthermore, we rescale the metric in the following way (conformal transformation):

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu} \ , \quad \sigma = -\ln F'(A) \ . \tag{22}$$

Hence, the Einstein frame action is obtained:

$$\begin{aligned} S_E &= \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left(R-\frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) \ , \\ V(\sigma) &= \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \end{aligned} \tag{23}$$

Here $g\left(\mathrm{e}^{-\sigma}\right)$ is given by solving the equation $\sigma = -\ln\left(1+f'(A)\right) = \ln F'(A)$ as $A = g\left(\mathrm{e}^{-\sigma}\right)$. Due to the scale transformation (22), there appears a coupling of the scalar field σ with usual matter. The mass of σ is given by

$$m_\sigma^2 \equiv \frac{1}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{1}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{\left(F'(A)\right)^2} + \frac{1}{F''(A)}\right\} \ . \tag{24}$$

Unless m_σ is very large, there appears the large correction to the Newton law. Naively, one expects the order of the mass m_σ could be that of the Hubble rate, that is, $m_\sigma \sim H \sim 10^{-33}\,\mathrm{eV}$, which is very light and the correction could be very large, which is the claim in [7].

We should note, however, that the mass m_σ depends on the detailed form of $F(R)$ in general [11]. Moreover, the mass m_σ depends on the curvature. The curvature on the earth R_{earth} is much larger than the average curvature R_{solar} in the solar system and R_{solar} is also much larger than the average curvature in the unverse, whose order is given by the square of the Hubble rate H^2, that is, $R_{\mathrm{earth}} \gg R_{\mathrm{solar}} \gg H^2$. Then if the mass becomes large when the curvature is large, the correction to the Newton law could be small. Such a mechanism is called the Chameleon mechanism and proposed for the scalar-tensor theory in [12]. In fact, the HS model [3] has this property and the correction to the Newton law can be very small on the earth or in the solar system. In the HS model, the mass m_σ is given by (see also [13])

$$m_\sigma^2 \sim \frac{m^2 c_2^2}{2n(n+1)c_1}\left(\frac{R}{m^2}\right)^{n+2} \ . \tag{25}$$

Here the order of the mass-dimensional parameter m^2 could be $m^2 \sim 10^{-64}\,\mathrm{eV}^2$. Then in solar system, where $R \sim 10^{-61}\,\mathrm{eV}^2$, the mass is given by $m_\sigma^2 \sim 10^{-58+3n}\,\mathrm{eV}^2$ and in the air on the earth, where $R \sim 10^{-50}\,\mathrm{eV}^2$, $m_\sigma^2 \sim 10^{-36+14n}\,\mathrm{eV}^2$. The order of the radius of the earth is $10^7\,\mathrm{m} \sim \left(10^{-14}\,\mathrm{eV}\right)^{-1}$. Therefore the scalar field σ could be heavy enough if $n \gg 1$ and the correction to the Newton law is not observed being extremely small. On the other hand, in the air on the earth, if we choose $n = 10$, for example, one gets the mass is extremely large:

$$m_\sigma \sim 10^{43}\,\mathrm{GeV} \sim 10^{29} \times M_{\mathrm{Planck}} \ . \tag{26}$$

Here M_{Planck} is the Planck mass. Hence, the Newton law correction should be extremely small.

Let us discuss the matter instability proposed in [8], which may appear when the energy density or the curvature is large compared with the average one in the universe, as is the case inside of the planet. Multiplying $g^{\mu\nu}$ with Eq.(13), one obtains

$$\Box R + \frac{F^{(3)}(R)}{F^{(2)}(R)}\nabla_\rho R\nabla^\rho R + \frac{F'(R)R}{3F^{(2)}(R)} - \frac{2F(R)}{3F^{(2)}(R)} = \frac{\kappa^2}{6F^{(2)}(R)}T \ . \tag{27}$$

Here T is the trace of the matter energy-momentum tensor: $T \equiv T_{(m)\rho}^{\ \ \rho}$. We also denote $d^nF(R)/dR^n$ by $F^{(n)}(R)$. Let us now consider the perturbation from the solution of the Einstein gravity. We denote the scalar curvature solution given by the matter density in the Einstein gravity by $R_b \sim (\kappa^2/2)\rho > 0$ and separate the scalar curvature R into the sum of R_b and the perturbed part R_p as $R = R_b + R_p$ ($|R_p| \ll |R_b|$). Then Eq.(27) leads to the perturbed equation:

$$\begin{aligned} 0 =& \Box R_b + \frac{F^{(3)}(R_b)}{F^{(2)}(R_b)}\nabla_\rho R_b \nabla^\rho R_b + \frac{F'(R_b)R_b}{3F^{(2)}(R_b)} \\ & - \frac{2F(R_b)}{3F^{(2)}(R_b)} - \frac{R_b}{3F^{(2)}(R_b)} + \Box R_p + 2\frac{F^{(3)}(R_b)}{F^{(2)}(R_b)}\nabla_\rho R_b \nabla^\rho R_p + U(R_b)R_p \ . \end{aligned} \tag{28}$$

Here $U(R_b)$ is given by

$$\begin{aligned} U(R_b) \equiv & \left(\frac{F^{(4)}(R_b)}{F^{(2)}(R_b)} - \frac{F^{(3)}(R_b)^2}{F^{(2)}(R_b)^2}\right)\nabla_\rho R_b \nabla^\rho R_b + \frac{R_b}{3} \\ & - \frac{F^{(1)}(R_b)F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} - \frac{F^{(1)}(R_b)}{3F^{(2)}(R_b)} + \frac{2F(R_b)F^{(3)}(R_b)}{3F^{(2)}(R_b)^2} - \frac{F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} \end{aligned} \tag{29}$$

It is convenient to consider the case that R_b and R_p are uniform, that is, they do not depend on the spatial coordinate. Hence, the d'Alembertian can be replaced with the second derivative with respect to the time coordinate: $\Box R_p \to -\partial_t^2 R_p$ and Eq.(29) has the following structure:

$$0 = -\partial_t^2 R_p + U(R_b)R_p + \text{const.} \ . \tag{30}$$

Then if $U(R_b) > 0$, R_p becomes exponentially large with time t: $R_p \sim \mathrm{e}^{\sqrt{U(R_b)}t}$ and the system becomes unstable. In the $1/R$-model [14], since the order of mass parameter m_μ is

$$\mu^{-1} \sim 10^{18}\text{sec} \sim \left(10^{-33}\text{eV}\right)^{-1} \ , \tag{31}$$

one finds

$$\begin{aligned} & U(R_b) = -R_b + \frac{R_b^3}{6\mu^4} \sim \frac{R_0^3}{\mu^4} \sim \left(10^{-26}\text{sec}\right)^{-2}\left(\frac{\rho_m}{\text{g cm}^{-3}}\right)^3 \ , \\ & R_b \sim \left(10^3\text{sec}\right)^{-2}\left(\frac{\rho_m}{\text{g cm}^{-3}}\right) \end{aligned} \tag{32}$$

Hence, the model is unstable and it would decay in 10^{-26} sec (for planet size). On the other hand, in $1/R + R^2$-model [11], we find

$$U(R_0) \sim \frac{R_0}{3} > 0 \ . \tag{33}$$

Then the system could be unstable again but the decay time is $\sim 1,000$ sec, that is, macroscopic. In HS model [3], $U(R_b)$ is negative[13]:

$$U(R_0) \sim -\frac{(n+2)m^2c_2^2}{c_1 n(n+1)} < 0 \ . \tag{34}$$

Therefore, there is no matter instability[13].

Let us discuss the critical claim against modified gravity in [9, 10]. As shown in (16), as an exact solution, there appears de Sitter-Schwarzschild spacetime in $F(R)$-gravity. The claim in [9, 10] is that the solution does not match onto the stellar interior solution. Since it is difficult to construct explicit solution describing the stellar configuration even in the Einstein gravity, we now proceed in the following way: First, we separate $F(R)$ into the sum of the Einstein-Hilbert part and other part as $F(R) = R + f(R)$. Then Eq.(13) has the following form:

$$\begin{aligned} & \frac{1}{2}g_{\mu\nu}R - R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Lambda + \frac{\kappa^2}{2}T_{(m)\mu\nu} \\ & = -\frac{1}{2}g_{\mu\nu}\left(f(R) + \Lambda\right) + R_{\mu\nu}f'(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) \ . \end{aligned} \tag{35}$$

Here $-\Lambda$ is the value of $f(R)$ in the present universe, that is, Λ is the effective cosmological constant: $\Lambda = -f(R_0)$. We now treat the r.h.s. in (35) as a perturbation. Then the last two derivative terms in (35) could be dangerous since there could be jump in the value of the scalar curvature R on the

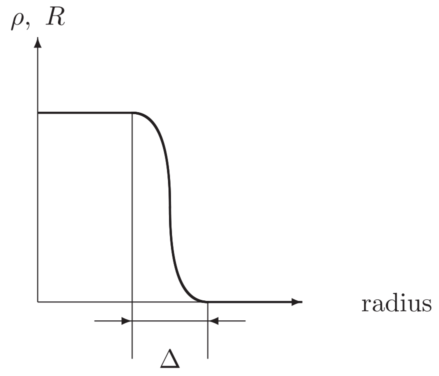

Figure 1: Typical behavior of R and ρ near the surface of the stellar configuration.

surface of stellar configuration. Of course, the density on the surface could change in a finite width Δ as in Figure 1 and the derivatives should be finite and the magnitude could be given by

$$\partial_\mu \sim \frac{1}{\Delta} \ . \tag{36}$$

One now assumes the order of the derivative could be the order of the Compton length of proton:

$$\partial_\mu \sim m_p \sim 1\,\mathrm{GeV} \sim 10^9\,\mathrm{eV} \tag{37}$$

Here m_p is the mass of proton. It is also assumed

$$R \sim R_e \sim 10^{-47}\,\mathrm{eV}^2 \ , \tag{38}$$

that is, the order of the scalar curvature R is given by the order of it inside the earth.

In case of the $1/R$ model [14], one gets

$$\Box f'(R) \sim \nabla_\mu \nabla_\nu f'(R) \sim \frac{m_p^2 \mu^4}{R^2} \sim 10^{-20}\,\mathrm{eV}^2 \gg R_e \ . \tag{39}$$

Then the perturbative part could be much larger than unperturbative part in (35), say, $R \sim R_e \sim 10^{-47}\,\mathrm{eV}^2$. Therefore, the perturbative expansion could be inconsistent.

In case of the model [3], however, we find

$$\Box f'(R) \sim \nabla_\mu \nabla_\nu f'(R) \sim \frac{m_p^2 \Lambda}{m^2} \left(\frac{R}{m^2}\right)^{-n-1} \sim 10^{-3-17n}\,\mathrm{eV}^2 \ . \tag{40}$$

Then if $n > 2$, we find $\Box f'(R)$, $\nabla_\mu \nabla_\nu f'(R) \ll R_e$ and therefore the perturbative expansion could be consistent. This indicates that such modified gravity model may pass the above test. Thus, it is demonstrated that some versions of modified gravity may easily pass above tests.

4 Unifying inflation and cosmic acceleration

In this section, we consider an extension of the HS model [3] to unify the early-time inflation and late-time acceleration, following proposals [4, 5].

In order to construct such models, we impose the following conditions:

- Condition that inflation occurs:
$$\lim_{R \to \infty} f(R) = -\Lambda_i \ . \tag{41}$$
Here Λ_i is an effective early-time cosmological constant.

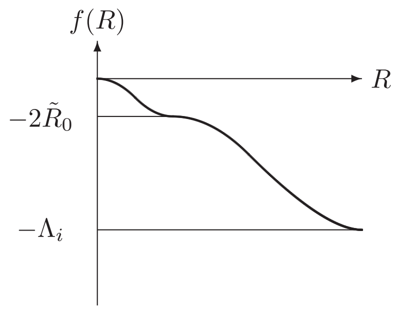

Figure 2: The typical behavior of $f(R)$ which satisfies the conditions (41), (44), and (45).

Instead of (41) one may impose the following condition

$$\lim_{R\to\infty} f(R) = \alpha R^m \ . \tag{42}$$

Here m and α are positive constants. Then as shown in (19), the scale factor $a(t)$ evolves as

$$a(t) \propto t^{h_0} \ , \quad h_0 \equiv \frac{2m}{3(w+1)} \ , \tag{43}$$

and $w_{\rm eff} = -1 + 2/3h_0$. Here w is the matter EoS parameter, which could correspond to dust or radiation. We assume $m \gg 1$ so that $\dot{H}/H^2 \gg 1$.

- The condition that there is flat spacetime solution is given as

$$f(0) = 0 \tag{44}$$

- The condition that late-time acceleration occurs should be

$$f(R_0) = -2\tilde{R}_0 \ , \quad f'(R_0) \sim 0 \ . \tag{45}$$

Here R_0 is the current curvature of the universe and we assume $R_0 > \tilde{R}_0$. Due to the condition (45), $f(R)$ becomes almost constant in the present universe and plays the role of the effective small cosmological constant: $\Lambda_l \sim -f(R_0) = 2\tilde{R}_0$.

The typical behavior of $f(R)$ which satisfies the conditions (41), (44), and (45) is given in Figure 2 and the behavior of $f(R)$ satisfying (42), (44), and (45) is given in Figure 3.

Some examples may be of interest. An example which satisfies the conditions (41), (44), and (45) is given by the following action[4]:

$$f(R) = -\frac{(R-R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\left\{(R-R_0)^{2n+1} + R_0^{2n+1}\right\}} \ . \tag{46}$$

Here n is a positive integer. The conditions (42) and (45) require

$$\frac{R_0^{2n+1}}{f_0 + f_1 R_0^{2n+1}} = 2\tilde{R}_0 \ , \quad \frac{1}{f_1} = \Lambda_i \ . \tag{47}$$

One can now investigate how the exit from the inflation could be realized in the model (46). It is easier to consider this problem in the scalar-tensor form (Einstein frame) in (23). In the inflationary epoch, when the curvature $R = A$ is large, $f(R)$ has the following form:

$$f(R) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 R^{2n+1}} \ . \tag{48}$$

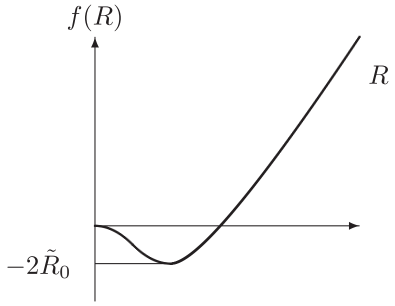

Figure 3: The typical behavior of $f(R)$ which satisfies the conditions (42), (44), and (45).

Hence, one gets

$$\sigma \sim \frac{(2n+1)f_0}{f_1^2 A^{2n+2}} , \tag{49}$$

and

$$V(\sigma) \sim \frac{1}{f_1} - \frac{2(n+1)f_0}{f_1^2} \left(\frac{f_1^2 \sigma}{(2n+1)f_0} \right)^{\frac{2n+1}{2n+2}} . \tag{50}$$

Note that the scalar field σ is dimensionless now. Let us check the condition for the slow roll, $|V'/V| \ll 1$. Since

$$\frac{V'(\sigma)}{V(\sigma)} \sim -f_1 \left(\frac{f_1^2 \sigma}{(2n+1)f_0} \right)^{-\frac{1}{2n+2}} , \tag{51}$$

if we start with $\sigma \sim 1$, one finds

$$\frac{V'(\sigma)}{V(\sigma)} \sim -\left(\frac{R_0}{\Lambda_i} \right)^{\frac{2n}{2n+1}} , \tag{52}$$

which is very small and the slow roll condition is satisfied. Thus, the value of the scalar field σ increases very slowly and the scalar curvature R becomes smaller. When σ becomes large enough and R becomes small enough, the inflation could stop. Another possibility to achieve the exit from the inflation is to add small non-local term to gravitational action.

We now consider another example, where $f(R)$ satisfies the conditions (42), (44), and (45) [5]:

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1+\gamma R^n} . \tag{53}$$

Here α, β, and γ are positive constants and n is a positive integer. When the curvature is large ($R \to \infty$), $f(R)$ behaves as

$$f(R) \to \frac{\alpha}{\gamma} R^n . \tag{54}$$

To achieve the exit from the inflation, more terms could be added in the action. Since the derivative of $f(R)$ is given by

$$f'(R) = \frac{nR^{n-1}\left(\alpha\gamma R^{2n} - 2\alpha R^n - \beta\right)}{(1+\gamma R^n)^2} , \tag{55}$$

we find the curvature R_0 in the present universe, which satisfies the condition $f'(R_0) = 0$, is given by

$$R_0 = \left\{ \frac{1}{\gamma} \left(1 + \sqrt{1 + \frac{\beta\gamma}{\alpha}} \right) \right\}^{1/n} , \tag{56}$$

and

$$f(R_0) \sim -2\tilde{R}_0 = \frac{\alpha}{\gamma^2} \left(1 + \frac{(1-\beta\gamma/\alpha)\sqrt{1+\beta\gamma/\alpha}}{2+\sqrt{1+\beta\gamma/\alpha}} \right) . \tag{57}$$

Let us check if we can choose parameters to reproduce realistic cosmological evolution. As a working hypothesis, we assume $\beta\gamma/\alpha \gg 1$, then

$$R_0 \sim (\beta/\alpha\gamma)^{1/2n} \quad , \quad f(R_0) = -2\tilde{R}_0 \sim -\beta/\gamma \tag{58}$$

We also assume $f(R_I) \sim (\alpha/\gamma) R_I^n \sim R_I$. Here R_I is the curvature in the inflationary epoch. As a result, one obtains

$$\alpha \sim 2\tilde{R}_0 R_0^{-2n}, \ \beta \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{n-1}, \ \gamma \sim 2\tilde{R}_0 R_0^{-2n} R_I^{n-1}. \tag{59}$$

Hence, we can confirm the assumption $\beta\gamma/\alpha \gg 1$ if $n > 1$ as

$$\frac{\beta\gamma}{\alpha} \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{2n-2} \sim 10^{228(n-1)} \gg 1 \ . \tag{60}$$

Thus, we presented modified gravity models which unify early-time inflation and late-time acceleration. One should stress that the above models (46) and (53) satisfy the cosmological constraints/local tests in the same way as in the HS model [3].

5 Reconstruction of $F(R)$-gravity

In this section, it is shown how we can construct $F(R)$ model realizing *any* given cosmology (including inflation, matter-dominated epoch, *etc*) using technique of ref.[15]. The general $F(R)$-gravity action with general matter is given as:

$$S = \int d^4x \sqrt{-g} \left\{ F(R) + \mathcal{L}_{\rm matter} \right\} \ . \tag{61}$$

The action (61) can be rewritten by using proper functions $P(\phi)$ and $Q(\phi)$ of a scalar field ϕ:

$$S = \int d^4x \sqrt{-g} \left\{ P(\phi) R + Q(\phi) + \mathcal{L}_{\rm matter} \right\} \ . \tag{62}$$

Since the scalar field ϕ has no kinetic term, one may regard ϕ as an auxiliary scalar field. By the variation over ϕ, we obtain

$$0 = P'(\phi) R + Q'(\phi) \ , \tag{63}$$

which could be solved with respect to ϕ as $\phi = \phi(R)$. By substituting $\phi = \phi(R)$ into the action (62), we obtain the action of $F(R)$-gravity where

$$F(R) = P(\phi(R)) R + Q(\phi(R)) \ . \tag{64}$$

By the variation of the action (62) with respect to $g_{\mu\nu}$, the equation of motion follows:

$$0 = -\frac{1}{2} g_{\mu\nu} \left\{ P(\phi) R + Q(\phi) \right\} - R_{\mu\nu} P(\phi) + \nabla_\mu \nabla_\nu P(\phi) - g_{\mu\nu} \nabla^2 P(\phi) + \frac{1}{2} T_{\mu\nu} \tag{65}$$

In FRW universe (14), Eq.(65) has the following form:

$$\begin{aligned} 0 &= -6H^2 P(\phi) - Q(\phi) - 6H \frac{dP(\phi(t))}{dt} + \rho \\ 0 &= \left(4\dot{H} + 6H^2 \right) P(\phi) + Q(\phi) + 2\frac{d^2 P(\phi(t))}{dt} + 4H \frac{dP(\phi(t))}{dt} + p \end{aligned} \tag{66}$$

By combining the two equations in (66) and deleting $Q(\phi)$, we obtain

$$0 = 2\frac{d^2 P(\phi(t))}{dt^2} - 2H \frac{dP(\phi(t))}{dt} + 4\dot{H} P(\phi) + p + \rho \ . \tag{67}$$

Since one can redefine ϕ properly as $\phi = \phi(\varphi)$, we may choose ϕ to be a time coordinate: $\phi = t$. Then assuming ρ, p could be given by the corresponding sum of matter with a constant EoS parameters w_i

and writing the scale factor $a(t)$ as $a = a_0 e^{g(t)}$ (a_0 : constant), we obtain the second rank differential equation:

$$0 = 2\frac{d^2P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi))}{d\phi} + 4g''(\phi)P(\phi) + \sum_i (1+w_i)\,\rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)} \ . \quad (68)$$

If one can solve Eq.(68), with respect to $P(\phi)$, one can also find the form of $Q(\phi)$ by using (66) as

$$Q(\phi) = -6\left(g'(\phi)\right)^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + \sum_i \rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)} \ . \quad (69)$$

Thus, it follows that any given cosmology can be realized by some specific $F(R)$-gravity.

We now consider the cases that (68) can be solved exactly. A first example is given by

$$g'(\phi) = g_0 + \frac{g_1}{\phi} \ . \quad (70)$$

For simplicity, we neglect the contribution from matter. Then Eq.(68) gives

$$0 = \frac{d^2P}{d\phi^2} - \left(g_0 + \frac{g_1}{\phi}\right)\frac{dP}{d\phi} - \frac{2g_1}{\phi^2}P \ . \quad (71)$$

The solution of (71) is given in terms of the Kummer functions or confluent hypergeometric functions:

$$P = z^\alpha F_K(\alpha,\gamma;z) \ , \quad z^{1-\gamma}F_K(\alpha-\gamma+1, 2-\gamma; z) \quad (72)$$

Here

$$z \equiv g_0\phi \ , \quad \alpha \equiv \frac{1+g_1 \pm \sqrt{g_1^2+10g_1+1}}{4} \ ,$$
$$\gamma \equiv 1 \pm \frac{\sqrt{g_1^2+10g_1+1}}{2} \ , \quad F_K(\alpha,\gamma;z) = \sum_{n=0}^\infty \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{z^n}{n!} \ . \quad (73)$$

Eq.(70) gives the following Hubble rate:

$$H = g_0 + \frac{g_1}{t} \ . \quad (74)$$

Then when t is small, H behaves as

$$H \sim \frac{g_1}{t} \ , \quad (75)$$

which corresponds to the universe with matter whose EoS parameter is given by

$$w = -1 + \frac{2}{3g_1} \ . \quad (76)$$

On the other hand, when t is large, we find

$$H \to g_0 \ , \quad (77)$$

that is, the universe is asymptotically deSitter space.

We now show how we could reconstruct a model unifying the early-time inflation with late-time acceleration. In principle, one may consider $g(\phi)$ satisfying the following conditions:

- The condition for the inflation ($t = \phi \to 0$):

$$g''(0) = 0 \ , \quad (78)$$

  which shows that $H(0) = g'(0)$ is almost constant, which corresponds to the asymptotically deSitter space.

- The condition fot the late-time acceleration (at $t = \phi \sim t_0$):

$$g''(t_0) = 0 \ , \quad (79)$$

  which corresponds to the asymptotically deSitter space again.

An example could be

$$g'(\phi) = g_0 + g_1 \frac{\left(t_0^2-\phi^2\right)^n - t_0^{2n}}{\left(t_0^2-\phi^2\right)^n + c} \ . \tag{80}$$

Here g_0, g_1, and c are positive constants and n is positive integer greater than 1. Note that $g'(\phi)$ is a monotonically decreasing function of ϕ if $0 < \phi < t_0$ We also assume

$$0 < g_0 - \frac{g_1 t_0^{2n}}{c} \ll g_0 \ . \tag{81}$$

One should note that $g'(0) = g_0$ corresponds to the large Hubble rate in the inflationary epoch and $g'(t_0) = g_0 - \frac{g_1 t_0^{2n}}{c}$ to the small Hubble rate in the present universe. It is very difficult to solve (68) with (80), so we expand $g(\phi)$ for small ϕ. For simplicity, we consider the case that $n = 2$ and no matter presents. Then

$$g(\phi) = g_0 - \frac{2g_1 t_0^2}{t_0^4+c}\phi^2 + \mathcal{O}\left(\phi^4 \text{ or } g_1^2\right) \ . \tag{82}$$

Hence, one gets

$$\begin{aligned} P(\phi) \ =& \ P_0 + P_1 \mathrm{e}^{g_0\phi} - \frac{2g_1 t_0^2}{t_0^4+c}\left[P_1\left\{\frac{\phi^3}{3} - \frac{3\phi^2}{g_0} + \frac{6\phi}{g_0^2} - \frac{6}{g_0^4}\right\}\mathrm{e}^{g_0\phi} + \left\{\frac{2\phi^2}{g_0} + \frac{4\phi}{g_0^2}\right\}P_0 \right. \\ & \left. - \frac{P_2}{g_0}\mathrm{e}^{g_0\phi} - P_3\right] + \mathcal{O}\left(g_1^2\right) \ . \end{aligned} \tag{83}$$

Here P_0, P_1, P_2, and P_3 are constants of integration. Using boundary conditions we can specify different modified gravities which unify the early-time inflation with late-time acceleration.

We may consider another model:

$$g'(\phi) = \frac{a+b\phi^2}{1+c\phi^2} \ . \tag{84}$$

Here a, b, and c are positive constants satisfying the condition:

$$\frac{b}{c} \ll a \ . \tag{85}$$

In the early time $\phi = t \sim 0$, we find

$$H(t) = g'(t) = a + (b - ac)\,\phi^2 + \mathcal{O}\left(\sqcup^{\triangle}\right) \ . \tag{86}$$

Then we may identify a as a cosmological constant which generates the inflation. On the other hand, in the late time $\phi = t \to \infty$, the Hubble rate H is given by

$$H(t) = g'(t) = \frac{b}{c} + \frac{a - b/c}{c\phi^2} + \mathcal{O}\left(\phi^{-4}\right) \ , \tag{87}$$

which tells that the effective cosmological constant generating accelerating expansion of the universe could be given by b/c.

When $\phi = t$ is small, by comparing (86) with (82), we may identify

$$a \leftrightarrow g_0 \ , \quad ac - b \leftrightarrow \frac{2g_1 t_0^2}{t_0^4+c} \ . \tag{88}$$

Then by using (83), we find that the corresponding P could be given by

$$\begin{aligned} P(\phi) \ =& \ P_0^I + P_1^I \mathrm{e}^{a\phi} - (ac-b)\left[P_1^I\left\{\frac{\phi^3}{3} - \frac{3\phi^2}{a} + \frac{6\phi}{a^2} - \frac{6}{a^4}\right\}\mathrm{e}^{a\phi} + \left\{\frac{2\phi^2}{a} + \frac{4\phi}{a^2}\right\}P_0^I \right. \\ & \left. - \frac{P_2^I}{g_0}\mathrm{e}^{a\phi} - P_3^I\right] + \mathcal{O}\left((ac-b)^2\right) \ . \end{aligned} \tag{89}$$

Here P_0^I, P_1^I, P_2^I, and P_3^I are constants of integration. On the other hand, when $\phi = t$ is large, we find

$$\begin{aligned} P(\phi) \ =& \ P_0^L + P_1^L \mathrm{e}^{b\phi/c} + \frac{a-b/c}{c}\left[4P_0^L \int^{\phi} d\phi' \mathrm{e}^{B\phi'} \int^{\phi'} \frac{d\phi''}{\phi''^3}\mathrm{e}^{-B\phi''} \right. \\ & \left. - P_1^L\left\{\int^{\phi} d\phi' \left(\frac{b}{c\phi'} - \frac{2}{\phi'^2}\right)\mathrm{e}^{b\phi'/c}\right\}\right] + P_2^L \mathrm{e}^{b\phi/c} \ . \end{aligned} \tag{90}$$

Here P_0^I, P_1^I, and P_2^I are constants of integration, again.

The important element of above reconstruction scheme is that it may be applied partially. For instance, one can start from the known model which passes local tests and describes the late-time acceleration. After that, the reconstruction method may be applied only at very small times (inflationary universe) to modify such a theory partially. As a result, we get the modified gravity with necessary early-time behavior and (or) vice-versa.

6 Dark Matter from $F(R)$-gravity

It is extremely interesting that dark matter could be explained in the framework of viable $F(R)$-gravity which was discussed in previous sections.

The previous considerations of $F(R)$-gravity suggest that it may play the role of gravitational alternative for dark energy. However, one can study $F(R)$-gravity as a model for dark matter. There have been proposed several scenarios to explain dark matter in the framework of $F(R)$-gravity. In most of such approaches[16], the MOND-like scenario or power-law gravity have been considered. In such scenarios, the field equations have been solved and the large-scale correction to the Newton law has been found and used as a source of dark matter.

There was, however, an observation [17] that the distribution of the matter is different from that of dark matter in a galaxy cluster. From this it has been believed that the dark matter can not be explained by the modification of the Newton law but dark matter should represent some (particles) matter.

It is known that $F(R)$-gravity contains a particle mode called 'scalaron', which explicitly appears when one rewrites $F(R)$-gravity in the the scalar-tensor form (23). In the Einstein gravity, when we quantize the fluctuations over the background metric, we obtain graviton, which is massless tensor particle. In case of $F(R)$-gravity, when one quantizes the fluctuations of the scalar field in the background metric, one gets the massive scalar particles in the addition to the graviton. Since the scalar particles in $F(R)$-gravity are massive, the pressure could be negligible and the strength of the interaction between such the scalar particles and usual matter should be that of the gravitational interaction order and therefore very small. Hence, such scalar particle could be a natural candidate for dark matter.

In the model [3] or our models (46) and (53), the mass of the effective scalar field depends on the curvature or energy density, in accord with so-called Chameleon mechanism. As our models (46) and (53) describe the early-time inflation as well as late-time acceleration, the 'scalaron' particles, that is, the scalar particles in $F(R)$-gravity, could be generated during the inflationary era. An interesting point is that the mass could change after the inflation due to Chameleon mechanism. Especially in the model (46), the mass decreases when the scalar curvature increases as shown in (49). Hence, in the inflationary era, when the curvature is large, one may consider the model where m_σ is large. After the inflationary epoch, the scalar particles, generated by the inflation, could lose their mass. Since the mass corresponds to the energy, the difference between the mass in the inflationary epoch and that after the inflation could be radiated as energy and could be converted into the matter and the radiation. This indicates that the reheating could be naturally realized in such model. Let the mass of σ in the inflationary epoch be m_I and that after inflation be m_A. Then for N particles, the radiated energy E_N may be estimated as

$$E = (m_I - m_A)\, N \ , \tag{91}$$

which could be converted into radiation, baryons and anti-baryons (and leptons). It is believed that the number of early-time baryons and anti-baryons is 10^{10} times of the number of baryons in the present universe. Since the density of the dark matter is almost five times of the density of the baryonic matter, we find

$$m_I > 10^{10} m_A \ . \tag{92}$$

In the solar system, one gets $A = R \sim 10^{-61}\,\mathrm{eV}^2$. Then if $n \gg 10 \sim 12$ and $\Lambda_i \sim 10^{20\sim 38}$, the order of the mass m_σ is given by

$$m_\sigma^2 \sim 10^{239\sim 295-10n}\,\mathrm{eV}^2 \ , \qquad (93)$$

which is large enough so that σ could be Cold (non-relativistic) Dark Matter. On the other hand, in $1/R$-model, the corresponding mass is given by

$$m_{1/R}^2 \sim \tfrac{\mu^4}{R} \sim 10^{-51}\,\mathrm{eV}^2 \ . \qquad (94)$$

Here μ is the parameter with dimension of mass and $\mu \sim 10^{-33}\,\mathrm{eV}$. The mass $m_{1/R}$ is very small and cannot be a Cold Dark Matter. The corresponding composite particles can be a Hot (relativistic) Dark Matter but Hot Dark Matter has been excluded due to difficulty to generate the universe structure formation.

In the inflationary era, the spacetime is approximated by the de Sitter space:

$$ds^2 = -dt^2 + \mathrm{e}^{2H_0 t}\sum_{i=1,2,3}\left(dx^i\right)^2 \ . \qquad (95)$$

Then the scalar particle σ could be Fourier-transformed as

$$\sigma = \int d^3k \tilde\sigma(\mathbf{k},t)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \ . \qquad (96)$$

Hence, the number of the particles with $\mathbf{k}$ created during inflation is proportional to $\mathrm{e}^{\nu\pi}$. Here

$$\nu \equiv \sqrt{\tfrac{m_\sigma^2}{H_0^2} - \tfrac{9}{4}} \ . \qquad (97)$$

Then if

$$\tfrac{m_\sigma^2}{H_0^2} > \tfrac{9}{4} \ , \qquad (98)$$

sufficient number of the particles could be created.

In the original $f(R)$-frame (7), the scalar field σ appears as composite state. The equation of motion in $f(R)$-gravity contains fourth derivatives, which means the existence of the extra particle mode or composite state. In fact, the trace part of the equation of motion (13) has the following Klein-Gordon equation-like form:

$$3\nabla^2 f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T\,. \qquad (99)$$

The above trace equation can be interpreted as an equation of motion for the non trivial 'scalaron' $f'(R)$. This means that the curvature itself propagates. In fact the scalar field σ in the scalar-tensor form of the theory can be given by 'scalaron', which is the combination of the scalar curvature in the original frame:

$$\sigma = -\ln\left(1 + f'(R)\right) \ . \qquad (100)$$

Note that the 'scalaron' is different mode from graviton, which is massless and tensor.

Eq.(49) shows that the mass, which depends on the value of the scalar field σ, is given by

$$m_\sigma^2 \sim \tfrac{f_0}{f_1^2}\left(\tfrac{2n+1}{2n+2}\right)\left(\tfrac{f_1^2}{(2n+1)f_0}\right)^{\frac{2n+1}{2n+2}}\sigma^{-\frac{2n+3}{2n+2}} \ . \qquad (101)$$

If the curvature becomes small, σ becomes large and m_σ^2 decreases. Then the scalar particles lose their masses after the inflation. The difference of the mass in the inflationary epoch and that after the inflation could be radiated as energy and can be converted into the matter and the radiation.

By substituting the expression of σ (49) into (101), one obtains

$$m_\sigma^2 \sim \tfrac{f_1^2 A^{2n+3}}{2(2n+1)(n+1)f_0} \ . \qquad (102)$$

Note that A corresponds to the scalar curvature. Let denote the value of A in the inflationary epoch by A_I and that after the inflation by A_A. Then the condition (92) shows

$$\frac{m_I}{m_A} \sim \left(\frac{A_I}{A_A}\right)^{n+3/2} > 10^{10} \ . \tag{103}$$

For the model with $n = 2$, the condition (92) or (103) could be satisfied if $A_I/A_A > 10^3$, which seems to indicate that the reheating could be easily realized in such a model.

Now we check if the condition (98) could be satisfied. Note $H_0^2 \sim \Lambda_i$. Eq.(102) also indicates that in the inflationary era, where $A = R \sim \Lambda_i$, the magnitude of the mass is given by

$$m_\sigma^2 \sim \frac{\Lambda_i^{2n+1}}{R_0^{2n}} \ , \tag{104}$$

which is large enough and the condition (98) is satisfied. Here Eq.(47) is used. Thus, sufficient number of σ-particles could be created.

Let us consider the rotational curve of galaxy. As we will see the shift of the rotational curve does not occur due to correction to the Newton law between visible matter (baryon or intersteller gas) but due to invisible (dark) matter, and the Newton law itself is not modified.

Let the temperature of the dark matter be $T = 1/k\beta$ where k is the Boltzmann constant. First, we assume the mass m_σ of the scalar particle σ is constant. As the total mass of dark matter is much larger than that of baryonic matter and radiation, we neglect the contributions from the baryonic matter and radiation just for simplicity. We now work in Newtonian approximation and the system is spherically symmetric. Let the gravitational potential, which can be formed by the sum of the dark matter particles, be $V(r)$. Then the gravitational force is given by $\mathcal{F}(r) = -mdV(r)/dr$. If we denote the number density of the dark matter particles by $n(r)$, in the Newtonian approximation, by putting $\kappa^2 = 8\pi G$, one gets

$$\mathcal{F}(r) = -\frac{Gm_\sigma^2}{r^2}\int_0^r 4\pi s^2 n(s)ds \tag{105}$$

and therefore $V(r)$ is given by

$$V(r) = 4\pi G m_\sigma \int^r \frac{ds}{s^2}\int_0^s u^2 n(u)du \ . \tag{106}$$

If one assumes the number density $n(r)$ of dark matter particles could obey the Boltzmann distribution, we find

$$n(r) = N_0 \mathrm{e}^{-\beta m_\sigma V(r)} \ . \tag{107}$$

Here N_0 is a constant, which can be determined by the normalization. Using (106) and (107) and deleting $n(r)$, the differential equation follows:

$$\left(r^2 V'(r)\right)' = 4\pi G m_\sigma N_0 r^2 \mathrm{e}^{-\beta m_\sigma V(r)} \ . \tag{108}$$

An exact solution of the above equation is given by

$$V(r) = \frac{2}{\beta m_\sigma}\ln\left(\frac{r}{r_0}\right) \ , \quad r_0^2 \equiv \frac{1}{2\pi G m_\sigma^2 N_0 \beta} \ . \tag{109}$$

As the general solution for the non-linear differential equation (108) is not known, we assume $V(r)$ could be given by (109). Then the rotational speed v of the stars in the galaxy could be determined by the balance of the gravitational force and the centrifugal force:

$$m_\star \frac{v^2}{r} = -\mathcal{F}(r) = m_\star V'(r) = \frac{2m_\star}{\beta m_\sigma r} \ . \tag{110}$$

Here $m_\star$ is the mass of a star. Hence,

$$v^2 = \frac{2}{m_\sigma \beta} \ , \tag{111}$$

that is, v becomes a constant, which could be consistent with the observation.

For the dark matter particles from $f(R)$-gravity, the mass m_σ depends on the scalar curvature or the value of the background σ as in (101). The scalar curvature is determined by the energy density ρ (if pressure could be neglected as in usual baryonic matter and cold dark matter) and if we neglect the contribution from the baryonic matter, the energy density ρ is given by

$$\rho(r) = m_\sigma n(r) \ . \tag{112}$$

Therefore it follows

$$m_\sigma = m_\sigma\left(\rho(r)\right) = m_\sigma\left(m_\sigma n(r)\right) \ , \tag{113}$$

which could be solved with respect to m_σ:

$$m_\sigma = m_\sigma\left(n(r)\right) \ . \tag{114}$$

Furthermore by combining (107) and (114), one may solve m_σ with respect to $V(r)$ and N_0 as

$$m_\sigma = m_\sigma\left(N_0, V(r)\right) \ . \tag{115}$$

Then (105) could be modified as

$$\mathcal{F}(r) = -\frac{G m_\sigma\left(N_0, V(r)\right)}{r^2}\int_0^r 4\pi s^2 m_\sigma\left(N_0, V(r)\right) n(s) ds \tag{116}$$

which gives, instead of (108),

$$\left(r^2 V'(r)\right)' = 4\pi G m_\sigma\left(N_0, V(r)\right) N_0 r^2 \mathrm{e}^{-\beta m_\sigma(N_0, V(r)) V(r)} \ . \tag{117}$$

Eq.(117) is rather complicated but at least numerically solvable.

For the model (46), if the curvature is large enough even around the galaxy, the mass m_σ is given by (102). The scalar curvature $A = R$ is proportional to the energy density (since the pressure could be neglected), $A \propto \rho$, and the energy density ρ is given by (112). Then

$$n(r) \sim \frac{1}{\kappa^2}\left\{\frac{2(n+1)(2n+1)f_0}{f_1^2}\right\}^{\frac{1}{2n+3}} \left(m_\sigma(r)\right)^{-\frac{2n+1}{2n+3}} \ . \tag{118}$$

Using (107), one also gets

$$V(r) = \frac{2n+1}{(2n+3)\beta m_\sigma(r)} \ln\frac{m_\sigma(r)}{m_0} \ , \quad m_0 \equiv \left(\kappa^2 N_0\right)^{-\frac{2n+3}{2n+1}} \left\{\frac{2(n+1)(2n+1)f_0}{f_1^2}\right\}^{\frac{1}{2n+1}} \ . \tag{119}$$

Here m_0 has mass dimension. By substituting (119) into (117), it follows

$$\begin{aligned} &\left(\frac{2n+1}{2n+3}\right)\frac{1}{\beta}\left\{r^2\left(1 - \ln\frac{m_\sigma(r)}{m_0}\right)\frac{m_\sigma''(r)}{m_\sigma(r)^2} - r^2\left(3 - 2\ln\frac{m_\sigma(r)}{m_0}\right)\frac{\left(m_\sigma'(r)\right)^2}{m_\sigma(r)^3}\right. \\ &\left. + 2r\left(1 - \ln\frac{m_\sigma(r)}{m_0}\right)\frac{m_\sigma'(r)}{m_\sigma(r)^2}\right\} = \frac{1}{2}\left\{\frac{2(n+1)(2n+1)f_0}{f_1^2}\right\}^{\frac{1}{2n+3}} r^2 \left(m_\sigma(r)\right)^{\frac{2}{2n+3}} \ . \end{aligned} \tag{120}$$

It is very difficult to find the exact solution of (120), although one may solve (120) numerically. Then we now consider the region where $m_\sigma \ll m_0$ but $\ln\left(m_\sigma/m_0\right)$ is slow varying function of r, compared with the power of r. In the region, we may treat $\ln\left(m_\sigma/m_0\right)$ as a large negative constant:

$$\ln\left(m_\sigma/m_0\right) \sim -C \ . \tag{121}$$

Then the following solution is obtained:

$$\begin{aligned} m_\sigma(r) &= m_0\left(\frac{r}{r_0}\right)^{-\frac{2(2n+3)}{2n+5}} \ , \\ r_0^2 &\equiv \frac{4(2n+1)(2n+9)C}{(2n+5)\beta}\left(\kappa^2 N_0\right)^{\frac{2n+5}{2n+1}}\left\{\frac{2(n+1)(2n+1)f_0}{f_1^2}\right\}^{-\frac{1}{2n+1}} \ . \end{aligned} \tag{122}$$

Note that r_0 can be real for any positive n. Eq.(119) shows that

$$V(r) = -\frac{2(2n+1)}{2n+5}\frac{1}{\beta m_0}\left(\frac{r}{r_0}\right)^{\frac{2(2n+3)}{2n+5}} \ln\frac{r}{r_0} . \tag{123}$$

Note that the potential (123) is obtained by assuming the Newton law by summing up the Newton potentials coming from the $f(R)$-dark matter particles ('scalaron') distributed around the galaxy. Eq.(122) indicates that the condition $m_\sigma \ll m_0$ requires $r \gg r_0$. Then by using the equation for the balance of the gravitational force and the centrifugal force, as in (110), we find

$$v \propto \left(\frac{r}{r_0}\right)^{\frac{2n+3)}{2n+5}} , \tag{124}$$

which is monotonically increasing function of r and the behavior is different from that in (111). If there is only usual baryonic matter without any dark matter, the velocity is the decreasing function of r, if there is also usual dark matter, as shown in (111), the velocity is almost constant, if dark matter originates from $f(R)$-gravity, as we consider here, there is a region where the velocity could be an *increasing* function of r. Of course, one should be more careful as these are qualitative considerations. The condition $m_\sigma \ll m_0$ requires $r \gg r_0$ but in the region faraway from galaxy, the scalar curvature becomes small and the approximation (102) could be broken. Anyway if there appears a region where velocity is the increasing function of r, this might be a signal of $f(R)$-gravity origin for dark matter. For more precise quantitative arguments, it is necessary to include the contribution from usual baryonic matter as well as to do numerical calculation. In any case, it seems very promising that composite particles from viable modified gravity which unifies inflation with late-time acceleration may play the role of dark matter.

7 Non-minimal modified gravity

In this section, we consider the theory with non-minimal gravitational coupling as an extension to the $F(R)$-gravity.

In [18], the non-minimal gravitational coupling of the scalar field ϕ was considered:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{f_1(A)}{2}\partial_\mu\phi\partial^\mu\phi - f_2(A)\right] . \tag{125}$$

Here A is a geometrical invariant like scalar curvature or Gauss-Bonnet invariant. In the FRW universe (14), by the variation over ϕ, one finds

$$\dot\phi = q f_1(A)^{-1}a^{-3} . \tag{126}$$

Here q is a constant. By combining (126) and the gravitational field equation, we delete the scalar field ϕ and obtain FRW-like equation:

$$\begin{aligned}
\frac{6}{\kappa^2}H^2 &= \rho_1 + \rho_2 , \\
\rho_1 &= \frac{q^2}{f_1 a^6}\left[\frac{1}{2} + 3\frac{f_1'}{f_1}(\dot H + 7H^2) + 6H\frac{(f_1')^2}{f_1^2}\dot R - 3H\frac{f_1''}{f_1}\dot R\right] , \\
\rho_2 &= f_2 - 6f_2'(\dot H + H^2) + 6Hf_2''\dot R .
\end{aligned} \tag{127}$$

In case $f_2 = 0$ and $f_1 = f_{HS}$ in (8), if the curvature R is small, the following solution is obtained:

$$a \propto t^{(k+1)/3} , \tag{128}$$

which gives the effective EoS parameter $w_{\rm eff}$ in (12) as

$$w_{\rm eff} = \frac{1-n}{1+n} . \tag{129}$$

Then the late-time acceleration of the universe $w_{\rm eff} < -1/3$ occurs when $n > 2$. Furthermore, the effective phantom regime $w_{\rm eff} < -1$ appears when $n < -1$. Within such theory for proper choice of gravitational part (like in previous sections) one achieves the unification of early-time inflation with late-time acceleration. Moreover, there is no extra problems to pass local tests.

One may also consider the theory with non-minimal coupling with electro-magnetic field [19]:

$$S = \int d^4x\sqrt{-g}\left[\frac{F(R)}{2\kappa^2} - \frac{I(R)}{4}F_{\mu\nu}F^{\mu\nu}\right] . \tag{130}$$

Here $I(R)$ is a proper function of the scalar curvature R. Even if gravity is the Einstein one ($F(R) = R$), if $I(R)$ is chosen as

$$I(R) = 1 + f_{HS}(R) , \tag{131}$$

($f_{HS}(R)$ is given in (8)), the power law inflation occurs, where

$$a(t) \propto t^{(n+1)/2} , \tag{132}$$

when $R/m^2 \gg 1$. Note that the large-scale magnetic fields are generated due to the breaking of the conformal invariance of the electromagnetic field through its non-minimal gravitational coupling. The main conclusion is that adding the non-minimal coupling with gravity, the qualitative results of previous sections remain valid. On the same time, new bounds for non-minimal gravitational couplings only restrict such couplings to tend to constant (or, sometimes, to zero) at current epoch. The important point of non-minimal modified gravity is that as purely gravitational part any $F(R)$ gravity[2, 20, 21] may be discussed in cosmological context. The only condition is that it should be realistic, i.e. to pass the local tests and cosmological bounds.

8 Discussion

In summary, we reviewed $F(R)$-gravity and demonstrated that some versions of such theory are viable gravitational candidates for unification of early-time inflation and late-time cosmic acceleration. It is explicitly shown that the known critical arguments against such theory do not work for those models. In other words, the modified gravity under consideration may pass the local tests (Newton law is respected, the very heavy positive mass for additional scalar degree of freedom is generated). The reconstruction of modified $F(R)$ gravity is considered. It is demonstrated that such theory may be reconstructed for any given cosmology. Moreover, the partial reconstruction (at early universe) may be done for modified gravity which complies with local tests and dark energy bounds. This leads to some freedom in the choice of modified gravity for the unification of given inflationary era compatible with astrophysical bounds and dark energy epoch. Moreover, non-minimal gravitational coupling with usual matter may be successfully included into above scheme. As a final very promising result it is shown that modified gravity under consideration may qualitatively well describe dark matter, using the composite scalar particle from $F(R)$ theory and Chameleon scenario.

Thus, modified gravity remains viable cosmological theory which is realistic alternative to standard Einstein gravity. Moreover, it suggests the universal gravitational unification of inflation, cosmic acceleration and dark matter without the need to introduce any exotic matter. Moreover, it remains enough freedom in the formulation of such theory which is very positive fact, having in mind, coming soon precise observational data.

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Separation of Variables in Gravitational Theory

Dedicated to the 60 year Jubilee of Professor I. L. Buchbinder

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Abstract
In order to solve the problem of exact integration of the field equations or equations of motion of matter one can use the class of Riemannian metrics for which the simplest equations of motion can be integrated by the method of complete separation of the variables.

1 Introduction.

One of the main problems of mathematical physics for the gravity theory is the problem of exact integration of the field equations or equations of motion of matter. To solve this problem one can use the class of Riemannian metrics for which the simplest equations of motion can be integrated by the complete separation of variables method. Apparently in this class the Stäckel metrics are of the same interest [1]. Recall that metric is called the Stäckel one if the Hamilton–Jacobi equation

$$g^{ij}S_{,i}S_{,j} = m^2 \qquad i,j,k,l = 1,...n \tag{1.1}$$

can be integrated by the complete separation of variables method. In this case the privileged coordinate set $\{u^i\}$ exists for which complete integral of eq. (1.1) can be shown in the form

$$S = \sum_{i=1}^{n} \phi_i(u^i, \lambda) \tag{1.2}$$

where λ_i – is the essential parameter.

It appears that the other important equations of motion (Klein–Gordon–Fock, Dirac, Weyl) can be integrated by complete separation of variables method only for the metrics, belonging to the class of Stäckel spaces [2]. That is why the research of this class of spaces belongs to the one of the important problems of the mathematical physics. In the present report we consider the following parts of this problem:

- The problem of complete separation of variables for the Hamilton–Jacobi equation.
- Integration of Einstein equations for the Stäckel spaces.
- Conformally Stäckel spaces.
- Homogeneous Stäckel space-times.

2 Stäckel spaces.

The theory of Stäckel spaces has been developed by many authors. Let us recall the main statements and enumerate the main theorems of the Stäckel spaces theory [1].

Definition 1 *Let V_n be a n–dimensional Riemannian space with metric tensor g_{ij}. The Hamilton – Jacobi equation can be integrated by complete separation of variables method if co-ordinate set $\{u^i\}$ exists for which complete integral can be presented in the form (1.2).*

Definition 2 *V_n is called the Stäckel space if the Hamilton–Jacobi equation (1.1) can be integrated by complete separation of variables method.*

The next theorem was proved by Shapovalov [3, 4].

Theorem 1 *Let V_n be the Stäckel space. Then g_{ij} in privileged co-ordinate set can be shown in the form*

$$\begin{aligned} g^{ij} &= (\Phi^{-1})^{\nu}_{n} G^{ij}_{\nu}, \\ G^{ij}_{\nu} &= G^{ij}_{\nu}(u^{\nu}), \qquad \Phi^{\nu}_{\mu} = \Phi^{\nu}_{\mu}(u^{\mu}) \\ G^{ij}_{\nu} &= \delta^i_{\nu}\delta^j_{\nu}\varepsilon_{\nu}(u^{\nu}) + (\delta^i_{\nu}\delta^j_{p} + \delta^j_{\nu}\delta^i_{p})G^{\nu p}_{\nu} + \delta^i_{p}\delta^j_{q}G^{pq}_{\nu}, \\ & p,q = 1,...N, \quad \nu,\mu = N+1,...,n. \end{aligned} \tag{2.1}$$

where $\Phi^{\nu}_{\mu}(u^{\mu})$ – is called the Stäckel matrix.

It is assumed that summation is performed over repeated superscripts and subscripts provided the symbol $ns(i,j,...)$ (no summation over the indices given in the brackets) does not occur on the right of the formula. One can show that the geodesic equations of Stäckel spaces admit the first integral that commutes pairwise with respect to the Poisson bracket

$$\underset{\mu}{X} = (\Phi^{-1})^{\nu}_{\mu} H_{\nu}, \quad H_{\nu} = \varepsilon_{\nu} p^2_{\nu} + 2G^{\nu p}_{\nu} p_p p_{\nu} + h^{pq}_{\nu} p_p p_q,$$

$$\underset{p}{Y} = \underset{p}{Y}{}^{i} p_i. \tag{2.2}$$

Thus for a covariant characterization of Stäckel spaces it is sufficient to find determining properties of the integrals (2.1) in an arbitrary co-ordinate system $\{x^i\}_n$. Let us write the functions X_{ν}, Y_P in the form

$$\underset{\nu}{X} = \underset{\nu}{X}{}^{ij} p_i p_j, \qquad \underset{p}{Y} = \underset{p}{Y}{}^{i} p_i. \tag{2.3}$$

Then

$$\underset{\nu}{X}{}_{(ij;k)} = \underset{p}{Y}{}_{(i;j)} = 0$$

(the semicolon denotes the covariant derivative and the brackets denote symmetrization). Therefore $Y_p{}^i$, $X_{\nu}{}^{ij}$ are the components of vector and tensor Killing fields respectively.

Definition 3 *Pairwais commuting vector $Y_p{}^i$, where $p = 1,...N$ and tensor $X_{\nu}{}^{ij}$, where $\nu = N+1,...n$ Killing fields form a complete set of the type $(N.N_0)$ if*

$$B^{pq} \underset{p}{Y}{}^{i} \underset{q}{Y}{}^{j} + B^{\nu} \underset{\nu}{X}{}^{ij} = 0 \quad \Longrightarrow \quad B^{pq} = B^{\nu} = 0 \tag{2.4}$$

$$rank||\underset{p}{Y}{}^{i} \underset{q}{Y}{}_{i}|| = N - N_0 \tag{2.5}$$

$$\underset{\nu}{X}{}^{ik} \underset{\mu}{X}{}^{j}{}_{k} = C^{pq}_{\nu\mu} \underset{p}{Y}{}^{i} \underset{p}{Y}{}^{j} + C^{\tau}_{\nu\mu} \underset{\tau}{X}{}^{ij}, \tag{2.6}$$

$$C^{\tau}_{\nu\mu} = \Phi^{\tau}_{\rho}(\Phi^{-1})^{\rho}_{\nu}(\Phi^{-1})^{\rho}_{\mu}/(\Phi^{-1})^{\rho}_{n}$$

$$\underset{\nu}{X}{}^{ij} \underset{p}{Y}{}_{j} = C^{q}_{\nu p} \underset{q}{Y}{}^{i}. \tag{2.7}$$

Theorem 2 *A necessary and sufficient geometrical criterion of a Stäckel space is the presence of a complete set of the type $(N.N_0)$.*

In other words the Hamilton - Jacobi equation can be integrated by the complete separation of variables method if and only if the complete set of the first integrals exists.

Now it is possible to give

Definition 4 *Space - time is called the Stäckel one of the type* $(N.N_0)$ *if the complete set of the type* $(N.N_0)$ *exists.*

All these theorems and definitions are valid for the case when free Hamilton–Jacobi equation is considered. Let us consider the Hamilton–Jacobi equation for the charged particle

$$g^{ij}(S_{,i}+A_i)(S_{,j}+A_j)=m^2. \tag{2.8}$$

Definition 5 *Equation (2.8) admits complete separation of variables if co-ordinate set* $\{u^i\}$ *for which complete integral can be presented in the form (1.2) exists.*

It is possible to prove

Theorem 3 *If eq.(2.8) admits complete separations of variables then* g^{ij} *is the metric tensor of the Stäckel space type* $(N.N_0)$.

Using this theorem one can show that the separation takes place for the same privileged co-ordinate set and

$$A^i=(\Phi^{-1})^\nu_n h^i_\nu(u^\nu), \qquad A_iA^i=(\Phi^{-1})^\nu_n h_\nu(u^\nu). \tag{2.9}$$

The last condition can be regarded as a functional equation. Note that up to now this equation has not been solved for the general case. It has been solved only for the case when A^i and g^{ij} obey the Einstein–Maxwell equations [11-14]

$$R_{ij}-\frac{1}{2}g_{ij}R=4\pi\kappa T_{ij}+\Lambda g_{ij}, \qquad \Lambda=const, \tag{2.10}$$

$$T_{ij}=\frac{1}{4\pi}(F_{il}F^l{}_j-\frac{1}{4}g_{ij}F_{kl}F^{kl}), \tag{2.11}$$

$$F_{ij}=A_{j,i}-A_{i,j}, \qquad F^{ij}{}_{;j}=0. \tag{2.12}$$

3 Stäckel spaces and field equations of the theories of gravity.

The metrics of the Stäckel spaces can be used for integrating the field equations of General Relativity and other theories of Gravity. Note that such famous solutions as Schwarcshild, Kerr, NUT, Friedman and others belong to the class of Stäckel spaces. Apparently the first papers devoted to the problem of classification of the Stäckel spaces satisfying the Einstein equations were published by B.Carter. Later in our paper it has been found the complete classification of the special Stäckel electrovac spaces. In other words all Stäckel spaces satisfying the Einstein–Maxwell equations (2.10–2.12) for the case when potentials A_i admit complete separation of variables for Hamilton–Jacobi equation (2.8) have been found. In our paper the classification problem has been solved for the case when A_i are arbitrary functions and spaces are null (types (N.1)). In our paper all electrovac space-times admitting diagonalization and complete separation of variables for the Dirac–Fock–Ivanenko equation were found [6-10].

One of the complicated problems of the modern mathematical physics is the integration problem of the Einstein–Dirac equations. Using the Newman–Penrose formalism [11] one can present these equations in the form

$$R_{ij}-\frac{1}{2}g_{ij}R=4\pi GT_{ij},$$
$$\nabla_{ab'}\xi^a=m_0\eta_{b'}, \qquad \nabla_{ab'}\eta^a=m_0\xi_{b'} \tag{3.1}$$

where

$$T_{ij}=Z^a_iZ^b_j\sigma_a^{AB'}\sigma_b^{CD'}T_{AB'CD'},$$
$$T_{AB'CD'}=ik(\xi_{D'}\nabla_{AB'}\xi_C+\xi_B\nabla_{CD'}\xi_A-$$
$$\xi_C\nabla_{AB'}\xi_{D'}-\xi_A\nabla_{CD'}\xi_{B'}-\eta_{D'}\nabla_{AB'}\eta_C-$$
$$\eta_B\nabla_{CD'}\eta_A+\eta_C\nabla_{AB'}\eta_{D'}+\eta_A\nabla_{CD'}\eta_{B'})$$

$$Z_i^a = (l_i, n_i, m_i, \bar{m}_i)$$

∇_{AB} - spinor derivative, $\sigma_a^{AB'}$ - Infeld-Wan der Varden symbols.

Using spaces for which equation (3.1) can be integrated by the complete separation of variables and separated solutions of the Dirac equation one can transform Einstein–Dirac equations to the set of functional equations. The first papers devoted to the classification problem for the Einstein–Dirac equations were done by Bagrov, Obukhov, Sakhapov [13]. The Stäckel spaces of type (3.1) for Einstein–Dirac and Einstein–Weyl equations have been studied. Appropriate solutions have been obtained. They contain arbitrary functions depending on null variable only.

The problem of classification of Stäckel spaces for other theories of gravity for the first time was considered in papers [14–16]. The next theories have been considered

1. Brans–Dicke theory. The field equations have the form

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{\phi}T_{ij} - \frac{\omega}{\phi^2}(\phi_{;i}\phi_{;j} - \\ - \frac{1}{2}g_{ij}\phi_{;k}\phi^{;k}) - \frac{1}{\phi}(\phi_{;ij} - g_{ij}\Box\phi) \tag{3.2}$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T^i{}_i, \quad \Box = g^{ij}\nabla_i\nabla_j, \quad \omega = const.$$

   The case when T_{ij} has the form (2.10) and the metric has type (N.1) has been considered in paper [14]. All appropriate solutions have been obtained.

2. The same problem for the multiscalar–tensor theory for which field equations have the form [15]

$$\overset{*}{R}_{\nu\mu} = 2 < \phi_{,\nu}\phi_{,\mu} > +8\pi G(\overset{*}{T}_{\nu\mu} - \frac{1}{2}\overset{*}{g}_{\nu\mu}\overset{*}{T}) \tag{3.3}$$

$$\overset{*}{\Box}\phi^A + \gamma^A{}_{BC}\phi^A_{,\nu}\phi^B_{,\mu}\overset{*}{g}^{\nu\mu} = 4\pi G\gamma^{AB}\frac{\partial\alpha}{\partial\phi^B}\overset{*}{T}$$

   was considered in paper [16]. Here $g^*{}_{\nu\mu}$, $R^*{}_{\nu\mu}$ are metric tensor and Ricci tensor of the space which conformal to space-time,

$$\overset{*}{g}_{\nu\mu} = \alpha^2 g_{\nu\mu}, \qquad \alpha = \alpha(\phi^A),$$

$$< \phi_\nu\phi_\mu >= \gamma_{AB}\phi^A_{,\nu}\phi^B_{,\mu},$$

   where ϕ^A - are scalar fields, $\gamma_{AB} = \gamma_{AB}(\phi^C)$ can be considered as a metric tensor of the n - dimensional space of scalars,

$$\gamma^C_{AB} = \frac{1}{2}\gamma^{CD}(\frac{\partial\gamma_{AD}}{\partial\phi^B} + \frac{\partial\gamma_{DB}}{\partial\phi^A} - \frac{\partial\gamma_{AB}}{\partial\phi^C}),$$

$$\overset{*}{\Box}\phi^A \equiv (\sqrt{|\overset{*}{g}|}\overset{*}{g}^{\nu\mu}\phi^A_{,\nu})_{,\mu}/\sqrt{|\overset{*}{g}|}$$

3. The classification problem for the Einstein–Vaidya equations. Let the stress–energy tensor have the form

$$T_{ij} = T^{(e)}_{ij} + a(x)l_il_j, \qquad l_il^i = 0 \tag{3.4}$$

   then Einstein equations can by written in the following way

$$R_{ij} - \frac{1}{2}g_{ij}R = 4\pi G(T^{(e)}_{ij} + al_il_j). \tag{3.5}$$

   If $T^{(e)}_{ij}$ has the form (2.11), and F_{ij} satisfies the Maxwell equations (2.12) the solutions of the equations (3.5), (2.12) are electrovac ones. For these equations the classification problem was solved in paper [12] for the case when the complete set has type (N.1) (null case). In other words all metrics and electromagnetic potentials satisfying equations (3.5) and (2.11–2.12) provided that Hamilton–Jacobi equation (1.1) or (2.8) can be integrated by the complete separation of variables method for null Stäckel spaces have been found.

4 Conformally Stäckel spaces.

Let us consider the Hamilton–Jacobi equation for a massless particle

$$g^{ij}S_{,i}S_{,j}=0 \tag{4.1}$$

Obviously this equation admits complete separation of variables for a Stäckel space. Yet one can verify that if g^{ij} has the form

$$g_{ij}=\tilde{g}_{ij}(x)\exp 2\omega(x) \tag{4.2}$$

where $\tilde{g}_{ij}$ is a metric tensor of the Stäckel space, then eq.(4.1) can be solved by complete separation of variables method too. In paper [5] it was proved that (4.2) is necessary and additional condition of the complete separation of variables. Note that conformally Stäckel spaces play important role when massless quantum equations are considered (f.e. conformal invariant Chernikov–Penrose equation, Weyl's equation etc.). That is why the problem of investigation of Einstein spaces which admits complete separation of variables in eq.(4.1) is exceptionally interesting. Apparently the first attempt to consider this problem was taken in paper [18]. The next step was done in paper [19], where some of metrics belonging to conformal Stäckel spaces of type (N.1) was studied. The problem of classification of conformally Stäckel spaces satisfying the Einstein equation

$$R_{ij}=\Lambda g_{ij}, \qquad \Lambda=const \tag{4.3}$$

is more difficult than appropriate problem for the Stäckel spaces. To obtain the functional equations from eq. (4.3) one has to use the compatible conditions. These conditions were found by Brinkman [17]. Let us recall the main Brinkman's results. Let us denote V_n the Riemannian space with metric tensor $g_{\alpha\beta}$, $\tilde{V}_n$ be an Einstein's space with metric tensor $\tilde{g}_{\alpha\beta}$. $\tilde{R}_{\alpha\beta}, \tilde{R}_{\alpha\beta\gamma\delta}, \tilde{R}$ are components of Ricci tensor , Rieman tensor and scalar curvature respectively for the space $\tilde{V}_n$, and $R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, R$ are those tensors for the space V_n. Moreover we denote:

$$\omega_{\alpha\beta}=\omega_{,\alpha;\beta}-\omega_{,\alpha}\omega_{,\beta}+\frac{1}{2}g_{\alpha\beta}(\nabla\omega)^2,$$

$$(\nabla\omega)^2=g^{\alpha\beta}\nabla_\alpha(\omega)\nabla_\beta(\omega),$$

$$T_{\alpha\beta}=\frac{1}{n-2}\left(R_{\alpha\beta}-\frac{Rg_{\alpha\beta}}{2(n-1)}\right),$$

$$W=\frac{1}{2}(\nabla\omega)^2-\frac{\Lambda}{2(n-1)}\exp 2\omega$$

where $\omega_{,\alpha}$ are the partial derivatives and $\omega_{;\alpha}\equiv\nabla_\alpha\omega$ are the covariant derivatives in V_n. It is easy to show that

$$\tilde{R}_{\alpha\beta}=R_{\alpha\beta}+(n-2)\omega_{\alpha\beta}+g_{\alpha\beta}g^{\gamma\delta}\omega_{\gamma\delta},$$

$$\tilde{R}=(R+2(n-1)g^{\gamma\delta}\omega_{\gamma\delta})\exp(-2\omega).$$

We can write (4.3) in the form

$$\omega_{,\alpha;\beta}-\omega_{,\alpha}\omega_{,\beta}+Wg_{\alpha\beta}+T_{\alpha\beta}=0. \tag{4.4}$$

Brinkman has shown that integrability conditions of eq. (4.4) have the form

$$\omega_{,\delta}C^{\delta}{}_{\alpha\beta\gamma}=S_{\alpha\beta\gamma} \tag{4.5}$$

$$S_{\alpha\beta\gamma}\equiv T_{\alpha\gamma;\beta}-T_{\alpha\beta;\gamma}$$

where $C_{\alpha\beta\gamma\delta}$ are the components of Weyl tensor. In paper [19] eqs.(4.5) have been presented in a more simple form. To simplify them we use the consequence from Bianchi identities

$$R^{\sigma}{}_{\alpha\beta\gamma;\sigma}=R_{\alpha\beta;\gamma}-R_{\alpha\gamma;\beta}. \tag{4.6}$$

One can present the right hand part of the eq. (4.6) in the form

$$R_{\alpha\beta;\gamma} - R_{\alpha\gamma;\beta} = (n-2)\left[\left(T_{\alpha\beta} + \frac{Rg_{\alpha\beta}}{2(n-1)(n-2)}\right)_{;\gamma} - \right.$$

$$\left. - \left(T_{\alpha\gamma} + \frac{Rg_{\alpha\gamma}}{2(n-1)(n-2)}\right)_{;\beta}\right] =$$

$$= (n-2)\left(T_{\alpha\beta;\gamma} - T_{\alpha\gamma;\beta} + \frac{g_{\alpha\beta}R_{,\gamma} - g_{\alpha\gamma}R_{,\beta}}{2(n-1)(n-2)}\right) =$$

$$= (n-2)\left(-S_{\alpha\beta\gamma} + \frac{g_{\alpha\beta}R_{,\gamma} - g_{\alpha\gamma}R_{,\beta}}{2(n-1)(n-2)}\right). \tag{4.7}$$

It follows from (4.7) that

$$R^{\sigma}{}_{\alpha\beta\gamma;\sigma} = -(n-2)S_{\alpha\beta\gamma} - \frac{1}{2(n-1)}(R_{,\gamma}g_{\alpha\beta} - R_{,\beta}g_{\alpha\gamma}). \tag{4.8}$$

Besides

$$T^{\alpha}{}_{\beta;\alpha} = \frac{1}{2(n-1)}R_{,\beta}. \tag{4.9}$$

In terms of $T_{\alpha\beta}$ Weyl tensor can be written as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}T_{\beta\delta} - g_{\alpha\delta}T_{\beta\gamma} + g_{\beta\delta}T_{\alpha\gamma} - g_{\beta\gamma}T_{\alpha\delta}, \tag{4.10}$$

then

$$C^{\alpha}{}_{\beta\gamma\delta;\alpha} = R^{\alpha}{}_{\beta\gamma\delta;\alpha} + T_{\beta\delta;\gamma} - T_{\beta\gamma;\delta} + g_{\beta\delta}T^{\alpha}_{\gamma;\alpha} - g_{\beta\gamma}T^{\alpha}_{\delta;\alpha}. \tag{4.11}$$

Using eqs. (4.5) and (4.9) one can take out from (4.11) the next correlation

$$C^{\alpha}{}_{\beta\gamma\delta;\alpha} = R^{\alpha}{}_{\beta\gamma\delta;\alpha} + S_{\beta\gamma\delta} - \frac{1}{2(n-1)}(g_{\beta\gamma}R_{,\delta} - g_{\beta\delta}R_{,\gamma}). \tag{4.12}$$

That is why

$$C^{\delta}{}_{\alpha\beta\gamma;\delta} = -(n-3)S_{\alpha\beta\gamma} \tag{4.13}$$

and Brinkman's conditions can be presented in the form

$$\omega_{,\delta}C^{\delta}{}_{\alpha\beta\gamma} = -\frac{1}{(n-3)}C^{\delta}{}_{\alpha\beta\gamma;\delta},$$

finally

$$\nabla_{\delta}\left(C^{\delta}{}_{\alpha\beta\gamma}\exp{(n-3)\omega}\right) = 0. \tag{4.14}$$

If dimension of the space V_n equals to 4, eq. (4.14) has the form

$$\nabla^{\delta}\left(C_{\delta\alpha\beta\gamma}\exp\omega\right) = 0. \tag{4.15}$$

Using (4.15) we have proved the following theorem [19]

Theorem 4 *Let g_{ij} be the metric tensor of the Stäckel space of type (N.1). Then Einstein space conformal to $\tilde{V}_4$ admits the same Killing vectors as V_4.*

Moreover one can prove the following statement.

Theorem 5 *Let $\tilde{V}_n$ is conformally Stäckel space of type (N.1) ($N \geq 2$) satisfying the Einstein equation (4.3). Then Hamilton–Jacobi equation (1.1) admits the complete separation of variables.*

In other words all null conformally Stäckel Einstein spaces belong to the class of the null Stäckel ones. Nontrivial null conformally Stäckel solutions of the Einstein equations belong only to (1.1)-type.

5 Homogeneous Stäckel space-times.

This section is devoted to the problem of classification of space-homogeneous models of spacetimes based on the presence of a complete set of Killing Vectors and Killing Tensors, which is a necessary and sufficient condition for integration of the Hamilton-Jacobi equation by the method of complete separation of variables.

Thus there is a problem of finding a subclass of homogeneous space-times admitting complete sets of integrals of motion. In other words, a spacetime with a complete set must admit a 3-parametrical transitive group of motions with space-like orbits. There are 7 types of complete sets for space-times with the signature $(-,+,+,+)$. This section is devoted to classification of space-homogenous space-times with a complete set of types (2.1) and (3.1).

In a privileged coordinate set the metric of (3.1) type has the form

$$\mathrm{g}^{ij} = \begin{pmatrix} 0 & 1 & b_2(x^0) & b_3(x^0) \\ 1 & 0 & 0 & 0 \\ b_2(x^0) & 0 & a_{22}(x^0) & a_{23}(x^0) \\ b_3(x^0) & 0 & a_{23}(x^0) & a_{33}(x^0) \end{pmatrix},$$

where x^0 is the wave-like variable.

This metric admit 3 commuting Killing vectors

$$X_1, X_2, X_3; \qquad [X_p, X_q] = 0, \qquad p, q, r = 1, 2, 3$$

with components

$$X_p{}^i = \delta^i_p.$$

The metric projection on orbits of this group of motions is degenerated. Thus, we need an additional Killing vector

$$X_4^i = \xi^i.$$

The commutation relations of group X_1 — X_4 have the form

$$\begin{array}{ll} [X_m, X_4] = \alpha_m X_4 + \beta_m{}^n X_n & \\ [X_1, X_4] = \alpha_1 X_4 + \beta_1{}^p X_p, & \qquad p, q = 1, 2, 3 \\ [X_p, X_q] = 0 & \end{array}$$

We can simplify it by using the transformations of Killing vectors, which do not break the group structure

1) $X_m = S_m{}^n \tilde{X}_n,\ m, n = 2, 3,$ therefore we have new structure constants

$$\tilde{\alpha}_1 = \alpha_1, \quad \tilde{\alpha}_n = (S_n{}^m)^{-1} \alpha_m, \quad \tilde{\beta}_1{}^1 = \beta_1{}^1,$$
$$\tilde{\beta}_1{}^m = \beta_1{}^n S_n{}^m, \quad \tilde{\beta}_n{}^m = \left(S_n{}^l\right)^{-1} \beta_l{}^k S_k{}^m.$$

2) $X_4 = \tilde{X}_4 + b^m X_m$ and new constants

$$\tilde{\alpha}_p = \alpha_p, \quad \tilde{\beta}_1{}^1 = \beta_1{}^1, \quad \tilde{\beta}_p{}^m = \beta_p{}^m + \alpha_p b^m.$$

We use also coordinate transformations, which do not break the form of the metric

$$\begin{array}{l} \tilde{x}^0 = a^0 x^0 \\ \tilde{x}^1 = \frac{1}{a^0} x^1 \\ \tilde{x}^n = a_p{}^n x^p \end{array}$$

¿From commutative relations and Jacobi identities we obtain the following form of Killing vector

$$\xi^i = \beta_p{}^q \delta_q{}^i x^p + f^i(x^0), \qquad \beta_m{}^1 = 0,$$
$$f^0 = -\beta_1{}^1 x^0 + \sigma^0, \quad f^1 = \sigma^1,$$
$$f^2 = f^3 = 0, \quad \sigma = \text{const.}$$

Further classification can be made by use of two following independent conditions:

1. According to the value of ${\beta_1}^1$. If ${\beta_1}^1 \neq 0$, then we may choose ${\beta_1}^1 = 1$, $\sigma^0 = \sigma^1 = 0$, if ${\beta_1}^1 = 0$, then we may choose $\sigma^0 = \sigma^1 = 1$. Thus, we have 2 types

Type A: $\xi^0 = -x^0$, $\xi^1 = x^1$;

Type B: $\xi^0 = 1$, $\xi^1 = 1$.

2. According to classes of matrix ${\beta_m}^n$. By use of transformations it can be brought to one of following 3 classes

$$1) \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} \; 2) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \; 3) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Finally, we have 6 classes A1, A2, A3, B1, B2, B3 with some subclasses.

The contravariant metric tensor of the Stäckel space of type (2.1) in a privileged coordinate system can be written as (signature $(-,+,+,+)$)

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & f(x^1) & 1 \\ 0 & f(x^1) & c(x^0,x^1) & b(x^0) \\ 0 & 1 & b(x^0) & a(x^0) \end{pmatrix}$$

where $\Delta = d_0(x^0) + d_1(x^1)$, $\quad c = c_0(x^0) + c_1(x^1)$

$$\det g^{ij} = -\frac{D}{\Delta^4}, \qquad D = a\,f^2 - 2\,b\,f + c > 0$$

The complete set of type (2.1) includes the following Killing vectors:

$$X_1 = (0,0,0,1)\;, \quad X_2 = (0,0,1,0)$$

The vector X_2 is space-like because $D > 0$; however, the restriction of the metric to the orbits of soubgroup X_1, X_2 is degenerated. It is a general property of Stäckel spaces with $N_0 \neq 0$, which are called null spaces. The breaking of space-likeness of the orbits of the complete set subgroup demands the introduction of an additional Killing vectors,

$$X_3 = \xi\;, \qquad X_4 = \eta.$$

The commutation relations are

$$\begin{aligned} &[X_1, X_2] = 0 \\ &[X_1, X_a] = \alpha_a\,X_1 + {\beta_a}^b\,X_b, \qquad a,b = 3,4 \\ &[X_2, X_a] = {\gamma_a}^2\,X_2 + {\gamma_a}^b\,X_b \\ &[X_3, X_4] = \gamma_5\,X_3 + \gamma_6\,X_4 + \gamma_7\,X_2 \end{aligned}$$

These relations lead to the following differential equations:

$$1. \begin{cases} {\xi^i}_{,3} = {\beta_3}^3\xi^i + {\beta_3}^4\eta^i + \alpha_3\,\delta^i_3 + {\beta_3}^2\,\delta^i_2 \\ {\eta^i}_{,3} = {\beta_4}^3\xi^i + {\beta_4}^4\eta^i + \alpha_4\,\delta^i_3 + {\beta_4}^2\,\delta^i_2 \end{cases}$$

$$2. \begin{cases} {\xi^i}_{,2} = {\gamma_3}^3\xi^1 + {\gamma_3}^4\eta^i + {\gamma_3}^2\,\delta^i_2 \\ {\eta^i}_{,2} = {\gamma_4}^3\xi^i + {\gamma_4}^4\eta^i + {\gamma_4}^2\,\delta^i_2 \end{cases}$$

$$3.\; \xi^j\,{\eta^i}_{,j} - \eta^j\,{\xi^i}_{,j} = \gamma_5\,\xi^i + \gamma^6\,\eta^i + \gamma_7\delta^i_2$$

To find uncrossed algebraic classes, the vectors of this subgroup can be subjected to the linear trnsformation

$$\tilde{X}_p = {S_p}^q\,X_q, \qquad p,q = 2..4$$

Also, one can use a coordinat transformation respecting the form of the metric tensor

$$\begin{aligned} &\tilde{x^p} = \alpha^p\,x^p \qquad\qquad p,q = 0,1 \\ &\tilde{x^\nu} = \alpha^\nu + {\beta_\mu}^\nu(x^p)\,x^\mu \qquad\qquad \mu,\nu = 2,3 \end{aligned}$$

There are 2 cases

$$1)\ g^{12} = f(x^1) = 0 \qquad 2)\ g^{12} = f(x^1) \neq 0$$

1. Cosider the case $f = 0$.

Applying the admissible transformations (5, 5) and the Jacobi identities to the commutation relations, one finds 6 types of dependences of the KV on the ignored variables x^2, x^3:

1) $\xi^i = \phi^i$
$\eta^i = \psi^i$

2) $\xi^i = \phi^i + x^2\,\delta^i_2$
$\eta^i = \psi^i$

3) $\xi^i = \phi^i + {\gamma_3}^2\,x^2\,{\delta_2}^i + x^3\,{\delta_3}^i$
$\eta^i = \psi^i$

4) $\xi^i = \phi^i + x^3\,\delta^i_3$
$\eta^i = \psi^i + x^2\,\delta^i_2$

5) $\xi^i = \phi^i + \psi^i\,x^2 + ({\gamma_3}^2\,x^2 + {\beta_3}^2\,x^3)\,\delta^3_2 + \alpha_3\,x^3\,\delta^i_3$
$\eta^i = \psi^i$

6) $\xi^i = \phi^i + \psi^i\,x^2$
$\eta^i = \psi^i + x^2\,\delta^i_2$,

where $\phi^i, \psi^i(x^0, x^1)$.

The finding of functions $\phi^i, \psi^i(x^0, x^1)$ is a difficulty leading to a large number of solutions.

2. If $f \neq 0$, the equations are integrated by the introduction of the new variable

$$\tilde{x^2} = x^2 - f\,x^3.$$

In the new coordinates, the block structure of the metric tensor is the following:

$$g^{\tilde{i}j} = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & D & B \\ 0 & 1 & B & a \end{pmatrix},$$

where $D = a\,f^2 - 2\,b\,f + c, \quad B = b - af - f'x^3$. This is why the equation is solved in a manner similar to the case $f = 0$.

In this section we are obtained next results:

1. Homogeneuos Stackel space-times of (2.1) type:

a) 29 types were found.

b) The solutions were classificated according Bianci, types I – VII are present.

c) Petrov types III and N are present, they are depend on conformal factor Δ:

$\Delta = d_1(x^1),\ R = 0,$ type N;

$\Delta = d_0(x^0),\ R = const < 0$, type D.

d) Space-times of (3.1) type allow an analogue of spherical wave or an analogue of homogeneous radiation for high-frequency radiation of general nature.

2. Homogeneuos Stackel space-times of (3.1) type:

a) 12 types were found.

b) The solutions were classificated according Bianci, types I – VII are present.

c) Petrov types D and N are present, they are depend on functions b_k:

$b_2 = b_3 = 0,\ R = 0,$ type N;

$R \neq 0$, type D.

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Hypermultiplet dependence of one-loop effective action in the $\mathcal{N}=2$ superconformal theories

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Abstract

I review the approach [1] to the one-loop low-energy effective action in the hypermultiplet sector for $\mathcal{N}=2$ superconformal models. Any such a model contains an $\mathcal{N}=2$ vector multiplet and some number of hypermultiplets. We found a general expression for the low-energy effective action in the form of a proper-time integral. The leading space-time dependent contributions to the effective action are derived and their bosonic component structure is analyzed. The component action contains terms with three and four space-time derivatives of component fields and has the Chern-Simons-like form.

1 Introduction

I am very glad to take part in this book devoted to celebration of the 60 birth day of remarkable scientist and my dear friend Ioseph L. Buchbinder.

Four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories are formulated in terms of $\mathcal{N}=2$ vector multiplet coupled to a massless hypermultiplets in certain representations R of the gauge group G. All such models possess only one-loop divergences [2] and can be made finite at certain restrictions on representations and field contents. In the model with n_σ hypermultiplets in representations R_σ of the gauge group G the finiteness condition has simple and universal form

$$C(G)=\sum_\sigma n_\sigma T(R_\sigma), \tag{1}$$

where $C(G)$ is the quadratic Casimir operator for the adjoint representation and $T(R_\sigma)$ is the quadratic Casimir operator for the representation R_σ. A simplest solution to Eq.(1) is $\mathcal{N}=4$ SYM theory where $n_\sigma=1$ and all fields are taken in the adjoint representation. It is evident that there are other solutions, e.g. for the case of SU(N) group and hypermultiplets in the fundamental representation one gets $T(R)=1/2$, $C(G)=N$ and $n_\sigma=2N$. A number of $\mathcal{N}=2$ superconformal models has been constructed in the context of AdS/CFT correspondence (see e.g. [3], the examples

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of such models and description of structure of vacuum states were discussed in details e.g. in Ref. [4]).

In this paper we study the structure of the low-energy one-loop effective action for the $\mathcal{N}=2$ superconformal theories. The effective action of the $\mathcal{N}=4$ SYM theory and $\mathcal{N}=2$ superconformal models in the sector of $\mathcal{N}=2$ vector multiplet has been studied by various methods. However a problem of hypermultiplet dependence of the effective action in the above theories was open for a long time.

The low-energy effective action containing both $\mathcal{N}=2$ vector multiplet and hypermultiplet background fields in $\mathcal{N}=4$ SYM theory was first constructed in Ref. [5] and studied in more details in [6]. In this paper we will consider the hypermultiplet dependence of the effective action for $\mathcal{N}=2$ superconformal models. Such models are finite theories as well as the $\mathcal{N}=4$ SYM theory and one can expect that hypermultiplet dependence of the effective action in $\mathcal{N}=2$ superconformal models is analogous to one in $\mathcal{N}=4$ SYM theory. However this is not so evident. The $\mathcal{N}=4$ SYM theory is a special case of the $\mathcal{N}=2$ superconformal models, however it possesses extra $\mathcal{N}=2$ supersymmetry in comparison with generic $\mathcal{N}=2$ models. As it was noted in [5] just this extra $\mathcal{N}=2$ supersymmetry is the key point for finding an explicit hypermultiplet dependence of the effective action in $\mathcal{N}=4$ SYM theory. Therefore a derivation of the effective action for $\mathcal{N}=2$ superconformal models in the hypermultiplet sector is an independent problem.

In this paper we derive the complete $\mathcal{N}=2$ supersymmetric one-loop effective action depending both on the background vector multiplet and hypermultiplet fields in a mixed phase where both vector multiplet and hypermultiplet have non-vanishing expectation values. The $\mathcal{N}=2$ supersymmetric models under consideration are formulated in harmonic superspace [7]. We develop a systematic method of constructing the lower- and higher-derivative terms in the one-loop effective action given in terms of a heat kernel for certain differential operators on the harmonic superspace and calculate the heat kernel depending on $\mathcal{N}=2$ vector multiplet and hypermultiplet background superfields. We study a component form of a leading quantum corrections for on-shell and beyond on-shell background hypermultiplets and find that they contain, among the others, the terms corresponding to the Chern-Simons-type actions. The necessity of such manifest scale invariant P-odd terms in effective action of $\mathcal{N}=4$ SYM theory, involving both scalars and vectors, has been pointed out in [8]. Proposal for the higher-derivative terms in the effective action of the $\mathcal{N}=2$ models in the harmonic superspace has been given in [9]. We show how the terms in the effective action assumed in P.C. Argyres at al. can be actually computed in supersymmetric quantum field theory.

2 The model and background field splitting

$\mathcal{N}=2$ harmonic superspace has been introduced in [10] extending the standard $\mathcal{N}=2$ superspace with coordinates $z^M=(x^m,\theta^\alpha_i,\overline{\bar\theta^i_{\dot\alpha}})$ $(i=1,2)$ by the harmonics $u^\pm_i$ parameterizing the two-dimensional sphere S^2: $u^{+i}u^-_i=1,\quad \overline{u^{+i}}=u^-_i$.

The main advantage of harmonic superspace is that the $\mathcal{N}=2$ vector multiplet and hypermultiplet can be described by unconstrained superfields over the analytic subspace with the coordinates $\zeta^M\equiv(x^m_A,\theta^{+\alpha},\bar\theta^+_{\dot\alpha},u^\pm_i)$, where the so-called analytic basis is defined by

$$x^m_A=x^m-i\theta^+\sigma\bar\theta^- - i\theta^-\sigma^m\bar\theta^+,\quad \theta^\pm_\alpha=u^\pm_i\theta^i_\alpha,\quad \bar\theta^\pm_{\dot\alpha}=u^\pm_i\bar\theta^i_{\dot\alpha}\ . \tag{2}$$

The $\mathcal{N}=2$ vector multiplet is described by a real analytic superfield $V^{++}=V^{++I}(\zeta)T_I$ taking values in the Lie algebra of the gauge group. A hypermultiplet, transforming in the representation R of the gauge group, is described by an analytic superfield $\mathbf{q}^+(\zeta)$ and its conjugate $\tilde{\mathbf{q}}^+(\zeta)$.

The classical action of $\mathcal{N}=2$ SYM theory coupled to hypermultiplets consist of two parts: the pure $\mathcal{N}=2$ SYM action and the q-hypermultiplet action in the fundamental or adjoint representation of the gauge group. Written in the harmonic superspace its action reads

$$S=\tfrac{1}{2g^2}\mathrm{tr}\int d^8z\,\mathcal{W}^2+\tfrac12\int d\zeta^{(-4)}q^{+f}_a(D^{++}+igV^{++})q^{+a}_f\ , \tag{3}$$

where we used the doublet notation $q_a^+ = (q^+, -\tilde{q}^+)$. By construction, the action (3) is manifestly $\mathcal{N}=2$ supersymmetric. Here $d\zeta^{(-4)} = d^4x d^4\theta^+ du$ denotes the analytic subspace integration measure and

$$\mathcal{D}^{++} = D^{++} + iV^{++}, \quad D^{++} = \partial^{++} - 2i\theta^+\sigma^m\bar{\theta}^+\partial_m, \quad \partial^{++} \equiv u^{+i}\frac{\partial}{\partial u^{-i}}$$

is the analyticity-preserving covariant harmonic derivative. It can be shown that V^{++} is the single unconstrained analytic, $D^+_{(\alpha,\dot{\alpha})}V^{++} = 0$, prepotential of the pure $\mathcal{N}=2$ SYM theory, and all other geometrical object are determined in terms of it. So,the covariantly chiral superfield strength $\mathcal{W}$

$$\mathcal{W} = -\tfrac{1}{4}(\bar{D}^+)^2 V^{--}, \quad \bar{\mathcal{W}} = -\tfrac{1}{4}(D^+)^2 V^{--}. \tag{4}$$

is expressed through the (nonanalytic) real superfield V^{--} satisfying the equation

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0.$$

This equation has a solution in form of the power series in V^{++} [11].

For further use we will write down also the superalgebra of gauge covariant derivatives with the notation $\mathcal{D}^{\pm}_{(\alpha,\dot{\alpha})} = \mathcal{D}^{i}_{(\alpha,\dot{\alpha})}u^{\pm}_i$:

$$\{\mathcal{D}^+_\alpha, \mathcal{D}^-_\beta\} = -2i\varepsilon_{\alpha\beta}\bar{\mathcal{W}}\ , \quad \{\bar{\mathcal{D}}^+_{\dot{\alpha}}, \bar{\mathcal{D}}^-_{\dot{\beta}}\} = 2i\varepsilon_{\dot{\alpha}\dot{\beta}}\mathcal{W}\ , \tag{5}$$

$$\{\bar{\mathcal{D}}^+_{\dot{\alpha}}, \mathcal{D}^-_\alpha\} = -\{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^-_{\dot{\alpha}}\} = 2i\mathcal{D}_{\alpha\dot{\alpha}}\ ,$$

$$[\mathcal{D}^\pm_\alpha, \mathcal{D}_{\beta\dot{\beta}}] = \varepsilon_{\alpha\beta}\bar{\mathcal{D}}^\pm_{\dot{\beta}}\bar{\mathcal{W}}\ , \quad [\bar{\mathcal{D}}^\pm_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \varepsilon_{\dot{\alpha}\dot{\beta}}\mathcal{D}^\pm_\beta\mathcal{W}\ ,$$

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \tfrac{1}{2i}\{\varepsilon_{\alpha\beta}\bar{\mathcal{D}}^+_{\dot{\alpha}}\bar{\mathcal{D}}^-_{\dot{\beta}}\bar{\mathcal{W}} + \varepsilon_{\dot{\alpha}\dot{\beta}}\mathcal{D}^-_\alpha\mathcal{D}^+_\beta\mathcal{W}\} = \tfrac{1}{2i}\{\varepsilon_{\alpha\beta}\bar{F}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}\}\ .$$

The operators $\mathcal{D}^+_\alpha$ and $\bar{\mathcal{D}}^+_{\dot{\alpha}}$ strictly anticommute

$$\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot{\alpha}}, \bar{\mathcal{D}}^+_{\dot{\beta}}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot{\alpha}}\} = 0\ . \tag{6}$$

A full set of gauge covariant derivatives includes also the harmonic derivatives ($\mathcal{D}^{++}$, $\mathcal{D}^{--}, \mathcal{D}^0$), which form the algebra $su(2)$ and satisfy the obviously commutation relations with $\mathcal{D}^\pm_\alpha$ and $\bar{\mathcal{D}}^\pm_{\dot{\alpha}}$.

The action (3) possesses the superconformal symmetry $SU(2,2|2)$ which is manifest in the harmonic superspace approach. The low energy effective action at a generic vacuum of $\mathcal{N}=2$ gauge theory includes only massless U(1) vector multiplets and massless neutral hypermultiplets, since charged vectors and charged hypermultiplets get masses by the Higgs mechanism. The moduli space of vacua for the theory under consideration is specified by the following conditions [12]:

$$[\bar{\phi}, \phi] = 0, \quad \phi f_i = 0, \quad \bar{f}^i\bar{\phi} = 0 \quad \bar{f}^{(i}T_I f^{j)} = 0\ . \tag{7}$$

Here the $\phi, \bar{\phi}$ are the scalar components of $\mathcal{N}=2$ vector multiplet and complex scalars f_i are the scalar components of the hypermultiplet.

The structure of a vacuum state is characterized by solutions to Eqs. (7). These solutions can be classified according to the phases or branches of the gauge theory under consideration. In the pure Coulomb phase $f_i = 0$, $\phi \neq 0$ and unbroken gauge group is $\mathrm{U}(1)^{\mathrm{rank(G)}}$. In the pure Higgs phase $f_i \neq 0$ and the gauge symmetry is completely broken; there are no massless gauge bosons. In the mixed phases, i.e. on the direct product of the Coulomb and Higgs branches (some number of $\phi, \bar{\phi}$ is not equal to zero and some number of f_i is not equal to zero) the gauge group is broken down to $\tilde{G} \times K$ where K is some Abelian subgroup.

Further we impose the special restrictions on the background $\mathcal{N}=2$ vector multiplet and hypermultiplet. They are chosen to be aligned along a fixed direction in the moduli space vacua; in particular, their scalar fields should solve Eqs. (7):

$$V^{++} = \mathbf{V}^{++}(\zeta)H, \quad q^+ = \mathbf{q}^+(\zeta)\Upsilon\ . \tag{8}$$

Here H is a fixed generator in the Cartan subalgebra corresponding to Abelian subgroup K, and Υ is a fixed vector in the R-representation space of the gauge group, where the hypermultiplet takes values, chosen so that $H\Upsilon = 0$ and $\bar{\Upsilon}T_I\Upsilon = 0$. Eq.(8) defines a single U(1) vector multiplet and a single hypermultiplet which is neutral with respect to the U(1) gauge subgroup generated by H.

At the tree level and energies below the symmetry breaking scale, we have free field massless dynamics of the $\mathcal{N}=2$ vector multiplet and the hypermultiplet aligned in a particular direction in the moduli space of vacua. Thus the low energy propagating fields are massless neutral hypermultiplets and U(1) vector which form the on shell superfields possessing the properties

$$(D^{\pm})^2\mathcal{W} = (\bar{D}^{\pm})^2\bar{\mathcal{W}} = 0 \ , \tag{9}$$

$$D^{++}q^{+a} = (D^{--})^2q^{+a} = D^{--}q^{-a} = 0, \quad q^{-a} = D^{--}q^{+a}, \quad D^{-}_{(\alpha,\dot\alpha)}q^{-a} = 0 \ .$$

The equations (9) eliminate the auxiliary fields and put the physical fields on shell.

At the quantum level, however, exchanges of virtual massive particles produce the corrections to the action of the massless fields. We quantize the $\mathcal{N}=2$ supergauge theory in the framework of the $\mathcal{N}=2$ supersymmetric background field method [13] by splitting the fields V^{++}, q^{+a} into the sum of the background fields V^{++}, q^{+a}, parameterized according to (8), and the quantum fields v^{++}, Q^{+a} and expanding the Lagrangian in a power series in quantum fields. Such a procedure allows us to find the effective action for arbitrary $\mathcal{N}=2$ supersymmetric gauge model in a form preserving the manifest $\mathcal{N}=2$ supersymmetry and classical gauge invariance in quantum theory.

In the background-quantum splitting, the classical action of the pure $\mathcal{N}=2$ SYM theory can be shown to be given by

$$S_{SYM}[V^{++}+v^{++}] = S_{SYM}[V^{++}] + \tfrac{1}{4}\int d\zeta^{(-4)}du v^{++}(D^{+})^2\mathcal{W}_\lambda \tag{10}$$

$$-\mathrm{tr}\int d^{12}z\sum_{n=2}^{\infty}\tfrac{(-ig)^{n-2}}{n}\int du_1...du_n\tfrac{v_\tau^{++}(z,u_1)...v_\tau^{++}(z,u_n)}{(u_1^+u_2^+)...(u_n^+u_1^+)} \ .$$

$\mathcal{W}_\lambda$ and v_τ^{++} denote the λ- and τ-frame forms of $\mathcal{W}$ and v^{++} respectively. The hypermultiplet action becomes

$$S_H(q+Q) = S_H[q] + \int d\zeta^{(-4)}du Q_a^+\mathcal{D}^{++}q^{+a} + \tfrac{1}{2}\int d\zeta^{(-4)}du q_a^+ iv^{++}q^{+a} \tag{11}$$

$$+\tfrac{1}{2}\int d\zeta^{(-4)}du\{Q_a^+\mathcal{D}^{++}Q^{+a} + Q_a^+iv^{++}q^{+a} + q_a^+iv^{++}Q^{+a} + Q_a^+iv^{++}Q^{+a}\} \ .$$

The terms linear in v^{++} and q^+ in (10), (11) determines the equation of motion and this term should be dropped when considering the effective action.

To construct the effective action, we will follow the Faddeev-Popov Ansatz. We write the final result for the effective action $\Gamma[V^{++}, q^+]$

$$e^{i\Gamma[V^{++},\,q^+]} = e^{iS_{cl}[V^{++},\,q^+]}\mathrm{Det}^{1/2}\widehat{\Box}_{(4,0)}\int\mathcal{D}v^{++}\mathcal{D}Q^+\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\mathcal{D}\varphi e^{iS_q}, \tag{12}$$

where $\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2$ and action S_q is as follows

$$S_q[v^{++}, Q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, q^+] = S_2[v^{++}, Q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, q^+] + S_{int},$$

$$S_2 = -\tfrac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du v^{++}\widehat{\Box}v^{++} + \mathrm{tr}\int d\zeta^{(-4)}du\mathbf{b}(\mathcal{D}^{++})^2\mathbf{c} \tag{13}$$

$$+\tfrac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\varphi(\mathcal{D}^{++})^2\varphi + \tfrac{1}{2}\int d\zeta^{(-4)}du\{Q_a^+\mathcal{D}^{++}Q^{+a}$$

$$+Q_a^+iv^{++}q^{+a} + q_a^+iv^{++}Q^{+a}\} \ ,$$

This equations completely determine the structure of the perturbation expansion for calculating the effective action $\Gamma[V^{++}, q^+]$ of the $\mathcal{N}=2$ SYM theory with hypermultiplets in a manifestly supersymmetric and gauge invariant form. The action S_2 defines the propagators depending on background fields. In the framework of the background field formalism in $\mathcal{N}=2$ harmonic superspace there appear three types of covariant matter and gauge field propagators. Associated with $\widehat{\Box}$ is a Green's function $G^{(2,2)}(z,z')$ which satisfies the equation $\widehat{\Box}\, G^{(2,2)}(1|2) = -\mathbf{1}\delta^{(2,2)}(1|2)$, is

$$G^{(2,2)}(1,2) = -\tfrac{1}{2\widehat{\Box}_1\widehat{\Box}_2}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\{\mathbf{1}\delta^{12}(z_1-z_2)(D_2^{--})^2\delta^{(-2,2)}(u_1,u_2)\}\ . \tag{14}$$

The Q^+ hypermultiplet propagator associated with the action (13) has the form

$$G_b^{a(1.1)}(1|2) = -\delta_b^a \tfrac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\tfrac{1}{\widehat{\Box}_1}\delta^{12}(z_1-z_2)\ . \tag{15}$$

It is not hard to see that this manifestly analytic expression is the solution of the equation $\mathcal{D}_1^{++}G^{(1,1)} = \delta_A^{(3,1)}(1|2)$. For the hypermultiplet of the second type described by a chargeless real analytic superfield $\omega(\zeta,u)$ the equation for Green' function is
$(\mathcal{D}_1^{++})^2G^{(0,0)}(1|2) = \delta_A^{(4,0)}(1|2)$. The suitable expression for $G^{(0,0)}$ is

$$G^{(0,0)}(1|2) = -\tfrac{1}{\widehat{\Box}_1}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\{\mathbf{1}\delta^{12}(z_1-z_2)\tfrac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\}. \tag{16}$$

The operator $\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2$ transforms each covariantly analytic superfield into a covariantly analytic and, using algebra (5), can be rewritten as second-order d'Alemberian-like differential operator on the space of such superfields. The coefficients of this operator depend on background superfields $\mathcal{W},\bar{\mathcal{W}}$.

3 Structure of the one-loop effective action

Consider the loop expansion of the effective action within the background field formulation. A formal expression of the one-loop effective action $\Gamma[V^{++},\ q^+]$ for the theory under consideration is written in terms of a path integral as follows (12), where the full quadratic action is defined in Eq. (13). Here v^{++} is a quantum vector superfield taking values in the Lie algebra of the gauge group and **b**, **c** are two real analytic Faddeev-Popov fermionic ghosts and φ is the bosonic Nielsen-Kallosh ghost, all in the adjoint representation of the gauge group.

In the vector sector of the $\mathcal{N}=2$ SYM theory where the matter hypermultiplet are integrated out, the one-loop effective action $\Gamma[V^{++}]$ reads

$$\Gamma[V^{++}] = \tfrac{i}{2}\mathrm{Tr}_{(2,2)}\ln\widehat{\Box} - \tfrac{i}{2}\mathrm{Tr}_{(4,0)}\ln\widehat{\Box} - \tfrac{i}{2}\mathrm{Tr}_{ad}\ln(\mathcal{D}^{++})^2 + i\mathrm{Tr}_{R_q}\ln\mathcal{D}^{++} + \tfrac{i}{2}\mathrm{Tr}_{R_\omega}\ln(\mathcal{D}^{++})^2.$$

Currently, the holomorphic and non-holomorphic parts of the low-energy effective action $\mathcal{N}=2,4$ SYM theory on the Coulomb branch, including Heisenberg-Euler type action in the presence of a covariantly constant vector multiplet, are completely known. The general structure of the low-energy effective action in $\mathcal{N}=2,4$ superconformal theories is [14]:

$$\Gamma = S_{cl} + c\int d^{12}z\ln\mathcal{W}\ln\bar{\mathcal{W}} + \int d^{12}z\ln\mathcal{W}\Lambda(\tfrac{D^4\ln\mathcal{W}}{\bar{\mathcal{W}}^2}) + c.c. + \int d^{12}z\Upsilon(\tfrac{\bar{D}^4\ln\bar{\mathcal{W}}}{\mathcal{W}^2},\tfrac{D^4\ln\mathcal{W}}{\bar{\mathcal{W}}^2}),$$

where Λ and Υ are holomorphic and real analytic function of the (anti)chiral superconformal invariants. The c-term is known to generate four-derivative quantum corrections at the component level which include an famous F^4 term.

The hypermultiplet dependent part of the effective action in $\mathcal{N}=4$ SYM theory in leading order is also known [15]. For further analysis of the effective action it is convenient to diagonalize the action of quantum fields $S^{(2)}$ using a special shift of hypermultiplet variables in the path integral

$$Q^{+a} = \xi^{+a} + i\int d\zeta_2^{(-4)}q^{+b}(2)v^{++}(2)G_b^{a(1.1)}(1|2), \tag{17}$$

$$Q_a^+ = \xi_a^+ - i\int d\zeta_2^{(-4)}G_a^{b(1.1)}(1|2)v^{++}(2)q_b^+(2)\ ,$$

where ξ^{+a}, ξ^{+}_{a} are the new independent variables in the path integral. It is evident that the Jacobian of the replacement (17) is equal to unity. Here $G^{a(1.1)}_{b}(1|2)$ is the background-dependent propagator (15) for the superfields Q^{+a}, Q^{+}_{b}. In terms of the new set of quantum fields we obtain for the following hypermultiplet dependent part of the quadratic action

$$S^{(2)}_{H} = -\tfrac{1}{2}\int d\zeta^{(-4)}\xi^{a+}\mathcal{D}^{++}\xi^{+}_{a} - \tfrac{1}{2}\int d\zeta^{(-4)}_{1}d\zeta^{(-4)}_{2}q^{+a}(1)v^{++}(1)G^{b(1.1)}_{a}(1|2)v^{++}(2)q^{+}_{b}(2)\ .$$

Then the vector multiplet dependent part of the quadratic action gets the following non-local extension

$$S^{(2)}_{v} = -\tfrac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}_{1}v^{++}_{1}\int d\zeta^{(-4)}_{2}\left(\widehat{\Box}\delta^{(2.2)}_{A}(1|2) + q^{+a}(1)G^{b(1.1)}_{a}(1|2)q^{+}_{b}(2)\right)v^{++}_{2}\ . \qquad (18)$$

Expression (18), written as an analytical nonlocal superfunctional, will be a starting point for our calculations of the one-loop effective action in the hypermultiplet sector. Our aim in the current and later sections is to find the leading low-energy contribution to the effective action for the slowly varying hypermultiplet when all derivatives of the background hypermultiplet can be neglected. We will show that for such a case the non-local interaction is localized.

Using the relation $v^{++}_{2} = \int d\zeta^{(-4)}_{3}\delta^{(2.2)}_{A}(2|3)v^{++}_{3}$ one can rewrite expression for $S^{(2)}_{v}$ (18) in the form

$$S^{(2)}_{v} = -\tfrac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}_{1}\, v^{++}_{1}\int d\zeta^{(-4)}_{2}(\widehat{\Box}\delta^{(2.2)}_{A}(1|2) \qquad (19)$$

$$+\int d\zeta^{(-4)}_{3}q^{+a}(1)G^{b(1.1)}_{a}(1|3)q^{+}_{b}(3)\delta^{(2.2)}_{A}(3|2)v^{++}_{2})\ .$$

Then we use the explicit form of the Green function (15) and the relation allowing us to express the $(\mathcal{D}^{+}_{1})^{4}(\mathcal{D}^{+}_{2})^{4}$ as a polynomial in powers of $(u^{+}_{1}u^{+}_{2})$ [16]:

$$(\mathcal{D}^{+}_{1})^{4}(\mathcal{D}^{+}_{2})^{4} \qquad (20)$$

$$= (\mathcal{D}^{+}_{1})^{4}\left((\mathcal{D}^{-}_{1})^{4}(u^{+}_{1}u^{+}_{2})^{4} - \tfrac{i}{2}\Delta^{--}_{1}(u^{+}_{1}u^{+}_{2})^{3}(u^{-}_{1}u^{+}_{2}) - \widehat{\Box}_{1}(u^{+}_{1}u^{+}_{2})^{2}(u^{-}_{1}u^{+}_{2})^{2}\right)\ ,$$

where the operator Δ^{--} is

$$\Delta^{--} = \mathcal{D}^{\alpha\dot\alpha}\mathcal{D}^{-}_{\alpha}\bar{\mathcal{D}}^{-}_{\dot\alpha} + \tfrac{1}{2}\mathcal{W}(\mathcal{D}^{-})^{2} + \tfrac{1}{2}\bar{\mathcal{W}}(\bar{\mathcal{D}}^{-})^{2} + (\mathcal{D}^{-}\mathcal{W})\mathcal{D}^{-} + (\bar{\mathcal{D}}^{-}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-}\ . \qquad (21)$$

The non-local term in (19) takes the form

$$\int d\zeta^{(-4)}_{3}q^{+a}(1)(\mathcal{D}^{+}_{3})^{4}\times$$

$$\times\left((\mathcal{D}^{-}_{3})^{4}(u^{+}_{3}u^{+}_{1})\tfrac{1}{\Box_{3}} - \tfrac{i}{2}\Delta^{--}_{3}(u^{-}_{3}u^{+}_{1})\tfrac{1}{\Box_{3}} - \tfrac{(u^{-}_{3}u^{+}_{1})^{2}}{(u^{+}_{3}u^{+}_{1})}\right)\delta^{12}(1|3)q^{+}_{a}(3)\delta^{(2.2)}_{A}(3|2)\ .$$

The large braces here contain three terms. It is easy to see that two first terms include the derivatives which will lead to derivatives of the hypermultiplet in the effective action. Since we keep only contributions without derivatives, the above terms can be neglected. As a result, is it sufficient to consider only the third term in the braces.

Now we apply the relation $\int d\zeta^{(-4)}_{3}(\mathcal{D}^{+}_{3})^{4} = \int d^{12}z$, allowing to integrate over z_3, and obtain

$$-\int du_{3}\, q^{+a}(1)\tfrac{(u^{-}_{3}u^{+}_{1})^{2}}{(u^{+}_{3}u^{+}_{1})}q^{+}_{a}(u_{3},z_{1})\delta^{(2.2)}_{A}(u_{3},z_{1}|2)\ .$$

Then one uses the on-shell harmonic dependence of hypermultiplet $q^{+a}(3) = u^{+}_{3i}q^{ia}$ and take the coincident limit $u_1 = u_3$ (conditioned by $\delta^{(2.2)}_{A}(u_3, z_1|2)$). After that we get $\int du_{3}\frac{u^{+}_{3i}}{u^{+}_{3}u^{+}_{1}} = -u^{-}_{1i}$. As a result, the term under consideration has the form

$$q^{+a}(1)q^{-}_{a}(1)\delta^{(2.2)}_{A}(1|2),$$

where the expression $q^{+a}(1)q_a^-(1) = q^{ia}q_{ia}$ is treated further as the slowly varying superfield and all its derivatives are neglected. Namely such an expression was obtained in [6] by summation of harmonic supergraphs.

Thus, the second term in (19) becomes local in the leading low-energy approximation. As a result, the operator in action $S_v^{(2)}$ determining the effective background covariant propagator of the quantum vector multiplet superfield v_I^{++} takes the form

$$\left(\widehat{\Box}_{IJ} + q^{+a}(z_1,u_1)\{T_I,T_J\}q_a^-(z_1,u_1)\right)\delta_A^{(2.2)}(1|2)\;, \tag{22}$$

where

$$\widehat{\Box}_{IJ} = \mathrm{tr}(T_{(I}\Box T_{J)} + \tfrac{i}{2}T_{(I}[\mathcal{D}^{+\alpha}\mathcal{W},T_{J)}]\mathcal{D}_\alpha^- + \tfrac{i}{2}T_{(I}[\bar{\mathcal{D}}_{\dot\alpha}^+\bar{\mathcal{W}},T_{J)}]\bar{\mathcal{D}}^{-\dot\alpha} + T_{(I}[\mathcal{W},[\bar{\mathcal{W}},T_{J)}]]. \tag{23}$$

Here $\Box = \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$ is the covariant d'Alemberian.

Thus, using the $\mathcal{N}=2$ harmonic superspace formulation of the $\mathcal{N}=2$ SYM theory with hypermultiplets and techniques of the non-local shift we obtained that the whole dependence on the background hypermultiplet is concentrated in the quantum vector multiplet sector with the modified quadratic action. Therefore the one-loop effective action is given by the expression

$$\Gamma^{(1)}[V^{++},q^+] = \Gamma_v^{(1)}[V^{++},q^+] + \widetilde{\Gamma}^{(1)}[V^{++}]\;, \tag{24}$$

where the first term in (24) is originated from quantum vector multiplet v_I^{++}

$$\Gamma_v^{(1)}[V^{++},q^+] = \tfrac{i}{2}\mathrm{Tr}\ln(\widehat{\Box}_{IJ} + q^{+a}\{T_I,T_J\}q_a^-)\;. \tag{25}$$

Second term in (24) is the contribution of ghosts and quantum hypermultiplet ξ_a^+ and does not depend on the background hypermultiplet.

As a result, the background hypermultiplet dependence of one-loop effective action is included into the operator (23), acting on v_I^{++} and containing the mass matrix of the vector multiplet

$$(\mathcal{M}_v^2)_{IJ} = \mathrm{tr}\left([T_I,\mathcal{W}][\bar{\mathcal{W}},T_J] + (I\leftrightarrow J)\right) + q^{+a}\{T_I,T_J\}q_a^-\;, \tag{26}$$

if q^+ is in the fundamental representation, and

$$(\mathcal{M}_v^2)_{IJ} = \mathrm{tr}\left([T_I,\mathcal{W}][\bar{\mathcal{W}},T_J] + [q^{+a},T_I][T_J,q_a^-]\right) + (I\leftrightarrow J)\;, \tag{27}$$

if q^+ in an arbitrary matrix representation.

In the above discussion, the gauge group structure of the superfields $\mathcal{W}, q_a^+$ has been completely arbitrary. Henceforth, the background superfields will be chosen to be aligned along a fixed direction in the moduli space of vacua in such a way that their scalar fields should solve Egs. (7). Then the hypermultiplet dependent effective action in the case under consideration takes the universal form

$$\Gamma_v^{(1)}[V^{++},q^+] \tag{28}$$
$$= \tfrac{i}{2}n(\Upsilon)\times\mathrm{Tr}\ln\left(\Box + \tfrac{i}{2}\alpha(H)(\mathcal{D}^+\mathcal{W}\mathcal{D}^- + \bar{\mathcal{D}}^+\bar{\mathcal{W}}\bar{\mathcal{D}}^-) + \alpha^2(H)\mathcal{W}\bar{\mathcal{W}} + r(\Upsilon)q^{+a}q_a^-\right)\;.$$

As the examples we list the values of $\alpha(H), r(\Upsilon)$ and $n(\Upsilon)$ for models considered in [4].

(**i**) $\mathcal{N}=4$ SYM theory with gauge groups SU(N), Sp($2N$) and SO(N). Here the hypermultiplet sector is composed of a single hypermultiplet in the adjoint representation of the gauge group. The background was chosen such that the gauge groups are broken down as follows SU(N) $\to$ SU($N-1$) $\times$ U(1), Sp($2N$) $\to$ Sp($2N-2$) $\times$ U(1), SO(N) $\to$ SO($N-2$) $\times$ U(1). All background fields aligned along element $H=$ U(1) of the Cartan subalgebra (with $\Upsilon = H$). The mass matrix becomes

$$(\mathcal{M}_v^2)_{IJ} = (\mathcal{W}\bar{\mathcal{W}} + \mathbf{q}^{+a}\mathbf{q}_a^-)(\alpha(H))^2\delta_{I,J}$$

and traces in Eq.(24) produce the coefficient $n(\Upsilon)$ which is equal to the number of roots with $\alpha(H)\neq 0$, i.e. to the number of broken generators

$$n(\Upsilon) = \begin{cases} 2(N-1) & \text{for SU}(N)\,,\\ 4N-2 & \text{for Sp}(2N)\text{ and SO}(2N+1)\,,\\ 4N-1 & \text{for SO}(2N)\,.\end{cases}$$

The form of the mass matrix shows that in this case $r(\Upsilon) = \alpha(H)$.

(ii) The model introduced in [17]. The gauge group is $\mathrm{USp}(2N) = \mathrm{Sp}(2N, C) \bigcap \mathrm{U}(2N)$. The model contains four hypermultiplets q^+_F in the fundamental and one hypermultiplet q^+_A in the antisymmetric traceless representation $\mathrm{USp}(2N)$. The background fields $\mathcal{W}$, q^+_F, q^+_A are chosen to solve Eqs. (7) with the unbroken maximal gauge subgroup $\mathrm{USp}(2N-2) \times \mathrm{U}(1)$:

$$\mathcal{W} = \tfrac{\mathcal{W}}{\sqrt{2}}\,\mathrm{diag}(1, \underbrace{0, ..., 0}_{N-1}, -1, \underbrace{0, ..., 0}_{N-1}), \quad q^+_F = 0\,,$$

$$(q^+_A)_\alpha{}^\beta = \tfrac{\mathbf{q}^+}{\sqrt{2N(N-1)}}\,\mathrm{diag}(N-1, \underbrace{-1, ..., -1}_{N-1}, N-1, \underbrace{-1, ..., -1}_{N-1})\,.$$

The mass matrix $(\mathcal{M}^2_v)_{IJ}$ has been calculated in [4] and it has $n(\Upsilon) = 4(N-1)$ eigenvectors with the eigenvalue $\mathcal{M}^2_v = \bar{\mathcal{W}}\mathcal{W} + \frac{N}{N-1}\bar{\mathbf{q}}^j \mathbf{q}_j$.

(iii) The $\mathcal{N} = 2$ superconformal model which is the simplest quiver gauge theory [18]. Gauge group is $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$. The model contains two hypermultiplets q^+, $\tilde{q}^+$ in the bifundamental representations $(N, \bar{N})$ and $(\bar{N}, N)$ of the gauge group. In [4] a solutions of (7) with non-vanishing hypermultiplet components that specifies the flat directions in massless $\mathcal{N} = 2$ SYM theories has been constructed. The moduli space of vacua for this model includes the following field configuration

$$\mathcal{W}_L = \mathcal{W}_R = \tfrac{\mathcal{W}}{N\sqrt{2(N-1)}}\,\mathrm{diag}(N-1, \underbrace{-1..., -1}_{N-1})\,,$$

$$q^+ = \tilde{q}^+ = \tfrac{\mathbf{q}^+}{\sqrt{2}}\,\mathrm{diag}(1, 0, ..., 0)\,,$$

which preserves an unbroken gauge group $\mathrm{SU}(N-1) \times \mathrm{SU}(N-1)$ together with the diagonal $\mathrm{U}(1)$ subgroup in $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ associated with the chosen $\mathcal{W}$. In such a background the mass matrix has eigenvalue $\mathcal{M}^2_v = \frac{1}{N-1}\bar{\mathcal{W}}\mathcal{W} + \frac{1}{N}\mathbf{q}^{+a}\mathbf{q}^-_a$ and the corresponding $n(\Upsilon) = 4(N-1)$.

As the result, the hypermultiplet dependent effective action is given by the expression (28). In the next section we will consider the evaluation of this expression.

4 Calculation of the one-loop effective action

The expression (28) is a basis for an analysis of the hypermultiplet dependence of the effective action. In the framework of the Fock - Schwinger proper-time representation, the effective action (28) is written as follows

$$\Gamma^{(1)}_v[V^{++}, q^+] = \tfrac{i}{2} n(\Upsilon) \int d\zeta^{(-4)} du \int_0^\infty \tfrac{ds}{s} e^{-s(\square + \frac{i}{2}\alpha(H)(\mathcal{D}^+\mathcal{W}\mathcal{D}^- + \bar{\mathcal{D}}^+\bar{\mathcal{W}}\bar{\mathcal{D}}^-) + \mathcal{M}^2_v)} \times \tag{29}$$

$$\times (\mathcal{D}^+)^4 \left(\delta^{12}(z - z')\delta^{(-2,2)}(u, u')\right)|_{z=z', u=u'} = \int_0^\infty \tfrac{ds}{s} \mathrm{Tr} K(s),$$

where $\mathcal{M}^2_v = \alpha^2(H)\mathcal{W}\bar{\mathcal{W}} + r(\Upsilon) q^{+a} q^-_a$. Here $K(s)$ is a superfield heat kernel, the operation Tr means the functional trace in the analytic subspace of the harmonic superspace $\mathrm{Tr} K(s) = \mathrm{tr} \int d\zeta^{(-4)} K(\zeta, \zeta|s)$, where tr denotes the trace over the discrete indices. Representation of the effective action (29) allows us to develop a straightforward evaluation of the effective action in a form of covariant spinor derivatives expansion in the superfield Abelian strengths $\mathcal{W}, \bar{\mathcal{W}}$. The leading low-energy terms in this expansion correspond to the constant space-time background $D^-_\alpha D^+_\beta \mathcal{W} = \mathrm{const}$, $\bar{D}^-_{\dot\alpha} \bar{D}^+_{\dot\beta} \bar{\mathcal{W}} = \mathrm{const}$ and on-shell background hypermultiplet. However, it does not mean that we miss all space-time derivatives in the component effective Lagrangian. Grassmann measure in the integral over harmonic superspace $d^4\theta^+ d^4\theta^-$ generates four space-time derivatives in component expansion of the superfield Lagrangian. Therefore the above assumption is sufficient to obtain a component effective Lagrangian including four space-time derivatives of the scalar components of the hypermultiplet.

Calculation of the effective action (29) is based on evaluating the superfield heat kernel $K(s)$ and lead to a final result for the hypermultiplet dependent low-energy one-loop effective action of the Heisenberg-Euler type. We remind that the whole background hypermultiplet is concentrated in $\mathcal{M}_v^2$. The explicit form of it is:

$$\Gamma^{(1)}[V^{++},q^+] = \tfrac{1}{(4\pi)^2} n(\Upsilon) \int d\zeta^{(-4)} du \int_0^\infty \tfrac{ds}{s^3} e^{-s(\alpha^2(H)\mathcal{W}\bar{\mathcal{W}}+r(\Upsilon)q^{+a}q_a^-)} \times \tag{30}$$

$$\times \tfrac{\alpha^4(H)}{16}(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2 \tfrac{s^2(\mathcal{N}^2-\bar{\mathcal{N}}^2)}{\cosh(s\mathcal{N})-\cosh(s\bar{\mathcal{N}})} \cdot \tfrac{\cosh(s\mathcal{N})-1}{\mathcal{N}^2} \cdot \tfrac{\cosh(s\bar{\mathcal{N}})-1}{\bar{\mathcal{N}}^2} .$$

Here $\mathcal{N}$ is given by $\mathcal{N} = \sqrt{-\frac{1}{2}D^4\mathcal{W}^2}$. It can be expressed in terms of the two invariants of the Abelian vector field $\mathcal{F} = \frac{1}{4}F^{mn}F_{mn}$ and $\mathcal{G} = \frac{1}{4}{}^\star F^{mn}F_{mn}$ as $\mathcal{N} = \sqrt{2(\mathcal{F}+i\mathcal{G})}$.It is easily to see that the integrand in (30) can be expanded in power series in the quantities $s^2\mathcal{N}^2$, $s^2\bar{\mathcal{N}}^2$. After change of proper time s to $s'\mathcal{W}\bar{\mathcal{W}}$ we get the expansion in power of $s'^2\frac{\mathcal{N}^2}{(\mathcal{W}\bar{\mathcal{W}})^2}$ and their conjugate. Since the integrand of (30) is already $\sim (D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2$, we can change in each term of expansion the quantities $\mathcal{N}^2$, $\bar{\mathcal{N}}^2$ by superconformal invariants Ψ^2 and $\bar{\Psi}^2$ [14] expressing these quantities from $\bar{\Psi}^2 = \frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W} = \frac{1}{2\bar{\mathcal{W}}^2}\{\frac{\mathcal{N}^\beta_\alpha\mathcal{N}^\alpha_\beta}{\mathcal{W}^2} + \mathcal{O}(D^+\mathcal{W})\}$ and its conjugate. After that, one can show that each term of the expansion can be rewritten as an integral over the full $\mathcal{N}=2$ superspace.

It is interesting and instructive to evaluate the leading part of the effective action (30) that exactly coincides, up to group factor Υ with the earlier results [5], [6], [15]:

$$\Gamma^{(1)}_{\rm lead} = \tfrac{1}{(4\pi)^2} n(\Upsilon) \int d^{12}z\,(\ln\mathcal{W}\ln\bar{\mathcal{W}} + {\rm Li}_2(X) + \ln(1-X) - \tfrac{1}{X}\ln(1-X)). \tag{31}$$

Here ${\rm Li}_2(X)$ is the Euler's dilogarithm function. Next-to-leading corrections to (31) can also be calculated. The remarkable feature of the low-energy effective action (31) is the appearance of the factor $r(\Upsilon)/\alpha(H)$ in argument X. This factor is conditioned by the vacuum structure of the model under consideration and depends on the specific features of the symmetry breaking.

Now we discuss some terms in the component Lagrangian corresponding to the effective action (31). Component structure of the effective action (31) has been studied [5] in the context of $\mathcal{N}=4$ SYM theory in bosonic sector for completely constant background fields $F_{mn}, \phi, \bar{\phi}, f^i, \bar{f}_i$. However, it was pointed out above that the superfield effective action (31) allows us to find the terms in the effective action up to fourth order in space-time derivatives of component fields. Now our aim is to find such terms in the hypermultiplet scalar component sector. To do that we omit all components of the background superfields besides the scalars $\phi, \bar{\phi}$ in the $\mathcal{N}=2$ vector multiplet and scalars $f, \bar{f}$ in the hypermultiplet and integrate over $d^4\theta^+ d^4\theta^- = (D^-)^4(D^+)^4$. To get the leading space-time derivatives of the hypermultiplet scalar components we should put exactly two spinor derivatives on each hypermultiplet superfield. It yields, after some transformations, to the following term with four space-time derivatives on $q^\pm$ in component expansion of effective action :

$$\Gamma^{(1)}_{\rm lead} = \int d^4x du \tfrac{n(\Upsilon)}{(4\pi)^2} \sum_{k=2}^{\infty} \tfrac{1}{16}\tfrac{k-1}{k(k+1)}\tfrac{X^{k-2}}{(\mathcal{W}\bar{\mathcal{W}})^2}$$

$$\{-\bar{D}^{+\dot{\alpha}}D^{+\alpha}q_b^-\,\bar{D}^+_{\dot{\alpha}}D^-_\beta q^{+(b}\,\bar{D}^{-\dot{\beta}}D^{-\beta}q^{+a)}\,\bar{D}^-_{\dot{\beta}}D^+_\alpha q_a^-$$

$$+\tfrac{1}{2}\bar{D}^{+\dot{\alpha}}D^{+\alpha}q_b^-\,\bar{D}^{-\dot{\beta}}D^{-\beta}q^{+b}\,\bar{D}^-_{\dot{\beta}}D^-_\beta q^{+a}\,\bar{D}^+_{\dot{\alpha}}D^+_\alpha q_a^-$$

$$+\tfrac{1}{2}\bar{D}^{-\dot{\beta}}D^{+\alpha}q_b^-\,\bar{D}^{+\dot{\alpha}}D^{-\beta}q^{+b}\,\bar{D}^+_{\dot{\alpha}}D^-_\beta q^{+a}\,\bar{D}^-_{\dot{\beta}}D^+_\alpha q_a^-\}|_{\theta=0}\,.$$

The straightforward calculation of the components in this expression shows that among the many terms with four derivatives there is an interesting term of the special type. As the first term in expansion over variable $X_0 = \frac{r(\Upsilon)\bar{f}^i f_i}{\alpha^2\bar{\phi}\phi}$ we have

$$\Gamma^{(1)}_{\rm lead} = -\tfrac{1}{48\pi^2} n(\Upsilon)\left(\tfrac{r(\Upsilon)}{\alpha(H)}\right)^2 \int d^4x \tag{32}$$

$$\times \tfrac{1}{(\bar{\phi}\phi)^2} i\varepsilon^{\mu\nu\lambda\rho}(\partial_\mu\bar{f}^i\partial_\nu f_i\partial_\lambda\bar{f}^j\partial_\rho f_j - \partial_\mu\bar{f}^i\partial_\nu\bar{f}_i\partial_\lambda f^j\partial_\rho f_j)$$

The expression (32) has a form of the Chern-Simons-like action for the multicomponent complex scalar filed. The terms of such form in the effective action were discussed in Refs. [8], [9] in context of $\mathcal{N}=4,2$ SYM models and in Refs. [19] for $d=6, \mathcal{N}=(2,0)$ superconformal models respectively. Here the expression (32) is obtained as a result of straightforward calculation in the supersymmetric quantum field theory.

5 Hypermultiplet dependent contribution to the effective action beyond the on-shell condition

In the above consideration a crucial point was the condition that the hypermultiplet q^+ satisfies the one-shell conditions (9) and the constraint $q^+ = D^{++}q^-$. Here we relax the on-shell conditions and study some of possible subleading contributions with the minimal number of space-time derivatives in the component effective action.

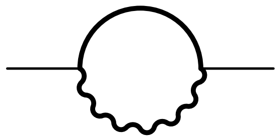

Figure 1: One-loop supergraph

We consider a supergraph given in Fig.1 with two external hypermultiplet legs and with all propagators depending on the background $\mathcal{N}=2$ vector multiplet. Here the wavy line stands for the $\mathcal{N}=2$ gauge superfield propagator and the solid external and internal lines stand for the background hypermultiplet superfields and quantum hypermultiplet propagator respectively. For simplicity we suppose that the background field is Abelian and omit all group factors. The corresponding contribution to effective action looks like

$$i\Gamma_2 = \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} du_1 du_2 \left(\frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4}{(u_1^+ u_2^+)^3} \frac{1}{\widehat{\Box}_1} \delta^{12}(1|2) \right) \times \tag{33}$$

$$\times \left(\frac{(\mathcal{D}_2^+)^4 (\mathcal{D}_1^+)^4}{\widehat{\Box}_2 \widehat{\Box}_1} \delta^{12}(2|1) (\mathcal{D}_1^{--})^2 \delta^{(-2,2)}(u_2, u_1) \right) \tilde{q}^+(z_1, u_1) q^+(z_2, u_2).$$

As usually, we extract the factor $(D^+)^4$ from the vector multiplet propagator for reconstructing the full $\mathcal{N}=2$ measure. Then we shrink a loop into a point by transferring the $\widehat{\Box}$ and $(\mathcal{D}^+)^4$ from first δ-function to another one and kill one integration. At this procedure the operator $\widehat{\Box}$ does not act on q^+ because we are interesting in the minimal number of space-time derivatives in the component form of the effective action. As a result, one obtains

$$i\Gamma_2 = \int \frac{d\zeta_1^{(-4)} du_1 du_2}{(u_1^+ u_2^+)^3} \frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4_2 (\mathcal{D}_1^+)^4}{\widehat{\Box}_2 \widehat{\Box}_1} \delta^{12}(z - z') \Big| \times$$

$$\times \left((\mathcal{D}_1^{--})^2 \delta^{(-2,2)}(u_2, u_1) \right) \tilde{q}^+(z_1, u_1) q^+(z_1, u_2) \ .$$

Further we use twice the relation (20) allowing us to express the $(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4$ as a polynomial in powers of $(u_1^+ u_2^+)$. Then after multiplying the $(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4 (\mathcal{D}_1^+)^4$ with the distribution $1/(u_1^+ u_2^+)^3$ we obtain a polynomial in $(u_1^+ u_2^+)$ containing the powers of this quantity from 5-th to 1-st. The first order is just a contribution of the type which we considered in the previous section, because one derivation $(D^{--})^2$ is used for transformation $(u_1^+ u_2^+)$ into $(u_1^+ u_2^-)|_{u_1=u_2} = 1$ in the coincident limit. Another D^{--} transforms q^+ into q^-. All that has been already done in Section 4.

Here we consider the new contribution to the effective action containing term $(u_1^+u_2^+)^2$ in the above polynomial:

$$\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{(u_1^+u_2^+)^3} = \tag{34}$$
$$...+(u_1^+u_2^+)^2(u_1^-u_2^+)(u_2^-u_1^+)(\mathcal{D}_1^+)^4\left(\tfrac{i}{2}\widehat{\Box}_1\Delta_2^{--}(u_2^+u_1^-)-\tfrac{i}{2}\Delta_1^{--}\widehat{\Box}_2(u_1^+u_2^-)\right)+...$$

The ellipsis means the terms with the powers of $(u_1^+u_2^+)$ other then 2. One can show that in the coincident limit they disappear. Now transferring $(D^{--})^2$ on $(u_1^+u_2^+)^2$ we obtain the expression:

$$i\Gamma_2 = i\int d\zeta^{(-4)}du(\mathcal{D}^+)^4\tfrac{1}{\widehat{\Box}^3}(\underbrace{\widehat{\Box}\Delta^{--}}_{\Gamma_2(1)}-\underbrace{\Delta^{--}\widehat{\Box}}_{\Gamma_2(2)})\delta^{12}(z-z')|_{z=z'}\tilde{q}^+(z,u)q^+(z,u)\,, \tag{35}$$

where Δ^{--} is defined in (21).

Let us consider each of the two underlined contributions separately. We use the representation

$$\tfrac{1}{\widehat{\Box}^2}\Delta^{--}\delta^{12}(z-z')| = \int ds\, se^{s\widehat{\Box}}\Delta^{--}\delta^{12}(z-z')|, \tag{36}$$

where $|$ means the coincident limit $z=z'$. Then we can apply a derivative expansion of the heat kernel. The goal is to collect the maximum possible number of factors of $\mathcal{D}^+,\mathcal{D}^-$ acting on $(\theta^+-\theta^{'+})^4(\theta^--\theta^{'-})^4$ and having the minimum order in s in the integral over s. Higher orders in s generate the higher spinor derivatives in the effective action. We take terms $\frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2+c.c.$ from Δ^{--} and expand the exponential so as to find $(\mathcal{D}^-)^4$. The Eq. (36) allows us to write the leading contribution to $\Gamma_2(1)$ as follows

$$\Gamma_2(1) = -\int d^{12}zdu\int_0^\infty ds\cdot s\int \tfrac{d^4p}{(2\pi)^4}e^{-sp^2}e^{s(\mathcal{W}\bar{\mathcal{W}}-\varepsilon)}\tfrac{s^2}{32}\bar{\mathcal{W}}(D^{+\alpha}\mathcal{W}D_\alpha^+\mathcal{W})\times$$
$$\times(D^-)^2(\bar{D}^-)^2\delta^8(\theta-\theta')|\tilde{q}^+q^+ + \text{ c.c.}$$

After trivial integration over p and s this contribution has the form

$$\Gamma_2(1) = \tfrac{i}{32\pi^2}\int d^{12}zdu\tfrac{D^+\mathcal{W}D^+\mathcal{W}}{\mathcal{W}\mathcal{W}^2}\tilde{q}^+(z,u)q^+(z,u)(\mathcal{D}^-)^4\delta^8(\theta-\theta')|+c.c. \tag{37}$$

Now we fulfil the same manipulations with the second underlined contribution $\Gamma_2(2)$ keeping the same order in s and $D^-,\bar{D}^-$ as in the expression (37). After that we see that the leading term of the form (37) is absent in $\Gamma_2(2)$. Then it is not difficult to show that the contribution (37) is rewritten as follows [we use $\int d^2\bar{\theta}^- = \bar{D}^{+2}$]

$$-\tfrac{i}{32\pi^2}\int d^4xd^4\theta^+d^2\theta^-du(\bar{D}^+)^2(D^+)^2\tfrac{\ln\mathcal{W}}{\mathcal{W}}\tilde{q}^+(z,u)q^+(z,u)(\mathcal{D}^-)^4\delta^8(\theta-\theta')|$$

The non-zero result arises when all D^+ - factors act only on the spinor delta-function. Thus, the contribution under consideration is written as an integral over the measure $d^4xdud^4\theta^+d^2\theta^-$ which looks like "3/4 - part" of the full $\mathcal{N}=2$ harmonic superspace measure $d^4xdud^4\theta^+d^4\theta^-$.

Therefore, the hypermultiplet dependent effective action contains the term

$$\Gamma_2 = -\tfrac{i}{32\pi^2}\int d^4xdud^4\theta^+d^2\theta^-\tfrac{1}{\mathcal{W}}\ln(\mathcal{W})\tilde{q}^+q^+|_{\bar{\theta}^-=0} \tag{38}$$
$$-\tfrac{i}{32\pi^2}\int d^4xdud^4\,\theta^+d^2\bar{\theta}^-\tfrac{1}{\bar{\mathcal{W}}}\ln(\bar{\mathcal{W}})\tilde{q}^+q^+|_{\theta^-=0}\,.$$

Presence of such a term in the effective action for $\mathcal{N}=2$ supersymmetric models in subleading order was proposed in [9]. Here we have shown how this term can be derived in the supersymmetric quantum field theory.

It is interesting and instructive to find a component form of such a non-standard superfield action (38). Here we consider only a purely bosonic sector of (38). After integration over anticommuting variables, which can be equivalently replaced by supercovariant derivatives evaluated at $\theta = 0$, we obtain a Chern-Simons-like contribution to the effective action containing three space-time derivatives

$$\Gamma_2 = -\tfrac{1}{2\pi^2} \int d^4x \tfrac{1}{\phi\bar{\phi}} \varepsilon^{mnab} \partial_m \bar{f}^i \partial_n f_i F_{ab} \ . \tag{39}$$

This expression is the simplest contribution to the hypermultiplet dependent effective action beyond the on-shell conditions (9) for the background hypermultiplet. Of course, there exist other, more complicated contributions including the hypermultiplet derivatives, they also can be calculated by the same method which led to (38). Here we only demonstrated a procedure which allows us to derive the contributions to the effective action in the form of integral over 3/4 - part of the full $\mathcal{N} = 2$ harmonic superspace.

6 Summary

We have studied the one-loop low-energy effective action in $\mathcal{N} = 2$ superconformal models. The models are formulated in harmonic superspace and their field content correspond to the finiteness condition (1). Effective action depends on the background Abelian $\mathcal{N} = 2$ vector multiplet superfield and background hypermultiplet superfields satisfying the special restrictions (7), (8) which define the vacuum structure of the models. The effective action is calculated on the base of the $\mathcal{N} = 2$ background field method for the background hypermultiplet on-shell (9) and beyond the on-shell conditions. For an on-shell hypermultiplet we found the universal expression for the effective active action. For hypermultiplet beyond on-shell, we calculated the special manifestly $\mathcal{N} = 2$ supersymmetric subleading contribution which is written as an integral over 3/4 of the full $\mathcal{N} = 2$ harmonic superspace. We believe that such contributions deserves a special study.

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$USp(4)$ harmonic superspace for $\mathcal{N}=4$ super Yang-Mills theory

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Abstract

We introduce the $\mathcal{N}=4$ harmonic superspace with harmonic variables on $USp(4)/(U(1)\times U(1))$ coset and apply it for solving the constraints in $\mathcal{N}=4$ SYM model and for constructing invariant actions in this superspace. In particular, we obtain the actions which are responsible for F^2 and F^4 terms in SYM model.

Foreword

It is high pleasure for me to consider myself as one of the disciples of professor I.L. Buchbinder, who is truly world-wide recognized theoretical physicist, devoted himself to the modern high energy theoretical physics. During last several decades he was particularly engaged in the study of supersymmetric field theories which are considered now as an important branch in the modern quantum field theory. I joined I.L. Buchbinder in 1998 being third year student at quantum field theory department in Tomsk state university. Since that time I have been involved in the studies of supersymmetric models with extended supersymmetry in harmonic superspace. The main idea of harmonic superspace approach is the use of such superspace formulation for $\mathcal{N}=2$ and $\mathcal{N}=3$ supersymmetric models, in which all supersymmetries are manifest and all properties of these models can be studied directly in harmonic superspace. However, the model of $\mathcal{N}=4$ supers Yang-Mills field, which, I think, is the most beautiful field theory due to its unique properties, does not possess manifestly $\mathcal{N}=4$ supersymmetric description in any harmonic superspace. In fact, the problem of unconstrained superfield formulation of this model in $\mathcal{N}=4$ superspace has been open for about thirty years, despite such an approach was always very desired. In our recent joint paper [13] we have got some new insight on this problem by introducing a new $\mathcal{N}=4$ harmonic superspace which appeared very useful for constructing some $\mathcal{N}=4$ supersymmetric actions directly in $\mathcal{N}=4$ superspace. The basic aspects of this new superspace are described in the present note, with some applications to the $\mathcal{N}=4$ supergauge theory. I hope that the development of these ideas will eventually lead to the long-wished-for unconstrained superfield description of this model and will open a new way for studying its quantum aspects directly in $\mathcal{N}=4$ superspace. I think it would be the most valuable present to I.L. Buchbinder, devoted to its sixtieth birthday, which I could make to express my deep respect to I.L. Buchbinder and its scientific achievements.

1 Introduction

The $\mathcal{N}=4$ super Yang-Mills (SYM) field theory, being the maximally extended rigid supersymmetric model, possesses many remarkable properties. The symmetry of this model is so large that the only freedom in the classical action is the choice of the gauge group, and the quantum dynamics is free of divergences. It worth pointing out that this theory has profound relations with superstring theory, particularly due to the AdS/CFT correspondence (see, e.g., [1]).

The problems of $\mathcal{N}=4$ SYM theory in the quantum domain are mainly related to the effective action and correlation functions of composite operators. The superfield approaches seem to be more efficient for these purposes, since they allow one to use the supersymmetries in explicit form. However, a description of $\mathcal{N}=4$ SYM theory in terms of unconstrained $\mathcal{N}=4$ superfields is still missing. For various applications, formulations in terms of $\mathcal{N}=1$ superfields (see, e.g., [2]), in terms of $\mathcal{N}=2$ superfields [3, 4], or in terms of $\mathcal{N}=3$ superfields [5] are used. All attempts to find an unconstrained $\mathcal{N}=4$ harmonic superfield formulation for the $\mathcal{N}=4$ SYM theory have been futile so far [6, 7, 8, 9, 10, 11], for a number of different types of harmonic variables originating from various cosets of the $SU(4)$ group. However, one may still hope that there exists some other harmonic superspace, not based on some $SU(4)$ coset, which is better suited for a superfield realization of $\mathcal{N}=4$ supergauge theory. In other words, we need new superfield representations of the known irreducible multiplets of the $\mathcal{N}=4$ superalgebra realized in an appropriate harmonic superspace.

In the present paper we introduce the $\mathcal{N}=4$ harmonic superspace with harmonic variables on $USp(4)/(U(1)\times U(1))$ coset. The main motivation for using this coset is the observation that both massless and massive BPS multiplets can be describes in such superspace. Indeed, the massive vector multiplet with $\mathcal{N}=4$ supersymmetry respects the $\mathcal{N}=4$ supersymmetry with central charge, which breaks the R-symmetry group of internal automorphisms of $\mathcal{N}=4$ superalgebra down to $USp(4)$. Therefore one can use only $USp(4)$ harmonics for such multiplets.

We show that the constraints of $\mathcal{N}=4$ SYM model are easily solved in the $USp(4)$ harmonic superspace resulting in six analytic superfields with different types of analyticity. All these superfields satisfy equations of motion with covariant harmonic derivatives and describe $\mathcal{N}=4$ supergauge multiplet on-shell. We apply these superfields for constructing various invariant actions in $USp(4)$ harmonic superspace which, in particular, describe F^2 and F^4 terms. We expect that it will help in the development of unconstraint $\mathcal{N}=4$ superfield approach to the $\mathcal{N}=4$ SYM model.

The paper is organized as follows. In Section 2 we introduce the $\mathcal{N}=4$ harmonic superspace with $USp(4)$ harmonic variables and review the basic constructions in it. In the next Section we present the solution of constraints in $\mathcal{N}=4$ SYM model in this superspace. The last section is devoted to the construction of various superfield actions in $USp(4)$ harmonic superspace. In Summary we discuss the results obtained and some ideas of their further application to $\mathcal{N}=4$ SYM theory.

2 $\mathcal{N}=4$ $USp(4)$ harmonic superspace

Consider standard $\mathcal{N}=4$ superspace with coordinates $Z^M=\{x^m,\theta_{i\alpha},\bar\theta^i_{\dot\alpha}\}$, where $m=0,1,2,3$ is the Lorentz group index, $\alpha,\dot\alpha=1,2$ denote the $SL(2,C)$ indices and $i,j,\ldots=1,2,3,4$ correspond to R-symmetry. There are supercovariant spinor derivatives in this superspace,

$$D^i_\alpha=\frac{\partial}{\partial\theta^\alpha_i}+i\bar\theta^{\dot\alpha i}\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m},\qquad \bar D_{\dot\alpha i}=-\frac{\partial}{\partial\bar\theta^{\dot\alpha i}}-i\theta^\alpha_i\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m},\tag{1}$$

which satisfy standard anticommutation relations,

$$\{D^i_\alpha,\bar D_{j\dot\alpha}\}=-2i\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m},\qquad \{D^i_\alpha,D^j_\beta\}=\{\bar D_{i\dot\alpha},\bar D_{j\dot\beta}\}=0.\tag{2}$$

As is well known, the $\mathcal{N}=4$ superalgebra without central charges respects the $U(4)$ R-symmetry group. Therefore one can apply the $\mathcal{N}=4$ harmonic superspace for $\mathcal{N}=4$ SYM model with harmonic variables in one of the coset of $SU(4)$ group. This issue was studied in details in many works, see, e.g., [6, 7, 8, 9, 10, 11]. However, these superspaces do not help to achieve the unconstraint superfield formulation for this model. Therefore in the present paper we introduce another $\mathcal{N}=4$ harmonic superspace, based on the $USp(4)/(U(1)\times U(1))$ coset, and develop the description of $\mathcal{N}=4$ SYM model in it. Such harmonic variables were introduced in [12] and recently applied in [13] for studying the $\mathcal{N}=4$ harmonic superparticle model. As was explained in [13], both massless and massive BPS multiplets of $\mathcal{N}=4$ supersymmetry can be described in such a harmonic superspace. Now we review the basic properties of harmonic variables on $USp(4)/U(1)\times U(1)$ coset.

The $USp(4)$ harmonic variables are 4×4 unitary matrices $u=(u^i{}_j)$ preserving constant anti-symmetric tensor Ω,

$$u\in USp(4) \quad\Rightarrow\quad u\,u^\dagger=1,\quad u\,\Omega\,u^{\mathrm T}=\Omega. \tag{3}$$

In our work we prefer the following form of Ω,

$$\Omega_{ij}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\qquad \Omega^{ij}=(\Omega_{ij})^{-1}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \tag{4}$$

although many other equivalent choices are possible. Note that it is not necessary to impose the constraint $\det u=1$ since it follows from (3).

Let us denote the elements of complex conjugate matrix as $u^*=(\bar u_i{}^j)$. Then the identities (3) can be written for the matrix elements as

$$u^i{}_j\bar u_k{}^j=\delta^i_k,\qquad u^i{}_j\Omega^{jk}u^l{}_k=\Omega^{il}. \tag{5}$$

As follows from (5),

$$\bar u_i{}^j=\Omega_{ik}u^k{}_l\Omega^{lj}, \tag{6}$$

the conjugated matrix in the $USp(4)$ group is not independent, but is expressed through the original one with the help of invariant tensor Ω. In other words, the fundamental and conjugated representations are equivalent, similarly as for the $SU(2)$ group. Hence, the invariant tensors Ω_{ij} and Ω^{ij} are used to lower and rise the $USp(4)$ indices, e.g.,

$$u^{ij}=u^i{}_k\Omega^{kj}=\Omega^{ik}\bar u_k{}^j,\qquad \bar u_{ij}=\Omega_{ik}u^k{}_j=\bar u_i{}^k\Omega_{kj}. \tag{7}$$

Here we assume $(\Omega_{ij})^*=-\Omega^{ij}$.

Now we introduce the $usp(4)$ algebra as a space spanned on the following differential operators

$$\begin{aligned}
&S_1=D^1_1-D^2_2, && S_2=D^3_3-D^4_4,\\
&D^{(++,0)}=D^1_2, && D^{(--,0)}=D^2_1,\\
&D^{(0,++)}=D^3_4, && D^{(0,--)}=D^4_3,\\
&D^{(+,+)}=D^1_4+D^3_2, && D^{(-,-)}=D^2_3+D^4_1,\\
&D^{(+,-)}=D^1_3-D^4_2, && D^{(-,+)}=D^2_4-D^3_1,
\end{aligned} \tag{8}$$

where

$$D^i_j=u^i{}_k\frac{\partial}{\partial u^j{}_k}. \tag{9}$$

It is easy to see that S_1, S_2 are Cartan generators in the $USp(4)$ group which measure the $U(1)$ charges of the other generators,

$$[S_1,D^{(s_1,s_2)}]=s_1D^{(s_1,s_2)},\qquad [S_2,D^{(s_1,s_2)}]=s_2D^{(s_1,s_2)}. \tag{10}$$

It is convenient to label the harmonic variables by their $U(1)$ charges as well,

$$u^1{}_i=u^{(+,0)}_i,\quad u^2{}_i=u^{(-,0)}_i,\quad u^3{}_i=u^{(0,+)}_i,\quad u^4{}_i=u^{(0,-)}_i. \tag{11}$$

The harmonic derivatives (8) can now be rewritten in the more useful form for practical calculations with harmonics (11),

$$\begin{aligned}
&S_1 = u_i^{(+,0)}\frac{\partial}{\partial u_i^{(+,0)}} - u_i^{(-,0)}\frac{\partial}{\partial u_i^{(-,0)}}, && S_2 = u_i^{(0,+)}\frac{\partial}{\partial u_i^{(0,+)}} - u_i^{(0,-)}\frac{\partial}{\partial u_i^{(0,-)}},\\
&D^{(\pm\pm,0)} = u_i^{(\pm,0)}\frac{\partial}{\partial u_i^{(\mp,0)}}, && D^{(0,\pm\pm)} = u_i^{(0,\pm)}\frac{\partial}{\partial u_i^{(0,\mp)}},\\
&D^{(\pm,\pm)} = u_i^{(\pm,0)}\frac{\partial}{\partial u_i^{(0,\mp)}} + u_i^{(0,\pm)}\frac{\partial}{\partial u_i^{(\mp,0)}}, && D^{(\pm,\mp)} = u_i^{(\pm,0)}\frac{\partial}{\partial u_i^{(0,\pm)}} - u_i^{(0,\mp)}\frac{\partial}{\partial u_i^{(\mp,0)}}. \qquad (12)
\end{aligned}$$

Using the notations (11), the basic relations for harmonics (5) can be written as orthogonality

$$u^{(+,0)i}u_i^{(-,0)} = u^{(0,+)i}u_i^{(0,-)} = 1, \qquad (13)$$

$$u_i^{(+,0)}u^{(0,+)i} = u_i^{(+,0)}u^{(0,-)i} = u_i^{(0,+)}u^{(-,0)i} = u_i^{(-,0)}u^{(0,-)i} = 0 \qquad (14)$$

and completeness conditions,

$$u^{(+,0)i}u_j^{(-,0)} - u_j^{(+,0)}u^{(-,0)i} + u^{(0,+)i}u_j^{(0,-)} - u_j^{(0,+)}u^{(0,-)i} = \delta_j^i. \qquad (15)$$

Apart from the usual complex conjugation there is the following conjugation for harmonics [12],

$$\widetilde{u_i^{(\pm,0)}} = u^{(0,\pm)i}, \quad \widetilde{u_i^{(0,\pm)}} = u^{(\pm,0)i}, \quad \widetilde{u^{(\pm,0)i}} = -u_i^{(0,\pm)}, \quad \widetilde{u^{(0,\pm)i}} = -u_i^{(\pm,0)}. \qquad (16)$$

It is the conjugation (16) which allows one to define real objects in harmonic superspace with $USp(4)$ harmonics.

Now we define the $\mathcal{N}=4$ $USp(4)$ harmonic superspace as a superspace with coordinates $Z_H = \{x^m, \theta^i_\alpha, \bar\theta_{i\dot\alpha}, u\}$, where the harmonic variables u are defined in (3). The main advantage of using the harmonic superspace Z_H is the possibility of passing to the harmonic projections for all objects with $USp(4)$ indices. For instance, for the Grassmann variables we have

$$\theta^I_\alpha = -u^{Ii}\theta_{i\alpha}, \qquad \bar\theta^I_{\dot\alpha} = u^I{}_i\bar\theta^i_{\dot\alpha}, \qquad (17)$$

where the index I takes the following values

$$I = \{(+,0),\ (-,0),\ (0,+),\ (0,-)\}. \qquad (18)$$

One can promote the conjugation (16) to such objects,

$$\widetilde{\theta_\alpha^{(\pm,0)}} = \bar\theta_{\dot\alpha}^{(0,\pm)}, \quad \widetilde{\theta_\alpha^{(0,\pm)}} = \bar\theta_{\dot\alpha}^{(\pm,0)}, \quad \widetilde{\bar\theta_{\dot\alpha}^{(0,\pm)}} = -\theta_\alpha^{(\pm,0)}, \quad \widetilde{\bar\theta_{\dot\alpha}^{(\pm,0)}} = -\theta_\alpha^{(0,\pm)}. \qquad (19)$$

Analogously, we project the Grassmann derivatives (1) with harmonics,

$$D^I_\alpha = u^I{}_iD^i_\alpha \qquad \bar D^I_{\dot\alpha} = -u^{Ii}\bar D_{i\dot\alpha}. \qquad (20)$$

They are also related by the conjugation as

$$\widetilde{D_\alpha^{(\pm,0)}} = -\bar D_{\dot\alpha}^{(0,\pm)}, \quad \widetilde{D_\alpha^{(0,\pm)}} = -\bar D_{\dot\alpha}^{(\pm,0)}, \quad \widetilde{\bar D_{\dot\alpha}^{(\pm,0)}} = D_\alpha^{(0,\pm)}, \quad \widetilde{\bar D_{\dot\alpha}^{(0,\pm)}} = D_\alpha^{(\pm,0)}. \qquad (21)$$

3 Solution of $\mathcal{N}=4$ SYM constraints in $USp(4)$ HSS

According to the generic procedure of superspace formulation of the extended supersymmetric models [14], one introduces the gauge connections for the covariant spinor derivatives (1),

$$D^i_\alpha \to \nabla^i_\alpha = D^i_\alpha + V^i_\alpha, \qquad \bar D_{i\dot\alpha} \to \bar\nabla_{i\dot\alpha} = \bar D_{i\dot\alpha} + \bar V_{i\dot\alpha} \qquad (22)$$

and defines the superfield strengths by the following anticommutators,

$$\{\nabla^i_\alpha, \nabla^j_\beta\} = 2\varepsilon_{\alpha\beta}W^{ij}, \qquad \{\bar\nabla_{i\dot\alpha}, \bar\nabla_{j\dot\beta}\} = 2\varepsilon_{\dot\alpha\dot\beta}\bar W_{ij}. \qquad (23)$$

It is well known that the following $\mathcal{N}=4$ SYM constraints put the superfield strengths on-shell [14],

$$\bar{D}_{i\dot{\alpha}}W^{jk} = \tfrac{1}{3}(\delta_i^j \bar{D}_{l\dot{\alpha}}W^{lk} - \delta_i^k \bar{D}_{l\dot{\alpha}}W^{lj}), \tag{24}$$

$$D^i_\alpha W^{jk} + D^j_\alpha W^{ik} = 0, \tag{25}$$

$$\overline{W^{ij}} = \bar{W}_{ij} = \tfrac{1}{2}\varepsilon_{ijkl}W^{kl}. \tag{26}$$

Let us project the strengths W^{ij} with harmonics,

$$W^{ij} \to W^{IJ} = u^I{}_i u^J{}_j W^{ij}, \tag{27}$$

where the indices I,J take the values (18). We denote these superfields also as

$$\begin{aligned} W_1 &= u_i^{(0,+)}u_j^{(0,-)}W^{ij}, \qquad W_2 = u_i^{(+,0)}u_j^{(-,0)}W^{ij},\\ W^{(+,+)} &= u_i^{(+,0)}u_j^{(0,+)}W^{ij}, \qquad W^{(-,-)} = u_i^{(-,0)}u_j^{(0,-)}W^{ij},\\ W^{(+,-)} &= u_i^{(+,0)}u_j^{(0,-)}W_{ij}, \qquad W^{(-,+)} = u_i^{(-,0)}u_j^{(0,+)}W_{ij}. \end{aligned} \tag{28}$$

Contracting equations (24,25) with harmonics we find a number of Grassmann analyticity constraints for these superfields,

$$\begin{aligned} &D^{(0,+)}_\alpha W_1 = D^{(0,-)}_\alpha W_1 = \bar{D}^{(+,0)}_{\dot{\alpha}} W_1 = \bar{D}^{(-,0)}_{\dot{\alpha}} W_1 = 0,\\ &D^{(+,0)}_\alpha W_2 = D^{(-,0)}_\alpha W_2 = \bar{D}^{(0,+)}_{\dot{\alpha}} W_2 = \bar{D}^{(0,-)}_{\dot{\alpha}} W_2 = 0,\\ &D^{(+,0)}_\alpha W^{(+,+)} = D^{(0,+)}_\alpha W^{(+,+)} = \bar{D}^{(+,0)}_{\dot{\alpha}} W^{(+,+)} = \bar{D}^{(0,+)}_{\dot{\alpha}} W^{(+,+)} = 0,\\ &D^{(-,0)}_\alpha W^{(-,-)} = D^{(0,-)}_\alpha W^{(-,-)} = \bar{D}^{(-,0)}_{\dot{\alpha}} W^{(-,-)} = \bar{D}^{(0,-)}_{\dot{\alpha}} W^{(-,-)} = 0,\\ &D^{(+,0)}_\alpha W^{(+,-)} = D^{(0,-)}_\alpha W^{(+,-)} = \bar{D}^{(+,0)}_{\dot{\alpha}} W^{(+,-)} = \bar{D}^{(0,-)}_{\dot{\alpha}} W^{(+,-)} = 0,\\ &D^{(-,0)}_\alpha W^{(-,+)} = D^{(0,+)}_\alpha W^{(-,+)} = \bar{D}^{(-,0)}_{\dot{\alpha}} W^{(-,+)} = \bar{D}^{(0,+)}_{\dot{\alpha}} W^{(-,+)} = 0. \end{aligned} \tag{29}$$

Moreover, by construction, the superfields (28) are annihilated by the following harmonic derivatives,

$$\begin{aligned} &D^{(++,0)}W_1 = D^{(--,0)}W_1 = D^{(0,++)}W_1 = D^{(0,--)}W_1 = (D^{(+,+)})^2 W_1 = 0,\\ &D^{(++,0)}W_2 = D^{(--,0)}W_2 = D^{(0,++)}W_2 = D^{(0,--)}W_2 = (D^{(-,-)})^2 W_2 = 0,\\ &D^{(++,0)}W^{(+,+)} = D^{(0,++)}W^{(+,+)} = D^{(+,+)}W^{(+,+)} = D^{(+,-)}W^{(+,+)} = D^{(-,+)}W^{(+,+)} = 0,\\ &D^{(--,0)}W^{(-,-)} = D^{(0,--)}W^{(-,-)} = D^{(-,-)}W^{(-,-)} = D^{(+,-)}W^{(-,-)} = D^{(-,+)}W^{(-,-)} = 0,\\ &D^{(++,0)}W^{(+,-)} = D^{(0,--)}W^{(+,-)} = D^{(+,-)}W^{(+,-)} = D^{(+,+)}W^{(+,-)} = D^{(-,-)}W^{(+,-)} = 0,\\ &D^{(--,0)}W^{(-,+)} = D^{(0,++)}W^{(-,+)} = D^{(-,+)}W^{(-,+)} = D^{(+,+)}W^{(-,+)} = D^{(-,-)}W^{(-,+)} = 0. \end{aligned} \tag{30}$$

Let us now consider the reality constraint (26). Applying the following identities with harmonics

$$\begin{aligned} u^{(+,0)i}u^{(-,0)j}\varepsilon_{ijkl} &= 2u^{(0,+)}_{[k}u^{(0,-)}_{l]},\\ u^{(0,+)i}u^{(0,-)j}\varepsilon_{ijkl} &= 2u^{(+,0)}_{[k}u^{(-,0)}_{l]},\\ u^{(+,0)i}u^{(0,-)j}\varepsilon_{ijkl} &= -2u^{(+,0)}_{[k}u^{(0,-)}_{l]},\\ u^{(0,+)i}u^{(-,0)j}\varepsilon_{ijkl} &= -2u^{(0,+)}_{[k}u^{(-,0)}_{l]},\\ u^{(+,0)i}u^{(0,+)j}\varepsilon_{ijkl} &= -2u^{(+,0)}_{[k}u^{(0,+)}_{l]},\\ u^{(-,0)i}u^{(0,-)j}\varepsilon_{ijkl} &= -2u^{(-,0)}_{[k}u^{(0,-)}_{l]}, \end{aligned} \tag{31}$$

we find that (26) leads to the reality properties of superfield strengths,

$$\begin{gathered} \widetilde{W_{1,2}} = W_{1,2}, \quad \widetilde{W^{(+,+)}} = W^{(+,+)}, \quad \widetilde{W^{(-,-)}} = W^{(-,-)},\\ \widetilde{W^{(+,-)}} = W^{(-,+)}, \quad \widetilde{W^{(-,+)}} = W^{(+,-)}. \end{gathered} \tag{32}$$

As a result, all the constraints (24)–(26) are solved by the superfields (28) satisfying (29,30,32).

4 Invariant actions in $USp(4)/(U(1)\times U(1))$ harmonic superspace

Each of the superfields (28) depends effectively on eight Grassmann variables rather than six, as is seen in each line in (29). In other words, there are six different analytic subspaces with eight Grassmann variables in full $\mathcal{N}=4$ harmonic superspace. Here we restrict ourself to one such subspace corresponding to the superfield W_1, the others can be studied in a similar way.

The superfield W_1 lies in the analytic superspace with the coordinates $Z_A=\{x^m_A,\theta^{(\pm,0)}_\alpha,\bar\theta^{(0,\pm)}_{\dot\alpha},u\}$, where

$$x^m_A=x^m-i\theta^{(0,-)}\sigma^m\bar\theta^{(0,+)}+i\theta^{(0,+)}\sigma^m\bar\theta^{(0,-)}-i\theta^{(+,0)}\sigma^m\bar\theta^{(-,0)}+i\theta^{(-,0)}\sigma^m\bar\theta^{(+,0)}. \tag{33}$$

In the coordinates (33), the covariant spinor derivatives (20) and harmonic derivatives (12) are given by

$$D^{(0,\pm)}_\alpha=\pm\frac{\partial}{\partial\theta^{(0,\mp)\alpha}},\qquad \bar D^{(\pm,0)}_{\dot\alpha}=\pm\frac{\partial}{\partial\bar\theta^{(\mp,0)\dot\alpha}}, \tag{34}$$

$$D^{(\pm\pm,0)}_A=D^{(\pm\pm,0)}+\theta^{(\pm,0)}\frac{\partial}{\partial\theta^{(\mp,0)}},\qquad D^{(0,\pm\pm)}_A=D^{(0,\pm\pm)}+\theta^{(0,\pm)}\frac{\partial}{\partial\theta^{(0,\mp)}}, \tag{35}$$

$$\begin{aligned}D^{(+,+)}_A&=D^{(+,+)}-2i(\theta^{(+,0)}\sigma^m\bar\theta^{(0,+)}-\theta^{(0,+)}\sigma^m\bar\theta^{(+,0)})\partial_m\\&\quad+\theta^{(0,+)}\frac{\partial}{\partial\theta^{(-,0)}}+\bar\theta^{(+,0)}\frac{\partial}{\partial\bar\theta^{(0,-)}}.\end{aligned} \tag{36}$$

Using the expressions (34)–(36), one can easily find the on-shell component structure of the superfield W_1 by solving the equations (29,30,32),

$$\begin{aligned}W_1&=\phi+if^{ij}(u^{(+,0)}_{[i}u^{(-,0)}_{j]}-u^{(0,+)}_{[i}u^{(0,-)}_{j]})\\&\quad+i\theta^{(+,0)\alpha}\psi^i_\alpha u^{(-,0)}_i-i\theta^{(-,0)\alpha}\psi^i_\alpha u^{(+,0)}_i+i\bar\theta^{(0,+)}_{\dot\alpha}\bar\psi^{i\dot\alpha}u^{(0,-)}_i-i\bar\theta^{(0,-)}_{\dot\alpha}\bar\psi^{i\dot\alpha}u^{(0,+)}_i\\&\quad+\theta^{(+,0)}_\alpha\theta^{(-,0)}_\beta F^{(\alpha\beta)}+\bar\theta^{(0,+)}_{\dot\alpha}\bar\theta^{(0,-)}_{\dot\beta}\bar F^{(\dot\alpha\dot\beta)}+\ldots,\end{aligned} \tag{37}$$

where dots stand for the terms with derivatives of fields. Here ϕ, $f^{[ij]}$ are six real scalar fields, ψ^i_α are four Weyl spinors and $F_{\alpha\beta}$, $\bar F_{\dot\alpha\dot\beta}$ correspond to the spinor components of the Maxwell field strength. All component fields here depend only on x^m_A and satisfy the corresponding free equations of motion. As a result, the $\mathcal{N}=4$ SYM multiplet is embedded into the on-shell superfield W_1.

Since the superfield W_1 is analytic, an action with this superfield should be given by the integral over the analytic subspace Z_A with the analytic measure (the details of integration over the harmonic variables are given in [13])

$$d\zeta=\tfrac{1}{2^8}d^4x_A du(D^{(+,0)})^2(D^{(-,0)})^2(\bar D^{(0,+)})^2(\bar D^{0,-})^2. \tag{38}$$

In general, one can consider arbitrary superpotential $\mathcal{F}(W_1)$ which gives us the manifestly supersymmetric and gauge invariant action,

$$S=\int d\zeta\,\mathcal{F}(W_1). \tag{39}$$

The following particular choices of the superpotential may by interesting for the physical applications,

$$S_4=g\int d\zeta\,(W_1)^4\sim g\int d^4x\,F^4+\ldots, \tag{40}$$

$$S_{conf}=\int d\zeta\,\ln(W_1/\Lambda)\sim\int d^4x\frac{F^4}{\phi^4}+\ldots. \tag{41}$$

Here g and Λ are constants with mass dimension -4 and $+1$, respectively. The leading terms in the component structure of these actions, given in rhs of (40,41), are obtained by substituting (37) into these actions. Here we see that (40) corresponds to the $\mathcal{N}=4$ supersymmetric generalization of quartic term in the Born-Infeld action (see, e.g., [16] for a review on this subject) while the action (41) is nothing but a scale-invariant generalization of this term. From the field theory point of view,

(41) can be interpreted as the leading term in the low-energy effective action in $\mathcal{N}=4$ SYM model, see [15].

Let us address the issue of $\mathcal{N}=4$ superfield description of the classical $\mathcal{N}=4$ SYM action in harmonic superspace. One can check that the action (39) does not contain the F^2 term in its component structure for any choice of the superpotential $\mathcal{F}$. As a way out, we construct the action reproducing the F^2 term in another analytic subspace with coordinates $Z'_A=\{x'^m_A,\theta^{(+,0)}_\alpha,\theta^{(0,+)}_\alpha,\bar\theta^{(+,0)}_{\dot\alpha},\bar\theta^{(0,+)}_{\dot\alpha},u\}$, where

$$x'^m_A=x^m-i\theta^{(-,0)}\sigma^m\bar\theta^{(+,0)}-i\theta^{(+,0)}\sigma^m\bar\theta^{(-,0)}-i\theta^{(0,-)}\sigma^m\bar\theta^{(0,+)}-i\theta^{(0,+)}\sigma^m\bar\theta^{(0,-)}. \quad (42)$$

The corresponding analytic measure is given by

$$d\zeta^{(-4,-4)}=\tfrac{1}{2^8}d^4x_A du(D^{(-,0)})^2(D^{(0,-)})^2(\bar D^{(-,0)})^2(\bar D^{(0,-)})^2. \quad (43)$$

Now we propose the following action, which depends manifestly not only on the superfield W_1 but also on the Grassmann variables,

$$S_2=\int d\zeta^{(-4,-4)}(D^{(+,+)}W_1)^2[(\theta^{(+,0)})^2-(\bar\theta^{(+,0)})^2][(\theta^{(0,+)})^2-(\bar\theta^{(0,+)})^2]\sim\int d^4x\,F^2+\ldots. \quad (44)$$

Here dots mean all component terms in classical action of $\mathcal{N}=4$ SYM model, which complement Maxwell F^2 term up to the full $\mathcal{N}=4$ supersymmetry.

Of course, the manifest presence of Grassmann variables in (44) means that this action is supersymmetric only if the superfield W_1 satisfy some constraints. Indeed, one can check that the action S_2 is invariant under supersymmetry, $\delta_\epsilon S_2=0$, if

$$D^{(++,0)}W_1=D^{(0,++)}W_1=0. \quad (45)$$

It is very important to realize that the equations (45) do not put the superfield W_1 on-shell but just reduce the component structure in it. It follows from the fact that the operators (35) do not have spatial derivatives in the corresponding analytic coordinates. The equations (45) mean that the superfield W_1 depends effectively on $USp(4)/(SU(2)\times SU(2))$ harmonic variables rather than on $USp(4)/(U(1)\times U(1))$ ones. Hence, the action (44) may serve as the off-shell $\mathcal{N}=4$ SYM classical action in the Abelian case, written in terms of superfield strength.

The action (44) requires further studies, to be useful for applications for the $\mathcal{N}=4$ SYM model. In particular, one has to introduce the prepotentials for the superfield strength W_1 and to express (44) in terms of these prepotentials, resulting in the unconstrained superfield action for $\mathcal{N}=4$ SYM model. We hope that it may open a way for the unconstrained $\mathcal{N}=4$ superfield quantization of this model.

5 Summary

In this note we develop the $\mathcal{N}=4$ harmonic superspace approach for the $\mathcal{N}=4$ SYM model, which is based on the use of $USp(4)/(U(1)\times U(1))$ harmonic variables. We have shown that the $\mathcal{N}=4$ SYM constraints are easily solved in this superspace resulting in six analytic superfield strengths. Each of this superfields lies in its own analytic subspace with eight Grassmann variables and satisfies definite equations of motion involving covariant harmonic derivatives. Using one of these superfields, W_1, we constructed a number of invariant actions in such harmonic superspace, which contain F^2 or F^4 terms in their component structure. The action reproducing F^2 term in components is interpreted as off-shell classical action for $\mathcal{N}=4$ SYM model. The action with F^4 term in components corresponds to superfield generalization of the quartic term in the Born-Infeld action. We have found also a scale-invariant generalization of the quartic action which gives the leading terms in low-energy effective action in $\mathcal{N}=4$ SYM model.

As a result, we show that the $USp(4)$ harmonic superspace, introduced in [13], appears very useful for the superfield formulation of the $\mathcal{N}=4$ SYM model. We expect that the further development of this formulation will lead to the unconstraint $\mathcal{N}=4$ superfield approach for this model, which was missing during last thirty years. Such unconstraint superfield formulation of $\mathcal{N}=4$ SYM model is very desirable for studying quantum aspects of this model, directly in $\mathcal{N}=4$ superspace.

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Geometry and dynamics of higher–spin triplets

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Abstract

We consider the frame–like formulation of so–called bosonic and fermionic higher–spin triplets in flat and AdS backgrounds of an arbitrary dimension. The relation of the fields of higher–spin triplets to the higher–spin vielbeins and connections is found. The gauge invariant actions are constructed including, in particular, the reducible higher–spin fermion (*i.e.* triplet) case in AdS space.

1 Introduction

It is a pleasure to write this contribution on the occasion of the 60th birthday anniversary of Joseph Buchbinder, our colleague and friend, who, among other important subjects in his fruitful scientific carrier, made an extensive contribution to the theory of higher–spin fields.

The construction of gauge invariant actions for single (irreducible) massless higher–spin fields usually requires that these fields are constrained to be double–traceless or traceless in the case of integer spins and triple–gamma–traceless or gamma–traceless in the case of half integer fields, depending on whether one uses the metric–like or frame–like formalism (see [1]–[4] for recent reviews and references). The trace constraints on the dynamical fields can be removed by introducing a number of auxiliary fields or allowing non–locality in the theory. Several versions of the unconstrained Lagrangian formulation of higher–spin fields have been proposed (see *e.g.* [5, 6, 7, 8, 9, 10, 11, 12, 13]).

A nice unconstrained two–derivative formulation was proposed by J. Buchbinder, A. Galajinsky and V. Krykhtin [11]. Their construction is closely related to (and is actually based on) the so called triplets of higher–spin fields, on which we focus in this contribution.

The bosonic triplet system describes the field of an integer spin s together with its descendants of spins $s-2, s-4, \ldots, 1$ or 0. The fermionic triplet system consists of the field of a half–integer spin s and its descendants of spins $s-1, s-2, \ldots, 1/2$.

The massless higher–spin triplets show up when one picks up certain states of the open string spectrum while taking its tensionless limit [14, 15, 17, 16, 9, 18] (see also [19] for further developments). So one can regard the triplets as fields which manifest their origin from massive higher–spin fields of the tension–full string. The study of these sets of fields may shed some light on a mechanism of higher–spin symmetry breaking resulting in the generation of mass in Higher Spin Theory, which is the necessary step in establishing a relationship of the Higher Spin Gauge Theory with String Theory.

However, the geometrical nature of triplet fields, *i.e.* their relation to higher–spin counterparts of metric (or vielbein) and connection, has not been clarified yet. Moreover, neither equations of motion nor the action for fermionic triplets in AdS space have been constructed. This hinders the study of the relation of the fermionic triplets to string states in AdS backgrounds and corresponding applications.

In this contribution we provide the answers to these problems using the frame–like formulation of higher–spin fields [20, 21, 22, 1, 4]. Upon establishing the geometrical meaning of the triplet fields and finding their proper gauge transformations, in particular in AdS, we obtain a relatively simple Lagrangian description of the bosonic and fermionic triplets in flat and AdS backgrounds, which should be useful for their applications, *e.g.* for studying interactions of triplets. In Conclusion we also mention how one can get a frame–like version of the unconstrained formulation of single higher–spin fields proposed in [11].

2 Frame–like action for bosonic higher–spin fields in flat space–time

In the frame–like formulation (see [20, 21, 1, 4] for details) a massless symmetric field of an integer spin s in flat space–time of dimension D is described by the higher–spin vielbein one–form[1]

$$e^{n_1\cdots n_{s-1}} = dx^m\, e_{m;}{}^{n_1\cdots n_{s-1}}, \tag{1}$$

by the one–form connection

$$\omega^{n_1\cdots n_{s-1},p} = dx^m\, \omega_{m;}{}^{n_1\cdots n_{s-1},p}, \tag{2}$$

and by extra fields which do not play a role in the free (linearized) field theory. In (1) and (2) the indices $n_1 \ldots n_{s-1}$ are symmetric, and $\omega^{n_1\cdots n_{s-1},p}$ has the property of the Young tableau $Y(s-1,1)$ [2], *i.e.* its part which is totally symmetric in all of the tangent space indices vanishes

$$\omega^{(n_1\cdots n_{s-1},p)} := \frac{1}{s}\,(\omega^{n_1\cdots n_{s-1},p} + \omega^{p\cdots n_{s-1},n_1} + s-2 \text{ terms}) = 0\,. \tag{3}$$

The brackets () and [] will define, respectively, the symmetrization and anti–symmetrization of indices with the unit weight.

The connection is the auxiliary field if, like in the case of the Einstein gravity, we impose the zero torsion condition

$$T^{n_1\cdots n_{s-1}} \equiv d\,e^{n_1\cdots n_{s-1}} - (s-1)\,dx^q\,\omega^{n_1\cdots n_{s-1},p}\,\eta_{pq} = 0\,. \tag{4}$$

If eq. (4) holds, the dynamical degrees of freedom of the massless higher–spin field are contained in the higher–spin vielbein (1) which also has pure gauge degrees of freedom because of the presence in the theory of the higher–spin gauge symmetry. In particular, the torsion (4) is invariant under the following gauge transformations of the vielbein and the connection

$$\delta\, e^{n_1\cdots n_{s-1}} = d\xi^{n_1\cdots n_{s-1}} - (s-1)\,dx^q\,\xi^{n_1\cdots n_{s-1},p}\,\eta_{pq}\,, \tag{5}$$

$$\delta\,\omega^{n_1\cdots n_{s-1},p} = d\xi^{n_1\cdots n_{s-1},p} - (s-2)\,dx^q\,\xi^{n_1\cdots n_{s-1},pr}\,\eta_{rq}\,. \tag{6}$$

The gauge parameters $\xi^{n_1\cdots n_{s-1}}$, $\xi^{n_1\cdots n_{s-1},p}$ and $\xi^{n_1\cdots n_{s-1},p_1p_2}$ are symmetric in each group of indices n and p. In addition, $\xi^{n_1\cdots n_{s-1},p}$ and $\xi^{n_1\cdots n_{s-1},p_1p_2}$ have the symmetry properties of the Young

[1] In flat space–time we shall not distinguish between the world and tangent space indices. Both kinds of indices will be denoted by lower case Latin letters. World indices will be separated from the tangent–space ones by ';'.

[2] To separate the sets of symmetric tangent–space indices we use coma. $Y(s-1,1)$ means that the Young tableau has $s-1$ cells in the first row and 1 cell in the second raw.

tableaux $Y(s-1,1)$ and $Y(s-1,2)$, respectively, which means like in (3) that the symmetrization of any s indices gives zero.

Note that so far we have not imposed traceless conditions on the higher–spin vielbein, connection and gauge parameters.

We would like to derive the zero torsion condition (4) from an action together with dynamical field equations on the physical components of $e^{n_1\cdots n_{s-1}}$. We construct such an action by analogy with the frame formulation of the action for (linearized) gravity.

The free higher–spin action has the following simple form

$$S=\int_{M^D} dx^{a_1}\dots dx^{a_{D-3}}\,\varepsilon_{a_1\dots a_{D-3}pqr}\,(d\,e^{n_1\dots n_{s-2}p}-\frac{s-1}{2}\,dx_m\,\omega^{n_1\dots n_{s-2}p,\,m})\,\omega_{n_1\dots n_{s-2}}{}^{q,\,r}\,. \quad (7)$$

The action is invariant under the transformations (5 – 6) provided the gauge parameters $\xi^{n_1\cdots n_{s-1},\,m}$ and $\xi^{n_1\cdots n_{s-1},\,ml}$ satisfy the constraints

$$\eta_{n_1 m}\,\xi^{n_1\dots n_{s-1},\,m}=0\,,\quad \eta_{n_1 m}\,\xi^{n_1\dots n_{s-1},\,ml}=0\,, \quad (8)$$

while the parameter $\xi^{n_1\cdots n_{s-1}}$ remains arbitrary.

To be consistent with (8) the connection ω is subject to the analogous trace constraint

$$\eta_{n_1 m}\,\omega^{n_1\dots n_{s-1},\,m}=0\,. \quad (9)$$

The consequence of (9) (and also of (8) for the gauge parameters) is that the trace of ω in a pair of the symmetric indices n has again definite $Y(s-3,1)$ Young–symmetry properties

$$\eta_{n_1 n_2}\,\omega^{n_1 n_2 (n_3\dots n_{s-1},\,m)}=0\,. \quad (10)$$

We should stress that the trace $\eta_{n_1 n_2}\,\omega^{n_1 n_2 n_3\cdots n_{s-1},\,m}$ can be *non–zero.* Therefore, the condition (9) is weaker than the conventional trace constraint on the quantities of the frame–like formulation of a single higher–spin field which corresponds to Fronsdal theory [24]. We shall call eq. (9) the relaxed traceless condition.

Note that the vielbein $e^{n_1\cdots n_{s-1}}$ remains traceful. Here we should point out, however, that in the case of the odd integer spins $s=2k+1$ the fully trace part of $e^{n_1\cdots n_{s-1}}$, *i.e.* $e^{n_1\cdots n_{2k}}\eta_{n_1 n_2}\cdots\eta_{n_{2k-1}n_{2k}}$ does not appear in the action (7) because of its differential form structure. Technically, the reason for this is that the one–form associated with the spin–1 field does not have external (tangent–space) indices required for the construction of the action as an integral of a differential form. Thus, the action (7) does not describe fields of spin one.

In the case of the even integer spins $s=2k$ the total trace component of the higher–spin vielbein $e_{n;}^{n_1\dots n_{2k+1}}\delta^n_{n_1}\eta_{n_2 n_3}\cdots\eta_{n_{2k}n_{2k+1}}$ is pure gauge, in view of the gauge transformations (5). Thus, the action (7) does not describe the kinematics of the scalar fields either.

To include the spin 0 and spin 1 fields into the system one should add to the action (7) the corresponding Klein–Gordon and Maxwell terms. This can be achieved by adding to the action the spin–one and spin–zero kinetic terms formulated in terms of the so-called Weyl zero-forms as discussed in [23] for the scalar case and in [26] for the spin–one case.

By virtue of (9), the general local variation of the action (7) can be presented in two forms, which are equivalent up to total derivatives,

$$\begin{aligned}\delta S\;=&\int_{M^D} dx^{a_1}\dots dx^{a_{D-3}}\,\varepsilon_{a_1\cdots a_{D-3}}{}^{pqr}\,\delta T_{n_1\cdots n_{s-2}p}\,\omega^{n_1\cdots n_{s-2}}{}_{q,\,r}\\ =&\int_{M^D} dx^{a_1}\dots dx^{a_{D-3}}\,\varepsilon_{a_1\cdots a_{D-3}}{}^{pqr}\,(T_{n_1\cdots n_{s-2}p}\,\delta\omega^{n_1\cdots n_{s-2}}{}_{q,\,r}-\delta e_{n_1\cdots n_{s-2}p}\,d\omega^{n_1\cdots n_{s-2}}{}_{q,\,r})\,,\end{aligned} \quad (11)$$

where the torsion $T^{n_1\cdots n_{s-1}}$ is defined in the left hand side of (4).

The first form of the variation is convenient for the chek of the gauge invariance of the action under (5), (6) and (8). The second line of (11) yields the field equations for ω and e, which follow from

$$\delta^m_{[b}\delta^n_c\delta^r_{d]}\,T_{mn;n_1\cdots n_{s-2}}{}^{b}\;\delta\,\omega_{r;}{}^{n_1\cdots n_{s-2}c,\,d}=0\,, \quad (12)$$

and

$$\delta^m_{[b}\delta^n_c\delta^r_{d]}\delta e_{m;n_1\cdots n_{s-2}}{}^b\,\partial_n\omega_{r;}{}^{n_1\cdots n_{s-2}c,\,d}=0\,. \tag{13}$$

As one can show, the equation (12) is equivalent to the zero–torsion condition (4), modulo its full trace in the tangent space indices in the case of spin $s=2k+1$. There is no condition on the full trace of the torsion, since, as we have explained above the corresponding trace of the vielbein does not appear in the action.

The zero torsion condition

$$(s-1)\omega_{[n;}{}^{n_1\cdots n_{s-1},b}\eta_{m]b}=\partial_{[m}\,e_{n];}{}^{n_1\cdots n_{s-1}} \tag{14}$$

expresses the higher–spin connection in terms of the first derivatives of the higher–spin vielbein up to the Stueckelberg gauge transformations (6). In the case of odd $s=2k+1$, $e_m{}^{n_1\cdots n_{s-1}}$ in (14) stands for the part of the vielbein whose total trace is zero $e_m{}^{n_1\cdots n_{2k}}\,\eta_{a_1a_2}\cdots\eta_{a_{2k-1}a_{2k}}=0$. In view of the relation (14) the equations which follow from eq. (13), namely

$$\delta^m_{(b}\partial^c\omega^{d;}{}_{n_1\cdots n_{s-2})[c,d]}+\partial_d\,\omega_{(b;n_1\cdots n_{s-2})}{}^{[m,d]}+\partial_{(b}\,\omega^{d;}{}_{n_1\cdots n_{s-2})[d,}{}^{m]}=0 \tag{15}$$

are the dynamical (second–order) equations of motion of the higher–spin vielbein field.

Let us analyze the field content of this model.

2.1 Fronsdal case

We first consider the case of an irreducible massless field. Following [20] we impose on the higher–spin vielbein and connection *maximal* trace constraints

$$\eta_{n_1n_2}\,\tilde e^{n_1\ldots n_{s-1}}=0\qquad \eta_{n_1n_2}\,\tilde\omega^{n_1\ldots n_{s-1},\,m}=0\,, \tag{16}$$

where we use $\tilde e$ and $\tilde\omega$ for the traceless objects to distinguish them from the original (relaxed) e and ω. Note that the condition (9) follows from (3) and (16), but not vice versa.

The parameters of the gauge transformations (5) and (6) acting on the traceless $\tilde e$ and $\tilde\omega$ are also traceless

$$\eta_{n_1n_2}\,\tilde\xi^{n_1\ldots n_{s-1}}=0,\qquad \eta_{n_1n_2}\,\tilde\xi^{n_1\ldots n_{s-1},\,m}=0\,,\qquad \eta_{n_1n_2}\,\tilde\xi^{n_1\ldots n_{s-1},\,mp}=0. \tag{17}$$

Using the gauge transformations (5) and (6) with the parameters $\tilde\xi$ one can gauge fix to zero the "antisymmetric" parts of the components of the vielbein $\tilde e$ and of the connection $\tilde\omega$. Then, taking into account the algebraic expression (14) we see that all the physical degrees of freedom are contained in the symmetric part of the vielbein

$$s\,\tilde e_{(n_s;\,n_1\cdots n_{s-1})}:=\tilde\phi_{n_1\cdots n_s}\,, \tag{18}$$

which is double traceless because the vielbein $\tilde e_{n_s;}{}^{n_1\cdots n_{s-1}}$ is traceless in the indices $n_1\cdots n_{s-1}$.

The remaining local symmetry is then just that of the Fronsdal metric–like formulation of the dynamics of a single symmetric bosonic higher–spin field in flat space–time [24] with the completely symmetric traceless parameter $\tilde\xi_{n_1\cdots n_{s-1}}$.

If we now substitute into the action (7) the connection $\tilde\omega$ with its expression (14) in terms of the symmetric and *double traceless* field (18), the resulting action will be quadratic in the derivatives of $\tilde\phi_{n_1\cdots n_s}$ and will be invariant under the local transformations

$$\delta\,\tilde\phi_{n_1\ldots n_s}=s\,\partial_{(n_1}\tilde\xi_{n_2\ldots n_{s-1})} \tag{19}$$

with traceless gauge parameters $\tilde\xi_{n_1\cdots n_{s-1}}$. As such, one can directly verify that this action is equivalent (up to a total derivative) to the Fronsdal action for a massless gauge field of spin s

$$S=\int d^Dx\left(\frac{1}{2}\tilde\phi^{m_1\cdots m_s}\,\mathcal F_{m_1\cdots m_s}-\frac{1}{8}\,s(s-1)\,\tilde\phi_n{}^{nm_3\cdots m_s}\,\mathcal F^p{}_{pm_3\cdots m_s}\right) \tag{20}$$

where

$$\mathcal F_{m_1\cdots m_s}(x)\equiv\partial^2\,\tilde\phi_{m_1\cdots m_s}-s\,\partial_{(m_1}\partial^n\,\tilde\phi_{m_2\cdots m_s)n}+\frac{s(s-1)}{2}\,\partial_{(m_1}\partial_{m_2}\,\tilde\phi^n{}_{m_3\cdots m_s)n} \tag{21}$$

is the so–called Fronsdal operator.

2.2 Triplet case

Let us now analyze the field content of the model described by the action (7) with the traceful higher–spin vielbein and with the higher–spin connection subject to the relaxed traceless condition (9). By representing the vielbein as a sum of traceless (lower rank) symmetric tensors, one can see that the action (7) is actually the sum of the actions for the *traceless* vielbeins $\tilde{e}^{a_1\cdots a_{t-1}}$ and connections $\tilde{\omega}^{a_1\cdots a_{t-1},b}$ with t taking the even or odd values ($t = 2, 4, \cdots, s$ or $t = 3, 5, \cdots, s$), depending on whether s is even or odd,

$$S = \sum_{k=1}^{[\frac{s}{2}]} \alpha(t,D) \int_{M^D} dx^{a_1} \ldots dx^{a_{D-3}} \, \varepsilon_{a_1\ldots a_{D-3}pqr} \, (d\, \tilde{e}^{n_1\ldots n_{t-2}p} - \tfrac{t-1}{2}\, dx_m \, \tilde{\omega}^{n_1\ldots n_{t-2}p,\, m}) \, \tilde{\omega}_{n_1\ldots n_{t-2}}{}^{q,\, r} \,, \tag{22}$$

In eq. (22) $t = 2k$ or $t = 2k+1$, $[\frac{s}{2}]$ denotes the integral part of $\frac{s}{2}$ when s is odd, $\alpha(t,D)$ are constants which depend on space–time dimension D and the rank t (spin) of the tensor fields. Thus, the sum in (7) is taken over even $t = 2, 4, \cdots, s-2, s$ or odd $t = 3, 5, \cdots, s-2, s$ depending whether s is even or odd. Each of the terms of the sum (22) with a given t is gauge invariant under the transformations eqs. (5), (6) with the traceless parameters (17).

As we have explained in Subsection 2.1, for a given t each term of (22) describes a free massless field of spin t. Thus the action (22), and hence (7), describes the family of massless fields of even integer spins $t = 2, 4, \cdots, s-2, s$ and of odd integer spins $t = 3, 5, \cdots, s-2, s$.

These field contents are similar to the field contents of the higher–spin triplets [15, 16, 9, 18] (except for the presence in the latter of the fields of the lowest spins 0 and 1). We shall now demonstrate that there is indeed the relation between the triplet fields and the components of the higher–spin vielbein e and connection ω, thus clarifying the geometrical meaning of the former.

Recall that the higher–spin triplet is described by the following three symmetric tracefull tensor fields of rank s, $s-1$ and $s-2$

$$\Phi_{n_1\cdots n_s}\,, \qquad C_{n_1\cdots n_{s-1}}\,, \qquad D_{n_1\cdots n_{s-2}}\,.$$

On the mass shell these fields satisfy the following equations

$$C_{n_1\cdots n_{s-1}} = \partial_m \Phi^m{}_{n_1\cdots n_{s-1}} - (s-1)\, \partial_{(n_{s-1}} D_{n_1\cdots n_{s-2})}\,, \tag{23}$$

$$\Box\, \Phi_{n_1\cdots n_s} = s\, \partial_{(n_s} C_{n_1\cdots n_{s-1})}\,, \qquad \Box := \partial_m \partial^m, \tag{24}$$

$$\Box\, D_{n_1\cdots n_{s-2}} = \partial_m\, C^m{}_{n_1\cdots n_{s-2}}\,. \tag{25}$$

Eqs. (23)–(25) are invariant under the gauge transformations

$$\delta\Phi_{n_1\cdots n_s} = s\, \partial_{(n_s} \xi_{n_1\cdots n_{s-1})} \tag{26}$$

$$\delta C_{n_1\cdots n_{s-1}} = \Box\, \xi_{n_1\cdots n_{s-1}} \tag{27}$$

$$\delta D_{n_1\cdots n_{s-2}} = \partial_m\, \xi^m{}_{n_1\cdots n_{s-2}} \tag{28}$$

where the unconstrained parameter $\xi_{n_1\cdots n_{s-1}}$ is completely symmetric.

Let us now compare the gauge transformations (26)–(28) with the gauge transformations (5) and (6) of the higher–spin vielbein and connection. This comparison suggests that the fields Φ and D of the triplet are just the symmetric components of the higher–spin vielbein[3]

$$\Phi_{n_1\cdots n_s} = s\, e_{(n_s;\, n_1\cdots n_{s-1})} \qquad D_{n_1\cdots n_{s-2}} = e_{p;\, n_1\cdots n_{s-2} n_{s-1}}\, \eta^{n_{s-1}p}\,. \tag{29}$$

It remains only to identify the field C. To this end let us have a look at the zero torsion condition (14). In (14) we first symmetrize the index n with $n_1, \ldots, n_{s-1}$ and then take the trace of n with m. In view of eqs. (3) and (9) we thus get

$$(s-1)\, \omega_{m;n_1\cdots n_{s-1},}{}^m = \partial_m \Phi^m{}_{n_1\cdots n_{s-1}} - (s-1)\, \partial_{(n_{s-1}} D_{n_1\cdots n_{s-2})} - \partial^m e_{m;n_1\cdots n_{s-1}}\,, \tag{30}$$

[3]Note that using the transformation (5) with the parameter $\xi^{n_1\cdots n_{s-1},p}$ one can gauge away from the vielbein $e_p{}^{n_1\cdots n_{s-1}}$ its part which corresponds to the hook Young tableau of $\xi^{n_1\cdots n_{s-1},p}$, *i.e.* the part which satisfies the 'antisymmetry' condition $e_{(p;\, n_1\cdots n_{s-1})} = 0$ and is subject to the relaxed trace constraint similar to (8).

where Φ and D are defined in (29). Comparing (30) with (23) we see that the triplet field C is actually composed of the trace of the higher–spin connection and the divergence of the higher–spin vielbein

$$C_{n_1\cdots n_{s-1}} = (s-1)\,\omega_{m;n_1\cdots n_{s-1},}{}^{m} + \partial^m e_{m;n_1\cdots n_{s-1}}\,. \tag{31}$$

We have thus identified the fields of the higher–spin triplet as components of the higher–spin vielbein and connection of the frame–like formulation with the relaxed trace constraints.

The comment on the lowest spin fields (*i.e.* the scalar and the vector) is now in order. As we have already mentioned, these fields are not contained in the frame–like action (7). In the case of the even integer spin $s=2k$ the complete trace component of the vielbein, which could be the scalar field, is a pure gauge. In the case of the odd integer spin $s=2k+1$, as we have explained earlier, the spin 1 part of the vielbein does not enter the action (7).

As a result, the zero torsion condition (14) and its consequence (30), which defines the field C (31), are applicable only to the components of the vielbein whose complete trace in the indices $n_1,\ldots,n_{s-1}$ is zero (*i.e.* do not contain the spin 1 field).

As we have already mentioned, to include the scalar and the vector field into the above scheme one should add to the action (7) corresponding kinetic terms, and a systematic way to do this is to use the zero-forms in the so-called twisted adjoint representation of the higher–spin algebra. (see e.g. [23, 26]).

To conclude this section we note that the zero torsion condition (14) and the dynamical field equations (15) indeed imply the equations of motion (23)–(25) of the triplet higher–spin fields defined by eqs. (29)–(31). Thus, we have shown that, up to a subtlety regarding the spin–0 and spin–1 field, the higher–spin system described by the frame–like action (7) for the unconstrained vielbein and the connection subject to the relaxed trace constraint (9) is equivalent to the higher–spin triplet. The triplet fields Φ, C and D have been thus endowed with a geometrical meaning to be certain components of the higher–spin vielbein and connection. By singling out these components in the action (7) and partially solving the zero–torsion condition (14) one should be able to reduce action (7) to the triplet actions of [18].

We shall now extend the results of this section to the description of bosonic higher–spin triplets in the AdS background.

3 Frame–like action for bosonic higher–spin fields in AdS

The AdS space is described by the vielbein $e^a = dx^m\, e^a_m$ and the connection $\omega^{ab} = dx^m\,\omega^{ab}_m$ which satisfy the following torsion and constant curvature conditions

$$T^a := de^a + \omega^a{}_b\, e^b := \nabla\, e^a = 0\,, \tag{32}$$

$$R^{ab}(\omega) := d\omega^{ab} + \omega^a{}_c\,\omega^{cb} = -\Lambda\, e^a e^b \qquad \text{or} \qquad [\nabla,\nabla] = R\,, \tag{33}$$

where $\nabla = d+\omega$ is the $O(1,D-1)$ covariant differential and Λ is the negative (cosmological) constant determining the AdS curvature. The indices from the beginning of the Latin alphabet now denote the tangent space indices transformed under the local $O(1,D-1)$ Lorentz transformations. The indices $m,n,\ldots$ from the middle of the alphabet denote curved world indices.

The frame–like action for a system of higher–spin fields which generalizes to AdS the flat space action (7) has the following form

$$\begin{aligned} S = \int_{AdS} e^{a_1}\ldots e^{a_{D-3}}\,\varepsilon_{a_1\ldots a_{D-3}cdf}\, \Big[\big(\nabla\, e^{b_1\ldots b_{s-2}c} - \tfrac{s-1}{2}\, e_k\,\omega^{b_1\ldots b_{s-2}c,\,k}\big)\,\omega_{b_1\ldots b_{s-2}}{}^{d,\,f} \\ +\Lambda\,\tfrac{s(D+s-4)}{2(s-1)(D-2)}\, e^{cb_1\cdots b_{s-2}}\, e^d{}_{b_1\cdots b_{s-2}}\, e^f - \Lambda\,\tfrac{(s-2)(s-3)}{2(D-2)(s-1)}\, e^{cb_1\cdots b_{s-4}j}{}_j\, e^d{}_{b_1\cdots b_{s-4}i}{}^i\, e^f\Big]\,. \end{aligned} \tag{34}$$

We observe that, apart from covariantization, the action (34) differs from the flat space action (7) by the last two mass–like terms proportional to the AdS space scalar curvature. Note that the last term in (7) contains the trace of the higher–spin vielbein. As is well known, in AdS space such terms are required to keep the number of the physical states of the higher–spin field equal to that of the massless field. The coefficients in front of these terms are fixed by the requirement of the invariance of this action under gauge transformations of the higher–spin vielbein and connection whose form we shall discuss in the next two subsections.

Here we only note that, as in the flat case, the higher–spin vielbein is unconstrained, while the variation of the action (34) with respect to the higher–spin connection produces the zero torsion condition

$$T^{a_1\cdots a_{s-1}}=0 \qquad \Leftrightarrow \qquad (s-1)\omega_{[n;}{}^{a_1\cdots a_{s-1},b}\,e_{m]b}=\nabla_{[m}\,e_{n];}{}^{a_1\cdots a_{s-1}}\,, \tag{35}$$

provided that the higher–spin connection obeys the relaxed traceless condition

$$\eta_{a_1 b}\,\omega^{a_1a_2\cdots a_{s-1},b}=0\,. \tag{36}$$

The dynamical field equation of the higher–spin vielbein in AdS gets modified by the contribution of the terms proportional to the AdS scalar curvature Λ and acquires the form

$$\Big(\nabla_n\omega_{r;(a_1\cdots a_{s-2}}{}^{c,\,d}-\Lambda\,\frac{s(D+s-4)}{(s-1)(D-2)}\,e_n^d\,e_{r;(a_1\cdots a_{s-2}}{}^{c} \tag{37}$$
$$+\Lambda\,\tfrac{(s-2)(s-3)}{2(D-2)(s-1)}\,e_n^d\,e_{r;(a_3\cdots a_{s-2}}{}^{c}\,\eta_{a_1a_2}\Big)\;e^m_{[b)}e^n_c\,e^r_{d]}=0\,,$$

where we first antisymmetrized the indices b,c,d and then symmetrized the index b with $a_1,\cdots,a_{s-2}$.

3.1 Fronsdal case

The frame-like action for irreducible massless fields in AdS_D was originally proposed in [28]. In fact, the action proposed in this reference was the first action [4] for symmetric massless fields in $D>4$. The action constructed in [28] is manifestly gauge invariant due to the use of higher connections called extra fields, which however do not contribute to the free field equations. A version of this approach which is manifestly $o(2,D-1)$ (rather than $o(1,D-1)$ invariant) was later proposed in [29]. We shall demonstrate how the $o(2,D-1)$–covariant approach works in the case of the higher–spin triplets in the forthcoming paper [30]. Here we restrict our consideration to the $o(1,D-1)$–invariant formulation.

If we impose on the higher–spin vielbein and connection an additional (conventional) traceless condition (as above we distinguish between the traceless and traceful quantities by putting tildes on the former)

$$\eta_{a_1a_2}\,\tilde e^{a_1a_2\cdots a_{s-1}}=0,\quad \eta_{a_1a_2}\,\tilde\omega^{a_1a_2\cdots a_{s-1},b}=0, \tag{38}$$

the action (34) reduces to

$$S=\int_{AdS}e^{a_1}\ldots e^{a_{D-3}}\,\varepsilon_{a_1\ldots a_{D-3}cdf}\,\Big[(\nabla\,\tilde e^{b_1\ldots b_{s-2}c}-\tfrac{s-1}{2}\,e_k\,\tilde\omega^{b_1\ldots b_{s-2}c,\,k})\,\tilde\omega_{b_1\ldots b_{s-2}}{}^{d,\,f} \tag{39}$$
$$+\Lambda\,\tfrac{s(D+s-4)}{2(s-1)(D-2)}\,\tilde e^{cb_1\cdots b_{s-2}}\,\tilde e^{d}{}_{b_1\cdots b_{s-2}}\,e^f\Big]\,.$$

It is invariant under the following gauge transformations of the higher–spin vielbein and connection

$$\delta\,\tilde e^{a_1\ldots a_{s-1}}=\nabla\tilde\xi^{a_1\ldots a_{s-1}}-(s-1)\,e^c\,\tilde\xi^{a_1\ldots a_{s-1},b}\,\eta_{bc}\,, \tag{40}$$

$$\delta\,\tilde\omega^{a_1\ldots a_{s-1},b}=\nabla\,\xi^{a_1\ldots a_{s-1},b}-(s-2)\,e^c\,\tilde\xi^{a_1\ldots a_{s-1},bd}\,\eta_{cd}-\Lambda\,(e^b\,\tilde\xi^{a_1\ldots a_{s-1}}-e^{(a_1}\,\tilde\xi^{a_2\ldots a_{s-1})b})\,, \tag{41}$$

[4] The metric-like formulation of Fronsdal was originally proposed in [24, 27] for the case of $D=4$. It turns out that the coefficients in front of different terms of the action are independent of D in Minkowski space but those of mass-like terms are D-dependent in AdS_D.

where the parameters $\tilde\xi^{a_1\dots a_{s-1}}$ and $\tilde\xi^{a_1\dots a_{s-1},b}$ are traceless and the parameter $\tilde\xi^{a_1\dots a_{s-1},bd}$ satisfies the following trace conditions

$$\eta_{a_1a_2}\,\tilde\xi^{a_1a_2\cdots a_{s-1},b_1b_2} = \tfrac{2\Lambda}{(s-1)(s-2)}\,\tilde\xi^{a_3\cdots a_{s-1}b_1b_2}\,,$$
$$\eta_{a_1b_1}\,\tilde\xi^{a_1a_2\cdots a_{s-1},b_1b_2} = -\tfrac{\Lambda}{s-1}\,\tilde\xi^{a_2\cdots a_{s-1}b_2}\,. \tag{42}$$

In the flat limit $\Lambda\to 0$, eqs. (40)–(42) reduce to the corresponding gauge transformations discussed in Subsection 2.1.

Action (39) describes in AdS space the dynamics of a single massless field of spin s. Upon solving $\tilde\omega^{a_1\cdots a_{s-1},b}$ in terms of $\tilde e^{a_1\cdots a_{s-1}}$ and partially fixing local higher–spin symmetry (40), (41) one can reduce (39) to the Fronsdal AdS action [25] for the double traceless field $\phi^{a_1\cdots a_s} := e^{m(a_1}\,\tilde e_{m;}{}^{a_2\cdots a_s)}$.

3.2 Triplet case

If the full traceless condition is not imposed, the higher–spin connection only satisfies the relaxed trace condition (36) and the higher–spin vielbein is unconstrained. Then the action (34) describes in AdS space a system of free massless fields of descending spins $s-2, s-4,\dots,3$ or 2 depending on whether s is odd or even. The analysis and the proof is the same as in the flat case (see Subsection 2.2). The only difference is that the gauge transformations of the higher–spin vielbein and connection which leave the action (34) invariant and which have the same form as eqs. (40) and (41) contain the unconstrained parameter $\xi^{a_1\dots a_{s-1}}$, while $\xi^{a_1\dots a_{s-1},b}$ satisfies the relaxed traceless condition

$$\xi^{a_1\dots a_{s-1},b}\,\eta_{a_1b} = 0\,, \tag{43}$$

while the parameter $\xi^{a_1\dots a_{s-1},bd}$ is subject to the following relaxed constraints

$$(s-1)\,\eta_{bc}\,\xi^{a_1\dots a_{s-2}b,cd} = \Lambda\,(\,\eta^{d(a_1}\,\xi^{a_2\dots a_{s-2})b}{}_b - \xi^{a_1\dots a_{s-2}d}) \quad\Rightarrow \tag{44}$$
$$\eta_{bc}\,\xi^{a_1\dots a_{s-1},bc} = \Lambda\,(\xi^{a_1\dots a_{s-1}} - \eta^{(a_1a_2}\,\xi^{a_3\dots a_{s-1})b}{}_b) \quad\Rightarrow \tag{45}$$
$$\eta_{bc}\,\xi^{bc(a_1\dots a_{s-3},a_{s-2})d} = \tfrac{2\Lambda}{(s-1)(s-2)}\,(\xi^{a_1\dots a_{s-2}d} - \eta^{d(a_1}\,\xi^{a_2\dots a_{s-2})b}{}_b) \tag{46}$$

instead of being traceless as in the flat space case (see eq. (8)) or 'partially' traceless in the Fronsdal AdS case (see eq. (42)).

Let us now identify the AdS higher–spin triplet in terms of components of the higher–spin vielbein and connection. In the AdS space the bosonic higher–spin triplet is defined (in our notation and convention) by the following equations [9, 18]

$$C_{n_1\cdots n_{s-1}} = \nabla_m\Phi^m{}_{n_1\cdots n_{s-1}} - (s-1)\,\nabla_{(n_{s-1}}\,D_{n_1\cdots n_{s-2})}\,, \tag{47}$$

$$\Box\,\Phi_{n_1\cdots n_s} = s\,\nabla_{(n_s}\,C_{n_1\cdots n_{s-1})} + \Lambda\,\big[(s-(s-2)(D+s-3))\,\Phi_{n_1\cdots n_s} + 2s(s-1)\,g_{(n_1n_2}(\Phi_{n_3\cdots n_s)ml}\,g^{ml} - 4D_{n_3\cdots n_s)})\big]\,, \tag{48}$$

$$\Box\,D_{n_1\cdots n_{s-2}} = \nabla_m\,C^m{}_{n_1\cdots n_{s-2}} - \Lambda\,\big[(s(D+s-2)+6)\,D_{n_1\cdots n_{s-2}} -4\,\Phi_{n_1\cdots n_{s-2}ml}\,g^{ml} - (s-2)(s-3)\,g_{(n_1n_2}\,D_{n_3\cdots n_{s-2})ml}\,g^{ml}\big]\,. \tag{49}$$

where $\Box := \nabla_m\,\nabla^m$ and $g_{mn} = e^a_m\,e^b_n\,\eta_{ab}$ is an AdS metric.

The equations (23)–(25) are invariant under the following gauge transformations

$$\delta\Phi_{n_1\cdots n_s} = s\,\nabla_{(n_s}\,\xi_{n_1\cdots n_{s-1})} \tag{50}$$

$$\delta D_{n_1\cdots n_{s-2}} = \nabla_m\,\xi^m{}_{n_1\cdots n_{s-2}}\,, \tag{51}$$

$$\delta C_{n_1\cdots n_{s-1}} = \Box\,\xi_{n_1\cdots n_{s-1}} - \Lambda(D+s-3)\,(s-1)\,\xi_{n_1\cdots n_{s-1}} +(s-1)(s-2)\Lambda\,g_{(n_1n_2}\,\xi_{n_3\cdots n_{s-1})lm}\,g^{lm}\,. \tag{52}$$

where the parameter $\xi_{n_1\cdots n_{s-1}}$ is completely symmetric and *traceful*.

As in the flat case, we can identify the fields $\Phi_{n_1\cdots n_s}$ and $D_{n_1\cdots n_s}$ with the completely symmetric part and a trace part of the higher–spin vielbein $e_{m;}^{a_1\cdots a_{s-1}}$, respectively,

$$\Phi_{n_1\cdots n_s} = s\, e_{(n_s;\, n_1\cdots n_{s-1})} \qquad D_{n_1\cdots n_{s-2}} = e_{p;\, n_1\cdots n_{s-2} n_{s-1}}\, g^{pn_{s-1}}\,, \tag{53}$$

where the tangent space indices of the higher–spin vielbein have been converted into the 'curved' world indices with the use of the AdS vielbein e^a_m. The field $C_{n_1\cdots n_{s-1}}$ is then identified by analyzing the zero torsion condition (35) and has the form similar to (31), namely

$$C_{n_1\cdots n_{s-1}} = (s-1)\,\omega_{m;n_1\cdots n_{s-1},}{}^{m} + \nabla^m\, e_{m;n_1\cdots n_{s-1}}\,. \tag{54}$$

As in the flat space case, one can show that the gauge transformations (50)–(52) and the equations of motion (47)–(49) of the triplet fields follow, respectively, from the transformations (40)–(41) and the equations of motion (35)–(37). For instance, the last two terms in the variation of the field C (52) come from the terms of the gauge variation of the higher–spin connection which are proportional to Λ (see eqs. (41) and (45)).

By singling out the fields (53) and (54) in the action (34) and partially solving the zero–torsion condition (35) one should be able to reduce the action (34) to the AdS triplet actions of [18] for $s \geq 2$. As has been already explained in the case of flat space–time, the scalar and the vector field are not part of the triplet spectrum in our formulation, but they can be included into the model by adding corresponding terms to the AdS action (34).

4 Frame–like action for fermionic higher–spin fields in flat space–time

The frame-like formulation of irreducible higher–spin fermions was originally developed in $D = 4$ Minkowski space in [20, 31] and then for AdS_D with any D in [22]).

It looks simpler than that of bosons because the field equations are of the first order and hence the free action does not contain auxiliary fields.

The flat space higher–spin field strength (torsion) is of the same form as in the bosonic case [22]

$$T_{a_1\ldots a_{s-\frac{3}{2}}} = d\psi_{a_1\ldots a_{s-\frac{3}{2}}} - e^b\psi_{a_1\ldots a_{s-\frac{3}{2}},b} \tag{55}$$

where $\psi^\alpha_{a_1\ldots a_{\sigma-1}} = dx^n\psi^\alpha_{n\,;a_1\ldots a_{\sigma-1}}$ and $\psi_{a_1\ldots a_{s-\frac{3}{2}},b}$ are one-form fermionic vielbein and connection (with respect to the index n) and rank $s-\frac{3}{2}$ and $s-\frac{1}{2}$ tensor-spinors, respectively, and α being a (usually implicit) index associated with a spinor representation of $Spin(1, D-1)$[5]). The field strength (55) is manifestly invariant under the gauge transformations

$$\delta\psi_{a_1\ldots a_{s-\frac{3}{2}}} = d\xi_{a_1\ldots a_{s-\frac{3}{2}}} - e^b\xi_{a_1\ldots a_{s-\frac{3}{2}},b}\,. \tag{56}$$

$$\delta\psi_{a_1\ldots a_{s-\frac{3}{2}},b} = d\xi_{a_1\ldots a_{s-\frac{3}{2}},b} - e^c\xi_{a_1\ldots a_{s-\frac{3}{2}},bc}\,.$$

As in the bosonic case the symmetry properties of the fermionic higher–spin fields and of the gauge parameters are governed by the Young tableaux. The tensor–spinor connection $\psi_{a_1\ldots a_{s-\frac{3}{2}},b}$ is the so–called extra field. It will not participate in the description of the free higher–spin fermionic system.

[5]In a generic D–dimensional space–time the spinors are of the Dirac type. In even dimensions one can restrict spinors to be Weyl and in certain dimensions, *e.g.* $D = 3, 4, 6, 10$ and 11, one can consider Majorana or symplectic Majorana tensor–spinors. In the case of the Dirac and Weyl spinors the actions which we consider below implicitly contain the hermitian conjugate part, which we shall skip for brevity.

Let us consider the following first order action for the fermionic higher–spin field

$$S=\int_{M^D} e^{a_1}\ldots e^{a_{D-3}}\,\varepsilon_{a_1\ldots a_{D-3}pqr}(\bar\psi_{d_1\ldots d_{s-\frac32}}\gamma^{pqr}d\psi^{d_1\ldots d_{s-\frac32}}-6(s-\tfrac32)\,\bar\psi_{d_1\ldots d_{s-\frac52}}{}^{p}\gamma^{q}\,d\,\psi^{d_1\ldots d_{s-\frac52}\,r}) \tag{57}$$

where $\gamma^{p_1\cdots p_k}\equiv\gamma^{[p_1}\cdots\gamma^{p_k]}$.

The coefficients in the action (57) are fixed by requiring its invariance under the gauge transformations (56). As in the bosonic case, the gauge invariance takes place provided that gauge parameters satisfy some constraints. In the fermionic case they have the following form

$$\gamma_b\,\xi^{a_1\ldots a_{s-\frac32},b}=0\,,\qquad \xi^{a_1\ldots a_{s-\frac52}b,}{}_b=0\,. \tag{58}$$

Note that eqs. (58) are weaker than the gamma-trace condition, and that the parameter $\xi^{a_1\ldots a_{s-\frac32}}$ remains unconstrained.

The variation of the action (57) with respect to ψ produces the following equations of motion

$$\begin{aligned}\tfrac{2}{s-\frac32}\,\gamma^{mqr}\partial_r\,\psi_{q;a_1\cdots a_{s-\frac32}} &= \gamma^q\,\partial^r\,\psi_{q;r(a_2\cdots a_{s-\frac32}}\,\delta^m_{a_1)}-\gamma^m\,\partial^r\,\psi_{(a_1;a_2\cdots a_{s-\frac32})r}\\ &+\gamma^r\,\partial_r\,\psi_{(a_1;a_2\cdots a_{s-\frac32})}{}^m-\gamma^r\,\partial_r\,\psi^{p;}{}_{p(a_2\cdots a_{s-\frac32}}\,\delta^m_{a_1)}\\ &+\gamma^m\,\partial_{(a_1}\,\psi^{q;}{}_{a_2\cdots a_{s-\frac32})q}-\gamma^q\,\partial_{(a_1}\,\psi^{m;}{}_{a_2\cdots a_{s-\frac32})q}\,.\end{aligned} \tag{59}$$

4.1 Fang–Fronsdal case

Let us now consider the case in which the fermionic higher–spin vielbein, the connections and the gauge parameters are (in addition) required to be gamma–transversal (or gamma–traceless) and hence traceless *in all* tangent space indices

$$\gamma^c\tilde\psi_{a_1\ldots a_{s-\frac32},b_1\ldots b_t c}=0\,,\qquad \gamma^c\tilde\psi_{a_1\ldots a_{s-\frac52}c,b_1\ldots b_t}=0, \tag{60}$$

$$\gamma^c\tilde\xi_{a_1\ldots a_{s-\frac32},b_1\ldots b_t c}=0\,,\qquad \gamma^c\tilde\xi_{a_1\ldots a_{s-\frac52}c,b_1\ldots b_t}=0. \tag{61}$$

As we shall now demonstrate, in this case the action (57) is the frame–like counterpart of the Fang–Fronsdal action [27] for a single fermionic field of half–integer spin s in flat space.

The gamma–tansversal higher–spin vielbein $\tilde\psi^{\alpha}_{m;\,n_1\cdots n_{s-\frac32}}$ contains the irreducible Lorentz tensors[6] described by the following gamma-transversal Young tableaux

$$\square\otimes\boxed{}\;{}_{s-\frac32}=\boxed{}\;{}_{s-\frac12}\oplus\boxed{}\;{}_{s-\frac32}\oplus\boxed{}\;{}_{s-\frac52}\;\oplus\;\begin{array}{l}\boxed{}\;{}_{s-\frac32}\\ \boxed{}\,{}_1\end{array}. \tag{62}$$

The first tableau of length s on the right hand side of (62) describes the totally symmetric and gamma–transversal part $\tilde\psi_{(m;\,n_1\cdots n_{s-\frac32})}$, the second and third tableaux of the length $s-\frac32$ and $s-\frac52$, respectively, corresponds to the contractions $\gamma^m\tilde\psi_{m;\,n_1\cdots n_{s-\frac32}}$ and $\eta^{mk}\tilde\psi_{m;\,n_1\cdots n_{s-\frac52}k}$, respectively. The hook tableau corresponds to the irreducible (gamma–transversal) part of $\tilde\psi$ that satisfies $\tilde\psi_{(m;\,n_1\cdots n_{s-\frac32})}=0$.

In virtue of the gauge transformations of the higher–spin vielbein (with gamma–traceless parameters)

$$\delta\,\tilde\psi^{n_1\cdots n_{s-\frac32}}=d\tilde\xi^{n_1\cdots n_{\sigma-1}}-e^m\,\tilde\xi^{n_1\cdots n_{s-\frac32},b}\,\eta_{mb}\,, \tag{63}$$

[6] As in the bosonic case, in flat space–time we do not distinguish between symmetric tangent space indices $a,b,\ldots$ and world indices $m,n,\ldots$. The latter are separated from the former by ';'.

the hook part of the higher–spin vielbein field can be gauge fixed to zero by the appropriate choice of the parameter $\tilde{\xi}^{a_1\cdots a_{s-\frac{3}{2}},b}$. As a result, the remaining part of the vielbein amounts to the combination of three totally symmetric gamma-transversal tensor–spinors of rank $s-\frac{1}{2}$, $s-\frac{3}{2}$ and $s-\frac{5}{2}$ which are equivalent to the Fang–Fronsdal symmetric tensor–spinor field $\Psi^{\alpha}_{n_1\cdots n_{s-\frac{1}{2}}}$ that satisfies the triple gamma–traceless condition

$$\gamma^l\,\gamma^m\,\gamma^p\,\Psi_{lmpn_1\cdots n_{s-\frac{5}{2}}} = \eta^{lm}\gamma^p\Psi_{lmpn_1\cdots n_{s-\frac{5}{2}}} = 0\,. \tag{64}$$

The remaining local symmetry is the gauge invariance of the Fang–Fronsdal metric–like formulation in flat space–time with the completely symmetric gamma–traceless tensor–spinor parameter $\tilde{\xi}_{m_1\cdots m_{s-\frac{3}{2}}}$

$$\gamma^n\,\tilde{\xi}_{m_1\cdots m_{s-\frac{5}{2}}n} = 0\,. \tag{65}$$

Thus, the action (57) with the fields restricted by the conditions (60) is equivalent to the Fang-Fronsdal action as was shown long time ago in [20, 31, 22].

4.2 Triplet case

Let us now consider the case in which the fermionic higher–spin vielbein $\psi^{a_1\ldots a_{s-\frac{3}{2}}}$ and the gauge parameter $\xi^{a_1\ldots a_{s-\frac{3}{2}}}$ are unconstrained while the parameter $\xi^{a_1\ldots a_{s-\frac{3}{2}},c}$ of the gauge transformation (56) (for $t=0$) is constrained by the relaxed conditions

$$\gamma_b\,\xi^{a_1\ldots a_{s-\frac{3}{2}},b} = 0\,, \qquad \xi^{a_1\ldots a_{s-\frac{5}{2}}b,}{}_b = 0 \quad \Rightarrow \quad [\gamma^c\,,\gamma^d]\xi_{a_1\ldots a_{\sigma-2}c\,,d} = 0\,. \tag{66}$$

In order to figure out what is the field spectrum of the model in this case we observe that, the one-form fermionic field $\psi_{a_1\ldots a_{s-\frac{3}{2}}} = dx^m\,\psi_{m;a_1\ldots a_{s-\frac{3}{2}}}$ is composed of the tensors characterized by the following *unrestricted* (*i.e.* gamma–traceful) Young tableaux

$$\square \otimes \boxed{}_{\,s-\frac{3}{2}} = \boxed{}_{\,s-\frac{1}{2}} \oplus \begin{array}{l}\boxed{}_{\,s-\frac{3}{2}}\\ \square_{\,1}\end{array}\,. \tag{67}$$

On the other hand, the parameter $\xi_{\alpha_1\ldots a_{\sigma-1}\,,b}$ of the Stueckelberg gauge symmetry that satisfies (66) has the following components

$$\begin{array}{l}\boxed{}_{\,s-\frac{3}{2}}\\ \square_{\,1}\end{array} \Big/ \left(\boxed{}_{\,s-\frac{3}{2}} \oplus \boxed{}_{\,s-\frac{5}{2}}\right) \tag{68}$$

where the subtracted (factored out) tensors take into account the two conditions (66). As a result, we find that, upon gauge fixing to zero the pure gauge part of $\psi_{a_1\ldots a_{s-\frac{3}{2}}}$ associated with the Stueckelberg symmetry, the remaining components of the fermionic field are described by the sum of the following unrestricted Young tableaux

$$\boxed{}_{\,s-\frac{1}{2}} \oplus \boxed{}_{\,s-\frac{3}{2}} \oplus \boxed{}_{\,s-\frac{5}{2}}\,. \tag{69}$$

Each term in (69) describes *unconstrained* totally symmetric spinor–tensors of ranks $s-\frac{1}{2}$, $s-\frac{3}{2}$ and $s-\frac{5}{2}$, respectively. Decomposing this set of fields into Lorentz irreducible gamma-traseless components, we get the set of Fang-Fronsdal massless fields of the half–integer spins descending from s down to $3/2$. Note that, analogously to the fields of spin one and zero in the bosonic case, the spin–1/2 field (being a zero–form) is not described by the action (57) and should be treated separately.

Up to this subtlety, the field content of the model under consideration is the same as that of the fermionic higher–spin triplets [9, 18]. Sine both models describe free fields, there should be a relation between them.

To find this relation let us look at the form of the equations and gauge transformations which define the triplet of unconstrained fermionic higher–spin fields $\Psi_{m_1\cdots m_{s-\frac{1}{2}}}$, $\chi_{m_1\cdots m_{s-\frac{3}{2}}}$ and $\lambda_{m_1\cdots m_{s-\frac{5}{2}}}$ in flat space–time [9, 18]. Their equations of motion are

$$\gamma^n\,\partial_n\,\Psi_{m_1\cdots m_{s-\frac{1}{2}}} = (s-\tfrac{1}{2})\,\partial_{(m_1}\,\chi_{m_2\cdots m_{s-\frac{1}{2}})}\,, \tag{70}$$

$$\partial^n\,\Psi_{nm_2\cdots m_{s-\frac{1}{2}}} - (s-\tfrac{3}{2})\,\partial_{(m_2}\,\lambda_{m_3\cdots m_{s-\frac{1}{2}})} = \gamma^n\,\partial_n\,\chi_{m_2\cdots m_{s-\frac{1}{2}}}\,, \tag{71}$$

$$\gamma^n\,\partial_n\,\lambda_{m_1\cdots m_{s-\frac{5}{2}}} = (s-\tfrac{5}{2})\,\partial^n\,\chi_{nm_1\cdots m_{s-\frac{5}{2}}}\,. \tag{72}$$

These equations are invariant under the following unconstrained gauge transformations of the fields

$$\delta\,\Psi_{m_1\cdots m_{s-\frac{1}{2}}} = (s-\tfrac{1}{2})\,\partial_{(m_1}\,\xi_{m_2\cdots m_{s-\frac{1}{2}})}\,, \tag{73}$$

$$\delta\,\chi_{m_1\cdots m_{s-\frac{3}{2}}} = \gamma^n\partial_n\,\xi_{m_1\cdots m_{s-\frac{3}{2}}}\,, \tag{74}$$

$$\delta\,\lambda_{m_1\cdots m_{s-\frac{5}{2}}} = \partial^n\,\xi_{nm_1\cdots m_{s-\frac{5}{2}}}\,. \tag{75}$$

The form of the gauge transformations (73)–(75) prompts us that the fermionic triplet fields are related to the components of the fermionic higher–spin 'vielbein' $\psi_{n;m_1\ldots m_{s-\frac{3}{2}}}$ as follows

$$\Psi_{m_1\cdots m_{s-\frac{1}{2}}} = (s-\tfrac{1}{2})\,\psi_{(m_1;m_2\ldots m_{s-\frac{1}{2}})}\,, \tag{76}$$

$$\chi_{m_1\cdots m_{s-\frac{3}{2}}} = \gamma^n\,\psi_{n;m_1\ldots m_{s-\frac{3}{2}}} \tag{77}$$

and

$$\lambda_{m_1\cdots m_{s-\frac{5}{2}}} = \eta^{nl}\,\psi_{n;lm_1\ldots m_{s-\frac{5}{2}}}\,. \tag{78}$$

Upon this identification the fermionic triplet field equations of motion (70)–(72) follow from eqs. (59). As such, upon substituting eqs. (76)–(78) for corresponding components of the fermionic frame–like field into the action (57) and gauge fixing to zero its Stueckelberg symmetry, one will reduce eq. (57) to the fermionic triplet action of [18] in flat space–time.

5 Frame–like action for fermionic higher–spin fields in AdS

In AdS space the gauge transformations (56) of the dynamical fermionic field $\psi^{\alpha}_{a_1\cdots a_{s-\frac{3}{2}}}$ get modified as follows

$$\delta\psi_{a_1\cdots a_{s-\frac{3}{2}}} = \mathcal{D}\,\xi_{a_1\cdots a_{s-\frac{3}{2}}} - e^b\,\xi_{a_1\cdots a_{s-\frac{3}{2}},b}\,, \tag{79}$$

where following [32] the generalized covariant differential $\mathcal{D}$ is defined as the sum of the conventional AdS covariant differential ∇ and the term $\frac{i\sqrt{-\Lambda}}{2}\,e^a\,\gamma_a$, namely,

$$\mathcal{D} = \nabla + \frac{i\sqrt{-\Lambda}}{2}\,e^a\,\gamma_a\,. \tag{80}$$

The external differential (80) is actually covariant with respect to the AdS isometry group $Spin(2, D-2)$, it is defined in such a way that its square vanishes when acting on spinor differential forms

$$\mathcal{D}^2\,\chi^\alpha = 0 \tag{81}$$

and it acts as ∇^2 on the tensor differential forms

$$\mathcal{D}^2\,T^{a_1\cdots a_t} = \nabla^2\,T_{a_1\cdots a_t} = -t\,\Lambda\,e^{(a_1}\,e_b\,T^{a_2\cdots a_t)b}\,. \tag{82}$$

Thus, in virtue of eq. (81), $\mathcal{D}^2$ acts on the tensor–spinor forms in the same way as on the tensors, *i.e.*

$$\mathcal{D}^2\,\psi^{a_1\cdots a_t} = -t\,\Lambda\, e^{(a_1}\, e_b\, \psi^{a_2\cdots a_t)b}. \tag{83}$$

Note also that

$$\mathcal{D}\,\gamma_a = -\frac{i\sqrt{-\Lambda}}{2}\, e^b\,[\gamma_b,\gamma_a] = -i\,\sqrt{-\Lambda}\, e^b\,\gamma_{ba}\,. \tag{84}$$

Eqs. (81)–(84) are useful when checking the gauge invariance of the action for the fermionic higher–spin fields in AdS under the transformations (79).

5.1 Fang-Fronsdal case in AdS

As in the flat–space, the fermionic dynamical higher–spin field $\tilde\psi^{a_1\cdots a_{s-\frac{3}{2}}}$ in AdS is subject to the gamma–trace condition

$$\gamma^c\tilde\psi_{a_1\ldots a_{s-\frac{5}{2}}c} = 0\,. \tag{85}$$

For this condition to be compatible with the gauge transformations (79) the gauge parameters must obey the following constraints

$$\gamma^c\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}c} = 0\,,\quad \gamma^c\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}c,b} = i\,\sqrt{-\Lambda}\,\gamma_{b}{}^{c}\,\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}c} = -i\,\sqrt{-\Lambda}\,\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}b} \tag{86}$$

$$\Longrightarrow\quad \gamma^b\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}c,b} = (s-\tfrac{3}{2})\,i\,\sqrt{-\Lambda}\,\tilde\xi_{a_1\ldots a_{s-\frac{5}{2}}c},\quad \bar{\tilde\xi}_{a_1\ldots a_{s-\frac{5}{2}}c,b}\,\gamma^b = -(s-\tfrac{3}{2})\,i\,\sqrt{-\Lambda}\,\bar{\tilde\xi}_{a_1\ldots a_{s-\frac{5}{2}}c}\,.$$

The action for γ–traceless $\tilde\psi_{a_1\ldots a_{s-\frac{3}{2}}}$ in AdS which is invariant under (79) with the parameters satisfying (86) has the following form

$$S = \int_{M^D} e^{a_1}\ldots e^{a_{D-3}}\,\varepsilon_{a_1\ldots a_{D-3}pqr}\Bigg[\bar{\tilde\psi}_{d_1\ldots d_{s-\frac{3}{2}}}\gamma^{pqr}\mathcal{D}\,\tilde\psi^{d_1\ldots d_{s-\frac{3}{2}}} - 6(s-\tfrac{3}{2})\,\bar{\tilde\psi}_{d_1\ldots d_{s-\frac{5}{2}}}{}^{p}\gamma^q\,\mathcal{D}\,\tilde\psi^{d_1\ldots d_{s-\frac{5}{2}}r}$$
$$+\frac{3i\,\sqrt{-\Lambda}\,(s-\frac{3}{2})}{D-2}\,e^r\,\bar{\tilde\psi}_{d_1\ldots d_{s-\frac{3}{2}}}\gamma^{pq}\tilde\psi^{d_1\ldots d_{s-\frac{3}{2}}} + \frac{6i\,\sqrt{-\Lambda}\,(s-\frac{3}{2})^2}{D-2}\,e^p\,\bar{\tilde\psi}^q{}_{d_1\ldots d_{s-\frac{5}{2}}}\tilde\psi^{rd_1\ldots d_{s-\frac{5}{2}}}\Bigg]\,. \tag{87}$$

The last two "mass–like" terms in (87) are proportional to the square root of the cosmological constant (which is also present in the covariant differential $\mathcal{D}$ (80). These terms insure the gauge invariance of the higher–spin system in AdS.

5.2 Fermionic triplets in AdS

Let us now consider the form of the action in AdS space for the fermionic higher–spin fields $\psi^{a_1\cdots a_{s-\frac{3}{2}}}$ which are not subject to the gamma–trace condition, *i.e.* describe fermionic triplets. By now the action and the equations of motion for the fermionic triplets in AdS space have been unknown. To demonstrate that such an action does exist, we first consider the simplest case of the spin $\frac{5}{2}$ field.

Spin–$\frac{5}{2}$ example

The one–form tensor–spinor field under consideration is the gamma–traceful field $\psi^a = dx^m\,\psi^a_m$. Its gauge transformations have the form

$$\delta\psi^a = \mathcal{D}\xi^a - e^b\xi^{a,b}, \tag{88}$$

where the parameter ξ^a is gamma–traceful, while the antisymmetric parameter $\xi^{a,b} = -\xi^{b,a}$ is required to satisfy the following relation

$$\gamma_b\,\xi^{a,b} = -i\sqrt{-\Lambda}\,\gamma^{ab}\xi_b = i\sqrt{-\Lambda}\,(\xi^a - \gamma^a\,\gamma^b\,\xi_b). \tag{89}$$

The condition (89), which reduces to the corresponding eq. (86) in the gamma–traceless case, ensures that the gamma trace of ψ^a transforms as a divergence, *i.e.* as a Rarita–Schwinger field of spin 3/2,

$$\delta(\gamma_a\,\psi^a) = \mathcal{D}(\gamma_a\,\xi^a). \tag{90}$$

The action for the field ψ^a which is invariant under the transformations (88)–(90) has the following form

$$\begin{aligned} S = \int_{M^D} e^{a_1}\dots e^{a_{D-3}}\,\varepsilon_{a_1\dots a_{D-3}bcd}\Big[&\bar\psi_f\gamma^{bcd}\mathcal{D}\,\psi^f - 6\,\bar\psi^b\gamma^c\,\mathcal{D}\,\psi^d + \frac{3i\sqrt{-\Lambda}}{D-2}\,e^d\,\bar\psi_f\gamma^{bc}\psi^f \\ &+\frac{6i\sqrt{-\Lambda}}{D-2}\,e^b\,\bar\psi^c\psi^d + \frac{6i\sqrt{-\Lambda}}{D-2}\,e^d\,(\bar\psi_f\gamma^f)\,\gamma^b\psi^c - \frac{3i\sqrt{-\Lambda}}{D-2}\,e^d\,(\bar\psi_f\gamma^f)\,\gamma^{bc}\,(\gamma_i\psi^i)\Big]\,. \end{aligned} \tag{91}$$

One can see that in comparison with the action (87) for a single spin–5/2 field, the action (91) contains two terms which depend on the gamma–trace of ψ^a. It can be shown that by splitting ψ^a into the gamma-traceless and gamma-trace parts

$$\psi^a = \tilde\psi^a - \frac{1}{D}\gamma^a\,\tilde\psi, \qquad \gamma_a\,\tilde\psi^a = 0, \qquad \tilde\psi = \gamma_a\,\psi^a\,, \tag{92}$$

the action (91) splits into the direct sum of the actions for the single spin–5/2 field $\tilde\psi^a$ and the spin–3/2 field $\tilde\psi$ in a way similar to the bosonic case (see Subsection 2.2). As mentioned above, the spin–1/2 field has not appeared in our construction. The above example is the simplest fermionic "triplet" (actually the doublet) of fields in AdS space.

Generic case of AdS higher–spin fermion triplets

The gamma–traceful one–form tensor–spinor field $\psi^{a_1\cdots a_{s-\frac32}}$ describing the fermionic triplet in AdS space undergoes the gauge transformations

$$\delta\psi^{a_1\cdots a_{s-\frac32}} = \mathcal{D}\,\xi^{a_1\cdots a_{s-\frac32}} - e_b\,\xi^{a_1\cdots a_{s-\frac32},b}\,, \tag{93}$$

with the unconstrained parameter $\xi^{a_1\cdots a_{s-\frac32}}$ and the parameter $\xi^{a_1\cdots a_{s-\frac32},b}$ satisfying the Young property, $\xi^{(a_1\cdots a_{s-\frac32},b)} = 0$, and the relaxed traceless condition (as in the case of the bosonic triplets)

$$\xi^{a_1\cdots a_{s-\frac52}\,c,b}\,\eta_{bc} = 0 \tag{94}$$

and the following relation

$$\begin{aligned} \gamma_b\xi^{a_1\dots a_{s-\frac32},b} &= -(s-\frac32)\,i\,\sqrt{-\Lambda}\gamma^{(a_1}{}_b\,\xi^{a_2\dots a_{s-\frac32})b}\,, \\ \bar\xi^{a_1\dots a_{s-\frac32},b}\,\gamma_b &= -(s-\frac32)\,i\,\sqrt{-\Lambda}\,\bar\xi^{(a_2\dots a_{s-\frac32}}{}_b\,\gamma^{a_1)b}\,. \end{aligned} \tag{95}$$

Eq. (95) reduces to (86) if the parameter $\xi^{a_1\cdots a_{s-\frac32}}$ was gamma–traceless and ensures that the gamma–trace of $\psi^{a_1\cdots a_{s-\frac32}}$ transforms as a spin–$(s-1)$ field, *i.e.* similar to (93) with $s\to s-1$.

The action which is invariant under the transformations (93)–(95) has the following form

$$
\begin{aligned}
S = \int_{M^D} e^{a_1} \dots e^{a_{D-3}}\, \varepsilon_{a_1\dots a_{D-3}abc} \Big[& \bar\psi_{d_1\dots d_{s-\frac32}}\gamma^{abc}\mathcal{D}\,\psi^{d_1\dots d_{s-\frac32}} \\
& -6(s-\tfrac32)\,\bar\psi_{d_1\dots d_{s-\frac52}}{}^{a}\gamma^b\,\mathcal{D}\,\psi^{d_1\dots d_{s-\frac52}c} \\
+\frac{3i\sqrt{-\Lambda}\,(s-\frac32)}{D-2}\Big(& e^c\,\bar\psi_{d_1\dots d_{s-\frac32}}\gamma^{ab}\psi^{d_1\dots d_{s-\frac32}} + 2(s-\tfrac32)\,e^a\,\bar\psi^b{}_{d_1\dots d_{s-\frac52}}\,\psi^{cd_1\dots d_{s-\frac52}}\Big) \\
& +\frac{3i\sqrt{-\Lambda}\,(s-\frac32)}{D-2}\Big(2\,e^c\,(\bar\psi_{d_1\dots d_{s-\frac52}f}\gamma^f)\,\gamma^a\psi^{bd_1\dots d_{s-\frac52}} \\
& -e^c\,(\bar\psi_{d_1\dots d_{s-\frac52}f}\gamma^f)\,\gamma^{ab}\,(\gamma_i\psi^{id_1\dots d_{s-\frac52}})\Big) \\
& -\frac{6i\sqrt{-\Lambda}\,(s-\frac32)(s-\frac52)}{D-2}\;e^a\,(\bar\psi^{bi}{}_{d_1\dots d_{s-\frac72}}\gamma_i)\,(\gamma_f\psi^{cfd_1\dots d_{s-\frac72}})\Big]\,.
\end{aligned}
\tag{96}
$$

It has one more (the last) term in comparison with the action (91) for the $s=\frac52$ triplet. The AdS analogues of the flat–space fermionic triplet fields of [9, 18] are extracted from $\psi_{a_1\cdots a_{s-\frac32}} = e^b\psi_{b;a_1\cdots a_{s-\frac32}}$ in a way similar to (76)–(78)

$$
\Psi_{a_1\cdots a_{s-\frac12}} = (s-\tfrac12)\,\psi_{(a_1;a_2\dots a_{s-\frac12})}\,, \tag{97}
$$

$$
\chi_{a_1\cdots a_{s-\frac32}} = \gamma^b\,\psi_{b;a_1\dots a_{s-\frac32}}\,, \tag{98}
$$

$$
\lambda_{a_1\cdots a_{s-\frac52}} = \eta^{bc}\,\psi_{b;ca_1\dots a_{s-\frac52}}\,. \tag{99}
$$

Their gauge transformations are easily derived from eqs. (93)–(95)

$$
\delta\,\Psi_{a_1\cdots a_{s-\frac12}} = (s-\tfrac12)\,\mathcal{D}_{(a_1}\,\xi_{a_2\cdots a_{s-\frac12})}\,, \tag{100}
$$

$$
\delta\,\chi_{a_1\cdots a_{s-\frac32}} = \gamma^b\mathcal{D}_b\,\xi_{a_1\cdots a_{s-\frac32}} - (s-\frac32)\,i\sqrt{-\Lambda}\gamma_{(a_1}{}^b\,\xi_{a_2\dots a_{s-\frac32})b}\,, \tag{101}
$$

$$
\delta\,\lambda_{a_1\cdots a_{s-\frac52}} = \mathcal{D}^b\,\xi_{ba_1\cdots a_{s-\frac52}} \equiv \nabla^b\,\xi_{ba_1\cdots a_{s-\frac52}} - \tfrac{i\sqrt{-\Lambda}}{2}\,\gamma^b\,\xi_{ba_1\cdots a_{s-\frac52}}\,, \tag{102}
$$

and the equations of motion, which generalize to AdS space eqs. (70)–(72), follow from the action (96).

Upon eliminating the Stueckelberg degrees of freedom and splitting the components of the spinor-tensor $\psi_{b;a_1\cdots a_{s-\frac32}}$ in its triplet constituents (97)–(99) one can reduce the action (96) to an action which describes the fermionic triplets in AdS in the metric–like formulation.

6 Concluding remarks

We have considered the frame–like Lagrangian formulation of free systems of bosonic and fermionic higher–spin fields in flat and AdS backgrounds of arbitrary dimension. We have shown that the higher–spin systems described by an unconstrained higher–spin vielbein and by the connections which are subject to weaker (gamma)–trace constraints than those required for the description of single Fronsdal and Fang–Fronsdal fields correspond to the higher–spin triplets whose fields are associated with certain components of the higher–spin vielbein and connection. We have thus endowed the triplet fields with a clear geometrical meaning. This allowed us to identify the appropriate form of the gauge transformations of the fermionic triplets in AdS space and construct the gauge invariant action which describes their dynamics.

It might be of interest to reformulate the unconstrained construction of irreducible higher–spin fields by [11] in the geometrical frame–like fashion. Actually, the additional constraints introduced in [11] (see *e.g.* their eqs. (10)) to relate triplet components to first derivatives of the compensator field $\alpha^{a_1\cdots a_{s-3}}$, in our construction amount to imposing the requirement that the trace of the higher–spin vielbein field is a pure gauge

$$e_{m;}{}^{a_1\cdots a_{s-3}b}{}_b = \nabla_m\, \alpha^{a_1\cdots a_{s-3}} \;-(s-1)\,\beta^{a_1\cdots a_{s-3},}{}_m\,, \tag{103}$$

where, for the higher–spin vielbein to obey the transformation rules (5) and (8) (or (40), (43)–(46) in the AdS case), the compensator fields $\alpha^{a_1\cdots a_{s-3}}$ and $\beta^{a_1\cdots a_{s-3},m}$ (the latter having the symmetry of the Young tableu $Y(s-3,1)$) are gauge transformed in the appropriate way

$$\delta\,\alpha^{a_1\cdots a_{s-3}} = \xi^{a_1\cdots a_{s-3}b}{}_b\,, \qquad \delta\,\beta^{a_1\cdots a_{s-3},m} = \xi^{a_1\cdots a_{s-3}}{}_b{}^{b,m}\,.$$

We expect that the frame–like version of [11] is obtained by adding to our triplet actions the Lagrange multiplier term which produces (103) on the mass shell.

Analogously, in the fermionic case the additional requirement that the gamma–trace of the higher–spin field $\psi_{m;\,a_1\ldots a_{s-\frac{3}{2}}}$ is a pure gauge, *i.e.*

$$\gamma^{a_{s-\frac{3}{2}}}\,\psi_{m;\,a_1\ldots a_{s-\frac{3}{2}}} = \mathcal{D}_m\,\alpha_{a_1\ldots a_{s-\frac{5}{2}}} - \beta_{a_1\ldots a_{s-\frac{5}{2}}\,,\,m} \tag{104}$$

(with $\alpha_{a_1\ldots a_{s-\frac{5}{2}}}$ and $\beta_{a_1\ldots a_{s-\frac{5}{2}}\,,\,m}$ being fermionic compensators) reduces the fermionic triplet to the single field of spin s in the unconstrained formulation.

The results obtained can be useful for further study of higher–spin triplets, their interactions and their generalization to mixed–symmetry fields, in particular in AdS backgrounds, in various contexts of higher–spin theory and its relation to string theory.

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Quantum superstring corrections and AdS/CFT

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Abstract

We review the result of the quantum $AdS_{\times}S^5$ superstring 2-loop computation of correction to the cusp anomalous dimension which is a function $f(\lambda)$ of the gauge coupling λ or the string tension $\frac{\sqrt{\lambda}}{2\pi}$. The result provides a check of the recent Bethe ansatz proposal for the string spectrum. This computation verified the cancellation of all 2-loop logarithmic divergences thus demonstrating the quantum consistency of the $AdS_5 \times S^5$ Green-Schwarz superstring action to this order.

Anomalous dimension of minimal twist large spin single trace operator or anomalous dimension of a Wilson line with a null cusp [2] was a subject of much attention in the context of the AdS/CFT duality for several years starting with the seminal work of [3] (see also [4, 5, 6]). In the planar limit this dimension is a function $f(\lambda)$ of the 't Hooft coupling λ or of the $AdS_5 \times S^5$ string tension $\frac{\sqrt{\lambda}}{2\pi}$. Finding this function exactly would be an important progress. A series of recent developments based on the apparent integrability of the theory culminated in a suggestion [7] of an integral equation that, in principle, determines $f(\lambda)$ for any value of λ.

To check the consistency of this equation and thus of the underlying asymptotic Bethe ansatz it is important compare its prediction with that of the quantum superstring theory in $AdS_5 \times S^5$. The perturbative string theory or the strong-coupling expansion of $f(\lambda)$ can be written as

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \left[\mathrm{a}_0 + \frac{\mathrm{a}_1}{\sqrt{\lambda}} + \frac{\mathrm{a}_2}{(\sqrt{\lambda})^2} + \frac{\mathrm{a}_3}{(\sqrt{\lambda})^3} + ... \right] , \tag{1}$$

where the tree-level [3] and the 1-loop [4] superstring predictions are

$$\mathrm{a}_0 = 1 \ , \qquad \mathrm{a}_1 = -3 \ln 2 \ . \tag{2}$$

The computation of the 2-loop superstring coefficient was initiated in [8] where it was found to be expressed in terms of the Catalan's constant $\mathrm{K} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159$.

The expansion of the BES [7] equation at strong coupling turned out to be a non-trivial problem [9, 10, 11, 12, 13]. The results for the three leading a_n coefficients 2 were first found only numerically

[9] (a_0 was later computed exactly [11]).[1] The numerical result for the third coefficient found in [9] was $a_3 \approx -0.9158 \pm 0.0039$.[2]

Very recently the analytic results for the coefficients in the strong coupling expansion of the solution of the BES equation for the cusp anomaly function 1 was found in a remarkable paper of [19], with the first few leading coefficients given by[3]

$$a_2 = -K , \tag{3}$$

$$a_3 = -\frac{1}{32}\left[27\zeta(3) + 96K\ln 2\right] , \tag{4}$$

$$a_4 = -\frac{1}{16}\left[84\beta(4) + 81\zeta(3)\ln 2 + 32K^2 + 144K(\ln 2)^2\right] , \tag{5}$$

$$a_5 = -\frac{9}{2048}\left[4785\zeta(5) + 10572\beta(4)\ln 2 + 4416\zeta(3)K + 5184\zeta(3)(\ln 2)^2 + 4096K^2\ln 2\right] \tag{6}$$

where

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} , \qquad \beta(k) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^k} , \qquad \beta(2) = K . \tag{7}$$

The expression for a_2 3 thus agrees with the numerical value found in [9] and matches precisely (the corrected version of) the result of the 2-loop superstring computation in [8].

The aim of [1] was to confirm the Catalan constant value of a_2 in 3 by an independent 2-loop superstring computation. The agreement of the results for a_2 obtained in [9] and [19] from the BES equation with our superstring expression provides an important test of the BES equation and thus of the underlying asymptotic Bethe ansatz. The significance of the result of the present paper is that it provides a highly non-trivial confirmation of the proposal for the all-order strong-coupling phase [18] and its weak-coupling continuation in [7]. Indeed, while the expressions for the tree-level [16] and the 1-loop [20, 21] terms in the strong-coupling expansion for the phase where essentially put into the Bethe ansatz expression from the known string theory results, the higher order terms in the phase where conjectured in [18] using the crossing symmetry condition [22] (which so far was not directly derived from string theory). The present computation demonstrates that the 2-loop term in the phase suggested in [18] is indeed in agreement with string theory.

The computation described in [1] resolved also a technical problem related to UV regularization present in the original approach of [8]. The manifest cancellation of the logarithmic UV divergences that we find here provides a direct demonstration of the quantum consistency of the $AdS_5 \times S^5$ Green-Schwarz (GS) action of [23]. This (together with the earlier 1-loop results [4, 24]) removes any doubt that this action can be used as a basis for non-trivial strong-coupling computations in the AdS/CFT. The agreement with the Bethe ansatz result provides also an implicit check of the quantum integrability of this $AdS_5 \times S^5$ superstring theory.

Another new result is the suggestion of a 2d Feynmann diagram (i.e. quantum superstring) interpretation to the higher-order coefficients 4–6, etc. found in [19]. In our computation $f(\lambda)$ appears in the quantum 2d effective action of the $AdS_5 \times S^5$ superstring sigma model expanded near a particular "homogeneous" string background in AdS_5

$$\Gamma = -\ln Z = \frac{1}{2}f(\lambda)V_2 . \tag{8}$$

Γ is proportional to the (large) volume factor V_2.[4] This 2d QFT interpretation of $f(\lambda)$ implies that

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[1] a_1 was also computed [17] from the "string" version of the Bethe ansatz, i.e. with the magnon scattering phase taken in the strong-coupling expanded form [18].

[2] The proximity of the absolute value of this number to the value of the Catalan's constant was noticed by the authors of [8] but the final result for the coefficient a_3 in the original version of [8] was incorrect due to several errors which were finally corrected in the revised version ([8],v4).

[3] The relation of the notation used in [19] to ours is: $\Gamma_{\rm cusp}(g) = \frac{1}{2}f(\lambda)$, $c_k = -\frac{1}{(4\pi)^k}a_k$, $g = \frac{\sqrt{\lambda}}{4\pi}$. We do not shift the argument of cusp anomaly function $\Gamma_{\rm cusp}(g)$ by c_1 as was done in [19].

[4] For a homogeneous backgrounds such as those considered in [8] and here there is no distinction between the 1-PI effective action and the logarithm of the partition function Z: connected but not 1-PI irreducible 2d Feynman graphs vanish.

different parts of the transcendental coefficients a_L appearing in 1,3-6 can be associated with the contributions of different L-loop Feynmann diagrams in the superstring sigma model.

In the 2-loop case both the bosonic and the fermionic "sunset" diagrams happen to contribute terms proportional to K (see [8] and below). Extending our superstring computation to the 3-loop order appears to be relatively straightforward. A qualitative analysis shows that $\zeta(3)$ term in a_3 in 4 should originate from the corresponding 3-loop diagrams like sunset one.

In general, it is natural to conjecture that the "maximally irreducible" terms $\zeta(2m+1)$ in the coefficients a_{2m+1} and $\beta(2m)$ in the coefficients a_{2m} [19] should originate, respectively, from the "maximally irreducible" odd-loop $L = 2m+1$ and even-loop $L = 2m$ superstring Feynman diagrams. This should apply starting with 2-loop order. Using this logic at the 1-loop order one would get $a_1 \sim \zeta(1)$ but this is logarithmically divergent; in fact, the 1-loop divergences cancel between bosons and fermions and the finite remainder happens to be proportional to $\ln 2$ [4, 25]. The 1-loop tadpoles adjoined to lower-loop topologies should perhaps be interpreted in this way.

This string world-sheet, i.e. 2d QFT interpretation of the function $f(\lambda)$ may help to clarify the meaning of the Borel non-summability of the strong-coupling expansion for $f(\lambda)$ as found from the BES equation in [19]. As was observed in [19], all coefficients a_k in 1 except the first one are *negative* and their values grow factorially (cf. 4–6). It appears that in contrast to sign-alternating Borel-summable series usually found in QM or QFT problems with perturbatively stable vacuum here we are dealing with an expansion near an unstable point. This is puzzling since the rotating folded string solution or the null cusp solution of [5] we consider below (which are closely related [25, 26]) are perturbatively stable.[5] One may contemplate the presence of some non-perturbative instability.

The computation in [1] was organized as follows. We start with setting up the computation of the the cusp anomaly function using the open-string (Wilson line [27, 28]) approach which is based on expansion near a Wilson line surface with a null cusp [5, 25]. As was explained in [25, 29, 26] it is equivalent to the closed-string approach used in [3, 4, 30, 8]. We use the $AdS_5 \times S^5$ GS superstring action in a special κ-symmetry gauge which becomes *quadratic* in fermions [31] after the T-duality along the 4 AdS_5 boundary directions in the Poincare coordinates. This action was found in [31] by starting with the action of [23] written in a special κ-symmetry gauge discussed in [32]. An equivalent action which also becomes quadratic in fermions after the T-duality was found in a similar κ-symmetry gauge ("S-gauge") in Appendix C of [33]. This action was already used in [25] for the computation of the 1-loop coefficient a_1 in 2. We utilize its simple structure (in particular, the absence of the quartic fermionic terms) to perform the computation of the 2-loop coefficient a_2.

We then compute the quantum corrections to string partition function expanded near the "null cusp" string background and discuss the issue of UV regularization, pointing out that the structure of the superstring action involving the ϵ^{ab} tensor in the fermionic term prohibits the use of a direct version of the 2d dimensional regularization. Its use is not actually necessary since we find that all the logarithmic 2-loop divergences cancel out separately in the sums of the bosonic and fermionic graphs computed directly in $d = 2$. The remaining power divergences can then be eliminated using a kind of analytic regularization which essentially amounts to setting $\delta^{(2)}(0) = 0$.[6] The same applies to the fermionic sector.[7] This should be considered as a regularization prescription that defines the

[5] In the conformal gauge we are using there is formally a ghost fluctuation mode corresponding to the time direction in AdS_5 but like in the flat Minkowski space case or in the AdS_3 WZW model the underlying string theory should be unitary: the Virasoro condition selects only physical on-shell modes. In our conformal-gauge partition function computation we are expanding near a consistent on-shell string background so the unphysical modes (a massless time-like (ghost) fluctuation mode and another massless longitudinal mode) should decouple and they actually do (their trivial 1-loop contribution cancels against that of the conformal gauge ghosts).

[6] In principle, one should be able to show the cancellation of all power-like divergent terms directly, by carefully including the contributions of all local factors (measure, κ-symmetry ghosts, Jacobians due to change of fluctuation bases, etc.). Bosonic power-like divergences are indeed cancelled by the invariant measure contribution [8].

[7] As was discussed in Appendices C and D.1 in [8], the cancellation of the 2-loop power-like divergences is required in order for the superstring partition function to be equal to 1 in supersymmetric cases such as the flat space GS action expanded near a long fundamental string background and the $AdS_5 \times S^5$ GS action expanded near a BMN geodesic.

quantum $AdS_5 \times S^5$ superstring theory in a way consistent with its classical symmetries, i.e. as a conformal quantum 2d field theory.

The resulting finite contributions to the 2-loop coefficient in 1 coming from the bosonic and from the fermionic 2-loop graphs in Figure 1 happen to be the same as found in the closed-string picture computation in [8]

$$\mathrm{a}_2 = \mathrm{a}_{2B} + \mathrm{a}_{2F} = \mathrm{K} - 2\mathrm{K} = -\mathrm{K} \ , \tag{9}$$

so that the total result matches the value in 3.

Going to higher, e.g. 3-loop, order is, in principle, straightforward. We again expect that all logarithmic divergences will cancel directly in $d = 2$ while power divergences can be unambiguously separated and regularized away.

Based on the spectrum of fluctuations and the form of the propagators and vertices in the string fluctuation action it is relatively straightforward to determine the general structure of the finite higher loop contributions to the effective action and thus to the strong-coupling expansion of cusp anomaly function as predicted by the string inverse tension expansion.

On dimensional grounds, the finite contribution to the effective action or cusp anomaly comes from momentum integrals of mass dimension -2. Most vertices in the action contain derivatives; employing partial fractioning and 2d Lorentz invariance these derivatives may be used to cancel some of the propagators. Since many of the fluctuation fields in the theory are massive, this leaves behind terms with uncanceled propagators and with the momenta in the numerators replaced by the mass values. Thus, the L-loop contribution to the effective action can be expressed in terms of *scalar* vacuum integrals whose topology is that of the initial Feynman diagrams as well as that of the "daughter" diagrams obtained by collapsing some of the propagators. It will then lead to $\zeta(n)$ coefficients.

One consequence of the strong-coupling solution of the BES equation found in [19] was that the coefficients $\mathrm{a}_1, \mathrm{a}_2, ...$ are all negative and grow factorially. The series in 1 is then not Borel summable, i.e. its summation is ambiguous and this might be suggesting adding to 1 exponentially small terms $\sim e^{-k\sqrt{\lambda}}$ for some positive k. By formally changing the sign of $\sqrt{\lambda}$

$$\sqrt{\lambda} \ \ \rightarrow \ \ -\sqrt{\lambda} \tag{10}$$

one finds that 1 becomes a sign-alternating and thus Borel summable series. This is puzzling since the *weak-coupling* expansion $f(\lambda) = b_1\lambda + b_2\lambda^2 + ...$ which is also described by the BES equation (and which has finite radius of convergence) is formally invariant under the sign change 10 and thus is "not aware" of the problem with summation of the strong-coupling expansion. The string theory interpretation of $f(\lambda)$ as a coefficient in the partition function expanded near a perturbatively stable string solution would also suggest a standard asymptotic but Borel-summable expansion in $\frac{1}{\sqrt{\lambda}}$. However, the string theory result for a_2 in $f(\lambda)$ found in [8] and here reproduces the negative sign of a_2 in 3 and thus appears to support the conclusion of [19] about the lack of Borel-summability of the strong coupling expansion of $f(\lambda)$.[8]

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On gauge invariant description of massive high spin fields

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Abstract

We present a mini review of gauge invariant description of massive high spin fields as well as some simple examples of its possible applications such as investigation of massive and partially massless particles in $(A)dS$ spaces, constructing massive high spin supermultiplets and search for high spin particles interactions.

Introduction

As is well known, Lorentz covariant description of massless high spin fields requires a theory to be gauge invariant. This, in particular, lead to so called constructive approach to investigation of consistent interactions of such fields when interaction Lagrangians and appropriate gauge transformations are constructed iteratively by the number of fields. In turn, common description of massive fields requires that some constraints must follow from equation of motion excluding all unphysical degrees of freedom. In this, at least two general problems appear then one tries to switch on interactions. First of all, a number of constraints could change thus leading to a change in a number of degrees of freedom and reappearing of unphysical ones. At second, even if a number of constraints remains to be the same as in free theory, interacting theory very often turns out to be non-casual, i.e. has solution corresponding to non-luminal propagation.

One of the possible solutions is to use gauge invariant description of massive high spin fields. There at least two basic approaches to such description. One of them based on the powerfull BRST method [1, 2, 3, 4, 5]. Another one appeared in attempt to generalize to high spins a very well known mechanism of spontaneous gauge symmetry breaking [6]. In such a breaking a set of Goldstone fields with non-homogeneous gauge transformations appear making gauge invariant description of massive gauge fields possible. In what follows, we give a short review of such approach and some of its applications.

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1 Gauge invariant description of massive fields

Let us start with some simple examples. The simplest and very well known example is a gauge invariant description for massive spin 1 particle using vector A_μ and scalar φ fields. The Lagrangian

$$\mathcal{L} = -\frac{1}{4}{A_{\mu\nu}}^2 + \frac{1}{2}(\partial_\mu\varphi)^2 - mA^\mu\partial_\mu\varphi + \frac{m^2}{2}{A_\mu}^2$$

is invariant under the following gauge transformations

$$\delta A_\mu = \partial_\mu\lambda \qquad \delta\varphi = m\lambda$$

In this, the possibility to have gauge invariance for massive particle depends on the non-trivail (shift) transformations of scalar Goldstone filed φ. Note also, that in a massless limit such theory describes just a sum of massless spin 1 and spin 0 particles.

Our next example deals with massive spin 2 particle. Usual non-gauge invariant description is given by the famous Fierz-Pauli Lagrangian for the symmetric second rank tensor $h_{\mu\nu}$:

$$\mathcal{L}_0 = \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - (\partial h)^\mu(\partial h)_\mu + (\partial h)^\mu\partial_\mu h - \frac{1}{2}\partial^\mu h\partial_\mu h - \frac{m^2}{2}(h^{\mu\nu}h_{\mu\nu} - h^2)$$

It's equations of motion give 5 conditions:

$$\partial^\mu h_{\mu\nu} = 0, \qquad h = h_{\mu\mu} = 0$$

leaving us with 10 - 5 = 5 physical degrees of freedom.

In the massless limit such theory has gauge invariance with vector parameter ξ_μ, so a natural first step to gauge invariant description is an introduction of vector field A_μ. And indeed it is easy to check that if one adds the following additional terms to the initial Lagrangian:

$$\Delta\mathcal{L} = -\frac{1}{4}(A_{\mu\nu})^2 + m\sqrt{2}[h^{\mu\nu}\partial_\mu A_\nu - h(\partial A)]$$

the result will be gauge invariant under

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \qquad \delta A_\mu = m\sqrt{2}\xi_\mu$$

But our vector Goldstone field A_μ is a gauge field itself and in the massless limit has it's own gauge invariance which was broken by our construction. So to achieve fully gauge invariant description of massive spin 2 we need one more Goldstone field, namely scalar φ. Total Lagrangian turns out to be:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0(h_{\mu\nu}) + \mathcal{L}_0(A_\mu) + \mathcal{L}_0(\varphi) + m\sqrt{2}[h^{\mu\nu}\partial_\mu A_\nu - h(\partial A)] - m\sqrt{3}A^\mu\partial_\mu\varphi - \\ &- \frac{m^2}{2}(h^{\mu\nu}h_{\mu\nu} - h^2) - m^2\sqrt{\frac{3}{2}}h\varphi + m^2\varphi^2\end{aligned}$$

In this, the Lagrangian is invariant under the following gauge transformations:

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \frac{m}{\sqrt{2}}g_{\mu\nu}\Lambda, \qquad \delta A_\mu = \partial_\mu\Lambda + m\sqrt{2}\xi_\mu, \qquad \delta\varphi = m\sqrt{3}\Lambda$$

Now we are ready to consider generalization of such construction for the arbitrary spin case. There are at least two possible approaches:

- Start from known(?) form of massive theory and "unHiggs" it introducing additional (Stukelberg) fields e.g. [1].
- Start from the appropriate set of massless fields with all their gauge invariances and obtain massive theory as a smooth deformation $\Rightarrow$ spontaneous symmetry breaking [7].

Here we will follow the second approach. In this, for the gauge invariant description of massive integer spin s particle we need tensor fields $\Phi_s, \Phi_{s-1}, \dots \Phi_0$ which are symmetric and double traceless. Here we use condensed notations where subscript on Φ_k denotes just a number of indices and not the indices themselves. The Lagrangian has the following general form:

$$\mathcal{L} \sim \sum_{k=0}^{s} [\Phi_k \partial^2 \Phi_k + m \Phi_k \partial \Phi_{k-1} + m^2 (\Phi_k \Phi_k + \Phi_k \Phi_{k-2})]$$

while gauge transformations leaving it invariant look like:

$$\delta \Phi_k \sim \partial \xi_{k-1} + m[\xi_k + g_2 \xi_{k-2}]$$

Note that double tracelessness condition severely restricts a number of possible terms here. In this, requirement of gauge invariance gives a number of algebraic equations on unknown parameters which could be easily solved. Note also that almost all fields play double role being gauge and Goldstone fields simultaneously. In the massless limit such Lagrangian reduces to the sum of massless Lagrangians for spin s, s-1, ... 0 particles.

Note that a set of double traceless tensors $\Phi_s, \Psi_{s-4}, \dots$ could be combined into one unconstrained tensor Φ_s. Doing the same with all other tensors one can rewrite total gauge invariant Lagrangian in terms of just four unconstrained fields Φ_s, Φ_{s-1}, Φ_{s-2} and Φ_{s-3}. At the same time, joining together $\xi_{s-1}, \xi_{s-3} \dots$ and $\xi_{s-2}, \xi_{s-4}, \dots$ one can rewrite all gauge transformations in terms of two unconstrained gauge parameters ξ_{s-1} and ξ_{s-2}. Such description was considered in [8] (see also [9]).

It is instructive to look how our gauge invariant description is related with non-gauge invariant one [10]. First of all, we represent our double traceless tensors in terms of two traceless ones: $\Phi_k = \Phi'_k + \frac{k-1}{4} g_{(12}\varphi_{k-2)}$. Then we can use gauge transformations in order to set the gauge where all $\Phi'_k = 0$, $0 \le k \le s-1$. This leaves us with our main double traceless tensor Φ_s (which is equivalent to two traceless tensors Φ'_s and φ_{s-2}) and a set of auxiliary traceless tensors φ_k, $0 \le k \le s-3$ exactly as in [10].

Such gauge invariant description of massive fields works well not only in flat Minkowski space-time, but in (anti) de Sitter space-times as well [11, 3]. All that one needs to do is to replace ordinary partial derivatives with the covariant ones and take into account commutator of these derivatives which is non-zero now. In particular, this formulation turns out to be very convenient for investigation of so called partially massless theories which appear in de Sitter space. Lets again take spin 2 case as an example. As we have seen, complete gauge invariant description of massive particle requires three fields: $h_{\mu\nu}$, A_μ and φ. But in de Sitter space for critical mass value $m^2 = \Lambda/3$ scalar fields φ completely decouples and we obtain theory with just two fields $h_{\mu\nu}$ and A_μ, describing partially massless particle with four physical degrees of freedom (helicities ± 2, ± 1).

In four-dimensional Minkowski space-time for the description of all irreducible representations of Poincare group it is enough to consider completely symmetric (spin)-tensors only. But in dimensions greater than four one faces the fact that many interesting and physically important theories such as supergravity, superstring and (supersymmetric) high spin theories, necessarily contain mixed symmetry (spin)-tensors. In the $(A)dS_d$ space the situation becomes even more complicated because besides general problems related with the existence of forbidden mass ranges and partially massless particles, we discover that some fields do not have massless limit at all [12] making the very definition of mass problematic.

Gauge invariant description of massive fields, which could be constructed for mixed symmetry bosonic fields as well [13, 5], allows one carefully investigate all related problems. Let us consider as an example simplest mixed symmetry tensor $\Phi_{[\mu\nu],\alpha}$ such that $\Phi_{[\mu\nu\alpha]} = 0$. In flat space free massless Lagrangian is invariant under two gauge transformations:

$$\delta \Phi_{\mu\nu,\alpha} = \partial_\mu x_{\nu\alpha} - \partial_\nu x_{\mu\alpha} + 2\partial_\alpha y_{\mu\nu} - \partial_\mu y_{\nu\alpha} + \partial_\nu y_{\mu\alpha}$$

where parameter $x_{(\alpha\beta)}$ — symmetric, while $y_{[\alpha\beta]}$ — antisymmetric on their indices. In turn, gauge invariant description of massive particle requires four fields: $\Phi_{\mu\nu,\alpha}$, symmetric tensor $h_{(\mu\nu)}$, antisymmetric one $B_{[\mu\nu]}$ and vector A_μ. Such formulation [13] could be easily deformed into $(A)dS$ space without introduction of any other fields. In this, it turns out that there is no massless limit at all (i.e. limit where both gauge symmetries become unbroken). Instead, depending on the sign of cosmological term one can get one of the two possible partially massless limits. In the AdS space fields $B_{\mu\nu}$ and A_μ decouple while $\Phi_{\mu\nu,\alpha}$ and $h_{\mu\nu}$ describes partially massless theory [12]. At the same time, in dS space $h_{\mu\nu}$ and A_μ could decouple leaving us with $\Phi_{\mu\nu,\alpha}$ and $B_{\mu\nu}$ fields describing one more partially massless theory.

At last but not least, gauge invariant formulation is possible for massive fermionic fields (both in flat Minkowski space and in $(A)dS$ spaces) as well [14, 2, 4]. Let us take simplest example — massive spin 3/2 particle. Gauge invariant formulation requires two fields: vector-spinor Ψ_μ and spinor χ. The Lagrangian

$$\mathcal{L} = \frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}\bar{\Psi}_\mu\gamma_5\gamma_\nu\partial_\alpha\Psi_\beta + \frac{i}{2}\bar{\chi}\hat{\partial}\chi + \frac{m}{2}\bar{\Psi}_\mu\sigma^{\mu\nu}\Psi_\nu + im\sqrt{\frac{3}{2}}(\bar{\Psi}\gamma)\chi + m\bar{\chi}\chi$$

is invariant under the following gauge transformations

$$\delta\Psi_\mu = (\partial_\mu + \frac{im}{2}\gamma_\mu)\xi \qquad \delta\chi = m\sqrt{\frac{3}{2}}\xi$$

and in the massless limit reduces to the sum of massless spin 3/2 and spin 1/2 particles.

In general, for the gauge invariant description of massive half-integer spin $s+1/2$ particle [14] one needs the spin-tensors $\Psi_s, \Psi_{s-1}, \ldots \Psi_0$ which are symmetric and triple γ-traceless. The Lagrangian has the following general form:

$$\mathcal{L} \sim \sum_{k=0}^{s}[\bar{\Psi}_k\hat{\partial}\Psi_k + m(\bar{\Psi}_k\Psi_k + i\bar{\Psi}_k\gamma\Psi_{k-1})]$$

and gauge transformations leaving it invariant look like:

$$\delta\Psi_k \sim (\partial + im\gamma)\xi_{k-1} + m\xi_k$$

Once again note correct massless limit of such description.

2 Massive high spin supermultiplets

It is evident that in any theory of high spin particles most of them have to be massive (and their gauge symmetries have to be spontaneously broken). It means that in any supersymmetric high spin theory like the superstring these particles must belong to some massive supermultiplet. It may seems strange but though explicit realization of massless supermultiplets with arbitrary spins were known for a long time [15] explicit construction for massive supermultiplets was not available until recently [16]. To make presentation as simple as possible here we restrict ourselves with the simplest case — $N = 1$ supersymmetry in flat $d = 4$ Minkowski space-time though the procedure works for extended supersymmetries and in AdS space as well [17, 18].

Massless $N = 1$ supermultiplets contain one bosonic and one fermionic fields which differ in spin by 1/2. In this, general structure of supertransformations is very simple:

$$\delta F \sim \partial B\eta, \qquad \delta B \sim \bar{F}\eta$$

where F - fermion while B - boson. Note that by choosing canonical dimensions of bosonic and fermionic fields and appropriate dimension of the parameter η all the coefficients can be made dimesionless. Surprisingly, the structure of supertransformations for massive supermultiplets turns out to be much more complicated. Even explicit realization of relatively low superspins such as 1 and 3/2 requires hard work [19, 20, 21].

Let us consider simplest nontrivial example — massive supermultiplet with superspin 1 containing four fields: vector-spinor Ψ_μ, two vector fields A_μ and B_μ and spinor χ. Starting with the known form of supertransformations for massless supermultiplets by usual "trial and error" method one can obtain following result:

$$\begin{aligned}
\delta\Psi_\mu &= -\tfrac{i}{4}\sigma^{\alpha\beta}\gamma_\mu C_{\alpha\beta}\eta + mC_\mu\eta - \tfrac{1}{m}(\partial_\mu + \tfrac{im}{2}\gamma_\mu)[\tfrac{1}{6}\sigma^{\alpha\beta}C_{\alpha\beta}\eta + \tfrac{2im}{3}\hat{C}\eta] \\
\delta\bar{C}_\mu &= 2(\bar{\Psi}_\mu\eta) + \tfrac{2i}{\sqrt{3}}(\bar{\chi}\gamma_\mu\eta) - \tfrac{2}{m\sqrt{3}}\partial_\mu(\bar{\chi}\eta) \qquad \delta\chi = -\tfrac{1}{2\sqrt{3}}\sigma^{\alpha\beta}C_{\alpha\beta}\eta + \tfrac{im}{\sqrt{3}}\hat{C}\eta
\end{aligned}$$

Here $C_\mu = A_\mu + \gamma_5 B_\mu$. One can see that there are three type of terms in these formulas. At first there are terms having the same form as in the massless case. Then there are corrections to the fermionic supertransformations containing bosonic fields without derivatives with the coefficients proportional to the mass m. At last, there are higher derivatives terms (up to two derivatives in the fermionic transformations and up to one derivative in the bosonic transformations in this simplest case) with coefficients proportional to inverse powers of mass m so that there is no any straightforward massless limit. Moreover, the higher superspin one tries to consider the higher and higher derivatives one will have to introduce. Doing carefull rearrangement of these higher derivative terms (as have been done in the formulas above) one can observe that these terms are just some field dependent gauge transformations and this in turn suggests that we can achieve much more straightforward and transparent construction by using gauge invariant description of massive high spin particles.

The main idea is that massive supermultiplet must be easily constructed out of the appropriate set of massless ones exactly in the same way as massive particle could be constructed using appropriate set of massless ones. Let us show how this general procedure works on the particular case of superspin 1 multiplet shown above.

- Determine an appropriate set of massless supermultiplets. Using the fact that in the massless limit massive spin 3/2 particle reduces to massless spin 3/2 and 1/2 particles, while massive spin 1 particle — to massless spin 1 and 0 ones, it is easy to see that we will get three massless supermultiplets:

$$\begin{pmatrix} & 3/2 & \\ 1 & & 1' \\ & 1/2 & \end{pmatrix} \Rightarrow \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1' \\ 1/2 \end{pmatrix} \oplus \begin{pmatrix} 1/2 \\ 0, 0' \end{pmatrix}$$

- Duality rotation. As is known two bosonic fields in the massive supermultiplet must have different parity. This allows one to consider duality transformations mixing massless supermultiplets, containing these particles. As a result, the most general supertransformations leaving sum of kinetic terms invariant have the following form:

$$\begin{aligned}
\delta\Psi_\mu &= -\tfrac{i}{2\sqrt{2}}\sigma^{\alpha\beta}[\cos(\theta)A_{\alpha\beta} - \sin(\theta)B_{\alpha\beta}\gamma_5]\gamma_\mu\eta \\
\delta A_\mu &= \sqrt{2}\cos(\theta)(\bar{\Psi}_\mu\eta) + i\sin(\theta)(\bar{\rho}\gamma_\mu\eta) \\
\delta B_\mu &= \sqrt{2}\sin(\theta)(\bar{\Psi}_\mu\gamma_5\eta) + i\cos(\theta)(\bar{\rho}\gamma_\mu\gamma_5\eta) \\
\delta\rho &= -\tfrac{1}{2}\sigma^{\alpha\beta}[\sin(\theta)A_{\alpha\beta} + \cos(\theta)B_{\alpha\beta}\gamma_5]\eta \\
\delta\chi &= -i\hat{\partial}(\varphi + \gamma_5\pi)\eta \quad \delta\varphi = (\bar{\chi}\eta) \quad \delta\pi = (\bar{\chi}\gamma_5\eta)
\end{aligned}$$

  where θ — mixing angle to be determined.

- Now we obtain massive theory as a smooth deformation of initial massless one by adding to the Lagrangian mass terms for all fields as well as corrections to the fermionic supertransformations, containing bosonic fields without derivatives. In this, mixing angle θ turns out to be fixed. Resulting supertransformations:

$$\delta\Psi_\mu = -\tfrac{i}{4}\sigma^{\alpha\beta}\gamma_\mu C_{\alpha\beta}\eta - D_\mu z\eta \qquad \delta\bar{C}_\mu = 2(\bar{\Psi}_\mu\eta) + i\sqrt{2}(\bar{\rho}\gamma_\mu\eta)$$

$$\delta\rho = -\tfrac{1}{2\sqrt{2}}\sigma^{\alpha\beta}C_{\alpha\beta}\eta \qquad \delta\chi = -i\hat{D}z\eta \qquad \delta\bar{z} = 2(\bar{\chi}\eta)$$

  where $C_\mu = A_\mu + \gamma_5 B_\mu$, $z = \varphi + \gamma_5\pi$, $D_\mu z = \partial_\mu z - mC_\mu$

- Apart from global supertransformations resulting theory is invariant under the following local gauge transformations:

$$\delta\Psi_\mu = (\partial_\mu - \tfrac{im}{2}\gamma_\mu)\xi \qquad \delta\rho = -\tfrac{m}{\sqrt{2}}\xi \qquad \delta\chi = m\xi$$

$$\delta C_\mu = \partial_\mu \Lambda \qquad \delta z = m\Lambda \qquad \Lambda = \lambda + \gamma_5\tilde{\lambda}$$

Note, that one can use these transformations to choose a gauge where all Goldstone fields equal to zero leaving us with just four physical massive fields. In this, supertransformations must be completed with the field dependent gauge transformations restoring the gauge. This explains the appearance of higher derivative terms we have shown above.

3 Interactions

Construction of consistent high spin particles interactions is one of the old, hard and still unsolved problems. For the massless particles it is possible to formulate constructive approach to this problem (for BRST formulation see [22]). In this approach one starts with free Lagrangian for the collection of massless fields with appropriate gauge transformations and tries to construct interacting Lagrangian and modified gauge transformations iteratively by the number of fields so that:

$$\mathcal{L} \sim \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \dots, \qquad \delta \sim \delta_0 + \delta_1 + \delta_2 + \dots$$

where $\mathcal{L}_1$ — cubic vertex, $\mathcal{L}_2$ — quartic one and so on, while δ_1 — corrections to gauge transformations linear in fields, δ_2 — quadratic in fields and so on.

The mere existence of gauge invariant formulation for massive high spin particles allows us to extend such constructive approach for any collection of massive and/or massless particles. Let us illustrate this idea using relatively simple but non-trivial example — interaction of massive spin 2 particles with gravity [23].

First of all let us denote a metric tensor as $g_{\mu\nu}$. Let us stress that it is not some fixed background here, but a dynamical filed with its own equations of motion:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 0$$

Here we assume that the cosmological term is absent, though it could be easily introduced. As usual in gravity, we also assume that connection is metric compatible $D_\alpha g_{\mu\nu} = 0$ and we have usual identities:

$$D^\mu R_{\mu\nu,\alpha\beta} = D_\alpha R_{\beta\nu} - D_\beta R_{\alpha\nu}, \qquad D^\mu R_{\mu\nu} = \tfrac{1}{2}D_\nu R$$

Now consider gauge invariant Lagrangian for massive spin 2 particle where all derivatives are replaced by the covariant ones:

$$\begin{aligned}\mathcal{L}_2 = {} & \tfrac{1}{2}D^\alpha h^{\mu\nu}D_\alpha h_{\mu\nu} - D^\alpha h^{\mu\nu}D_\mu h_{\nu\alpha} + (Dh)^\mu D_\mu h - \tfrac{1}{2}D^\mu h D_\mu h - \tfrac{1}{2}(D_\mu A_\nu - D_\nu A_\mu)^2 + \\ & + 3D^\mu\varphi D_\mu\varphi + 2m[h^{\mu\nu}D_\mu A_\nu - h(DA)] - 6mA^\mu D_\mu\varphi - \\ & - \tfrac{m^2}{2}[h^{\mu\nu}h_{\mu\nu} - h^2] - 3m^2 h\varphi + 6m^2\varphi^2\end{aligned}$$

as well as appropriate gauge transformations:

$$\delta h_{\mu\nu} = D_\mu\xi_\nu + D_\nu\xi_\mu + mg_{\mu\nu}\lambda, \qquad \delta A_\mu = D_\mu\lambda + m\xi_\mu, \qquad \delta\varphi = m\lambda$$

Here we use slightly different normalization of Goldstone fields compatible with that in [23]. Covariant derivatives do not commute and, as a result, this Lagrangian is not invariant under these gauge transformations any more. But gauge invariance could be restored if one adds to the Lagrangian:

$$\Delta\mathcal{L} = -2R^{\mu\nu}h_{\mu\alpha}h_{\nu\alpha} + R^{\mu\nu}h_{\mu\nu}h + \tfrac{1}{2}Rh^{\alpha\beta}h_{\alpha\beta} - \tfrac{1}{4}Rh^2$$

and requires that metric $g_{\mu\nu}$ has non-trivial transformations

$$\delta g_{\alpha\beta} = 2(D_\mu h_{\alpha\beta} - D_\alpha h_{\beta\mu} - D_\beta h_{\alpha\mu})\xi^\mu + 2m[A_\alpha\xi_\beta + A_\beta\xi_\alpha - g_{\alpha\beta}(A\xi)]$$

Note that in any case when the number of derivatives in the interactions is equal or greater than that in free Lagrangian, the theory is always determined up to the possible field redefinitions. In the case at hands, cubic vertex and concrete form of gauge transformations are determined up to the redefinitions of metric tensor:

$$g_{\mu\nu} \Rightarrow g_{\mu\nu} + \kappa_1 h_{\mu\alpha} h_{\alpha\nu} + \kappa_2 h h_{\mu\nu} + \kappa_3 g_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} + \kappa_4 g_{\mu\nu} h^2$$

It is instructive to compare our results here with the investigations of massive spin 2 particle in gravitational background [24, 25, 26].

Conclusion

We hope that a few simple examples given here clearly shows that gauge invariant description of massive high spin particles really provides powerfull approach to investigation of the whole number related problems including possible interactions of such particles.

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Supersymmetric Chern-Simons models in harmonic superspaces

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Abstract

We review harmonic superspaces of the $D = 3, N{=}3$ and 4 supersymmetries and gauge models in these superspaces. Superspaces of the $D{=}3, N{=}5$ supersymmetry use harmonic coordinates of the $SO(5)$ group. The superfield $N{=}5$ actions describe the off-shell infinite-dimensional Chern-Simons supermultiplet.

1 Introduction

Supersymmetric extensions of the $D{=}3$ Chern-Simons theory were discussed in [1]-[10]. A superfield action of the $D{=}3, N{=}1$ Chern-Simons theory can be interpreted as the superspace integral of the differential Chern-Simons superform $dA + \frac{2i}{3}A^3$ in the framework of our theory of superfield integral forms [3]-[6].

The Abelian $N{=}2$ CS action was first constructed in the $D{=}3, N{=}1$ superspace [1]. The corresponding non-Abelian action was considered in the $D{=}3, N{=}2$ superspace with the help of the Hermitian superfield $V(x^m, \theta^\alpha, \bar\theta^\alpha)$, where θ^α and $\bar\theta^\alpha$ are the complex conjugated spinor coordinates [3]. The unusual dualized form of the $N{=}2$ CS Lagrangian contains the second vector field instead of the scalar field[7].

The $D{=}3, N{=}3$ CS theory was first analyzed by the harmonic-superspace method [8, 9]. Supersymmetric action of the $D{=}3, N{=}4$ Yang-Mills theory can also be constructed in the $D{=}3, N{=}3$ superspace, but the alternative formalism exists in the $N{=}4$ superspace [15].

The authors of [18] propose using the $SO(5)/U(2)$ harmonic superspace for the superfield description of the $D{=}3, N{=}5$ Chern-Simons theory. The detailed analysis of the superfield formalism of the Chern-Simons theory in this harmonic superspace was presented in our recent paper [19]. The alternative formalism of this theory using the $SO(5)/U(1)\times U(1)$ harmonics and additional harmonic conditions was considered in [20]. It was shown that the action of this model is invariant with respect to the $D{=}3, N{=}6$ superconformal group. The superfield action without harmonic constraints describes additional matter fields [21].

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2 $N = 4$ and $N = 3$ harmonic superspaces

We consider the following coordinates of the $D = 3, N = 4$ superspace:

$$z = (x^m, \theta^\alpha_{i\hat{k}}), \tag{1}$$

where i and $\hat{k}$ are two-component indices of the automorphism groups $SU_L(2)$ and $SU_R(2)$, respectively, α is the two-component index of the $SL(2,R)$ group and $m = 0,1,2$ is the 3D vector index. The $N = 4$ supersymmetry transformations are

$$\delta x^m = -i(\gamma^m)_{\alpha\beta}(\epsilon^\alpha_{j\hat{k}}\theta^{\beta j\hat{k}} - i\epsilon^\beta_{j\hat{k}}\theta^{\alpha j\hat{k}}), \tag{2}$$

where γ^m are the 3D γ matrices.

The $SU_L(2)/U(1)$ harmonics $u^\pm_i$ [11] can be used to construct the left analytic superspace [15] with the LA coordinates

$$\zeta_L = (x^m_L, \theta^{+\hat{k}\alpha}). \tag{3}$$

The L-analytic prepotential $V^{++}(\zeta_L, u)$ describes the left $N = 4$ vector multiplet $A_m, \phi_{\hat{k}\hat{l}}, \lambda^\alpha_{i\hat{k}}, D^{ik}$. The $D = 3, N = 4$ SYM action can be constructed in terms of this prepotential by analogy with the $D = 4, N = 2$ SYM action [14].

Let us introduce the new notation for the left harmonics $u^\pm_i = u^{(\pm1,0)}_i$ and the analogous notation $v^{(0,\pm1)}_{\hat{k}}$ for the right $SL_R(2)/U(1)$ harmonics. The biharmonic $N = 4$ superspace uses the Grassmann coordinates [15]

$$\theta^{(\pm1,\pm1)\alpha} = u^{(\pm1,0)}_i v^{(0,\pm1)}_{\hat{k}} \theta^{i\hat{k}\alpha}. \tag{4}$$

In this representation, we have

$$\zeta_L = (x^m_L, \theta\theta^{(1,\pm1)\alpha}), \theta V^{++} \equiv V^{(2,0)}, \tag{5}$$

$$D^{(1,\pm1)}_\alpha V^{(2,0)} = 0, \theta D^{(0,2)}_v V^{(2,0)} = 0. \tag{6}$$

The right analytic $N = 4$ coordinates are

$$\zeta_R = (x^m_R, \theta\theta^{(\pm1,1)\alpha}), \theta x^m_R = x^m_L - 2i(\gamma^m)_{\alpha\beta}\theta^{(-1,1)\alpha}\theta^{(1,1)\beta}. \tag{7}$$

The mirror R analytic prepotential $\hat{V}^{(0,2)}$

$$D^{(\pm1,1)}\hat{V}^{(0,2)} = 0, \theta D^{(2,0)}_u \hat{V}^{(0,2)} = 0 \tag{8}$$

describes the right $N = 4$ vector multiplet $\hat{A}_m, \hat{\phi}_{ij}, \hat{\lambda}^\alpha_{i\hat{k}}, \hat{D}_{ik}$, where $\hat{A}_m$ is the mirror vector gauge field using the independent gauge group. The right $N = 4$ SYM action is similar to the analogous left action. These multiplets can be formally connected by the map $SU_L(2) \leftrightarrow SU_R(2)$.

The $N = 4$ superfield Chern-Simons type (or BF-type) action for the gauge group $U(1) \times U(1)$ connects two mirror vector multiplets

$$\int dud^3x_l d\theta^{(-4,0)} V^{(2,0)}(\zeta_L, u) D^{(1,1)\alpha} D^{(1,1)}_\alpha \hat{V}^{(0,-2)}, \tag{9}$$

where the right connection satisfies the equation

$$\mathcal{D}^{(0,2)}_v \hat{V}^{(0,-2)} = \mathcal{D}^{(0,-2)}_v \tilde{V}^{(0,2)}, \theta\mathcal{D}^{(2,0)}_u \hat{V}^{(0,-2)} = 0. \tag{10}$$

The component form of this action was considered in [16, 17].

We can identify the left and right isospinor indices in the $N = 4$ spinor coordinates

$$\theta^\alpha_{j\hat{k}} \;\rightarrow\; \theta^\alpha_{jk} = \theta^\alpha_{(jk)} + \frac{1}{2}\varepsilon_{jk}\theta^\alpha, \tag{11}$$

where the Grassmann coordinates $\theta^\alpha_{(jk)}$ describe the $N=3$ superspace. The harmonic superspace of the $D=3,N=3$ supersymmetry uses the standard harmonics $u^\pm_i$[8, 9]

$$x_3^{\alpha\beta}=x^{\alpha\beta}+i(\theta^{\alpha++}\theta^{\beta\mu}+\theta^{\beta++}\theta^{\alpha\mu})\ , \tag{12}$$

$$\theta^{++\alpha}=u^+_iu^+_k\theta^{(ik)\alpha}\ ,\quad \theta^{--\alpha}=u^-_iu^-_k\theta^{(ik)\alpha}\ ,\quad \theta^{\alpha\sigma0}=u^+_iu^-_k\theta^{(ik)\alpha}\ , \tag{13}$$

$$\delta x_3^{\alpha\beta}=-2i\epsilon^{--\alpha}\theta^{++\beta}-2i\epsilon^{--\beta}\theta^{++\alpha}+2i\epsilon^{0\alpha}\theta^{0\beta}.$$

The vector $N=3$ supermultiplet is described by the analytic superfield $V^{++}(x_3,\theta^{++},\theta^0,u)$ and the corresponding analytic $N=3$ superfield strength is

$$W^{++}(x_3,\theta^{++},\theta^0,u)=-\frac{1}{2}D^{++\alpha}D^{++}_\alpha V^{--}. \tag{14}$$

The action of the corresponding CS-theory can be constructed in the full or analytic $N=3$ superspaces [8, 9].

3 Harmonic superspaces for the group $SO(5)$

The homogeneous space $SO(5)/U(2)$ is parametrized by elements of the harmonic 5×5 matrix

$$U^K_a=(U^{+i}_a,U^0_a,U^-_{ia})=(U^{+1}_a,U^{+2}_a,U^0_a,U^-_{1a},U^-_{2a}), \tag{15}$$

where $a=1,\dots 5$ is the vector index of the group $SO(5)$, $i=1,2$ is the spinor index of the group $SU(2)$, and $U(1)$-charges are denoted by symbols $+,-,0$. The basic relations for these harmonics are

$$U^{+i}_aU^{+k}_a=U^{+i}_aU^0_a=0,\theta U^-_{ia}U^-_{ka}=U^-_{ia}U^0_a=0,\theta U^{+i}_aU^-_{ka}=\delta^i_k,\theta U^0_aU^0_a=1,$$

$$U^{+i}_aU^-_{ib}+U^-_{ia}U^{+i}_b+U^0_aU^0_b=\delta_{ab}. \tag{16}$$

We consider the $SO(5)$ invariant harmonic derivatives with nonzero $U(1)$ charges

$$\partial^{+i}=U^{+i}_a\frac{\partial}{\partial U^0_a}-U^0_a\frac{\partial}{\partial U^-_{ia}},\theta\partial^{+i}U^0_a=U^{+i}_a,\theta\partial^{+i}U^-_{ka}=-\delta^i_kU^0_a,$$

$$\partial^{++}=U^+_{ia}\frac{\partial}{\partial U^-_{ia}},\theta[\partial^{+i},\partial^{+k}]=\varepsilon^{ki}\partial^{++},\theta\partial^{+i}\partial^+_i=\partial^{++},$$

$$\partial^-_i=U^-_{ia}\frac{\partial}{\partial U^0_a}-U^0_a\frac{\partial}{\partial U^{+i}_a},\theta\partial^-_iU^0_a=U^-_{ia},\theta\partial^-_iU^{+k}_a=-\delta^k_iU^0_a, \tag{17}$$

$$\partial^{--}=U^{-i}_a\frac{\partial}{\partial U^{+i}_a},\theta[\partial^-_i,\partial^-_k]=\varepsilon_{ki}\partial^{--},\theta\partial^{-k}\partial^-_k=-\partial^{--},$$

where some relations between these harmonic derivatives are defined. The $U(1)$ neutral harmonic derivatives form the Lie algebra $U(2)$

$$\partial^i_k=U^{+i}_a\frac{\partial}{\partial U^{+k}_a}-U^-_{ka}\frac{\partial}{\partial U^-_{ia}},\theta[\partial^{+i},\partial^-_k]=-\partial^i_k, \tag{18}$$

$$\partial^0\equiv\partial^k_k=U^{+k}_a\frac{\partial}{\partial U^{+k}_a}-U^-_{ka}\frac{\partial}{\partial U^-_{ka}},\theta[\partial^{++},\partial^{--}]=\partial^0,$$

$$\partial^i_kU^{+l}_a=\delta^l_kU^{+i}_a,\theta\partial^i_kU^-_{la}=-\delta^i_lU^-_{ka}. \tag{19}$$

The operators $\partial^{+k},\partial^{++},\partial^-_k,\partial^{--}$ and ∂^i_k satisfy the commutation relations of the Lie algebra $SO(5)$.

One defines an ordinary complex conjugation on these harmonics

$$\overline{U^{+i}_a}=U^-_{ia},\theta\overline{U^0_a}=U^0_a, \tag{20}$$

however, it is convenient to use a special conjugation in the harmonic space

$$(U^{+i}_a)^\sim=U^{+i}_a,\theta(U^-_{ia})^\sim=U^-_{ia},\theta(U^0_a)^\sim=U^0_a. \tag{21}$$

All harmonics are real with respect to this conjugation.

The full superspace of the D=3, N=5 supersymmetry has the spinor CB coordinates $\theta^\alpha_a,\theta(\alpha=1,2;\theta a=1,2,3,4,5)$ in addition to the coordinates x^m of the three-dimensional Minkowski space.

The group $SL(2,R)\times SO(5)$ acts on the spinor coordinates. The superconformal transformations of these coordinates are considered in Appendix.

The $SO(5)/U(2)$ harmonics allow us to construct projections of the spinor coordinates and the partial spinor derivatives

$$\begin{aligned} &\theta^{+i\alpha}=U^{+i}_a\theta^\alpha_a, \theta\theta^{0\alpha}=U^0_a\theta^\alpha_a, \theta\theta^{-\alpha}_i=U^-_{ia}\theta^\alpha_a,\\ &\partial^-_{i\alpha}=\partial/\partial\theta^{+i\alpha}, \theta\partial^0_\alpha=\partial/\partial\theta^{0\alpha}, \theta\partial^{+i}_\alpha=\partial/\partial\theta^{-\alpha}_i. \end{aligned} \tag{22}$$

The analytic coordinates (AB-representation) in the full harmonic superspace use these projections of 10 spinor coordinates $\theta^{+i\alpha},\theta^{0\alpha},\theta^{-\alpha}_i$ and the following representation of the vector coordinate:

$$x^m_{\sigma A}\equiv y^m=x^m+i(\theta^{+k}\gamma^m\theta^-_k)=x^m+i(\theta_a\gamma^m\theta_b)U^{+k}_aU^-_{kb}. \tag{23}$$

The analytic coordinates are real with respect to the special conjugation.

The harmonic derivatives have the following form in AB:

$$\begin{aligned} &\mathcal{D}^{+k}=\partial^{+k}-i(\theta^{+k}\gamma^m\theta^0)\partial_m+\theta^{+k\alpha}\partial^0_\alpha-\theta^{0\alpha}\partial^{+k}_\alpha,\\ &\mathcal{D}^{++}=\partial^{++}+i(\theta^{+k}\gamma^m\theta^+_k)\partial_m+\theta^{+\alpha}_k\partial^{+k}_\alpha,\\ &\mathcal{D}^k_l=\partial^k_l+\theta^{+k\alpha}\partial^-_{l\alpha}-\theta^{-\alpha}_l\partial^{-k}_\alpha. \end{aligned} \tag{24}$$

We use the commutation relations

$$[\mathcal{D}^{+k},\mathcal{D}^{+l}]=-\varepsilon^{kl}\mathcal{D}^{++}, \theta\mathcal{D}^{+k}\mathcal{D}^+_k=\mathcal{D}^{++}. \tag{25}$$

The AB spinor derivatives are

$$\begin{aligned} &D^{+i}_\alpha=\partial^{+i}_\alpha, \theta D^-_{i\alpha}=-\partial^-_{i\alpha}-2i\theta^{-\beta}_i\partial_{\alpha\beta},\\ &D^0_\alpha=\partial^0_\alpha+i\theta^{0\beta}\partial_{\alpha\beta}. \end{aligned} \tag{26}$$

The coordinates of the analytic superspace $\zeta=(y^m,\theta^{+i\alpha},\theta^{0\alpha},U^K_a)$ have the Grassmann dimension 6 and dimension of the even space 3+6. The functions $\Phi(\zeta)$ satisfy the Grassmann analyticity condition in this superspace

$$D^{+k}_\alpha\Phi=0. \tag{27}$$

In addition to this condition, the analytic superfields in the $SO(5)/U(2)$ harmonic superspace possess also the $U(2)$-covariance. This subsidiary condition looks especially simple for the $U(2)$-scalar superfields

$$\mathcal{D}^k_l\Lambda(\zeta)=0. \tag{28}$$

The integration measure in the analytic superspace $d\mu^{(-4)}$ has dimension zero

$$d\mu^{(-4)}=dUd^3x_{\sigma A}(\partial^0_\alpha)^2(\partial^-_{i\alpha})^4=dUd^3x_{\sigma A}d\theta^{(-4)}. \tag{29}$$

The $SO(5)/U(1)\times U(1)$ harmonics can be defined via the components of the real orthogonal 5×5 matrix [20, 21]

$$U^K_a=\left(U^{(1,1)}_a,U^{(1,-1)}_a,U^{(0,0)}_a,U^{(-1,1)}_a,U^{(-1,-1)}_a\right) \tag{30}$$

where a is the SO(5) vector index and the index $K=1,2,\dots 5$ corresponds to given combinations of the U(1)×U(1) charges.

We use the following harmonic derivatives

$$\begin{aligned} &\partial^{(2,0)}=U^{(1,1)}_b\partial/\partial U^{(-1,1)}_b-U^{(1,-1)}_b\partial/\partial U^{(-1,-1)}_b,\\ &\partial^{(1,1)}=U^{(1,1)}_b\partial/\partial U^{(0,0)}_b-U^{(0,0)}_b\partial/\partial U^{(-1,-1)}_b,\\ &\partial^{(1,-1)}=U^{(1,-1)}_b\partial/\partial U^{(0,0)}_b-U^{(0,0)}_b\partial/\partial U^{(-1,1)}_b,\\ &\partial^{(0,2)}=U^{(1,1)}_b\partial/\partial U^{(1,-1)}_b-U^{(-1,1)}_b\partial/\partial U^{(-1,-1)}_b,\\ &\partial^{(0,-2)}=U^{(1,-1)}_b\partial/\partial U^{(1,1)}_b-U^{(-1,-1)}_b\partial/\partial U^{(-1,1)}_b. \end{aligned} \tag{31}$$

We define the harmonic projections of the $N{=}5$ Grassmann coordinates

$$\theta^K_\alpha = \theta_{a\alpha}U^K_a = (\theta^{(1,1)}_\alpha, \theta^{(1,-1)}_\alpha, \theta^{(0,0)}_\alpha, \theta^{(-1,1)}_\alpha, \theta^{(-1,-1)}_\alpha). \quad (32)$$

The $SO(5)/U(1)\times U(1)$ analytic superspace contains only spinor coordinates

$$\zeta = (x^m_{\sigma A}, \theta^{(1,1)}_\alpha, \theta^{(1,-1)}_\alpha, \theta^{(0,0)}_\alpha), \quad (33)$$
$$x^m_{\sigma A} = x^m + i\theta^{(1,1)}\gamma^m\theta^{(-1,-1)} + i\theta^{(1,-1)}\gamma^m\theta^{(-1,1)},$$
$$\delta_\epsilon x^m_{\sigma A} = -i\epsilon^{(0,0)}\gamma^m\theta^{(0,0)} - 2i\epsilon^{(-1,1)}\gamma^m\theta^{(1,-1)} - 2i\epsilon^{(-1,-1)}\gamma^m\theta^{(1,1)}, \quad (34)$$

where $\epsilon^{K\alpha} = \epsilon^\alpha_a U^K_a$ are the harmonic projections of the supersymmetry parameters.

General superfields in the analytic coordinates depend also on additional spinor coordinates $\theta^{(-1,1)}_\alpha$ and $\theta^{(-1,-1)}_\alpha$. The harmonized partial spinor derivatives are

$$\partial^{(-1,-1)}_\alpha = \partial/\partial\theta^{(1,1)\alpha}, \theta\partial^{(-1,1)}_\alpha = \partial/\partial\theta^{(1,-1)\alpha}, \theta\partial^{(0,0)}_\alpha = \partial/\partial\theta^{(0,0)\alpha}, \quad (35)$$
$$\partial^{(1,1)}_\alpha = \partial/\partial\theta^{(-1,-1)\alpha}, \theta\partial^{(1,-1)}_\alpha = \partial/\partial\theta^{(-1,1)\alpha}.$$

We use the special conjugation $\sim$ in the harmonic superspace

$$\widetilde{U^{(p,q)}_a} = U^{(p,-q)}_a, \theta\widetilde{\theta^{(p,q)}_\alpha} = \theta^{(p,-q)}_\alpha, \theta\widetilde{x^m_{\sigma A}} = x^m_{\sigma A},$$
$$(\theta^{(p,q)}_\alpha\theta^{(s,r)}_\beta)^\sim = \theta^{(s,-r)}_\beta\theta^{(p,-q)}_\alpha, \theta\widetilde{f(x_{\sigma A})} = \bar{f}(x_{\sigma A}), \quad (36)$$

where $\bar{f}$ is the ordinary complex conjugation. The analytic superspace is real with respect to the special conjugation.

The analytic-superspace integral measure contains partial spinor derivatives (35)

$$d\mu^{(-4,0)} = -\tfrac{1}{64}dUd^3x_{\sigma A}(\partial^{(-1,-1)})^2(\partial^{(-1,1)})^2(\partial^{(0,0)})^2 = dUd^3x_{\sigma A}d^6\theta^{(-4,0)}, \quad (37)$$
$$\int d^6\theta^{(-4,0)}(\theta^{(1,1)})^2(\theta^{(1,-1)})^2(\theta^{(0,0)})^2 = 1.$$

The harmonic derivatives of the analytic basis commute with the generators of the $N{=}5$ supersymmetry

$$\mathcal{D}^{(1,1)} = \partial^{(1,1)} - i\theta^{(1,1)}_\alpha\theta^{(0,0)}_\beta\partial^{\alpha\beta} - \theta^{(0,0)\alpha}\partial^{(1,1)}_\alpha + \theta^{(1,1)\alpha}\partial^{(0,0)}_\alpha,$$
$$\mathcal{D}^{(1,-1)} = \partial^{(1,-1)} - i\theta^{(1,-1)}_\alpha\theta^{(0,0)}_\beta\partial^{\alpha\beta} - \theta^{(0,0)\alpha}\partial^{(1,-1)}_\alpha + \theta^{(1,-1)\alpha}\partial^{(0,0)}_\alpha = -(\mathcal{D}^{(1,1)})^\dagger,$$
$$\mathcal{D}^{(2,0)} = [\mathcal{D}^{(1,-1)}, \mathcal{D}^{(1,1)}] = \partial^{(2,0)} - 2i\theta^{(1,1)}_\alpha\theta^{(1,-1)}_\beta\partial^{\alpha\beta} - \theta^{(1,-1)\alpha}\partial^{(1,1)}_\alpha + \theta^{(1,1)\alpha}\partial^{(1,-1)}_\alpha,$$
$$\mathcal{D}^{(0,2)} = \partial^{(0,2)} + \theta^{(1,1)\alpha}\partial^{(-1,1)}_\alpha - \theta^{(-1,1)\alpha}\partial^{(1,1)}_\alpha$$
$$\mathcal{D}^{(0,-2)} = -(\mathcal{D}^{(0,2)})^\dagger = \partial^{(-2,0)} + \theta^{(1,-1)\alpha}\partial^{(-1,-1)}_\alpha - \theta^{(-1,-1)\alpha}\partial^{(1,-1)}_\alpha.$$

It is useful to define the AB-representation of the U(1) charge operators

$$\mathcal{D}^0_1 A^{(p,q)} = p\,A^{(p,q)}, \theta\mathcal{D}^0_2 A^{(p,q)} = q\,A^{(p,q)}, \quad (38)$$

where $A^{(p,q)}$ is an arbitrary harmonic superfield in AB.

The spinor derivatives in the analytic basis are

$$D^{(-1,-1)}_\alpha = \partial^{(-1,-1)}_\alpha + 2i\theta^{(-1,-1)\beta}\partial_{\alpha\beta}, \theta D^{(-1,1)}_\alpha = \partial^{(-1,1)}_\alpha + 2i\theta^{(-1,1)\beta}\partial_{\alpha\beta},$$
$$D^{(0,0)}_\alpha = \partial^{(0,0)}_\alpha + i\theta^{(0,0)\beta}\partial_{\alpha\beta}, \theta D^{(1,1)}_\alpha = \partial^{(1,1)}_\alpha, \theta D^{(1,-1)}_\alpha = \partial^{(1,-1)}_\alpha. \quad (39)$$

4 $N = 6$ Chern-Simons theory in harmonic superspaces

The harmonic derivatives $\mathcal{D}^{+k}, \mathcal{D}^{++}$ together with the spinor derivatives D^{+k}_α determine the CR-structure of the harmonic $SO(5)/U(2)$ superspace. This $U(2)$-covariant CR-structure is invariant with respect to the N=5 supersymmetry. This CR-structure should be preserved in the superfield gauge theory.

The gauge superfields (prepotentials) $V^{+k}(\zeta)$ and $V^{++}(\zeta)$ in the harmonic $SO(5)/U(2)$ superspace satisfy the following conditions of the Grassmann analyticity and $U(2)$-covariance:

$$D^{+k}_\alpha V^{+k} = D^{+k}_\alpha V^{++} = 0, \theta \mathcal{D}^i_j V^{+k} = \delta^k_j V^{+i}, \theta \mathcal{D}^i_j V^{++} = \delta^i_j V^{++}. \quad (40)$$

In the gauge group $SU(n)$, these traceless matrix superfields are anti-Hermitian

$$(V^{+k})^\dagger = -V^{+k}, \theta (V^{++})^\dagger = -V^{++}, \quad (41)$$

where operation $\dagger$ includes the transposition and $\sim$-conjugation.

Analytic superfield parameters of the gauge group $SU(n)$ satisfy the conditions of the generalized CR analyticity

$$D^{+k}_\alpha \Lambda = \mathcal{D}^i_j \Lambda = 0, \quad (42)$$

they are traceless and anti-Hermitian $\Lambda^\dagger = -\Lambda$.

We treat these prepotentials as connections in the covariant gauge derivatives

$$\begin{aligned}
&\nabla^{+i} = \mathcal{D}^{+i} + V^{+i}, \theta \nabla^{++} = \mathcal{D}^{++} + V^{++},\\
&\delta_\Lambda V^{+i} = \mathcal{D}^{+i}\Lambda + [\Lambda, V^{+i}], \theta \delta_\Lambda V^{++} = \mathcal{D}^{++}\Lambda + [\Lambda, V^{++}],\\
&D^{+k}_\alpha \delta_\Lambda V^{+k} = D^{+k}_\alpha \delta_\Lambda V^{++} = 0, \theta \mathcal{D}^i_j \delta_\Lambda V^{+k} = \delta^k_j \delta_\Lambda V^{+i}, \theta \mathcal{D}^i_j \delta_\Lambda V^{++} = \delta^i_j \delta_\Lambda V^{++},
\end{aligned} \quad (43)$$

where the infinitesimal gauge transformations of the gauge superfields are defined. These covariant derivatives commute with the spinor derivatives D^{+k}_α and preserve the CR-structure in the harmonic superspace.

We can construct three analytic superfield strengths off the mass shell

$$\begin{aligned}
F^{++} &= \frac{1}{2}\varepsilon_{ki}[\nabla^{+i}, \nabla^{+k}] = V^{++} - \mathcal{D}^{+k} V^+_k - V^{+k} V^+_k,\\
F^{(+3)k} &= [\nabla^{++}, \nabla^{+k}] = \mathcal{D}^{++} V^{+k} - \mathcal{D}^{+k} V^{++} + [V^{++}, V^{+k}].
\end{aligned}$$

The superfield action in the analytic $SO(5)/U(2)$ superspace is defined on three prepotentials V^{+k} and V^{++} by analogy with the off-shell action of the SYM^3_4 theory [12]

$$S_1 = \tfrac{ik}{12\pi} \int d\mu^{(-4)} \mathrm{Tr}\{V^{+j}\mathcal{D}^{++}V^+_j + 2V^{++}\mathcal{D}^+_j V^{+j} + (V^{++})^2 + V^{++}[V^+_j, V^{+j}]\}, \quad (44)$$

where k is the coupling constant, and a choice of the numerical multiplier guarantees the correct normalization of the vector-field action. This action is invariant with respect to the infinitesimal gauge transformations of the prepotentials (43). The idea of construction of the superfield action in the harmonic $SO(5)/U(2)$ was proposed in [18], although the detailed construction of the superfield Chern-Simons theory was not discussed in this work. The equivalent superfield action was considered in the framework of the alternative superfield formalism [20]. The superconformal $N = 5$ invariance of this action was proven in [19].

The action S_1 yields superfield equations of motion which mean triviality of the superfield strengths of the theory

$$\begin{aligned}
&F^{(+3)}_k = \mathcal{D}^{++} V^+_k - \mathcal{D}^+_k V^{++} + [V^{++}, V^+_k] = 0,\\
&F^{++} = V^{++} - \mathcal{D}^{+k} V^+_k - V^{+k} V^+_k = 0.
\end{aligned} \quad (45)$$

These classical superfield equations have pure gauge solutions for the prepotentials only

$$V^{+k} = e^{-\Lambda}\mathcal{D}^{+k}e^\Lambda, \theta V^{++} = e^{-\Lambda}\mathcal{D}^{++}e^\Lambda, \quad (46)$$

where Λ is an arbitrary analytic superfield.

The transformation of the sixth supersymmetry can be defined on the analytic N=5 superfields

$$\delta_6 V^{++} = \epsilon_6^\alpha D_\alpha^0 V^{++}, \theta\delta_6 V^{+k} = \epsilon_6^\alpha D_\alpha^0 V^{+k}, \quad (47)$$
$$\delta_6 \mathcal{D}^{+k} V^{+l} = \epsilon_6^\alpha D_\alpha^0 \mathcal{D}^{+k} V^{+l}, \theta\delta_6 \mathcal{D}^{++} V^{+l} = \epsilon_6^\alpha D_\alpha^0 \mathcal{D}^{++} V^{+l}, \quad (48)$$

where ϵ_6^α are the corresponding odd parameters. This transformation preserves the Grassmann analyticity and $U(2)$-covariance

$$\{D_\alpha^0, D_\beta^{+k}\} = 0, \theta[\mathcal{D}_l^k, D_\alpha^0] = 0, \theta[\mathcal{D}^{+k}, D_\alpha^0] = D_\alpha^{+k}, \theta[\mathcal{D}^{++}, D_\alpha^0] = 0. \quad (49)$$

The action S_1 is invariant with respect to this sixth supersymmetry

$$\delta_6 S_1 = \int d\mu^{(-4)} \epsilon_6^\alpha D_\alpha^0 L^{(+4)} = 0. \quad (50)$$

In the $SO(5)/U(1)\times U(1)$ harmonic superspace, we can introduce the D=3, N=5 analytic matrix gauge prepotentials corresponding to the five harmonic derivatives

$$V^{(p,q)}(\zeta, U) = [V^{(1,1)}, \theta V^{(1,-1)}, \theta V^{(2,0)}, \theta V^{(0,\pm 2)}],$$
$$(V^{(1,1)})^\dagger = -V^{(1,-1)}, \theta(V^{(2,0)})^\dagger = V^{(2,0)}, \theta V^{(0,-2)} = [V^{(0,2)}]^\dagger, \quad (51)$$

where the Hermitian conjugation $\dagger$ includes $\sim$ conjugation of matrix elements and transposition.

We shall consider the restricted gauge supergroup using the supersymmetry-preserving harmonic (H) analyticity constraints on the gauge superfield parameters

$$H1: \qquad \mathcal{D}^{(0,\pm 2)}\Lambda = 0. \quad (52)$$

These constrains yield additional reality conditions for the component gauge parameters.

We use the harmonic-analyticity constraints on the gauge prepotentials

$$H2: \qquad V^{(0,\pm 2)} = 0, \theta\mathcal{D}^{(0,-2)} V^{(1,1)} = V^{(1,-1)}, \theta\mathcal{D}^{(0,2)} V^{(1,1)} = 0 \quad (53)$$

and the conjugated constraints combined with relations (51).

The superfield CS action can be constructed in terms of these H-constrained gauge superfields [21]

$$S = -\tfrac{2ik}{12\pi} \int d\mu^{(-4,0)} \mathrm{Tr}\{V^{2,0}\mathcal{D}^{(1,-1)} V^{(1,1)} + V^{1,1}\mathcal{D}^{(2,0)} V^{(1,-1)}$$
$$+ V^{1,-1}\mathcal{D}^{(1,1)} V^{(2,0)} + V^{2,0}[V^{(1,-1)}, V^{(1,1)}] - \frac{1}{2} V^{(2,0)} V^{(2,0)}\}. \quad (54)$$

Note, that the similar harmonic superspace based on the $USp(4)/U(1) \times U(1)$ harmonics was used in [22] for the harmonic interpretation of the $D = 4, N = 4$ super Yang-Mills constraints.

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2001–2004	Grant of the European Community (INTAS), Tomsk team leader in framework of International project.
2003–2005	Grant of the Russian Foundation for Basic Research Group leader.
2003–2005	President grant for Leading Scientific Schools, Head of research collective.
2004–2007	Grant of the European Community (INTAS), Tomsk team leader in framework of International project.

2005–2008	Joint grant of the Russian Foundation for Basic Research and the German Research Foundation (DFG), together with Joint Institute for Nuclear Research, Tomsk team leader in framework of International project.
2006–2007	President grant for Leading Scientific Schools, Head of research collective.
2006–2008	Grant of the Russian Foundation for Basic Research, Group leader.
2006–2008	Grant of the European Community (INTAS), Tomsk team leader in framework of International project.
2008–2009	President grant for Leading Scientific Schools, Head of research collective.

HONORS AND AWARDS:

1994–1999	Soros Professor (title given by the International Science Foundation)
1995	Named "Honored Scientist of the Russian Federation"
1996	Acting Member of the International Academy of Higher Schools
1997–2003	Presidential Grant for Eminent Scientists of Russia
2000	Laureate of Tomsk Region Prize in the area of education and science
2000	Medal “"For Services to the Homeland"” of the II degree
2004	Finalist of Tomsk Region Competition “"Man of the Year"” in the “"Scientist of the Year"” category
2006	Finalist of Tomsk Region Competition “"Man of the Year"” in the “"Scientist of the Year"” category
2007	Laureate of Tomsk Region Prize in the area of education and science

VISITS FOR JOINT RESEARCH AND LECTURING:

1992	Visiting Scientist, Trinity College and Department of Applied Mathematics and Theoretical Physics University of Cambridge, Cambridge, UK
1993	Visiting Scientist, Institute of Theoretical Physics, Chalmers University of Technology Geteborg, Sweden
1995	Visiting Scientist, Institute of Physics Humboldt Berlin University, Berlin, Germany
1996	Visiting Scientist, Department of Physics University of Pennsylvania, Philadelphia, PA, USA
1997, 1998	Visiting Scientist, Institute of Physics Humboldt Berlin University, Berlin, Germany
1999	Visiting Scientist, Newton Institute for Mathematical Sciences, Cambridge, UK
1999–2000	Visiting Professor, Institute of Physics University of Sao Paulo, Brazil
2001	Visiting Scientist, Institute of Physics, Humboldt Berlin University, Germany
2001	Visiting Scientist, Department of Physics and Astronomy University of Maryland, USA
2001	Visiting Scientist, Department of Physics University of Rome II, Italy
2002	Visiting Scientist, Institute of Physics Humboldt Berlin University, Germany

2002	Visiting Scientist, Spinoza Institute for Theoretical Physics, Utrecht University, Netherlands
2002	Visiting Scientist, Department of Physics and Astronomy University of Maryland, USA
2003	Visiting Scientist, Institute of Physics Humboldt Berlin University, Germany
2003	Visiting Scientist, Institute of Theoretical Physics Hannover University, Germany
2004	Visiting Scientist, National Physical Laboratory Frascati, Italy
2005	Visiting Professor, Trinity College and Department of Applied Mathematics and Theoretical Physics University of Cambridge, UK
2006	Visiting Scientist, Department of Physics Crete University, Greece
2006	Visiting Professor, Department of Physics University of Juiz de Fora, Brazil
2007	Visiting Scientist, Department of Physics University of Western Australia, Crawley, Australia
2007	Visiting Professor, Department of Chemistry and Physics University of North Carolina at Pembroke, USA
2008	Visiting Scientist, Institute of Theoretical Physics Hannover University, Germany

Number of Main Publications: 238
Number of citations (1986 – 2006): 3042

Published Books:

I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro. Effective Action in Quantum Gravity, IOP Publishing, Bristol and Philadelphia, 1992, 413 pages

I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity, IOP Publishing, Bristol and Philadelphia, 1995, 640 p.; second edition, revised and supplemented, 1998, 656 p.

Supervision of 18 PhD theses

LECTURE COURSES TAUGHT (at different time):

Mathematical Analysis, Differential Equations, Theory of Probability, Classical Mechanics, Electrodynamics, Quantum Mechanics, Statistical Physics, Nuclear Physics, Elementary Particles Physics, Mathematical Physics, General Relativity, Quantum Field Theory, Supersymmetry, Group Theory.

ORGANIZING ACTIVITY:

Member of Committee on Physical Education in Pedagogical Universities

at Russian Ministry of Education and Science.
Member of editorial board of the International Journal
“”Gravity and Cosmology””
Chairman and Member of Organizing Committees of various
International Scientific Conferences

Reviewer of International Scientific Journals:
Physical Review D,
Nuclear Physics B,
Physics Letters B,
Classical and Quantum Gravity,
Europhysics Letters,
Journal of Physics A,
International Journal of Modern Physics A,
Modern Physics Letters A,
Journal of High Energy Physics

Publication List of I.L. Buchbinder

Books

1. I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro.
   EFFECTIVE ACTION IN QUANTUM GRAVITY.
   IOP Publishing, Bristol and Philadelphia, 1992, 413 pages.
2. I.L. Buchbinder, S.M. Kuzenko.
   IDEAS AND METHODS OF SUPERSYMMETRY AND SUPERGRAVITY OR A WALK THROUGH SUPERSPACE.
   IOP Publishing, Bristol and Philadelphia, 1995, 640 pages, Revised Edition, 1998, 656 pages.

Review papers

1. I.L. Buchbinder. Progress in study of $\mathcal{N} = 4$ SYM effective action.
   Proceedings of International Workshop "Supersymmetries and Quantum Symmetries (SQS'03)" Dubna, Russia, July 24-29, 2003, ed. by E. Ivanov and A. Pashnev, Dubna - 2004, pp. 290 - 301.
2. E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut. Low-Energy Effective Action in $\mathcal{N} = 2$ Supersymmetric Field Theories.
   Physics of Particles and Nuclei, Vol. 32, No 5, pp. 641 - 674, 2001
3. I. Buchbinder, S. Kuzenko, B. Ovrut. Covariant Harmonic Supergraphity for $\mathcal{N} = 2$ Super Yang-Mills Theories.
   Supersymmetries and Local Symmetries, Proceedings of the International Seminar Dedicated to Victor I. Ogievetsky, ed. by E. Ivanov and J. Wess, Dubna, Russia, 22 - 26 July, 1997. Springer, 1999, pp. 21 - 36.
4. I.L. Buchbinder. Introduction to Perturbative Einstein Gravity.
   Lecture Notes in Theoretical and Mathematical Physics, Vol. 1, part 1, pp. 195-255. Kazan State University Publ., 1996.
5. I.L. Buchbinder, D.M. Gitman, E.S. Fradkin. Quantum Electrodynamics in Curved Space-Time.
   Proceedings of P.N. Lebedev Physical Institute 1990, Vol. 201, pp. 33-73.

6. I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro. Renormalization Group Approach to Quantum Field Theory in Curved Space - Time.
   Rivista dell'a Nuovo Cimento, Vol. 12, N 10, pp. 1 - 112, 1989.
7. I.L. Buchbinder, S.D. Odintsov. Effective Action in Multidimensional (Super)Gravities and Spontaneous Compactification (Quantum Aspects of Kaluza - Klein Theories).
   Fortschritte der Physik, Vol. 37, N 4, pp. 225 - 259, 1989.
8. I.L. Buchbinder. Renormalization of Quantum Field Theory in Curved Space - Time and Renormalization Group Equations.
   Fortschritte der Physik, Vol. 34, N 9, pp. 605 - 628, 1986.

Popular articles

1. I.L. Buchbinder. String theory and unification of fundamental interactions.
   Soros Educational Journal (in Russian), Vol. 7, No 7, pp. 95-101, 2001
2. I.L. Buchbinder. Introducing Supermathematics.
   Soros Educational Journal (in Russian), Vol. 4, No 8, pp. 115-120, 1998
3. I.L. Buchbinder. Fundamental Interactions.
   Soros Educational Journal (in Russian), Vol. 3, No 5, pp. 66-73, 1997

Scientific papers

1. I.L. Buchbinder, V.A. Krykhtin, H. Takata. Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields.
   Physics Letters, Vol. B656, No 4-5, pp. 253 - 264, 2007
2. I.L. Buchbinder, V.A. Krykhtin, A.A. Reshetnyak. BRST approach to Lagrangian construction for fermionic higher spin fields in AdS space.
   Nuclear Physics, Vol. B787, No 3, pp. 211-240, 2007
3. I.L. Buchbinder, P.M. Lavrov. Classical Becchi-Rouet-Stora-Tyutin charge for nonlinear algebras.
   Journal of Mathematical Physics, Vol. 48, No 8, pp. 082306-1 – 082306-15, 2007
4. I.L. Buchbinder, A.V. Galajinsky, V.A. Krykhtin. Quartet unconstrained formulation for massless higher spin fields.
   Nuclear Physics, Vol. B779, No 3, pp. 155 – 177, 2007
5. I.L. Buchbinder, G. de Berredo-Peixoto, I.L. Shapiro. Quantum effects in softly broken gauge theories in curved space-time.
   Physics Letters, Vol. B649, No 5-6, pp. 454 – 462, 2007.
6. I.L. Buchbinder, N.G. Pletnev. Hypermultiplet dependence of one-loop effective action in the $\mathcal{N} = 2$ superconformal theories.
   Journal of Higher Energy Physics, No 0704, pp. 096-0 – 096-31, 2007.
7. I.L. Buchbinder, P.M. Lavrov, V.D. Krykhtin. Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space.
   Nuclear Physics, Vol. B762, No 3, pp. 344 – 376, 2007

8. I.L. Buchbinder, A. Fotopoulos, A.C. Petkou, M. Tsulaia. Constructing the cubic interaction vertex of higher spin gauge fields.
   Physical Review, Vol. D74, No 10, pp. 105018-1 – 105018-16, 2006
9. I.L. Buchbinder, O. Lechtenfeld, I.B. Samsonov. Vector multiplet effective action in the non-anticommutative charged hypermultiplet model.
   Nuclear Physics, Vol. B758, No 1-2, pp. 185 – 203, 2006
10. I.L. Buchbinder, V.A. Krykhtin, L.L. Ryskina, H. Takata. Gauge invariant Lagrangian construction for massless higher spin fermionic fields.
    Physics Letters, Vol. B641, No 5, pp. 386 – 392, 2006
11. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. On quantum properties of the four-dimensional generic chiral superfield model.
    Physical Review, Vol. D74, No 4, pp. 045010-1 – 045010-10, 2006
12. I.L. Buchbinder, E.A. Ivanov, O. Lechtenfeld, I.B. Samsonov, B.M. Zupnik. Renormalization of $\mathcal{N} = (1,1)$ nonanticommutative theories with singlet deformation.
    Nuclear Physics, Vol. B740, No 3, pp. 358 – 385, 2006
13. O.D. Azorkina, A.T. Banin, I.L. Buchbinder, N.G. Pletnev. One-loop effective potential in $\mathcal{N} = 1/2$ nonanticommutative generic chiral superfield model.
    Physics Letters, Vol. B635, No 1, pp. 50 – 55, 2006
14. O.D. Azorkina, A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Construction of the effective action in nonanticommutative supersymmetric field theories.
    Physics Letters, Vol.B633, No 2-3, pp. 389 – 396, 2006
15. I.L. Buchbinder, V.A. Krykhtin. Gauge invariant Lagrangian construction for massive bosonic higher-spin fields in D-dimensions.
    Nuclear Physics, Vol. B727, No 3, pp. 537 – 563, 2005
16. I.L. Buchbinder, N.G. Pletnev. Construction of one-loop $\mathcal{N} = 1/2$ SYM effective action in the harmonic superspace approach.
    Journal of High Energy Physics, No 0905, pp. 073-0 – 073-32, 2005
17. O.D. Azorkina, A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Generic Chiral Superfield Model on Non-Anticommutative $\mathcal{N} = 1/2$ Superspace.
    Modern Physics Letters, Vol. A20, No 19, pp. 1423 - 1436, 2005
18. I.L. Buchbibder, S.J. Gates, S.M. Kuzenko, J. Phillips. Massive 4D, $\mathcal{N} = 1$ Superspin 1 and 3/2 Multiplets and Duality.
    Journal of High Energy Physics, No 0502, pp. 056-0 – 056-14, 2005
19. I.L. Buchbinder, V.A. Krykhtin, A. Pashnev. BRST approach to Lagrangian construction for fermionic massless higher spin fields.
    Nuclear Physics, Vol. B711, No 1-2, pp. 367 - 391, 2005
20. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Chiral effective potential in $\mathcal{N} = 1/2$ non-commutative Wess-Zumino model.
    Journal of High Energy Physics, No 0407, pp. 011-0 – 011-29, 2004
21. I.L. Buchbinder, S.J. Gates, Hock-Seng Goh, W.D. Linch, M.A. Luty, Siew-Phang Ng, J. Phillips. Supergravity loop contributions to brane world supersymmetry breaking.
    Physical Review, Vol. D70, No 2, pp. 025009-1 – 025009-6, 2004
22. S. Bellucci, I.L. Buchbinder, V.A. Krykhtin. Renormalization of energy-momentum tensor in noncommutative complex scalar field theory.
    Nuclear Physics, Vol. B693, No 1-3, pp. 57 - 75, 2004

23. I.L. Buchbinder, E.A. Ivanov, I.B. Samsonov, B.M. Zupnik. Scale invariant low-energy effective action in $\mathcal{N} = 3$ SYM theory.
Nuclear Physics, Vol. B689, No 1-2, pp. 108 - 126, 2004
24. X. Bekaert, I.L. Buchbinder, A. Pashnev, M. Tsulaia. On higher spin theory: strings, BRST, dimensional reduction.
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25. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. One-loop effective action for $\mathcal{N} = 4$ SYM theory in the hypermultiplet sector: leading low-energy approximation and beyond.
Physical Review, Vol. D68, No 6, pp. 065024-1 – 065024-13, 2003
26. S. Bellucci, I.L. Buchbinder, V.A. Krykhtin. Renormalization of the energy-momentum tensor in non-commutative scalar field theory.
Nuclear Physics, Vol. B665, No 1, pp. 402-424, 2003
27. I.L. Buchbinder, V.A. Krykhtin. One-loop renormalization of general noncommutative Yang-Mills field model coupled to scalar and spinor fields.
International Journal of Modern Physics, Vol. A18, No 17, pp. 3057-3088, 2003
28. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Effective action in $\mathcal{N} = 2, 4$ supersymmetric Yang-Mills theories.
Gravitation and Cosmology, Vol. 9, No 1-2, pp. 2-10, 2003
29. I.L. Buchbinder, E.A. Ivanov, A.Yu. Petrov. Complete low-energy effective action in $\mathcal{N} = 4$ SYM: a direct $\mathcal{N} = 2$ supergraph calculation.
Nuclear Physics, Vol. B653, No 1-2, pp. 64-84, 2003
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International Journal of Theoretical Physics, Vol. 41, No 4, pp. 753-790, 2002
31. I.L. Buchbinder, S. James Gates Jr., W.D. Linch III, J. Phillips. Dynamical Theory of free massive superspin-1 multiplet.
Physics Letters, Vol. B549, No 1-2, pp. 229-236, 2002
32. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Low-energy effective action in $\mathcal{N} = 2$ super Yang-Mills theories on a non-Abelian background.
Physical Review, vol. D66, No 4, pp. 045021-1 – 045021-13, 2002
33. I.L. Buchbinder, I.B. Samsonov. Nocommutative $\mathcal{N} = 2$ Supersymmetric Theories in Harmonic Superspace.
Gravitation and Cosmology, Vol. 8, No 1/2, pp. 17-30, 2002
34. I.L. Buchbinder, A. Pashnev, M. Tsulaia. Massless Higher Spin Fields in the AdS Background and BRST Construction for Nonlinear Algebras.
Proceedings of XVI Max Born Symposium "Supersymmetries and Quantum Symmetries", Karpacz, Poland, 21-25 September, 2001, Dubna 2002, pp. 3-10.
35. I.L. Buchbinder, S.J. Gates, W.D. Linch, J. Phillips. New $4D, N = 1$ superfield theory: model of free massive superspin- 3/2 multiplet.
Physics Letters, Vol. B535, No 1-4, pp. 280 - 288, 2002
36. I.L. Buchbinder, E.A. Ivanov. Complete $\mathcal{N} = 4$ structure of low-energy effective action in N=4 super Yang-Mills theories.
Physics Letters, Vol. B524, No 1-2, pp. 208 - 216, 2002
37. I.L. Buchbinder, A.Yu. Petrov, A.A. Tseytlin. Two-loop N=4 super Yang-Mills effective action and interaction between D3-branes.
Nuclear Physics, Vol. B621, No 1-2, pp. 179 - 207, 2002

38. I.L. Buchbinder. Construction of low-energy effective action in N=4 super Yang-Mills theories.
    Proceedings of the IX International Conference "Supersymmetry and Unification of Fundamental Interactions". World Scientific, 2002, pp. 337 - 344.
39. I.L. Buchbinder. Low-Energy Effective Action in N=4 Super Yang-Mills Theories.
    Proceedings of the International Conference "Quantization, Gauge Theory and Strings", World Scientific, 2001, Vol. 2, pp. 162 - 170.
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    Physics Letters, Vol. B523, No 3,4 pp. 338 - 346, 2001
41. I.L. Buchbinder, M. Gomes, A.Yu. Petrov, V.O. Rivelles. Superfield Effective Action in the Noncommutative Wess-Zumino Model.
    Physics Letters, Vol. B517, No 1-2, pp. 191 - 202, 2001
42. A.T. Banin, I.L. Buchbinder, N.G. Pletnev. Effective Action of N=2 Gauge Multiplet Induced by Hypermultiplet Matter.
    Nuclear Physics, Vol. B598, No 102, pp. 371 - 399, 2001
43. I.L. Buchbinder, V.D. Pershin. Gravitational Interaction of Higher Spin Massive Fields and String Theory.
    Proceedings of the International Conference "Geometrical Aspects of Quantum Fields", World Scientific, 2001, pp. 11 - 30
44. I.L. Buchbinder, A.Yu. Petrov. N=4 Super Yang-Mills Low-Energy Effective Action at Three and Four Loops.
    Physics Letters, Vol. B482, No 4, pp. 429 - 439, 2000
45. I.L. Buchbinder, I.B. Samsonov. The Holomorphic Effective Action in N=2 D=4 Supergauge Theories with Variouis Gauge Groups.
    Teoreticheskaya i Matematicheskaya Fisica (Theoretical and Mathematical Physics), Vol. 122, No 3, pp 444-455, 2000
46. I.L. Buchbinder. Structure of Low-Energy Effective Action in N=4 Supersymmetric Yang-Mills Theories.
    Preprint, hep-th/0001142, 11 pp., 2000. Talk given at International Workshop "Supersymmetries and Quantum Symmetries", Dubna, Russia, July 21-31, 1999, published in the Proceedings.
47. I.L. Buchbinder, S.M. Kuzenko, A.A. Tseytlin. On Low-Energy Effective Actions in N = 2, 4 Superconformal Theories in Four Dimensions.
    Physical Review, Vol. D62, No 4, 045001, 2000 (9 pages)
48. I.L. Buchbinder, D.M. Gitman, V.A. Krykhtin, V.D. Pershin. Equations of Motion for Massive Spin 2 Field Coupled to Gravity.
    Nuclear Physics, Vol. B584, No 1-2, pp. 615-640, 2000
49. I.L. Buchbinder, I.B. Samsonov. On Holomorphic Effective Actions of Hypermultiplets Coupled to External Gauge Superfields.
    Modern Physics Letters, Vol. 14A, No 36, pp. 2537 - 2544, 1999.
50. I.L. Buchbinder, M. Cvetic, A.Yu. Petrov. One-loop effective potential of N=1 supersymmetric theory and decoupling effects.
    Nuclear Physics, Vol. 571B, No 1-2, pp. 358 - 418, 2000
51. I.L. Buchbinder, M. Cvetic, A.Yu. Petrov. Implications of decoupling effects for one-loop corrected effective actions from superstring theory.
    Modern Physics Letters, Vol. 15A, No 11-12, pp. 783 - 790, 2000

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Proceedings of the 32nd International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, Germany, Sept. 1-5, 1998. Fortshritte der Physik, Vol. 48, No 1-3, pp. 45-48, 2000.

53. I.L. Buchbinder, V.A. Krykhtin, V.D. Pershin. Consistent equations for massive spin-2 field coupled to gravity from string theory.
Physics Letters, Vol. B466, No 2-4, pp. 216 - 226, 1999.

54. I.L. Buchbinder, A.Yu. Petrov. Holomorphic effective potential in general chiral superfield model.
Physics Letters, Vol. B461, No 3, pp. 209 - 217, 1999.

55. E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko. Non-Holomorphic Effective Potential in N=4 SU(n) SYM.
Physics Letters 1999, Vol. B446, No 3-4, pp. 216 - 223.

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Physics Letters, Vol. B433, No 3-4, pp. 335 - 345, 1998.

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Modern Physics Letters, Vol. A13, No 13, pp. 1071 - 1082, 1998.

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Physics Letters, Vol. B417, No 1-2, pp. 61 - 71, 1998.

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63. I.L. Buchbinder, B.R. Mistchuk, V.D. Pershin. Canonical Analysis of Quantum Theory of Bosonic String in Massless Background Fields.
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Modern Physics Letters, Vol. A12, N 30, pp. 2271 - 2278, 1997.

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Physics Letters, Vol. B412, N 3-4, pp. 309 - 319, 1997.

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Nuclear Physics B (Proceedings, Supplements), Vol. 57, pp. 280 - 283, 1997.
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Classical and Quantum Gravity, Vol. 13, N 8, pp. 2081 - 2096, 1996.
75. I.L. Buchbinder, A.Yu. Petrov. On Quantum Model of Supergravity Compensator.
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