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One-loop superstring amplitude from integrals on pure spinors space

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Abstract: In the Type II superstring the 4-point function for massless NS-NS bosons at one-loop is well known [1, 14]. The overall constant factor in this amplitude is very important because it needs to satisfy the unitarity and S-duality conditions [14]. This coefficient has not been computed in the pure spinor formalism due to the difficulty to solve the integrals on the pure spinors space. In this paper we compute it by using the non-minimal pure spinor formalism and we will show that the answer is in perfect agreement with the one given in [14].

Keywords: Superstrings and Heterotic Strings, Differential and Algebraic Geometry, Field Theories in Higher Dimensions

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1 Introduction

The pure spinor formalism has many advantages for computing scattering amplitudes compared to the RNS and the GS formalism. For example, it does not have to deal with world-sheet spin structures [2, 25], it has manifest Super-Poincaré invariance and incorporate in a natural way the Ramond sectors. Nevertheless the formalism presents some difficulties, for example, the normalization of the integration measure in the pure spinors space, the computational difficulty to solve the integrals in this space and the S matrix unitarity has not been demostrated yet.

In this paper we will compute the one-loop scattering amplitude in the non-minimal pure spinor formalism for Type II superstrings and we will show that the overall constant factor is the same as the one given in [14]. Let's remember that this factor was also computed from the unitarity condition [1]. So, showing that the non-minimal pure spinor formalism predicts the same result as the RNS formalism is a direct test of unitarity.

To compute the scattering amplitude we normalize the integration measure of the pure spinors space in the same way as the phase space in quantum mechanics is normalized in the path integral, this is because the pure spinor formalism is a first order formalism.

To compute the integral on pure spinors space we use some tools of algebraic geometry. We also show that this normalization in the amplitude does not require computing functional determinants at all. This implies that computations using pure spinor formalism are easier than the ones done in RNS or GS formalism.

This paper is organized as follows. In section 2, the non-minimal pure spinor formalism will be reviewed and the space time units will be defined. We will normalize the massless vertex operator of the pure spinors formalism to coincide with the RNS normalization. In section 3, the 4-point one-loop scattering amplitude will be computed in the NS-NS sector using the non-minimal pure spinor formalism, up to an integration on pure spinors space. In the subsection 3.1 we will give a review to the $x^m(z,\bar{z})$ fields contribution and we justify the normalization of the path integral measures. In the subsection 3.2 we compute the contribution of the others fields and discuss biefly the modular invariance of the scattering amplitude. We use some results found in [4, 16, 21, 22] in which the authors showed: 1) the equivalence between the kinematic factor of the non-minimal pure spinor formalism and the minimal pure spinors formalism, 2) the equivalence between the kinematic factor of the minimal pure spinors formalism and the RNS formalism. At the end of the section we find all the factors in the 1-loop scattering amplitude, up to an integration over pure spinors space. In the last section, we will compute the integral on the pure spinors space. This is the most important section of the paper and we suggest the reader check the appendix beforehand, in which we apply the tools used to compute the integral in the pure spinors space in lower dimensions ($D = 2n < 10$). The aim is to be more familiar with the concepts of algebraic geometry involved in the computation. In this section we arrive to the following result

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-a\,\lambda\bar{\lambda}} = (2\pi)^{11}(a^8 \cdot 12 \cdot 5)^{-1}\,, \qquad a \in \mathbb{R}^+$$

where $\mathcal{O}(-1)$ is the line bundle blow-up at the origin with base space SO(10)/U(5). In others words, $\mathcal{O}(-1)$ is the pure spinors space. Finally, with this result we find the overall constant factor, which is called C_1 [14].

Our future goal is to compute the overall constants factors at tree level, which we call C_0, and at two loops, called C_2, in the non-minimal pure spinor formalism [31] and to show that the S-duality constraint ($C_1^2 = 2\pi^2 C_0 C_2$)[14] is a consequence of the identities for massless four-point kinematic factors [20].

2 Review on the non-minimal pure spinor formalism

We will give a brief review of the non-minimal pure spinor formalism. The idea is to introduce our own conventions and to normalize the massless vertex operator in the same way as in the D'Hoker, Phong and Gutperle's paper [14].

The superstring theory action in the right sector of the non-minimal pure spinor formalism [3] is given by

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma_g} \mathrm{d}^2 z\, \left(\partial x^m \bar{\partial} x_m + \alpha' p_\alpha \bar{\partial}\theta^\alpha - \alpha'\omega_\alpha \bar{\partial}\lambda^\alpha - \alpha'\bar{\omega}^\alpha \bar{\partial}\bar{\lambda}_\alpha + \alpha' s^\alpha \bar{\partial} r_\alpha\right) \tag{2.1}$$

where we define the space time dimensions of the variables and coupling constant α' as follows

$$[x^m] = 1, \quad [\alpha'] = 2, \quad [p_\alpha] = [\omega_\alpha] = [\bar{\lambda}_\alpha] = [r_\alpha] = -1/2, \tag{2.2}$$

$$[\theta^\alpha] = [\lambda^\alpha] = [\bar{\omega}^\alpha] = [s^\alpha] = 1/2. \tag{2.3}$$

The OPE's for the matter variables are easily computed

$$x^m(z)x_n(w) \sim -\frac{\alpha'}{2}\delta^m_n \ln|z-w|^2, \quad p_\alpha(z)\theta^\beta(w) \sim \frac{\delta^\beta_\alpha}{z-w}. \tag{2.4}$$

The complex bosonic spinors λ^α and $\bar{\lambda}_\alpha$ satisfy[1] the pure spinor constraint

$$\lambda\gamma^m\lambda = \bar{\lambda}\gamma^m\bar{\lambda} = 0, \qquad m = 0,\, 1,\, 2, \ldots, 9 \tag{2.5}$$

and the fermionic spinor r_α satisfies the constraint

$$\bar{\lambda}\gamma^m r = 0. \tag{2.6}$$

Because of the constraints on λ^α, $\bar{\lambda}_\alpha$ and r_α, their conjugate momenta ω_α, $\bar{\omega}^\alpha$ and s^α are defined up to a gauge tranformation,

$$\delta\omega_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha \tag{2.7}$$

$$\delta\bar{\omega}^\alpha = \bar{\Lambda}_m(\gamma^m\bar{\lambda})^\alpha - \phi_m(\gamma^m r)^\alpha, \quad \delta s^\alpha = \phi_m(\gamma^m\bar{\lambda})^\alpha\,, \tag{2.8}$$

for arbitrary Λ_m, $\bar{\Lambda}_m$ and ϕ_m.

In the U(5) variables the pure spinor constraints takes the following form [3]

$$2\lambda^+\lambda^a - \frac{1}{4}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \qquad a,\, b,\, c,\, d,\, e = 1,\, 2, \ldots, 5 \tag{2.9}$$

$$2\lambda^b\lambda_{ab} = 0. \tag{2.10}$$

where just five equations are linearly independent. In the chart $U_{+++++} = \{\lambda^+ \neq 0\}$ these equations are solved by [33]

$$\lambda^+ = \gamma, \quad \lambda_{ab} = \gamma u_{ab}, \quad \lambda^a = \frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de}. \tag{2.11}$$

As the u_{ab} variables parametrize the projective pure spinors space, then it is clear that the pure spinors space is the total space of the $\mathcal{O}(-1)$ bundle over the projective pure spinors space with blow-up at the origin ($\gamma = 0$) [7, 10, 11].

In this chart, we can take the gauge $\omega_a = \bar{\omega}^a = 0$ and the parametrization

$$\omega_+ = \beta - \frac{1}{2\gamma}v^{ab}u_{ab}, \quad \omega^{ab} = \frac{v^{ab}}{\gamma}, \tag{2.12}$$

$$\bar{\omega}^+ = \bar{\beta} - \frac{1}{2\bar{\gamma}}\bar{v}_{ab}\bar{u}^{ab}, \quad \bar{\omega}_{ab} = \frac{\bar{v}_{ab}}{\bar{\gamma}}, \tag{2.13}$$

so the pure spinors action takes the form

$$S_{PS} = \frac{1}{2\pi}\int d^2z \left(\beta\bar{\partial}\gamma + \frac{1}{2}v_{ab}\bar{\partial}u^{ab} + \bar{\beta}\bar{\partial}\bar{\gamma} + \frac{1}{2}\bar{v}^{ab}\bar{\partial}\bar{u}_{ab}\right). \tag{2.14}$$

With this action it is easy to get the OPE's

$$\beta(z)\gamma(w) \to (z-w)^{-1}, \quad v^{ab}(z)u_{cd}(w) \to \delta^a_{[c}\delta^b_{d]}(z-w)^{-1}, \tag{2.15}$$

$$\bar{\beta}(z)\bar{\gamma}(w) \to (z-w)^{-1}, \quad \bar{v}_{ab}(z)\bar{u}^{cd}(w) \to \delta^c_{[a}\delta^d_{b]}(z-w)^{-1}. \tag{2.16}$$

For the s^α, r_α fields the procedure is similar.

[1] The $\bar{\lambda}_\alpha$ spinor is treated as the complex conjugate of the λ^α spinor.

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From the previous definitions of the space-time dimensions of the fields and their OPEs we can get the following OPE's [23]

$$d_\alpha = p_\alpha - \frac{1}{\alpha'}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m - \frac{1}{4\alpha'}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta, \quad \Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta,$$

$$d_\alpha(z)d_\beta(w) \sim -\frac{2}{\alpha'}\frac{\gamma^m_{\alpha\beta}\Pi_m}{z-w}, \quad d_\alpha(z)\Pi^m(w) \sim \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta}{z-w},$$

$$d_\alpha(z)f(\theta(w),x(w)) \sim (z-w)^{-1}D_\alpha f(\theta(w),x(w)),$$

where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta\gamma^m_{\alpha\beta}\partial_m,$$

is the covariant super-derivate on $\mathbb{R}^{10}$. The supersymmetry generator is

$$q_\alpha = \int \mathrm{d}z\left(p_\alpha + \frac{1}{\alpha'}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m + \frac{1}{12\alpha'}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta\right)$$

and it satisfies the algebra

$$\{q_\alpha, q_\beta\} = \frac{2}{\alpha'}\gamma^m_{\alpha\beta}\int \mathrm{d}z\,\partial x_m, \quad [q_\alpha, \Pi^m(z)] = 0, \quad \{q_\alpha, d_\beta(z)\} = 0. \tag{2.17}$$

The construction of the b-ghost is such that [3, 29]

$$\{Q, b(z)\} = T(z),$$

where

$$Q = \int \mathrm{d}z\,(\lambda^\alpha d_\alpha + \bar{\omega}^\alpha r_\alpha), \quad T(z) = -\frac{1}{\alpha'}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + \omega_\alpha\partial\lambda^\alpha + \bar{\omega}^\alpha\partial\bar{\lambda}_\alpha - s^\alpha\partial r_\alpha.$$

Since Q and T are space time dimensionless so is b, which is given by

$$\begin{aligned} b = s^\alpha\partial\bar{\lambda}_\alpha &+ \frac{\bar{\lambda}_\alpha(2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\lambda\bar{\lambda})} \\ &+ \frac{(\bar{\lambda}\gamma^{mnp}r)(\frac{\alpha'}{2}d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\lambda\bar{\lambda})^2} - \frac{\frac{\alpha'}{2}(r\gamma_{mnp}r)(\bar{\lambda}\gamma^m d)N^{np}}{16(\lambda\bar{\lambda})^3} \\ &+ \frac{\frac{\alpha'}{2}(r\gamma_{mnp}r)(\bar{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\lambda\bar{\lambda})^4}. \end{aligned}$$

In order to build the vertex operators we use the following $\mathcal{N}=1$ SYM θ expansions [22–24]

$$A_\alpha(x,\theta) = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \cdots \tag{2.18}$$

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \cdots \tag{2.19}$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \cdots \tag{2.20}$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \cdots. \tag{2.21}$$

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Here $\xi^\alpha(x) = (2/\alpha')^{1/2}\chi^\alpha e^{ik\cdot x}$, where $[\chi^\alpha] = 1/2$ and $a_m = e_m e^{ik\cdot x}$, where $[e_m] = 0$. $F_{mn} = 2\partial_{[m}a_{n]}$ is the curvature and $[F_{mn}] = -1$. The dimensions of the superfields are

$$[A_\alpha] = 1/2, \quad [A_m] = 0, \quad [W^\alpha] = -1/2, \quad [\mathcal{F}_{mn}] = -1,$$

hence the massless vertex operators have the following dimensions

$$[V] = [\lambda^\alpha A_\alpha] = 1,\ [U] = [\partial\theta^\alpha A_\alpha + A_m \Pi^m + \frac{\alpha'}{2} d_\alpha W^\alpha + \frac{\alpha'}{4} N_{mn}\mathcal{F}_{mn}] = 1\,, \tag{2.22}$$

where U satisfies $QU = \partial(\lambda^\alpha A_\alpha)$. These vertex operators have the same normalization as the vertex operators of [14], therefore we can compare the amplitudes in a straight forward way. For example, the closed superstring massless operator in the NS-NS sector is [14]

$$V = e_m \bar{e}_n \int \mathrm{d}^2 z\, (\partial x^m + ik\cdot\psi_+\psi_+^m)(\bar{\partial} x^n + ik\cdot\psi_-\psi_-^n) e^{ik\cdot x}\,, \tag{2.23}$$

where the dimension of V is two if the dimension of the polarization vectors is zero.

3 Four point 1-loop massless amplitude

Using the normalization of the previous section we will compute the one loop amplitude for 4-massless vertex operator in the NS-NS sector. Although the general structure of this section can be found in the references [16, 22, 23, 27], we include it to justify the normalization of the measures and to find the overall constant factor for the amplitude, which has not been computed.

As non-minimal pure spinor formalism is a critical topological string, then one can use the bosonic string prescription for computing scattering amplitudes [3, 10]. So the four points 1-loop massless amplitude is given by

$$\mathcal{A} = \frac{1}{2}\kappa^4 \int_{\mathcal{M}_1} \mathrm{d}^2\tau\ \langle \Big|\mathcal{N}\ (b,\mu)\ V(z_1)\ \prod_{i=2}^{4}\int \mathrm{d}z_i\, \mathrm{U}(z_i)\Big|^2 \rangle\,, \tag{3.1}$$

where $\mathcal{M}_1 = \mathrm{H}/\mathrm{PSL}(2,\mathbb{Z})$ is the fundamental region, μ is the Beltrami differential, $\mathcal{N}$ is a regulator, z_1 is a fixed point and finally, κ is the normalization constant of the massless vertex operator. Its precise value will not be needed here. The 1/2 factor is needed because the total group of automorphism on the torus is $\mathrm{SL}(2,\mathbb{Z})$ instead of $\mathrm{PSL}(2,\mathbb{Z})$ [26, 27]. As the amplitude is computed using the bosonic string prescription, we must take in account the normalization of the inner product between the b-ghost and the Beltrami differential in the same way as in bosonic string theory [27]

$$(b,\mu) = \frac{1}{2\pi}\int \mathrm{d}^2 z\ b(z)\mu_{\bar{z}}^{z} = \frac{1}{\pi} b(0), \tag{3.2}$$

where $\mu_{\bar{z}\bar{z}} = 1/2\tau_2$.

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3.1 Review of the $x^m(z,\bar z)$ fields contribution

In this subsection we compute the $x^m(z,\bar z)$ contribution and justify, in a natural way, the normalization of the integration measures.

In order to compute the $x^m(z,\bar z)$ contribution we expand it in terms of a complete set $X_I(z,\bar z)$ of eigenfunctions of the worldsheet Laplacian operator

$$\begin{aligned} x^m(z,\bar z) &= \sum_I x_I^m X_I(z,\bar z), \\ \partial\bar\partial X_I(z,\bar z) &= -\lambda_I^2 X_I(z,\bar z) \\ \int_{\Sigma_g} \mathrm{d}^2 z\, X_I(z,\bar z) X_J(z,\bar z) &= \delta_{IJ}. \end{aligned}$$

The bosonic contribution is given by [27]

$$\left\langle \prod_{i=1}^4 : e^{k_i\cdot x} : \right\rangle = \prod_{Im} \int \frac{\mathrm{d}x_I^m}{\sqrt{2\pi^2\alpha'}} \exp\left[-\frac{1}{2\pi\alpha'} \sum_{I\neq 0} \left(\lambda_I^2 x_I\cdot x_I - 2\pi\alpha' i x_I \cdot J_I\right) + i x_0 \cdot J_0 \right] \tag{3.3}$$

$$= (2\pi)^{10} \delta^{(10)}(J_0) \left(2\pi^2\alpha' \det{}'\partial\bar\partial\right)^{-5} \exp\left[-\sum_{I\neq 0} \frac{\pi\alpha'}{2\lambda_I^2} J_I\cdot J_I \right] \tag{3.4}$$

where

$$J^m(z,\bar z) = \sum_{i=1}^4 k_i^m \delta^{(2)}(z,\bar z) = \sum_I J_I^m X_I(z,\bar z) \tag{3.5}$$

$$J_I^m = \int_{\Sigma_g} \mathrm{d}^2 z\, J^m(z,\bar z) X_I(z,\bar z). \tag{3.6}$$

In particular

$$J_0^m = X_0 \int_{\Sigma_g} \mathrm{d}^2 z\, J^m(z,\bar z) = X_0 \sum_{i=1}^4 k_i^m,$$

thus, we have

$$\left\langle \prod_{i=1}^4 : e^{k_i x} : \right\rangle = (2\pi)^{10} \delta^{(10)}(X_0 k) \left(2\pi^2\alpha' \det{}'\partial\bar\partial\right)^{-5} \exp\left[-\frac{1}{2} \sum_{i\neq j} k_i \cdot k_j \sum_{I\neq 0} \frac{\pi\alpha'}{\lambda_I^2} X_I(z_i,\bar z_i) X_I(z_j,\bar z_j) \right]$$

where $k = \sum_{i=1}^4 k_i^m$. The term

$$\sum_{I\neq 0} \frac{\pi\alpha'}{\lambda_I^2} X_I(z_i,\bar z_i) X_I(z_j,\bar z_j)$$

is the Green's function and it satisfies the differential equation

$$-\frac{1}{\pi\alpha'} \partial\bar\partial G(z,w) = \sum_{I\neq 0} X_I(z,\bar z) X_I(w,\bar w) \tag{3.7}$$

$$= \delta^{(2)}(z-w) - X_0^2. \tag{3.8}$$

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In the torus we have defined the normalization of the X_0 mode to be

$$X_0^2 = (2\tau_2)^{-1}, \tag{3.9}$$

such that

$$||X_0||^2 = X_0^2 \int_{\Sigma_g} \mathrm{d}^2 z \; = 1 \tag{3.10}$$

where $\int_{\Sigma_g} \mathrm{d}^2 z = 2\tau_2$.

With this normalization, the Green's function for the torus is given by[25]

$$\begin{aligned} G(z, w, \tau) &= -\frac{\alpha'}{2} \ln|E(z, w)|^2 + \frac{\alpha'\pi}{4\tau_2}(z - \bar{z} - w + \bar{w})^2 \\ &= -\frac{\alpha'}{2} \ln|E(z, w)|^2 + \frac{2\alpha'\pi}{\tau_2} \mathrm{Im} z \, \mathrm{Im} w, \end{aligned}$$

and therefore the final expression for the bosonic contribution is [14, 25]

$$\begin{aligned} \left\langle \prod_{i=1}^{4} : e^{k_i \cdot x} : \right\rangle &= (2\pi)^{10} (2\tau_2)^5 \delta^{(10)}(k) \left(2\pi^2 \alpha' \mathrm{det}' \partial\bar{\partial}\right)^{-5} \\ &\prod_{i<j} |E(z_i, z_j)|^{\alpha' k_i \cdot k_j} \exp\left[-k_i \cdot k_j \frac{2\pi\alpha'}{\tau_2} \mathrm{Im} z_i \, \mathrm{Im} z_j\right]. \end{aligned}$$

The factors $(2\pi^2\alpha')^{-1/2}$ of the integration measure of (3.3) come from treating the $x^m(z, \bar{z})$ action in a first order formalism [28]. To see this, let's take the action

$$S = \frac{1}{\pi\alpha'} \int \mathrm{d}^2 z \, (g^{i\bar{j}} p_i p_{\bar{j}} + p_i \bar{\partial} x^i + p_{\bar{i}} \partial x^{\bar{i}}) \tag{3.11}$$

where the index $i, \bar{i} = 1, \ldots, 5$, p_i and $p_{\bar{i}}$ are (1,0) and (0,1) forms with conformal weight (1,0) and (0,1) respectively and $g^{i\bar{j}} = \delta^{i\bar{j}}$.

In this first order action we can easily see that the conjugate momenta of the x^i and $x^{\bar{i}}$ fields are $P_i := p_i/\pi\alpha'$ and $P_{\bar{i}} := p_{\bar{i}}/\pi\alpha'$ respectively, so the Dirac brackets (DB) are

$$\begin{aligned} \left[P_i(\sigma), x^j(\sigma')\right]_{DB} &= \left[\frac{p_i(\sigma)}{\pi\alpha'}, x^j(\sigma')\right]_{DB} = i\delta_i^j \delta(\sigma - \sigma'), \\ \left[P_{\bar{i}}(\sigma), x^{\bar{j}}(\sigma')\right]_{DB} &= \left[\frac{p_{\bar{i}}(\sigma)}{\pi\alpha'}, x^{\bar{j}}(\sigma')\right]_{DB} = i\delta_{\bar{i}}^{\bar{j}} \delta(\sigma - \sigma'). \end{aligned}$$

In quantum mechanics, because of the commutator relation $[p, x] = i$ one has the identity

$$\int \frac{\mathrm{d}x}{\sqrt{2\pi}} \frac{\mathrm{d}p}{\sqrt{2\pi}} e^{-ipx} = 1, \tag{3.12}$$

and the integration measure on the phase space in the path integral is [27]

$$\frac{\mathrm{d}x}{\sqrt{2\pi}} \frac{\mathrm{d}p}{\sqrt{2\pi}}. \tag{3.13}$$

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In the same way, the measure on the phase space in the path integral for the action (3.11) is

$$\prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}} \frac{\mathrm{d}P_i}{\sqrt{2\pi}}\frac{\mathrm{d}P_{\bar{i}}}{\sqrt{2\pi}}\frac{\mathrm{d}x^j}{\sqrt{2\pi}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi}} = \prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}} \frac{\mathrm{d}p_i}{\pi\alpha'\sqrt{2\pi}}\frac{\mathrm{d}p_{\bar{i}}}{\pi\alpha'\sqrt{2\pi}}\frac{\mathrm{d}x^j}{\sqrt{2\pi}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi}}$$
$$= \prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}} \frac{\mathrm{d}p_i}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}p_{\bar{i}}}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}x^j}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi^2\alpha'}},$$

hence if we compute the integral by p_i, $p_{\bar{i}}$ fields we get (3.3).

We can see that the p^m fields have $10g$ zero modes in a Riemann surface of genus g and their normalizations do not affect the answer.

Note that in this first order formalism the number of zero modes of the bosonic fields, including the pure spinors at the right and left sectors, is equal to zero modes of the fermionic fields

$$\text{bosonic} \quad \begin{cases} x^m & p^m & \lambda^\alpha & \omega_\alpha & \bar{\lambda}_\alpha & \bar{\omega}^\alpha & \hat{\lambda}^\alpha & \hat{\omega}_\alpha & \hat{\bar{\lambda}} & \hat{\bar{\omega}}^\alpha \\ 10 & 10g & 11 & 11g & 11 & 11g & 11 & 11g & 11 & 11g \end{cases} \tag{3.14}$$
$$\text{fermionic} \quad \begin{cases} \theta^\alpha & p_\alpha & \hat{\theta}^\alpha & \hat{p}_\alpha & r_\alpha & s^\alpha & \hat{r}_\alpha & \hat{s}^\alpha \\ 16 & 16g & 16 & 16g & 11 & 11g & 11 & 11g \end{cases}.$$

3.2 Pure spinors and p_α, θ^α fields contribution

First, we will compute the contribution of the non zero modes to the amplitude and we will show explicity that in the non-minimal formalism it is not necessary to compute functional determinants.

The action of the pure spinors in a chart is given by [2]

$$S = -\frac{1}{2\pi}\int_{\Sigma_g} \mathrm{d}^2z \left(\beta\bar{\partial}\gamma + \frac{1}{2}v^{ab}\bar{\partial}u_{ab} + \bar{\beta}\bar{\partial}\bar{\gamma} + \frac{1}{2}\bar{v}_{ab}\bar{\partial}\bar{u}^{ab}\right).$$

When the Riemann surface Σ_g is the torus, all the elements of the set $\{\gamma, \bar{\gamma}, \beta, \bar{\beta}, u_{ab}, \bar{u}_{ab}, v^{ab}, \bar{v}^{ab}\}$ have one zero mode only [25].

The contribution of the non-zero modes is given by

$$\prod_{I\neq 0}\int \frac{[\mathrm{d}\beta_I]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\gamma_I]}{\sqrt{4\pi^2}} \bigwedge_{a<b,\, c<d} \frac{[\mathrm{d}v_I^{ab}]}{\sqrt{4\pi^2}}\frac{[\mathrm{d}u_{cd\,I}]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\bar{\beta}_I]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\bar{\gamma}_I]}{\sqrt{4\pi^2}} \bigwedge_{e<f,\, g<h} \frac{[\mathrm{d}\bar{v}_{ef\,I}]}{\sqrt{4\pi^2}}\frac{[\mathrm{d}\bar{u}_I^{gh}]}{\sqrt{4\pi^2}}$$
$$\exp\left(-\frac{1}{2\pi}\sum_{I\neq 0}\lambda_I(\beta_I\gamma_I + \frac{1}{2}v_I^{ab}u_{ab\,I} + \bar{\beta}_I\bar{\gamma}_I + \frac{1}{2}\bar{v}_{ab\,I}\bar{u}_I^{ab})\right) \tag{3.15}$$

where the $\{\lambda_I\}$ are the eigenvalues of the $\bar{\partial}$ operator and we write the measure in the same way as in the previous section. We can write the argument of the exponential function in the following form (for example for the (γ, β) fields)

$$\exp\left(-\frac{1}{2\pi}\sum_{I\neq 0}\lambda_I(\beta_I\gamma_I + \bar{\beta}_I\bar{\gamma}_I)\right) = \exp\left(\frac{1}{2\pi}V^\dagger M V\right), \tag{3.16}$$

where $V^T = (\gamma_I, \bar{\beta}_I)$, M is the matrix

$$M := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} , \tag{3.17}$$

and A is the matrix $A := \mathrm{diag}(\lambda_I)$. The same happens for the (v^{ab}, u_{cd}) fields. Therefore the non-zero modes contribution of the pure spinors is $(\det \bar{\partial})^{-22}$.

Although we computed the path integral of the pure spinors in a particular chart and gauge, the answer is correct because the $\{\gamma = 0\} = \mathrm{SO}(10)/\mathrm{U}(5)$ space has measure zero with respect to the pure spinors space.

The integration measure for the I^{th} mode can be written in a covariant way [21, 22] as follows

$$\begin{aligned} &[\mathrm{d}\omega^I] = \frac{(4\pi^2)^{-11/2}}{11!5!}(\lambda_I\gamma^m)_{\alpha_1}(\lambda_I\gamma^n)_{\alpha_2}(\lambda_I\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\mathrm{d}\omega^I_{\delta_1}\wedge\cdots\wedge\mathrm{d}\omega^I_{\delta_{11}} \\ &[\mathrm{d}\lambda_I](\lambda_I\gamma^m)_{\alpha_1}(\lambda_I\gamma^n)_{\alpha_2}(\lambda_I\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{(4\pi^2)^{-11/2}}{11!}\epsilon_{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\mathrm{d}\lambda_I^{\rho_1}\wedge\cdots\wedge\mathrm{d}\lambda_I^{\rho_{11}}, \end{aligned} \tag{3.18}$$

from which we can easily see that $[\mathrm{d}\lambda_I]$ and $[\mathrm{d}\omega^I]$ have ghost number 8 and -8, respectively. Taking the wedge product we get

$$[\mathrm{d}\lambda_I]\wedge[\mathrm{d}\omega^I] = \frac{(4\pi^2)^{-11}}{11!}\mathrm{d}\lambda_I^{\alpha_1}\wedge\mathrm{d}\omega^I_{\alpha_1}\wedge\cdots\wedge\mathrm{d}\lambda_I^{\alpha_{11}}\wedge\mathrm{d}\omega^I_{\alpha_{11}}. \tag{3.19}$$

In the chart $U_{+++++} = \{\lambda^+ \neq 0\}$

$$\begin{aligned} \lambda^\alpha &= (\lambda^+, \lambda^{ab}, \lambda^a)\,, \\ \lambda^+ &= \gamma\,, \quad \lambda_{ab} = \gamma u_{ab}\,, \quad \lambda^a = -\frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de} \end{aligned}$$

and in the gauge $\omega_a = 0$

$$\begin{aligned} \omega_\alpha &= (\omega_+, \omega^{ab}, \omega_a) \qquad a,\, b = 1,\, 2, \ldots, 5 \\ \omega_+ &= \beta - \frac{1}{2\gamma}v^{ab}u_{ab}\,, \quad \omega^{ab} = \frac{1}{\gamma}v^{ab}\,, \quad \omega_a = 0, \end{aligned}$$

we have the measure in the form desired

$$[\mathrm{d}\lambda_I]\wedge[\mathrm{d}\omega^I] = (4\pi^2)^{-11}\bigwedge_{a<b,\,c<d}\mathrm{d}\gamma_I\mathrm{d}u^I_{ab}\mathrm{d}\beta_I\mathrm{d}v_I^{cd}. \tag{3.20}$$

For the $\bar{\lambda}_\alpha$ and $\bar{\omega}^\alpha$ fields we define the measures $[\mathrm{d}\bar{\lambda}^I]$ and $[\mathrm{d}\bar{\omega}_I]$ for the I^{th} mode in the following form

$$\begin{aligned} &[\mathrm{d}\bar{\omega}_I](\lambda_I\gamma^m)_{\alpha_1}(\lambda_I\gamma^n)_{\alpha_2}(\lambda_I\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = (4\pi^2)^{-11/2}\frac{(\lambda_I\bar{\lambda}^I)^3}{11!}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\mathrm{d}\bar{\omega}_I^{\delta_1}\wedge\cdots\wedge\mathrm{d}\bar{\omega}_I^{\delta_{11}} \\ &[\mathrm{d}\bar{\lambda}^I] = \frac{(4\pi^2)^{-11/2}}{11!5!(\lambda_I\bar{\lambda}^I)^3}(\lambda_I\gamma^m)_{\alpha_1}(\lambda_I\gamma^n)_{\alpha_2}(\lambda_I\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\mathrm{d}\bar{\lambda}^I_{\rho_1}\wedge\cdots\wedge\mathrm{d}\bar{\lambda}^I_{\rho_{11}}, \end{aligned} \tag{3.21}$$

so

$$[\mathrm{d}\bar{\lambda}^I]\wedge[\mathrm{d}\bar{\omega}_I]=\frac{(4\pi^2)^{-11}}{11!}\mathrm{d}\bar{\lambda}^I_{\alpha_1}\wedge\mathrm{d}\bar{\omega}_I^{\alpha_1}\wedge\cdots\wedge\mathrm{d}\bar{\lambda}^I_{\alpha_{11}}\wedge\mathrm{d}\bar{\omega}_I^{\alpha_{11}}, \tag{3.22}$$

as expected.

The contribution of the fields of opposite worldsheet chirality is $(\det'\partial)^{-22}$. So, the contribution of the non zero modes of the pure spinors is $(\det'\partial\bar{\partial})^{-22}$.

From the action of the p_α, θ^α fields

$$S_{p\theta}=\frac{1}{2\pi}\int \mathrm{d}^2z\; p_\alpha\bar{\partial}\theta^\alpha\,, \tag{3.23}$$

we get the anticommutation relation

$$\left\{P_\alpha(\sigma),\theta^\beta(\sigma')\right\}_{DB}:=\left\{\frac{p_\alpha(\sigma)}{2\pi},\theta^\beta(\sigma')\right\}_{DB}=\delta^\beta_\alpha\delta(\sigma-\sigma'). \tag{3.24}$$

Therefore, the measure of the phase space in the path integral is [27]

$$\prod_{z,\bar{z}}\prod_{\alpha\beta}\mathrm{d}P_\alpha\,\mathrm{d}\theta^\beta=\prod_{z,\bar{z}}\prod_{\alpha\beta}(2\pi\mathrm{d}p_\alpha)\,\mathrm{d}\theta^\beta=\prod_{z,\bar{z}}\prod_{\alpha\beta}\left(\sqrt{2\pi}\mathrm{d}p_\alpha\right)\left(\sqrt{2\pi}\mathrm{d}\theta^\beta\right), \tag{3.25}$$

and the contribution of the non zero modes of p_α and θ^α fields is given by

$$\begin{aligned}&\prod_{\alpha\,\beta}\int[\mathrm{d}P_\alpha]'[\mathrm{d}\theta^\beta]'\exp\left(-\frac{1}{2\pi}\int_{\Sigma_g}\mathrm{d}^2z\,p_\alpha\bar{\partial}\theta^\alpha\right)\\&=\prod_{\alpha\,\beta\,I\neq 0}\int\left(\sqrt{2\pi}\mathrm{d}p_{\alpha\,I}\right)\left(\sqrt{2\pi}\mathrm{d}\theta^\beta_I\right)\exp\left(-\frac{1}{2\pi}\sum_{I\neq 0}\lambda_I p_{\alpha\,I}\theta^\alpha_I\right)\\&=\left[\det{}'\left(\bar{\partial}\right)\right]^{16},\end{aligned}$$

where $p_{\alpha\,I}$, θ^α_I are Grassmann numbers. As in the previous case, the contribution of the fields of opposite worldsheet chirality is $(\det'\partial)^{16}$. Thus the total contribution of the fermions p_α and θ^β is

$$[\det{}'(\partial\bar{\partial})]^{16}.$$

For the r_α and s^α Grassmann fields we can define the covariant measure in the path integral for the I^{th} mode as [22]

$$[\mathrm{d}r^I]=\frac{(2\pi)^{11/2}}{11!5!}(\bar{\lambda}^I\gamma^m)^{\alpha_1}(\bar{\lambda}^I\gamma^n)^{\alpha_2}(\bar{\lambda}^I\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\partial^{\delta_1}_{r^I}\cdots\partial^{\delta_{11}}_{r^I} \tag{3.26}$$

$$[\mathrm{d}s_I](\bar{\lambda}^I\gamma^m)^{\alpha_1}(\bar{\lambda}^I\gamma^n)^{\alpha_2}(\bar{\lambda}^I\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}=\frac{(2\pi)^{11/2}}{11!}\epsilon^{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\partial^{s_I}_{\rho_1}\cdots\partial^{s_I}_{\rho_{11}}$$

so

$$[\mathrm{d}r^I][\mathrm{d}s_I]=(2\pi)^{11}\partial^1_{r^I}\partial^{s_I}_1\cdots\partial^{11}_{r^I}\partial^{s_I}_{11}. \tag{3.27}$$

In an analogous way as the previous case we get the contribution from the non-zero modes

$$\left(\det{}'\partial\bar{\partial}\right)^{11}. \tag{3.28}$$

Finally, the total contribution of the non-zero modes of the (λ^α, ω_α, $\bar{\lambda}_\beta$, $\bar{\omega}^\beta$, r_β, s^β) fields is

$$\left(\det{}'\partial\bar{\partial}\right)^{-11}\left(\det{}'\partial\bar{\partial}\right)^{-11}\left(\det{}'\partial\bar{\partial}\right)^{16}\left(\det{}'\partial\bar{\partial}\right)^{11}=\left(\det{}'\partial\bar{\partial}\right)^{5}. \tag{3.29}$$

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3.2.1 Modular invariance

Before to compute the zero mode contribution we discuss briefly the modular invariance. This subject is important because the zero modes normalization of the vertex operators and the b-ghost contains modular parameters.

With all the contributions that we have computed up to now, our 4-points 1-loop amplitude has the form

$$\mathcal{A}=\frac{(2\pi)^{10}\delta^{(10)}(k)\kappa^4}{2\pi^2\left(2\pi^2\alpha'\right)^5}\int_{\mathcal{M}_1}\mathrm{d}^2\tau(2\tau_2)^5\prod_{k=1}^{3}\int\mathrm{d}^2z_k\prod_{i<j=1}^{4}|E(z_i,z_j)|^{\alpha' k_i\cdot k_j}\exp\left[-k_i\cdot k_j\frac{2\pi\alpha'}{\tau_2}\mathrm{Im}z_i\,\mathrm{Im}z_j\right]$$
$$\left|\left(\frac{\alpha'}{2}\right)^4\int[\mathrm{d}r][\mathrm{d}s][\mathrm{d}d][\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar\lambda][\mathrm{d}\omega][\mathrm{d}\bar\omega]e^{(-\lambda\bar\lambda-\bar\omega\omega-r\theta+sd)}\frac{(d\gamma^{pqr}d)}{192(\lambda\bar\lambda)^2}(\bar\lambda\gamma_{pqr}r)(\lambda A_1dW_2dW_3dW_4)\right|_0^2 \tag{3.30}$$

where the subindex "0" means that only the zero modes will be computed.

Is clear that (3.30) is not modular invariant since the scattering amplitude needs a $(\tau_2)^{-5}$ factor instead of the $(\tau_2)^5$ factor. The reason for this is that we have not introduced yet the zero modes normalization of the vertex operators (so as in the $x^m(z,\bar z)$ fields case). We will show that by introducing it we get the $(\tau_2)^{-5}$ factor and the scattering amplitude will be modular invariant.

On the torus all the fields have one zero mode, so we can do the following expansion on a complete set of eigenfunctions of the world-sheet operators $\bar\partial$ and ∂

$$\begin{aligned}
\theta^\alpha(z,\bar z) &= \theta_0^\alpha\Lambda_0+\sum_{I\neq0}\theta_I^\alpha\Lambda_I(z,\bar z), \quad p_\alpha(z,\bar z)=p_\alpha^0\Omega_0+\sum_{I\neq0}p_\alpha^I\Omega_I(z,\bar z)\\
\lambda^\alpha(z,\bar z) &= \lambda_0^\alpha\Lambda_0+\sum_{I\neq0}\lambda_I^\alpha\Lambda_I(z,\bar z), \quad \bar\lambda_\alpha(z,\bar z)=\bar\lambda_\alpha^0\Lambda_0+\sum_{I\neq0}\bar\lambda_\alpha^I\Lambda_I(z,\bar z)\\
\bar\omega^\alpha(z,\bar z) &= \bar\omega_0^\alpha\Omega_0+\sum_{I\neq0}\bar\omega_I^\alpha\Omega_I(z,\bar z), \quad \omega_\alpha(z,\bar z)=\omega_\alpha^0\Omega_0+\sum_{I\neq0}\omega_\alpha^I\Omega_I(z,\bar z),\\
r_\alpha(z,\bar z) &= r_\alpha^0\Lambda_0+\sum_{I\neq0}r_\alpha^I\Lambda_I(z,\bar z), \quad s^\alpha(z,\bar z)=s_0^\alpha\Omega_0+\sum_{I\neq0}s_I^\alpha\Omega_I(z,\bar z),
\end{aligned}$$

where

$$\begin{aligned}
\int\mathrm{d}^2z\Omega_I(z,\bar z)\bar\Omega_J(\bar z,z) &= \delta_{IJ}\\
\int\mathrm{d}^2z\Lambda_I(z,\bar z)\bar\Lambda_J(\bar z,z) &= \delta_{IJ},
\end{aligned}$$

in particular $||\Lambda_0||^2=||\Omega_0||^2=(2\tau_2)^{-1}$. From the previous section we know that only the term $\frac{(\bar\lambda\gamma^{mnp}r)(\frac{\alpha'}{2}d\gamma_{mnp}d)}{192(\lambda\bar\lambda)^2}$ of the b-ghost can saturate the d_α zero modes. Since our interests are the zero modes then we write this term as

$$\frac{\alpha'}{2}\frac{(1/2\tau_2)^2}{(1/2\tau_2)^2}\frac{(\bar\lambda^0\gamma^{mnp}r^0)(d^0\gamma_{mnp}d^0)}{192(\lambda_0\bar\lambda^0)^2}=\frac{\alpha'}{2}\frac{(\bar\lambda^0\gamma^{mnp}r^0)(d^0\gamma_{mnp}d^0)}{192(\lambda_0\bar\lambda^0)^2}. \tag{3.31}$$

To saturate the 11 zero modes of r_α we need $10r_\alpha$ zero modes. The regulator

$$e^{(-\lambda_0\bar\lambda^0-\bar\omega_0\omega^0-r^0\theta_0+s_0d^0)} \tag{3.32}$$

supplies the $10r_\alpha$ zero modes plus $10\theta^\alpha$ zero modes. The $6\theta^\alpha$ zero modes necessary to saturate the $16\theta^\alpha$ zero modes come from the vertex operator

$$(\lambda A_1 dW_2 dW_3 dW_4), \tag{3.33}$$

so, these $6\theta^\alpha$ zero modes contribute with a factor $(2\tau_2)^{-3}$ and the $3d_\alpha$ and λ^α fields contribute with $(2\tau_2)^{-2}$. In this way the factor in the right sector is $(2\tau_2)^{-5}$. In the left sector the analysis is the same, so the total factor is $(2\tau_2)^{-10}$ and the amplitude

$$\mathcal{A}=\frac{(2\pi)^{10}\delta^{(10)}(k)\kappa^4}{2\pi^2\,(2\pi)^{10}\,(\alpha')^5}\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\frac{\mathrm{d}^2\tau}{(\tau_2)^5}\prod_{k=1}^{3}\int \mathrm{d}^2 z_k\prod_{i<j=1}^{4}|E(z_i,z_j)|^{\alpha' k_i\cdot k_j}\exp\left[-k_i\cdot k_j\frac{2\pi\alpha'}{\tau_2}\mathrm{Im}z_i\,\mathrm{Im}z_j\right]$$
$$\left|\int[\mathrm{d}r][\mathrm{d}s][\mathrm{d}d][\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega-r\theta+sd)}\frac{(d\gamma^{pqr}d)}{192(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{pqr}r)(\lambda A_1 dW_2 dW_3 dW_4)\right|_0^2 \tag{3.34}$$

is modular invariant.

3.2.2 Contribution of the zero modes

Now we are going to compute the zero mode contribution in the NS-NS sector where we use some of the results given in [21, 22]. This calculation is totally algebraic and easy to follow due to our choice of the integration measures.

We rewrite the integration measure in the following way

$$\begin{aligned}
[\mathrm{d}\lambda_0] &= (4\pi^2)^{-11/2}[\mathrm{d}\lambda], & [\mathrm{d}\omega^0] &= (4\pi^2)^{-11/2}[\mathrm{d}\omega],\\
[\mathrm{d}\bar{\lambda}^0] &= (4\pi^2)^{-11/2}[\mathrm{d}\bar{\lambda}], & [\mathrm{d}\bar{\omega}_0] &= (4\pi^2)^{-11/2}[\mathrm{d}\bar{\omega}],\\
[\mathrm{d}r^0] &= (2\pi)^{11/2}[\mathrm{d}r], & [\mathrm{d}s_0] &= (2\pi)^{11/2}[\mathrm{d}s],\\
[\mathrm{d}\theta_0] &= (2\pi)^{16/2}[\mathrm{d}\theta], & [\mathrm{d}d^0] &= (2\pi)^{16/2}[\mathrm{d}d],
\end{aligned} \tag{3.35}$$

where the measures $[\mathrm{d}\cdot]$ are defined from the previous subsection, for example

$$[\mathrm{d}\lambda](\lambda_0\gamma^m)_{\alpha_1}(\lambda_0\gamma^n)_{\alpha_2}(\lambda_0\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}=\frac{1}{11!}\epsilon_{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\mathrm{d}\lambda_0^{\rho_1}\wedge\cdots\wedge\mathrm{d}\lambda_0^{\rho_{11}}, \tag{3.36}$$

and similarly for the others measures. For the rest of this paper the subindex "0" will be dropped out. In this new notation the scattering amplitude has the form

$$\begin{aligned}
\mathcal{A} &= \frac{(2\pi)^{10}\delta^{(10)}(k)\kappa^4}{2\pi^2\,(2\pi)^{10}\,(\alpha')^5}\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\mathrm{d}^2\tau\,(\tau_2)^{-5}\prod_{k=1}^{3}\int \mathrm{d}^2 z_k\prod_{i<j=1}^{4}|E(z_i,z_j)|^{\alpha' k_i\cdot k_j}\\
&\quad \exp\left[-k_i\cdot k_j\frac{2\pi\alpha'}{\tau_2}\mathrm{Im}z_i\,\mathrm{Im}z_j\right]\left|(2\pi)^{-17}\mathcal{K}\right|^2\\
&= \frac{(2\pi)^{10}\delta^{(10)}(k)\kappa^4}{2\pi^2\,(2\pi)^{44}\,(\alpha')^5}\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\mathrm{d}^2\tau\,(\tau_2)^{-5}\prod_{k=1}^{3}\int \mathrm{d}^2 z_k\\
&\quad \prod_{i<j=1}^{4}|E(z_i,z_j)|^{\alpha' k_i\cdot k_j}\exp\left[-k_i\cdot k_j\frac{2\pi\alpha'}{\tau_2}\mathrm{Im}z_i\,\mathrm{Im}z_j\right]|\mathcal{K}|^2
\end{aligned}$$

where we have defined $\mathcal{K}$ to be

$$\mathcal{K}=\int[\mathrm{d}r][\mathrm{d}s][\mathrm{d}d][\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega-r\theta+sd)}\frac{(d\gamma^{pqr}d)}{192(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{pqr}D)(\lambda A_1 dW_2 dW_3 dW_4). \tag{3.37}$$

In order to compute the $\mathcal{K}$ factor let's remember that the measures of r_α and s^α are given by

$$[\mathrm{d}r]=\frac{1}{11!5!}(\bar{\lambda}\gamma^m)^{\alpha_1}(\bar{\lambda}\gamma^n)^{\alpha_2}(\bar{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\partial_r^{\delta_1}\cdots\partial_r^{\delta_{11}}$$

$$[\mathrm{d}s](\bar{\lambda}\gamma^m)^{\alpha_1}(\bar{\lambda}\gamma^n)^{\alpha_2}(\bar{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}=\frac{1}{11!}\epsilon^{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\partial^s_{\rho_1}\cdots\partial^s_{\rho_{11}}.$$

We rewrite the $[\mathrm{d}s]$ measure as

$$[\mathrm{d}s]=\frac{1}{2^6\cdot 11!5!}\frac{1}{(\lambda\bar{\lambda})^3}(\lambda\gamma^r)_{\alpha_1}(\lambda\gamma^s)_{\alpha_2}(\lambda\gamma^q)_{\alpha_3}(\gamma_{rsq})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\partial^s_{\rho_1}\cdots\partial^s_{\rho_{11}}. \tag{3.38}$$

Integrating the r_α, s^α and d_α variables in $\mathcal{K}$ we get

$$\mathcal{K}=\frac{1}{11!11!5!\cdot 2^9\cdot 3\cdot 5}\int[\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega)}(\bar{\lambda}\gamma^r)^{\alpha_1}(\bar{\lambda}\gamma^s)^{\alpha_2}(\bar{\lambda}\gamma^t)^{\alpha_3}(\gamma_{rst})^{\alpha_4\alpha_5}$$
$$\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\theta^{\delta_1}\cdots\theta^{\delta_{11}}\frac{(2^4\cdot 11!5!3!)}{2^6\cdot 3(\lambda\bar{\lambda})^5}(\bar{\lambda}\gamma_{mnp}D)(\lambda A_1(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)).$$

In [22] the following identity was proven

$$(\bar{\lambda}\gamma_{mnp}D)((\lambda A_1)(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4))=40(\lambda\bar{\lambda})(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}, \tag{3.39}$$

and the $\mathcal{K}$ factor takes the form

$$\mathcal{K}=\frac{40}{11!11!5!\cdot 2^9\cdot 3\cdot 5}\frac{(2^4\cdot 11!5!3!)}{2^6\cdot 3}\int[\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]\frac{e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega)}}{(\lambda\bar{\lambda})^5}$$
$$(\bar{\lambda}\gamma^r)^{\alpha_1}(\bar{\lambda}\gamma^s)^{\alpha_2}(\bar{\lambda}\gamma^t)^{\alpha_3}(\gamma_{rst})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\theta^{\delta_1}\cdots\theta^{\delta_{11}}(\lambda\bar{\lambda})(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}.$$

Since we are interested in the NS-NS sector, we can use the following result found in [22]

$$\langle\,(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\,\rangle=\frac{1}{2^3\cdot 2880}K_0, \tag{3.40}$$

where K_0 is the Kinematic factor of [14]

$$K_0=(e_1\cdot e_2)\,[2tu(e_3\cdot e_4)-4t(e_3\cdot k_1)(e_4\cdot k_2)]+\text{perm}. \tag{3.41}$$

But as (3.40) was computed using the normalization $\langle\,(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\,\rangle=1$, we can write the following equality for NS-NS sector

$$(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\Big|_{\text{NS-NS}}=(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\frac{K_0}{2^3\cdot 2880}. \tag{3.42}$$

Now, we can integrate the θ^α variable in the $\mathcal{K}$ factor

$$\mathcal{K}=\frac{40}{11!11!5!\cdot 2^9\cdot 3\cdot 5}\frac{(2^4\cdot 11!5!3!)}{2^6\cdot 3\cdot 2^3\cdot 2880}\int[\mathrm{d}\theta][\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]\frac{e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega)}}{(\lambda\bar{\lambda})^5}(\lambda\bar{\lambda})$$
$$(\bar{\lambda}\gamma^r)^{\alpha_1}(\bar{\lambda}\gamma^s)^{\alpha_2}(\bar{\lambda}\gamma^t)^{\alpha_3}(\gamma_{rst})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\theta^{\delta_1}\cdots\theta^{\delta_{11}}(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)K_0$$
$$=\frac{5}{2^4\cdot 3}K_0\int[\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]\frac{e^{(-\lambda\bar{\lambda}-\bar{\omega}\omega)}}{(\lambda\bar{\lambda})}. \tag{3.43}$$

4 Integration on pure spinors space

In order to get the full expression of the one loop amplitude we need to compute the integral on the pure spinors space. It is not a trivial integral. Actually, if we try to solve it in a straight forward way or using computational methods maybe we could not do it. We will use some tools of algebraic geometry to solve it and we suggest the reader to read before the appendix for a better understanding of this section.

Let's remember that the measures $[\mathrm{d}\lambda]$ and $[\mathrm{d}\omega]$ were defined in (3.18) and (3.35)

$$
\begin{aligned}
[\mathrm{d}\omega] &= \frac{1}{11!5!}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\mathrm{d}\omega_{\delta_1}\wedge\cdots\wedge\mathrm{d}\omega_{\delta_{11}},\\
[\mathrm{d}\lambda](\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} &= \frac{1}{11!}\epsilon_{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\mathrm{d}\lambda^{\rho_1}\wedge\cdots\wedge\mathrm{d}\lambda^{\rho_{11}}
\end{aligned}
$$

and the measures $[\mathrm{d}\bar{\lambda}]$ and $[\mathrm{d}\bar{\omega}]$ were defined in (3.21) and (3.35)

$$
\begin{aligned}
[\mathrm{d}\bar{\omega}](\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} &= \frac{(\lambda\bar{\lambda})^3}{11!}\epsilon_{\alpha_1\cdots\alpha_5\delta_1\cdots\delta_{11}}\mathrm{d}\bar{\omega}^{\delta_1}\wedge\cdots\wedge\mathrm{d}\bar{\omega}^{\delta_{11}},\\
[\mathrm{d}\bar{\lambda}] &= \frac{1}{11!5!(\lambda\bar{\lambda})^3}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\cdots\alpha_5\rho_1\cdots\rho_{11}}\mathrm{d}\bar{\lambda}_{\rho_1}\wedge\cdots\wedge\mathrm{d}\bar{\lambda}_{\rho_{11}}.
\end{aligned}
$$

With these measures it follows

$$
[\mathrm{d}\omega]\wedge[\mathrm{d}\bar{\omega}] = \frac{(\lambda\bar{\lambda})^3}{11!}\mathrm{d}\omega_{\alpha_1}\wedge\mathrm{d}\bar{\omega}^{\alpha_1}\wedge\cdots\wedge\mathrm{d}\omega_{\alpha_1}\wedge\mathrm{d}\bar{\omega}^{\alpha_1} = (\lambda\bar{\lambda})^3\mathrm{d}\omega_+\wedge\mathrm{d}\bar{\omega}^+\bigwedge_{a<b,\,c<d}\mathrm{d}\omega^{ab}\mathrm{d}\bar{\omega}_{cd},
$$

where we have taken the gauge $\omega^a = \bar{\omega}_a = 0$. Now the integral (3.43) on the ω and $\bar{\omega}$ variables is trivial

$$
\begin{aligned}
\int[\mathrm{d}\omega][\mathrm{d}\bar{\omega}]\;e^{-\omega\bar{\omega}} &= (\lambda\bar{\lambda})^3\int\mathrm{d}\omega_+\wedge\mathrm{d}\bar{\omega}^+\bigwedge_{a<b,\,c<d}\mathrm{d}\omega^{ab}\mathrm{d}\bar{\omega}_{cd}\;e^{-\omega_+\bar{\omega}^+-\frac{1}{2}\omega^{ab}\bar{\omega}_{ab}}\\
&= (\lambda\bar{\lambda})^3(2\pi)^{11}.
\end{aligned}
$$

Integral on pure spinors space. From the above result we can write the integral (3.43) in the following form

$$
\begin{aligned}
\int[\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}][\mathrm{d}\omega][\mathrm{d}\bar{\omega}]\;\frac{e^{-\lambda\bar{\lambda}-\omega\bar{\omega}}}{\lambda\bar{\lambda}} &= (2\pi)^{11}\int[\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}]\,(\lambda\bar{\lambda})^2e^{-\lambda\bar{\lambda}}\\
&= (2\pi)^{11}\lim_{a\to1}\frac{\partial^2}{\partial a^2}\int[\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}]\,e^{-a\lambda\bar{\lambda}}. \qquad (4.1)
\end{aligned}
$$

Thus the integral of our interest is simply

$$
\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]\,e^{-a\lambda\bar{\lambda}} \qquad (4.2)
$$

and next we will show that it is equal to

$$
\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]\,e^{-a\lambda\bar{\lambda}} = \frac{(2\pi)^{11}}{a^8\cdot 60}. \qquad (4.3)
$$

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We can easily see that the measure $[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]$ can be written as

$$
\begin{aligned}
[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}] &= \frac{1}{11!(\lambda\bar{\lambda})^3} \mathrm{d}\lambda^{\alpha_1} \wedge \mathrm{d}\bar{\lambda}_{\alpha_1} \wedge \cdots \wedge \mathrm{d}\lambda^{\alpha_{11}} \wedge \mathrm{d}\bar{\lambda}_{\alpha_{11}} \\
&= \frac{1}{11!(\lambda\bar{\lambda})^3} \partial\bar{\partial}(\lambda\bar{\lambda}) \wedge \cdots \wedge \partial\bar{\partial}(\lambda\bar{\lambda}) \\
&= \frac{\Omega^{11}}{11!} ,
\end{aligned}
$$

where

$$
\Omega = \frac{1}{(\lambda\bar{\lambda})^{3/11}} \partial\bar{\partial}(\lambda\bar{\lambda})
$$

is the Kähler form[2] on the pure spinors space in $D = 2n = 10$ dimension. In the parametrization (2.11) the integration measure on pure spinors space is

$$
\frac{\Omega^{11}}{11!} = \gamma^7 \mathrm{d}\gamma \bigwedge_{a<b} \mathrm{d}u_{ab} \wedge \bar{\gamma}^7 \mathrm{d}\bar{\gamma} \bigwedge_{c<d} \mathrm{d}\bar{u}^{cd}. \tag{4.4}
$$

The Kähler form of the pure spinors space in any dimension is given by

$$
\Omega_{D=2n} = \frac{1}{(\lambda\bar{\lambda})^{\frac{\dim_{\mathbb{C}} PS - c_1}{\dim_{\mathbb{C}} PS}}} \partial\bar{\partial}(\lambda\bar{\lambda}), \tag{4.5}
$$

where $c_1 = 2n - 2$ is the first Chern class of the tangent bundle over $\mathrm{SO}(2n)/\mathrm{U}(n)$ [5] and $\dim_{\mathbb{C}} PS = \frac{n(n-1)}{2} + 1$ is the complex dimension of the pure spinors space.

Writing (4.2) in the coordinates (2.11) we get

$$
\int [\mathrm{d}\lambda][\mathrm{d}\bar{\lambda}] e^{-a\lambda\bar{\lambda}} = \int (\gamma\bar{\gamma})^7 \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \bigwedge_{a<b,c<d} \mathrm{d}u_{ab} \mathrm{d}\bar{u}^{cd} e^{-a\gamma\bar{\gamma}\left(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi}\right)} . \tag{4.6}
$$

The γ, $\bar{\gamma}$ variables can be integrated easily

$$
\int (\gamma\bar{\gamma})^7 \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\; e^{-b\gamma\bar{\gamma}} = -\frac{\partial^7}{\partial b^7} \int \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\; e^{-b\gamma\bar{\gamma}} = (2\pi) \cdot 7! \cdot \frac{1}{b^8} , \tag{4.7}
$$

where

$$
b := a \left(1 + \frac{1}{2} u_{ab}\bar{u}^{ab} + \frac{1}{8^2} \epsilon^{abcde} \epsilon_{afghi} u_{bc} u_{de} \bar{u}^{fg} \bar{u}^{hi} \right), \tag{4.8}
$$

so (4.2) has now the form

$$
\int [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-a\lambda\bar{\lambda}} = \frac{(2\pi) \cdot 7!}{a^8} \int_{\mathrm{SO}(10)/\mathrm{U}(5)} \alpha \;, \tag{4.9}
$$

where

$$
\alpha := \frac{\bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab} \mathrm{d}\bar{u}^{cd}}{(1 + \frac{1}{2} u_{ab}\bar{u}^{ab} + \frac{1}{8^2} \epsilon^{abcde} \epsilon_{afghi} u_{bc} u_{de} \bar{u}^{fg} \bar{u}^{hi})^8} \tag{4.10}
$$

[2]Easily we can see that $(\lambda\bar{\lambda})$ is a scalar function (global) on the pure spinors space.

is a global form on SO(10)/U(5), therefore it belongs to the $H^{20}_{DR}(\mathrm{SO}(10)/\mathrm{U}(5))$ de-Rham cohomology group [7, 8]. Note that the number 8 is the first Chern class of the tangent bundle over SO(10)/U(5).

The α-form can be written as

$$\alpha = \frac{\omega^{10}}{10!}, \tag{4.11}$$

where

$$\omega = -\partial\bar{\partial}\,\ln(\lambda\bar{\lambda}) \tag{4.12}$$

and λ and $\bar{\lambda}$ are projective pure spinors, in others words

$$\lambda\bar{\lambda} = 1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi}, \tag{4.13}$$

where $\{u_{ab}\}$ is a complex parametrization on SO(10)/U(5). The 2-form ω is the Kähler form, so $\ln(\lambda\bar{\lambda})$ is the Kähler potential [7]. From the identity

$$\partial\bar{\partial} = \frac{1}{2}\mathrm{d}(\partial - \bar{\partial})$$

we can see that $\mathrm{d}\omega = 0$ is closed, therefore SO(10)/U(5) is a Kähler manifold.

From the algebraic geometry point of view, the projective pure spinors space in $d = 2n = 10$ is a variety (manifold) on the projective space $\mathbb{C}P^{15}$, then its Kälher form is the pullback of the Kähler form of $\mathbb{C}P^{15}$ given by [7, 9]

$$\omega = f^*\Omega, \tag{4.14}$$

where Ω is the Fubini-Study [7] metric of $\mathbb{C}P^{15}$ and

$$f:\ \mathrm{SO}(10)/\mathrm{U}(5)\ \to\ \mathbb{C}P^{15} \tag{4.15}$$

is the corresponding map. It is given locally on the chart $U = \{\lambda^+ \neq 0\}$ by the following five holomorphic homogeneous polynomials [2, 33]

$$2\lambda^+\lambda^a - \frac{1}{4}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \quad a = 1,\ldots,5. \tag{4.16}$$

As SO(10)/U(5) is a closed manifold on $\mathbb{C}P^{15}$, then it belongs to the $H_{20}(\mathbb{C}P^{15}) = \mathbb{Z}$ homology group [34], so the projective pure spinors space is proportional to the $[\mathbb{C}P^{10}]$ homology class because $\mathbb{C}P^{10}$ is the generator of the $H_{20}(\mathbb{C}P^{15})$ homology group [32, 34]. The proportionality factor is called the " degree" of a variety and it is a integer number since $H_{20}(\mathbb{C}P^{15}) = \mathbb{Z}$. The degree of projective pure spinors is given by

$$\mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5)) = {}^{\#}(\mathrm{SO}(10)/\mathrm{U}(5)\cdot\mathbb{C}P^5), \tag{4.17}$$

where ${}^{\#}(\mathrm{SO}(10)/\mathrm{U}(5)\cdot\mathbb{C}P^5)$ are the intersection numbers between SO(10)/U(5) and $\mathbb{C}P^5$ inside $\mathbb{C}P^{15}$, hence the previous integral can be written as

$$\int_{\mathrm{SO}(10)/\mathrm{U}(5)}\frac{\omega^{10}}{10!} = \mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5))\ \int_{\mathbb{C}P^{10}}\left.\frac{\Omega^{10}}{10!}\right|_{\mathbb{C}P^{10}}. \tag{4.18}$$

Remember that the pure spinors space is identified with the total space of the line bundle $\mathcal{O}(-1)$; which is the inverse of the line bundle $\mathcal{L} = \mathcal{O}(1)$ [7, 11]. The first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ is simply the pullback of the hyperplane class H [7, 8]

$$c_1(\mathcal{L}) = f^* H \tag{4.19}$$

and the degree of the projective pure spinors space is given by

$$\begin{aligned}\int_{\mathrm{SO}(10)/\mathrm{U}(5)} c_1(\mathcal{L})^{10} &= \mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5)) \int_{\mathbb{C}P^{10}} H^{10}\Big|_{\mathbb{C}P^{10}} \\ &= \mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5)) \int_{\mathbb{C}P^{10}} \frac{c_{10}(T\mathbb{C}P^{10})}{11} \\ &= \mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5)),\end{aligned} \tag{4.20}$$

where $\int_{\mathbb{C}P^{10}} c_{10}(T\mathbb{C}P^{10})$ is the Euler characteristic of $\mathbb{C}P^{10}$. We will compute this degree using the pure spinors character at zero level. The Riemann-Roch formula gives us an expression for the pure spinors character at level zero [5]

$$Z_{10}(t) = \int_{\mathrm{SO}(10)/\mathrm{U}(5)} \frac{1}{1 - te^{-c_1(\mathcal{L})}} Td(T(\mathrm{SO}(10)/\mathrm{U}(5))), \tag{4.21}$$

where $Td(T(\mathrm{SO}(10)/\mathrm{U}(5)))$ is the Todd genus

$$Td(T(\mathrm{SO}(10)/\mathrm{U}(5))) = 1 + \frac{1}{2} c_1(T(\mathrm{SO}(10)/\mathrm{U}(5)) + \cdots \ . \tag{4.22}$$

Expanding $Z_{10}(t)$ near to $t = 1$ or near to $\epsilon = 1 - t = 0$, the most singular term is [11]

$$\frac{1}{\epsilon^{11}} \int_{\mathrm{SO}(10)/\mathrm{U}(5)} c_1(\mathcal{L})^{10}. \tag{4.23}$$

The pure spinors character can also be computed with the reducibility method, in this case the result is [5, 12]

$$Z_{10}(t) = \frac{1 + 5t + 5t^2 + t^3}{(1-t)^{11}}. \tag{4.24}$$

Again, expanding near $\epsilon = 0$ we get that the most singular term is

$$\frac{12}{\epsilon^{11}}. \tag{4.25}$$

Comparing both results we conclude that the projective pure spinors degree is

$$\mathrm{degree}(\mathrm{SO}(10)/\mathrm{U}(5)) = \int_{\mathrm{SO}(10)/\mathrm{U}(5)} c_1(\mathcal{L})^{10} = 12. \tag{4.26}$$

Therefore we have solved in a easy way the integral (4.2)

$$\begin{aligned}\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-a\lambda\bar{\lambda}} &= \frac{(2\pi)\cdot 7!}{a^8} \int_{\mathrm{SO}(10)/\mathrm{U}(5)} \frac{(f^*\Omega)^{10}}{10!} \\ &= \frac{(2\pi)\cdot 7! \cdot 12}{a^8 \cdot 10!} \cdot \int_{\mathbb{C}P^{10}} \Omega^{10}\Big|_{\mathbb{C}P^{10}} \\ &= \frac{(2\pi)^{11} \cdot 7! \cdot 12}{a^8 \cdot 10!} \\ &= \frac{(2\pi)^{11}}{a^8 \cdot 60}.\end{aligned} \tag{4.27}$$

Actually, we can compute (4.2) for any dimension using the Kähler form (4.5) (see appendix)

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}] e^{-a\lambda\bar{\lambda}} = \frac{(2\pi)^{c_1(T\mathbb{C}P^{n(n-1)/2})}}{a^{c_1(T\mathcal{Q}_{2n})}} \cdot \frac{c_1(T\mathcal{Q}_{2n})!}{c_1(T\mathbb{C}P^{n(n-1)/2})!} \cdot \frac{c_1(T\mathbb{C}P^{n(n-1)/2})}{c_1(T\mathcal{Q}_{2n})} \cdot \mathrm{degree}(\mathcal{Q}_{2n})$$

where $c_1(T\mathcal{Q}_{2n}) = 2n-2$ is the first Chern class of the tangent bundle over projective pure spinors space $\mathcal{Q}_{2n} \equiv \mathrm{SO}(2n)/\mathrm{U}(n)$, $c_1(T\mathbb{C}P^{n(n-1)/2}) = (n(n-1)+2)/2$ is the first Chern class of the tangent bundle over projective space $\mathbb{C}P^{n(n-1)/2}$ and $\mathrm{degree}(\mathcal{Q}_{2n})$ is the degree of the projective pure spinors space

$$[\mathcal{Q}_{2n}] = \mathrm{degree}(\mathcal{Q}_{2n})[\mathbb{C}P^{n(n-1)/2}]. \tag{4.28}$$

With this result, we finally have that the 4-points scattering amplitude is

$$\mathcal{A} = (2\pi)^{10}\delta^{(10)}(k)\frac{1}{2^7\pi^2(\alpha')^5}\left[\left(\frac{\alpha'}{2}\right)^2\kappa\right]^4 K_0\overline{K}_0\int_{\mathcal{M}_1}\frac{\mathrm{d}^2\tau}{(\tau_2)^5}\prod_{k=1}^{3}\int \mathrm{d}^2 z_k \prod_{i<j=1}^{4}|E(z_i,z_j)|^{\alpha' k_i\cdot k_j}$$
$$\exp\left[-k_i\cdot k_j\frac{2\pi\alpha'}{\tau_2}\mathrm{Im}z_i\mathrm{Im}z_j\right] \tag{4.29}$$

This answer is in perfect agreement with the result found by D'hoker, Phong and Gutperle in [14] up to a $(\alpha'/2)^8$ factor. Is easy to see that this factor is needed in order to have the right space-time dimensions [30]. Hence the amplitude found in [14] by D'hoker, Phong and Gutperle missed this term.

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A Pure spinors in lower dimensions and partition function

The aim of studying pure spinors in lower dimensions ($D = 2n < 10$) is to have a better feeling of some algebraic properties of the pure spinors space. At the end of the appendix we make some remarks and give a nice geometric interpretation of the character of pure spinors.

We know that in $D = 4,6,8$ the projective pure spinors space are $\mathbb{C}P^1$, $\mathbb{C}P^3$ and a quadric variety embedded in $\mathbb{C}P^7$, respectively.

$\mathbb{C}P^1$ and $\mathbb{C}P^3$ are the trivial cases because in $D = 4,6$ the pure spinors don't have any constraints and the pure spinors space is the simple blow-up of the origin [9] (the pure spinors space is the total space of the line bundle $\mathcal{O}(-1)$). In these cases the Kälher form of the pure spinors space is simply

$$\Omega = \partial\bar{\partial}(\lambda\bar{\lambda}), \tag{A.1}$$

where we have used the general formula (4.5)

$$\Omega_{D=2n} = (\lambda\bar{\lambda})^{-\frac{\dim_{\mathbb{C}} PS - c_1}{\dim_{\mathbb{C}} PS}} \partial\bar{\partial}(\lambda\bar{\lambda})$$

and the notation

$$\lambda\bar{\lambda} = \gamma\bar{\gamma}(1+z\bar{z}), \qquad \text{for } D=4\,, \tag{A.2}$$
$$\lambda\bar{\lambda} = \gamma\bar{\gamma}(1+z\bar{z}+u\bar{u}+v\bar{v}), \qquad \text{for } D=6\,, \tag{A.3}$$

where $\{z\}$ parametrize $\mathbb{C}P^1$, $\{z,\, u,\, v\}$ parametrize $\mathbb{C}P^3$, $\{\gamma\}$ is the fiber and c_1 is the first Chern class of projective pure spinors space. From [5] we can see that in $D=4,6$ the first Chern class of the tangent bundle over the projective pure spinors space is

$$\begin{aligned} c_1(T\mathbb{C}P^1) &= 2, \\ c_1(T\mathbb{C}P^3) &= 4 \end{aligned} \tag{A.4}$$

and it has the same value of the complex dimension of the pure spinors space ($\dim_{\mathbb{C}} PS$).

The integration measures for the pure spinors space in $D=4,6$ are given by

$$\frac{\Omega^2}{2!} = \omega\wedge\bar{\omega} \qquad \text{for } D=4\,, \tag{A.5}$$
$$\frac{\Omega^4}{4!} = \omega\wedge\bar{\omega} \qquad \text{for } D=6\,, \tag{A.6}$$

where

$$\omega = \gamma\, \mathrm{d}\gamma\wedge \mathrm{d}z \qquad \text{for } D=4 \tag{A.7}$$
$$\omega = \gamma^3\, \mathrm{d}\gamma\wedge \mathrm{d}z\wedge \mathrm{d}u\wedge \mathrm{d}v \qquad \text{for } D=6 \tag{A.8}$$

are the holomorphic top forms, which agree with the ones of [11]. To compute (A.4) is very easy from the following exact sequence of bundles (the Euler sequence)[7, 9]

$$0 \longrightarrow \mathbb{C} \longrightarrow H^{\oplus(n+1)} \longrightarrow T\mathbb{C}P^n \longrightarrow 0\ , \tag{A.9}$$

where $\mathbb{C}$ is a trivial bundle, H is the hyperplane class and $T\mathbb{C}P^n$ is the tangent bundle on $\mathbb{C}P^n$. This sequence implies that

$$H^{\oplus(n+1)} = T\mathbb{C}P^n \oplus \mathbb{C}. \tag{A.10}$$

Therefore, the total Chern class of the tangent bundle on $\mathbb{C}P^n$ is

$$c(T\mathbb{C}P^n) = (1+H)^{n+1} \tag{A.11}$$

where we have denoted the first Chern class of the hyperplane bundle H with the same letter H. Now it is clear that $c_1(T\mathbb{C}P^n) = (n+1)H$ and that $c_n(T\mathbb{C}P^n) = (n+1)H^n$. As the Euler characteristic of a complex manifold M of complex dimension n is [7]

$$\chi(M) = \int_M c_n(TM)\ , \tag{A.12}$$

then we have that

$$\int_{\mathbb{C}P^n} H^n = 1, \tag{A.13}$$

which was used in (4.20) and (4.27). Let's apply the previous results to the pure spinors space in $D = 4$.

We know that the integration measure on the pure spinors space in $D = 4$ is

$$\frac{\Omega^2}{2!} = -\gamma\bar{\gamma}\,\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{A.14}$$

Let's integrate the function $\exp\{-a\lambda\bar{\lambda}\}$, with $a \in \mathbb{R}^+$,

$$\begin{aligned}\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-a\lambda\bar{\lambda}} &= -\int_{\mathbb{C}^2} \gamma\bar{\gamma}\,\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}\, e^{-a\gamma\bar{\gamma}(1+z\bar{z})} \\ &= \frac{\pi}{a^2 i}\int_{\mathbb{C}} \frac{2}{(1+z\bar{z})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.\end{aligned} \tag{A.15}$$

We can see that $g_{z\bar{z}} = 2/(1+z\bar{z})^2$ is the metric of S^2 with radius 1 on a chart homeomorphic to $\mathbb{C}$. The area of a sphere with radius R is $4\pi R^2$, so the integral (A.15) is $4\pi^2/a^2$. Nevertheless we want to show how to compute the integral (A.15) using simple topological properties of the projective pure spinors space (S^2). Let's remember that the first Chern class of a complex manifold $\mathcal{M}$ is given by the expression

$$c_1(T\mathcal{M}) = \frac{i}{2\pi}\partial\bar{\partial}\ln\det(g_{i\bar{j}}), \tag{A.16}$$

so, in our example we have

$$c_1(TS^2) = \frac{2}{2\pi i}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2}. \tag{A.17}$$

Note that the number 2 on the numerator, which comes of the exponent of $(1+z\bar{z})^2$, is simply the first Chern class of the tangent bundle with respect to the hyperplane bundle H ($c_1(TS^2) = 2H$),[3] hence

$$H = \frac{1}{2\pi i}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2} \tag{A.18}$$

on the chart. Now, using (A.13) we can easily compute (A.15)

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-a\lambda\bar{\lambda}} = \frac{2\pi}{a^2}2\pi\int_{\mathbb{C}}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi i(1+z\bar{z})^2} = \frac{4\pi^2}{a^2}\int_{\mathbb{C}P^1} H = \frac{4\pi^2}{a^2}, \tag{A.19}$$

as expected.

We can get the same result (A.13) from the partition function, for example, computing the partition function for $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ in the zero level with the reducibility method [13] we have

$$Z_{\mathcal{O}(-1)}(t) = \frac{1}{(1-t)^{n+1}}. \tag{A.20}$$

Expanding around to $\epsilon = 1 - t = 0$ the most singular term is

$$\frac{1}{\epsilon^{n+1}}, \tag{A.21}$$

[3] This is the same argument by which the number 8 is in the 20-form (4.10).

and by comparing with the Riemann-Roch formula (4.21) we get (A.13).

Now we discuss some aspects of intersection theory. It is clear that in $\mathbb{C}P^n$ we have a set $\{\mathbb{C}P^m\}$ with $m \leq n$ which is embedded it. It is easy to see that these $\mathbb{C}P^m$'s intersect transversally of a point [7], i.e

$$^{\#}(\mathbb{C}P^m \cdot \mathbb{C}P^{n-m}) = 1\,, \qquad m \leq n. \tag{A.22}$$

As the homology groups of $\mathbb{C}P^n$ are [34]

$$H_{2i}(\mathbb{C}P^n) = \mathbb{Z}\,, \qquad i = 1, 2, \ldots, n \tag{A.23}$$

then by (A.22) we can take the homology generators to be the $[\mathbb{C}P^i]$ classes. With this, we define the degree of a closed variety V of complex dimension m by

$$\text{degree}(V) = \,^{\#}(V \cdot \mathbb{C}P^{n-m}). \tag{A.24}$$

This is a topological number because it depends only on the homology class.

Now we compute the degree for projective pure spinors in $D = 8$.The projective pure spinors space in $D = 8$ ($\mathcal{Q}_8$) is a hypersurface in $\mathbb{C}P^7$. It is given in terms of homogeneous coordinates $\{\lambda^+, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23},$
$\lambda_{24}, \lambda_{34}, \lambda_{1234}\}$ on $\mathbb{C}P^7$ as the zero locus of [33]

$$\lambda^+\lambda_{1234} - \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{23}\lambda_{14} = 0. \tag{A.25}$$

Since degree($\mathcal{Q}_8$) is the number of points where $\mathcal{Q}_8$ and $\mathbb{C}P^1$ are intersected, if we take $\mathbb{C}P^1$ as the locus $\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = \lambda_{34} = 0$, the degree($\mathcal{Q}_8$) will be the number of solutions of the homogeneous polynomial

$$\lambda^+\lambda_{1234} = 0. \tag{A.26}$$

The solutions of this polynomial are the points $[1, 0, 0, 0, 0, 0, 0, 0]$ and $[0, 0, 0, 0, 0, 0, 0, 1]$, therefore

$$\text{degree}(\mathcal{Q}_8) = 2. \tag{A.27}$$

Using the partition function we get the same answer, i.e, the partition function for $\mathcal{O}(-1)$ over $\mathcal{Q}_8$ is given by [13]

$$Z_{\mathcal{Q}_8}(t) = \frac{1+t}{(1-t)^7}\,. \tag{A.28}$$

Expanding near to $\epsilon = 1 - t = 0$, the most singular term of $Z_{\mathcal{Q}_8}(t)$ is

$$\frac{2}{\epsilon^7}, \tag{A.29}$$

so, by comparing with the Riemann-Roch formula (4.21) we get

$$\int_{\mathcal{Q}_8} c_1(\mathcal{L})^6 = 2. \tag{A.30}$$

Actually this result was expected, since $\mathcal{Q}_8$ is a hypersurface given by a homogeneous polynomial of degree 2, then the first Chern class of the divisor $[\mathcal{Q}_8]$ is

$$c_1([\mathcal{Q}_8]) = 2H, \tag{A.31}$$

which is Poincaré dual to $\mathcal{Q}_8$ [7, 8]. So

$$\int_{\mathcal{Q}_8} c_1(\mathcal{L})^6 = \int_{\mathcal{Q}_8} (f^* H)^6 = \int_{\mathbb{C}P^7} H^6 \wedge c_1([\mathcal{Q}_8]) = 2\int_{\mathbb{C}P^7} H^7 = 2. \tag{A.32}$$

where $f : \mathcal{Q}_8 \to \mathbb{C}P^7$ is the embeding.

We now have a geometric interpretation to the result found in [5]. In [5] it was shown that the partition function of pure spinors can be written as a rational function[4]

$$Z_{\mathcal{O}(-1)}(t) = \frac{P(t)}{Q(t)}\,, \tag{A.33}$$

where $P(t)$ and $Q(t)$ are polynomials. In $D = 2n$ the $Q(t)$ polynomial has the form [5, 12]

$$Q(t) = (1-t)^{\dim_{\mathbb{C}} PS}. \tag{A.34}$$

In [5] it was also shown that $Z_{\mathcal{O}(-1)}(t)$ can be written as an infinite product ($ghost-ghost$)

$$Z_{\mathcal{O}(-1)}(t) = \prod_{n=1}^{\infty} (1-t^n)^{-N_n}. \tag{A.35}$$

The N_n coefficients contain the information about the Virasoro central charge, ghost number anomaly, etc

$$\frac{1}{2} c_{\text{vir}} = \sum_n N_n\,, \tag{A.36}$$

$$a_{\text{ghost}} = \sum_n n N_n. \tag{A.37}$$

From (A.33) and (A.35) we have

$$\ln(-x)\sum_n N_n + \sum_n \ln(n) N_n + \frac{x}{2}\sum_n n N_n + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g} \sum_n n^{2g} N_n = -\ln P(e^x) + \ln Q(e^x), \tag{A.38}$$

where $\{B_g\}$ are the Bernoulli numbers. Replacing (A.34) in the previous expression we get

$$\ln(1-e^x)^{\dim_{\mathbb{C}} PS} = (\dim_{\mathbb{C}} PS)\ln(-x) + \frac{\dim_{\mathbb{C}} PS}{2} x + \frac{\dim_{\mathbb{C}} PS}{24} x^2 + \cdots\,. \tag{A.39}$$

Without loss of generality we can suppose that

$$P(e^x) = y + a\,e^x + b\,e^{2x} + c\,e^{3x} + \cdots\,, \tag{A.40}$$

[4]We are only interested in the zero level.

so

$$
\begin{aligned}
-\ln P(e^x) &= -\ln P(1) - \partial_x \ln P(x)|_{x=1}\, x + \cdots && \text{(A.41)}\\
&= -\ln P(1) - \frac{\partial_x P(x)|_{x=1}}{P(1)} x + \cdots && \text{(A.42)}\\
&= -\ln(y + a + b + c + \cdots) - \frac{a + 2\,b + 3\,c + \cdots}{y + a + b + c + \cdots} x + \cdots && \text{(A.43)}\\
& && \text{(A.44)}
\end{aligned}
$$

and therefore we have

$$
\begin{aligned}
\frac{1}{2} c_{\text{vir}} &= \sum_n N_n = \dim_{\mathbb{C}} PS\,, && \text{(A.45)}\\
a_{\text{ghost}} &= \sum_n n N_n = \dim_{\mathbb{C}} PS - 2\frac{\partial_x P(x)|_{x=1}}{P(1)}\,, && \text{(A.46)}\\
\ln P(1) &= -\sum_n \ln(n) N_n = \ln(\text{degree}\,\mathcal{Q}_{2n}), \qquad \mathcal{Q}_{2n} := \mathrm{SO}(2n)/\mathrm{U}(n). && \text{(A.47)}
\end{aligned}
$$

From the Riemann-Roch formula (4.21) and by expanding (A.33) with (A.34) near to $\epsilon = 1 - t = 0$ it is clear than degree$(\mathcal{Q}_{2n}) = P(1)$.

We know that a_{ghost} is the first Chern class of $T\mathcal{Q}_{2n}$ and that the degree$(\mathcal{Q}_{2n})$ gives the homology class

$$
[\mathcal{Q}_{2n}] = \text{degree}(\mathcal{Q}_{2n})[\mathbb{C}P^{n(n-1)/2}]\,, \qquad \text{(A.48)}
$$

in others words, the degree$(\mathcal{Q}_{2n})$ gives us the Poincaré dual class of $\mathcal{Q}_{2n}$. Noting that the homology class of $\mathcal{Q}_{2n}$ is an integer number times the homology class of $\mathbb{C}P^{n(n-1)/2}$, we can interpret $\dim_{\mathbb{C}} PS = 1 + n(n-1)/2$ as the first Chern class of $T\mathbb{C}P^{n(n-1)/2}$. Thus we have

$$
\begin{aligned}
c_1(T\mathbb{C}P^{n(n-1)/2}) &= \sum_n N_n\,, && \text{(A.49)}\\
c_1(T\mathcal{Q}_{2n}) &= \sum_n n N_n\,, && \text{(A.50)}\\
\text{degree}(\mathcal{Q}_{2n}) &= \exp(-\sum_n \ln(n) N_n) = \left(\prod_n n^{N_n}\right)^{-1}. && \text{(A.51)}
\end{aligned}
$$

With these geometric interpretation we get a geometric constraint on the coefficients of the $P(t)$ polynomial

$$
\text{degree}(\mathcal{Q}_{2n})\,\{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})\} = 2\,\partial_x P(x)|_{x=1}. \qquad \text{(A.52)}
$$

We can also rewrite the integration measure of the pure spinors space (4.5) as

$$
\Omega_{D=2n} = (\lambda\bar\lambda)^{-\frac{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})}{c_1(T\mathbb{C}P^{n(n-1)/2})}} \partial\bar\partial(\lambda\bar\lambda) = (\lambda\bar\lambda)^{\frac{-2\,\partial_x P(x)|_{x=1}}{\text{degree}(\mathcal{Q}_{2n}) c_1(T\mathbb{C}P^{n(n-1)/2})}} \partial\bar\partial(\lambda\bar\lambda)\,, \qquad \text{(A.53)}
$$

where we interpret the term $\{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})\}$ as a topological deviation and find a relationship between the integration measure and the character of the pure spinors space.

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