Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

# Generalized Proca action for an Abelian vector field

**Erwan Allys,**[a,b] **Patrick Peter**[a,b] **and Yeinzon Rodríguez**[c,d,e]

[a]Institut d'Astrophysique de Paris, UMR 7095,
98 bis bd Arago, 75014 Paris, France
[b]UPMC Université Paris 6 et CNRS,
Sorbonne Universités, Paris, France
[c]Centro de Investigaciones en Ciencias Básicas y Aplicadas, Universidad Antonio Nariño,
Cra 3 Este # 47A-15, Bogotá D.C. 110231, Colombia
[d]Escuela de Física, Universidad Industrial de Santander,
Ciudad Universitaria, Bucaramanga 680002, Colombia
[e]Simons Associate at The Abdus Salam International Centre for Theoretical Physics,
Strada Costiera 11, I-34151, Trieste, Italy

E-mail: allys@iap.fr, peter@iap.fr, yeinzon.rodriguez@uan.edu.co

Received November 11, 2015
Accepted January 11, 2016
Published February 1, 2016

**Abstract.** We revisit the most general theory for a massive vector field with derivative self-interactions, extending previous works on the subject to account for terms having trivial total derivative interactions for the longitudinal mode. In the flat spacetime (Minkowski) case, we obtain all the possible terms containing products of up to five first-order derivatives of the vector field, and provide a conjecture about higher-order terms. Rendering the metric dynamical, we covariantize the results and add all possible terms implying curvature.

**Keywords:** gravity, modified gravity, particle physics - cosmology connection

**ArXiv ePrint:** 1511.03101



doi:10.1088/1475-7516/2016/02/004

## Contents





## 1 Introduction

Modifying general relativity in order to account for otherwise unresolved issues like, e.g., the cosmological constant or even dark matter, has become common practice lately. One way of doing so, following the scalar-tensor proposal of the 1930s, is the so-called galileon method, which provides a means to write down the most general theory allowing for Ostrogradski-instability-free [1, 2] second-order equations of motion [3–7], which happens to be equivalent [8], in the single-field case[1] and in four dimensions, to that proposed much earlier by Horndeski [9]. The latter author, indeed, had generalized this idea to what, today, should appropriately be named "vector Galileon", an Abelian vector field, with an action including sources, and with the assumption of recovering the Maxwell equations in flat spacetime [10]. Related works were developed recently that extended the applicability of the word "vector Galileon": not invoking gauge invariances but having several vector fields and sticking to purely second-order field equations [11], coupling an Abelian vector field with a scalar field in the framework of Einstein gravity [12], or invoking an Abelian gauge invariance for just one vector field in flat spacetime and sticking to purely second-order field equations [13]. The latter work became a no-go theorem that, however, is not the end of the vector Galileon

[1]It is a subset in the multi-field case.

theories: it suffices to drop the U(1) invariance hypothesis to obtain more terms in a non-trivial and viable theory. This procedure thus naturally generalizes the Proca theory for a massive vector field in the sense that it describes a vector field with second-order equations of motion for three propagating degrees of freedom, as required for the finite-dimensional representation of the Lorentz group leading to a massive spin 1 field.

In previous works [14, 15], such a theory was elaborated, introducing some fruitful ideas. A thorough examination of these works revealed that some terms that have been proposed are somehow redundant, while others appear to be missing. The purpose of the present article is thus to complete these pioneering works, investigating in a hopefully exhaustive way all the terms which can be included in such a generalized Proca theory under our assumptions.[2] The most crucial constraint in the building up of the theory is the demand that only three polarizations should be able to propagate, namely two transverse and one longitudinal, the latter being reducible to the scalar formulation of the Galileon, thus providing the required link between the vector and scalar theories. On the other hand, considering the transverse modes, we found numerous differences between the two theories, as in particular we obtained many more terms in the vector case. Indeed, whereas the scalar Lagrangian comprises a finite number of possibilities, we conjecture that the vector one is made up of an infinite tower of such terms.

The structure of the rest of the article goes as follows. First, we recall previous works by introducing the generalized Abelian Proca theory, and explicit the assumptions under which our Lagrangian is constructed. Recalling as a starting point the terms already obtained in refs. [14] and [15], we use their results to understand what kind of other terms could also be compatible with our assumptions. We then move on to these extra terms in section 3, first introducing a general method of investigation of these new terms, then applying the latter to terms containing products of up to five first-order derivatives of the vector field. We conclude this part by discussing what the complete generalized Proca theory could be. Finally, in section 4, we consider the extension of the Proca theory to a curved spacetime. We need to add yet more terms necessary to render the Lagrangian healthy (in the sense defined above), and this also leads to new interactions with gravity.

## 2 Generalized Abelian Proca theory

The Proca theory describes the dynamics of a vector field with a mass term in its Lagrangian, thus explicitly breaking the U(1) gauge invariance usually associated with the Maxwell field $A_\mu$. The Proca action therefore reads

$$\mathcal{S}_{\text{Proca}} = \int \mathcal{L}_{\text{Proca}}\, \mathrm{d}^4x = \int \left( -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m_A^2 X \right) \mathrm{d}^4x, \tag{2.1}$$

where the antisymmetric Faraday tensor is $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and we have defined, for further convenience, the shorthand notation $X \equiv A_\mu A^\mu$. Following and extending ref. [14], we want to provide the broadest possible generalization of this action under a set of specific assumptions to which we now turn.

Before we move on to curved spacetime in section 4, we shall in what follows consider that our field lives in a non-dynamical Minkowski metric $g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

[2] Some general works on general ghost-free Lagrangians published recently, see refs. [16, 17], can also be applied to the generalization of the Proca theory.



JCAP02(2016)004

We shall also make use of the notation $(\partial \cdot A) = \partial_\mu A^\mu$ for simplification and notational convenience.

### 2.1 Theoretical assumptions

The idea is to add to the minimal Proca action in eq. (2.1) all acceptable terms containing not only functions of $X$ but also derivative self-interactions under a set of suitable conditions. In order to explicit those conditions, let us split the vector into a scalar/vector decomposition

$$A_\mu = \partial_\mu \pi + \bar{A}_\mu \quad \text{or} \quad A = \mathrm{d}\pi + \bar{A}, \tag{2.2}$$

where $\pi$ is a scalar field, called the Stückelberg field, and $\bar{A}_\mu$ is a divergence-free vector ($\partial_\mu \bar{A}^\mu = 0$), containing the curl part of the field, i.e. that for which the Faraday tensor is non vanishing (in differential form terms, it is the non-exact part of the form, with the Faraday form now equal to $F = \mathrm{d}A = \mathrm{d}\bar{A}$). The conditions we want to impose on the theory in order that it makes (classical) sense are reminiscent of the galileon conditions, namely we demand

a) at most second-order equations of motion for all physical degrees of freedom, i.e., for both $A_\mu$ and $\pi$,

b) at most second-order derivative terms in $\pi$ in the action, and first-order derivative terms for $A_\mu$,

c) only three propagating degrees of freedom for the vector field; in other words, there should be no propagation of its zeroth component.

The first condition ensures stability, as discussed e.g. in ref. [2].[3] Because we demand first-order derivatives in $A_\mu$, the second condition implies that the first is automatically satisfied for $A_\mu$; the first condition is therefore necessary to implement only for the scalar part $\pi$ of the vector field. As for the third condition, it stems from the definition of a vector as a unit spin Lorentz-group representation, a spin $s$ object having $(2s + 1)$ propagating degrees of freedom. In ref. [14], an extra condition was implicit, namely that the longitudinal mode should not have trivial total derivative interactions; the terms we obtain below and that were not written down in this reference stem precisely from our relaxing of this condition.

For a given Lagrangian function $\mathcal{L}(A_\mu)$, the number of actually propagating degrees of freedom must be limited to three. To enforce this requirement, we compute the Hessian matrix $\mathcal{H}^{\mu\nu}$ associated with the Lagrangian term considered through

$$\mathcal{H}^{\mu\nu} = \frac{\partial^2 \mathcal{L}}{\partial(\partial_0 A_\mu)\partial(\partial_0 A_\nu)}, \tag{2.3}$$

and demand that it should have a vanishing eigenvalue associated to the absence of propagation of the time component of $A_\mu$. A sufficient condition to achieve this goal is to ask that $\mathcal{H}^{00} = 0$ and $\mathcal{H}^{0i} = 0$. One can show that this condition is the unique one permitting a vanishing eigenvalue: since different derivatives of the vector field are independent, the only way to diagonalize the Hessian without any vanishing eigenvalue is to cancel its symmetric terms with one another, and this, in turn, is not possible by means of a Lorentz transformation.

[3] This result is indirect: accepting a third-order derivative in the equations of motion would imply at least a second-order time derivative in the Lagrangian, which is not degenerate but yields a Hamiltonian which is unbounded from below.



JCAP02(2016)004

## 2.2 The Heisenberg action

Let us summarize ref. [14], switching to notations agreeing with those we use here. The Lagrangian terms which appear in addition to the standard kinetic term will be called $\mathcal{L}_n$, a notation coming from that which is usual in the Galileon theory [7]. There, $N = n + 2$ is the number of scalar field factors appearing in each Lagrangian, and $M = n - 2$ is the number of second derivatives of the Galileon field (we do not count the arbitrary functions of $X$ in this power counting). Keeping the same notation, now applied to the scalar part $\pi$ of the vector field, one can write the general Lagrangian as the usual Maxwell Lagrangian with a generalized mass term, i.e., one replaces the Lagrangian in eq. (2.1) by

$$\mathcal{L}^{\text{gen.}}_{\text{Proca}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{5} \mathcal{L}_n, \tag{2.4}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}),\\
\mathcal{L}_3 &= f_3^{\text{Gal}}(X)(\partial\cdot A),\\
\mathcal{L}_4 &= f_4^{\text{Gal}}(X)\left[(\partial_\mu A_\nu \partial^\nu A^\mu) - (\partial\cdot A)(\partial\cdot A)\right],\\
\mathcal{L}_5 &= f_5^{\text{Gal}}(X)\left[(\partial\cdot A)^3 - 3(\partial\cdot A)(\partial_\nu A_\rho \partial^\rho A^\nu) + 2(\partial_\mu A^\nu \partial_\nu A^\rho \partial_\rho A_\mu)\right]\\
&\quad + f_5^{\text{Perm}}(X)\big[(\partial\cdot A)\left[(\partial_\rho A_\nu \partial^\rho A^\nu) - (\partial_\nu A_\rho \partial^\rho A^\nu)\right]\\
&\quad + (\partial_\mu A_\rho \partial^\nu A^\mu \partial^\rho A_\nu) - (\partial^\nu A^\mu \partial_\rho A_\nu \partial^\rho A_\mu)\big],
\end{aligned} \tag{2.5}$$

where $f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu})$ is an arbitrary function of all scalars which can be constructed from $A_\mu$, $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, the latter being the Hodge dual of the Faraday tensor.[4] All other functions $f$ are independent arbitrary functions of $X$ only.

These expressions actually differ from those given in ref. [14]. First, we did not add a term of the form $f(X)\left[(\partial_\nu A_\mu \partial^\nu A^\mu) - (\partial_\mu A_\nu \partial^\nu A^\mu)\right]$ in $\mathcal{L}_4$, as such a term is merely equal to $\frac{1}{2} f(X) F_{\mu\nu}F^{\mu\nu}$ and is therefore already contained in $f_2$. Moreover, both terms in $\mathcal{L}_5$ have different arbitrary functions as prefactors. In fact, this general form is sufficient to verify the conditions given in section 2.1; this will be explained in section 3.2.

The Lagrangians given by eqs. (2.5) contain different kinds of terms with various origins. The first contributors, with prefactors given by the arbitrary functions $f_n^{\text{Gal}}$, originate from the scalar part of the vector field: setting $\bar{A} \to 0$, i.e. $A_\mu \to \partial_\mu \pi$, they are indeed nonvanishing. As, for consistency, they must verify the hypothesis of Galileon theory, they should thus come straightforwardly from the Galileon theory and therefore ought to be equivalent to the only Lagrangians satisfying such properties as they have been classified in ref. [6]. One way of defining them explicitly consists, for instance, in taking the specific form chosen e.g. in ref. [6, 7] (where it was called $\mathcal{L}_N^3$)

$$\mathcal{L}_N^{\text{Gal}} = \left[\mathcal{A}_{(2n)}^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n}\pi_\lambda\pi^\lambda\right]\pi_{\mu_1\nu_1}\ldots\pi_{\mu_n\nu_n}, \tag{2.6}$$

with

$$\mathcal{A}_{(2m)}^{\mu_1\mu_2\ldots\mu_m\nu_1\nu_2\ldots\nu_m} = \frac{1}{(D-m)!}\epsilon^{\mu_1\mu_2\ldots\mu_m\sigma_1\sigma_2\ldots\sigma_{D-m}}\epsilon^{\nu_1\nu_2\ldots\nu_m}{}_{\sigma_1\ldots\sigma_{D-m}}, \tag{2.7}$$

[4] Here and in what follows, we denote by $\epsilon^{\sigma_1\sigma_2\ldots\sigma_D}$ the totally antisymmetric Levi-Civita tensor in $D$ dimensions (in the case $D = 4$ we are concerned with in the rest of this paper, we write $\epsilon^{\mu\nu\alpha\beta}$).



JCAP02(2016)004

where $\pi_{\mu\nu\cdots} \equiv \partial_\mu\partial_\nu\cdots\pi$ is a convenient shorthand notation for the derivatives. The definition which appears to be unambiguous then arises from the following procedure: we substitute, in eq. (2.6), the first derivative of the scalar field by a component of the vector field ($\pi_{\mu\nu\cdots} \to A_{\mu,\nu\cdots}$), keeping all indices the way they appear in the original form. From the point of view of the scalar part of the Lagrangian, this leads to the same Galileon action, and it would also if we were to permute derivatives (indices in this case). For example, we could take in $\mathcal{L}_5$, a term having the special form

$$f_5^{\text{Gal,alt}}(X)\left[(\partial\cdot A)^3 - (\partial\cdot A)\left[(\partial_\nu A_\rho \partial^\rho A^\nu) + 2\left(\partial_\rho A_\nu \partial^\rho A^\nu\right)\right] + 2\left(\partial^\nu A^\mu \partial_\rho A_\nu \partial^\rho A_\mu\right)\right], \quad (2.8)$$

which would also reduce to eq. (2.6) in the scalar case. This term is also obtained by taking $f_5^{\text{Perm}} = -2f_5^{\text{Gal}}$. Both terms contained in $\mathcal{L}_5$ could thus be obtained by using the Galileon Lagrangian and shuffling around the second-order derivatives before doing the replacement by the vector field. The choice proposed above however provides a uniquely defined action when it comes to the vector field case.

The second category of terms, including $f_2$ (apart from terms containing only $X$) and $f_5^{\text{Perm}}$, gives a vanishing contribution when going to the scalar sector, due to the fact that $\partial_\mu\partial_\nu\pi = \partial_\nu\partial_\mu\pi$, which is not verified by $\partial_\mu\bar{A}_\nu$ by definition. This is trivial for the terms containing the Faraday tensor, since $A_\mu \to \partial_\mu\pi$ implies $F_{\mu\nu} \to 0$ identically, and can be verified explicitly for the term including $f_5^{\text{Perm}}$. These terms can also be seen to lead to only three propagating degrees of freedom for the vector field. Given this fact, one is naturally led to ask whether similar kind of terms, including more derivatives, could be possible. They are discussed in the following section.

## 3 Additional terms

### 3.1 Procedure of investigation

In order to find all the possible terms satisfying the conditions of section 2.1, we developed a systematic procedure, which we explain below by considering the term $\mathcal{L}_n$ of the Lagrangian, containing $(n-2)$ first-order derivatives of the vector field.[5] The final action will then consist of a sum over all such possible terms, weighted by arbitrary functions of $X$, as this can in no way change anything in the discussion of the Hessian.

We first list all the possible terms which can be written as contractions of $(n-2)$ first-order derivatives of $A_\mu$. We call them $\mathcal{L}_{n,i}^{\text{test}}$, where $i$ labels the different terms which can be written for a given $n$. These test Lagrangians are then linearly combined to provide the most general term at a given order $n$, whose Hessian, in eq. (2.3), is computed. We apply our requirements ($\mathcal{H}^{0\mu} = 0$ for all $\mu = 0, \cdots, 3$) to derive relations among the coefficients of the linear combination, and to finally obtain the relevant terms giving only three propagating degrees of freedom. Those fall into two distinct categories, namely those whose scalar sector vanishes or not. The latter necessarily reduce to the Galileon action, while the former are new, purely vectorial, terms which, in turn, can either be constructed from $F_{\mu\nu}$ only, or be the terms we are interested in. Any term leading to a non-trivial dynamics for the scalar part that would be nonvanishing should be then set to zero in order to comply with the requirement that the scalar action is that provided by the Galileon.

[5]The index $n$ appearing in our Lagrangians is inconvenient but we keep it in order to follow previous conventions developed historically from the Galileon action.

We are investigating terms that contain only derivatives and not the vector field itself in the contractions over Lorentz indices, as all possible such terms can always be reduced to those studied below and a total derivative. The typical example we have in mind is a term of the form

$$A_\alpha A^\mu \partial_\mu A^\alpha = \frac{1}{2} \left[\partial_\mu \left(X A^\mu\right) - X \left(\partial \cdot A\right)\right],$$

which is then equivalent to those we discuss below, up to a total derivative. Note that a similar term including an arbitrary function of $X$ in front would lead to second-order derivatives in $A$, and is thus excluded by construction.

Finally, we would like to mention at this point that we are here only considering terms involving the metric, i.e., we build scalars for the Lagrangian by contracting the indices of quantities such as $\partial_\mu A_\nu$ using the metric $g^{\mu\nu}$. Below, in section 3.4, we will consider another possible case, that for which scalars are also built through contractions with the completely antisymmetric tensor $\epsilon^{\alpha\beta\mu\nu}$; these extra terms and those first considered produce independent Hessian variations and can thus be studied independently.

### 3.2 From $\mathcal{L}_3$ to $\mathcal{L}_5$

The first term appearing in our expansion is $\mathcal{L}_3$, whose only test Lagrangian can be written as $(\partial \cdot A)$, which has a vanishing Hessian, and is nothing but the Galileon vector term

$$\mathcal{L}_3^{\rm Gal} = \left(\partial \cdot A\right). \tag{3.1}$$

The case of $\mathcal{L}_4$ is slightly more involved, the test Lagrangians reading

$$\mathcal{L}_{4,1}^{\rm test} = \left(\partial \cdot A\right)^2, \quad \mathcal{L}_{4,2}^{\rm test} = \left(\partial_\nu A_\mu \partial^\nu A^\mu\right), \quad \text{and} \quad \mathcal{L}_{4,3}^{\rm test} = \left(\partial_\mu A_\nu \partial^\nu A^\mu\right). \tag{3.2}$$

Setting the full test Lagrangian at this order to be

$$\mathcal{L}_4^{\rm test} = x_1 \mathcal{L}_{4,1}^{\rm test} + x_2 \mathcal{L}_{4,2}^{\rm test} + x_3 \mathcal{L}_{4,3}^{\rm test}, \tag{3.3}$$

we find the following relevant Hessian terms,

$$\mathcal{H}_4^{00} = 2 \left(x_1 + x_2 + x_3\right) \quad \text{and} \quad \mathcal{H}_4^{0i} = 0, \tag{3.4}$$

so that the solutions ensuring only three propagating degrees of freedom are

$$\mathcal{L}_4^{\rm Gal} = \left(\partial_\mu A_\nu \partial^\nu A^\mu\right) - \left(\partial \cdot A\right)\left(\partial \cdot A\right), \tag{3.5}$$

and

$$\mathcal{L}_{FF} = F_{\mu\nu} F^{\mu\nu} = \left(\partial_\nu A_\mu \partial^\nu A^\mu\right) - \left(\partial_\mu A_\nu \partial^\nu A^\mu\right). \tag{3.6}$$

We identify the first term with the Galileon vector term, and the second to the only term at this order which can be built from the field tensor. This term was expected, and we will not consider it any further since it is already included in $\mathcal{L}_2$. We also check that, as these terms independently verify the required conditions, they are effectively independent.

Let us move on to the following order, namely $\mathcal{L}_5$. The test Lagrangians are

$$\begin{aligned}
\mathcal{L}_{5,1}^{\rm test} &= \left(\partial \cdot A\right)^3, \\
\mathcal{L}_{5,2}^{\rm test} &= \left(\partial \cdot A\right)\left(\partial_\rho A_\nu \partial^\rho A^\nu\right), \\
\mathcal{L}_{5,3}^{\rm test} &= \left(\partial \cdot A\right)\left(\partial_\nu A_\rho \partial^\rho A^\nu\right), \\
\mathcal{L}_{5,4}^{\rm test} &= \left(\partial_\mu A_\rho \partial^\nu A^\mu \partial^\rho A_\nu\right), \\
\mathcal{L}_{5,5}^{\rm test} &= \left(\partial^\nu A^\mu \partial_\rho A_\nu \partial^\rho A_\mu\right).
\end{aligned} \tag{3.7}$$

Using

$$\mathcal{L}_5^{\rm test} = x_1 \mathcal{L}_{5,1}^{\rm test} + x_2 \mathcal{L}_{5,2}^{\rm test} + x_3 \mathcal{L}_{5,3}^{\rm test} + x_4 \mathcal{L}_{5,4}^{\rm test} + x_5 \mathcal{L}_{5,5}^{\rm test}, \tag{3.8}$$

we obtain the relevant Hessian terms[6]

$$\begin{aligned} \mathcal{H}_5^{00} &= -2\left(-3x_1 - x_2 - x_3\right)\left(\partial \cdot A\right) - 2\left[2x_2 + 2x_3 + 3\left(x_4 + x_5\right)\right]\left(\partial^0 A^0\right), \\ \mathcal{H}_5^{0i} &= -2\left(x_2 + x_5\right)\left(\partial^0 A^i\right) - \left(2x_3 + 3x_4 + x_5\right)\left(\partial^i A^0\right), \end{aligned} \tag{3.9}$$

which, when vanishing, provide the following solutions:

$$\mathcal{L}_5^{\rm Gal} = \left(\partial \cdot A\right)^3 - 3\left(\partial \cdot A\right)\left(\partial_\nu A_\rho \partial^\rho A^\nu\right) + 2\left(\partial_\mu A^\nu \partial_\nu A^\rho \partial_\rho A_\mu\right), \tag{3.10}$$

and

$$\mathcal{L}_5^{\rm Perm} = \frac{1}{2}\left(\partial \cdot A\right) F_{\mu\nu} F^{\mu\nu} + \partial_\rho A_\nu \partial^\nu A_\mu F^{\mu\rho}. \tag{3.11}$$

The first, eq. (3.10), is the Galileon scalar Lagrangian, verifying all the imposed conditions, leading to second-order equations of motion, even multiplied by any function of $X = \partial_\mu \pi \partial^\mu \pi$; this is shown in ref. [6]. The second one, eq. (3.11), gives no dynamics for the Galileon scalar, since it vanishes in the scalar sector, as can be seen from the fact that it can be factorized in terms of the strength tensor $F_{\mu\nu}$ (but not exclusively, since it would otherwise be gauge invariant and thus reducible to functions of $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$). It does however also satisfy our conditions, even multiplied by any function of $X$.

This complete the proof that eq. (2.5), as originally obtained in ref. [14], fulfills the required constraints for a generic classical second-order action for a vector field. We shall now discuss the possibility of extra terms, not present in previous works.

### 3.3 Higher-order actions

Since the scalar Galileon action stops at the level discussed above, it sounds reasonable to assume the same to apply for the vector field, in particular in view of the fact that this field contains a scalar part, and to consider eq. (2.5) to provide the most general second-order classical vector theory. This is not what happens, in practice, as we show below, as one can indeed find extra terms, which do vanish in the limit $\mathcal{A}_\mu \to \partial_\mu \pi$, leaving a non-trivial dynamics for the divergence-free part $\bar{A}_\mu$.

#### 3.3.1 Fourth power derivatives: $\mathcal{L}_6$

We begin our examination of the higher powers in derivatives by concentrating on $\mathcal{L}_6$, which involves therefore four powers of the field gradient. Since the actual calculations imply very large and cumbersome terms, and because those are not so important for the understanding of this work, we have regrouped this calculation in appendix A.

We find that there are four Lagrangians verifying the Hessian condition. They are

$$\begin{aligned} \mathcal{L}_6^{\rm Gal} = & \left(\partial \cdot A\right)^4 - 2\left(\partial \cdot A\right)^2\left[\left(\partial_\rho A_\sigma \partial^\sigma A^\rho\right) + 2\left(\partial_\sigma A_\rho \partial^\sigma A^\rho\right)\right] + 8\left(\partial \cdot A\right)\left(\partial^\rho A^\nu \partial_\sigma A_\rho \partial^\sigma A_\nu\right) \\ & - \left(\partial_\mu A_\nu \partial^\nu A^\mu\right)^2 + 4\left(\partial_\nu A_\mu \partial^\nu A^\mu\right)\left(\partial_\rho A_\sigma \partial^\sigma A^\rho\right) - 2\left(\partial_\nu A^\sigma \partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu\right) \\ & - 4\left(\partial^\nu A^\mu \partial^\rho A_\mu \partial_\sigma A_\rho \partial^\sigma A_\nu\right), \end{aligned} \tag{3.12}$$

[6] Although the Hessian looks different from that of ref. [14], it actually is equivalent, the difference stemming from the fact that in the latter work, $\mathcal{H}_5^{00}$ was decomposed on $\partial_i A^i$ and $\partial_0 A^0$, while we chose to decompose it on $\partial_\mu A^\mu$ and $\partial_0 A^0$.



JCAP02(2016)004

which should a priori give a dynamics for the scalar part, and

$$\begin{aligned}\mathcal{L}_6^{\rm Perm} =& \,(\partial\cdot A)^2\, F^{\mu\nu}F_{\mu\nu} - (\partial_\rho A_\sigma \partial^\sigma A^\rho)\, F^{\mu\nu}F_{\mu\nu} + 4\,(\partial\cdot A)\,\partial^\rho A^\nu \partial^\sigma A_\rho F_{\nu\sigma}\\ &+ \partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - 4\,\partial^\mu A_\nu \partial^\nu A_\rho \partial^\rho A_\sigma F^\sigma{}_\mu .\end{aligned} \tag{3.13}$$

The first term, eq. (3.12), contains four second-order derivatives of the scalar part and, thus, because it is of order higher than 3, cannot verify the conditions of section 2.1 [3, 6]. We shall consequently discard it.

There are two more terms which satisfy the required conditions, namely $\mathcal{L}_{FF\cdot FF} = (F_{\mu\nu}F^{\mu\nu})^2$ and $\mathcal{L}_{FFFF} = F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu$. Since these terms are built out only of the field strength tensor, they do give a dynamics to the vector field, but are already included in $\mathcal{L}_2$, and we therefore will not consider them any further.

The terms in eq. (3.13) exhibit explicit powers of $(\partial\cdot A)$, and therefore cannot be built out of simple products of the field tensor. The Lagrangian $\mathcal{L}_6^{\rm Perm}$ vanishes in the scalar sector and verifies all the conditions we want to impose, even multiplied by an arbitrary function of $X$. We consider the general theory of a vector field, and we thus must add to eq. (2.5) the term

$$\mathcal{L}_6 = f_6^{\rm Perm}(X)\mathcal{L}_6^{\rm Perm}, \tag{3.14}$$

with $f_6^{\rm Perm}$ being an arbitrary function of $X$.

#### 3.3.2 Fifth power derivatives: $\mathcal{L}_7$

Having found an extra non-trivial dynamical term, we now move on to including yet one more power in the derivatives. Even more cumbersome than those leading to $\mathcal{L}_6$, the calculations yielding $\mathcal{L}_7$ are reproduced in appendix B. We obtain

$$\begin{aligned}\mathcal{L}_7^{\rm Gal} =& \,3\,(\partial\cdot A)^5 - (\partial\cdot A)^3\, 10\,[(\partial_\sigma A_\gamma \partial^\gamma A^\sigma) - 2\,(\partial_\gamma A_\sigma \partial^\gamma A^\sigma)] + 60\,(\partial\cdot A)^2\,(\partial^\sigma A^\rho \partial_\gamma A_\sigma \partial^\gamma A_\rho)\\ &+ 15\,(\partial\cdot A)\,\big[\,(\partial_\nu A_\rho \partial^\rho A^\nu)\,(\partial_\sigma A_\gamma \partial^\gamma A^\sigma) + 2\,(\partial_\rho A_\nu \partial^\rho A^\nu)\,(\partial_\sigma A_\gamma \partial^\sigma A^\gamma)\\ &- 4\,(\partial_\rho A^\gamma \partial^\rho A^\nu \partial_\sigma A_\gamma \partial^\sigma A_\nu) - 4\,(\partial^\rho A^\nu \partial^\sigma A_\nu \partial_\gamma A_\sigma \partial^\gamma A_\rho)\,\big]\\ &+ 20\,(\partial_\rho A_\gamma \partial^\sigma A^\rho \partial^\gamma A_\sigma)\,[(\partial_\nu A_\mu \partial^\nu A^\mu) - (\partial_\mu A_\nu \partial^\nu A^\mu)]\\ &- 60\,(\partial_\nu A_\mu \partial^\nu A^\mu)\,(\partial^\sigma A^\rho \partial_\gamma A_\sigma \partial^\gamma A_\rho) + 60\,(\partial^\nu A^\mu \partial_\rho A^\gamma \partial^\rho A_\mu \partial_\sigma A_\gamma \partial^\sigma A_\nu)\\ &+ 60\,(\partial^\nu A^\mu \partial^\rho A_\mu \partial^\sigma A_\nu \partial_\gamma A_\sigma \partial^\gamma A_\rho) - 60\,(\partial^\nu A^\mu \partial_\rho A_\gamma \partial^\rho A_\mu \partial^\sigma A_\nu \partial^\gamma A_\sigma)\\ &+ 12\,(\partial_\mu A_\gamma \partial^\nu A^\mu \partial^\rho A_\nu \partial^\sigma A_\rho \partial^\gamma A_\sigma)\,,\end{aligned} \tag{3.15}$$

which, similarly to eq. (3.12), must be discarded as it leads to higher-order equations of motion for the scalar part, and two extra terms, namely

$$\begin{aligned}\mathcal{L}_7^{\rm Perm,1} =&\, (\partial\cdot A)^3\, F^{\mu\nu}F_{\mu\nu} + 6\,(\partial\cdot A)^2\, \partial^\mu A^\nu \partial^\rho A_\mu F_{\nu\rho}\\ &+ 3\,(\partial\cdot A)\,\big[\,(\partial_\nu A_\rho \partial^\rho A^\nu)^2 - (\partial_\rho A_\nu \partial^\rho A^\nu)^2\,\big]\\ &+ 3\,(\partial\cdot A)\,(\partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - 4\partial^\mu A_\nu \partial^\nu A_\rho \partial^\rho A_\sigma F^\sigma{}_\mu)\\ &+ 4\,(\partial_\mu A_\nu \partial^\mu A^\nu)\,(\partial^\rho A^\sigma \partial_\gamma A_\rho \partial^\gamma A_\sigma)\\ &- 4\,(\partial_\mu A_\nu \partial^\nu A^\mu)\,(\partial^\rho A^\sigma \partial_\gamma A_\rho \partial_\sigma A^\gamma) + 2\,(\partial_\mu A_\nu \partial^\mu A^\nu)\,(\partial^\rho A_\sigma \partial_\gamma A_\rho F^{\gamma\sigma})\\ &- 6\partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\gamma F^\gamma{}_\mu + 12\partial^\mu A_\nu \partial^\nu A_\rho \partial^\rho A_\sigma \partial^\sigma A_\gamma F^\gamma{}_\mu,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\mathcal{L}_7^{\rm Perm,2} =& \frac{1}{4}\,(\partial\cdot A)\,\big[\,(F_{\mu\nu}F^{\mu\nu})^2 - 4\partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu\big]\\ &+ (F^{\mu\nu}F_{\mu\nu})\,\partial^\sigma A^\rho \partial^\gamma A_\sigma F_{\rho\gamma} + 2\,(\partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\gamma F^\gamma{}_\mu)\,,\end{aligned} \tag{3.17}$$

both of which vanish in the limit $\mathcal{A}_\mu \to \partial_\mu \pi$. It can be checked explicitly that they independently verify all the conditions given in section 2.1, so the final theory will contain

$$\mathcal{L}_7 = f_7^{\text{Perm},1}(X)\mathcal{L}_7^{\text{Perm},1} + f_7^{\text{Perm},2}(X)\mathcal{L}_7^{\text{Perm},2}, \tag{3.18}$$

with $f_7^{\text{Perm},1}$ and $f_7^{\text{Perm},2}$ being two independent arbitrary functions of $X$.

### 3.4 Antisymmetric $\epsilon$ terms

As argued in section 3.1, one can also build scalars out of a vector field by contractions with the completely antisymmetric Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$. In order for these new terms to be effectively independent on the previously derived ones, it is necessary that two such tensors never appear contracted with one another, since there exists relations such as $\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\rho\sigma} = 2\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right)$, all such contractions will give back our previous terms. This produces only a limited number of independent new terms at each order.

The first order, quadratic in the derivative terms, contains only one possible contraction, namely $\mathcal{L}_4^\epsilon = \epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma = \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}$, which is not a new term, being included by construction in $\mathcal{L}_2$. At this order, this is the only possibility with vanishing Hessian constraints.

Cubic terms produce two test Lagrangians, namely

$$\begin{aligned} \mathcal{L}_{5,1}^{\text{test}} &= \epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma \left(\partial \cdot A\right) = \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\left(\partial \cdot A\right) \\ \text{and} \quad \mathcal{L}_{5,2}^{\text{test}} &= \epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A_\alpha \partial^\alpha A^\sigma = \tilde{F}_{\rho\sigma}\partial^\rho A_\alpha \partial^\alpha A^\sigma, \end{aligned} \tag{3.19}$$

which, once combined in such a way as to ensure the vanishing of the Hessian $\mathcal{H}^{0\mu}$, yield

$$\mathcal{L}_5^\epsilon = F_{\mu\nu}\tilde{F}^{\mu\nu}\left(\partial \cdot A\right) - 4\left(\tilde{F}_{\rho\sigma}\partial^\rho A_\alpha \partial^\alpha A^\sigma\right). \tag{3.20}$$

Increasing the powers of the derivatives, as usual by now, also increases the complexity of the possible new terms. Quartic test terms are found to be given by

$$\begin{aligned}
\mathcal{L}_{6,1}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma\right)^2 = \frac{1}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2, \\
\mathcal{L}_{6,2}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma\right)\left(\partial \cdot A\right)^2 = \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\left(\partial \cdot A\right)^2, \\
\mathcal{L}_{6,3}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma\right)\left(\partial^\alpha A^\beta \partial_\alpha A_\beta\right), \\
\mathcal{L}_{6,4}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma\right)\left(\partial^\alpha A^\beta \partial_\beta A_\alpha\right), \\
\mathcal{L}_{6,5}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A_\alpha \partial^\alpha A^\sigma\right)\left(\partial \cdot A\right) = \left(\tilde{F}_{\rho\sigma}\partial^\rho A_\alpha \partial^\alpha A^\sigma\right)\left(\partial \cdot A\right), \\
\mathcal{L}_{6,6}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A_\alpha \partial^\sigma A_\beta \partial^\alpha A^\beta\right) = \left(\tilde{F}_{\rho\sigma}\partial^\rho A_\alpha \partial^\sigma A_\beta \partial^\alpha A^\beta\right), \\
\mathcal{L}_{6,7}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial_\alpha A^\rho \partial_\beta A^\sigma \partial^\alpha A^\beta\right) = \left(\tilde{F}_{\rho\sigma}\partial_\alpha A^\rho \partial_\beta A^\sigma \partial^\alpha A^\beta\right), \\
\mathcal{L}_{6,8}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A_\alpha \partial_\beta A^\sigma \partial^\alpha A^\beta\right) = \left(\tilde{F}_{\rho\sigma}\partial^\rho A_\alpha \partial_\beta A^\sigma \partial^\alpha A^\beta\right), \\
\mathcal{L}_{6,9}^{\epsilon,\text{test}} &= \left(\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A_\beta \partial_\alpha A^\sigma \partial^\alpha A^\beta\right) = \left(\tilde{F}_{\rho\sigma}\partial^\rho A_\beta \partial_\alpha A^\sigma \partial^\alpha A^\beta\right),
\end{aligned} \tag{3.21}$$



JCAP02(2016)004

leading to one new solution with vanishing Hessian constraint, namely

$$\mathcal{L}_6^{\epsilon} = \tilde{F}_{\rho\sigma} F^{\rho}{}_{\beta} F^{\sigma}{}_{\alpha} \partial^{\alpha} A^{\beta} \,, \tag{3.22}$$

together with $\mathcal{L}_{F\tilde{F}\times F\tilde{F}} = (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$ and $\mathcal{L}_{F\tilde{F}\times FF} = (F_{\mu\nu}\tilde{F}^{\mu\nu})\,(F_{\rho\sigma}F^{\rho\sigma})$, both of which are already part of $\mathcal{L}_2$. As before, all these extra Lagrangians can be multiplied by arbitrary functions $f^{\epsilon}(X)$ without modifying our conclusions.

### 3.5 Final Lagrangian in flat spacetime

Up to this point, we have considered explicit terms involving products of up to five derivatives of the vector field. Higher-order terms can be derived by continuing along the same lines of calculation; this would imply an important number of long terms. We have not found a general rule allowing to derive a generic action at any given order, so we are merely led to conjecture, given the above calculations, that there is no reason the higher-order terms to be vanishing (assuming they do not lead to trivially vanishing equations of motion [13]).

Considering the result previously derived, we conjecture that terms like $\mathcal{L}_n^{\text{Perm}}$ will continue to show up at higher order, and in fact, we speculate that the higher order the more numerous terms one will find. A generic $\mathcal{L}_n^{\text{Perm}}$ will yield vanishing $\mathcal{H}^{00}$ and $\mathcal{H}^{0i}$ and vanish when going to the scalar limit $A_\mu \to \partial_\mu \pi$ of the theory. Including the Levi-Civita terms, we propose that the most general Lagrangian leading to second-order equations of motion for three vector propagating degrees of freedom contains an infinite number of terms, whose general formulation takes the form

$$\mathcal{L}_{\text{gen}}\,(A_\mu) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \sum_{n\geq 2} \mathcal{L}_n + \sum_{n\geq 5} \mathcal{L}_n^{\epsilon}, \tag{3.23}$$

with

$$\begin{aligned}
\mathcal{L}_2 =& f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}),\\
\mathcal{L}_3 =& f_3^{\text{Gal}}(X)\mathcal{L}_3^{\text{Gal}},\\
\mathcal{L}_4 =& f_4^{\text{Gal}}(X)\mathcal{L}_4^{\text{Gal}},\\
\mathcal{L}_5 =& f_5^{\text{Gal}}(X)\mathcal{L}_5^{\text{Gal}} + f_5^{\text{Perm}}(X)\mathcal{L}_5^{\text{Perm}},\\
\mathcal{L}_6 =& f_6^{\text{Perm}}(X)\mathcal{L}_6^{\text{Perm}},\\
\mathcal{L}_7 =& f_7^{\text{Perm},1}(X)\mathcal{L}_7^{\text{Perm},1} + f_7^{\text{Perm},2}(X)\mathcal{L}_7^{\text{Perm},2},\\
\mathcal{L}_{n\geq 8} =& \sum_i f_n^{\text{Perm},i}(X)\mathcal{L}_n^{\text{Perm},i},\\
\mathcal{L}_n^{\epsilon} =& \sum_i g_n^{\epsilon,i}(X)\mathcal{L}_n^{\epsilon,i},
\end{aligned} \tag{3.24}$$

where the terms in $\mathcal{L}_3$ to $\mathcal{L}_5$ are given in section 3.2, those in $\mathcal{L}_6$ and $\mathcal{L}_7$ can be found respectively in section 3.3.1 and 3.3.2, and those in $\mathcal{L}_n^{\epsilon}$ are shown in section 3.4. Once again, $f_2$ is an arbitrary scalar function of all the possible contractions among $A_\mu$, $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, and all the other $f_n$ and $g_n^{\epsilon}$ are arbitrary functions of $X$.



JCAP02(2016)004

## 4 Curved space-time

Coupling the vector with the metric and rendering the latter dynamical implies many new possible terms satisfying our conditions. Those are explored below. Some of the terms have already been discussed in previous works [10, 14, 18], and we introduce new ones below.

### 4.1 Covariantization

In order to take into account the metric $g_{\mu\nu}$ itself as dynamical in the most general way, one needs to impose its equation of motion to be also of order two at most. First, we transform partial derivatives into covariant derivatives. This satisfies the constraints in the vector sector since the additional terms, coming from the connection, are only first order in the metric derivatives and thus ensures that no derivative higher than second order will appear in the equations of motion of the vector field or the metric.

On the other hand, when going to the scalar sector, one has to pay special attention to the higher derivative terms which can appear due to the commutations of derivatives of the scalar field. This can be problematic only when one considers terms reducing to the scalar Galileon Lagrangians because the property $\nabla_\mu \nabla_\nu \pi = \nabla_\nu \nabla_\mu \pi$ remains valid in curved spacetime. As for the extra terms, called $\mathcal{L}_n^{\text{Perm}}$ above, as well as the U(1)-invariants, they keep vanishing in the scalar formulation of the theory. The terms reducing to the scalar Galileon Lagrangians have already been studied, and their curved space-time extension can be found in refs. [4, 6, 14]. In conclusion, the only terms to be modified in eq. (3.23) are

$$\begin{aligned} \mathcal{L}_4 &= f_4^{\text{Gal}}(X) R - 2 f_{4,X}^{\text{Gal}}(X) \mathcal{L}_4^{\text{Gal}}, \\ \mathcal{L}_5 &= f_5^{\text{Gal}}(X) G_{\mu\nu} \nabla^\mu A^\nu + 3 f_{5,X}^{\text{Gal}}(X) \mathcal{L}_5^{\text{Gal}} + f_5^{\text{Perm}}(X) \mathcal{L}_5^{\text{Perm}}, \end{aligned} \tag{4.1}$$

where the notation $f_{,X}$ stands for a derivative with respect to $X$, i.e. $f_{,X} \equiv \mathrm{d}f/\mathrm{d}X$.

### 4.2 Additional curvature terms

This last part is dedicated to all the additional terms which can appear from the coupling contractions of curvature terms with terms implying the vector field. A similar study was already proposed for the Abelian case, and it was shown that the only possibility was to contract the field tensor with divergence-free objects built from curvature [19, 20], i.e. the Einstein tensor $G_{\mu\nu}$ and the following fourth-rank divergence-free tensor

$$L_{\mu\nu\rho\sigma} = 2R_{\mu\nu\rho\sigma} + 2(R_{\mu\sigma} g_{\rho\nu} + R_{\rho\nu} g_{\mu\sigma} - R_{\mu\rho} g_{\nu\sigma} - R_{\nu\sigma} g_{\mu\rho}) + R(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\rho\nu}). \tag{4.2}$$

This tensor has the same symmetry properties as the Riemann tensor $R_{\mu\nu\rho\sigma}$, i.e. it is anti-symmetric in $(\mu, \nu)$ and $(\rho, \sigma)$, and symmetric in the exchange of $(\mu\nu)$ and $(\rho\sigma)$. The use of divergence-free tensors permits us to avoid higher-order derivatives of the metric in the equations of motion: contracting such a tensor with another dynamical object of the required derivative order will naturally lead, in the equations of motion, to the divergence of this tensor, and hence will vanish in the chosen case.

In order to derive the relevant generalizing terms for the Proca theory, we proceed in the same way as in the previous section. We first consider contractions of both $G_{\mu\nu}$ and $L_{\mu\nu\rho\sigma}$ with $A_\mu$ only, and then these specific contractions with $A_\mu$ and its first derivative that vanish in the scalar part. We then apply the Hessian condition given in section 3.1 to obtain the required terms.

For the contractions with the vector field only, the only term we can have is

$$\mathcal{L}_1^{\text{Curv}} = G_{\mu\nu} A^\mu A^\nu, \tag{4.3}$$

with all possible contractions with $L_{\mu\nu\rho\sigma}$ being vanishing due to its antisymmetry properties. Note that in this case, we cannot multiply this Lagrangian by a scalar function of the vector field $X$: in the scalar sector, this would lead to first-order derivatives of the scalar component $\pi$ which would subsequently yield terms involving $\nabla_\alpha G_{\mu\nu}$ in the equations of motion, terms which are third order in the metric and thus excluded.

Contractions with the field tensor only have already been studied in ref. [20], where it was shown that the only available term satisfying our requirements can be written as

$$\mathcal{L}_2^{\text{Curv}} = L_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{4.4}$$

the other possible contraction over different indices, $L_{\mu\nu\rho\sigma} F^{\mu\rho} F^{\nu\sigma}$, being equivalent to eq. (4.4) because of the symmetries of $L_{\mu\nu\rho\sigma}$. Similarly, one could consider the term built out of the dual of the field tensor; this actually also reduces to $\mathcal{L}_2^{\text{Curv}}$ [20]. Moreover, this last term can be multiplied by any scalar function of the vector field only without losing its properties, since it vanishes in the scalar sector.

Last on the list are those non-U(1) invariant terms with first-order derivatives of the vector field; these must vanish in the scalar sector in order to comply with our demands. In this case, in order to have at most a second-order equation of motion of the metric from the equation of motion of the vector field, it is sufficient to contract only the indices of the derivatives with the divergence-free curvature tensors. At this point, we see that we need to introduce a term in $\nabla^\mu A^\nu$, if we want to commute derivatives in order to have a vanishing term in the scalar sector, and that this term will thus involve the field strength tensor. As a result, we can consider terms with either $L_{\mu\nu\alpha\beta} F^{\mu\nu}$ or $L_{\mu\alpha\rho\beta} F^{\mu\rho}$ contracted with $(\nabla^\alpha A^\beta)$, $(\nabla^\beta A^\alpha)$, $(\nabla^\alpha A^\gamma \nabla^\beta A_\gamma)$ or $(A^\alpha A^\beta)$. A close examination of these possibilities reveals that such terms are either vanishing, or else that they reduce to $\mathcal{L}_2^{\text{Curv}}$.

We summarize the set of all possible additional terms containing the curvature in the following Lagrangian

$$\mathcal{L}^{\text{Curv}} = f_1^{\text{Curv}} G_{\mu\nu} A^\mu A^\nu + f_2^{\text{Curv}}(X) L_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{4.5}$$

where $f_1^{\text{Curv}}$ is a constant, and $f_2^{\text{Curv}}$ is an arbitrary function of $X$ only. We did not consider higher-order products of curvature terms since our aim is to focus on the the vector part of the theory.

### 4.3 Final Lagrangian in curved spacetime

We can finally write the complete expression of the generalized Abelian Proca theory in curved spacetime, which reads

$$\mathcal{L}_{\text{gen}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}^{\text{Curv}} + \sum_{n\geq 2} \mathcal{L}_n + \sum_{n\geq 5} \mathcal{L}_n^\epsilon, \tag{4.6}$$



JCAP02(2016)004

where

$$
\begin{aligned}
\mathcal{L}^{\rm Curv} &= f_1^{\rm Curv} G_{\mu\nu} A^\mu A^\nu + f_2^{\rm Curv}(X) L_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \\
\mathcal{L}_2 &= f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}), \\
\mathcal{L}_3 &= f_3^{\rm Gal}(X) \mathcal{L}_3^{\rm Gal}, \\
\mathcal{L}_4 &= f_4^{\rm Gal}(X) R - 2 f_{4,X}^{\rm Gal}(X) \mathcal{L}_4^{\rm Gal}, \\
\mathcal{L}_5 &= f_5^{\rm Gal}(X) G_{\mu\nu} \nabla^\mu A^\nu + 3 f_{5,X}^{\rm Gal}(X) \mathcal{L}_5^{\rm Gal} + f_5^{\rm Perm}(X) \mathcal{L}_5^{\rm Perm}, \\
\mathcal{L}_6 &= f_6^{\rm Perm}(X) \mathcal{L}_6^{\rm Perm}, \\
\mathcal{L}_7 &= f_7^{\rm Perm,1}(X) \mathcal{L}_7^{\rm Perm,1} + f_7^{\rm Perm,2}(X) \mathcal{L}_7^{\rm Perm,2}, \\
\mathcal{L}_{n\geq 8} &= \sum_i f_n^{\rm Perm,\it i}(X) \mathcal{L}_n^{\rm Perm,\it i}, \\
\mathcal{L}_n^\epsilon &= \sum_i g_n^{\epsilon,i}(X) \mathcal{L}_n^{\epsilon,i},
\end{aligned}
\tag{4.7}
$$

all $f$ and $g$ being arbitrary functions of $X$, except $f_1^{\rm Curv}$ which is a constant, and $f_2$ which is an arbitrary function of $A_\mu$, $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$. The complete expression of the Lagrangians which appears in eq. (4.7) are given in section 3.2 to 3.4 where partial derivatives must be replaced by covariant ones.

As in the previous, flat-space case, we conjecture the series to contain an infinite number of terms.

## 5 Conclusion and discussion

In this paper, we investigated the terms which can be included in the generalized Proca theory. After obtaining those which contain products of up to five first derivatives, we proposed a conjecture concerning the following terms. We discuss a crucial difference between the generalized scalar and vector theories: in contrast with the Galileon theory, the generalization of the Proca theory seems to depend on an infinite number of arbitrary functions, either in front of $\mathcal{L}_n^{\rm Perm,\it i}$ or in front of all the possible contractions among field tensors.

On the other hand, one could ask if this theory is the most general one can write, in flat as in curved spacetime.[7] A first extension would, for instance, consist in including terms containing higher-order derivatives leading still to second-order equations of motion; this can be found for instance in ref. [11]. A second extension would be to investigate the terms additional to those appearing in the covariantization of the action, through the equation of motion rather than the Lagrangian, as it is done in ref. [24] following the original Horndeski's procedure [9]; this could be interesting in order to have a full understanding of the model.

[7] The most general multiple-scalar field theory in flat spacetime was given in ref. [21] and covariantized in ref. [22]. In the latter paper, it was suggested that the ensuing theory would be the multi-scalar generalization of Horndeski's theory. However, as shown in ref. [23], the covariantization procedure does not guarantee that the resulting action is the most general; indeed, as presented in ref. [24], a better procedure is that devised originally by Horndeski in ref. [9].

In the future, this work could be completed along many directions. For instance, a Hamiltonian analysis of the relevant degrees of freedom would allow us to determine the relevant symmetries. Moreover, one could study how a general theory such as that defined here can be obtained from an effective U(1) invariant initial model. Pioneering work on this direction, along the line of spontaneous symmetry breaking to obtain extensions of Proca theory, has been done and can be found in, e.g., refs. [25, 26]. An extension to the non-Abelian situation is currently undergoing, with a much richer phenomenology. Finally, one could consider the effects of such a theory in a cosmological context, be it by means of implementing an inflation model [20, 27] or by suggesting new solutions to the cosmological constant problem [25].


## Acknowledgments

Throughout this paper, we have used xTensor, xCoba and TexAct, all parts of the Mathematica package xAct available in ref. [28]. We wish to thank L. Bernard, C. Deffayet, G. Esposito-Farèse, L. Heisenberg and G. Faye for illuminating discussions. This work was supported by COLCIENCIAS - ECOS NORD grant number RC 0899-2012 with the help of ICETEX, and by COLCIENCIAS grant numbers 110656933958 RC 0384-2013 and 123365843539 RC FP44842-081-2014.


## A Hessian condition for the quartic terms: $\mathcal{L}_6$

The different test Lagrangians that can be written down at the fourth-order derivative level are

$$\begin{aligned}
\mathcal{L}_{6,1}^{\text{test}} &= (\partial \cdot A)^4 \,, \\
\mathcal{L}_{6,2}^{\text{test}} &= (\partial \cdot A)^2 \left(\partial_\sigma A_\rho \partial^\sigma A^\rho\right) , \\
\mathcal{L}_{6,3}^{\text{test}} &= (\partial \cdot A)^2 \left(\partial_\rho A_\sigma \partial^\sigma A^\rho\right) , \\
\mathcal{L}_{6,4}^{\text{test}} &= (\partial \cdot A) \left(\partial_\nu A_\sigma \partial^\rho A^\nu \partial^\sigma A_\rho\right) , \\
\mathcal{L}_{6,5}^{\text{test}} &= (\partial \cdot A) \left(\partial^\rho A^\nu \partial_\sigma A_\rho \partial^\sigma A_\nu\right) , \\
\mathcal{L}_{6,6}^{\text{test}} &= \left(\partial_\mu A_\sigma \partial^\nu A^\mu \partial^\rho A_\nu \partial^\sigma A_\rho\right) , \\
\mathcal{L}_{6,7}^{\text{test}} &= \left(\partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu \partial^\sigma A_\nu\right) , \\
\mathcal{L}_{6,8}^{\text{test}} &= \left(\partial_\nu A^\sigma \partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu\right) , \\
\mathcal{L}_{6,9}^{\text{test}} &= \left(\partial^\nu A^\mu \partial^\rho A_\mu \partial_\sigma A_\rho \partial^\sigma A_\nu\right) , \\
\mathcal{L}_{6,10}^{\text{test}} &= \left(\partial_\nu A_\mu \partial^\nu A^\mu\right)^2 , \\
\mathcal{L}_{6,11}^{\text{test}} &= \left(\partial_\mu A_\nu \partial^\nu A^\mu\right) \left(\partial_\sigma A_\rho \partial^\sigma A^\rho\right) , \\
\mathcal{L}_{6,12}^{\text{test}} &= \left(\partial_\mu A_\nu \partial^\nu A^\mu\right) \left(\partial_\rho A_\sigma \partial^\sigma A^\rho\right) ,
\end{aligned} \tag{A.1}$$

so that setting

$$\mathcal{L}_6^{\text{test}} = \sum_{k=1}^{12} x_k \, \mathcal{L}_{6,k}^{\text{test}} , \tag{A.2}$$

we obtain for the (00) component of the Hessian

$$\begin{aligned}
\mathcal{H}_6^{00} &= 2\left(6x_1+x_2+x_3\right)\left(\partial\cdot A\right)^2 + 2\left(2x_{10}+x_{11}+x_2\right)\left(\partial_\mu A_\nu \partial^\mu A^\nu\right) \\
&+ 2\left(x_3+x_{11}+2x_{12}\right)\left(\partial^\mu A^\nu \partial_\nu A_\mu\right) - 2\big[4x_2+4x_3+3\left(x_4+x_5\right)\big]\left(\partial\cdot A\right)\left(\partial^0 A^0\right) \\
&+ 4\left[x_6+x_7+x_8+x_9+2\left(x_{10}+x_{11}+x_{12}\right)\right]\left(\partial^0 A^0\right)^2 \\
&- 2\left(x_5+x_7+2x_8+x_9\right)\left(\partial_\mu A^0\right)\left(\partial^\mu A^0\right) - \left(\partial^0 A_\mu\right)\big\{2\left(x_5+x_7+2x_8+x_9\right)\left(\partial^0 A^\mu\right) \\
&+ 2\big[3x_4+x_5+2\left(2x_6+x_7+x_9\right)\big]\left(\partial^\mu A^0\right)\big\}\,,
\end{aligned} \tag{A.3}$$

and the $(0i)$ component

$$\begin{aligned}
\mathcal{H}_6^{0i} &= 2\left(x_7+2x_8+x_9+4x_{10}+2x_{11}\right)\left(\partial^0 A^0\right)\left(\partial^0 A^i\right) \\
&+ 2\left(2x_6+x_7+x_9+2x_{11}+4x_{12}\right)\left(\partial^0 A^0\right)\left(\partial^i A^0\right) \\
&- \left(\partial\cdot A\right)\left[2\left(2x_2+x_5\right)\left(\partial^0 A^i\right)+\left(4x_3+3x_4+x_5\right)\left(\partial^i A^0\right)\right] \\
&- \left(3x_4+4x_6+x_7\right)\left(\partial^i A_\mu\right)\left(\partial^\mu A^0\right) - \left(x_5+x_7+4x_8\right)\left(\partial_\mu A^i\right)\left(\partial^\mu A^0\right) \\
&- \left(\partial^0 A_\mu\right)\left(x_5+x_7+2x_9\right)\left(\partial^i A^\mu + \partial^\mu A^i\right).
\end{aligned} \tag{A.4}$$

Canceling these two functions of $x_k$ provides the solutions exhibited in section 3.3.1.

# B Hessian condition for the fifth order: $\mathcal{L}_7$

In the last case we considered, for which the highest power in derivatives is five, we can write the various test Lagrangians as

$$\begin{aligned}
\mathcal{L}_{7,1}^{\rm test} &= \left(\partial\cdot A\right)^5\,, \\
\mathcal{L}_{7,2}^{\rm test} &= \left(\partial\cdot A\right)^3\left(\partial_\gamma A_\sigma \partial^\gamma A^\sigma\right), \\
\mathcal{L}_{7,3}^{\rm test} &= \left(\partial\cdot A\right)^3\left(\partial_\sigma A_\gamma \partial^\gamma A^\sigma\right), \\
\mathcal{L}_{7,4}^{\rm test} &= \left(\partial\cdot A\right)^2\left(\partial_\rho A_\gamma \partial^\sigma A^\rho \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,5}^{\rm test} &= \left(\partial\cdot A\right)^2\left(\partial^\sigma A^\rho \partial_\gamma A_\sigma \partial^\gamma A_\rho\right), \\
\mathcal{L}_{7,6}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial_\nu A_\gamma \partial^\rho A^\nu \partial^\sigma A_\rho \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,7}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial^\rho A^\nu \partial_\sigma A_\gamma \partial^\sigma A_\nu \partial^\gamma A_\rho\right), \\
\mathcal{L}_{7,8}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial_\rho A^\gamma \partial^\rho A^\nu \partial_\sigma A_\gamma \partial^\sigma A_\nu\right), \\
\mathcal{L}_{7,9}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial^\rho A^\nu \partial^\sigma A_\nu \partial_\gamma A_\sigma \partial^\gamma A_\rho\right), \\
\mathcal{L}_{7,10}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial_\rho A_\nu \partial^\rho A^\nu\right)^2, \\
\mathcal{L}_{7,11}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial_\nu A_\rho \partial^\rho A^\nu\right)\left(\partial_\gamma A_\sigma \partial^\gamma A^\sigma\right), \\
\mathcal{L}_{7,12}^{\rm test} &= \left(\partial\cdot A\right)\left(\partial_\nu A_\rho \partial^\rho A^\nu\right)\left(\partial_\sigma A_\gamma \partial^\gamma A^\sigma\right), \\
\mathcal{L}_{7,13}^{\rm test} &= \left(\partial_\mu A_\gamma \partial^\nu A^\mu \partial^\rho A_\nu \partial^\sigma A_\rho \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,14}^{\rm test} &= \left(\partial^\nu A^\mu \partial_\rho A_\gamma \partial^\rho A_\mu \partial^\sigma A_\nu \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,15}^{\rm test} &= \left(\partial^\nu A^\mu \partial^\rho A_\mu \partial^\sigma A_\nu \partial_\gamma A_\sigma \partial^\gamma A_\rho\right), \\
\mathcal{L}_{7,16}^{\rm test} &= \left(\partial^\nu A^\mu \partial_\rho A^\gamma \partial^\rho A_\mu \partial_\sigma A_\gamma \partial^\sigma A_\nu\right), \\
\mathcal{L}_{7,17}^{\rm test} &= \left(\partial_\nu A_\mu \partial^\nu A^\mu\right)\left(\partial_\rho A_\gamma \partial^\sigma A^\rho \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,18}^{\rm test} &= \left(\partial_\nu A_\mu \partial^\nu A^\mu\right)\left(\partial^\sigma A^\rho \partial_\gamma A_\sigma \partial^\gamma A_\rho\right), \\
\mathcal{L}_{7,19}^{\rm test} &= \left(\partial_\mu A_\nu \partial^\nu A^\mu\right)\left(\partial_\rho A_\gamma \partial^\sigma A^\rho \partial^\gamma A_\sigma\right), \\
\mathcal{L}_{7,20}^{\rm test} &= \left(\partial_\mu A_\nu \partial^\nu A^\mu\right)\left(\partial^\sigma A^\rho \partial_\gamma A_\sigma \partial^\gamma A_\rho\right).
\end{aligned} \tag{B.1}$$



JCAP02(2016)004

We again set

$$\mathcal{L}_6^{\rm test} = \sum_{k=1}^{20} x_k\, \mathcal{L}_{7,k}^{\rm test}, \tag{B.2}$$

leading to the following Hessian matrix elements:

$$\begin{aligned}
\mathcal{H}_7^{00} = {}& 2\,(10x_1+x_2+x_3)\,(\partial\cdot A)^3 + 2\,(x_4+x_{17}+x_{19})\,(\partial^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu) \\
& + 2\,(x_5+x_{18}+x_{20})\,(\partial^\mu A^\nu \partial_\rho A_\mu \partial^\rho A_\nu) - 6\,(2x_2+2x_3+x_4+x_5)\,(\partial\cdot A)^2\left(\partial^0 A^0\right) \\
& - 2\big[4x_{10}+2x_{11}+3\,(x_{17}+x_{18})\big]\,(\partial_\mu A_\nu \partial^\mu A^\nu)\left(\partial^0 A^0\right) \\
& - 2\big[2x_{11}+4x_{12}+3\,(x_{20}+x_{19})\big]\,(\partial^\mu A^\nu \partial_\nu A_\mu)\left(\partial^0 A^0\right) \\
& + (\partial\cdot A)\,\big\{2\,(3x_2+2x_{10}+x_{11})\,(\partial_\mu A_\nu \partial^\mu A^\nu) + 2\,(3x_3+x_{11}+2x_{12})\,(\partial^\mu A^\nu \partial_\nu A_\mu) \\
& + 4\,[x_6+x_7+x_8+x_9+2\,(x_{10}+x_{11}+x_{12})]\left(\partial^0 A^0\right)^2 \\
& - 2\,(2x_5+x_7+2x_8+x_9)\left(\partial_\mu A^0\right)\left(\partial^\mu A^0\right) - \left(\partial^0 A_\mu\right)\big[2\,(2x_5+x_7+2x_8+x_9)\left(\partial^0 A^\mu\right) \\
& - 4\,(3x_4+x_5+2x_6+x_7+x_9)\left(\partial^\mu A^0\right)\big]\big\} \\
& + \left(\partial^0 A^0\right)\big\{2\,[x_{14}+x_{15}+2\,(x_{16}+x_{18}+x_{20})]\,\big[\left(\partial_\mu A^0\right)\left(\partial^\mu A^0\right) + \left(\partial^0 A_\mu\right)\left(\partial^0 A^\mu\right)\big] \\
& + 2\,(5x_{13}+3x_{14}+3x_{15}+x_{16}+6x_{17}+2x_{18}+6x_{19}+2x_{20})\left(\partial^\mu A^0\right)\left(\partial^0 A_\mu\right)\big]\big\} \\
& - (\partial^\mu A^\nu)\left(\partial^0 A_\mu\right)\big\{2\big[x_7+x_{14}+x_{16}+2\,(x_9+x_{15})\big]\left(\partial^0 A_\nu\right) \\
& - 2\,(4x_6+x_7+5x_{13}+2x_{14}+x_{15})\left(\partial_\nu A^0\right)\big\} \\
& - (\partial^\mu A^\nu)\left(\partial_\mu A^0\right)\big\{2\,(x_7+4x_8+x_{14}+3x_{16})\left(\partial^0 A_\nu\right) \\
& + 2\big[x_7+x_{14}+x_{16}+2\,(x_9+x_{15})\big]\left(\partial_\nu A^0\right)\big\},
\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}
\mathcal{H}_7^{0i} = {}& (x_{14}+x_{16}+2x_{20})\left(\partial^0 A_\mu\right)\left(\partial^0 A^\mu\right)\left(\partial^i A^0\right) \\
& - (\partial\cdot A)^2\,\big[2\,(3x_2+x_5)\left(\partial^0 A^i\right) + (6x_3+3x_4+x_5)\left(\partial^i A^0\right)\big] \\
& + (\partial_\mu A_\nu \partial^\mu A^\nu)\,\big[2\,(2x_{10}+x_{18})\left(\partial^0 A^i\right) + (2x_{11}+3x_{17}+x_{18})\left(\partial^i A^0\right)\big] \\
& + (\partial^\mu A^\nu \partial_\nu A_\mu)\,\big[2\,(x_{11}+x_{20})\left(\partial^0 A^i\right) + (4x_{12}+3x_{19}+x_{20})\left(\partial^i A^0\right)\big] \\
& + (5x_{13}+x_{14}+2x_{15}+6x_{19}+2x_{20})\left(\partial^0 A_\mu\right)\left(\partial^i A^0\right)\left(\partial^\mu A^0\right) \\
& + (x_{14}+x_{15}+2x_{20})\left(\partial^i A^0\right)\left(\partial_\mu A^0\right)\left(\partial^\mu A^0\right) + \left(\partial^0 A^i\right)\big[(2x_{16}+2x_{18})\left(\partial^0 A_\mu\right)\left(\partial^0 A^\mu\right) \\
& + (2x_{14}+x_{15}+x_{16}+6x_{17}+2x_{18})\left(\partial^0 A_\mu\right)\left(\partial^\mu A^0\right) + (x_{15}+x_{16}+2x_{18})\left(\partial_\mu A^0\right)\left(\partial^\mu A^0\right)\big] \\
& + (\partial\cdot A)\,\big\{(2x_7+4x_8+2x_9+8x_{10}+4x_{11})\left(\partial^0 A^0\right)\left(\partial^0 A^i\right) \\
& + (4x_6+2x_7+2x_9+4x_{11}+8x_{12})\left(\partial^0 A^0\right)\left(\partial^i A^0\right) \\
& - (6x_4+4x_6+x_7)\left(\partial^i A^\mu\right)\left(\partial_\mu A^0\right) - (2x_5+x_7+4x_8)\left(\partial_\mu A^0 \partial^\mu A^i\right) \\
& - \left(\partial^0 A_\mu\right)\big[(2x_5+x_7+2x_9)\left(\partial^i A^\mu\right) - (2x_5+x_7+2x_9)\left(\partial^\mu A^i\right)\big]\big\} \\
& + \left(\partial^0 A^0\right)\big[(5x_{13}+2x_{14}+x_{15}+6x_{17}+6x_{19})\left(\partial^i A^\mu\right)\left(\partial_\mu A^0\right) \\
& + (x_{14}+3x_{16}+2x_{18}+2x_{20})\left(\partial_\mu A^0\right)\left(\partial^\mu A^i\right) \\
& + \left(\partial^0 A_\mu\right)\big\{(x_{14}+2x_{15}+x_{16}+2x_{18}+2x_{20})\left(\partial^i A^\mu\right) \\
& + \big[x_{14}+x_{15}+x_{16}+2\,(x_{18}+2x_{20})\big]\left(\partial^\mu A^i\right)\big\}\big] \\
& - (\partial^\mu A^\nu)\,\big\{(x_7+x_{14}+x_{16})\left(\partial^i A_\nu\right)\left(\partial_\mu A^0\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\partial^0 A_\nu\right)\left[\left(x_7+x_{14}+x_{15}\right)\left(\partial^i A_\mu\right)+2\left(2x_8+x_{16}\right)\left(\partial_\mu A^i\right)\right] \\
&+\left(4x_6+5x_{13}+x_{14}\right)\left(\partial^i A_\mu\right)\left(\partial_\nu A^0\right)+\left(x_7+x_{14}+x_{16}\right)\left(\partial_\mu A^i\right)\left(\partial_\nu A^0\right) \\
&+\left(2x_9+x_{15}+x_{16}\right)\ \left(\partial_\mu A^0\right)\left(\partial_\nu A^i\right) \\
&+\left(\partial^0 A_\mu\right)\left[2\left(x_9+x_{15}\right)\left(\partial^i A_\nu\right)+\left(x_7+x_{14}+x_{15}\right)\left(\partial_\nu A^i\right)\right]\Big\}, \qquad \text{(B.4)}
\end{aligned}
$$

whose vanishing leads to the solutions presented in section 3.3.2.

## References


[1] M. Ostrogradski, *Memoires sur les equations differentielles relatives au probleme des isoperimetres*, *Mem. Ac. St. Petersbourg* **VI** (1850) 385.

[2] R.P. Woodard, *Avoiding dark energy with* $1/r$ *modifications of gravity*, *Lect. Notes Phys.* **720** (2007) 403 [astro-ph/0601672] [INSPIRE].

[3] A. Nicolis, R. Rattazzi and E. Trincherini, *The galileon as a local modification of gravity*, *Phys. Rev.* **D 79** (2009) 064036 [arXiv:0811.2197] [INSPIRE].

[4] C. Deffayet, G. Esposito-Farese and A. Vikman, *Covariant galileon*, *Phys. Rev.* **D 79** (2009) 084003 [arXiv:0901.1314] [INSPIRE].

[5] C. Deffayet, S. Deser and G. Esposito-Farese, *Generalized galileons: all scalar models whose curved background extensions maintain second-order field equations and stress-tensors*, *Phys. Rev.* **D 80** (2009) 064015 [arXiv:0906.1967] [INSPIRE].

[6] C. Deffayet, X. Gao, D.A. Steer and G. Zahariade, *From k-essence to generalised Galileons*, *Phys. Rev.* **D 84** (2011) 064039 [arXiv:1103.3260] [INSPIRE].

[7] C. Deffayet and D.A. Steer, *A formal introduction to Horndeski and Galileon theories and their generalizations*, *Class. Quant. Grav.* **30** (2013) 214006 [arXiv:1307.2450] [INSPIRE].

[8] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *Generalized G-inflation: inflation with the most general second-order field equations*, *Prog. Theor. Phys.* **126** (2011) 511 [arXiv:1105.5723] [INSPIRE].

[9] G.W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, *Int. J. Theor. Phys.* **10** (1974) 363 [INSPIRE].

[10] G.W. Horndeski, *Conservation of charge and the Einstein-Maxwell field equations*, *J. Math. Phys.* **17** (1976) 1980 [INSPIRE].

[11] C. Deffayet, S. Deser and G. Esposito-Farese, *Arbitrary p-form galileons*, *Phys. Rev.* **D 82** (2010) 061501(R) [arXiv:1007.5278] [INSPIRE].

[12] P. Fleury, J.P. Beltrán-Almeida, C. Pitrou and J.-P. Uzan, *On the stability and causality of scalar-vector theories*, *JCAP* **11** (2014) 043 [arXiv:1406.6254] [INSPIRE].

[13] C. Deffayet, A.E. Gümrükçüoğlu, S. Mukohyama and Y. Wang, *A no-go theorem for generalized vector Galileons on flat spacetime*, *JHEP* **04** (2014) 082 [arXiv:1312.6690] [INSPIRE].

[14] L. Heisenberg, *Generalization of the Proca action*, *JCAP* **05** (2014) 015 [arXiv:1402.7026] [INSPIRE].

[15] G. Tasinato, *Cosmic acceleration from abelian symmetry breaking*, *JHEP* **04** (2014) 067 [arXiv:1402.6450] [INSPIRE].

[16] W. Lin, *A unifying framework for ghost-free Lorentz-invariant Lagrangian field theories*, arXiv:1508.03247.

[17] W. Lin, *Unifying ghost-free Lorentz-invariant Lagrangians*, arXiv:1510.05496.

[18] M. Hull, K. Koyama and G. Tasinato, *Covariantised vector galileons*, arXiv:1510.07029 [INSPIRE].

[19] C. de Rham and L. Heisenberg, *Cosmology of the galileon from massive gravity*, *Phys. Rev.* **D 84** (2011) 043503 [arXiv:1106.3312] [INSPIRE].

[20] J.B. Jiménez, R. Durrer, L. Heisenberg and M. Thorsrud, *Stability of Horndeski vector-tensor interactions*, *JCAP* **10** (2013) 064 [arXiv:1308.1867] [INSPIRE].

[21] V. Sivanesan, *Generalized multiple-scalar field theory in Minkowski space-time free of Ostrogradski ghosts*, *Phys. Rev.* **D 90** (2014) 104006 [arXiv:1307.8081] [INSPIRE].

[22] A. Padilla and V. Sivanesan, *Covariant multi-galileons and their generalisation*, *JHEP* **04** (2013) 032 [arXiv:1210.4026] [INSPIRE].

[23] T. Kobayashi, N. Tanahashi and M. Yamaguchi, *Multifield extension of G inflation*, *Phys. Rev.* **D 88** (2013) 083504 [arXiv:1308.4798] [INSPIRE].

[24] S. Ohashi, N. Tanahashi, T. Kobayashi and M. Yamaguchi, *The most general second-order field equations of bi-scalar-tensor theory in four dimensions*, *JHEP* **07** (2015) 008 [arXiv:1505.06029] [INSPIRE].

[25] G. Tasinato, *A small cosmological constant from Abelian symmetry breaking*, *Class. Quant. Grav.* **31** (2014) 225004 [arXiv:1404.4883] [INSPIRE].

[26] M. Hull, K. Koyama and G. Tasinato, *A Higgs mechanism for vector galileons*, *JHEP* **03** (2015) 154 [arXiv:1408.6871] [INSPIRE].

[27] J.D. Barrow, M. Thorsrud and K. Yamamoto, *Cosmologies in Horndeski's second-order vector-tensor theory*, *JHEP* **02** (2013) 146 [arXiv:1211.5403] [INSPIRE].

[28] J. Martin-Garcia, http://www.xact.es, (2002).

JCAP02(2016)004