MUC-NOTE-COOL_THEORY-250

1 Cooling Efficiency Factor

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Abstract

A local cooling efficiency parameter Q6 is defined by

Q6(z)= (relative change of emittance)/(relative change of flux)

Approximate equations for Q6 are given. The required average value of Q6 for the Collider Feasibility Study parameters is 20. It is shown that this value is achieved briefly in an RFOFO Ring without windows. Many rings, or continuous cooling with tapering, would appear needed to meet this requirement for a complete system.

1.1 Introduction

It has become popular to judge cooling systems by their "Merit Factor":

$$\text{Merit} \;=\; \frac{\text{6D emittance In}}{\text{6D Emittance Out}} \;\times\; \text{Transmission} \tag{1}$$

Merit depends on the initial beam size. A maximum Merit (as used by Harold Kirk) can be obtained by starting with a distribution that uniformly fills the lattice acceptance, but this is not a realistic distribution and thus gives an unrealistic Merit.

Merit is also maximized by cooling for as long as possible in a given system, even if the gains are slow. This will not give the best final performance for multiple systems.

A better criterion, perhaps, is a local parameter: "Cooling Efficiency" ($Q_6(z)$).

1.2 Definition of Efficiency Q_6

Define

$$Q_6(z) = \frac{d\epsilon_6/\epsilon_6}{dN/N} \tag{2}$$

Note, if $Q_6(z)$=constant, then

$$\int_o^n \frac{d\epsilon_6}{\epsilon_6} \;=\; Q_6 \;\int_o^n \frac{dN}{N}$$

$$\mathrm{Ln}\left(\frac{\epsilon_6(n)}{\epsilon_6(o)}\right) = Q_6 \ \mathrm{Ln}\left(\frac{N(n)}{N(o)}\right)$$

$$\frac{N(n)}{N(o)} = \left(\frac{\epsilon_6(n)}{\epsilon_6(o)}\right)^{1/Q_6} \tag{3}$$

This suggests a definition for an effective average $Q_6(ave)$:

$$Q_6(ave) = \frac{\mathrm{Ln}\left(\frac{\epsilon_6(n)}{\epsilon_6(o)}\right)}{\mathrm{Ln}\left(\frac{N(n)}{N(o)}\right)}$$

1.3 Requirement for a collider

The Luminosity of a collider is given by

$$\mathcal{L} \propto \frac{\mathrm{N}^2\ \mathrm{f}}{\sigma_\mathrm{x}\ \sigma_\mathrm{y}} \propto \frac{\mathrm{N}^2}{\epsilon_\perp\ \beta} \propto \frac{\mathrm{N}^2}{\epsilon_6^{2/3}}$$

If N^2 and $\epsilon_6^{2/3}$, instead of N and ϵ_6, are taken as the criterion for cooling then:

$$\frac{N^2(n)}{N^2(o)} = \left(\frac{\epsilon_6(n)^{2/3}}{\epsilon_6(o)^{2/3}}\right)^{3/Q_6}$$

Giving just a factor of 3 in Q. If there were no limit to how small the emittance can be, then any fixed $Q > 3$ will raise luminosity without limit, as the emittance is reduced and the N^2 does not fall faster than $\epsilon^{2/3}$.

This raises an orthogonal issue: what is the minimum limit to 6D emittance and whether that is related to N. This we do not discuss.

We are concerned here with the cooling requirements to reach pre-determined goals. e.g. In the Collider Feasibility Study, we specified:

$$\frac{\epsilon_6(n)}{\epsilon_6(o)} = 10^{-6}, \qquad \frac{N(n)}{N(o)} = 0.5$$

$$\text{Requiring} \qquad Q_6 = \frac{\mathrm{Ln}(10^{-6})}{\mathrm{Ln}(0.5)} = 20$$

1.4 Theoretical Expectation

With no heating:

$$\frac{d\epsilon_i}{\epsilon_i} = J_i \frac{dp}{p} = J_i \frac{dE}{E} \frac{1}{\beta_v^2}$$

Where i is x, y or z, and J_i are the partition functions, and dp and dE are the momentum and energy loss in the material.

If there are no wedges:

$$J_x = J_y = 1, \quad J_z \approx 0$$

but with wedges and dispersion we can distribute the J's as long as

$$J_x + J_y + J_z \approx 2$$

Defining Q_i in x,y and z, where $Q_6 = Q_x + Q_y + Q_z$, and τ_μ is the muon lifetime:

$$Q_i = \frac{\frac{d\epsilon_i}{\epsilon_i}}{\frac{dn}{n}} = \frac{J_i \frac{dE}{E} \frac{1}{\beta_v^2}}{\frac{d\ell}{(c\gamma\beta_v\tau_\mu)}} \left(\frac{\text{Decay Loss}}{\text{All Losses}}\right)$$

If the average energy loss in the material $dE/d\ell$ is set equal to the average maximum acceleration gradient $\mathcal{E}_{\text{rf}}$ times the rf phase ϕ, then

$$Q_i = \left(J_i \frac{c\ \tau_\mu}{m_\mu}\right) \ \mathcal{E}_{\text{rf}} \ \sin(\phi) \ \frac{1}{\beta_{\text{v}}} \left(\frac{\text{Decay Loss}}{\text{All Losses}}\right) \tag{4}$$

If heating is included, then the cooling Rate is given by:

$$\frac{d\epsilon}{\epsilon} = \left(1 - \frac{\epsilon_{\text{min}}}{\epsilon}\right) J_i \frac{dp}{p}$$

so

$$Q_i = \left(1 - \frac{\epsilon_{\text{min}}}{\epsilon}\right) \left(J_i \frac{c\ \tau_\mu}{m_\mu}\right) \ \mathcal{E}_{\text{rf}} \ \sin(\phi) \ \frac{1}{\beta_{\text{v}}} \left(\frac{\text{Decay Loss}}{\text{All Losses}}\right) \tag{5}$$

where for the **transverse directions**:

$$\epsilon_{x,y}(min) = \frac{\beta_\perp}{J_i\ \beta_v} C(mat, E) \tag{6}$$

and

$$C(mat, E) = \frac{1}{2} \left(\frac{14.1\ 10^6}{(m_\mu)}\right)^2 \frac{1}{L_R\ d\gamma/ds} \tag{7}$$

(For Hydrogen, $C(mat,E) \approx 38\ 10^{-4}$) .

and in the **longitudinal direction**:

$$(\epsilon_z)_{\rm min} \;=\; \beta_v \gamma \left(\frac{\sigma_p}{p}\right)^2_{\rm min} \beta_{\|} \tag{8}$$

where $\beta_{\|}$ is set by the rf:

$$\beta_{\|} \;=\; \frac{\beta_v}{2\pi} \sqrt{\frac{\beta_v \gamma\ \lambda_{rf}\ (m_\mu)\ \alpha}{\mathcal{E}_{\rm rf}\ \cos(\phi)}} \tag{9}$$

where, in a linear lattice, or at low energies

$$\alpha \;\approx\; \frac{1}{\gamma^2} \tag{10}$$

and

$$\left(\frac{\sigma_p}{p}\right)_{\rm min} \;=\; D(mat) \sqrt{\frac{\gamma}{\beta_v^2} \left(1 - \frac{\beta_v^2}{2}\right) \frac{1}{J_z}} \tag{11}$$

$$D(mat) \;=\; \left(\left(\frac{m_e}{m_\mu}\right) \sqrt{\frac{0.06\ Z\ \rho}{2\ A\ (d\gamma/ds)}}\right)$$

(For Hydrogen, $D(mat,E) \approx 1.45$ %).

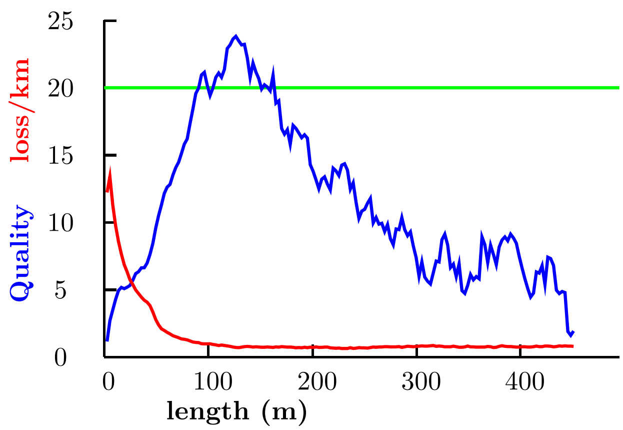
We see that:

- The efficiency Q_6 is strongly (linearly, for the same phase) dependent on the accelerating gradient $\mathcal{E}_{\rm rf}$.
- The cooling rate per unit of length is strongly dependent on the cooling momentum $\propto 1/(\gamma\beta^2)$, but Q_6 is only weakly dependent on the cooling momentum ($Q_6 \propto 1/\beta$): at higher momenta the cooling is slower but the decay rate is also slower.
- The efficiency is greatly reduced if the particle loss from scraping exceeds the decay loss.
- The efficiency is a function of the ratio of emittances to their equilibrium values
$$Q_i \propto \left(1 \;-\; \frac{\epsilon_{\rm min}}{\epsilon}\right)_{x,y,z}$$
- In the transverse direction the smallness of $\epsilon_{\rm min}/\epsilon$ will be limited by the lattice angular acceptance ($\sigma_\theta \;\propto\; \sqrt{\epsilon/\beta} \;\propto\; \sqrt{\epsilon/\epsilon_{\rm min}}$) needed to assure negligible scraping.

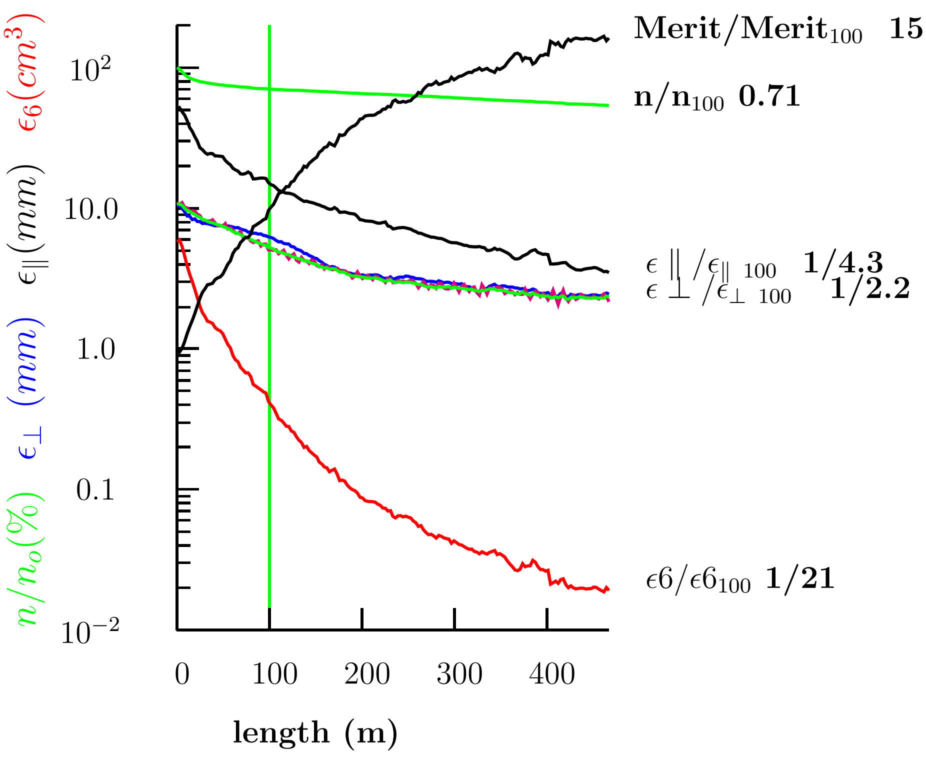
- In the longitudinal direction, $\epsilon_{\rm min}/\epsilon$ will be limitted by the bucket size needed to avoid significant losses.
- Maintaining $\epsilon/\epsilon(min.)$ in the transverse direction implies tapering down the transverse beta with distance.
- Maintaining $\epsilon/\epsilon(min.)$ in the longitudinal direction implies tapering down the longitudinal beta, so as to compress the bunch and maintain a high rms dp/p. That can be done by raising the rf phase, or lowering the rf frequency.

1.5 RFOFO Ring Example

e.g. for RFOFO Ring with Hydrogen wedges, without windows or Injection/extraction



We see that the losses are only small compared with decay loss after about 100 m of cooling in this ring.



	len	trans	$\epsilon_\perp$	$\epsilon_\parallel$	ϵ_6	max Q	merit
	m	%	mm	mm	cm^3		
final	468	54	2.3	3.5	0.019	24	
after 100 m	100		5.0	15	0.375		
ratio 100m→			2.2	4.3	21	24	**15**
initial			10.7	50.1	5.787		
ratio from start			4.6	14.4	302.0		**162**

We Note:

- Initially, the emittances are falling, but Q is low because particles are being lost by scraping or falling out of the bucket.
- After about 100 m, particle loss, other than decay, has become negligible and Q reaches a maximum that exceeds the above collider specification of 20.
- At this "negligible loss" point $\epsilon_\perp/\epsilon_{\perp \rm min} = 2.2$ and $\epsilon_\parallel/\epsilon_{\parallel \rm min} = 4.3$. These will be reduced when rf and absorber windows are included.
- Later, as the emittances approach their equilibria, the efficiency falls. But if the beam were transfered between a sequence of rings, or if the lattice were a tapered continuous spiral, then with falling transverse and longitudinal betas, the efficiency could be maintained.
- As the longitudinal emittance falls and higher rf phases, frequencies, and accelerating gradients can be used, the achievable efficiency may rise. But lower transverse emittances will require lower longitudinal betas that will restrict the length of absorbers, possibly require the use of Li or LiH, and thus reduce the achievable efficiency. Only further detailed studies will determine if the required efficiencies can be obtained over the required range of emittances.

1.6 Required Acceptances for low loss

At 100 m, in the 4D transverse planes: $\sigma_r = \sqrt{2}\,\sigma_\perp = 4.58 cm$, compared with the aperture of 18 cm, i.e. 3.9 sigma. In terms of the x, y acceptance of $A_r = \beta_v\gamma\, r^2\, \beta_\perp = 52$ mm which is 10.2 $\times\ \epsilon_\perp$.

In the Longitudinal plane, the $\sigma_z = 10$ cm compared with an approximate bucket width of ± 32 cm, or 3.2 sigma. Or in terms of the longitudinal acceptance of $A_{\parallel} = \beta_v \gamma \, \sigma_z \, \sigma_p/p = 2 \times .32 \times 0.24 =$ 153 mm, which is also approximately $10 \times \epsilon_{\parallel}$.

1.7 Conclusion

- Efficiency Q_6 is a useful criterion for judging cooling systems
- Its maximum value does not depend on unrealistic initial beams
- It can be applied to linear or ring systems equally
- The theoretical expression for this efficiency is a useful guide to system design.

- A goal of $Q_6 \geq 20$ needed for the 'Collider Feasibility Study' parameter, provides a useful benchmark
- It is shown that this value is achieved briefly in an RFOFO Ring without windows.
- Many rings, or continuous cooling with tapering, would appear needed to meet this requirement over the required range of emittances. Only detailed studies will determine if it can be achieved.