Exact Three-Point Functions of Determinant Operators in Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

Yunfeng Jiang,[1] Shota Komatsu,[2] and Edoardo Vescovi[3]
[1]Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland
[2]School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA
[3]The Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom

(Received 5 August 2019; published 7 November 2019)

We introduce a nonperturbative approach to correlation functions of two determinant operators and one nonprotected single-trace operator in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Based on the gauge-string duality, we propose that they correspond to overlaps on the string world sheet between an integrable boundary state and a state dual to the single-trace operator. We determine the boundary state using the symmetry and integrability of the dual superstring σ model and write expressions for the correlators at finite coupling, which we conjecture to be valid for operators of arbitrary size. The proposal is put to the test at weak coupling.

DOI: 10.1103/PhysRevLett.123.191601

Introduction.—To advance our understanding of nonperturbative dynamics in gauge theories, it is useful to study simple models with rich enough structures. $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions is one of the leading candidates for the following reasons: First it admits the planar large N_c limit, which makes it amenable to analytical studies. Second it is a conformal field theory, and all of the correlation functions can be decomposed into two- and three-point functions. Third it can be described alternatively in terms of two-dimensional string world sheets which we can analyze exactly using integrability. The application of integrability led to a complete determination of two-point functions of local operators [1]. It was applied also to the three-point function [2], but the result is still unsatisfactory since it is given by a series expansion which one needs to resum.

In this Letter, we present the first fully nonperturbative result for the three-point function valid for a large class of operators [3]. Specifically, we study the correlator of two determinant operators and one nonprotected single-trace operator. By interpreting this correlator as an overlap on the string world sheet between a boundary state and a state dual to the single-trace operator, we write nonperturbative expressions using the framework of the thermodynamic Bethe ansatz (TBA) [4].

Setup and basic strategy.—The main subject is the three-point function of a nonprotected single-trace operator O and two determinant operators $\mathcal{D}_{1,2} \equiv \det \mathfrak{Z}(a_{1,2})$ [5] with

$$\mathfrak{Z}(a) \equiv \frac{(1+a^2)\Phi^1 + i(1-a^2)\Phi^2 + 2ia\Phi^4}{\sqrt{2}}\bigg|_{\substack{x^\mu=\\(0,a,0,0)}}, \tag{1}$$

where $\Phi^{1,2,4}$ are real scalar fields in $\mathcal{N} = 4$ SYM theory. Owing to the superconformal symmetry, its spacetime dependence is fixed at [2,6]

$$\langle \mathcal{D}_1 \mathcal{D}_2 \mathsf{O}(0) \rangle = \left(\frac{a_1 - a_2}{a_1 a_2}\right)^{\Delta - J} \mathfrak{D}_{\mathsf{O}}, \tag{2}$$

where $\mathfrak{D}_{\mathsf{O}}$ is the structure constant, while Δ and J are the conformal dimension and the R charge of O.

The goal of this Letter is to compute $\mathfrak{D}_{\mathsf{O}}$ nonperturbatively using the gauge-string duality. As discussed in Refs. [7–9], the duality maps Eq. (2) to a closed string in $\mathrm{AdS}_5 \times S^5$, which ends on a geodesic of a D-brane dual to the determinant operators (see Fig. 1). On the string world sheet, this corresponds to an overlap between a

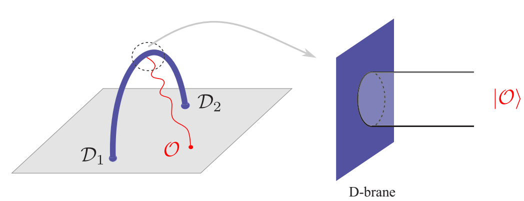

FIG. 1. The anti–de Sitter (AdS) description of the three-point function. The thick curve represents a geodesic of the D-brane dual to the determinant operators, while the wavy line denotes a closed string dual to the single-trace operator. On the world sheet, it corresponds to an overlap with a boundary state.

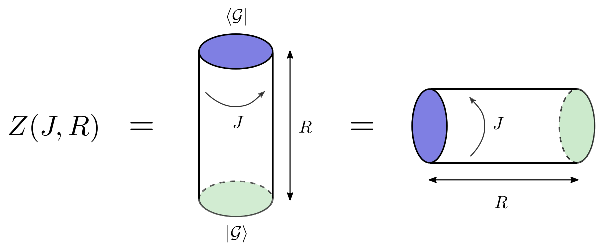

FIG. 2. The partition function $Z(J,R)$ evaluated in two different channels. The left (right) panel denotes the closed- (open-) string channel of the same partition function. To compute the overlap $\langle\mathcal{G}|\Omega\rangle$, we take the limit where J is finite and $R\to\infty$.

boundary state $\langle\mathcal{G}|$ and a state dual to O. To evaluate such an overlap, we first consider a partition function $Z(J,R)$ of a cylinder world sheet whose ends are capped off by the boundary states (see Fig. 2). In the limit $R\to\infty$, the expansion of $Z(J,R)$ in the closed string channel is dominated by the ground state $|\Omega\rangle$,

$$Z(J,R)=\sum_{\psi_c}\langle\mathcal{G}|\psi_c\rangle e^{-E_{\psi_c}R}\langle\psi_c|\mathcal{G}\rangle \\ \xrightarrow{R\to\infty}|\langle\mathcal{G}|\Omega\rangle|^2e^{-E_\Omega R}. \quad (3)$$

By contrast, in the open string channel, $Z(J,R)$ can be viewed as the thermal free energy, and the limit corresponds to the thermodynamic limit in which the volume of the space becomes infinite. This allows us to compute $\langle\mathcal{G}|\Omega\rangle$ using the TBA. The result for excited states can be obtained from $\langle\mathcal{G}|\Omega\rangle$ by analytic continuation [10].

Constraints on boundary states.—To apply the aforementioned strategy, we first determine the boundary state $\langle\mathcal{G}|$ in the infinite-volume ($J\to\infty$) limit. For this, we assume that $\langle\mathcal{G}|$ is an *integrable boundary state*—namely, a state corresponding to a boundary condition which preserves infinitely many conserved charges [11]. The assumption is justified *a posteriori* by agreement with weak-coupling computations, as we shall see later. For integrable boundary states, the overlap in the $J\to\infty$ limit can be factorized into two-particle overlaps

$$F_{\mathbf{AB}}(u)\equiv\langle\mathcal{G}|\mathcal{X}_{\mathbf{A}}(u)\mathcal{X}_{\mathbf{B}}(\bar{u})\rangle|_{J\to\infty}, \quad (4)$$

where the $\mathcal{X}$'s are magnons in the $\mathcal{N}=4$ SYM spin chain, and $\mathbf{A}=A\dot{A}$ and $\mathbf{B}=B\dot{B}$ are in the bifundamental representation of the $\mathfrak{psu}(2|2)^2$ symmetry [12]. The rapidities u and $\bar{u}$ are parity conjugate to each other and satisfy $x^{\pm}(\bar{u})=-x^{\mp}(u)$, where $f^{\pm}(u)\equiv f(u\pm i/2)$ and

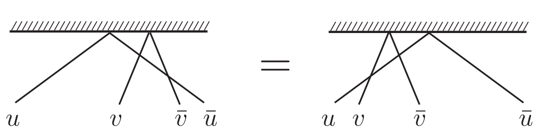

FIG. 3. The boundary Yang-Baxter equation.

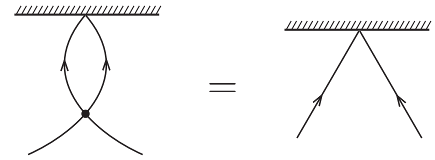

FIG. 4. Watson's equation for the two-particle overlap.

the Zhukovsky variable $x(u)$ is defined by $x(u)\equiv u+\sqrt{u^2-4g^2}/2g$, with $g\equiv\sqrt{\lambda}/4\pi$ and λ being the 't Hooft coupling constant.

Boundary Yang-Baxter equation: The integrable boundary state satisfies the so-called boundary Yang-Baxter equation, which reads (see Fig. 3)

$$\langle\mathcal{G}|\mathbb{S}_{24}\mathbb{S}_{34}|\mathcal{X}_1(u)\mathcal{X}_2(v)\mathcal{X}_3(\bar{v})\mathcal{X}_4(\bar{u})\rangle|_{J\to\infty} \\ =\langle\mathcal{G}|\mathbb{S}_{13}\mathbb{S}_{12}|\mathcal{X}_1(u)\mathcal{X}_2(v)\mathcal{X}_3(\bar{v})\mathcal{X}_4(\bar{u})\rangle|_{J\to\infty}, \quad (5)$$

where $\mathbb{S}_{kl}$ is the bulk S matrix [12] between $\mathcal{X}_k$ and $\mathcal{X}_l$.

Watson's equation: The second constraint is Watson's equation, which states that an exchange of particles is equivalent to a multiplication of the S matrix. Explicitly, it reads (see Fig. 4)

$$F_{\mathbf{AB}}(u)=\mathbb{S}_{\mathbf{AB}}^{\mathbf{CD}}(u,\bar{u})F_{\mathbf{CD}}(\bar{u}). \quad (6)$$

Decoupling equation: The last condition is the decoupling condition, which is equivalent to the boundary unitarity in Ref. [11]. It states that a pair of particle-antiparticle pairs must decouple from the rest of the overlap (see Fig. 5) and reads

$$F_{\mathbf{AB}}(u)\mathfrak{C}^{\mathbf{BB}'}F_{\mathbf{B}'\mathbf{C}'}(\bar{u}^{2\gamma})\mathfrak{C}^{\mathbf{C}'\mathbf{C}}=\delta_{\mathbf{A}}^{\mathbf{C}}, \quad (7)$$

where $\mathfrak{C}$ is the charge conjugation matrix [13], and $u^{2\gamma}$ is the crossing transformation defined by $x^{\pm}(u^{2\gamma})=[1/x^{\pm}(u)]$.

Solution: Solving these constraints, the two-particle overlap is fixed at

$$F_{\mathbf{AB}}(u)=\frac{x^+}{x^-}\frac{u-\frac{i}{2}}{u}\frac{\sigma_B(u)}{\sigma(u,\bar{u})}(-1)^{|\dot{A}||B|}M_{\mathbf{A},\mathbf{B}}, \quad (8)$$

where $|\bullet|$ is the grading of the index $\bullet$ and $\sigma(u,v)$ is the bulk dressing phase [14]. There are two choices for the matrix part $M_{\mathbf{A},\mathbf{B}}$ [15], and we conjecture that the

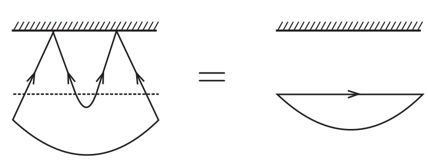

FIG. 5. Decoupling equation for the two-particle overlap.

three-point function is given by a sum of the two overlaps, $\mathfrak{D}_{\mathsf{O}} = \langle \mathcal{G}_{z=i}|\mathsf{O}\rangle + \langle \mathcal{G}_{z=-i}|\mathsf{O}\rangle$. $\sigma_B(u)$ is the *boundary dressing phase* satisfying

$$\sigma_B(\bar{u}) = \sigma_B(u), \qquad \sigma_B(u)\sigma_B(u^{2\gamma}) = \frac{u^2}{u^2+\frac{1}{4}}. \tag{9}$$

A solution is given by $\sigma_B(u) = 4^{1+x^+x^-/1-x^+x^-}[G(x^+)/G(x^-)]$ with

$$\log G(x) = \oint \frac{dz}{2\pi i} \frac{\log \mathfrak{G}[g(z+z^{-1})]}{x-z}, \tag{10}$$

and $\mathfrak{G}(u) \equiv \{[\Gamma(\frac{1}{2}-iu)\Gamma(1+iu)]/[\Gamma(\frac{1}{2}+iu)\Gamma(1-iu)]\}$.

g function for ground state.—We now discuss the ground-state overlap $\langle \mathcal{G}|\Omega\rangle$ for finite J. For this, we consider $Z(J,R)$ in the open string channel (also known as the *mirror channel*) and take the limit $R\to\infty$:

$$Z(J,R) = \sum_{\psi_o} e^{-\tilde{E}_{\psi_o} J} \overset{R\to\infty}{=} \mathcal{N} \int \mathcal{D}\rho\, e^{-RS_{\mathrm{eff}}[\rho]}. \tag{11}$$

As shown above, in the limit $R\to\infty$, one can replace the sum over ψ_o with a path integral of densities ρ.

Bethe equation in the mirror channel: The crucial input for writing S_{eff} is the boundary asymptotic Bethe equation (BABA), which constrains the rapidities of magnons. Schematically, it reads (see Fig. 6)

$$\mathbf{1} = e^{2i\tilde{p}_j R}\mathbb{R}_L(u_j)\mathbb{R}_R(u_j)\prod_{k\neq j}\mathbb{S}(u_j,u_k)\mathbb{S}(u_j,\bar{u}_k), \tag{12}$$

where $\mathbb{R}_{L(R)}$ is the left or right reflection matrix. The reflection matrices are related to the infinite-volume overlap (8) by $[\mathbb{R}_L]^{\mathbf{B}}_{\mathbf{A}}(u) = [\mathbb{R}_R]^{\mathbf{B}}_{\mathbf{A}}(\bar{u}) = F_{\mathbf{AC}}(u^\gamma)\mathfrak{C}^{\mathbf{CB}}$, with u^γ being the mirror transformation defined by $x^+(u^\gamma) = 1/x^+(u)$ and $x^-(u^\gamma) = x^-(u)$. As a result, we find [16]

$$[\mathbb{R}_L]^{B\dot{B}}_{A\dot{A}}(u) = \frac{u-\frac{i}{2}}{u}\frac{\sigma_B(u^\gamma)}{\sigma(\bar{u}^\gamma,u^\gamma)}\mathcal{S}^{B\dot{B}}_{A\dot{A}}(\bar{u}^\gamma,u^\gamma), \tag{13}$$

where $\mathcal{S}$ is a single copy of the $\mathfrak{psu}(2|2)$ S matrix [12]. The structure of $\mathbb{R}_{L,R}$ allows for the *unfolding* interpretation; the BABA (12) can be mapped to an ABA of a closed string with a single $\mathfrak{psu}(2|2)$ symmetry (see Fig. 6).

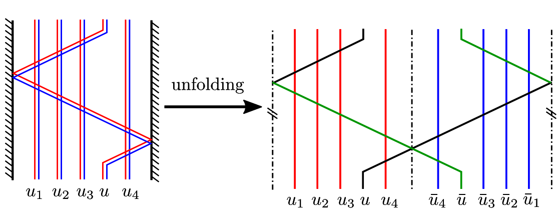

FIG. 6. The BABA and its unfolding interpretation. The structure of the reflection matrix allows us to map the BABA to an ABA of a closed chain with a single $\mathfrak{psu}(2|2)$ symmetry.

TBA equation: S_{eff} can be derived from an ABA following the standard derivation of the TBA. In the $R\to\infty$ limit, Eq. (11) can be approximated by the saddle point $\delta S_{\mathrm{eff}}/\delta\rho = 0$. Owing to the unfolding structure, the saddle-point equations coincide with the standard TBA for the spectrum [17–19] with the identification $Y_{a,s}(\bar{u}) = Y_{a,-s}(u)$. They take a form of $\log Y_{a,s} = \varphi^+_{a,s}$. For instance, $\varphi^+_{a,0}$ reads (the full equations are given in Ref. [20])

$$\begin{aligned}\varphi^+_{a,0} \equiv{}& -J\tilde{E}_a + \log(1+Y_{b,0}) * (\mathcal{K}_+)^{\bullet\bullet}_{b,a}\\ &+\log(1+Y_{m,1})\star(\mathcal{K}_+)^{\rhd\bullet}_{m-1,a} + \log(1+1/Y_{2,2})\hat{\star}(\mathcal{K}_+)^{y\bullet}_{+a}\\ &+\log(1+Y_{1,1})\hat{\star}(\mathcal{K}_+)^{y\bullet}_{-a}.\end{aligned} \tag{14}$$

Here we follow the notations in Ref. [21], and $\star$, $*$, and $\hat{\star}$ denote the convolutions along $[-\infty,\infty]$, $[0,\ \infty]$, and $[-2g, 2g]$, respectively. $\mathcal{K}_+$ is a symmetrized kernel defined by $K(u,v)+K(u,\bar{v})$, with K being the standard TBA kernel.

g function: The saddle-point value of S_{eff} gives only the exponentially decaying piece in Eq. (3), $e^{-E_\Omega R}$. To read off the overlap, we need to consider the one-loop fluctuation around the saddle point and the $O(1)$ normalization factor $\mathcal{N}$ in Eq. (11). Such an analysis has been performed in the literature [22–26], and the application to our problem leads to an expression for the ground states, which corresponds to Bogomol'nyi-Prasad-Sommerfield operators in $\mathcal{N}=4$ SYM,

$$\langle \mathcal{G}|\Omega\rangle = e^{\sum_a \int_0^\infty (du/2\pi)\Theta_a \log(1+Y_{a,0})}\frac{\sqrt{\mathrm{Det}(1-\hat{G})}}{\mathrm{Det}(1-\hat{G}_+)}. \tag{15}$$

Here a runs from 1 to ∞ and $\Theta_a(u) = i\partial_u \log r_a(\bar{u}) - \pi\delta(u) + (i/2)\partial_u \log S^{\bullet\bullet}_{aa}(u,\bar{u})$, where $S^{\bullet\bullet}_{ab}$ is the bound-state S matrix, and r_a is the bound-state reflection factor given in Ref. [15]. Det denotes the Fredholm determinant [27], and $\hat{G}_+$ is an integral kernel defined by $[\hat{G}_+]^{(a,s)}_{(b,t)}(u,v) \equiv [\delta\varphi^+_{a,s}(u)/\delta\log Y_{b,t}(v)]$. Similarly, $\hat{G}$ is given by $\delta\varphi_{a,s}(u)/\delta\log Y_{b,t}(v)$, where $\varphi_{a,s}$ are the "right-hand sides" of the standard TBA, $\log Y_{a,s} = \varphi_{a,s}$, without identification of Y functions.

Conjecture for SL(2) sector.—We now generalize Eq. (15) to excited states in the SL(2) sector using the analytic continuation trick [10] following the standard TBA analysis.

g function for excited states: After the analytic continuation, poles of $1/(1+1/Y_{1,0})$ cross the integration contour and modify the overlap (15). As a result, we find that the structure constant $\mathfrak{D}_{\mathsf{O}}$ is given by

$$\mathfrak{D}_{\mathsf{O}} = -\frac{i^J + (-i)^J}{\sqrt{J}} e^{\sum_a \int_0^\infty (du/2\pi)\Theta_a \log(1+Y_{a,0})} \times \sqrt{\prod_{1\le s\le \frac{M}{2}} \frac{u_s^2 + \frac{1}{4}}{u_s^2} \sigma_B^2(u_s)} \frac{\sqrt{\mathrm{Det}(1-\hat{G}^\bullet)}}{\mathrm{Det}(1-\hat{G}^\bullet_+)}. \tag{16}$$

This is the main result of this Letter, which we conjecture to be valid for any length J and at finite λ. Here the J-dependent prefactor reflects the fact that the true boundary state is a sum of two boundary states, as mentioned below Eq. (8). $\hat{G}^\bullet$ and $\hat{G}^\bullet_+$ are given by [28]

$$\hat{G}^\bullet \cdot f = \sum_{k=1}^{M} \frac{iK_{1,x}^{\bullet,X}(u_k,u)}{\partial_u \log Y_{1,0}(u_k)} f(u_k) + \hat{G}\cdot f,$$
$$\hat{G}^\bullet_+ \cdot f = \sum_{k=1}^{M/2} \frac{i(\mathcal{K}_+)_{1,x}^{\bullet,X}(u_k,u)}{\partial_u \log Y_{1,0}(u_k)} f(u_k) + \hat{G}_+\cdot f. \tag{17}$$

Here and below, x and X take various indices and symbols which represent different bound states. The sum in Eq. (17) come from the poles crossing the contours, and the u_k's are the magnon rapidities satisfying the parity condition,

$$u_{M/2+k} = \bar{u}_k \quad (1 \le k \le M/2). \tag{18}$$

They are the solutions to the *exact Bethe equations* [17],

$$\phi(u_j) = 2\pi i\left(n_j + \frac{1}{2}\right), \qquad n_j \in \mathbb{Z}, \tag{19}$$

with

$$\phi(u) \equiv -J\tilde{E}_1 + \sum_{k=1}^{M} \log S_{11}^{\bullet\bullet}(u,u_k) + \sum_{x,X} \log(1+Y_{X,x})\star K_{x,1}^{X,\bullet}. \tag{20}$$

Exact Gaudin determinants: The result, Eq. (17), can be rewritten into a form similar to the so-called Gaudin determinants. For this, we first split the kernel $\hat{G}^\bullet$ into a sum S and an integral I [see Eq. (17)], and rewrite $\mathrm{Det}(1-\hat{G}^\bullet)$ as $\mathrm{Det}(1-\mathsf{S}-\mathsf{I}) = \mathrm{Det}(1-\tilde{\mathsf{S}})\times \mathrm{Det}(1-\mathsf{I})$ with $\tilde{\mathsf{S}} \equiv \mathsf{S}/(1-\mathsf{I})$. Similarly, $\mathrm{Det}(1-\hat{G}^\bullet_+)$ can be split into a sum S_+ and an integral I_+ and can be reexpressed as $\mathrm{Det}(1-\hat{G}^\bullet_+) = \mathrm{Det}(1-\tilde{\mathsf{S}}_+)\times \mathrm{Det}(1-\mathsf{I}_+)$ with $\tilde{\mathsf{S}}_+ \equiv \mathsf{S}_+/(1-\mathsf{I}_+)$.

Next we consider $\partial_{u_k}\phi(u_j)$ $(j,k = 1,\ldots,M)$. The derivative ∂_{u_k} can act on one of the following, $p(u_k)$, $\log S_{11}^{\bullet\bullet}$, or $Y_{X,x}$, in Eq. (20). We then eliminate $\partial_{u_k} Y_{X,x}$ by considering the *excited state TBA* (see Ref. [17] for the full set of equations)

$$\log Y_{a,0} = -J\tilde{E}_a + \sum_{k=1}^{M} \log S_{a,1}^{\bullet\bullet}(u,u_k) + \sum_{x,X} \log(1+Y_{X,x})\star K_{x,a}^{X,\bullet}, \tag{21}$$

taking a derivative ∂_{u_k} of both sides, and solving for $\partial_{u_k} Y_{X,x}$. The parity condition (18) is imposed only at the end of the computation. As a result of these manipulations, we find that $\det[\partial_{u_k}\phi(u_j)] \propto \mathrm{Det}(1-\tilde{\mathsf{S}})$ up to some constant of proportionality. The relation shows in particular that $\mathrm{Det}(1-\tilde{\mathsf{S}})$ is actually a finite-dimensional determinant, although it was initially defined as the Fredholm determinant. Details of the rewriting are explained in a toy example in Ref. [15].

On the other hand, if we first impose the parity condition (18) and compute the derivatives $\partial_{u_k}\phi(u_j)$, we find that $\det[\partial_{u_k}\phi(u_j)]$ $(j,k = 1,\ldots,M/2)$ is now proportional to $\mathrm{Det}(1-\tilde{\mathsf{S}}_+)$. Upon taking the ratio, the constants of proportionality cancel out and we obtain

$$\frac{\sqrt{\mathrm{Det}(1-\tilde{\mathsf{S}})}}{\mathrm{Det}(1-\tilde{\mathsf{S}}_+)} = \frac{\sqrt{\det[\partial_{u_k}\phi(u_j)]}}{\det[\partial_{u_k}\phi^+(u_j)]}, \tag{22}$$

where ϕ^+ denotes that we are imposing the parity condition before computing derivatives. These determinants can be viewed as the finite-volume version of the Gaudin determinants for the norm of the spin chain. They also resemble the finite-volume one-point functions in the sin(h)-Gordon model [29–31].

Using this rewrite, we obtain an alternative representation for the Fredholm determinants in Eq. (16),

$$\frac{\sqrt{\mathrm{Det}(1-\hat{G}^\bullet)}}{\mathrm{Det}(1-\hat{G}^\bullet_+)} = \frac{\sqrt{\det[\partial_{u_k}\phi(u_j)]}}{\det[\partial_{u_k}\phi^+(u_j)]} \frac{\sqrt{\mathrm{Det}(1-\tilde{\mathsf{I}})}}{\mathrm{Det}(1-\tilde{\mathsf{I}}_+)}. \tag{23}$$

Asymptotic formula: Using the representation (23), one can take the *asymptotic limit* of Eq. (16), in which the size of the operator becomes large, $J \gg 1$. In this limit, the middle-node Y functions are exponentially suppressed, $Y_{a,0} \to 0$, and one can show that both

$$e^{\sum_a \int_0^\infty (du/2\pi)\Theta_a \log(1+Y_{a,0})} \quad \text{and} \quad \frac{\sqrt{\mathrm{Det}(1-\tilde{\mathsf{I}})}}{\mathrm{Det}(1-\tilde{\mathsf{I}}_+)}$$

tend to unity. We thus obtain the following expression for the structure constant in the asymptotic limit:

$$\mathfrak{D}_{\mathsf{O}}^{\mathrm{asym}} = -\frac{i^J + (-i)^J}{\sqrt{J}} \sqrt{\prod_{1\le s\le \frac{M}{2}} \frac{u_s^2 + \frac{1}{4}}{u_s^2} \sigma_B^2(u_s)} \times \frac{\sqrt{\det[\partial_{u_k}\phi(u_j)]}}{\det[\partial_{u_k}\phi^+(u_j)]}. \tag{24}$$

TABLE I. The squared structure constants $(\mathfrak{D}_{\mathcal{O}_S})^2$ for the spin-S twist-2 operators.

S	$(\mathfrak{D}_{\mathcal{O}_S})^2$
2	$\frac{1}{3} - 4g^2 + g^4(56 - 24\zeta_3)$
4	$\frac{1}{35} - \frac{205g^2}{441} + g^4(\frac{70219}{9261} - \frac{20\zeta_3}{7})$
6	$\frac{1}{462} - \frac{1106g^2}{27\,225} + g^4(\frac{772\,465\,873}{1\,078\,110\,000} - \frac{14\zeta_3}{55})$
8	$\frac{1}{6435} - \frac{14\,380\,057g^2}{4\,509\,004\,500} + g^4(\frac{5\,048\,546\,158\,688\,587}{85\,305\,405\,235\,050\,000} - \frac{1522\zeta_3}{75\,075})$

Note that the determinants on the second line are the standard Gaudin-like determinants since all of the finite-size corrections can be dropped. For a generalization of Eq. (24) to operators outside the SL(2) sector, see Ref. [20]. A similar formula was found at weak coupling for the defect one-point functions [32,33].

Weak-coupling test.—To test formula (24), we computed the four-point function of $\mathcal{D}_{1,2}$ and two $\mathbf{20}'$ operators $\mathcal{O}_{\mathbf{20}'}$ up to $O(\lambda^2)$. We then performed the operator product expansion to read off the conformal data of the spin-S twist-2 operators $\mathcal{O}_S$. The details are given in Ref. [20].

The results of the computation are summarized in Table I. We compared them against the integrability prediction (24) and observed a perfect match. This is quite a nontrivial test of our formalism since the results contain the transcendental number ζ_3 and include the contributions from the boundary dressing phase $\sigma_B(u)$. Further tests at weak and strong couplings are provided in Eq. [20].

We also found that the structure constants exhibit a simple large spin behavior up to two loops,

$$\log\left(\frac{\mathfrak{D}_{\mathcal{O}_S}}{\mathfrak{D}_{\mathcal{O}_S}|_{\text{tree level}}}\right) = f_1 \log S' + f_2 + O(1/S'), \qquad (25)$$

with $\log S' \equiv \log S + \gamma_E$, where γ_E is the Euler-Mascheroni constant, and

$$f_1 = -4g^2 \log 2 + 8g^4\left[\zeta_2 \log 2 + \frac{9}{2}\zeta_3\right] + O(g^6),$$

$$f_2 = -2g^2\zeta_2 + 8g^4\left[\frac{4}{5}(\zeta_2)^2 + \frac{3}{2}\zeta_3 \log 2\right] + O(g^6). \qquad (26)$$

Conclusion.—In this Letter, we applied the TBA formalism to write a nonperturbative expression for the structure constant of two determinant operators and a single-trace operator in an SL(2) sector of arbitrary size. Our result could provide a foundation for future developments, such as the reformulation in terms of the quantum spectral curve [34], as was the case with the TBA for the spectrum. It would also be worth trying to extract various interesting physics from our formula. We also hope that our approach gives useful insight into the three-point functions of single-trace operators [2].

We thank Amit Sever for the helpful discussions. S. K. is supported by U.S. DOE Grant No. DE-SC0009988. E. V. is funded by FAPESP Grants No. 2014/18634-9 and No. 2016/09266-1, and by STFC Grant No. ST/P000762/1.

[1] N. Beisert *et al.*, Lett. Math. Phys. **99**, 3 (2012).
[2] B. Basso, S. Komatsu, and P. Vieira, arXiv:1505.06745.
[3] Some three-point functions are already known at finite coupling. For instance, the structure constant of a Lagrangian operator and two identical operators is given by a derivative of the conformal dimension $\partial_\lambda\Delta$ [M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, J. High Energy Phys. 11 (2010) 141]. However, it is determined solely by the conformal dimension and does not provide truly new conformal data.
[4] A. B. Zamolodchikov, Phys. Lett. B **253**, 391 (1991).
[5] We chose a configuration suitable for analyzing the symmetry. More general configurations can be obtained through conformal and R-symmetry transformations. These do not affect the structure constant $\mathfrak{D}_{\mathcal{O}}$.
[6] N. Drukker and J. Plefka, J. High Energy Phys. 04 (2009) 052.
[7] A. Bissi, C. Kristjansen, D. Young, and K. Zoubos, J. High Energy Phys. 06 (2011) 085.
[8] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, J. High Energy Phys. 04 (2002) 034.
[9] J. McGreevy, L. Susskind, and N. Toumbas, J. High Energy Phys. 06 (2000) 008.
[10] P. Dorey and R. Tateo, Nucl. Phys. **B482**, 639 (1996).
[11] S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. A **09**, 3841 (1994); **09**, 4353(E) (1994).
[12] N. Beisert, Adv. Theor. Math. Phys. **12**, 948 (2008).
[13] R. A. Janik, Phys. Rev. D **73**, 086006 (2006).
[14] N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. (2007) P0 1021.
[15] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.191601 for details on the two-particle overlap and Fredholm determinants.
[16] Naively, it seems that the reflection matrix has a pole at $u = 0$. However, it is actually absent since $\sigma_B(u^\gamma)$ has a simple zero at $u = 0$.
[17] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, Lett. Math. Phys. **91**, 265 (2010).
[18] D. Bombardelli, D. Fioravanti, and R. Tateo, J. Phys. A **42**, 375401 (2009).
[19] G. Arutyunov and S. Frolov, J. High Energy Phys. 05 (2009) 068.
[20] Y. Jiang, S. Komatsu, and E. Vescovi, arXiv:1906.07733.
[21] Z. Bajnok, Lett. Math. Phys. **99**, 299 (2012).
[22] A. LeClair, G. Mussardo, H. Saleur, and S. Skorik, Nucl. Phys. **B453**, 581 (1995).
[23] F. Woynarovich, Nucl. Phys. **B700**, 331 (2004).
[24] P. Dorey, D. Fioravanti, C. Rim, and R. Tateo, Nucl. Phys. **B696**, 445 (2004).
[25] B. Pozsgay, J. High Energy Phys. 08 (2010) 090.
[26] I. Kostov, D. Serban, and D.-L. Vu, arXiv:1809.05705.

[27] Here Det denotes the Fredholm determinant, while det denotes the determinant of a finite-dimensional matrix.

[28] The kernels in this Letter are related to the ones in Ref. [20] by the transposition.

[29] M. Jimbo, T. Miwa, and F. Smirnov, Lett. Math. Phys. **96**, 325 (2011).

[30] S. Negro and F. Smirnov, Nucl. Phys. **B875**, 166 (2013).

[31] Z. Bajnok and F. Smirnov, Nucl. Phys. **B945**, 114664 (2019).

[32] M. de Leeuw, C. Kristjansen, and K. Zarembo, J. High Energy Phys. 08 (2015) 098.

[33] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, Phys. Rev. Lett. **119**, 261604 (2017).

[34] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, Phys. Rev. Lett. **112**, 011602 (2014).