## Applications of the spin networks and spin foam models in quantum gravity



by

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Thesis written under the supervision of prof. Jerzy Lewandowski submitted to the Faculty of Physics, University of Warsaw for the degree of Doctor of Philosophy in Physics

Warsaw, August 2014

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# Preface

A quantum theory of gravity is one of the greatest mysteries of modern physics. One of the several approaches to find it is the Loop Quantum Gravity framework, within which the Spin-Foam models provide a path integral picture of quantisation.

In this thesis we study four aspects of Spin-Foam models.

Firstly we address and answer the question of the class of 2-complex, that ensure that Spin-Foam models are compatible with the kinematic sector of Loop Quantum Gravity. While researching this issue, we developed a framework of diagrammatic representation of spin-foams, called Operator Spin-network Diagrams (OSDs). The OSDs allow to express a spin-foam as a collection of graphs, connected by certain relations. Each graph captures the local structure of one of spin-foam vertices, i.e. nodes of a graph correspond to edges and links of a graph correspond to faces incident to a vertex. The relations between graphs in OSDs represent the way, in which edges and faces connect vertices. We have proven, that for each OSD there is an unambiguous way to construct a 2-complex with cells labelled by a spin-foam coloring, so that one can calculate the spin-foam transition amplitude. We developed a procedure to glue OSDs along their boundary, being an equivalent of composing quantum processes. Moreover we characterised all possible OSDs in terms of gluing of basic diagrams representing zero or one interaction vertex each. We claim, that the appropriate class of 2-complexes for Spin-Foam models is given by all the 2-complexes that can be obtained out of one of OSDs. The chapter 3 presents this part of our results.

Our second aim was to apply OSDs to find a solution of so called *boundary problem*: given certain initial and final states of Loop Quantum Gravity we want to find all spinfoams which have these states as boundary. As our answer we provide an algorithm that finds a series of all OSDs with a given fixed boundary. The series is ordered by the number of internal edges of the corresponding spin-foam. We test our algorithm by applying it to (recently introduced) Dipole Cosmology model. We find all diagrams, that contribute to Dipole Cosmology amplitude, which have the minimal number of internal edges. Moreover, we studied the contribution to transition amplitude coming from the diagrams we found. In order to do this we adopted the techniques developed for spinfoams based on 2-complexes and obtained simple rules (similar to Feynman Rules of QFT) to read the components of the amplitude out of an OSD. Our study of the transition amplitudes of Dipole Cosmology diagrams led us to conclusion, that all the diagrams except from one gives amplitudes that are exponentially suppressed in the semiclassical limit, thus their presence does not spoil the result of authors of Dipole Cosmology model. The chapter 4 presents these results.

The third issue addressed in this thesis were the divergent amplitudes in Spin-Foam models. We focused on one of the sources of divergences, which are *bubbles* in spin-foam 2-complexes (i.e. subcomplexes forming closed surfaces). Within the framework of 2-complexes it is relatively hard to find the bubble part of a spin-foam. Thanks to OSDs we found a procedure that unambiguously identifies the bubble subdiagram. Moreover we introduced a notion of the rank of bubble, counting the number of elementary bubbles that the considered bubble consist of, and we presented a method to calculate it for each given OSD. We present a study of several possible simple cases of diagrams containing bubbles, that illustrate our algorithms. These results are presented in chapter 5.

The fourth question we posed and answered within this thesis is related to detailed study of one particular case of a spin-foam bubble, called *melonic bubble*. The melonic bubble is in a fact spin-foam analogue of self-energy renormalization in Quantum Field Theory. Recent research led to a conclusion, that in the first order the self-energy correction is proportional to some operator  $\mathbb{T}$ , however the operator  $\mathbb{T}$  was not known. We studied this operator in semiclassical limit. After some elaborate calculations we found the exact form of the leading order of  $\mathbb{T}$ : for fixed spin labels it is proportional to the identity operator, with the proportionality constant dependent on the spin labels. The calculations can are presented in chapter 6.

The presentation of our result is preceded by two chapters of introduction. First, in chapter 1, we recall the main ideas of Loop Quantum Gravity and Spin-Foam models. Then, in chapter 3, we provide a detailed presentation of each question addressed in the further part of the thesis, together with a brief statement of each result. The thesis is concluded in chapter 7, where we listed some directions of further research, that are opened by our results, or that still need to be done. To make the thesis selfcontained we followed the main text by two appendices: in Appendix A we provide an overview of mathematical notions, that we use in the main text, concerning graphs, CW-complexes, differential geometry and harmonic analysis on SU(2) and  $SL(2, \mathbb{C})$  groups. In Appendix B we present some proofs of non standard theorems, that although being technically complicated, are not of great physical relevance.

Part I Introduction

### Chapter 1

# General introduction to Loop Quantum Gravity and Spin-Foams

All the research presented in this thesis was done in Loop Quantum Gravity framework and Spin-foam models. This chapter provides a brief introduction to main points of these approaches.

Loop Quantum Gravity provides tools to describe the configuration space of General Relativity in such a way, that is relatively easy to translate into a quantum language. It bases on an assumption, that the gravitational field is the geometry of the space itself (in contrary to the approaches, where the gravitational field lives on a manifold with some background metric structure), and thus it gives a quantum descriptions of geometrical notions, such as area or volume. Loop Quantum Gravity is a framework of canonical quantisation, thus it provides a Hilbert space of quantum states over the configuration space of General Relativity together with an evolution operator - the Hamiltonian constraint. There are attempts to incorporate matter fields to Loop Quantum Gravity framework ([1], chapter 12) and to construct Dirac observable for its states (for example [2, 3]), however, since these problems are not related to this thesis, we do not explain them in this introduction. The issues related to the subject of the thesis are presented in section §1.1.

Spin-foam models give a formalism to calculate transition amplitudes for theories of dynamical connections. They are based on a concept of discretisation of the path integral formula. The discretisation leads to operators acting on states, that can be interpreted as kinematic states of Loop Quantum Gravity. An appropriate choice of interaction amplitude makes Spin-foam models a discretised version of a theory with the action equivalent to the action of Loop Quantum Gravity - thus the Spin-foam transition amplitudes are considered to describe the quantum evolution of Loop Quantum Gravity states. The introduction to Spin-foam models covering the points that are necessary in the further chapters of this thesis is presented in section §1.2.

This chapter is a brief introduction rather then a technical review of the subject. Detailed derivations and proofs are omitted. No discussion on former dead-ends of the research is given. The point of this chapter is to give the background to understand the further ones. Moreover, no open issues of the theory are discussed here. The presentation of the problems that were addressed in this thesis can be found in chapter 2.

All the mathematical conventions used in this thesis can be found in Appendix A.

### 1.1 Loop Quantum Gravity as canonical quantisation of General Relativity

Loop Quantum Gravity (LQG) is a promising attempt to quantise General Relativity. It started being developed in 1980's based on a discovery of new convenient variables that can be used to describe gravitational field [4, 5]. It is based on canonical quantisation framework.

Loop Quantum Gravity was widely described in several review papers and books, for example [6, 1]. This section is to provide a brief historical and technical review of the field, that is necessary to understand the further parts of the thesis. It is based mainly on [6].

The subsection 1.1.1 briefly recalls the way from Einstein's formulation of General Relativity to Ashtekar Variables for Gravity. The subsection 1.1.2 describes the kinematic Hilbert space of quantum states of Loop Quantum Gravity. Finally the subsection 1.1.3 presents the basic issues of quantum dynamics of the theory in it's canonical approach. The path-integral approach to the dynamics of LQG is described in section §1.2.

#### 1.1.1 General Relativity as a theory of dynamical connections

General relativity was originally formulated as a theory of a pseudo-metric field  $g_{\mu\nu}$  on a 4-dimensional Riemannian manifold  $\mathcal{M}$  [7]. There were many attempts to quantise the field  $g_{\mu\nu}$  itself or to decompose it into a perturbation around a flat metric  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , but none of them succeeded. Thus LQG is an attempt to quantise it using the canonical formalism.

The canonical formalism for General Relativity was given by Arnowitt, Deser and Misner [8, 9]. The gravitational field is expressed in terms of a metric tensor  $q_{ab}$  on a 3-dimensional manifold  $\Sigma$  and its conjugate momentum  $\pi^{ab}$ . An attempt to quantise such theory was done by deWitt in [10]. However, within this formalism there was never found a kinematic Hilbert space of the theory [11].

Instead of considering a metric tensor as a dynamical field of General Relativity one may consider a tetrad of 1-forms  $e^{I}$  and a connection  $\omega_{J}^{I}$ , as it was done in Plebański formalism [12]. Such approach with a slightly modified action [13] leads to new variables for General Relativity, known as Ashtekar Variables [4, 5].

In this subsection we start with a brief recall of the ADM formalism (in subsection 1.1.1.1), then we present the connection formalism (subsection 1.1.1.2) and finally we introduce the Ashtekar variables for General Relativity (subsection 1.1.1.3).

#### 1.1.1.1 Canonical formulation of General Relativity - ADM formalism

The Einstein-Hilbert action for General Relativity is given by

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-\det(g)} \overset{(4)}{R}$$
(1.1)

where  $\overset{(4)}{R}$  is the Ricci scalar of the metric tensor  $g_{\mu\nu}$  and  $\kappa = 8\pi G$  (we assume g has the signature (-, +, +, +)) [14, 7].

Consider a foliation of the manifold  $\mathcal{M} = \mathbb{R} \times \Sigma$  into 3-dimentonal Cauchy surfaces  $\Sigma$ . Let  $n^{\mu}$  be a timelike vector generating this foliation, so that for each  $t \in \mathbb{R}$  the space  $\{t\} \times \Sigma$  is the 3-submanifold of  $\mathcal{M}$  orthogonal to  $n^{\mu}$  at each point. Let  $q_{ab}$  be the restriction of the metric tensor  $g_{\mu\nu}$ , i.e.  $q = g|_{\Sigma}$  (the Latin indices from the beginning

of the alphabet  $a, b, c, \cdots$  will be used for manifold directions parallel to  $\Sigma$ , the Greek indices  $\mu, \nu, \cdots$  will be used for manifold directions in whole  $\mathcal{M}$ ). Let us introduce three more fields: a density  $\pi^{ab}$ , a scalar N and a vector  $N^a$ . The density density is defined as

$$\pi^{ab} := -\sqrt{q} \left( K^{ab} - q^{ab} K \right) \tag{1.2}$$

where  $K^{ab} := {}^{4}\nabla^{(a}n^{b)}$  is the second fundamental form of  $q_{ab}$  (i.e. the external curvature of q, the covariant derivative  ${}^{4}\nabla$  is taken with respect to 4-dimensional metric g) and  $K = K^{ab}q_{ab}$ . The fields N and  $N^{a}$  are such that

$$g_{\mu\nu} = \begin{pmatrix} -\left(N^2 - N_a N^a\right) & N_a \\ N_a & q_{ab} \end{pmatrix}$$
(1.3)

where  $N_a N^a = q_{ab} N^a N^b$ . Then the Einstein-Hilbert Lagrangian can be expressed as

$$\mathcal{L} = \sqrt{-g} \overset{(4)}{R} = -q_{ab} \partial_0 \pi^{ab} - N\mathcal{C} - N_a \mathcal{C}^a - 2\partial_a \left(\pi^{ab} N_b - \frac{1}{2}\pi N^a + \sqrt{\det\left(q\right)} \nabla^a N\right)$$
(1.4)

where  $\pi = \pi^{ab} q_{ab}$ ,  $\mathcal{C}^a := -2\nabla_b \pi^{ab}$  and

$$\mathcal{C} = -\sqrt{\det\left(q\right)} \left[ R + \frac{1}{\det\left(q\right)} \left( \frac{1}{2} \pi^2 - \pi^{ab} \pi_{ab} \right) \right]$$
(1.5)

with R being the Ricci scalar for q metric and all covariant derivatives taken with respect to q (see [8, 9]).

The last term is the divergence of a vector field, thus it can be integrated by parts to a boundary term (which vanish for  $\Sigma$  without boundary or for asymptotically flat gravitational field). The terms N and  $N_a$  are Lagrange multipliers, because no timederivatives of them appear in the Lagrangian. Thus the terms C and  $C^a$  are constraints of the theory (called *scalar* and *vector constraint* respectively).

Applying the Legendre transform to (1.4) we obtain the ADM Hamiltonian for General Relativity

$$H = \int_{\Sigma} \mathrm{d}^{3}x \left[ N\left(x\right) \mathcal{C}\left(x\right) + N_{a}\left(x\right) \mathcal{C}^{a}\left(x\right) \right]$$
(1.6)

#### 1.1.1.2 Palatini action for General Relativity

An alternative way to define General Relativity is to consider a tetrad of one-forms  $e := (e^I)_{I=0,...,3}$  instead of the metric field  $g_{\mu\nu}$ . Assuming, that the one-forms form a coframe (i.e. for each point the form  $d\mathcal{V} = \epsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L \neq 0$  for  $\epsilon_{IJKL}$  being the alternating tensor) the relation to the metric formulation is as follows: let  $\eta_{IJ} = \text{diag}(-1,1,1,1)$ , then  $g = \eta_{IJ}e^I \otimes e^J$ . Given a coordinate system  $(x^{\mu})$  we have  $e^I = e^I_{\mu}dx^{\mu}$  and thus  $g_{\mu\nu} = \eta_{IJ}e^I_{\mu}e^J_{\nu}$ . Now let  $\Gamma^I_J$  be the connection one-form of the Levi-Civita connection of the metric g and let  $\Omega^I_J$  be the curvature 2-form of  $\Gamma^I_J$ . Then the action

$$S[e] = \frac{1}{4\kappa} \int_{\mathcal{M}} e^{I} \wedge e^{J} \wedge \Omega^{KL}[e] \epsilon_{IJKL}$$
(1.7)

imposes (via variational principle) Einstein equations on the metric  $g_{\mu\nu}$ .

The action (1.7) is called *second-order tetrad action*, since the equations it imposes on the fields are of second order. There is a physically equivalent, but mathematically different formulation, called *first order*. Consider an action

$$S[e,\omega] = \frac{1}{4\kappa} \int_{\mathcal{M}} e^{I} \wedge e^{J} \wedge \Omega^{KL}[\omega] \epsilon_{IJKL}$$
(1.8)

depending on the tetrad field and a kinematically independent field  $\omega_J^I$ . Vanishing of variation of the action (1.8) with respect to  $\omega$  impose the requirement, that the curvature form is flat, while vanishing of variation of  $S[e, \omega]$  with respect to e requires  $\omega$  to be a connection compatible with the metric  $g = \eta_{IJ} e^I \otimes e^J$ . These two conditions put together makes 1st order the Palatini formulation of General Relativity physically equivalent to Einstein's General Relativity [15].

#### 1.1.1.3 Ashtekar Connection

Let us now introduce another variables. Consider the manifold  $\mathcal{M}$  foliated as  $\mathbb{R} \times \Sigma$ . The 3-dimensional metric is  $q = g|_{\Sigma} = \eta_{ij}e^i \otimes e^j$ , where the small indices i, j = 1, 2, 3, so  $\eta_{ij} = \text{diag}(1, 1, 1)$ , and the frame  $e^i$  is the projection of the frame  $e^I$  at  $\Sigma$ , i.e  $e^i_a = P^{\alpha}_a P^i_I e^I_{\alpha}$  for  $P^{\alpha}_a$  and  $P^i_I$  being appropriate projections<sup>1</sup>. Let us define the following two fields at  $\Sigma$ :

- a connection  $A = \Gamma + \gamma K$  (for  $\Gamma$  being the Levi-Civita connection of q and K being the extrinsic curvature of  $\Sigma$  and  $\gamma$  being a real positive number called the Barbero-Imirzi parameter),
- a triad of vector densities  $E_i = \sqrt{\det(q)}e_i$ .

It was shown [6, 5, 4] that there is a canonical transformation between (A, E) and  $(q, \pi)$  fields. Moreover, the fields (A, E) can be obtained from a canonical analysis of the action (1.8) modified by a so called Holst term [13]:

$$S_{\text{Holst}}\left[e,\omega\right] = \int_{\mathcal{M}} \left(\frac{1}{4\kappa} e^{I} \wedge e^{J} \wedge \Omega^{KL}\left[\omega\right] \epsilon_{IJKL} - \frac{1}{2\kappa\gamma} e^{I} \wedge e^{J} \wedge \Omega_{IJ}\left[\omega\right]\right)$$
(1.9)

The Holst term does not change the classical dynamics, because it vanishes on the solution of equation of motion generated by the action (1.7), so the action (1.9) describes the same physical theory, as the Palatini action, and thus it is Einstein's General Relativity, but expressed in different variables.

The connection A is called the Ashtekar connection and the triad E is it's canonically conjugated momentum. The momentum E can be expressed in coordinates as

$$E_i^a = \sqrt{\det\left(q\right)} e_i^a \tag{1.10}$$

where  $e_i = e_i^a \partial_a$  is a triad of vector fields dual to  $e^i = e_a^i dx^a$ , i.e  $e_a^i e_j^a = \delta_j^i$  and  $e_a^i e_i^b = \delta_a^b$ . The Ashtekar connection can be also expressed in coordinates. Let  $n = n^\alpha \partial_\alpha := e_0^\alpha \partial_\alpha$ be a tangent vector field normal to  $\Sigma$  at each point, then the external curvature map is  $K_a^b = P_\beta^b \left( P_a^{\alpha \, 4} \nabla_\alpha \right) n^\beta$ , and the external curvature tensor is  $K_a^i = K_a^b e_b^i$ . The Levi-Civita connection is given by the one-form valued in  $\mathfrak{so}(3)$ ,<sup>2</sup> i.e.  $\Gamma = \Gamma_a dx^a = \Gamma_{aj}^i dx^a w_i w^j$ (for  $w_i$ ,  $w^j$  being the basis vectors in  $\mathbb{R}^3$  and  $(\mathbb{R}^3)^*$  respectively). A choice of basis  $\tau_i$  in  $\mathfrak{so}(3)$  gives a map  $\mathfrak{so}(3) \to \mathbb{R}^3$ , i.e.  $\mathfrak{so}(3) \ni X = X^i \tau_i \mapsto X^i w_i \in \mathbb{R}^3$ . The

<sup>&</sup>lt;sup>1</sup>Let  $(\tilde{x}^{\alpha})_{\alpha=0,1,2,3}$  be a coordinate system on  $\mathcal{M}$  and  $(x^{\alpha})$  be a coordinate system on  $\Sigma$ , then  $P_a^{\alpha} := d\tilde{x}^{\alpha} \lrcorner \frac{\partial}{\partial x^{\alpha}}$ . Let  $(v_I)_{I=0,1,2,3}$  be a basis of the fibre vector space  $\mathbb{R}^4$  over  $\mathcal{M}$  and let  $(w_i)_{i=1,2,3}$  be a basis of the  $\mathbb{R}^3$ -subspace of the fibre, being the fibre over  $\Sigma$ , then  $P_I^i = v_I \eta^{ij} w_j$ . For simplicity we assume, that  $P_0^i = 0$  and  $P_I^i = \delta_I^i$ .

<sup>&</sup>lt;sup>2</sup>In fact it takes values in the anti-hermitian Lie algebra  $\mathfrak{so}(3)^A = i\mathfrak{so}(3)$  - see Remark A.3 in Appendix A.2.1.3. Similarly when saying, that the Ashtekar connection takes values in  $\mathfrak{su}(2)$  we mean  $\mathfrak{su}(2)^A$ . Since the hermitian and anti-hermitian Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}^A$  are isomorphic, the choice is only the matter of convention.

antisymmetric tensor  $\epsilon_{ijk}$  together with the metric  $\eta_{ij}$  indicates the choice of basis via  $\tau_i = \epsilon_{ijk} w^k w_l \eta^{jl} = \epsilon^j_{ik} w_j w^k$ . One can decompose  $\Gamma$  in the  $\tau_i$  basis obtaining  $\Gamma_a = \Gamma^i_a \tau_i$ . So that finally the Ashtekar connection is  $A = A^i_a dx^a \tau_i$  with

$$A_a^i = \Gamma_a^i + \gamma K_a^i \tag{1.11}$$

The Poisson bracket between A and E is

$$\left\{A_{a}^{i}\left(x\right), E_{j}^{b}\left(y\right)\right\} = \kappa\gamma\delta_{j}^{i}\delta_{a}^{b}\delta\left(x, y\right)$$

$$(1.12)$$

In the original General Relativity the connection  $\omega$  is SO(1,3)-connection. In order to admit coupling the gravitational field with fermionic fields one can consider the spinconnection, i.e. connection of the gauge group being the universal cover of SO(1,3), namely a  $SL(2,\mathbb{C})$ -connection. When doing the canonical analysis one picks a timelike vector  $n \perp \Sigma$ , which selects a SU(2) subgroup of  $SL(2,\mathbb{C})$ , that does not change the vector n (namely the *little group* of n). The Ashtekar connection can be naturally treated as SU(2)-connection. Indeed, although the Levi-Civita connection is SO(3)-connection, the Lie algebra  $\mathfrak{so}(3)$  is isomorphic to  $\mathfrak{su}(2)$ . One can choose a basis  $\tau_i = \tau_i^A w^B w_A$  in  $\mathfrak{su}(2) \equiv \mathfrak{so}(3)$  (where  $w_A$  and  $w^B$  are the basis vectors of  $\mathbb{C}^2$  and  $(C^2)^*$  respectively) and identify the components of the spin-connection by decomposing  $\Gamma$  in the 3-dimentional representation of  $\tau_i$ .

#### 1.1.2 Spin-network states

Spin-network states form the basis in the kinematic Hilbert space of Loop Quantum Gravity. They are complex functionals on the space of Ashtekar connections, and since the space of Ashtekar connections is the configuration space of Ashtekar formulation of General Relativity, spin-network states are wave-functions in the Schrödinger picture. Spin-network states represent singular configurations of the gravitational field i.e. they are eigenstates of quantum  $E_j^b$  operator, and thus, because of Heisenberg uncertainty principle, they give infinite uncertainty of the conjugate observable.

In this subsection we present a construction of spin-network functions. First in subsection 1.1.2.1 we study quantum kinematics of a free particle on a Lie group. This gives us a starting point to introduce spin-network functions, which are appropriate collections of wave-functions on many copies of a group (see subsection 1.1.2.2). In subsection 1.1.2.3 we introduce basic quantum operators acting on spin-network states, which leads us in subsection 1.1.2.4 to the physical interpretation of these states.

#### 1.1.2.1 Quantisation of a free particle on a group

Consider a free particle on a configuration space G being a compact Lie group [6]. In order to quantise it let us consider a Hilbert space  $\mathcal{H}^G := L^2(G, d\mu)$ , where  $d\mu$  is the Haar measure.

From the Peter-Weyl theorem we know, that  $\mathcal{H}^G = \bigoplus_{\rho} (\mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho}^*)$  where the sum goes through all the irreducible representations  $\rho$  of G and each  $\mathcal{H}_{\rho}$  is the carrier space of the corresponding representation. A natural basis in  $\mathcal{H}^G$  is given by vectors

$$\psi_{A,\rho}^B = |e_A\rangle \langle e_B|_{\rho} \in \mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho}^{\star} \tag{1.13}$$

for  $e_A, e_B$  - an orthonormal basis in  $\mathcal{H}_{\rho}$ . Considered as a wave functions on G they act as

$$\psi_{A,\rho}^{B}\left(g\right) := \left\langle e_{B}\right|\rho\left(g\right)\left|e_{A}\right\rangle_{\rho} = \rho_{A}^{B}\left(g\right) \tag{1.14}$$

which we shell often denote as  $\langle e_B | \rho(g) | e_A \rangle_{\rho} =: \langle e_B | g | e_A \rangle_{\rho}$ .

Quantum observables of such particle are generated by the operators  $\widehat{X}_i$  and  $\widehat{f}$ :

• Given an arbitrary smooth function  $f : G \to \mathbb{C}$  we define an operator (possibly unbounded)  $\hat{f}$  acting as follows: let  $\psi \in \mathcal{H}^G$ , then

$$\hat{f}: \psi \mapsto f\psi \quad (f\psi)(g) = f(g)\psi(g) \tag{1.15}$$

For G-compact all such operators are bounded operators. We call such operators *position operators*. Indeed, for f being a matrix element of G in defining representation the wave-function is multiplied by coordinates, and each other function is a function of these matrix elements.

• Given the Lie algebra  $\mathfrak{g}$  of G and a basis  $X_i$  of  $\mathfrak{g}$  let  $X_i^L$  be the left-invariant vector field on G. The operators  $\widehat{X_i}$  act on a function  $\psi(g)$  via the Lie derivative in direction  $X_i$ :

$$\widehat{X}_{i}\psi\left(g\right) := i\mathcal{L}_{X_{i}^{L}}\psi\left(g\right) \tag{1.16}$$

for the basis functions  $\psi^A_{B,\rho}$  the action of  $\widehat{X}_i$  can be expressed in terms of the representation  $\rho$ :

$$\widehat{X}_{i}\psi_{A,\rho}^{B}\left(g\right) = i\mathcal{L}_{X_{i}^{L}}\psi_{A,\rho}^{B}\left(g\right) = \psi_{C,\rho}^{B}\left(g\right)T\rho\left(X_{i}\right)_{A}^{C}$$
(1.17)

where  $T\rho$  is the representation of  $\mathfrak{g}$  tangent to  $\rho$ . The operators  $\widehat{X}_i$  are considered momenta operators. Indeed, they are differentiations with respect to position on  $G^{3}$ .

Given two functions f and f', the commutator

$$\left[\hat{f}, \hat{f}'\right] = 0 \tag{1.18}$$

The commutator of  $\widehat{X}_i$  operators is given by the commutator of the Lie algebra  $\mathfrak{g}$  elements:

$$\left[\widehat{X_i}, \widehat{X_j}\right]_{L^2(G)} = \widehat{[X_i, X_j]}_{\mathfrak{g}}$$
(1.19)

The commutator of  $\widehat{f}$  and  $\widehat{X}_i$  operators is given by the action of  $\widehat{X}_i$  on the function f:

$$\left[\widehat{X_i}, \widehat{f}\right] = i\left(\widehat{\mathcal{L}}_{X_i^L} \widehat{f}\right) \tag{1.20}$$

An important operator in  $\mathcal{H}^G$  is the Casimir operator. It is given by

$$\widehat{X^2} := \widehat{X_i} \widehat{X_j} k^{ij} \tag{1.21}$$

for  $k^{ij}$  being a *G*-invariant metric tensor (for example the Killing form).

$$\left(T\rho\left(X^{A}\right)\right)^{\dagger} = -T\rho\left(X^{A}\right)$$

In the anti-hermitian convention the  $\widehat{X}_i$  operator would act as

$$\widehat{X_{i}}\psi_{A,\rho}^{B}\left(g\right) = i\psi_{C,\rho}^{B}\left(g\right)T\rho\left(X_{i}^{A}\right)_{A}^{C}$$

The difference between the hermitian and anti-hermitian convention is in the choice of the exponent map from  $\mathfrak{g}$  to G: in the hermitian convention it is  $\mathfrak{g} \ni X \mapsto e^{iX} \in G$ , while in the anti-hermitian convention it is  $\mathfrak{g}^A \ni X^A \mapsto e^{X^A} \in G$ . It is discussed with more details in Appendix A.2.1.3.

<sup>&</sup>lt;sup>3</sup>Here we choose the convention in which for a unitary representation  $\rho$  of a group G the Lie algebra is represented by hermitian operators  $T\rho(\mathfrak{g})$ . One can as well choose the convention in which the Lie algebra is represented by anti-hermitian operators - we would denote it by  $\mathfrak{g}^A \ni X^A$ , such that

If G = SU(2), the representations  $\rho$  can be labelled by half-integer spin-labels (i.e.  $j \in \frac{\mathbb{Z}_+}{2} \cup \{0\}$ ). The basis in the Lie algebra  $\mathfrak{su}(2)$  is given by  $(L_i)_{i=1,2,3}$  being generators of rotations. The Casimir operator is

$$L^2 = \sum_{i=1}^3 L_i^2 \tag{1.22}$$

and it's eigenspaces are spanned by the matrix elements  $\psi^A_{B,j}$  of the representations j:

$$\widehat{L^2}\psi^A_{B,j} = j\,(j+1)\,\psi^A_{B,j} \tag{1.23}$$

#### 1.1.2.2 Spin-network states as quantum states for general relativity

Consider now a manifold  $\Sigma$  equipped with a SU(2)-connection A. Given a graph  $\mathcal{G}$  embedded in  $\Sigma$  one can calculate a holonomy of A along  $\mathcal{G}$  (see Appendix A.3.3.1):

$$\operatorname{Hol}_{\mathcal{G}}: A \mapsto (U_{\ell_1}[A], \dots, U_{\ell_2}[A]) \in SU(2)^{\mathbf{L}_{\mathcal{G}}}$$

$$(1.24)$$

For each link  $\ell \in \mathbf{L}_{\mathcal{G}}$  we introduce a Hilbert space  $\mathcal{H}_{\ell} = L^2(SU(2))$ , as it was discussed in previous subsection, obtaining  $L^2(SU(2)^{\mathbf{L}_{\mathcal{G}}}) = \bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \mathcal{H}_{\ell}$ . As it is shown in Appendix A.3.3.1, the gauge transformations act on  $\operatorname{Hol}_{\mathcal{G}}[A]$  only at the nodes of  $\mathcal{G}$  and the space of gauge-invariant holonomies along  $\mathcal{G}$  is  $SU(2)^{\mathbf{L}_{\mathcal{G}}}/SU(2)^{\mathbf{N}_{\mathcal{G}}}$ . Wave functions on such configuration space form a Hilbert space  $\mathcal{H}_{\mathcal{G}} = L^2(SU(2)^{\mathbf{L}_{\mathcal{G}}}/SU(2)^{\mathbf{N}_{\mathcal{G}}})$ .

In the Appendix A.3.3.2 we recalled the structure of the Hilbert space  $\mathcal{H}_{\mathcal{G}}$ . The orthonormal basis in  $\mathcal{H}_{\mathcal{G}}$  is given by so called *spin-network functions*:

$$\mathcal{N}_{\mathcal{G}, j_{\ell}, \iota_{n}}\left(\{U_{\ell}\}_{\ell \in \mathbf{L}}\right) = \prod_{\ell \in \mathbf{L}} \sqrt{2j_{\ell} + 1} D^{j_{\ell}} \left(U_{\ell}\right)_{m_{\ell^{+}}}^{m_{\ell^{-}}} \cdot \prod_{n \in \mathbf{N}} \iota_{n \, m_{\ell^{-}_{1}} \cdots m_{\ell^{-}_{l}}}^{m_{\ell^{+}_{1}} \cdots m_{\ell^{+}_{k}}}$$
(1.25)

for  $j_{\ell}$  - a collection of spin-labels (one per each link of  $\mathcal{G}$ ),  $\iota_n$  - a collection of SU(2)invariant tensors (belonging to appropriate Hilbert space  $\mathcal{H}_n$  called *node-Hilbert space*, one per each node of  $\mathcal{G}$ ) and  $D^j(U)$ -Wigner matrices. We will study interpretations of these states later.

If the connection A is the Ashtekar connection, the holonomy  $\operatorname{Hol}_{\mathcal{G}}[A]$  captures a finite number of degrees of freedom of the gravitational field. One could imagine a singular configuration of the gravitational field, such that the connection A vanishes everywhere outside the graph  $\mathcal{G}$  - given such constraints the holonomy  $\operatorname{Hol}_{\mathcal{G}}[A]$  would capture all degrees of freedom of A and the kinematic Hilbert space of a quantum theory of such connections should be  $\mathcal{H}_{\mathcal{G}}$ .

In order to capture all degrees of freedom of physical field configurations, one has to consider all possible graphs embedded in all possible ways in  $\Sigma$ . Moreover, two different graphs that overlap at least a part of a link cannot be considered separately, because the same degree of freedom would be counted twice, thus the kinematic Hilbert space cannot be simply  $\bigoplus_{\mathcal{G}} \mathcal{H}_{\mathcal{G}}$ . The kinematic Hilbert space of quantum states is the closure of the *inductive limit* of graph Hilbert spaces (being dual to the projective limit presented in [16]):

$$\mathcal{H}_{LQG} = \overline{\lim_{\substack{\mathcal{G} \to \infty \\ \supset}} \mathcal{H}_{\mathcal{G}}}$$
(1.26)

The projective limit is defined as follows. Let us introduce the following partial order on graphs: the graph  $\mathcal{G}$  is *later* then  $\mathcal{G}'$  if  $\mathcal{G} \supset \mathcal{G}'$ . Obviously two graphs  $\mathcal{G}_1, \mathcal{G}_2$  might be

incomparable, if they have nonintersecting parts, but there is always a graph later than both  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , namely  $\mathcal{G}_1 \cup \mathcal{G}_2$ . Each Hilbert space  $\mathcal{H}_{\mathcal{G}}$  is a subspace of the projective limit. For each Hilbert space state  $\mathcal{H}_{\mathcal{G}}$  there is a unique injection map into each  $\mathcal{H}_{\mathcal{G}'}$  for  $\mathcal{G}'$  later then  $\mathcal{G}$ , namely each wave-function  $\psi \in \mathcal{H}_{\mathcal{G}}$  has its equivalent  $\tilde{\psi} \in \mathcal{H}_{\mathcal{G}'}$  being a function that depends only on these holonomies, that are along the links common for  $\mathcal{G}$ and  $\mathcal{G}'$ . Each element of  $\mathcal{H}_{LQG}$  is a linear combination of elements of  $\mathcal{H}_{\mathcal{G}}$ s or is a limit of a Cauchy series of such elements.

Let  $\mathcal{A}$  be the space of all SU(2)-connections on  $\Sigma$ . The kinematic Hilbert space of a quantum theory of Ashtekar connection should be  $L^2(\mathcal{A}, d\mu)$  for an appropriate measure. It was proven [17], that there is a unique measure  $d\mu_{AL}$  on  $\mathcal{A}$  (called the Ashtekar-Lewandowski measure), that is diffeomorphism invariant and that is compatible with the action of the group. It is precisely the one, that is induced by the  $\mathcal{H}_{LQG}$  Hilbert space. Thus the spin-network states form an orthonormal basis of kinematic Hilbert states of Loop Quantum Gravity.

#### 1.1.2.3 Quantum operators

In order to define a quantum theory one has to introduce the algebra of canonical operators on the kinematic Hilbert space. In case of Loop Quantum Gravity these operators are the flux of a triad E and the holonomy of a connection A. We shall describe them for a connection defined on a graph (i.e. for  $\mathcal{H}_{\mathcal{G}}$ ), because generalisation of their action to the whole  $\mathcal{H}_{LQG}$  is straightforward.

Recalling, that  $\mathcal{H}_{\mathcal{G}}$  inherits its structure from  $\bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \mathcal{H}_{\ell}^{G}$ , the operators acting on  $\mathcal{H}_{\mathcal{G}}$  should be functions of group elements and acting of left-invariant fields. Indeed, the algebra of operators on  $\mathcal{H}_{\mathcal{G}}$  is spanned by  $\widehat{f(U_{\ell})}$  and  $\widehat{L_{i,\ell}}$ . They act intuitively: given  $\psi \in \mathcal{H}_{\mathcal{G}}$  we have  $(\widehat{f}\psi)(U_{\ell}) := f(U_{\ell})\psi(U_{\ell})$  and the operator  $\widehat{L_{i,\ell}}$  is a Lie derivative applied to  $U_{\ell}$  argument of  $\psi$ .

The operators of a connection A(x) and of a triad E(y) are ill defined on  $\mathcal{H}_{\mathcal{G}}$  [18]. One has to considered smeared operators. The connection A has to be smeared along a path  $\ell$ , obtaining  $U_{\ell}[A]$ . Quantum operator representing this quantity is  $\widehat{U_{\ell}[A]}$  which is obviously one of  $\widehat{f(U_{\ell})}$  operators. The triad has to be smeared along a surfaces S and it was shown [18], that the corresponding operator is proportional to  $\widehat{L_{i,\ell}}$  if the S crosses the link  $\ell$  and gives 0 if it does not. To be precise: let  $\vec{n}$  be a vector normal to the surface, then

$$\int_{S} \widehat{E_{i}} \, \mathbf{J} \vec{n} \, \mathrm{d}S = \hbar \kappa \gamma \sum_{\ell \in \mathbf{L}_{\mathcal{G}}: \ell \cap S \neq \emptyset} \widehat{L_{i,\ell}} \, \mathrm{sgn}\left(\vec{n} \cdot \vec{\ell}\right) \tag{1.27}$$

for l being the velocity vector of the link  $\ell$ . We will also introduce a shortcut notation for a triad operator smeared along a surface crossing only one link:  $\widehat{E_{i,\ell}} := \hbar \kappa \gamma \widehat{L_{i,\ell}}$ .

One can easily calculate the commutator of  $U_{\ell}$  and  $E_{i,\ell'}$ . It can be simply derived from the fact, that  $U_{\ell}$  is in fact a (very simple) spin-network function, thus

$$\left[\widehat{U_{\ell}}, \widehat{E_{i,\ell'}}\right] = -\widehat{E_{i,\ell'}} \left(U_{\ell}\right)$$
(1.28)

For  $\ell \neq \ell'$  we have obviously

$$\left[\widehat{U_{\ell}}, \widehat{E_{i,\ell'}}\right] = 0 \tag{1.29}$$

For  $\ell = \ell'$  we have

$$\left[\widehat{U_{\ell}}, \widehat{E_{i,\ell}}\right] = -\hbar\kappa\gamma\widehat{L_{i,\ell}}\left(U_{\ell}\right) \tag{1.30}$$

#### 1.1.2.4 Physical interpretation of spin-network states.

The constant in front of the triad operator  $\hbar\kappa$  has units of squared length. In fact it equals precisely  $\hbar\kappa = 8\pi \ell_{\rm Pl}^2$ . Thus the triad operator is a natural candidate to introduce geometric interpretations of spin-network states. Two basic geometric operators (area and volume) were quantised in [19] and [20]. They commute and thus they generate the physical quantum numbers for states of Loop Quantum Gravity. We discuss them below. The length and angle operators were introduced later [21, 22].

All the operators are considered for a spin-network state on a fixed graph  $\mathcal{G}$ . The generalisation to whole  $\mathcal{H}_{LQG}$  space is straightforward.

#### Area

The area of a surface S is  $S_S := \int_S \sqrt{\det(q|_S)}$ . Remembering, that matrix elements of q can be expressed in terms of the triad field E one can derive

$$S_{\mathcal{S}} = \gamma \int_{\mathcal{S}} \left( \sqrt{E_i E_j k^{ij}} \, \lrcorner \vec{n} \right) \mathrm{d}S \tag{1.31}$$

for  $\vec{n}$  - the vector normal to the surface. To find the quantum area operator one regularise the operator by dividing S into little fragments, so that each fragment has at most one intersection point with the graph  $\mathcal{G}$ . Since quantum  $\widehat{E}_i$  operator is proportional to  $\widehat{L}_i$ , the integrand can be quantised as the Casimir  $\widehat{L}_{\ell}^2$ . At the end of the day the quantum area operator acting on a spin-network state is

$$\widehat{S_{\mathcal{S}}} = \gamma \sum_{\ell \in \mathbf{L}_{\mathcal{G}}: \ell \cap \mathcal{S} \neq \emptyset} \sqrt{\widehat{E_{i,\ell}} \widehat{E_{j,\ell}} k^{ij}} = 8\pi \gamma \ell_{\mathrm{Pl}}^2 \sum_{\ell \in \mathbf{L}_{\mathcal{G}}: \ell \cap \mathcal{S} \neq \emptyset} \sqrt{\widehat{L_{\ell}^2}}$$
(1.32)

Since the spin-network states  $\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}$  are eigenstates of the Casimir operator  $\widehat{L_{\ell}^2}$ , one can easily see the area of a surface for the singular configuration of the gravitational field given by  $\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}$ :

$$\widehat{S_{\mathcal{S}}} |\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}\rangle = 8\pi\gamma \ell_{\mathrm{Pl}}^2 \sum_{\ell \in \mathbf{L}_{\mathcal{G}}: \ell \cap \mathcal{S} \neq \emptyset} \sqrt{j_{\ell} (j_{\ell}+1)} |\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}\rangle$$
(1.33)

So only these surfaces, that intersect the links of  $\mathcal{G}$  has any area, and each link contributes a quantum of area equal to  $8\pi\gamma\ell_{\rm Pl}^2\sqrt{j_\ell(j_\ell+1)}$ . Moreover, since there is a minimal nonzero eigenvalue of the Casimir operator (for spin  $j = \frac{1}{2}$ ), there is also a minimal eigenvalue of the area operator, equal to

$$a_0 = 4\sqrt{3}\pi\gamma\ell_{\rm Pl}^2\tag{1.34}$$

The details of this derivation can be found in [19].

This leads us to conclusion, that the physical area is concentrated at the links of spin-network states.

#### Volume

Given a region  $\mathcal{R} \subset \Sigma$  its volume is given by  $V_{\mathcal{R}} = \int_{\mathcal{R}} \sqrt{\det(q)}$ . Again one can express it in terms of triad field as  $V_{\mathcal{R}} = \int_{\mathcal{R}} \sqrt{\det(E)}$ . Regularisation and quantisation of this expression (done in [20]) leads to an operator

$$\widehat{V_{\mathcal{R}}} = \sum_{x \in \mathcal{R}} \sqrt{|\widehat{q}_x|} \tag{1.35}$$

where the operator

$$\widehat{q}_x = \frac{1}{48} \sum_{\ell,\ell',\ell''} \epsilon^{ijk} \epsilon(\ell,\ell',\ell'') \widehat{E_{i,\ell}} \widehat{E_{j,\ell'}} \widehat{E_{k,\ell''}}$$
(1.36)

where  $\epsilon(\ell, \ell', \ell'')$  is the orientation of the triad of velocity vectors of these links and the sum goes through all the germs of links crossing the point x. Obviously the operator gives o for points with no links as well as for the points with one link - thus two germs of link. Thus the operator  $\hat{q}_x$  vanishes for all points that are not nodes of  $\mathcal{G}$ , so

$$\widehat{V_{\mathcal{R}}} = \sum_{n \in \mathbf{N}_{\mathcal{G}}} \sqrt{|\widehat{q}_n|} \tag{1.37}$$

Given a spin-network state  $\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}$  one can consider a region containing only one node. Then obviously the operator  $\widehat{V_n}$  acts only on the tensor  $\iota_n$ . It was shown, that for a 3-valent node it gives always identically zero, but it acts nontrivially for 4- and higher valent nodes.

One can introduce a basis of eigenstates of  $\widehat{V_{\mathcal{R}}}$ , for example  $\iota_{v_1}, \iota_{v_2}, \ldots$  However, the exact form of these eigenstates (or their eigenvalues) is not known yet. Nevertheless, we conclude that the physical volume is concentrated at the nodes of spin-network states. It is expected (but not proven), that there is also a minimal nonzero eigenvalue of  $\widehat{V_n}$ , similarly to the area operator.

Volume operator commutes with the area operator.

#### Geometric interpretation of a spin-network

Knowing the action of volume and area operators the following picture of quantum geometry represented by a spin-network state arises. Consider a spin-network state  $\mathcal{N}_{\mathcal{G},j_{\ell},\iota_n}$ with each  $\iota_n$  being an eigenstate of the  $\widehat{V_n}$ . Each node  $n \in \mathbf{N}_{\mathcal{G}}$  represents a quantum polyhedron of volume given by the eigenvalue of  $\iota_n$ , with the number of faces given by the number of links incident to n and each face having the area  $S_{\ell} = 8\pi \gamma \ell_{\mathrm{Pl}}^2 \sqrt{j_{\ell}(j_{\ell}+1)}$ . Two polyhedra share a face if there is a link between corresponding nodes.

One can introduce length and angle operators by similar procedures [21, 22]. Since they are not necessary in later part of this thesis, we do not discuss here.

#### 1.1.3 Dynamics of Loop Quantum Gravity

Dynamics of Loop Quantum Gravity can be obtained by canonical analysis of the Holst action (1.9). It was presented in for example in [6, 1]. The canonical analysis gives

$$S_{\text{Holst}} = \int \mathrm{d}t \int_{\Sigma} \mathrm{d}^3 x \left( E_i^a \mathcal{L}_t A_a^i - \Lambda^i G_i - N^a \mathcal{C}_a - NC \right)$$
(1.38)

where  $\Lambda^i$ ,  $N^a$  and N are Lagrange multipliers, thus  $G_i$ ,  $C_a$  and C are constraints. The constraints are called: Gauss constraint, diffeomorphism constraint and scalar constraint, respectively. They are shortly discussed in what follows.

The constraints provide the dynamics for the classical theory. Quantising them is one of the most important open issues of Loop Quantum Gravity. Several strategies were already addressed with promising results [23, 24, 25, 26, 27], however, none of them leads to a computable theory (we are not going to discuss this issue further because it is loosely related to the topic of the thesis). It was one of the reasons to set up an alternative framework, namely the spin-foam models, described in section §1.2.

#### 1.1.3.1 Gauss constraint

The Gauss constraint  $G_i$  equals

$$G_i = \partial_a E_i^a + \epsilon_{ij}^k A_a^j E_k^a \tag{1.39}$$

which can be shortly written as  $G_i = \stackrel{(A)}{\nabla} E_i$ . So vanishing of this constraint imposes the condition

$$\nabla E = 0 \tag{1.40}$$

It ensures, that the theory is invariant under local SU(2)-transformations. This constraint was absent in the ADM formulation of General Relativity, however it is present in theories of dynamical connections [28]. It forces the triad to be divergence-free and it reduces the dimension of the phase space of connections so that it is equal to the dimension of phase space of metric tensors.

In quantum theory this constraint is solved by applying local SU(2)-symmetry, which leads to considering only gauge-invariant spin-network states (see subsection 1.1.2.2).

#### 1.1.3.2 Diffeomorphism constraint

The diffeomorphism constraint is

$$\tilde{\mathcal{C}}_a = E^b_i F^i_{ab} + \frac{1+\gamma}{\gamma} K^i_a G_i \tag{1.41}$$

which (for the Gauss constraint satisfied) is equivalent to

$$\mathcal{C}_a = E_i^b \stackrel{(A)}{\nabla}_a A_b^i \tag{1.42}$$

which generates diffeomorphism of fields on  $\Sigma$ .

This constraint is solved in quantum theory by averaging spin-network states  $|\mathcal{N}_{\mathcal{G}}\rangle$  over all possible diffeomorphisms of  $\phi$  of  $\Sigma$ . Each diffeomorphism defines a unitary map acting by transforming the underlying graph:

$$\mathcal{U}_{\phi} : |\mathcal{N}_{\mathcal{G}}\rangle \mapsto |\mathcal{N}_{\phi(\mathcal{G})}\rangle \tag{1.43}$$

Averaging a state  $|\mathcal{N}_{\mathcal{G}}\rangle$  over all possible diffeomorphisms  $\phi \in \text{Diff}(\Sigma)$  results with a vector which is not normalisable in  $\mathcal{H}_{\text{LQG}}$ . However, it is a well defined element of the space of linear (but unbounded) maps  $\mathcal{H}_{\text{LQG}} \to \mathbb{C}$ . We denote it as  $(\mathcal{N}_{\mathcal{G}}| \text{ and call it abstract}$ spin-network state. Abstract spin-network states act on an arbitrary spin-network as follows: given a spin-network  $\mathcal{N}'_{\mathcal{G}'}$  if there is a diffeomorphism  $\phi$  such that  $\phi(\mathcal{G}) = \mathcal{G}'$ , then  $(\mathcal{N}_{\mathcal{G}}| |\mathcal{N}'_{\mathcal{G}'}\rangle = \langle \mathcal{U}_{\phi}\mathcal{N}_{\mathcal{G}}| |\mathcal{N}'_{\mathcal{G}'}\rangle$ , otherwise  $(\mathcal{N}_{\mathcal{G}}| |\mathcal{N}'_{\mathcal{G}'}\rangle = 0$ . All the operators defined on  $\mathcal{H}_{\text{LQG}}$  can act on abstract spin-network states by duality: given  $\hat{A} : \mathcal{H}_{\text{LQG}} \to \mathcal{H}_{\text{LQG}}$  we have  $\hat{A}^{\dagger} : (\mathcal{N}_{\mathcal{G}}| \mapsto (\widetilde{\mathcal{N}_{\mathcal{G}}}| \text{ such that } (\widetilde{\mathcal{N}_{\mathcal{G}}}| |\mathcal{N}'_{\mathcal{G}'}\rangle = (\mathcal{N}_{\mathcal{G}}| \hat{A} |\mathcal{N}'_{\mathcal{G}'}\rangle$ .

Spin-network states are labelled by embedded graphs  $\mathcal{G} \subset \Sigma$ . Abstract spin-networks are labelled by abstract graphs  $\mathcal{G}$ , i.e. equivalence classes of all graphs that can be mapped one to another via a diffeomorphism.

#### 1.1.3.3 Scalar constraint and the Hamiltonian constraint

Scalar constraint is given by

$$\tilde{\mathcal{C}} = \frac{E_i^a E_j^b}{2\kappa\sqrt{\det\left(q\right)}} \left[ \epsilon_k^{ij} F_{ab}^k - \left(1 + \gamma^2\right) 2K_{[a}^i K_{b]}^i \right] + \left(1 + \gamma^2\right) \partial_a \left(\frac{E_i^a}{\sqrt{\det\left(q\right)}}\right) G^i \qquad (1.44)$$

Assuming the Gauss constraint satisfied it simplifies to

$$\mathcal{C} = \frac{E_i^a E_j^o}{2\kappa\sqrt{\det\left(q\right)}} \left[ \epsilon_k^{ij} F_{ab}^k - \left(1 + \gamma^2\right) 2K_{[a}^i K_{b]}^i \right]$$
(1.45)

This constraint generates the time evolution of Loop Quantum Gravity. Indeed, applying the Legendre transform to the Holst action (1.38) we obtain the Hamiltonian

$$H = \int_{\Sigma} \mathrm{d}^3 x \left( \Lambda^i G_i + N^a \mathcal{C}_a + N \mathcal{C} \right)$$
(1.46)

Assuming that we can quantise these operators, we would obtain

$$\hat{H} = \int_{\Sigma} \mathrm{d}^3 x \left( \Lambda^i \hat{G}_i + N^a \hat{\mathcal{C}}_a + N \hat{\mathcal{C}} \right) \tag{1.47}$$

Consider now an abstract, gauge-invariant spin-network state  $(\mathcal{N}_{\mathcal{G}}|$ . Obviously  $\hat{\mathcal{C}}_a(\mathcal{N}_{\mathcal{G}}| = 0$ and  $\hat{G}_i(\mathcal{N}_{\mathcal{G}}| = 0$ , so

$$\hat{H}\left(\mathcal{N}_{\mathcal{G}}\right) = \int_{\Sigma} \mathrm{d}^{3}x \, N\left(x\right) \widehat{\mathcal{C}\left(x\right)}\left(\mathcal{N}_{\mathcal{G}}\right) \tag{1.48}$$

thus quantisation of the scalar constraint is essential to understand the Hamiltonian constraint of Loop Quantum Gravity.

There were several attempts to regularise and quantise this constraint (for example [23, 29, 30, 24, 27]). However, the operator obtained as a result is hard to apply even to relatively simple quantum states. Nevertheless, there is an alternative, path-integral-based approach to describe dynamics of Loop Quantum Gravity, namely the spin-foam models.

### 1.2 Spin-foams as a way to calculate transition amplitudes of Loop Quantum Gravity

In this section we shall briefly present the spin-foam models being an alternative way to describe the dynamics of Loop Quantum Gravity. The spin-foam models, starting from another approach to quantisation of classical theories, end up with a theory analogous to Loop Quantum Gravity at the kinematic level, but with a build in path-integral formula for quantum amplitudes.

In subsection 1.2.1 we briefly discuss the relation between spin-foams and Loop Quantum Gravity providing a little historical and conceptual introduction to the theory. In subsection 1.2.2 we present the EPRL model, being currently the most popular one, and being the base to most of the calculation in this thesis. In subsection we discus some technical issues of the EPRL model.

#### 1.2.1 Spin-foams as another way to approach dynamics of Loop Quantum Gravity

The Loop Quantum Gravity provide a beautiful formalism to describe the kinematics of quantum theory of dynamical connections. It also gives tools to describe its dynamics, but solving the evolution equations appeared to be a difficult problem. Some attempts to derive the path-integral formalism from the Hamiltonian constraint of Loop Quantum Gravity were made - and they are briefly recalled in subsection 1.2.1.1, but they did not succeed so far.

However, there was another attempt to quantise a theory of dynamical connection, called the BF-theory - described in subsection 1.2.1.2. The BF-theory is a topological theory, not capturing the differential degrees of freedom of General Relativity. Nevertheless it gave a starting point to the concept obtaining the General Relativity by imposing so called *simplicity constraints* on the B-field of the BF-theory (see subsection 1.2.1.3 and [12]), which lead to a path-integral formulation of a theory with the Holst action - which is precisely the action of Loop Quantum Gravity.

#### 1.2.1.1 Motivation I: Spin-foam as a history of a spin-network

#### Problem of time

The Hamiltonian in General Relativity is in fact a constraint (see (1.6)). Thus time evolution in GR is a gauge transformation. This leads to so called problem of time in General Relativity, which can be solved by for example coupling gravity with a matter field (we call it *clock field* and denote by T) of appropriate properties (see for example [31, 2, 32]) which adds to the Hamiltonian constraint a term  $\partial_T^2$ , so that

$$0 = H = H_{\rm GR} - \partial_T^2 + H_{\rm matter} \tag{1.49}$$

which can be solved to

$$\partial_T = \sqrt{H_{\rm GR} + H_{\rm matter}} \tag{1.50}$$

The field T is called *emergent time*.

We are not going to discuss the framework of emergent time here. However, the problem of time leads to the following conclusion. If one does not use the emergent time framework, the time evolution in the coordinate time *is* a gauge transformation, and thus the traditional interpretation of quantum transition amplitudes

$$W_{\psi_{\rm in}\psi_{\rm out}}\left(t\right) = \left\langle\psi_{\rm out}\right| e^{-\frac{i\hat{H}t}{\hbar}} \left|\psi_{\rm in}\right\rangle \tag{1.51}$$

is not valid. Instead of evolution operator  $e^{-\frac{i\hat{H}t}{\hbar}}$  one has to consider the projection on the kernel of the Hamiltonian:

$$W_{\psi_{\rm in}\psi_{\rm out}} = \langle \psi_{\rm out} | \hat{P} | \psi_{\rm in} \rangle = \lim_{T \to \infty} \langle \psi_{\rm out} | e^{-\hat{H}T} | \psi_{\rm in} \rangle$$
(1.52)

#### History of a spin-network

In order to evaluate matrix elements of the  $\hat{P}$  operator (1.52) one can perform a similar procedure, as when deriving a path integral in ordinary quantum mechanics.

The exponent in the transition amplitude is  $-\hat{H}T = -\int_0^T dt \int d^3x \, \widehat{H(x)} = -\int_0^T dt \, \hat{H}$ . Using diffeomorphism invariance one can set T = 1 obtaining

$$W_{\psi_{\rm in}\psi_{\rm out}} = \langle \psi_{\rm out} | e^{-\int_0^1 \mathrm{d}t \, \hat{H}} | \psi_{\rm in} \rangle \tag{1.53}$$

$$\hat{H} \left| \begin{array}{c} j_{3} \\ j_{1} \\ j_{1} \\ j_{2} \end{array} \right\rangle = \sum_{j_{4}, j_{5}, j_{6}} \sum_{\iota_{1}, \iota_{2}, \iota_{3}} c_{\iota}^{\iota_{1}\iota_{2}\iota_{3}} \cdot \left| \begin{array}{c} j_{3} \\ j_{5} \\ \iota_{1} \\ j_{1} \\ j_{6} \\ \iota_{2} \\ j_{2} \end{array} \right\rangle$$

Figure 1.1: A scheme of action of Loop Quantum Gravity Hamiltonian on a node of a spin-network state (based on [33]).

Then one can split the "time" period into N intervals and insert N copies of identity operator

$$W_{\psi_{\rm in}\psi_{\rm out}} = \langle \psi_{\rm out} | \, \mathbb{1}e^{-\frac{\hat{H}}{N}} \, \mathbb{1}e^{-\frac{\hat{H}}{N}} \, \mathbb{1} \cdots \, \mathbb{1}e^{-\frac{\hat{H}}{N}} \, \mathbb{1} | \psi_{\rm in} \rangle \tag{1.54}$$

Note, that the identity operator is a sum over all spin-network states  $\mathbb{1} = \sum_{\mathcal{N}} |\mathcal{N}\rangle (\mathcal{N}|$ , not an integral. Thus the amplitude is

$$W_{\psi_{\rm in}\psi_{\rm out}} = \sum_{\{\mathcal{N}_i\}_{i=0,\dots,N}} \langle \psi_{\rm out} | |\mathcal{N}_0\rangle \langle \mathcal{N}_N | |\psi_{\rm in}\rangle \cdot \prod_{i=0}^{N-1} A_i^{(N)}$$
(1.55)

for  $A_i^{(N)} = \langle \mathcal{N}_i | e^{-\frac{\hat{H}}{N}} | \mathcal{N}_{i+1} \rangle$ . So the transition amplitude is given by a (discrete) sum over histories (each history given by a series of spin-networks) weighted by amplitude factors, not by an integral. For  $N \gg 1$  the operator  $e^{-\frac{\hat{H}}{N}}$  can be approximated by  $\mathbb{1} - \frac{\hat{H}}{N} + O\left(\frac{1}{N^2}\right)$ , so that each path can be approximated by a sum of the path with one less change of state and a matrix element  $\langle \mathcal{N}_i | \hat{H} | \mathcal{N}_{i+1} \rangle$ . Thus the sum over histories can be decomposed into parts with precisely M changes of the state:

$$W_{\psi_{\rm in}\psi_{\rm out}}^N = \sum_{M=0}^{\infty} \left(-\frac{1}{N}\right)^n \sum_{\{\mathcal{N}_i\}_{i=0,\dots,M}} \left\langle \psi_{\rm out} \right| \left|\mathcal{N}_0\right\rangle \left\langle \mathcal{N}_M\right| \left|\psi_{\rm in}\right\rangle \cdot \prod_{i=0}^{M-1} A_i \tag{1.56}$$

for  $A_i = \langle \mathcal{N}_i | \widehat{H} | \mathcal{N}_{i+1} \rangle$ .

The Hamiltonian operator was analysed in for example in [27] and it was shown, that it acts only at nodes of spin-network (see figure 1.1). Thus spin-networks contributing to a transition amplitude must form a series such that each two neighbouring spin-network have graphs differing either by one node splitting into several new nodes or by some neighbouring nodes joined into a node. The derivation above is based on[33], taking into account the remarks of [34].

Such history of spin-networks was called *spin-foam*, since it may be graphically presented as a foam (see figure 1.2). One could derive the dynamics of Loop Quantum Gravity by analysing all such spin-foams, where changes between two neighbouring states are governed by the Hamiltonian constraint (see for example [29, 30]). However, this would require detailed analysis of the Hamiltonian constraint, which is far non-trivial. Nevertheless, the concept of spin-foam representation of history of a spin-network leads to other approaches.



Figure 1.2: A spin-foam: a history of a spin-network.

#### 1.2.1.2 Motivation II: BF theory

Let us consider a so called BF-theory, i.e. a theory of a dynamical G-connection  $\omega$  and 2-form B valued in the lie algebra  $\mathfrak{g}$ . The action of the BF-theory is

$$S[B,\omega] = \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ}[\omega]$$
(1.57)

where F is the curvature 2-form of the connection  $\omega$ . The path-integral approach says, that given a boundary state  $|\psi\rangle = |\psi_{\rm in}\rangle \otimes |\psi_{\rm out}\rangle$  the transition amplitude between  $|\psi_{\rm in}\rangle$  and  $|\psi_{\rm out}\rangle$  is

$$A_{\psi_{\rm in} \to \psi_{\rm out}} = (Z | |\psi\rangle \tag{1.58}$$

for (Z) being the partition function generalized state, i.e.

$$Z_{BF} = \int DB \, D\omega \, e^{-i \int_{\mathcal{M}} S[B,\omega]} \tag{1.59}$$

Integrating over DB gives  $\delta(F[\omega])$ :

$$Z_{BF} = \int D\omega \,\delta\left(F\left[\omega\right]\right) \tag{1.60}$$

so the only connections that contribute are the flat connections.

Consider now a triangulation  $\Delta$  of a manifold  $\mathcal{M}$ . Let the 2-complex<sup>4</sup>  $\kappa = \Delta^*$  be a 2complex dual to the triangulation  $\Delta$  (see Appendix A.1.2.6). One can discretise the path integral (1.60) as follows. Given two neighbouring 4-simplices  $s_1$  and  $s_2$  let  $v_1$  and  $v_2$  be their middle points (i.e. their equivalents in  $\kappa = \Delta^*$ ). The 2-simplices  $s_1$  and  $s_2$  share a tetrahedron  $\theta$ , which is represented by an edge e in  $\kappa$ . Let  $g_e$  be the holonomy of  $\omega$  along this edge (let us fix an orientation of each edge of  $\kappa$ , obviously the holonomy calculated in the opposite direction is  $g_e^{-1}$ ). These holonomies will determine our discretised connection  $\omega$ , so the integral  $\int D\omega$  will be substituted by  $\int \prod_e dg_e$ . Now consider a triangle t of  $\Delta$ (and its dual face  $f \in \kappa$ ). It is shared by a number of tetrahedra  $\theta_i$ , such that the edges dual to them form a cycle  $e_1, e_2, \ldots, e_k$ . One can consider a holonomy  $g_f$  around this face being the holonomy along the loop  $e_1, e_2, \ldots, e_k$ . Let us introduce a notation  $g_{e,f}$ being the holonomy along e calculated in the direction induced by the orientation of the face (obviously  $g_{e,f} = g_e$  if the orientation of e induced by f agrees with the orientation of e we fixed above, and  $g_{e,f} = g_e^{-1}$  in the opposite case).

Since the connection  $\omega$  must be flat, the holonomy must be trivial. Thus let us introduce the discretised partition function

$$Z_{BF}^{\Delta} = \int \prod_{e} \mathrm{d}g_{e} \prod_{f} \delta\left(g_{e_{1},f}g_{e_{2},f}\cdots g_{e_{k},f}\right)$$
(1.61)

The integral goes over all internal edges of the 2-complex  $\kappa$ , so that the partition function becomes a function of the holonomies along the boundary links of  $\kappa$ :

$$Z_{BF}^{\Delta} = Z_{BF}^{\Delta}\left(g_{\ell}\right) \tag{1.62}$$

the boundary of  $\kappa$  is a graph with naturally induced structure of a spin-network. Thus for arbitrary spin-network state  $\mathcal{N}$  defined at  $\partial \kappa$  the transition amplitude is

$$A(\mathcal{N}) := \int \prod_{\ell \in \partial \kappa} \mathrm{d}g_{\ell} \mathcal{N}(g_{\ell}) \,\overline{Z_{BF}^{\Delta}(g_{\ell})} \tag{1.63}$$

<sup>&</sup>lt;sup>4</sup>By 2-complex we mean 2-CW-complex, for simplicity we drop CW- in what follows. For definitions - see Appendix A.1.2.

It appears [35], that the partition function is independent on the triangulation and it is equal to the smooth partition function:

$$Z_{BF}^{\Delta} = Z_{BF}^{\Delta'} = Z_{BF} \tag{1.64}$$

The  $\delta$ -functional on a group can be represented in Fourier transformed representation as a sum over irreducible representations:

$$\delta(g) = \sum_{\rho} \dim(\rho) \operatorname{Tr}(\rho(g))$$
(1.65)

and each integral over a group element can be represented as a projection on irreducible representations:

$$\int_{G} \mathrm{d}g \prod_{i=1}^{n} \rho_{i}\left(g\right) = P_{\mathrm{Inv}\left(\mathcal{H}_{\rho_{1}} \otimes \cdots \otimes \mathcal{H}_{\rho_{n}}\right)} = \sum_{\iota_{I}} \left|\iota_{I}\right\rangle \left\langle\iota_{I}\right| \tag{1.66}$$

for  $\iota_I$  forming an orthonormal basis in  $\operatorname{Inv}(\mathcal{H}_{\rho_1} \otimes \cdots \otimes \mathcal{H}_{\rho_n})$ . This leads to another representation of the discretised partition function in terms of colorings of elements of the 2-complex  $\kappa$  dual to  $\Delta$ :

$$Z_{BF}^{\kappa} = \sum_{\left\{\rho_{f}\right\}} \sum_{\left\{\iota_{e}\right\}} \prod_{f} \dim\left(\rho_{f}\right) \prod_{e} \left|\iota_{e}\right\rangle \left\langle\iota_{e}\right| \prod_{v} A_{v}$$
(1.67)

where  $A_v$  is a tensor that contracts the indices of  $\iota_e$ s in appropriate way in order to obtain the  $\delta$ -functionals present in (1.61) (in case of SO(4) - BF-theory it is 15*j*-symbol - see [36]).

#### 1.2.1.3 Concept: constrained BF theory

Note, that the Holst action for General Relativity (1.9) is precisely the *BF*-action (1.57), but with a constraint imposed on the *B*-field, namely

$$B^{IJ} = \epsilon^{IJ}{}_{KL}e^K \wedge e^L + \frac{1}{\gamma}e^I \wedge e^J$$
(1.68)

Thus one can try to translate this constraint to the language of BF-partition function and obtain a candidate for a partition function for gravity.

In order to understand the constraint (1.68) let us perform canonical analysis of the BF-action (see [37]). Let us pick a timelike vector field  $n = n^I e_I$  defining the foliation of  $\mathcal{M}$  into  $\Sigma$ , i.e.  $n \downarrow e|_{\Sigma} = 0$ . The momentum conjugate to  $\omega_{IJ}$  is  $B^{IJ}|_{\Sigma}$ . Now note, that in gravity the gauge group is  $SL(2, \mathbb{C})$ , thus  $B^{IJ}$  is a  $\mathfrak{sl}(2, \mathbb{C})^*$ -valued 2-form. A constraint on B can be thus expressed in terms of  $SL(2, \mathbb{C})$ -generators. Let us split B into generators of the little group preserving the timelike vector n (i.e. rotations) and the dual part (i.e. boosts). The rotations are

$$\dot{L} := -n \lrcorner \star B \tag{1.69}$$

and the boosts are

$$\vec{K} = n \lrcorner B \tag{1.70}$$

where  $(\star B)^{IJ} = \epsilon^{IJ}{}_{KL}B^{KL}$  is the Hodge dual. Applying the condition  $n \lrcorner e|_{\Sigma} = 0$  we get

$$\vec{K} = n \lrcorner \star (e \land e) \quad \text{and} \quad \vec{L} = -\frac{1}{\gamma} n \lrcorner \star (e \land e)$$
 (1.71)

$$\vec{K} + \gamma \vec{L} = 0 \tag{1.72}$$

This is the so called *simplicity constraint*. They can be solved by the EPRL-map, which is discussed in the next subsection.

#### 1.2.2 The Spin-foam models

The EPRL spin-foam model is based on the concept presented above, i.e. to impose the simplicity constraints on the *BF*-theory. There were several approaches to derive such model [38, 39, 40, 41, 42, 43, 44, 45, 46], here we present the result known as EPRL-FK or EPRL-KKL model (based mainly on [43, 42, 41], we will often shortly call it EPRL).

The subsection 1.2.2.1 presents the EPRL-map - the key object in calculating the vertex amplitude in the model. The subsection 1.2.2.2 shows the general construction of a spin-foam and a prescription, how to calculate the transition amplitude of a spin-foam. The subsection 1.2.2.3 presents the interpretation of the spin-foam amplitude as a matrix element of the evolution operator.

#### 1.2.2.1 The EPRL-map - a solution to the simplicity constraint

The integration over the *B*-field in (1.61) generates the  $\delta$ -distribution on a group. The  $\delta$ -distribution on a group is given by

$$\delta(g) = \sum_{\rho \in \operatorname{Irrep}(G)} \sum_{|v\rangle_{\rho} \in \mathcal{H}_{\rho}} \langle v | g | v \rangle_{\rho}$$
(1.73)

for  $|v\rangle_{\rho}$  being the orthonormal basis in the  $\rho$ -representation carrier space  $\mathcal{H}_{\rho}$ . One can rewrite it in terms of a big carrier Hilbert space  $\mathcal{H}_{\mathrm{Irrep}(G)} = \bigoplus_{\rho \in \mathrm{Irrep}(G)} \mathcal{H}_{\rho}$ 

$$\delta(g) = \sum_{|v\rangle \text{ basis in } \mathcal{H}_{\mathrm{Irrep}(G)}} \langle v | g | v \rangle = \mathrm{Tr}_{\mathcal{H}_{\mathrm{Irrep}(G)}}(g)$$
(1.74)

The simplicity constraint (1.72) has it's counterpart in  $\mathcal{H}_{Irrep(G)}$ . Indeed, the Lie algebra  $\mathfrak{g}$  has a natural action in  $\mathcal{H}_{Irrep(G)}$ . The idea presented in [41, 42, 40, 39] was to solve the simplicity constraint by replacing the  $\delta$ -functional by a constraint  $\delta$ -functional being the trace on the subspace of  $\mathcal{H}_{Irrep(G)}$  that satisfy the simplicity constraint. An attempt to solve this constraint strongly was done [45, 46], but it appeared, that the model build on such assumption does not capture enough degrees of freedom to be gravity. Thus in the EPRL model the simplicity constraint is solved weakly.

In case of General Relativity the gauge group is  $SL(2, \mathbb{C})$ . The unitary irreducible representations of  $SL(2, \mathbb{C})$  are labelled by a pair (p, k) of real positive number p and a half-integer number k (see (A.4)). Given a timelike vector n there is a SU(2) subgroup  $SL(2, \mathbb{C})$  being the little group not changing this vector. This choice of SU(2) subgroup introduce a basis in the carrier space of the representation (p, k) such that

$$\mathcal{H}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j^{(p,k)} \tag{1.75}$$

where the little group SU(2) acts at each  $\mathcal{H}_{j}^{(p,k)}$  via the Wigner matrices of spin j (the spins j differ from k by a an integer number, not a general half-integer). The basis in

 $\mathcal{H}^{(p,k)}$  is given by the vectors  $|(p,k),(j,m)\rangle$ . Thus an arbitrary matrix element of the (p,k) representation is

$$\langle g | | (p,k); (j,m), (j',m') \rangle = D^{(p,k)} (g)^{jm}_{j'm'}$$
 (1.76)

and the  $\delta$ -functional is

$$\delta(g) = \int dp \sum_{k \in \frac{\mathbb{Z}_+ \cup \{0\}}{2}} \sum_{j=k}^{\infty} \sum_{m=-j}^{j} \langle (p,k); (j,m) | g | (p,k); (j,m) \rangle \, \mu(p,k)$$
(1.77)

for  $\mu(p,k)$  being a weight of the irreducible representation.

It was shown in [47, 48], that the simplicity constraint are weakly satisfied for the vectors of carrier space such that  $p = \gamma k$  and j = k. Thus we substitute the  $\delta$ -functional by the functional

$$\delta^{\gamma}(g) = \sum_{k \in \frac{\mathbb{Z}_{+} \cup \{0\}}{2}} \sum_{m=-k}^{k} \left\langle \left(\gamma k, k\right); \left(k, m\right)\right| g \left|\left(\gamma k, k\right); \left(k, m\right)\right\rangle \mu\left(\gamma k, k\right)$$
(1.78)

For each representation  $(p = \gamma k, k)$  the carrier vectors satisfying the simplicity constraint span a subspace of the subspace  $\mathcal{H}_{j=k}^{(p,k)} \subset \mathcal{H}^{(p,k)}$ . All these spaces for all possible ks span the Hilbert space

$$\mathcal{H}^{\gamma} = \bigoplus_{k \in \frac{\mathbb{Z}_{+} \cup \{0\}}{2}} \left( \left( \mathcal{H}_{k}^{(\gamma k, k)} \right)^{\star} \otimes \mathcal{H}_{k}^{(\gamma k, k)} \right)$$
(1.79)

which is isomorphic to  $L^2(SU(2))$ . Note, that the structure of a  $\delta^{\gamma}$ -functional is also very similar the  $\delta$ -functional on the SU(2)-group:

$$\delta^{SU(2)}(h) = \sum_{k \in \frac{\mathbb{Z}_+ \cup \{0\}}{2}} \sum_{m=-k}^{k} (2j+1) \langle m | h | m \rangle_k$$
(1.80)

The idea of the EPRL model is to consider spin-foams labelled by SU(2)-tensors, which are mapped to corresponding  $SL(2, \mathbb{C})$ -tensors via the so-called EPRL-map

$$Y: L^2\left(SU(2)\right) \to \mathcal{H}^\gamma \tag{1.81}$$

This map can is determined by its action on the carrier space  $\mathcal{H}_{\mathrm{Irrep}(SU(2))}$ :

$$Y: |m\rangle_{j} \mapsto |(\gamma j, j); (j, m)\rangle \tag{1.82}$$

and can be generalised to arbitrary tensor product of representations of SU(2).

The next subsubsection explains, how the EPRL map is used in calculation of spinfoam transition amplitude.

#### 1.2.2.2 The general construction

A spin-foam is a 2-complex colored in an appropriate way, such that it represents a transition amplitude of a quantum process (this interpretation will be discussed further in subsection 1.2.2.3). The 2-complex  $\tilde{\kappa} = (\mathbf{F}, \tilde{\mathbf{E}}, \tilde{\mathbf{V}})$  consists of the set of faces  $\mathbf{F}$ , edges  $\mathbf{E}$  and vertices  $\mathbf{V}$ . The boundary  $\partial \kappa = (\mathbf{L}, \mathbf{N})$  is a graph (build of the set of links  $\mathbf{L} \subset \tilde{\mathbf{E}}$ 

and nodes  $\mathbf{N} \subset \mathbf{V}$ ). The bulk part of the 2-complex is the complex minus its boundary, so that  $\kappa = \operatorname{int}(\tilde{\kappa}) = (\mathbf{F}, \mathbf{E}, \mathbf{V})$  for  $\mathbf{E} = \tilde{\mathbf{E}} \setminus \mathbf{L}$  and  $\mathbf{V} = \tilde{\mathbf{V}} \setminus \mathbf{N}$  (more detailed introduction to CW-complex conventions used in this thesis can be found Appendix A.1). In the original papers [39, 42, 40, 41] the complex  $\tilde{\kappa}$  was assumed to be dual to some triangulation of the manifold  $\mathcal{M}$ , later the theory was generalised (see section §2.1), which was one of the issues addressed in this thesis (see chapter 3).

The bulk 2-complex is colored as follows:

- Each face  $f \in \mathbf{F}$  is colored by a representation of SU(2)-group (i.e. by a half-integer spin  $j_f$ ).
- Each edge  $e \in \mathbf{E}$  is colored by an invariant tensor  $|\iota_e\rangle \in \mathcal{H}_{e,v}^{\mathrm{Inv}}$  where v is one of the ending points of e,  $\mathcal{H}_{e,v}^{\mathrm{Inv}} = \mathrm{Inv}\left(\bigotimes_f \mathcal{H}_{j_f}^{\epsilon_f}\right)$ , the product is over the faces incident to e and  $\epsilon$ s mark the relative orientation of faces and the edge (assuming, that e points at the vertex v, if the orientation induced by f is consistent with the one assumed, then  $\epsilon_f$  disappears, and if the induced orientation is opposite to the one assumed, then  $\epsilon_f = \star$  stands for the dual representation). In fact the tensor  $|\iota_e\rangle$  appears in two copies: as  $|\iota_e\rangle$  at the starting point of e and as the dual  $\langle\iota_e|$  at the ending point of e (note, that for v and v' being the ending points of e we have  $\mathcal{H}_{e,v}^{\mathrm{Inv}} = \left(\mathcal{H}_{e,v'}^{\mathrm{Inv}}\right)^{\star}$ , thus the pair  $\{\langle\iota_e|, |\iota_e\rangle\}$  is independent on the fact which ending point we choose).

The coloring of the bulk induces the coloring of the boundary:

- Each link  $\ell \in \mathbf{L}$  is a boundary of precisely one face f, thus it is colored by the spin  $j_{\ell} = j_f$ .
- Each node n ∈ N is an endpoint (or startpoint) of precisely one edge e, thus it is colored by |ι<sub>n</sub>⟩ being either |ι<sub>e</sub>⟩ or ⟨ι<sub>e</sub>| respectively to the coloring of e.

Thus the boundary graph together with the above coloring determines a spin-network function  $\mathcal{N}_{\partial\kappa,j_{\ell,l_n}}$ .

The most general amplitude of a spin-foam is given by the following formula:

$$A_{\kappa}(j_f, \iota_e) = \left(\prod_{f \in \mathbf{F}} A_f \prod_{e \in \mathbf{E}} A_e \prod_{v \in \mathbf{V}} A_v\right) \cdot \left(\prod_{\ell \in \mathbf{L}} A_\ell \cdot \prod_{n \in \mathbf{N}} A_n\right)$$
(1.83)

In the EPRL model the edge- and node- amplitudes are fixed to be 1, thus the amplitude simplifies:

$$A_{\kappa}(j_f, \iota_e) = \left(\prod_{f \in \mathbf{F}} A_f \prod_{v \in \mathbf{V}} A_v\right) \cdot \left(\prod_{\ell \in \mathbf{L}} A_\ell\right)$$
(1.84)

ant the face-, link- and vertex-amplitudes are as follows:

- The face amplitude is  $A_f = \mu(\rho_f)$  is the weight of the representation carried by the face. It is disputed, whether one should use the SU(2)-weight [37] or the weight of the image of the EPRL map [44] (so  $SL(2, \mathbb{C})$ -weight for the Lorentzian case and Spin (4) for the Euclidean case, which shall be discussed in subsection 1.2.3.2). For most of the calculations presented in this thesis it is straightforward to substitute one or another approach. If it is not, we use the SU(2)-weight, i.e  $A_f = 2j_f + 1$ .
- The link amplitude is  $A_l = \frac{1}{\sqrt{A_f}}$  where f is the face containing the link  $\ell$ .

• The vertex amplitude  $A_v$  is given as follows.

Each edge ending at v contributes by the tensor  $|\iota_e\rangle$  (if it is the dual tensor  $\langle \iota_e|$ , we treat it as an ordinary tensor of a dual space). Each such tensor has one index  $m_{e,f}$  corresponding to each face incident to the edge (being either upper or lower index, depending on their relative orientation). Thus overall in the tensor product  $\bigotimes_{e \in v} |\iota_e\rangle$  there is one upper and one lower SU(2)-index per each face incident to v. We map the tensor product using the EPRL map, obtaining a tensor  $\bigotimes_{e \in v} Y |\iota_e\rangle$ , in which there is one upper and one lower  $SL(2, \mathbb{C})$ -index per each face incident to v. We project the obtained tensor onto the  $SL(2, \mathbb{C})$ -invariant tensors and then contract each pair of indices corresponding to the same face and obtain the amplitude:

$$A_{v} = \operatorname{Tr}_{SL(2,\mathbb{C})} \left[ \hat{P}_{SL(2,\mathbb{C})}^{\operatorname{Inv}} \left( \bigotimes_{e \in v} Y | \iota_{e} \right) \right]$$
(1.85)

This procedure can be also viewed in other way. The tensor  $\bigotimes_{e \in v} |\iota_e\rangle$  is in fact a SU(2)-spin-network state, let us denote is as  $\mathcal{N}_v$  (more discussion on the graph, on which it is defined, can be found in section §2.1 and in [43]). The vertex amplitude is

$$A_{v} = \left(\hat{P}_{SL(2,\mathbb{C})}^{\mathrm{Inv}} Y \mathcal{N}_{v}\right) (\mathbb{1})$$
(1.86)

The partition function of a given 2-complex  $\kappa$  is given by sum over all possible colorings:

$$Z_{\text{EPRL}}^{\kappa} = \sum_{\{j_f\}} \sum_{\{\iota_e\}} A_{\kappa} \left( j_f, \iota_e \right)$$
(1.87)

where the summation over the representations goes for each face from  $j_f = 0$  to  $j_f = \infty$ with the step of  $\frac{1}{2}$ , and having fixed the spin of the faces, the summation over the invariant tensors  $\iota_e$  goes for each edge through an orthonormal basis of the edge Hilbert space  $\mathcal{H}_e^{\text{Inv}} = \text{Inv}\left(\bigotimes_f \mathcal{H}_{j_f}^{\epsilon_f}\right)$  (it does not depend on the choice of the basis).

#### 1.2.2.3 Spin-foam transition amplitude as the evolution operator

As it was already stated in subsection 1.2.1.2, the partition function of a spin-foam can be treated as a function  $Z^{\kappa}(g_{\ell})$  of the holonomies on the boundary graph  $\partial \kappa$ . Thus given a spin-network function  $\mathcal{N}_{\partial \kappa}$  supported on the boundary graph, one can calculate the scalar product as:

$$(Z^{\kappa}||\mathcal{N}_{\partial\kappa}\rangle = \int_{G^{\mathbf{L}}} \mathrm{d}g_{\ell} Z^{\kappa} (g_{\ell}) \,\mathcal{N}_{\partial\kappa} (g_{\ell}) \tag{1.88}$$

The partition function in general does not have to be a spin-network function, because it does not have to be normalisable in the spin-network scalar product. In fact the partition function is an element of the algebraic dual  $\mathcal{H}^*_{\partial\kappa}$ , i.e. the linear (but not necessarily bounded) functionals on  $\mathcal{H}_{\partial\kappa}$ .

Consider a graph build of two disconnected subgraphs  $\mathcal{G} = \mathcal{G}_{in} \sqcup \mathcal{G}_{out}^{\star}$  (for later convenience we assume that the second component  $\mathcal{G} \setminus \mathcal{G}_{in}$  is dual to some graph  $\mathcal{G}_{out}$  - which is always true, since  $\mathcal{G}^{\star\star} = \mathcal{G}$ ; each of these two subgraphs may have arbitrary number of connected components). The graph Hilbert space containing such spin-network states can be decomposed as

$$\mathcal{H}_\mathcal{G} = \mathcal{H}_{\mathcal{G}_{\mathrm{in}}} \otimes \mathcal{H}^{\star}_{\mathcal{G}_{\mathrm{out}}}$$

Let us now consider a spin-foam defined on a 2-complex  $\kappa$  such that  $\partial \kappa = \mathcal{G}$ . Then for each coloring EPRL-coloring of  $\kappa$  the partition function  $Z_{\text{EPRL}}^{\kappa} \in \mathcal{H}_{\mathcal{G}}^{*}$ , thus it defines an operator  $\widehat{P_{\kappa}} : \mathcal{H}_{\mathcal{G}_{\text{in}}} \to \mathcal{H}_{\mathcal{G}_{\text{out}}}$ , by the following condition: let

$$|\psi\rangle = |\mathcal{N}_{\rm in}\rangle \otimes \langle \mathcal{N}_{\rm out}| \in \mathcal{H}_{\mathcal{G}_{\rm in}} \otimes \mathcal{H}_{\mathcal{G}_{\rm out}}^{\star} = \mathcal{H}_{\mathcal{G}}$$
(1.89)

then

$$\langle \mathcal{N}_{\text{out}} | \widehat{P_{\kappa}} | \mathcal{N}_{\text{in}} \rangle := (Z_{\text{EPRL}}^{\kappa} | \psi \rangle$$
 (1.90)

(obviously since  $Z_{\text{EPRL}}^{\kappa}$  does not have to be a bounded functional, the operator  $\widehat{P_{\kappa}}$  can be also unbounded).

The matrix elements of the operator  $\widehat{P_{\kappa}}$  are interpreted as transition amplitudes for the quantum process between the state  $|\mathcal{N}_{in}\rangle$  and  $|\mathcal{N}_{out}\rangle$  described by the spin-foam  $\kappa$ . The partition function already contains summation over all possible colorings of the bulk degrees of freedom. In order to obtain the transition amplitude concerning all possible quantum processes one has to sum over all possible 2-complexes  $\kappa$  (or take the limit with respect to refining the 2-complex  $\kappa$  - see [49]):

$$\left\langle \mathcal{N}_{\text{out}} \right| \left| \mathcal{N}_{\text{in}} \right\rangle_{\text{phys}} = \left\langle \mathcal{N}_{\text{out}} \right| \sum_{\kappa} \widehat{P_{\kappa}} \left| \mathcal{N}_{\text{in}} \right\rangle = \lim_{\kappa \to \infty^{n}} \left\langle \mathcal{N}_{\text{out}} \right| \widehat{P_{\kappa}} \left| \mathcal{N}_{\text{in}} \right\rangle$$
(1.91)

The question, what is the appropriate class of the 2-complexes one has to sum over is one of the problems addressed and answered within this thesis (see section §2.1 and chapter 3).

#### **1.2.3** Technical issues

Here let us focus on three technical details of the EPRL model, that are not of key importance to understand the construction of the theory, but are quite significant to follow the details of calculations in what follows.

Firstly, in subsection 1.2.3.1, we introduce a framework called *operator spin-foam models*, which simplifies some formulas and clarifies some physical interpretations of spin-foam models. Then in subsection 1.2.3.2 we present a short discussion of so called Euclidean version of the model. Finally, in subsection 1.2.3.3, we recall an argument that one can drop one of integrals over the  $SL(2, \mathbb{C})$  group per each vertex of the spin-foam.

#### 1.2.3.1 Operator spin-foam models

An useful reformulation of the partition function was given in [38], called *operator spin-foam models*. The authors realised, that instead of summing over an orthonormal basis of  $\mathcal{H}_e^{\text{Inv}}$ , one can use the fact, that each intertwiner  $\iota_e$  appears twice, once as  $\iota_e$  and once as  $\iota_e^{\text{inv}}$  and replace it with the projection operator  $P_e^{\text{Inv}} : \mathcal{H}_e \to \mathcal{H}_e^{\text{Inv}}$ , so that

$$Z_{\text{EPRL}}^{\kappa} = \sum_{\{j_f\}} A_{\kappa} \left( j_f, P_e^{\text{Inv}} \right)$$
(1.92)

The missing sum is build in the form of the projector:

$$P_e^{\text{Inv}} := \sum_{\iota_e \text{ - orthonormal basis}} |\iota_e\rangle \langle \iota_e|$$
(1.93)

The amplitude  $A_{\kappa}(j_f, P_e^{\text{Inv}})$  can be obtained by introducing any basis in each of  $\mathcal{H}_e$ -spaces, decomposing each of  $P_e^{\text{Inv}}$ -operators in the corresponding basis and applying the formula (1.85) at each vertex. The merit of the calculation does not change.

There are two benefits of the operator spin-foam models approach. Firstly it gives a clear interpretation of an edge of a spin-foam: it is an operator, that maps the initial state of a node of a spin-network into either an interaction, or to the final state. It can be seen as a *free propagator* of the state of a node. Secondly it allows to easily extend the the model to other dynamics not only by changing the vertex amplitude, but also by coloring the edges with the operators other then the projector - which may be useful when considering issues related to renormalization.

#### 1.2.3.2 Lorentzian versus Euclidean version of Spin-foams

The first papers presenting the spin-foam models of our concern there were considered a simplified version of the theory, namely the so called Euclidean gravity. This theory is defined by the Holst action, but with an assumption, that the metric 2-form in the fibres of is  $\eta_{IJ} = \text{diag}(1, 1, 1, 1)$ , and thus the gauge group is Spin (4) being the double cover of SO(4) instead of  $SL(2, \mathbb{C})$  being the double cover of  $SO(1, 3)^+$ .

Since Spin (4) =  $SU(2) \times SU(2)$ , each irreducible representation of Spin (4) is a tensor product of two representations of SU(2), thus they are labelled by two spins  $(j_+, j_-)$ . The Spin (4)-vectors are the tensor products of SU(2)-vectors, i.e:

$$\mathcal{H}_{(j_+,j_-)} = \mathcal{H}_{j_+} \otimes \mathcal{H}_{j_-} \ni |m_+\rangle_{j_+} \otimes |m_-\rangle_{j_-}$$
(1.94)

The Euclidean EPRL spin-foam model is defined in by the same construction, which was described in the previous subsections, with the only difference: the EPRL map Y injects SU(2) into SO(4)-vectors according to the formula

$$Y : |m\rangle_j \mapsto C^m_{m_+m_-} |m_+\rangle_{j_+} \otimes |m_-\rangle_{j_-} \quad \text{for } j_{\pm} = \frac{|1 \pm \gamma|}{2}j$$
 (1.95)

where  $C_{m_+m_-}^m$  are the Clebsh-Gordan coefficients [41].

#### **1.2.3.3** Omitting of one of $SL(2, \mathbb{C})$ -integrals

Consider a spin-foam vertex such that the vertex spin-network is  $\mathcal{N}$ . As we have noted in (1.86), the vertex amplitude is given by

$$A = \left(\hat{P}_G^{\text{Inv}} Y \mathcal{N}\right) (\mathbb{1}) \tag{1.96}$$

(where we consider a generalized EPRL map with the image in representations of arbitrary group G). The projection is

$$\hat{P}_{G}^{\text{Inv}} = \int_{G^{\mathbf{N}}} \mathrm{d}g_n \ g_n \tag{1.97}$$

We shall study the amplitude (1.96) and show, that for a compact group G one can skip one of integrals per each connected component of  $\mathcal{G}$ , and if G is non-compact (for example  $G = SL(2, \mathbb{C})$ ) the amplitude is infinite unless one regularizes it by skipping one integral per a connected component of  $\mathcal{G}$ . This derivation was already presented in [46, 50].

A spin-network state can be decomposed into a basis of tensor products of intertwiners in node Hilbert spaces (see Appendix A.3). Thus let us consider only a spin-network state of the form

$$|\mathcal{N}\rangle = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} |\iota_n\rangle \tag{1.98}$$

we can write it emphasising the index structure of the tensors  $\iota_n$  as

$$\mathcal{N} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\iota_n]_{n_{\ell_{k+1}^-}^{-} \cdots n_{\ell_{k+k'}^+}^{-}}^{m_{\ell_1}^+ \cdots m_{\ell_k}^+}$$
(1.99)

where k and k' are the numbers of incoming and outgoing half-links at n respectively (of course this numbers may be different for each node n). Acting by the EPRL map on  $\mathcal{N}$  gives

$$Y\mathcal{N} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\iota_{n}]_{n_{\ell_{k+1}^{-}} \cdots n_{\ell_{k+k'}^{-}}}^{m_{\ell_{k}^{+}}} Y_{m_{\ell_{1}^{+}}}^{A_{\ell_{1}^{+}}} \cdots Y_{m_{\ell_{k}^{+}}}^{A_{\ell_{k}^{+}}} Y_{B_{\ell_{k+1}^{-}}}^{n_{\ell_{k+1}^{-}}} \cdots Y_{B_{\ell_{k+k'}^{-}}}^{n_{\ell_{k+k'}^{-}}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{1}^{+}} \cdots A_{\ell_{k}^{+}}} Y_{B_{\ell_{k+1}^{-}}}^{n_{\ell_{k+1}^{-}}} \cdots Y_{B_{\ell_{k+k'}^{-}}}^{n_{\ell_{k+k'}^{-}}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{k}^{+}}} Y_{B_{\ell_{k+1}^{-}}}^{n_{\ell_{k+1}^{-}}} \cdots Y_{B_{\ell_{k+k'}^{-}}}^{n_{\ell_{k+k'}^{-}}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{k}^{+}}} Y_{B_{\ell_{k+1}^{-}}}^{n_{\ell_{k+1}^{-}}} \cdots Y_{B_{\ell_{k+k'}^{-}}}^{n_{\ell_{k+k'}^{-}}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{k}^{+}}}$$

$$(1.100)$$

for some  $\tilde{\iota}_n$  (not necessarily *G*-invariant). Applying the action of the projection onto the *G*-invariant states one gets

$$\hat{P}_{G}^{\text{Inv}}Y\mathcal{N} = \int_{G^{\mathbf{N}}} \mathrm{d}g_{n} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}}^{-} \cdots B_{\ell_{k+k'}}^{-}}^{A_{\ell_{1}^{+}} \cdots A_{\ell_{k}^{+}}}$$

$$\rho^{\ell_{1}^{+}} (g_{n})_{A_{\ell_{1}^{+}}^{\ell_{1}^{+}}}^{C_{\ell_{1}^{+}}} \cdots \rho^{\ell_{k}^{+}} (g_{n})_{A_{\ell_{k}^{+}}^{\ell_{k}^{+}}}^{C_{\ell_{k}^{+}}} \rho^{\ell_{k-1}^{-}} (g_{n}^{-1})_{D_{\ell_{k+1}}^{\ell_{k+1}}}^{B_{\ell_{k+1}}} \cdots \rho^{\ell_{k+k'}^{-}} (g_{n}^{-1})_{D_{\ell_{k+k'}}^{\ell_{k+k'}}}^{B_{\ell_{k+k'}}}$$

$$(1.101)$$

where  $\rho$ s are appropriate irreducible representations of G. By gathering together the term corresponding to the same link one gets

$$\hat{P}_{G}^{\text{Inv}}Y\mathcal{N} = \int_{G^{\mathbf{N}}} \mathrm{d}g_{n} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} [\tilde{\iota}_{n}]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{1}^{+}} \cdots A_{\ell_{k}^{+}}} \bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \rho^{\ell} \left(g_{t(\ell)}^{-1}\right)_{D_{\ell^{-}}}^{B_{\ell^{-}}} \rho^{\ell} \left(g_{s(\ell)}\right)_{A_{\ell^{+}}}^{C_{\ell^{+}}}$$
(1.102)

where all A and B indices of  $\rho$ s are contracted with indices of  $\tilde{\iota}$ s. The evaluation at unity means contracting the above formula with  $\bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \delta_{C_{\ell^+}}^{D_{\ell^-}}$ , which leads to

$$\left(\hat{P}_{G}^{\mathrm{Inv}}Y\mathcal{N}\right)(\mathbb{1}) = \int_{G^{\mathbf{N}}} \mathrm{d}g_{n} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \left[\tilde{\iota}_{n}\right]_{B_{\ell_{k+1}^{-}} \cdots B_{\ell_{k+k'}^{-}}}^{A_{\ell_{1}^{+}} \cdots A_{\ell_{k}^{+}}} \bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \rho^{\ell} \left(g_{t(\ell)}^{-1}g_{s(\ell)}\right)_{A_{\ell^{+}}}^{B_{\ell^{-}}} \tag{1.103}$$

Let us now pick a node  $n_0$  and insert an extra term  $g_{n_0}g_{n_0}^{-1}$  between each pair  $g_{t(\ell)}^{-1}g_{s(\ell)}$ . The result obviously does not change. Now using the invariance of the Haar measure let us shift all the group elements  $g_n$  for  $n \neq n_0$  to  $\tilde{g}_n := g_{n_0}^{-1}g_n$ . One can then rewrite the integral as

$$\begin{pmatrix} \hat{P}_{G}^{\mathrm{Inv}}Y\mathcal{N} \end{pmatrix} (\mathbb{1}) = \left( \int_{G} \mathrm{d}g_{n_{0}} \right) \int_{G^{\mathbf{N}-1}} \mathrm{d}\tilde{g}_{n\neq n_{0}} \bigotimes_{n\in\mathbf{N}_{G}} [\tilde{\iota}_{n}]_{B_{\ell^{-}}}^{A_{\ell^{+}}\cdots A_{\ell^{+}_{k}}} \\ \bigotimes_{\substack{\ell\in\mathbf{L}_{G}:s(\ell)=t(\ell)=n_{0}}} \rho^{\ell} (\mathbb{1})_{A_{\ell^{+}}}^{B_{\ell^{-}}} \\ \bigotimes_{\ell\in\mathbf{L}_{G}:s(\ell)=n_{0}\wedge t(\ell)\neq n_{0}} \rho^{\ell} \left( \tilde{g}_{t(\ell)}^{-1} \right)_{A_{\ell^{+}}}^{B_{\ell^{-}}} \\ (1.104) \\ \bigotimes_{\substack{\ell\in\mathbf{L}_{G}:s(\ell)\neq n_{0}\wedge t(\ell)=n_{0}}} \rho^{\ell} \left( \tilde{g}_{s(\ell)} \right)_{A_{\ell^{+}}}^{B_{\ell^{-}}} \\ \bigotimes_{\ell\in\mathbf{L}_{G}:\ell\cap n_{0}=\emptyset} \rho^{\ell} \left( \tilde{g}_{t(\ell)}^{-1} \tilde{g}_{s(\ell)} \right)_{A_{\ell^{+}}}^{B_{\ell^{-}}} \end{cases}$$
where we can factorize the integral  $\int_G dg_{n_0}$  thanks to the fact that the integrand does not depend on  $g_{n_0}$  any more. We can simplify this formula by introducing  $\tilde{g}_{n_0} := 1$  and putting it in the places, where there was  $g_{n_0}$ , so that we have

$$\begin{pmatrix} \hat{P}_{G}^{\mathrm{Inv}}Y\mathcal{N} \end{pmatrix} (\mathbb{1}) = \left( \int_{G} \mathrm{d}g \right) \int_{G^{\mathbf{N}-1}} \mathrm{d}\tilde{g}_{n \neq n_{0}} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \begin{bmatrix} \tilde{\iota}_{n} \end{bmatrix}_{B_{\ell_{k+1}}^{-} \cdots B_{\ell_{k+k'}}}^{A_{\ell_{1}}^{+} \cdots A_{\ell_{k}}^{+}} \\ \bigotimes_{\ell \in \mathbf{L}_{\mathcal{G}}} \rho^{\ell} \left( \tilde{g}_{t(\ell)}^{-1} \tilde{g}_{s(\ell)} \right)_{A_{\ell+1}}^{B_{\ell-1}} \Big|_{\tilde{g}_{n_{0}} = \mathbb{1}}$$

$$(1.105)$$

If the group G is compact, the Haar measure is normalized, so that the integral  $\int_G dg = 1$ and we can skip it. If the group G is not compact, the integral  $\int_G dg = \infty$ , thus in order to obtain a finite amplitude one has to regularize it by dropping one integral over G. Such regularization does not guarantee, that the result is finite, but it is necessary to make any chances for the amplitude to not diverge.

If the graph  $\mathcal{G}$  is not connected, all the above formulae factorize into a product over connected components of  $\mathcal{G}$ , thus one can perform the same derivation for each connected component separately, concluding that one can drop one integral over G per each connected component of  $\mathcal{G}$ .

# Chapter 2

# Presentation of the issues addressed in the thesis

This chapter is devoted to present the questions that was addressed an answered within this thesis.

There are three main topics that contribute to this thesis. The first one was the question about a class of 2-complex that should be used in defining the spin-foam models. It is presented in section §2.1, and answered in chapter 3. The second one is the problem of higher order contributions to the transition amplitude calculated in Dipole Cosmology model - one of basic applications of spin-foam models. This question led to more general problem, i.e. finding all spin-foams with a given fixed boundary, at some given order of expansion. We present this question, together with a brief review of Dipole Cosmology model, in section §2.2, and answer it in chapter 4. The last topic is the problem of divergences and renormalization of spin-foam models. In this thesis we focus on two aspects of it: finding the bubble part of a spin-foam and calculating one transition amplitude related to one of bubbles. The issues are presented in section §2.3 and answered in chapter 5 and chapter 6. Each section of this chapter contains a brief review of research in the field, that was done already, together with fixing some conventions of notation.

# 2.1 What class of 2-complexes should we use in Spin-foam models?

The EPRL spin-foam models were designed in analogy to the BF-model on spin-foams defined on 2-complexes dual to triangulations of a 4-manifold  $\mathcal{M}$  [51]. This construction was based on a concept of discretisation of an appropriate action (either BF or Holst). Such approach restricts the topology of the 2-complexes - and respectively their boundary graphs - to a certain class. On the contrary Loop Quantum Gravity admits all the spin-networks with the support being an arbitrary graph. This leads to an inconsistency of these two theories.

Original EPRL-vertex amplitude was defined in terms of geometrical data of a quantum 4-simplex (i.e. as a 15-j symbol, being a function of areas of triangles at the 4-simplex). Further study [43, 38] allowed to express it in terms of more general objects, so that it was easy to extend to case of vertices with more general structure - other then ones dual to 4-simplices. It was a major step to extend the EPRL-spin-foam formalism the one compatible with Loop Quantum Gravity kinematics: one can calculate spin-foam

amplitude for 2-complexes that are not necessarily dual to any triangulation, thus their boundaries can be arbitrary.

However, a question arose: what class of 2-complexes one should consider when summing over all possible processes? Obviously 2-complexes dual to triangulations are too narrow class, but on the other hand the class of all general 2-complexes seams to be too wide. The appropriate class should admit a natural way to calculate the transition amplitudes and should be compatible with the kinematic structure of LQG on the boundary. We addressed this question in our research.

The subsection 2.1.1 emphasises differences between the topological structures dual to triangulations and the general ones. The subsection 2.1.2 presents the KKL construction, being the starting point to our answer to the above question. The subsection 2.1.3 states the question formally and guides to the answer.

# 2.1.1 Incompatibility of topological structures of Loop Quantum Gravity and spin-foam models

As it was mentioned above, in canonical Loop Quantum Gravity the physical states are spin-network functions defined on closed graphs. On the other hand the underlying topological structure of spin-foam models are 2-complexes dual to triangulations of 4dimensional manifolds. Boundary graphs of these 2-complexes determine the possible boundary spin-network states of the spin-foam theory.

Consider a 2-complex  $\kappa$  dual to a 4-dimensional triangulation  ${}^{4}\Delta$  of a manifold  $\mathcal{M}$ . Its boundary  $\partial \kappa$  is a graph dual to a 3-triangulation  $\mathcal{G} = {}^{3}\Delta^{\star}$ . If the boundary manifold  $\Sigma = \partial \mathcal{M}$  has no boundary, the graph  $\mathcal{G}$  is closed and thus one can define a spin-network on the graph  $\mathcal{G}$ .

However, the kinematics of Loop Quantum Gravity considers spin-networks defined for all closed graphs. The difference is in allowed valence of nodes of graphs - a graph dual to a triangulation of a 3-manifold has only 4-valent nodes, while a graph of a general spin-network may have nodes of arbitrary valence (greater then 2).

A general graph might be dual to an arbitrary cellular decomposition of  $\Sigma$  - namely a 3-CW-complex  ${}^{3}C$  (not necessarily a  $\Delta$ -complex), such that  $X_{{}^{3}C} = \Sigma$  (for more details - see Appendix A.1.2.6). However, a graph dual to a general cellular decomposition of a closed manifold may be incompatible with Loop Quantum Gravity kinematics, which requires graphs to be closed. One can introduce a degenerate cellular decomposition of a closed manifold, in which there are 2-cells with only one faces - see figure 2.1, which would lead to a graph with 1-valent nodes, what is forbidden for gauge-invariant spin-network states. In this sense general cellular decompositions of  $\Sigma$  generate too wide class of graphs for Loop Quantum Gravity. Thus let us admit only these cellular decompositions, for which the graph dual to them is closed.

In this interpretation nodes of spin-networks becomes quantum polyhedra, glued together via quantum polygons - represented by faces - see [52].

Within the original EPRL framework there is a wide class of boundary spin-network states for which the dynamics is not defined. An extension of a spin-foam formalism is needed. Such an extension requires two steps: generalisation of the formula for transition amplitude (already done in [43], discussed in subsection 2.1.2) and a strict definition of the class of 2-complexes to be used in the computations.

A natural candidate for a class of 2-complexes, that would make spin-foam models compatible with Loop Quantum Gravity is the class of all 2-complexes dual to all 4dimensional cellular decompositions of  $\mathcal{M}$ . However, this class has the same drawback, that considered above class of graphs dual to cellular decompositions of  $\Sigma$  - it is too wide



Figure 2.1: Degenerate cellular decompositions. (a) - A 3-cell with one face, no edges and one vertex on its boundary. In the dual graph it is represented by a 1-valent node and a half-link. (b) - A cellular decomposition of a 3-sphere into two 3-balls glued by their boundary; each 3-ball is a cell like at (a). (c) - A graph dual to the cellular decomposition presented at (b); it is build of two 1-valent nodes connected by a single link.

for Loop Quantum Gravity. An immediate example is a cellular decomposition of  $\mathcal{M}$  such that it's boundary is a cellular decomposition of  $\Sigma = \partial \mathcal{M}$  with the dual graph containing 1-valent nodes. It is not clear, whether putting a restriction, that the boundary of the 4-celural decomposition is compatible with Loop Quantum Gravity is enough to avoid all non-physical behaviours in the bulk, so the question on the class of 2-complexes requires further investigation (see subsection 2.1.3 and chapter 3).

# 2.1.2 KKL construction

Recently a generalisation of spin-foam transition amplitude was proposed in [43]. The generalized prescription allows to calculate spin-foam amplitudes for 2-complexes with vertices more general, than dual to 4-simplex. The key concept of this generalization is to assign to each vertex a graph (called vertex graph), that encodes the local structure of the 2-complex at this vertex. Then one colors this graph using the coloring of the 2-complex in a neighbourhood of the vertex, obtaining a so called vertex spin-network. The transition amplitude is then read from the vertex spin-network function.

This procedure is the starting point of our research for the proper class of 2-complexes, thus let us recall it here with more details. In subsection 2.1.2.1 we recall the definition of a vertex graph for an internal vertex of a 2-complex. In subsection 2.1.2.2 we present the exact calculation of spin-foam transition amplitude using the vertex spin-networks.

#### 2.1.2.1 Vertex graph

Consider a 2-complex  $\kappa$  dual to some cellular decomposition (we assume, that the 2complex satisfies all required properties, not stating yet, what they are). Let us focus on one internal vertex v of  $\kappa$ . We shall define a graph  $\Gamma_v$ , that captures the local structure of  $\kappa$  at v, as follows. Let  $\Gamma_v = (\mathbf{N}, \mathbf{L})$ , then:



Figure 2.2: An illustration of the KKL procedure of obtaining the vertex graph. (a) - A spin-foam with a sphere drawn around the interaction vertex. (b) - Zoom at the interaction vertex. Note the relation between orientation of faces of the spin-foam and orientation of links the vertex graph. (c) - The vertex graph.

- For each edge e incident to v there is precisely one node  $n_e \in \mathbf{N}$  (and there are no other nodes).
- For each face f incident to v there is a link  $\ell_f \in \mathbf{L}$  as follows. Let us go along the boundary of f. At some point we arrive to v through the edge e, and leave it though the edge e' (it may be the same edge). We introduce a link  $\ell_f$  such that  $s(\ell_f) = n_{e'}$  and  $t(\ell_f) = n_e$ . If the face f is multi-incident to v, we introduce one link per each appearance of v at the boundary of f.

There is a geometrical interpretation of the above procedure. Let us embed  $\kappa$  in  $\mathbb{R}^4$  and draw a sphere  $S^3$  centred at v with the radius small enough that there is no other vertex of  $\kappa$  inside the sphere. The graph  $\Gamma_v$  is precisely the intersection of  $\kappa$  and the sphere (see figure 2.2).

# 2.1.2.2 Vertex spin-network and vertex amplitude

Given a spin-foam, i.e. a 2-complex  $\kappa$  with faces colored by irreducible representations of SU(2) and edges colored by invariant SU(2)-tensors (see subsection 1.2.2.2), one can define a spin-network state for each internal vertex of  $\kappa$ . Indeed, let v be an internal vertex and  $\Gamma_v$  its vertex-graph (as it was defined above). Each link  $\ell_f$  of  $\Gamma_v$  corresponds to a face f of  $\kappa$ , thus it can be colored by the irreducible representation, as the face, i.e.  $j_{\ell_f} := j_f$ . For each node  $n_e$  of  $\Gamma_v$  the invariant node Hilbert space  $\mathcal{H}_{n_e}^{\text{Inv}}$  is the same, as  $\mathcal{H}_{e,v}^{\text{Inv}}$ , thus the tensor  $|\iota_e\rangle$  is a good coloring of the node  $n_e$  (in the operator spin-foam framework one decomposes each operator  $P_e$  in a basis, and then obtain appropriate  $|\iota_e\rangle$ tensors - see subsection 1.2.3.1). The spin-network  $\mathcal{N}_{\Gamma_v, j_{\ell}, \iota_n}$  obtained by the  $j_{\ell_f} := j_f$ and  $\iota_{n_e} = \iota_e$  is precisely the one, that is used to calculate the vertex amplitude according to the formula (1.86):

$$A_{v} = \left(\hat{P}_{SL(2,\mathbb{C})}^{\mathrm{Inv}} Y \mathcal{N}_{\Gamma_{v}, j_{\ell}, \iota_{n}}\right) (1)$$

$$(2.1)$$

This construction allows to decompose a spin-foam into a set of spin-network states (each per each vertex). Note, that for two vertices v and v' connected by a face f the links of  $\Gamma_v$  and  $\Gamma_{v'}$  corresponding to the face f are colored by the same spin  $j_f$ . Similarly, for two vertices v and v' connected by an edge e the nodes of  $\Gamma_v$  and  $\Gamma_{v'}$  corresponding to the edge e are colored by dual invariant tensors  $|\iota_e\rangle$  and  $\langle\iota_e|$ . These properties will be the base for construction of operator spin-network diagrams framework (see chapter 3).

# 2.1.3 The question: What class of 2-complexes one should use in spinfoam models?

One of technical details of the KKL model was still unclear, namely: what class of 2complexes should be summed over in the formula (1.91)? The following properties of this class are suggested by physical intuition:

- The boundary of a 2-complex should be an arbitrary graph
- The boundary graph  $\mathcal{G}$  should have a neighbourhood of the topology  $\mathcal{G} \times [0, 1]$  in the 2-complex, i.e. there should not be any interaction vertex on the boundary.
- There should be a well defined operation of composition of 2-complexes, corresponding to composition of evolution process.

We found a class of 2-complexes, that satisfy the above properties and which is in natural way compatible with the dynamics described by the KKL model. We give our definition constructively, i.e. we provide a procedure of constructing a 2-complex, and our claim is that each 2-complex that can be a result of our procedure belongs to our class.

Our attempt to study this issue lead to the concept of Operator Spin-network Diagrams - a framework that far exceeded the answer to the question presented above. We developed an approach to note the dynamical processes in quantum gravity in terms of certain diagrams. The elements of these diagrams are graphs, each graph representing an interaction vertex of a spin-foam. Our framework allows both to reconstruct the 2-complex picture and to easily read out the formula for a transition amplitude of a quantum process. It is also easily extendible for more general models of spin-foam dynamics. Moreover, the Operator Spin-network Diagrams framework is a basic tool used in all other issues addressed in this thesis.

The Operator Spin-network Diagrams framework, being our answer to this question, was presented in [53] and it is described in chapter 3.

# 2.2 What is a contribution of higher order transition amplitudes in the Dipole Cosmology model?

A KKL construction, recalled in previous section, allowed to study a toy-model of Spinfoam theory, called the Dipole Cosmology model [54, 55, 56, 57, 58, 59]. It is based on several approximation assumptions:

- boundary states are assumed to be spin-networks on a fixed graph, called the Dipole Graph,
- interaction is calculated in large-*j*-limit,
- there is only one interaction vertex,
- the interaction vertex is of very specific type.

We will introduce the model in subsection 2.2.1, explaining in detail the assumptions presented above.

We are interested in generalizing the Dipole Cosmology by relaxing the above assumptions, especially by relaxing the last one. We do it by applying the Operator Spin-network Diagrams framework and finding all diagrams at certain level of expansion. Moreover, we found a general algorithm to find all diagrams (thus all spin-foams) with an arbitrary fixed boundary at all levels of expansion. The precise formulation of issues we addressed at this stage of research is presented in subsection 2.2.2.

# 2.2.1 Description of the model

The Dipole Cosmology model was introduced in [55, 54], then it was extended and studied in detail in [56, 57, 58]. It is based on an assumption, that a space of *in* and *out* states of the Universe is restricted to the ones supported on the *Dipole graph* (see subsection 2.2.1.1) and that it is a coherent state (see subsection 2.2.1.2). The quantum process itself is assumed to follow the simplest non-trivial spin-foam (see subsection 2.2.1.3). The overview on the derivation of the amplitude can be found in subsection 2.2.1.4, and the approximations taken in the limit of large volume of the Universe are recalled in subsection 2.2.1.5.

# 2.2.1.1 Dipole graph

In Dipole Cosmology one assumes that the boundary spin-network state of the considered spin-foam decomposes into *in* and *out* state, so that the underlying graph has two connected components. It is assumed that the topology of the graph is not changed during evolution, so the *in* and *out* part of the graph look the same.

The Dipole graph  $\mathcal{G}_{dipole} = \Gamma_{dipole}^{in} \sqcup \Gamma_{dipole}^{out}$  is presented at figure 2.3a. Let us focus on only one of its connected components. Note, that it is a graph dual to a triangulation of some 3-dimensional manifold (see Appendix A.1.2.6). Indeed, consider a 3-sphere. It can be decomposed into a pair of 3-dimensional balls glued along their boundaries. Each node of  $\Gamma_{dipole}$  is dual to a tetrahedron, which is homeomorphic to a 3-ball, thus the  $\Gamma_{dipole}$  is dual to a decomposition of a 3-sphere into two tetrahedra glued along their boundaries (see figure 2.3b).

Recalling the fact, that nodes of spin-network can be labelled by eigenvalues of the volume operator (see subsection 1.1.2.4), the Dipole graph captures the total volume of the Universe and the volume of each of the tetrahedra of the triangulation. Thus one can encode the dipole-moment of the volume, which is the origin of its name (see [54]).

# 2.2.1.2 Coherent states

In Dipole Cosmology model one assumes that the *in* and *out* states are coherent states of Loop Quantum Gravity, designed according to the procedure presented in [60].

To define a general coherent state one peaks for each link  $\ell$  of  $\Gamma_{\text{dipole}}^{\text{in/out}}$  an element  $H_{\ell} \in SL(2, \mathbb{C})$  and then define the state as a spin-network function given by

$$\psi_{H_{\ell}}(U_{\ell}) = \int_{SU(2)^{N}} \mathrm{d}h_{n} \prod_{\ell} K_{t} \left( h_{s(\ell)}^{-1} U_{\ell} h_{t(\ell)} H_{\ell}^{-1} \right)$$
(2.2)

where  $K_t: SU(2) \to \mathbb{C}$  is the heat-kernel function:

$$K_t(h) = \sum_{j=0}^{\infty} (2j+1) e^{-2t\hbar j(j+1)} \operatorname{Tr} \left[ D^j(h) \right]$$
(2.3)



Figure 2.3: The Dipole Graph. (a) - The graph and it's decomposition into  $\Gamma_{\text{dipole}}^{\text{in}} \sqcup \Gamma_{\text{dipole}}^{\text{out}}$ . (b) - The triangulation of a 3-sphere dual to one of  $\mathcal{G}_{\text{dipole}}$  connected components.

where the sum goes every half-integer j, the parameter t determines the spread of the coherent state and the Planck constant  $\hbar$  is kept to ensure the formulae to agree with [55, 56] (for  $D^j(H_\ell)$  we use the analytic continuation of the Wigner matrices, which are in fact the nonunitary representations of  $SL(2,\mathbb{C})$ ). One can show, that for  $H_\ell = e^{i(iX_\ell)}h_\ell$ , where  $h_\ell \in SU(2)$  and  $X_\ell \in \mathfrak{su}(2)$ , the expectation values are

$$\frac{\langle \psi_{H_{\ell}} | \widehat{U_{\ell}} | \psi_{H_{\ell}} \rangle}{\langle \psi_{H_{\ell}} | | \psi_{H_{\ell}} \rangle} = h_{\ell}$$
(2.4)

and

$$\frac{\langle \psi_{H_{\ell}} | \widehat{E_{\ell}} | \psi_{H_{\ell}} \rangle}{\langle \psi_{H_{\ell}} | | \psi_{H_{\ell}} \rangle} = \frac{\kappa \hbar \gamma}{4t} X_{\ell}$$
(2.5)

and that their spread is small [55, 60, 61].

In Dipole Cosmology model one chooses the coherent states such that their expectation values agree with the classical quantities calculated for a homogeneous isotropic space, so

$$h_{\ell} = n_{\ell} e^{i\alpha c \frac{\sigma_3}{2}} n_{\ell}^{-1} \quad \text{and} \quad X_{\ell} = n_{\ell} \beta p \frac{\sigma_3}{2} n_{\ell}^{-1}$$
 (2.6)

where (c, p) are symetry-reduced canonical coordinates - equivalents of the Ashtekar variables on an isotropic  $S^3$ -Universe (see [62]),  $\alpha$  and  $\beta$  are two real constants (irrelevant in further considerations) and  $n_\ell$  is the SU(2)-element such that  $n_\ell \sigma_3 n_\ell^{-1} = \vec{n_\ell} \cdot \vec{\sigma}$  for  $\vec{n_\ell}$  the vector normal to the triangle dual to the link  $\ell$  (obviously there are many such  $n_\ell$ elements, but the ambiguous part of  $n_\ell$  commutes with  $\sigma_3$ , thus  $h_\ell$  and  $X_\ell$  are determined unambiguously). Introducing a complex parameter  $z := \alpha c + i\beta p$  one can express

$$H_{\ell}(z) = n_{\ell} e^{i z \frac{\sigma_3}{2}} n_{\ell}^{-1}$$
(2.7)

and then label in/out states of Dipole Cosmology model by the phase-space point on which they are peaked:

$$\psi_z = \psi_{H_\ell(z)} \tag{2.8}$$

thus the full boundary state is

$$\psi_{z_{\rm in}, z_{\rm out}} := \psi_{z_{\rm in}} \cdot \psi_{z_{\rm out}} \tag{2.9}$$



Figure 2.4: A spin-foam used in dipole cosmology model. The structure of the interaction vertex is determined by the vertex graph being identical with the boundary graph.

# 2.2.1.3 Spin-foam

In Dipole Cosmology model one considers only one spin-foam, presented at figure 2.4. This spin-foam consists of precisely one interaction vertex, connected by edges with the boundary nodes. There are no other edges. Each face has a boundary link and forms a triangle build of the link and the edges connecting the nodes of the link with the interaction vertex. There are no other faces.

Although it is the most natural nontrivial spin-foam connecting the *in* and *out* part of  $\mathcal{G}_{dipole}$ -graph, it is not the only one. We have shown (see chapter 4), that even for only one internal vertex there are many more possible spin-foams.

# 2.2.1.4 Amplitude

The Dipole Cosmology vertex transition amplitude was calculated using the formula (1.86). Originally it was calculated in the Euclidean EPRL scheme [55], later it was generalized to the Lorentzian case [57]. The calculations presented below comes from the quoted papers. We recall them with details as a reference point for notation conventions, because very similar calculations will be performed in chapter 4.

Thanks to the very simple structure of the interaction vertex, the transition amplitude is

$$A(z_{\rm in}, z_{\rm out}) = \left(\hat{P}_G^{\rm Inv} Y \psi_{z_{\rm in}, z_{\rm out}}\right) (1)$$
(2.10)

where G = SO(4) in the Euclidean case and  $G = SL(2, \mathbb{C})$  in the Lorentzian case (the face amplitudes are omitted).

The EPRL map Y acts at  $\psi_z$  as follows (see (2.2))

$$(Y\psi_z)(g_\ell) = \int_{SU(2)^N} dh_n \prod_{\ell} K_t \left( h_{s(\ell)}^{-1} Y^{\dagger} g_\ell Y h_{t(\ell)} H_{\ell}^{-1}(z) \right)$$
(2.11)

where we generalize the notation of the heat kernel  $K_t$ , such that in presence of Y-map it is defined as follows:

$$K_t \left( h Y^{\dagger} g Y h' \right) := \sum_{j=0}^{\infty} \left( 2j+1 \right) e^{-2t\hbar j (j+1)} D^j \left( h \right)_{m'}^m Y^{\dagger m'}_A D^{Y(j)} \left( g \right)_B^A Y_{m''}^B D^j \left( h' \right)_m^{m''}$$
(2.12)

where Y(j) is the unitary representation of G corresponding to j according to Y map (it is  $(j^+, j^-)$  in the Euclidean case and  $(\gamma j, j)$  in the Lorentzian case), the indices m, m', m''label the coefficients in  $\mathcal{H}_j$  and the indices A, B label the coefficients in  $\mathcal{H}_{Y(j)}$ .

The projection  $\hat{P}_G^{\rm Inv}$  can be written in terms of the representation matrices:

$$\hat{P}_{G}^{\mathrm{Inv}}\psi\left(g_{\ell}\right) = \int_{G^{N}} \mathrm{d}g_{n}\psi\left(g_{s(\ell)}^{-1}g_{\ell}g_{t(\ell)}\right)$$
(2.13)

which composed with (2.10) and (2.11) gives

$$A(z_{\rm in}, z_{\rm out}) = \int_{G^N} \mathrm{d}g_n \int_{SU(2)^N} \mathrm{d}h_n \prod_{\ell} K_t \left( h_{s(\ell)}^{-1} Y^{\dagger} g_{s(\ell)}^{-1} \,\mathbb{1} \,g_{t(\ell)} Y \,h_{t(\ell)} H_{\ell}^{-1} \right) \tag{2.14}$$

the SU(2)-integrals were omitted, because each  $h \in SU(2)$  commute with the Y-map (see Appendix A.4.3), and then one can shift each integral  $\int_G dg_n$  to  $\int_G d\tilde{g}_n$  for  $\tilde{g}_n = g_n h_n$ , so that the integrals over SU(2) result in multiplication by the group volume (which is 1 due to normalisation of the Haar measure). Thus

$$A(z_{\rm in}, z_{\rm out}) = \int_{G^N} \mathrm{d}g_n \prod_{\ell} K_t \left( Y^{\dagger} g_{s(\ell)}^{-1} g_{t(\ell)} Y H_{\ell}^{-1} \right)$$
(2.15)

Notting the fact, that for  $\ell = 1, 2, 3, 4$  the nodes  $s(\ell) = N$  and  $t(\ell) = S$ , and for  $\ell = 5, 6, 7, 8$  the nodes  $s(\ell) = N'$  and  $t(\ell) = S'$  one can easily factorise the amplitude into

$$A(z_{\rm in}, z_{\rm out}) = A(z_{\rm in}) A(z_{\rm out})$$
(2.16)

where

$$A(z) = \int_{G^2} \mathrm{d}g_N \mathrm{d}g_S \prod_{\ell} K_t \left( Y^{\dagger} g_N^{-1} g_S Y H_{\ell}^{-1}(z) \right)$$
(2.17)

Again one of the integrals may be omitted (by shifting the other one and using the normalisation of the Haar measure in the Euclidean case or by using the property of  $SL(2, \mathbb{C})$ -integrals recalled in subsection 1.2.3.3), thus finally we have

$$A(z) = \int_{G} \mathrm{d}g \prod_{\ell} K_t \left( Y^{\dagger} g Y H_{\ell}^{-1}(z) \right)$$
(2.18)

# 2.2.1.5 Large-j-limit

Here let us again recall the details of calculations done in [55], which we shall refer to when doing some calculations in chapter 4.

The transition amplitude in Dipole Cosmology model is calculated by assuming, that the volume of the Universe is far bigger than  $\ell_{\rm Pl}^2$ . It is represented by the requirement, that  $\Im(z) \gg 1$ . This assumption allows to approximate the Wigner representation of  $H_{\ell}$ by

$$D^{j}(H_{\ell}^{-1}(z)) = D^{j}(H_{\ell}(-z)) \cong e^{-izj}P_{\ell}^{(j)}$$
(2.19)

where  $P_{\ell}^{(j)} = n_{\ell} |j\rangle_j \langle j|_j n_{\ell}^{-1} =: |n_{\ell}\rangle \langle n_{\ell}|_j$  is the projector on the state with maximum spin-projection on the direction given by  $\vec{n}_{\ell}$  (all the other matrix elements are suppressed by  $e^{-\Im(z)}$  factor). This simplify the formula for heat kernel:

$$K_t\left(Y^{\dagger}gY H_{\ell}^{-1}(z)\right) \cong \sum_j \left(2j+1\right) e^{-2t\hbar j(j+1)-izj} \left\langle n_{\ell}\right| Y^{\dagger}gY \left|n_{\ell}\right\rangle$$
(2.20)

The exponent is a 2nd order polynomial in j, with maximum at  $j = j_0 \cong \frac{-iz}{4t\hbar}$  (we assume that the real part of z is negligible when compared to it's imaginary part). For  $\Im(z) \gg 1$  the sum can be approximated by a Gaussian integral around the maximum point, so that

$$K_t\left(Y^{\dagger}gY H_{\ell}^{-1}(z)\right) \cong \sqrt{\frac{\pi}{2t\hbar}} e^{-\frac{z^2}{8t\hbar}} 2j_0 \left\langle n_{\ell} \right| Y^{\dagger}gY \left| n_{\ell} \right\rangle_{j_0}$$
(2.21)

(we approximate (2j + 1) by 2j), so that

$$A(z) \cong \left(\sqrt{\frac{\pi}{2t\hbar}}e^{-\frac{z^2}{8t\hbar}}2j_0\right)^4 \int_G \mathrm{d}g \prod_{\ell} \langle n_{\ell}| Y^{\dagger}gY | n_{\ell} \rangle_{j_0}$$
(2.22)

In the Euclidean case (i.e. G = SO(4)) the integral gives the volume of the Livine-Speziale coherent tetrahedron [63, 55], which is approximately  $N_0 j^{-3}$ . Recalling, that  $j_0 \cong \frac{-iz}{4t\hbar}$  one gets finally

$$A(z) = -\frac{i\pi^2}{t^3\hbar^3} N_0 z e^{-\frac{z^2}{2t\hbar}} = N z e^{-\frac{z^2}{2t\hbar}}$$
(2.23)

and the total amplitude is

$$A(z_{\rm in}, z_{\rm out}) = N^2 z_{\rm in} z_{\rm out} e^{-\frac{z_{\rm in}^2 + z_{\rm out}^2}{2t\hbar}}$$
(2.24)

In the Lorentzian case the integral  $\int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{\ell} \langle n_{\ell} | Y^{\dagger}gY | n_{\ell} \rangle_{j_0}$  requires techniques that are developed in chapter 6.

# 2.2.2 Formulation of the question

We are interested in finding the contribution to the Dipole Cosmology transition amplitude from spin-foams with more general interaction vertices that the one presented in subsection 2.2.1.3. An attempt to such generalization was already done in [59], where some of the possible interaction vertices were found. We go further: we want to find all spin-foams with this certain boundary, and order them in a well behaving expansion.

In order to do that we apply the operator spin-network diagrams formalism, which was developed as an answer to the question posed in section §2.1, and which is presented in chapter 3 and in [53]. We used it to develop a general algorithm to find all diagrams with some specified properties, such as: total number of vertices, total number of edges, structure of the boundary graph. Moreover the diagrammatic structure of our framework allows to introduce convenient Feynman-like rules to read the transition amplitude out of a diagram.

We applied the algorithm to find all diagrams with one vertex and minimal number of internal edges and with the boundary being the dipole graph. The Feynman-EPRL rules allowed us to estimate the contribution of most of them to the total amplitude in large-j-limit. These results, together with the algorithm to find all diagrams with given properties, were published in [64] and they are presented in chapter 4.

# 2.3 How to identify and compute the divergent part of Spin-Foam amplitudes?

In spin-foam models it might happen, that some foams give infinite contribution to the transition amplitude. This is usually caused by so called *bubbles* - fragments of 2-complexes forming closed surfaces. The bubbles are spin-foam analogues of loops in Feynman diagrams.

In this thesis we focus on two aspects of bubbles. First is to identify the spinfoams containing bubbles - which is discussed more in subsection 2.3.1 and answered in chapter 5. Second is an issue of regularisation one specific example of bubble - which is discussed in subsection 2.3.2 and answered in chapter 6.

# 2.3.1 A spin-foam with a bubble

As we have already noticed, a total spin-foam transition amplitude is a sum over all possible 2-complexes of amplitudes of corresponding spin-foams:

$$Z = \sum_{\kappa} Z_{\kappa} \tag{2.25}$$

It appears, that some of these amplitude may diverge, when summed over all possible colorings. This situation highlights the necessity of applying renormalization techniques to the theory. Developing a renormalization of spin-foam models is a huge project, which far exceeds the scope of this thesis. Nevertheless, one of the necessary steps in this direction was done: we found a way to identify the part of the spin-foam that admit infinite number of colorings giving nontrivial amplitude and thus is a candidate for the divergent part.

We start with some general remarks about the state of the research (in subsection 2.3.1.1). Then in subsection 2.3.1.2 we present the problem that we have studied.

# 2.3.1.1 Why bubbles cause divergence

The divergent parts of spin-foam models was already studied [65, 66, 67, 68, 69, 70], however so far the research were limited only to spin-foams dual to a triangulation of a 4-manifold. Since we have shown, that spin-foam models should be defined for more general class of 2-complexes, one should generalize the research to the extended case.

Most of bubbles admit infinite number of colorings giving non-trivial contribution to the transition amplitude for a fixed coloring of the boundary. This is not the case for bubbleless spin-foams, where the spins coloring of the internal faces is limited (via triangle-inequalities) by combinations of the boundary links spin-labels. One could attempt to solve this divergence problem by coloring spin-foams with representations of the quantum group  $SU(2)_q$  for  $q \in [0, 1[$ , where the total number of representations is limited by the q-parameter [71, 72, 73, 74], i.e. let  $q = e^{-\kappa}$ , then the maximum spin is less then  $\frac{4\pi}{\kappa\gamma}$ , where  $\gamma$  is the Barbero-Imirzi parameter. However, the deformation parameter has interpretation of exponent of the cosmological constant in Planck units:

$$q = e^{-\frac{\ell_{\rm Pl}^2}{\ell_c^2}} \quad \text{where} \quad \ell_c^2 := \frac{1}{\Lambda} \tag{2.26}$$

(see [73]) so the maximum spin is bounded by

$$j_{\rm max} < \frac{1}{\Lambda} \frac{4\pi}{\ell_{\rm Pl}^2 \gamma} \tag{2.27}$$

Nevertheless, since  $\Lambda \approx 10^{-120} \ell_{\rm Pl}^{-2}$  [75, 76], such regularization in general does not help to the problem of huge contribution of the order of  $10^{120}$  from the bubble parts. Thus the issue of renormalization is still unresolved.

#### 2.3.1.2 How to find a bubble sub-foam?

The first step to renormalize the theory is to identify and characterize the possible sources of divergences. As we have noticed, a natural candidate to the source of divergence is a bubble. So far bubbles were defined as a *subset of faces of a spin-foam, that form a closed surface* [77, 69, 65]. Such definition is neither precise nor applicable to operator spin-network diagrams framework, which we introduced in chapter 3.

We introduced a strict definition of a bubble in terms of operator spin-network diagrams framework, together with an procedure that allows to find the bubble subdiagram. Moreover, we introduce a notion of *rank of a bubble* - the number simple bubbles for which the bubble can be decomposed - together with another procedure to find it. We study several examples showing, how our algorithms works. Our results are presented in chapter 5 (the publication is in preparation).

# 2.3.2 A study of an example of a bubble

One example of spin-foam with bubbles was already studied in details [78, 79]. It is so called *melonic bubble*. It can be interpreted as a self-energy-like correction to the transition amplitude of a 4-valent edge of a spin-foam. The recent research led to a regularization of the transition amplitude together with a derivation showing, that the operator corresponding to the edge with a bubble is proportional to

$$W_{\text{melonic}}^{\Lambda} = \ln\left(\Lambda\right) \cdot \mathbb{T}^2 \tag{2.28}$$

for some operator  $\mathbb{T}$ , where  $\Lambda$  is the regularization constant. We studied the leading order of the operator  $\mathbb{T}$ .

A brief presentation of the regularization procedure is recalled in subsection 2.3.2.1. The problem of the operator  $\mathbb{T}$  is presented in subsection 2.3.2.2, and our solution is described in chapter 6.

# 2.3.2.1 Riello's melonic bubble

Consider a spin-network containing a 4-valent node  $n_0$ . This node may evolve into a spin-network in which all nodes and links are unchanged, except n, which is split into 4 new nodes  $n_1, \ldots, n_4$  - see figure 2.5a. Now if in the next step of evolution the nodes  $n_1, \ldots, n_4$  join into one node  $n'_0$ , the spin-foam representing these two steps would contain the so called *melonic bubble*. - see figure 2.5b.

Let us now fix the spins at the boundary spin-network, but not at the internal faces of the bubble (i.e. the spins  $j_{12}, j_{13} \ldots, j_{34}$  can be arbitrary) and consider an Operator Spin-foam of such process. It is easy to see, that it factorises into product over the nodes:

$$W_{\kappa} = \bigotimes_{n \in \Gamma} \widehat{P_n} \tag{2.29}$$

The operators related to all the nodes  $n \neq n_0$  are simply the projectors onto the invariant node Hilbert space  $\hat{P}_n^{\text{Inv}}$ . The operator  $\widehat{P}_{n_0}$  is a composition of the bubble amplitude  $W_{\text{bubble}}$  sandwiched by  $\hat{P}_{n_0}^{\text{Inv}}$ -operators:

$$\widehat{P_{n_0}} = \hat{P}_{n_0}^{\text{Inv}} W_{\text{bubble}} \hat{P}_{n_0}^{\text{Inv}}$$
(2.30)



Figure 2.5: An example of a quantum process represented by a spin-foam with the melonic bubble. (a) - A 4-valent node  $n_0$  splits into four 4-valent nodes  $n_1, \ldots, n_4$ ; the nodes  $n_5, \ldots, n_8$  evolve unchanged to  $n'_5, \ldots, n'_8$ . (b) - The nodes  $n_1, \ldots, n_4$  join back to  $n''_0$ ; the other nodes evolve unchanged to  $n''_5, \ldots, n''_8$ .

where

$$W_{\text{bubble}} = \sum_{\{j_{12},\dots,j_{34}\}} W_{\text{bubble}} (j_{12},\dots,j_{34})$$
(2.31)

The regularization introduced in [78] introduces a cutoff on the internal spins, i.e. one assumes that  $j_{12}, \ldots, j_{34} < \Lambda$ . Moreover one assumes, that the spins on the external faces of the bubble are also high, but they are much smaller, than the cutoff, i.e.  $1 \ll j_1, j_2, j_3, j_4 \ll \Lambda$ . Under such assumption one can estimate the  $W^{\Lambda}_{\text{hubble}}$ . The details of this estimation can be found in [78], they are not relevant to the further parts of the thesis, so let us recall only the result, which is

$$W_{\text{bubble}}^{\Lambda} \sim f^{\mu}(\Lambda) \int_{SL(2,\mathbb{C})^2} \mathrm{d}g \mathrm{d}g' \prod_{\ell=1}^{4} \langle m_{\ell} | Y^{\dagger}g Y Y^{\dagger}g' Y | n_{\ell} \rangle_{j_{\ell}}$$
(2.32)

where  $\mu$  is the degree of the polynomial  $A_f(j)$  of the face amplitude in the model considered and

$$f^{\mu}(\Lambda) := C \sum_{x=0}^{\Lambda} x^{6(\mu-1)-1} \cong \begin{cases} \tilde{C} \cdot \Lambda^{6(\mu-1)} & \mu \neq 1\\ \tilde{C} \cdot \ln(\Lambda) & \mu = 1 \end{cases}$$
(2.33)

for some constants C and  $\tilde{C}$  (since we consider  $\Lambda \gg 1$ , only the leading order in  $\Lambda$  is taken into account in (2.33)). For the standard face amplitude  $A_f(j) = 2j + 1$ , so the parameter  $\mu = 1$  and we have

$$W_{\text{bubble}}^{\Lambda} \sim \ln\left(\Lambda\right) \int_{SL(2,\mathbb{C})^2} \mathrm{d}g \mathrm{d}g' \prod_{\ell=1}^{4} \left\langle m_{\ell} \right| Y^{\dagger} g Y Y^{\dagger} g' Y \left| n_{\ell} \right\rangle_{j_{\ell}}$$
(2.34)

#### 2.3.2.2Lorentzian Polyhedra Propagator

The integral, that appears in (2.34), is a matrix element of a certain operator acting at  $\mathcal{H}_n$ . One can easily separate the integrals and obtain

$$W_{\text{bubble}}^{\Lambda} \sim \ln(\Lambda) \langle m_1, m_2, m_3, m_4 | \mathbb{T}^2 | m_1, m_2, m_3, m_4 \rangle_{j_1, j_2, j_3, j_4}$$
(2.35)

for

$$\mathbb{T} := \int_{SL(2,\mathbb{C})} \mathrm{d}g \,\prod_{\ell=1}^{4} \left[ Y^{\dagger} g Y \right]_{j_{\ell}} \tag{2.36}$$

where  $[Y^{\dagger}gY]_j := D^{(\gamma j,j)}(g)_{j,n}^{j,m}$ . The  $\mathbb{T}$  operator seams to be of key importance to deal with the melonic bubble at an edge of a spin-foam. Knowing it's exact form one could renormalize the edge operator by modifying the bare operator  $\widehat{P_n}$  by a "radiative correction" coming from the bubble. Thus we decided to investigate the properties of  $\mathbb{T}$ .

We calculated the leading order of  $\mathbb{T}$  operator under the assumption, that the spins on the external faces are much bigger then 1. It appeared to be proportional to the identity operator, with the proportionality factor being a function of the spins  $j_1 \ldots, j_4$ . The details of our calculation are presented in chapter 6.

Note, that the same operator's matrix elements appeared in the Dipole Cosmology transition amplitude in the Lorentzian model - see subsection 2.2.1.5. Indeed, our calculation can be applied also to this model.

Part II My own work

# Chapter 3

# **Operator Spin-network Diagrams**

The KKL procedure (recalled in subsection 2.1.2, first presented in [38, 43]) of calculating spin-foam vertex amplitudes provides a tool to decompose a spin-foam into a set of graphs with links colored by spins and nodes colored by operators. In this decomposition each vertex of a spin-foam is represented by a graph, each edge ending at this vertex is represented by a node of this graph, and each face sharing this edge is represented by a link incident to the corresponding node.

An extra structure emerges from this procedure. Consider an *internal* edge, i.e. an edge that has two internal vertices at its ends (in contrary to an *external* edge - namely an edge with one internal and one boundary vertex at its ends, and a *boundary* edge - an edge with two boundary vertices at its ends). Such edge is represented by precisely two nodes of graphs. Since both nodes represent the same edge, their structure must be related - i.e. if a node n has k incoming and l outgoing links, the node n' related to it has to have l incoming and k outgoing links. These links represent the faces incident to the edge we consider, thus each link incident to n is strictly associated with a link incident to n' representing the same face. The *external* edges in this language are represented by a single node of a graph, because they have only one internal vertex on their ends.

Thus the structure that emerges from KKL procedure is a set of graphs with some extra relations between nodes and links of these graphs. Appropriate coloring of this structure provides all data necessary to compute the transition amplitude of a spin-foam.

The key concept of our framework was to inverse this procedure: since all the data one need to compute a spin-foam amplitude is encoded in such structure, why not to start from such structure an reconstruct the spin-foams out of it? We gave a name of graph diagram to the topological structure of our framework, and the structure colored by SU(2)-tensors is named operator spin-network diagram. We developed the tools necessary to operate on them.

Very soon a question arose: are there any restrictions on graph diagrams? In other words: given a general graph diagram - defined as an arbitrary collection of graphs with nodes and links connected in some way (that will be precised soon) - is it always a graph diagram of some 2-complex? We addressed this question and answered it by providing an algorithm showing how to reconstruct a 2-complex out of an arbitrary graph diagram - no hidden assumptions on graph diagram are taken, thus all diagrams are admissible.

The similar question arises for coloring of diagrams and spin-foams. Here the answer is: our framework is more general than the standard spin-foam models. It admits the BF and EPRL models if certain constraints on the coloring are taken. However one can easily generalise it to other schemes of computing transition amplitudes. Our framework gives one of possible answers to the question raised in section §2.1: what class of 2-complexes should be used in spin-foam models? Our answer is: all 2complexes that can be constructed out of an operator spin-network diagram - it will be explained in the discussion at the end of this chapter.

This chapter is organised as follows. First of all we provide a brief mathematical introduction: definitions of most needed notions from CW-complex theory and from harmonic analysis on SU(2) group (the full mathematical context - the definitions, theorems and references to the textbooks - can be found in Appendix A). Then in section  $\S3.2$  we study the topological aspects of our framework. We introduce the definition of graph diagrams and describe their structure and basic properties. Then we explain the algorithm of constructing a 2-complex out of a graph diagram. In section §3.3 we study the computational properties of our framework. First we describe colorings of graph diagrams by SU(2)-tensors, then we explain the procedure of diagram-driven contraction of tensors, which lead us to a formula on a transition amplitude operator of a diagram. Finally we explain, how to obtain a transition amplitude compatible with various spin-foam models. In section §3.4 we study some special cases of graph diagrams: trivial diagram and one-interaction diagram. We also explain the procedure of gluing the diagrams. In fact each diagram can be decomposed into a number of gluings of trivial diagrams and oneinteraction diagrams, thus these examples cover all possible cases. In the next chapter we will use these examples to develop more sophisticated techniques, that will be used in finding higher order corrections to transition amplitudes of the Dipole Cosmology model. Finally, in section \$3.5, we conclude and discuss the answer to the question raised in section  $\S2.1$ .

This chapter is based mainly on [53] (some partial results were also presented in [80, 81]). The work was done in collaboration with mgr Marcin Kisielowski. The research on the topological structure of the framework in section §3.2 was done mainly by the Author. The analysis of tensor structure of Operator Spin-network Diagrams in section §3.3 is mostly the result of Marcin Kisielowski's work, thus they are recalled with less details. The examples in section §3.4 are the common result.

# 3.1 Introduction

# 3.1.1 The idea: to inverse the KKL procedure

The key concept of the KKL procedure is to take a little neighbourhood of each vertex (i.e. a contractible neighbourhood containing only one vertex, and having nonvanishing intersection only with edges and faces, that are incident to the vertex). Each such neighbourhood can be treated as a 2-complex with a boundary, and the boundary of this 2-complex is a graph that encodes the structure of the vertex.

This procedure can be visualised as follows. Since each vertex of a 2-complex has a neighbourhood, that can be embedded in  $\mathbb{R}^4$ , let us embed it and then draw a sphere  $S_r^3$  around it (r is the radius of the sphere). The topology of the intersection of  $S_r^3$  and the 2-complex may depend on r, however there is  $r_0 > 0$  such that for  $r < r_0$  the topology of the intersection does not depend on r and it is the topology of some graph. This graph is precisely the graph defined above.

Consider now the inversed procedure: given a set of graphs, draw each graph on a sphere  $S^3$ . Then shrink each sphere radially to its middle point. The trace of the graph will determine a fragment of a 2-complex being a neighbourhood of a vertex. Moreover the structure of the vertex will be indicated by the graph - it will be precisely the same,

as if the graph were obtained from this vertex via KKL procedure.

Now if some nodes of the graphs we had were gathered in pairs, and the structure of nodes in each pair is compatible (to be defined later), one can glue the fragments of 2-complexes obtained in above shrinking procedure along fragments of their boundary - neighbouring the nodes in a pair (a procedure of relating links incident to the node is also needed). One can perform such gluing for each such pair, resulting a more complicated 2-complex.

The concept of graph-diagrams concretises the intuitions presented above: it gives precise definition of *compatible* nodes and of relating links incident to the nodes. It describes in detail the glueing procedure and proves uniqueness of the result of glueing. As a result we end up extracting the minimal set of data necessary to construct a spinfoam. We find the structure of this data more convenient, then the traditional 2-complex approach, for at least two reasons. First of all: even though graphs do not have to be planar, they are far easier to draw then 2-complexes - the later ones require 3d- and sometimes even 4d-imagination (there are 2-complexes nonembeddable in  $\mathbb{R}^3$ ). Moreover the diagram structure is very closely related to the index structure of the coloring of a diagram - thus reading the transition amplitude and finding, which indices should be contracted with each other is natural. Of course there is a drawback of our framework. A spin-foam has its natural interpretation as a time evolution of a spin-network, which is not explicit in graph-diagrams approach.

# 3.1.2 Mathematical introduction

The detailed mathematical introduction can be found in Appendix A. Here let us fix some conventions and briefly recall definitions of non-standard notions.

#### 3.1.2.1 Graphs

Graphs will be denoted by  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$ , where  $\mathbf{N} = \{n_1, \ldots, n_N\}$  is the set of nodes and  $\mathbf{L} = \{\ell_1, \ldots, \ell_L\}$  is the set of links. All graphs are oriented, unless explicitly stated otherwise. Given a link  $\ell$  its source node is denoted by  $n = s(\ell)$  and its target node by  $n' = t(\ell)$  (some links might be loops, i.e. such that  $s(\ell) = t(\ell)$ ). All graphs (except so called squids) are closed, which means that each node has at least two links incident to it. In general graphs are not required to be connected. However for each graph one can define its decomposition into connected components. We will write  $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_K\}$ if  $\mathcal{G} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_K$  and each of graphs  $\Gamma_i$  is connected itself. General graphs will be denoted by  $\mathcal{G}$  letter, connected graphs will be denoted by  $\Gamma$ . A graph dual to a graph  $\mathcal{G}$  is a graph  $\mathcal{G}^*$  with the same set of nodes and links, as  $\mathcal{G}$ , but orientation of each link reversed.

A useful notion of a half-link of a graph is introduces in Appendix A.1.1. A half-link is a pair of a link and a sign  $\ell^{\epsilon} = (\ell, \epsilon) \in \mathbf{L} \times \{+, -\}$ . A half-link defines a decomposition of a link into two halves, without introducing any extra nodes. A half-link is incident to only one node, called its boundary  $n = \partial \ell^{\epsilon}$ . For the positive half-link we have  $\partial \ell^+ = s(\ell)$ and for the negative half-link we have  $\partial \ell^- = t(\ell)$ . Set of all half-links of a node n is denoted by  $\mathcal{L}_n$  (possibly with a superscript + or -). A set of all half-links of a graph  $\mathcal{G}$ is denoted by  $\mathcal{L}_{\mathcal{G}}$ .

Half-links allows to introduce a notion of duality of nodes: two nodes n, n' are dual, iff number of positive half-links of n equals to number of negative half-links of n' and vice versa. For each pair of dual nodes there is at least one *duality map*  $\phi_{n,n'}$ :  $\mathcal{L}_n \to \mathcal{L}_{n'}$ , i.e.



Figure 3.1: Graphs. (a) - A graph  $\mathcal{G} = \{\Gamma_1, \Gamma_2\}$ . The source of the link  $\ell_1$  is  $s(\ell_1) = n_I$ , the target is  $t(\ell_1) = n_{II}$  (b) - Half-links of the node  $n_{III}$  are highlighted (positive by red, negative by blue). (c) - The nodes  $n_{IV}$  and  $n_V$  are dual. The dotted line describe a duality map between them.

a bijective map such that  $\phi_{n,n'}(\mathcal{L}_n^+) = \mathcal{L}_{n'}^-$  and vice versa (for nodes of valency higher then 2 there are more than one such map).

A graphical explanation of the above definition can be found at figure 3.1.

Given a graph  $\mathcal{G}$  there is a group Aut ( $\mathcal{G}$ ) of automorphisms of the graph. In order to be an automorphism a map  $\phi = (\phi^{(0)}, \phi^{(1)})$  must be a pair of a bijection  $\phi^{(0)}$  on the set of nodes  $\mathbf{N}_{\mathcal{G}}$  and a bijection  $\phi^{(1)}$  on the set of links  $\mathbf{L}_{\mathcal{G}}$  and it must satisfy the following condition:

$$\forall_{\ell \in \mathbf{L}_{\mathcal{G}}} \ s\left(\phi^{(1)}(\ell)\right) \phi^{(0)}\left(s(\ell)\right) \wedge t\left(\phi^{(1)}(\ell)\right) = \phi^{(0)}\left(t(\ell)\right)$$
(3.1)

Each automorphism has the following property (called the *node-consistency property*): each node of  $\mathcal{G}$  is mapped onto a node with the same structure of incident links. Obviously there is always at least one automorphism of  $\mathcal{G}$ , namely the trivial automorphism. Other automorphisms represent the symmetries of  $\mathcal{G}$ .

Consider now a map  $\tilde{\phi} : \mathcal{G} \to \mathcal{G}^*$  being a composition of an automorphism of  $\mathcal{G}$  with the inversion of all the links of  $\mathcal{G}$ . From the node-consistency property one can see, that  $\tilde{\phi}$  maps each node of  $\mathcal{G}$  onto a dual node of  $\mathcal{G}^*$ . We call such map a *duality map of a* graph induced by an automorphism. Obviously  $\tilde{\phi}$  induces a duality map of half-links of each pair of nodes mapped one to another.

#### Squids and squid-graphs

A squid is a very specific kind of a graph defined as  $\lambda = (\mathbf{N}_{\lambda}, \mathbf{L}_{\lambda})$  with  $\mathbf{N}_{\lambda} = \{n\} \cup \{x_1, \ldots, x_k\}$  and  $\mathbf{L}_{\lambda} = \{\ell_1, \ldots, \ell_k\}$  such that each link  $\ell_i$  connects the node n and the node  $x_i$ . Example of a squid is on the figure 3.2a. The node n is called the *head* of the squid. The links  $\ell_i$  are called *legs* of the squid, the nodes  $x_i$  are called *leg-nodes*. A leg  $\ell_i$  is called *outgoing* iff  $s(\ell_i) = n \wedge t(\ell_i) = x_i$ . A leg is called *incoming* iff  $s(\ell_i) = x_i \wedge t(\ell_i) = n$ . Valence of a squid is the number of its legs k. One can also introduce the positive valence  $k^+$  being the number of outgoing legs, and the negative valence  $k^-$  being the number of incoming legs. (see Appendix A.1.1.4). Squids are always open graphs.

Given a graph  $\mathcal{G}$  one can define the squid-graph  $\gamma_{\mathcal{G}}$  as the pair  $(\mathcal{G}^{(s)}, S)$ , where

•  $\mathcal{G}^{(s)}$  (called the *split graph*) is a graph obtained by splitting each link  $\ell_i$  of  $\mathcal{G}$  into two links  $\ell_i^s$  and  $\ell_i^t$  by putting an extra node  $x_{\ell_i}$  in the interior of each link (the



Figure 3.2: Squid-graphs. (a) - An example of 6-valent squid, with  $k^+ = 4$  and  $k^- = 2$ . (b) - A graph  $\Gamma$  that we will turn into a squid-graph. (c) - The split graph  $\Gamma^{(s)}$  of the graph  $\Gamma$ . One of the squids, i.e. the squid  $\lambda_n$  of the node n of  $\Gamma$ , is emphasised.

link  $\ell_i^s$  starts at  $s(\ell_i)$  and ends at  $x_{\ell_i}$ , the link  $\ell_i^t$  starts at  $x_{\ell_i}$  and ends at  $t(\ell_i)$ ).

• S is the squid-set of the graph  $\mathcal{G}$ , i.e. for each node n of the graph  $\mathcal{G}$  there is one squid  $\lambda_n \in S$  such that n is the head of  $\lambda_n$ , and the legs of  $\lambda_n$  are links of  $\mathcal{G}^{(s)}$  incident to n (in fact they are in 1-to1 correspondence with half-links of  $\mathcal{G}$ ).

A procedure of obtaining a squid-graph of a graph is illustrated on figures 3.2b-3.2c.

**Remark 3.1.** Later we will need to know the precise graph structure of  $\mathcal{G}^{(s)}$ . Let  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$ , then

$$\mathcal{G}^{(s)} = (\mathbf{N} \cup \mathbf{X}, \mathbf{L}^+ \cup \mathbf{L}^-) \tag{3.2}$$

where  $\mathbf{X} = \{x_{\ell_i} \in \ell_i : \ell_i \in \mathbf{L}\}$  is the set of middle points of the links of  $\mathcal{G}$ , and the sets  $\mathbf{L}^{+/-}$  contain the links of  $\mathcal{G}^{(s)}$  representing the positive/negative half-links of  $\mathcal{G}$  respectively, i.e.  $\mathbf{L}^+ = \{\ell_i^s : \ell_i \in \mathbf{L}\}$  and  $\mathbf{L}^- = \{\ell_i^t : \ell_i \in \mathbf{L}\}$ .

### 3.1.2.2 2-complexes

The 2-complexes are CW-complexes with maximal dimension of cells equal 2. We shell use the definition of CW-complexes consistent with [82]. It can be also find in the appendix of [53] (some minor changes were done in the definition, they are explained in Appendix A.1.2).

A 2-complex  $\kappa = (\mathbf{F}, \mathbf{E}, \mathbf{V}; \mathbf{f}_2, \mathbf{f}_1)$  is a triple of sets: of faces  $\mathbf{F}$ , edges  $\mathbf{E}$  and vertices  $\mathbf{V}$ , together with a pair of boundary maps  $\mathbf{f}_1 : \partial \mathbf{E} \to \mathbf{V}$  and  $\mathbf{f}_2 : \partial \mathbf{F} \to \mathbf{E}$  defining how the 1- and 2-cells are glued onto lower dimensional skeletons. There are two so called boundary relations  $\sim_m (m = 1, 2)$  defined by the boundary functions  $\mathbf{f}_m$ . Two points  $x, y \in \partial \mathbf{E}$  are in  $\sim_1$  relation iff  $\mathbf{f}_1(x) = \mathbf{f}_1(y)$  (and likewise for  $x, y \in \partial \mathbf{F}$  with  $\sim_2$  and  $\mathbf{f}_2$ ). The topological space  $X_{\kappa}$  of a 2-complex  $\kappa$  is  $X_{\kappa} := (\bigsqcup_{\mathbf{F}} f) / \sim_2$  (see Appendix A.1.2.1).

A face  $f \in \mathbf{F}$  is called *incident* to an edge  $e \in \mathbf{E}$  iff  $e \subset \mathbf{f}_2(\partial f)$ . An edge  $e \in \mathbf{E}$  is called *incident* to a vertex  $v \in \mathbf{V}$  iff  $v \in \mathbf{f}_1(\partial e)$ . The *incidency* relation is symmetric (if a face is incident to an edge, than the edge is incident to this face) and transitive (i.e. a



Figure 3.3: Removable cells. (a) - A simple oriented 2-CW-complex. Orientations of faces is given by black round arrows. Induced orientation of edges is given by green arrows. The edge  $e_1$  can be oriented with respect to  $f_1$  (see the upper arrow) and with respect to  $f_2$  (the lower arrow). The edge  $e_2$  can be oriented with respect to  $f_2$  (the left arrow) and with respect to  $f_3$  (the right arrow). The boundary edges are oriented uniquely. (b) - The faces  $f_2$  and  $f_3$  are oriented consistently, because the orientations induced by them on  $e_2$  are opposite. Thus  $e_2$  is a removable cell. The edge  $e_1$  is not removable. The vertex V was not removable when the edge  $e_2$  was present, but now it is removable (we can call it pre-removable, as it was defined in Appendix A.1.2.3, but we will often call it also removable).

face may be incident to a vertex - if there is an edge incident to both of them). Two faces are called *adjacent* iff there is an edge incident to both of them. Similarly two edges are *adjacent* iff there is a vertex incident to both of them.

An edge is called a *boundary edge* iff there is precisely one face incident to it. A vertex is a *boundary vertex* iff it is incident to a boundary edge. A boundary  $\partial \kappa$  is a subcomplex of  $\kappa$  build of all boundary edges and vertices.

All 2-complexes we shall use are oriented, which means that for each face  $f \in \mathbf{F}$  we introduce its orientation (each face treated as a topological manifold is a disc - thus it is orientable). Orientation of an edge is defined only with respect to one of the faces incident to it (except for the boundary edges - they can be oriented uniquely, and thus the boundary graph is an oriented graph). We say, that two adjacent faces f, f' are oriented consistently iff for each edge e incident to both of them the orientation of e induced by f is opposite to the orientation of e induced by f'.

We say, that a vertex is removable iff it is incident to precisely two edges. We say, that an edge is removable iff it is incident to precisely two faces, these two faces are oriented consistently and none of its ending vertices is removable (see figure 3.3). There is a strict procedure of removing such cells (see Appendix A.1.2.3).

Whenever in this section we will say that two 2-complexes  $\kappa$  and  $\kappa'$  are equivalent, we will mean that they are the same oriented 2-complex up to a number of removable cells. In other words the topological spaces  $X_{\kappa}$  and  $X_{\kappa'}$  are homeomorphic and the orientations of 2-cells agrees. It is so called *weak equivalence*, of CW-complexes, in contrary to *strong equivalence*, which requires the same structure of the skeleton of 2-complex at all dimensions (see definitions A.19 and A.20).

Consider a graph  $\mathcal{G}$  and its squid-graph  $\gamma_{\mathcal{G}} = (\mathcal{G}^{(s)}, S)$  and a 2-complex  $\kappa$  such that  $\partial \kappa = \mathcal{G}^{(s)}$ . We can naturally introduce an extra structure on the boundary of  $\kappa$  - the decomposition into squids. The pair  $(\kappa, S)$  will be called a *squid-2-complex*.

# **3.1.2.3** SU(2)-tensors

Graph diagrams will be colored by SU(2)-tensors. These tensors are described in Appendix A.3.2. Here let us recall some basic objects and introduce some shortcut names for mostly used notions.

First of all: we will often call the representation Hilbert space  $\mathcal{H}_j$  shortly as *j*-representation.

Consider now a node n with  $k^+$  positive half-links and  $k^-$  negative half-links, each half-link labelled by a representation  $j_{\ell}$ . The Hilbert space  $\bigotimes_{\ell^+} \mathcal{H}_{j^+} \otimes \bigotimes_{\ell^-} \mathcal{H}_{\ell^-}$  will be called the *node Hilbert space* and denoted by  $\mathcal{H}_n$ . The SU(2)-invariant part of  $\mathcal{H}_n$  (i.e. Inv  $[\bigotimes_{\ell^+} \mathcal{H}_{j^+} \otimes \bigotimes_{\ell^-} \mathcal{H}_{\ell^-}]$  - see Appendix A.3.2) we will call *invariant node Hilbert space* and denote by  $\mathcal{H}_n^{\text{Inv}}$ . For each  $\mathcal{H}_n$  there is a natural projection map  $\widehat{P_n^{\text{Inv}}} : \mathcal{H}_n \to \mathcal{H}_n^{\text{Inv}}$ , called the *invariant projection*. Of course the projection becomes the unity operator, when its domain is restricted to  $\mathcal{H}_n^{\text{Inv}}$ , however we will often call it anyway the *projection*, even if the domain will be restricted (this is to keep our notation consistent with [53]).

Consider now a spin-network state with the underlying graph  $\Gamma$ . For fixed spins on the links the Hilbert space of such states is a product of the invariant node Hilbert spaces of each node of this graph and is denoted by

$$\mathcal{H}_{\Gamma,\vec{j}} := \bigotimes_{n \in \mathbf{N}_{\Gamma}} \mathcal{H}_n^{\mathrm{Inv}}$$
(3.3)

(we will not consider any non-invariant spin-network states in this chapter). The total Hilbert space of all spin-network states of this graph is  $\mathcal{H}_{\Gamma} = \bigoplus_{\vec{j}} \mathcal{H}_{\Gamma,\vec{j}}$ . The Hilbert space dual to  $\mathcal{H}_{\Gamma}$  will be called the *contractor space* of the graph  $\Gamma$ , and elements of it we will call *contractors*  $\langle A |$ :

$$\langle A | \in \mathcal{H}_{\Gamma}^* \tag{3.4}$$

Of course for fixed spins there is a *fixed-spin contractor space*  $\mathcal{H}^*_{\Gamma,\vec{j}}$ .

Given a spin-network Hilbert space  $\mathcal{H}_{\Gamma,\vec{j}}$  and an automorphism  $\phi$  of the graph  $\Gamma$  we say, that  $\phi$  is a *spin-network automorphism* iff each link of  $\Gamma$  is mapped by  $\phi$  onto a link colored by the same spin. The group of spin-network automorphisms is smaller than the corresponding group of graph-automorphisms, since there are more constraints on them. However there is always at least the trivial one.

# 3.2 Graph diagram - the underlying topological structure

# 3.2.1 Graph Diagram - definition

The original definition of graph diagram [53] was given in terms of squid-graphs and maps between squids. Here we will first provide an improved definition, based on the notion of half-links of a graph. Then we will explain, how to translate it to the language of squidgraphs. Finally we will define two relations that can be read out of a graph diagram (so called edge- and face- relation), which play key role in translating graph diagrams to spin-foams (however they are not necessary in defining the diagrams).

# 3.2.1.1 The definition

# **Definition 3.1.** Graph diagram

Let us define a graph diagram  $\mathcal{D} = (\mathcal{G}, \mathcal{R})$  as a graph  $\mathcal{G}$  and a family of relations  $\mathcal{R} = (\mathcal{R}_{node}, \mathcal{R}_{link})$ , where

- $\mathcal{R}_{node}$  is a symmetric relation in the set of nodes N of the graph  $\mathcal{G}$ , such that for each node  $n \in \mathbb{N}$  one of the following possibilities take place:
  - either n is in relation with precisely one node n', such that  $n' \neq n$  and n' is dual to n
  - or n is not in relation with any node.

In other words:

$$\forall_{n \in \mathbf{N}} \left( \exists !_{n' \in N \setminus \{n\}}(n, n') \in \mathcal{R}_{\text{node}} \right) \lor \left( \nexists_{n' \in N}(n, n') \in \mathcal{R}_{\text{node}} \right) \quad \text{and} \quad \forall_{n \in \mathbf{N}} \left( n, n \right) \notin \mathcal{R}_{\text{node}}$$
(3.5)

•  $\mathcal{R}_{link}$  is a family of relations:

$$\mathcal{R}_{\text{link}} = \left\{ \mathcal{R}_{\text{link}}^{(n,n')} : (n,n') \in \mathcal{R}_{\text{node}} \right\}$$
(3.6)

where each relation  $\mathcal{R}_{link}^{(n,n')}$  is a relation on the sets  $\mathcal{L}_n$  and  $\mathcal{L}_{n'}$  induced by one of the duality maps between the nodes n and n'. In other words for each pair of nodes (n, n') related by the node relation we pick one of the duality maps  $\phi_{n,n'}$  and define

$$\mathcal{R}_{\text{link}}^{(n,n')} := \left\{ (\ell^{\epsilon}, \ell^{\prime-\epsilon}) \in \mathcal{L}_n \times \mathcal{L}_{n'} : \phi_{n,n'}(\ell^{\epsilon}) = \ell^{\prime-\epsilon} \right\}$$
(3.7)

where  $-\epsilon = +$  iff  $\epsilon = -$  and vice versa.

An example of a graph diagram is presented an described at figure 3.4.

### 3.2.1.2 Translation to squid-graphs

Given a graph-diagram  $(\mathcal{G}, \mathcal{R})$  one can naturally translate it to the language of squidgraphs. A squid-graph diagram is defined as follows:

#### **Definition 3.2.** Squid-graph diagram

A squid-graph diagram is a pair  $(\gamma, \Phi)$  of a squid-graph  $\gamma$  and a collection of a duality maps between pairs of squids, such that each squid  $\lambda \in S$  is either a domain or an image of at most one duality map  $\phi \in \Phi$ .

The graph  $\mathcal{G}$  has its squid-graph  $\gamma_{\mathcal{G}}$  - see subsection 3.1.2.1. Each pair of nodes in node relation indicates a pair of dual squids. The link relation indicates the duality map of the pair of squids representing these nodes. Thus instead of a graph and a family of relations one can consider a squid-graph and a family of duality maps between the squids. This definition is used in [53].



Figure 3.4: An example of a graph diagram. Dashed lines show the node-relation, dotted lines show the link-relations.

# 3.2.1.3 Edge- and face-relations

It is convenient to introduce two more relations on a graph-diagram: the edge-relation  $\mathcal{R}_{edge}$  and the face-relation  $\mathcal{R}_{face}$ . They are named after faces and edges of 2-complex, because equivalence classes of them will be in 1-to-1 correspondence with the internal cells of the 2-complex constructed out of a graph diagram (see subsection 3.2.2).

### **Definition 3.3.** Edge-relation

The edge-relation  $\mathcal{R}_{edge}$  of a graph-diagram is the smallest equivalence relation on the set **N** of nodes of the graph  $\mathcal{G}$ , such that the node relation  $\mathcal{R}_{node}$  is a sub-relation of it.

The simplest way to characterise a relation is to list its equivalence classes. In case of  $\mathcal{R}_{edge}$  there are two types of equivalence classes:

- 1. For each pair of nodes (n, n') being in node relation there is a two-element equivalence class  $\{n, n'\}$  of edge relation. We will call it an *internal* equivalence class, or an *internal edge*.
- 2. For each node n that do not belong to any pair of the node relation there is a single element equivalence class  $\{n\}$  of the edge relation. We will call it an *external* equivalence class, or an *external edge*.

The names internal and external edge are explained in subsection 3.2.3.1.

Knowing  $\mathcal{R}_{edge}$  one can reconstruct  $\mathcal{R}_{node}$  and vice versa. The node-relation is easier to define and to use in construction of the 2-complex out of the graph diagram, however the edge relation will be more convenient in further use.

# **Definition 3.4.** Face-relation

The face relation  $\mathcal{R}_{face}$  of a graph diagram is the smallest equivalence relation on the set of half-links of the graph  $\mathcal{G}$ , such that

- each two half-links of the same link are in face relation,
- whenever there is a relation  $\mathcal{R}_{link}^{(n_1,n_2)}$  such that  $\ell_1^+$  is in relation with  $\ell_2^-$ , then  $\ell_1^+$  and  $\ell_2^-$  are in face-relation.

Again it is easier to characterise  $\mathcal{R}_{\text{face}}$  by listing the equivalence classes of it. There are two major types of classes: *open* and *closed*. The *closed* classes are these, where there is a cyclic series of connected half-links. The *open* classes are these, where the series of connected half-links is not cyclic. The following examples clarify these definitions:

- 1. If there is a link  $\ell$  of  $\mathcal{G}$  such that none of its half-links is related to any other half-link by any of  $\mathcal{R}_{link}^{(n,n')}$ -relations, then the pair  $\{\ell^+, \ell^-\}$  is an equivalence class of  $\mathcal{R}_{face}$ relation. We say it is an *open* equivalence class.
- 2. Consider a link  $\ell$  of  $\mathcal{G}$  such that  $n = s(\ell)$  and  $n' = t(\ell)$  and the pair (n, n') is related by the node relation  $\mathcal{R}_{node}$ . It may happen, that  $\ell^+$  is related to  $\ell^-$  by  $\mathcal{R}_{link}^{(n,n')}$ . In such case the half-links  $\{\ell^+, \ell^-\}$  form an equivalence class of  $\mathcal{R}_{face}$ . We call it a *closed* equivalence class.
- 3. Consider now a pair of links  $(\ell_1, \ell_2)$  such, that  $\ell_1^+$  and  $\ell_2^-$  are related by one of relations  $\mathcal{R}_{\text{link}}^{(n,n')}$ , but  $\ell_1^-$  and  $\ell_2^+$  are not related with any other half-links. Then the quadruple  $\{\ell_1^-, \ell_1^+, \ell_2^-, \ell_2^+\}$  form an equivalence class of  $\mathcal{R}_{\text{face}}$ . We call such a class an *open* class.

- 4. Consider a pair of links  $(\ell_1, \ell_2)$  like in previous example, however now assume, that  $\ell_1^-$  and  $\ell_2^+$  are also related by another link-relation  $\mathcal{R}_{\text{link}}^{(n'',n''')}$ . Then the equivalence class of the  $\mathcal{R}_{\text{face}}$  relation is the same quadruple  $\{\ell_1^-, \ell_1^+, \ell_2^-, \ell_2^+\}$ , but now we will call it a *closed* equivalence class.
- 5. The example 3 can be generalised to arbitrary number of links: consider a k-tuple of links  $(\ell_1, \ell_2, \ldots, \ell_k)$  such that each pair  $\ell_i^+$  and  $\ell_{i+1}^-$  is related by one of  $\mathcal{R}_{\text{link}}^{(n,n')}$ -relations (for  $i = 1, \ldots, k-1$ ), but  $\ell_1^-$  and  $\ell_k^+$  are not related to any other half-link. Than the 2k-tuple  $\{\ell_1^-, \ell_1^+, \ldots, \ell_k^-, \ell_k^+\}$  is an *open* equivalence class of  $\mathcal{R}_{\text{face}}$ .
- 6. Analogously the example 4 can be generalised to arbitrary number of links: consider a k-tuple of links  $(\ell_1, \ell_2, \ldots, \ell_k)$  such that each pair  $\ell_i^+$  and  $\ell_{i+1}^-$  is related by one of  $\mathcal{R}_{\text{link}}^{(n,n')}$ -relations (for  $i = 1, \ldots, k-1$ ), and  $\ell_1^-$  and  $\ell_k^+$  are related by another  $\mathcal{R}_{\text{link}}^{(n,n')}$ relation. Than the 2k-tuple  $\{\ell_1^-, \ell_1^+, \ldots, \ell_k^-, \ell_k^+\}$  is a *closed* equivalence class of  $\mathcal{R}_{\text{face}}$ .

The half-links that are not related to any other half-link (in the *open* equivalence classes) are called *boarder* members of the equivalence class. There are always two boarder members of each open equivalence class.

Note, that although  $\mathcal{R}_{face}$  is fully determined by  $\mathcal{R}_{link}$ , the opposite is not true. The information about openness and closeness of equivalence classes of  $\mathcal{R}_{face}$  is not encoded in this relation itself.

Some examples of equivalence classes of  $\mathcal{R}_{\text{face}}$  relation can be found at figure 3.5. The edge relation, thanks to its simplicity, does not require extra illustration.

# 3.2.2 How to construct a 2-complex out of a graph diagram

As we noted at the beginning of this chapter, given a spin-foam we can construct a graphdiagram out of it. Now we shell show, that for each graph diagram we can construct a spin-foam out of it.

The procedure will be as follows. First for each connected component  $\Gamma_I$  of the graph  $\mathcal{G}$  of a graph diagram  $\mathcal{D}$  we shell construct a 2-complex, that represents a fragment of a spin-foam being a small neighbourhood of a vertex with structure encoded in the graph  $\Gamma_I$  (see subsection 3.2.2.1). Then we glue these fragments of 2-complexes into bigger complexes according to the procedure, that we shell define in subsection 3.2.2.2: for each pair of nodes, that are in the node relation we glue the complexes by identifying corresponding half-links in a certain way. Finally we realise, that some elements of the resulting 2-complex are in fact redundant, and thus we remove them (see subsection 3.2.3).

The procedure was first presented in [53]) and in [81].

# 3.2.2.1 From a graph to 1-vertex foam

Our construction is reversing the procedure presented in subsection 2.1.2, where we were drawing a sphere around a vertex of 2-complex to capture the graph representing its structure. Here we draw a graph on a sphere and then shrink the sphere radially to a point in order to obtain a 2-complex. Given a connected graph  $\Gamma$  we will obtain a 2-complex containing precisely one internal vertex (we shell call it a 1-vertex foam  $\kappa_{\Gamma}$ ). Resulting 2-complex is characterised by the following prescription (which can be treated as its definition).











Figure 3.5: On figures (a) - (f) there are highlighted examples of equivalence classes of types 1-6 respectively.

# **Definition 3.5.** 1-vertex foam

Consider a closed connected graph  $\Gamma = (\mathbf{N}, \mathbf{L})$ . Its squid-graph is  $\gamma_{\Gamma} = (\Gamma^{(s)}, S)$  with  $\Gamma^{(s)} = (\mathbf{N} \cup \mathbf{X}, \mathbf{L}^+ \cup \mathbf{L}^-)$  - see Remark 3.1 in subsection 3.1.2.1. The 1-vertex foam of the graph  $\Gamma$  is a squid-2-comples  $\kappa_{\Gamma} = (\mathbf{F}, \mathbf{E}, \mathbf{V}; \mathbf{f}_2, \mathbf{f}_1; S)$  such that:

- The set of vertices is  $\mathbf{V} = \{v\} \cup \mathbf{N} \cup \mathbf{X}$ .
- The set of edges is  $\mathbf{E} = \mathbf{L}^+ \cup \mathbf{L}^- \cup \mathbf{N}_{\mathbf{I}} \cup \mathbf{X}_{\mathbf{I}}$ , where  $\mathbf{N}_{\mathbf{I}} = \{n_i \times \mathbf{I} : n_i \in \mathbf{N}\}$  where  $\mathbf{I} = [0, 1]$  is an interval, and similarly  $\mathbf{X}_{\mathbf{I}} = \{x_i \times \mathbf{I} : x_i \in \mathbf{X}\}$
- The set of faces is  $\mathbf{F} = \mathbf{L}_{\Delta}^+ \cup \mathbf{L}_{\Delta}^-$ , where  $\mathbf{L}_{\Delta}^{+/-} = \{\Delta_{\ell} : \ell \in \mathbf{L}^{+/-}\}$ , where  $\Delta$  is a 2-simplex (triangle).
- The map  $\mathbf{f}_1$  is defined as follows:
  - for each edge  $e_{\ell} \in \mathbf{L}^+ \cup \mathbf{L}^-$  the function  $\mathbf{f}_1$  is induced by the adjacency relation of the graph  $\Gamma^{(s)}$ , i.e

$$\mathbf{f}_1(e_\ell(0)) = s(\ell) \in \mathbf{N} \cup \mathbf{X} \subset \mathbf{V} \quad \text{and} \quad \mathbf{f}_1(e_\ell(1)) = t(\ell) \in \mathbf{N} \cup \mathbf{X} \subset \mathbf{V} \quad (3.8)$$

- for an edge  $e_n \in \mathbf{N}_{\mathbf{I}}$  the function  $\mathbf{f}_1$  maps its beginning onto the central vertex v, and its end onto the node n:

$$\mathbf{f}_1(e_n(0)) = v \quad \text{and} \quad \mathbf{f}_1(e_n(1)) = n \in \mathbf{N} \subset \mathbf{V}$$
(3.9)

and analogously for  $e_x \in \mathbf{X}_{\mathbf{I}}$ .

- The map f<sub>2</sub> is defined as follows. Boundary of each triangle consist of three intervals, let us name them VN, NX and XV. The boundary map acting on ∂f<sub>ℓ</sub> ∈ L<sup>+</sup><sub>Δ</sub> maps the NX interval onto the e<sub>ℓ</sub> edge f<sub>2</sub>(NX ⊂ ∂f<sub>ℓ</sub>) = e<sub>ℓ</sub> ∈ L<sup>+</sup> ⊂ E, the VN interval onto the edge of the source node of the link: f<sub>2</sub>(VN ⊂ ∂f<sub>ℓ</sub>) = e<sub>s(ℓ)</sub> ∈ N<sub>I</sub> ⊂ E and the interval XV onto the edge of the target node of the link: f<sub>2</sub>(XV ⊂ ∂f<sub>ℓ</sub>) = e<sub>t(ℓ)</sub> ∈ X<sub>I</sub> ⊂ E. For a face ∂f<sub>ℓ</sub> ∈ L<sup>-</sup><sub>Δ</sub> the action is analogous, the only change is that VN is mapped onto the edge of the target node e<sub>t(ℓ)</sub>, and XV is mapped onto the edge of the source node e<sub>s(ℓ)</sub>.
- Each link  $e_{\ell}$  is a boundary link, thus it can be oriented. We orient it in agreement with the link  $\ell$  of the graph  $\Gamma^{(s)}$  (i.e. in such a way, that the boundary graph  $\partial \kappa_{\Gamma} \leftrightarrow \Gamma^{(s)}$  including orientation). The orientation of each face  $f_{\ell}$  by definition agrees with the orientation of the edge of the link  $e_{\ell}$ .

Since the graph  $\Gamma$  is closed, the boundary of  $\kappa_{\Gamma}$  is precisely  $\Gamma^{(s)}$ , thus the squid-decomposition S is directly induced from  $\gamma_{\Gamma}$ .

An illustration of the construction is presented at figure 3.6.

Given a graph  $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_K\}$  we perform this construction separately for each connected component  $\Gamma_i$  and obtain a squid-2-complex  $\kappa_{\mathcal{G}} = \kappa_{\Gamma_1} \sqcup \cdots \sqcup \kappa_{\Gamma_K}$ .

# 3.2.2.2 Gluing along squids

Consider a squid-2-complex  $(\kappa, S)$  and a duality map  $\phi$  between two squids  $\lambda_1, \lambda_2 \in S$ . We will show now, how to glue this 2-complex along this pair of squids.



Figure 3.6: Illustration of the construction of the 1-vertex foam. (a) - A connected component  $\Gamma$ . (b) - Its split graph  $\Gamma^{(s)}$ . (c) - The 1-vertex foam  $\kappa_{\Gamma}$ .

**Definition 3.6.** Glueing a 2-comlex  $\kappa$  along a duality map  $\phi : \lambda_1 \to \lambda_2$ 

The squid-2-complex  $\kappa = (\mathbf{F}, \mathbf{E}, \mathbf{V}; \mathbf{f}_2, \mathbf{f}_1; S)$  glued along a pair of squids  $\lambda_1, \lambda_2 \in S$ ,  $\lambda_1 \neq \lambda_2$ , according to the duality map  $\phi : \lambda_1 \to \lambda_2$  is a squid-2-complex  $\kappa/\phi$  given by  $(\tilde{\mathbf{F}}, \tilde{\mathbf{E}}, \tilde{\mathbf{V}}; \tilde{\mathbf{f}}_2, \tilde{\mathbf{f}}_1; \tilde{S})$  defined as follows:

- The set of faces does not change:  $\mathbf{F} = \mathbf{F}$ .
- The set of edges is the original set of edges divided by a relation  $\tilde{\mathbf{E}} = \mathbf{E} / \sim_{\text{edge}}^{\phi}$ , where two edges  $e, e' \in \mathbf{E}$  are in the relation  $\sim_{\text{edge}}^{\phi}$  iff  $e \in \lambda_1$ ,  $e' \in \lambda_2$  (or opposite) and  $\phi(e) = e'$  (or  $\phi^{-1}(e) = e'$  in the opposite case).
- The set of vertices is defined in the analogous way:  $\tilde{\mathbf{V}} = \mathbf{V} / \sim_{\text{vertex}}^{\phi}$ , where two vertices  $v, v' \in \mathbf{V}$  are in the relation  $\sim_{\text{vertex}}^{\phi}$  iff  $v \in \lambda_1$ ,  $v' \in \lambda_2$  (or opposite) and  $\phi(v) = v'$  (or  $\phi^{-1}(v) = v'$  in the opposite case).
- The function f
  <sub>1</sub> coincides with the function f<sub>1</sub>, but it must be followed by the projection π<sup>φ</sup><sub>0</sub> onto equivalence classes of the relation ~<sup>φ</sup><sub>vertex</sub>:

$$\tilde{\mathbf{f}}_1: \partial \tilde{\mathbf{E}} \ni x \mapsto \pi_0^{\phi}(\mathbf{f}_1(x)) \in \tilde{\mathbf{V}}$$
(3.10)

One needs to check consistency of  $\tilde{\mathbf{f}}_1$  with the relation  $\sim_{\text{edge}}^{\phi}$ , i.e. check, if  $x \sim_{\text{edge}}^{\phi} x'$ implies  $\tilde{\mathbf{f}}_1(x) = \tilde{\mathbf{f}}_1(x')$ . Outside the glued squids it is obviously satisfied, since in this regime equivalence classes of  $\sim_{\text{edge}}^{\phi}$  are one-element sets. Assume therefore, that  $x \in \partial e$  for  $e \in \lambda_1$  and we have  $x' \neq x$  such that  $x' \sim_{\text{edge}}^{\phi} x$ . If it is so, x'must be in  $\lambda_2$  and  $\phi(x) = x'$ . We have  $\tilde{\mathbf{f}}_1(x) = \pi_0^{\phi} \circ \mathbf{f}_1(x)$  and  $\tilde{\mathbf{f}}_1(x') = \pi_0^{\phi} \circ \mathbf{f}_1(x')$ . However, since  $\phi$  is a morphism of graphs, the condition  $\phi(x) = x'$  must follow  $\phi(\mathbf{f}_1(x)) = \mathbf{f}_1(x')$ , and thus  $\mathbf{f}_1(x) \sim_{\text{vertex}}^{\phi} \mathbf{f}_1(x')$ , so  $\pi_0^{\phi}(\mathbf{f}_1(x)) = \pi_0^{\phi}(\mathbf{f}_1(x'))$ , what ends the proof.

 The function f<sub>2</sub> coincides with the function f<sub>2</sub>, but it must be followed by the projection π<sup>φ</sup><sub>1</sub> onto equivalence classes of the relation ~<sup>φ</sup><sub>edge</sub>:

$$\tilde{\mathbf{f}}_2: \partial \tilde{\mathbf{F}} \ni x \mapsto \pi_0^{\phi}(\mathbf{f}_2(x)) \in \tilde{\mathbf{E}}$$
 (3.11)

Since  $\tilde{\mathbf{F}} = \mathbf{F}$ , no consistency check is needed.



Figure 3.7: Illustration of the 2-complexes. (a) - Fragments of graphs that will be glued. (b) - Fragment of 2 complex before gluing. (c) - The glued 2-complex.

• The new set of boundary squids is  $\tilde{S} = S \setminus \{\lambda_1, \lambda_2\}.$ 

Each leg of the glued squids becomes an internal edge, because after gluing it is adjacent to precisely two faces. Thus the new squid set  $\tilde{S}$  covers all the boundary graph, and thus the resulting 2-complex  $\kappa/\phi$  is a proper squid-2-complex.

Example of the gluing procedure is presented at figure 3.7.

Given a graph diagram  $(\mathcal{G}, \mathcal{R})$  and a squid-2-complex  $\kappa_{\mathcal{G}}$  of its 1-vertex foams we can associate a duality map of squids  $\phi_i$  which each pair of nodes  $(n_i, n'_i)$  which are in the node relation (see subsection 3.2.1.2). Then we can perform gluing along each pair of dual squids obtaining a 2-complex  $\kappa/(\phi_1, \phi_2, \ldots, \phi_{\#\mathcal{R}_{node}}) = (\cdots (\kappa_{\mathcal{G}}/\phi_1)/\phi_2 \cdots)/\phi_{\#\mathcal{R}_{node}}$ . The resulting 2-complex does *not* depend on the order in which we perform gluings (for proof see Appendix B.1 of this thesis or Theorem 1 in [53, appendix]). So there is an unambiguous procedure of gluing a squid-2-complex along a set of maps, and the result is well defined:  $\kappa_{\mathcal{R}} := \kappa/\{\phi_1, \ldots, \phi_{\#\mathcal{R}}\}$  for an arbitrary order of gluings.

Thus we define the 2-complex of the graph diagram as

$$\kappa_{\mathcal{D}} = \kappa_{\mathcal{G}} / \mathcal{R} \tag{3.12}$$

and we know that the resulting 2-complex is well defined.

# 3.2.3 Properties of the 2-complexes and removing the unnecessary cells

Let us now characterise the resulting 2-complex  $\kappa_{\mathcal{D}}$ . We shell describe all types of cells (vertices, edges, faces) that can appear in the complex. We will start with describing the edges, because knowing, which types of edges are *removable* will make easier to understand, which vertices are removable and why. Then we will deal with faces, and finally we will characterise the boundary graph of  $\kappa_{\mathcal{D}}$ .

In subsection 3.2.2.1 we introduced a notation in which each face is a triangle VNX. At that stage the vertices N were heads of the squids of the boundary graphs of the 1-vertex foams, and the vertices X were the leg-nodes of these squids. Note, that through all the gluing procedure the vertices N can be glued only to other vertices N' and the vertices X are also glued only to other vertices X'. The vertices V are never subjects of the gluing procedure. Thus it is convenient to introduce three classes of vertices -V-type, N-type and X-type vertices. Those classes allow us also to characterise the edges as VN-type, VX-type and NX-type edges. We shell use these classification in what follows.

# 3.2.3.1 Edges

There are four types of edges: VN, VX, internal NX and boundary NX.

- 1. Each VN-type edge is a history of a *head* of a squid (the head itself corresponds to the point N). It is always shared by at least two faces (actually: the number of faces equals to the number of legs of the squid it was build from).
- 2. Each VX-type edge is a history of a *leg-node* of a squid (the leg-node itself corresponds to the point X). It is always shared by precisely two faces (coming from the links that were meeting at the node). These faces are oriented consistently.
- 3. Each NX-type edge is a leg of a squid. It is a boundary edge if and only if the squid it belongs to was not glued to another squid (i.e. the head of this squid is a type 2 equivalence class of the edge relation  $\mathcal{R}_{edge}$ , and the leg itself is a boundary member of an open equivalence class of  $\mathcal{R}_{face}$  see subsection 3.2.1.3).
- 4. An edge of type NX is an internal edge of the complex if and only if the squid it belongs to was glued to another squid. In such case this edge is shared by precisely two faces. These faces are oriented consistently (i.e. the head of this squid is a member of a type 1 equivalence class of the edge relation  $\mathcal{R}_{edge}$ , and the leg itself is a member of a closed equivalence class of  $\mathcal{R}_{face}$  - see subsection 3.2.1.3).

The edges of type VX and the internal edges of type NX are always *removable* edges (see definition subsection 3.1.2.2).

It may happen, that an edge of type VN is removable (if the squid of the vertex N is bivalent, and it has one incoming and one outgoing leg) - however in most cases we will not remove them. They constitute the skeleton of the spin-foam.

Removing the removable edges and anticipating from subsection 3.2.3.2, that the internal *N*-type nodes are removable, we are left with three types of edges:

- boundary edges one for each pair of the boundary NX-edges sharing the same boundary X-vertex. After removing the X-vertex they become a single NN-edge.
- internal edges with one boundary end the VN-edges of the nodes that has not been glued. There is one such edge for each external equivalence class of  $\mathcal{R}_{edge}$ -relation.
- internal edges one for each pair of VN-edges sharing the same N-vertex. Removing all NX-edges starting at this N-vertex makes the vertex removable, so the final edge is VV-type. There is one such edge for each pair  $\{n, n'\}$  being an internal equivalence class of the  $\mathcal{R}_{edge}$ -relation.

# 3.2.3.2 Vertices

There are six classes of vertices to describe: the V-type vertices, the internal N-type vertices, the boundary N-type vertices, the internal X-type vertices, the simple boundary X-type vertices and the complex boundary X-type vertices.

- 1. Vertices of type V are always internal vertices. There is one such vertex for each connected component  $\Gamma_I \in \mathcal{G}$ . The structure of this vertex is encoded in the split graph  $\Gamma^{(s)}$ .
- 2. Vertices of type N are heads of squids. Such a vertex is an internal vertex if and only if the squid, that the vertex came from, was glued with another squid. In such case the N-type vertex looks like it were the vertex of type V coming from the  $\theta$ -like graph (see figure 3.8 for illustration and subsection A.1.1.3 for definition). Of course the similarity is only local.

There are always two edges of type VN ending at such a vertex, and a number of internal NX-type edges. Since all those NX - edges are removable (what we shall explain shortly), after removing them the internal N-type vertex also becomes removable.

- 3. The vertex of type N coming from a non-glued squid is a boundary vertex. It is then a node of the boundary graph.
- 4. Vertices of type X are leg-nodes of squids. Such a vertex is an internal vertex if and only if the squid-leg it came from is a half-link that belongs to a *closed* equivalence class of the face relation (i.e. equivalence class of type 2,4 or 6 see subsection 3.2.1.3)

Since all edges ending at such X-type vertex are removable (i.e. VX-edges and internal NX-edges), the vertex itself also becomes removable (after removing all but last two edges).

5. If none of the squids, to whom a vertex of type X belongs, is glued to any other squid, it is a simple boundary vertex (it is a leg-node in the middle point of a link being unrelated to any other link by  $\mathcal{R}_{\text{link}}$  relation, so it belongs to a type 1 equivalence class of the face relation  $\mathcal{R}_{\text{face}}$  - see subsection3.2.1.3)

Removing the VX-edge ending at such vertex makes it a boundary bivalent node. Both links incident to it have consistent orientation. Thus such vertex is a removable cell.

6. The last possibility for the X-type vertex is that it is the middle of a link  $\ell$  being an element of a non-trivial *open* equivalence class of the face relation  $\mathcal{R}_{\text{face}}$  (i.e. an equivalence class of type 3 or 5 - see subsection 3.2.1.3).

Such a vertex is also removable, because all internal edges ending at it are either VX- or internal NX-edges, thus they are removable (see subsection 3.2.3.1). Removing all of them leaves us again with a simple bivalent boundary node, like in the previous case.

In what follows we will call the vertices of type 5 and 6 *boundary X-vertices* and treat them as the same type. The vertices of type 4 will be called *internal X-vertices*.

The only non-removable vertices are vertices of type V and boundary vertices of type N. Each boundary N-type vertex has a structure determined by the squid it came from. Each V-type vertex at first had structure indicated by the graph  $\Gamma^{(s)}$  it came from, however each node  $x_{\ell} \in \mathbf{X} \subset \mathbf{N}_{\Gamma^{(s)}}$  is represented by a VX-edge, that was already marked removable. Thus the structure of each vertex of type V simplifies while the removable parts are removed. Anticipating that the non-removable edges will be only the VN-type, we conclude, that each  $x_{\ell}$ -node of  $\Gamma^{(s)}$  should be removed. Removing them leads us to the original graph  $\Gamma$ , and thus after removing all removable parts the structure of each V-type vertex is indicated precisely by the connected graph  $\Gamma$  it came from.

# 3.2.3.3 Faces

The set of faces do not change during the gluing procedure, so the final set  $\mathbf{F}$  of the 2-complex  $\kappa_{\mathcal{D}}$  is the same, as the set  $\mathbf{F}$  of the 2-complex  $\kappa_{\mathcal{G}}$ . All the faces are triangular. Each face has two internal edges (VN and VX) and the third edge (NX-type), which is either also internal iff the squid that the face came from was glued to another squid, or it is a boundary edge iff the squid was not glued to anything. Topologically each face is a disk, placed onto some skeleton. Given a face, none of its edges are glued with other edges of the same face (see subsection 3.2.3.1).

Since the VX and internal NX edges are all removable, it is convenient to introduce generalised faces - being collections of faces sharing the same X-vertex. There are two types of generalised faces, determined by the type of X-vertex they contain.

• The boundary generalised face is a generalised face containing a boundary X-vertex. There is one such face for each open equivalence class of  $\mathcal{R}_{\text{face}}$ -relation. The face contains precisely one boundary edge (being the boundary  $N_1N_2$ -link) and the orientation of the edge and the face are consistent. Other edges of that face are in order: the  $N_2V_1$ -edge (where  $N_2$  is the ending of the boundary link), then a sequence of internal edges  $V_1V_2, \ldots, V_{k-1}V_k$  (however k may be equal to 1) and then  $V_kN_1$ .

Some of the  $V_i$  vertices may be equal, in that case it effects the topology of the generalised face. Moreover it may happen, that  $N_1 = N_2$  (the boundary link is a loop), and thus  $V_1 = V_k$ . In such case the edges  $N_2V_1$  and  $V_kN_1$  are the same - with all the consequences for the topology of the generalised face (i.e. the face is either a cylinder of a cone).

• The internal generalised face is a generalised face containing an internal X-vertex. There is one such face for each closed equivalence class of  $\mathcal{R}_{\text{face}}$ -relation. All edges of such face are internal VV-edges.

Orientation of such face is determined by orientation of links in the graph diagram. Each simple (triangular) face is oriented consistently with the half-link it came from. Then all the way through the gluing procedure the new neighbours of each face were oriented consistently with it. Thus after removing the removable edges, the orientation remain the same.

In other words we orient the generalised face in such a way, that if one considers a small neighbourhood of any internal vertex V, then the boundary of this neighbourhood agrees with the connected component  $\Gamma$  of the graph  $\mathcal{G}$  of which the vertex came from, including the orientation of the graph.

The edges of such face form a cyclic sequence. Some elements of this sequence (edges or vertices) may appear more then once.

After removing the removable edges and nodes each generalised face become a face of the 2-complex. Examples of generalised faces are shown at 3.9.

Note, that even though the interior of each face is a disc, its boundary may be glued in a topologically nontrivial way. A suitable example is shown and explained at figure 3.10.


Figure 3.8: Types of vertices: The vertices  $V_1$  and  $V_2$  represent the type 1 of the enumeration of subsection 3.2.3.2. The vertex  $N_1$  represents the type 2. The vertices  $N_2$  and  $N_3$ represent type 3. The vertices  $X_2$  and  $X_3$  might be of type 4, iff the dashed green edges are the internal edges of the graph 2-complex. Otherwise they are of type 6, and so does  $X_1$ . The vertices  $X_4, X_5, X_6, X_7$  are of type 5. One can also read the possible types of edges : the edges  $V_iN_j$  and  $V_iX_j$  are straightforward. The edges  $N_1X_1, N_1X_2$  and  $N_1X_3$ are the internal NX-edges (of type 4 of the enumeration of subsection 3.2.3.1), the other NX-edges are the boundary NX-edges (of type 3).



Figure 3.9: Examples of generalised faces. The removed edges are represented by dashed lines, the removed points are denoted by weak dots and grey letters. (a) - The face  $N_2V_1N_2V_2N_3X_1$  of figure 3.8, which after removing the removable cells becomes  $N_2V_1V_2N_3$ . (b) - The same face after gluing the boundary nodes corresponding to the vertices  $N_2$  and  $N_3$ .

#### 3.2.3.4 Boundary

The boundary of the 2-complex  $\kappa_{\mathcal{D}}$  is a so called boundary graph of a graph diagram (we shell denote it as  $\partial \mathcal{D}$ ). It can be read out of  $\mathcal{D}$  by a following simple algorithm:

- 1. Take the graph  $\mathcal{G}$  and remove each node, that is related with another node by  $\mathcal{R}_{node}$ -relation.
- 2. Remove the nodes together with the half-links incident to them.
  - (a) If both half-links of  $\ell$  are removed, we say that all the link is removed.
  - (b) If only one half-link of  $\ell$  is removed an open half-link is left. We shell deal with them in the next point.
- 3. Each open half-link is a boarder member of an open equivalence class of  $\mathcal{R}_{\text{face}}$ relation. Each such class has two boarder members, of the opposite signs so one
  can connect them into one new link. Thus we do connect them.

This algorithm can be illustrated graphically. An example is drawn and explained at figure 3.11.

The graph  $\partial \mathcal{D}$  can be characterised by description of its nodes and links:

- There is one node for each node of  $\mathcal{G}$  not related to any other node (one for each one-element equivalence class of  $\mathcal{R}_{edge}$ ).
- There is one link for each boundary generalised face (for each open equivalence class of  $\mathcal{R}_{\text{face}}$ ).



Figure 3.10: A face with the topology of the projective plane. (a) - Fragments of graph diagram that will be glued. The half-links a, b, c and d are related to a', b' c' and d' respectively. (b) - Fragments of one-vertex-foams. The edges  $\alpha = V_1 X'$  and  $\beta = V_2 X''$  are drawn in two copies each in order to make figure more legible. In fact each copy of  $\alpha$  (or  $\beta$  respectively) represent the same cell of 2-complex. (c) - The same fragment of 2-complex but with the edges  $\alpha$  and  $\beta$  drawn in one copy. The green arrows do not mean the orientation of the links, but the pattern of gluing (for example a and a' are glued in such a way, that  $N_2$  is glued with  $N'_2$  and X is glued with X''' - and the interior of the edges is glued continuously). (d) - The projective-plane-face. Each edge  $N_1 X$  is removable. The arrows on the edges  $N_1 X$  represent, in which way they are glued. Removing them leads to the topological space homeomorphic to a projective plane  $\mathbb{R}P(2)$ . However, the series of edges  $N_1 V_1 N_2 V_2 N_1$  is not removable - it is the part of the core skeleton of the 2-complex. Thus our generalised face is in fact a disk glued onto this skeleton in such a way, that the topology of the result of gluing is nontrivial. The topology of the face itself is trivial.



Figure 3.11: An example of the boundary graph of a graph diagram. (a) - A graph diagram. (b) - Step 1: the pairs of nodes related by  $\mathcal{R}_{node}$ -relation are removed. (c) - Step 2a: four links had both of their half-links removed, thus we remove them completely. (d) - Steps 2b and 3: connecting the remaining open half-links result with the boundary graph of the graph diagram.

# 3.3 Operator Spin-network Diagrams - colored graph diagrams

Operator Spin-network Diagram is a Graph Diagram colored by some SU(2) tensors. The brief recall of types of SU(2)-tensors we are going to use was presented in subsection 3.1.2.3. The detailed definitions and discussion can be found in Appendix A.3.2.

In the next subsection we will describe in detail the rules of general coloring of a graph diagram. Then in subsection 3.3.2 we will explain, how to calculate the transition amplitude of a colored diagram by performing so called *diagram-driven contraction* of the tensors used in the coloring. Then in subsection 3.3.3 we will explain, what specific coloring should be taken in order to obtain the transition amplitude compatible with one of standard spin-foam models amplitude. Finally, in subsection 3.3.4, we will discuss, what is the influence of the objects, that were marked as *removable* in the previous section, on the transition amplitude calculation.

#### 3.3.1 The proper coloring of a graph diagram

In order to obtain an Operator Spin-network Diagram one have to color a graph diagram  $\mathcal{D}$  in the following way:

- Each link of each graph is colored by a spin  $j_{\ell}$ .
- Each node of each graph is colored by an operator  $\widehat{P_n} \in \mathcal{H}_n^{\text{Inv}*} \otimes \mathcal{H}_n^{\text{Inv}}$ , where  $\mathcal{H}_n^{\text{Inv}}$  is the invariant node space defined by the spins of links incident to n.
- Each graph is colored by a contractor  $\langle A | \in \mathcal{H}^*_{\Gamma, \vec{i}}$ .

The coloring cannot be arbitrary, the following consistency conditions must be held:

- If two half-links are related by one of link relations, the links they come from must be colored by the same spin.
- If two nodes n and n' are related by the node relation, their operators must be hermitian conjugations, i.e.  $\widehat{P_n} = \widehat{P_{n'}^{\dagger}}$ .

The first condition ensures that all equivalence class of  $\mathcal{R}_{\text{face}}$  relation has the same spin coloring. Thanks to this fact, the second condition is self consistent: since n' is a dual node to n and each half-link of n has the same spin, as its corresponding half-link of n', the Hilbert spaces are dual:  $\mathcal{H}_n^{\text{Inv}} = \mathcal{H}_{n'}^{\text{Inv*}}$ , and thus the operators  $\widehat{P_n}$  and  $\widehat{P_{n'}}$  belong to the same space.

An Operator Spin-network Diagram will be denoted by  $(\mathcal{D}; \vec{j}, \vec{P}, \vec{A})$  or shortly  $(\mathcal{D}, C)$  with  $C = (\vec{j}, \vec{P}, \vec{A})$  being the coloring. An example can be found at figure 3.12.

#### Boundary

The coloring of a graph diagram indicates also a so called *boundary Hilbert space*. Each link of the boundary graph can be assigned to a single equivalence class of the  $\mathcal{R}_{\text{face}}$  relation. Thus for each link of the boundary there is a uniquely determined spin  $j_{\ell}$ . This fixes a spin-network Hilbert space for the graph  $\partial \mathcal{D}$  with spin labels  $\vec{j_{\ell}}$ . Let us then define the boundary Hilbert space of a graph diagram as this spin-network Hilbert space  $\mathcal{H}_{\partial \mathcal{D},\vec{j}}$ .



Figure 3.12: An example of operator spin-network diagram

#### 3.3.2 The amplitude operator of OSD

Operator Spin-network diagrams are in fact designed to be a tool to compute transition amplitude for boundary states  $|\psi\rangle \in \mathcal{H}_{\partial \mathcal{D},\vec{j}}$ . Let us now explain, how to construct a functional  $\langle A | \in \mathcal{H}^*_{\partial \mathcal{D},\vec{j}}$  such that the transition amplitude will be  $\langle A | |\psi\rangle$ . The functional  $\langle A |$  will be called *amplitude operator of the diagram*. The name *operator* comes from the fact, that given a diagram with a disjoint boundary graph  $\partial \mathcal{D} = \mathcal{G}_1 \sqcup \mathcal{G}_2^*$  (where  $\mathcal{G}^*$  is the *inversed graph* i.e a graph  $\mathcal{G}$  with each link's orientation reversed - see subsection 3.1.2.1) and a boundary state  $|\psi\rangle = |\psi_{\mathcal{G}_1}\rangle \otimes |\psi_{\mathcal{G}_2^*}\rangle$ , the transition amplitude can be treated as

$$\langle A_{\mathcal{D}} | | \psi \rangle = \langle \psi_{\mathcal{G}_2} | \widehat{A_{\mathcal{D}}} | \psi_{\mathcal{G}_1} \rangle \tag{3.13}$$

and the operator  $\widehat{A_{\mathcal{D}}}$  has interpretation of the evolution operator (see subsection 1.2.2.3).

The amplitude operator of a diagram  $(\mathcal{D}, C)$  will be given by certain contraction of indices of the tensors  $\langle A_{\Gamma} |$  and  $\widehat{P}_n$ , called *diagram-driven contraction*  $\lrcorner_{\mathcal{D}}$ , multiplied by *face-* and *link-*amplitudes (from equivalence classes of  $\mathcal{R}_{\text{face}}$  relation and from links of the boundary graph):

$$\langle A_{\mathcal{D}} | = \left( \bigotimes_{\Gamma \in \mathcal{G}} \langle A_{\Gamma} | \right) \, \lrcorner_{\mathcal{D}} \left( \bigotimes_{[n] \in \mathcal{R}_{\text{edge}}} \widehat{P_n} \right) \cdot \left( \prod_{f \in \mathcal{R}_{\text{face}}} A_f \right) \cdot \left( \prod_{\ell \in \partial \mathcal{D}} A_\ell \right)$$
(3.14)

The following subsections explain the diagram-driven contraction and other terms in the above formula.

#### 3.3.2.1 Diagram-driven contraction

The diagram-driven contraction gives us the so called *unweighted* diagram operator

$$A_{\mathcal{D}}^{\text{unweighted}} = "\bigotimes \langle A_{\Gamma} | \lrcorner_{\mathcal{D}} \bigotimes \widehat{P_n} "$$
(3.15)

Before defining it, let us point out some properties of tensor objects appearing in the coloring.

First let us remind that given a spin-network state  $|\mathcal{N}_{\Gamma}\rangle \in \mathcal{H}_{\Gamma}$  supported on a graph  $\Gamma$  for each link  $\ell$  there are two indices in spin  $j_{\ell}$  representation: the upper one (related to the positive half-link  $\ell^+$ ) and the lower one (related with the negative half-link  $\ell^-$ )

- see subsection A.3.3, i.e.  $|\mathcal{N}_{\Gamma}\rangle = \sum_{\left\{\overrightarrow{m_{\ell^+}, \overrightarrow{n_{\ell^-}}}\right\}} \mathcal{N}_{n_{\ell_1^-}\cdots n_{\ell_L^-}}^{m_{\ell_1^+}\cdots m_{\ell_L^+}} |\overrightarrow{m, n}\rangle$ . Since the contractors belong to the dual space  $\langle A_{\Gamma}| \in \mathcal{H}_{\Gamma}^*$ , it is opposite for them: the lower indices are related with positive half-links and the upper indices are related with the negative half-links:  $\langle A_{\Gamma}| = \sum_{\left\{\overrightarrow{m_{\ell^+}, \overrightarrow{n_{\ell^-}}}\right\}} \mathcal{A}_{m_{\ell_1^+}\cdots m_{\ell_L^+}}^{n_{\ell_1^-}} \langle \overrightarrow{m, n}|.$ 

Consider now an operator  $\widehat{P_n} \in \mathcal{H}_n^{\mathrm{Inv}} \otimes \mathcal{H}_n^{\mathrm{Inv}*}$ . Since

$$\mathcal{H}_{n}^{\mathrm{Inv}} \subset \mathcal{H}_{n} = \left(\bigotimes_{\ell^{+} \in \mathcal{L}_{n}^{+}} \mathcal{H}_{j_{\ell}}\right) \otimes \left(\bigotimes_{\ell^{-} \in \mathcal{L}_{n}^{-}} \mathcal{H}_{j_{\ell}}^{*}\right)$$
(3.16)

the operator  $\widehat{P_n}$  has two spin- $j_{\ell}$  indices for each half-link incident to n. For a positive half-link it has one upper index from  $\mathcal{H}_n$  space and one lower index from  $\mathcal{H}_n^*$ . In case of a negative half-link the upper index come from  $\mathcal{H}_n^*$  and the lower index come from  $\mathcal{H}_n$ . The indices of  $\mathcal{H}_n$ -space will be denoted by  $m_{\ell}$  for positive and  $n_{\ell}$  for negative half-links (and they will be called *small* indices). The indices of  $\mathcal{H}_n^*$  will be denoted by  $A_{\ell}$  for positive and  $B_{\ell}$  for negative half-links (and called *big* indices), thus

$$\widehat{P_{n}} = P_{n} \frac{m_{\ell_{1}^{+}} \cdots m_{\ell_{k}^{+}} B_{\ell_{1}^{-}} \cdots B_{\ell_{k}^{-}}}{n_{\ell_{1}^{-}} A_{\ell_{1}^{+}} \cdots A_{\ell_{k}^{+}}} = P_{n} \frac{\overrightarrow{m_{\ell_{n}^{+}}} \overrightarrow{B_{\ell_{n}^{-}}}}{\overrightarrow{n_{\ell_{n}^{+}}} A_{\ell_{n}^{+}}}$$
(3.17)

If the node *n* is related with another node n', than we can say, that  $\widehat{P_n} \in \mathcal{H}_n \otimes \mathcal{H}_{n'}$ , and then we translate the *big* indices coming from  $\mathcal{H}_n^*$  into *small* indices from  $\mathcal{H}_{n'}$ , and denote it as

$$\widehat{P_n} = P_n \frac{\overline{m_{\ell_n^+}} \overline{m_{\ell_n^+}}}{\overrightarrow{m_{\ell_n^-}} n_{\ell_{n'}^-}}$$
(3.18)

In this case all the indices will be called *small* indices. Note, that now each positive index  $A_{\ell_n^+}$  becomes a negative index  $n_{\ell_{n'}^-}$  and vice versa:  $B_{\ell_n^-}$  becomes  $m_{\ell_{n'}^+}$ .

The unweighted diagram operator is obtained by appropriate contraction of indices of all  $\widehat{P_n}$  operators and all contractors  $\langle A_{\Gamma} |$ . Let us make it more precise. Consider the tensor product or all contractors  $\langle \mathbf{A} | := \bigotimes_{\Gamma \in \mathcal{G}} \langle A_{\Gamma} |$  and the tensor product of all operators (the product is taken over the equivalence classes of the edge relation):  $\widehat{\mathbf{P}} := \bigotimes_{[n] \in \mathcal{R}_{edge}} \widehat{P_n}$ . The tensors  $\langle \mathbf{A} |$  and  $\widehat{\mathbf{P}}$  are very multi-index objects. However for each half-link of each graph of the diagram  $\mathcal{D}$  there is precisely one index in  $\langle \mathbf{A} |$ , one small index in  $\mathbf{\hat{P}}$  and zero or one big index in **P**. The big index appears if and only if the half-link is not related to any other half-link by the link relations. If the half-link is positive, the index of  $\langle \mathbf{A} |$  is lower, the small index of  $\widehat{\mathbf{P}}$  is upper and the possible big index of  $\widehat{\mathbf{P}}$  is lower. If the halflink is negative, the index of  $\langle \mathbf{A} |$  is upper, the small index of  $\widehat{\mathbf{P}}$  is lower and the possible big index of **P** is upper. The procedure of diagram-driven contraction<sub> $\mathcal{D}$ </sub> is following: each small index of  $\langle \mathbf{A} |$  is contracted with the small index of **P** corresponding to the same half-link. The big indices remain uncontracted, thus the result of this construction is a tensor having one lower index per each positive half-link of the boundary graph and one upper index per each negative half-link of the boundary graph. Such tensor structure is precisely the same, as the tensor structure of functional of  $\mathcal{H}^*_{\partial \mathcal{D},\vec{i}}$  Hilbert space.

Graphical illustration of the diagram-driven contraction can be found at figure 3.13 Given a boundary state  $|\psi\rangle \in \mathcal{H}_{\partial D, \vec{i}}$  the unweighted transition amplitude is

$$A_{\psi}^{\text{unweighted}} := \langle A_{\mathcal{D}} | | \psi \rangle \tag{3.19}$$



(c)

Figure 3.13: Graphical illustration of diagram-driven contraction. (a) - The contractor  $A_1$  and relation between its indices and the half-links of the diagram (only four of them, in order to keep the figure clear). (b) - An example of an operator  $\hat{P}$  of a pair of nodes related by the  $\mathcal{R}_{node}$ -relation. Some of indices of this operator refer to half-links at the node  $n_1$ , and the others refer to half-links at  $n'_{\cdot}$ . (c) - An example of an operator of a boundary node n' - only half of the indices refer to half-links of the diagram, the other half is left uncontracted.

#### 3.3.2.2 Face and link amplitudes

In the spin-foam models the amplitude is weighted by the so called *face amplitudes*. In fact one can introduce an amplitude for each cell-element of the diagram:  $A_f$  for faces (i.e. equivalence classes of  $\mathcal{R}_{\text{face}}$  relation),  $A_e$  for edges,  $A_v$  for vertices,  $A_n$  for nodes of the boundary graph and  $A_\ell$  for links of the boundary graph. However vertex amplitudes can be obviously incorporated to the contractors  $\langle A_{\Gamma} |$  and edge- and node-amplitudes can be incorporated to the operators  $\widehat{P}_n$ . For face- and boundary link-amplitudes (we will call them *weight* amplitudes) it is not possible in a general case. Thus finally the Operator Spin-network Diagram transition amplitude for a boundary state  $|\psi\rangle$  is

$$A_{\mathcal{D}}(\psi) := \left(\prod_{f \in \mathbf{F}} A_f\right) \cdot \left(\prod_{\ell \in \partial \mathcal{D}} A_\ell\right) \cdot \langle A_{\mathcal{D}} | |\psi\rangle$$
(3.20)

#### 3.3.3 Relation to the spin-foam transition amplitudes

In order to obtain the diagram transition amplitude equal to the BF transition amplitude the following assumptions on the coloring and weight amplitudes must be done:

• Each contractor  $\langle A_{\Gamma} |$  must be the natural contractor  $\langle A^{\mathrm{Tr}} |$  given by the following formula:

$$\left\langle A^{\mathrm{Tr}}\right| = \prod_{\ell \in \mathbf{L}_{\Gamma}} \delta_{m_{\ell^+}}^{n_{\ell^-}} \tag{3.21}$$

- Each operator  $\widehat{P_n}$  must be the identity operator on the invariant node Hilbert space  $\mathcal{H}_n^{\text{Inv}}$  (or in terms of  $\mathcal{H}_n$  it is a projector  $\widehat{P_n^{\text{Inv}}}$ ).
- The face amplitudes must be  $A_f = 2j_f + 1$  and the boundary link amplitudes must be  $A_\ell = \frac{1}{\sqrt{2j_\ell + 1}}$ .

In order to obtain the diagram transition amplitude equal to the EPRL spin-foam transition amplitude, the following assumptions on the coloring and weight amplitudes must be done:

- The nodes operators  $\widehat{P_n}$  and the weight amplitudes  $A_f$  and  $A_\ell$  are like in BF model.
- Each contractor  $\langle A_{\Gamma} |$  must be either the EPRL contractor  $\langle A^{\text{EPRL}} |$  or the natural contractor  $\langle A^{\text{Tr}} |$ . However the later one is allowed only in case of the  $\theta$ -like graphs. The EPRL contractor is given by the following formula:

$$\left\langle A^{\text{EPRL}} \right| = \left( \prod_{n \in \mathbf{N}_{\Gamma}} Y_{\overrightarrow{m_{\ell^+}} \overrightarrow{B_{\ell^-}}}^{\overrightarrow{n_{\ell^-}} \overrightarrow{A_{\ell^+}}} \left[ P_n^{\text{Inv}_G} \right]_{\overrightarrow{A_{\ell^+}} \overrightarrow{D_{\ell^-}}}^{\overrightarrow{B_{\ell^-}} \overrightarrow{C_{\ell^+}}} \right) \cdot \left( \prod_{\ell \in \mathbf{L}_{\Gamma}} \delta_{C_{\ell^+}}^{D_{\ell^-}} \right)$$
(3.22)

where G stands for a group (either SO(4) in case of Euclidean theory or  $SL(2, \mathbb{C})$ in case of Lorentzian theory), the tensor Y is the EPRL map to the group G (see subsection 1.2.2.1), the tensor  $\widehat{P_n^{\text{Inv}_G}}$  is a projection onto the invariant tensors in the representation of G-group, and the big A, B, C and D indices are indices of G-group action. For more details see Appendix A.4.3.

#### 3.3.4 The redundant parts of a 2-complex of an OSD

Note, that the  $\theta$ -like graphs labelled by  $\langle A^{\text{Tr}} |$  correspond to the fragments of spin-foam that do not change the transition amplitude. Indeed, they are (locally) fragments of *BF*-foams, which are invariant under removing bivalent edges and vertices (see [38], see also the calculation in subsection 3.4.1). Thus the only meaning of the  $\theta$ -like graphs labelled by  $\langle A^{\text{Tr}} |$  may by topological: they ensure some topological properties of the 2-complex and of the boundary graph.

Note then, that each of the vertices, that we marked as removable in subsection 3.2.3, had a structure of  $\theta$ -like graphs. Moreover the necessary criteria of choosing them was not changing the topology of the 2-complex. Thus if they were colored by  $\langle A^{\text{Tr}} |$ , they would indeed be irrelevant from the point of view of the transition amplitude calculation. On the other hand in the KKL procedure each vertex referring to some dynamical process was represented by one of graphs of the diagram - so all vertices other then V-type (i.e. other then coming from graphs of the graph diagram) should *not* refer to any dynamical process, and thus there is no reason to color them by anything else then  $\langle A^{\text{Tr}} |$ . So in the end we see, that each removable vertex of 2-complex of a graph diagram is in fact redundant from the point of view of the transition amplitude operator.

Let us thus summarise the redundant parts of a 2-complex of an Operator Spinnetwork Diagram.

The redundant edges are:

- All VX-type edges, i.e. all edges that are traces of the leg-nodes of squids.
- All internal *NX*-type edges, i.e. all edges that are legs of squids glued with another legs of squids.

The redundant vertices are

- All X-type vertices, i.e. leg-nodes of the squids.
- All internal N-type vertices i.e. heads of squids, that were glued to other squids.

## 3.4 Examples and properties of OSDs

Let us now provide some examples of operator spin-network diagrams to illustrate our framework in use. First of all we will describe a diagram corresponding to the amplitude operator equal to identity, called the *trivial* diagram. Such operator can be constructed for an arbitrary boundary spin-network Hilbert space and can be seen as a *free propagator*. Then we will describe a diagram referring to a spin-foam with precisely one nontrivial (interaction) vertex inside. Finally we will show, how to glue two diagrams into a bigger one. In fact this operation allow to construct arbitrary complicated diagrams out of the building blocs presented in first two steps.

#### 3.4.1 The Trivial diagram

Let us now show, how to construct a *trivial* diagram, i.e. such diagram, that the transition amplitude operator related to it will be the identity operator.

Consider a graph  $\mathcal{G} = (\mathbf{N}_{\mathcal{G}}, \mathbf{L}_{\mathcal{G}})$  with every link  $\ell$  colored by a spin  $j_{\ell}$ . This defines a spin-network Hilbert space  $\mathcal{H}_{\mathcal{G},\vec{j}}$ . In order to construct an operator on such Hilbert space we need a diagram with its boundary graph being  $\mathcal{G} \sqcup \mathcal{G}^*$ . Indeed, then the transition



Figure 3.14: Construction of  $\theta$ -like graphs

amplitude operator is an element of a space  $\mathcal{H}_{\mathcal{G},\vec{j}} \otimes \mathcal{H}^*_{\mathcal{G},\vec{j}}$ , so it can be treated as an operator.  $\widehat{A} : \mathcal{H}_{\mathcal{G},\vec{j}} \to \mathcal{H}_{\mathcal{G},\vec{j}}$ .

To make the construction more transparent, let us first explain how to construct a  $\theta$ -like graph of a node n (called  $\theta_n$ ).

- Given a node n let us take a squid  $\lambda_n$  of this node, and a dual squid  $\lambda_n^*$ .
- Each leg-node of  $\lambda_n$  has a corresponding leg-node of  $\lambda_n^*$ . We connect them into one node.
- We introduce the following naming of elements of this graphs:
  - The head node of  $\lambda_n$  will be called n, the head node of  $\lambda_n^*$  will be called  $n^*$ .
  - For each half-link  $\ell^{\epsilon}$  incident to n in the original graph (the sign  $\epsilon$  comes from its relation with n, not with  $n^*$ ) there are two links and one node in the  $\theta$ -like graph. The link incident to the head of  $\lambda_n$  we shell call  $\ell_n^{\epsilon}$ , the link incident to the head of  $\lambda_{n^*}$  will be called  $\ell_{n^*}^{\epsilon}$ . The node between them will be called  $x_{\ell^{\epsilon}}$ .

An example of a  $\theta_n$  graph is presented at figure 3.14.

The trivial diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Id}}$  of the graph  $\mathcal{G}$  is constructed as follows:

- For each node  $n \in \mathbf{N}_{\mathcal{G}}$  we construct a  $\theta$ -like graph  $\theta_n$ . The graph of  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Id}}$  is  $\bigsqcup_{n \in \mathbf{N}} \theta_n$ .
- We introduce the node relation  $\mathcal{R}_{node}$  as follows: For each link  $\ell \in \mathbf{L}_{\mathcal{G}}$  we have two nodes of type  $x_{\ell^{\epsilon}}$ : we have  $x_{\ell^{+}}$  of  $\theta_{s(\ell)}$  and  $x_{\ell^{-}}$  of  $\theta_{t(\ell)}$ . Each pair  $(x_{\ell^{+}}, x_{\ell^{-}})$  is in the  $\mathcal{R}_{node}$  relation.
- We introduce the link relation as follows: for each pair  $(x_{\ell^+}, x_{\ell^-})$  there are two links  $\ell_n^+$  and  $\ell_{n^*}^+$  incident to  $x_{\ell^+}$  and two links  $\ell_{n'}^-$  and  $\ell_{n'^*}^-$  incident to  $x_{\ell^-}$ . The relation  $\mathcal{R}_{\text{link}}^{(x_{\ell^+}, x_{\ell^-})}$  connects  $\ell_n^+$  with  $\ell_{n'}^-$  and  $\ell_{n^*}^+$  with  $\ell_{n'^*}^-$ .

Example of such diagram - together with its interpretation in terms of 2-complexes - can be found at figure 3.15. Note, that the 2-complex is in fact a cylinder over the graph  $\mathcal{G}$ , with an extra set of horizontal internal edges, forming a subcomplex homeomorphic with  $\mathcal{G}$ .

The coloring of the diagram is following:

• Each of links  $\ell_n^{\epsilon}$  and  $\ell_{n^*}^{\epsilon}$  is labelled by the spin  $j_{\ell}$ .



Figure 3.15: Example of a trivial diagram. (a) - The diagram. (b) - Reading the boundary graph of the diagram. (c) - The spin-foam corresponding to the diagram. (d) - Since each horizontal edge is bivalent and labelled i *BF*-manner, thus they can be removed and the resulting foam is a simple cylinder.

- Each node *n* is labelled by the identity operator  $\widehat{P_n} = \mathbb{1}_{\vec{j}}$  on  $\mathcal{H}_n^{\text{Inv}}$ . The node  $n^*$  is labelled by the same operator  $\widehat{P_{n^*}} = \widehat{P_n}$ .
- For each node  $x_{\ell^{\epsilon}}$  the invariant node Hilbert space is  $\mathcal{H}_{x_{\ell}}^{\mathrm{Inv}} = \mathrm{Inv}\left(\mathcal{H}_{j_{\ell}} \otimes \mathcal{H}_{j_{\ell}}^{*}\right) \equiv \mathbb{C}$ . We label each pair  $(x_{\ell^{+}}, x_{\ell^{-}})$  by the identity operator on this space  $\widehat{P_{x_{\ell}}} = \mathbb{1}|_{\mathcal{H}_{x_{\ell}}^{\mathrm{Inv}}}$ .
- Each  $\theta$ -like graph  $\theta_n$  is labelled by the natural contractor  $\langle A^{\mathrm{Tr}} |$ .
- The face amplitude is  $A_f = 2j_f + 1$  and the boundary link amplitude  $A_\ell = \frac{1}{\sqrt{2j_\ell + 1}}$ .

The transition amplitude operator calculated according to subsection 3.3.2 is precisely

$$\widehat{A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}}} = \mathbb{1}$$
(3.23)

Indeed, the following calculation proves above result.

Let us write explicitly the matrix elements of the tensors in the spin basis (i.e. in terms of elements of  $\mathcal{H}_n$ ).

• The contractors  $\langle A_{\theta_n}^{\text{Tr}} |$  expressed in the spin-basis are given by set of deltas:

$$\left\langle A_{\theta_n}^{\rm Tr} \right| = \prod_{\ell^{+/-} \in \mathcal{L}_n^{+/-}} \delta_{m_{\ell^{+/-}}}^{n_{\ell^{+/-}}} \delta_{m_{\ell^{+/-}}^*}^{n_{\ell^{+/-}}^*} \tag{3.24}$$

The  $m_{\ell^{+/-}}$  indices are always associated with the starting node of the link  $\ell^{+/-}$  in  $\theta_n$ , the indices  $n_{\ell^{+/-}}$  are associated with the target node of the link  $\ell^{+/-}$  in  $\theta_n$ . The indices  $m^*_{\ell^{+/-}}$  and  $n^*_{\ell^{+/-}}$  refer to the links  $\ell^{+/-}_{n^*}$ .

• The  $\widehat{P_{x_{\ell}}}$  operators expressed in the spin  $j_{\ell}$  basis are  $\widehat{P_{x_{\ell}}} = \frac{1}{2j_{\ell+1}} \delta_{n_{\ell-}^*}^{m_{\ell-}} \delta_{n_{\ell+}^*}^{m_{\ell+}^*}$  (the naming of the indices is the same as in case of contractors, however now indices  $m_{\ell+}^*$  and  $n_{\ell+}$  refer to the graph  $\theta_n$ , while the indices  $m_{\ell-}$  and  $n_{\ell-}^*$  refer to the graph  $\theta_{n'}$ , where  $n = s(\ell)$  and  $n' = t(\ell)$ ).

• The  $\widehat{P_n}$  operators expressed in spin basis are projection operators  $\widehat{P_n} = \left[P_n^{\text{Inv}}\right] \xrightarrow{\overline{m_{\ell^+}} \overline{B_{\ell^-}}}_{\overline{n_{\ell^-}} \overline{A_{\ell^+}}}$ . The  $\widehat{P_{n^*}}$  operators are the same operators, however their indices are named differently:  $\widehat{P_{n^*}} = \left[P_{n^*}^{\text{Inv}}\right] \xrightarrow{\overline{m_{\ell^-}} \overline{C_{\ell^+}}}_{\overline{n_{\ell^+}} \overline{D_{\ell^-}}}$ .

The unweighted transition amplitude operator is

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Inweighted}}}^{\mathrm{unweighted}} = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left( \prod_{\ell^{+/-} \in \mathcal{L}_{n}^{+/-}} \delta_{m_{\ell^{+/-}}}^{n_{\ell^{+/-}}} \delta_{m_{\ell^{+/-}}}^{n_{\ell^{+/-}}^{*}} \right) \sqcup_{\mathcal{D}} \\ \sqcup_{\mathcal{D}} \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left( [P_{n}^{\mathrm{Inv}}] \xrightarrow{\overrightarrow{m_{\ell^{+}}} \overrightarrow{B_{\ell^{-}}}} \cdot [P_{n^{*}}^{\mathrm{Inv}}] \xrightarrow{\overrightarrow{m_{\ell^{+}}} \overrightarrow{C_{\ell^{+}}}} \right) \cdot$$
(3.25)
$$\cdot \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \left( \frac{1}{2j_{\ell}+1} \delta_{n_{\ell^{-}}}^{m_{\ell^{-}}} \delta_{n_{\ell^{+}}}^{m_{\ell^{+}}^{*}}} \right)$$

the weight amplitude is: since there are two faces per each link, the face amplitude factor is  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} (2j_{\ell} + 1)^2$ , and since there are two boundary links per each link of  $\mathcal{G}$ , the link amplitude factor is  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \frac{1}{(\sqrt{2j_{\ell}+1})^2}$ , thus overall weight factor is  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} (2j_{\ell} + 1)$  and cancels with the factor  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \frac{1}{2j_{\ell}+1}$  coming from the  $\widehat{P_{x_{\ell}}}$  operators. Thus we are left with the index contractions:

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}} = \left[ \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left( \prod_{\ell^{+/-} \in \mathcal{L}_{n}^{+/-}} \delta_{m_{\ell^{+}/-}}^{n_{\ell^{+}/-}} \delta_{m_{\ell^{+}/-}}^{n_{\ell^{+}/-}^{*}} \right) \right] \lrcorner \mathcal{D}$$
$$\lrcorner \mathcal{D} \left[ \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left( \left[ P_{n}^{\mathrm{Inv}} \right]_{\overline{n_{\ell^{+}}} \overline{B_{\ell^{-}}}}^{\overline{m_{\ell^{+}}}} \cdot \left[ P_{n^{*}}^{\mathrm{Inv}} \right]_{\overline{n_{\ell^{+}}} \overline{D_{\ell^{-}}}}^{\overline{m_{\ell^{+}}}} \right) \cdot \left( 3.26 \right)$$
$$\cdot \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \left( \delta_{n_{\ell^{-}}}^{m_{\ell^{-}}} \delta_{n_{\ell^{+}}}^{m_{\ell^{+}}^{*}} \right) \right]$$

Note, that the terms coming from  $\widehat{P_{x_{\ell}}}$  operators can be decomposed into the product over nodes:  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \left( \delta_{n_{\ell^{-}}^{*}}^{m_{\ell^{+}}^{*}} \right) = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left( \prod_{\ell^{+} \in \mathcal{L}_{n}^{+}} \delta_{n_{\ell^{+}}}^{m_{\ell^{+}}^{*}} \right) \cdot \left( \prod_{\ell^{-} \in \mathcal{L}_{n}^{-}} \delta_{n_{\ell^{-}}^{*}}^{m_{\ell^{-}}^{*}} \right)$ , so we end up with

Performing the contraction we can group the deltas:

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}} = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left[ \left[ P_{n}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell}} \to \overrightarrow{A_{\ell+}}}^{\overrightarrow{m_{\ell+}}} \cdot \left( \prod_{\ell^{+} \in \mathcal{L}_{n}^{+}} \delta_{m_{\ell^{+}}}^{n_{\ell^{+}}} \delta_{n_{\ell^{+}}}^{n_{\ell^{+}}^{*}} \delta_{n_{\ell^{+}}}^{n_{\ell^{+}}^{*}} \right) \cdot \left( \prod_{\ell^{-} \in \mathcal{L}_{n}^{-}} \delta_{m_{\ell^{-}}}^{n_{\ell^{-}}} \delta_{n_{\ell^{-}}}^{n_{\ell^{-}}^{*}} \delta_{n_{\ell^{-}}}^{n_{\ell^{-}}^{*}} \right) \cdot \left[ P_{n^{*}}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell^{+}}^{*} \to D_{\ell^{-}}^{*}}}^{\overrightarrow{m_{\ell^{+}}}} \right]$$

$$(3.28)$$

Each triple of deltas in the brackets simplify to either  $\delta_{m_{\ell^+}}^{n_{\ell^+}^*}$  or  $\delta_{m_{e^-}}^{n_{\ell^-}}$ :

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}} = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left[ \left[ P_{n}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell}} - \overrightarrow{A_{\ell}}}^{\overrightarrow{m_{\ell}} + \overrightarrow{B_{\ell}}} \cdot \left( \prod_{\ell^{+} \in \mathcal{L}_{n}^{+}} \delta_{m_{\ell^{+}}}^{n_{\ell^{+}}^{*}} \right) \cdot \left( \prod_{\ell^{-} \in \mathcal{L}_{n}^{-}} \delta_{m_{\ell^{-}}^{*}}^{n_{\ell^{-}}^{*}} \right) \cdot \left[ P_{n^{*}}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell^{+}}^{*} + D_{\ell^{-}}^{*}}}^{\overrightarrow{m_{\ell^{+}}^{*} + A_{\ell^{+}}^{*}}} \\ = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left[ \left[ P_{n}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell^{-}}^{*} + A_{\ell^{+}}^{*}}}^{\overrightarrow{m_{\ell^{+}}^{*} + A_{\ell^{+}}^{*}}} \cdot \delta_{\overrightarrow{m_{\ell^{+}}^{*} + k}}^{\overrightarrow{m_{\ell^{+}}^{*} + k}} \cdot \left[ P_{n^{*}}^{\mathrm{Inv}} \right]_{\overrightarrow{n_{\ell^{+}}^{*} + D_{\ell^{-}}^{*}}}^{\overrightarrow{m_{\ell^{+}}^{*} + D_{\ell^{-}}^{*}}} \right]$$
(3.29)

so at the end of the day, knowing that  $\widehat{P_n^{\text{Inv}}} = \widehat{P_{n^*}^{\text{Inv}}}$ , for each node *n* of  $\mathcal{G}$  we have a product of  $\widehat{P_n^{\text{Inv}}}$  operators

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}} = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left[ \left[ \left( P_{n}^{\mathrm{Inv}} \right)^{2} \right]_{\mathcal{A}_{\ell}^{+} \mathcal{D}_{\ell}^{*}}^{\overrightarrow{B_{\ell}^{-}} \mathcal{C}_{\ell}^{*}} \right]$$
(3.30)

now knowing that in the spin basis  $P_n^{\text{Inv}}$  is a projection operator, we have

$$A_{\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}} = \prod_{n \in \mathbf{N}_{\mathcal{G}}} \left[ \left[ P_{n}^{\mathrm{Inv}} \right]_{\overrightarrow{A_{\ell} + D_{\ell}^{*}}}^{\overrightarrow{B_{\ell} - C_{\ell}^{*}}} \right]$$
(3.31)

which is precisely the identity operator on the gauge invariant spin-network Hilbert space  $\mathcal{H}_{\mathcal{G},\vec{j}}$  (see subsection A.3.3).

The coloring of the diagram we introduced here is in fact the *BF*-coloring. Note, that the *BF*-model has a very convenient property, namely each bivalent edge in *BF*-spin-foam can be removed with no consequences for the transition amplitude (see [38]). Indeed, it can be derived from the form of contractor. Thus the transition amplitude of  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  is equal to the transition amplitude of the foam being a simple cylinder over a graph  $\mathcal{G}$  (see figure 3.15).

#### 3.4.2 The one-interaction diagram

Consider now a following process. The initial state is given by a spin-network state supported on a graph  $\mathcal{G}_{in}$ . There is precisely one interaction vertex represented by a connected graph  $\Gamma$ . We pick some nodes  $n_1, \ldots, n_k$  of  $\mathcal{G}_{in}$  and relate them with some nodes  $n'_1, \ldots, n'_k$  respectively of  $\Gamma$ . This process is represented by a spin-foam with precisely one interaction vertex. It can be treated as a Feynman diagram with a very simple interaction.

The Operator Spin-network Diagram of such process is called *one-interaction diagram*. We will characterise such diagrams in this subsection. They are also studied in section 6.3 of [53]. The one-interaction Operator Spin-network Diagram can be always constructed by the following procedure.

- 1. Construct a trivial diagram  $\mathcal{D}_{\mathcal{G}_{in}}^{\mathrm{Tr}}$  of a graph  $\mathcal{G}_{in}$ .
- 2. Add the graph  $\Gamma$  to the set of graphs of  $\mathcal{D}_{\mathcal{G}_{in}}^{\mathrm{Tr}}$ .
- 3. Extend the node relation as follows. For each node  $n_i$  of the nodes  $n_1, \ldots, n_k$  that participate in the interaction take the node  $n_i^*$  of the graph  $\theta_{n_i}$ . Extend the  $\mathcal{R}_{node}$  relation by the pairs  $(n_i^*, n_i')$ , where  $n_i' \in \mathbf{N}_{\Gamma}$  (note, that  $n_i'$  must be dual to  $n_i^*$ , and thus it must be isomorphic to  $n_i$ ).
- 4. Extend the link relation as follows. For each pair  $(n_i^*, n_i')$  introduce a duality map  $\phi_{(n_i^*, n_i')}$  and the link relation induced by this map. The maps may be arbitrary, however one has to keep an eye on the consistency of the coloring of the diagram (see 5b).
- 5. Color the graph as follows:
  - (a) The trivial diagram is colored as it was indicated in the previous subsection.
  - (b) The links of the graph  $\Gamma$  that are incident to the nodes  $n'_i$  has spins indicated by the spins of links of  $\mathcal{G}_{in}$ . Note, that this indicates a constraint on the duality maps chosen in 4: in order to obtain non-vanishing transition amplitude, one has to ensure that two half-links of the same link are never connected to two links with different spins.
  - (c) The links of  $\Gamma$  that are not incident to any of  $n'_i$  are colored by arbitrary spins.
  - (d) The nodes of Γ are colored by the invariant projectors of an appropriate node Hilbert spaces.
  - (e) The graph  $\Gamma$  itself is colored by  $\langle A^{\text{EPRL}} |$  contractor.

We will denote such diagram by  $\mathcal{D}_{\mathcal{G}_{in},\Gamma}$ . An example of a one-interaction diagram is presented at figure 3.16

It is interesting to study the boundary of one-interaction diagrams. The boundary graph of  $\mathcal{D}_{\mathcal{G}_{in},\Gamma}$  can be always decomposed into two disjoint components:

$$\partial \left( \mathcal{D}_{\mathcal{G}_{\text{in}},\Gamma} \right) = \mathcal{G}_{\text{in}} \sqcup \mathcal{G}_{\text{out}} \tag{3.32}$$

The first disjoint component is precisely the initial graph  $\mathcal{G}_{in}$  (of course in the general case it does not have to be connected, what we emphasised by using the letter  $\mathcal{G}$ ). The second component  $\mathcal{G}_{out}$  is determined by the following algorithm:

- 1. Take the graph  $\mathcal{G}_{in}$  and reverse its links, obtaining  $\overline{\mathcal{G}_{in}}$ .
- 2. For each node  $n_i \in \mathbf{N}_{\mathcal{G}_{in}}$  that was connected to the interaction vertex, there is a node  $\overline{n_i} \in \mathbf{N}_{\overline{\mathcal{G}_{in}}}$ . Remove each  $\overline{n_i}$  together with the half-links incident to it.
- 3. Remove each node  $n'_i \in \mathbf{N}_{\Gamma}$  that was connected to one of initial graph's nodes. Remove it together with the half-links incident to it.
- 4. Whenever in step 2 of in step 3 two half-links of a link are removed consider all the link removed. If only one half-link is removed consider it as a temporary open half-link.



Figure 3.16: Two examples of one-interaction diagrams. (a) - A very simple example. (b) - The spin-foam corresponding to the diagram presented at (a). (c) - A more complicated example (link relations between the trivial part of the diagram and the interaction graph are indicated by letters, i.e. a goes with a', b with b' etc.). The spin-foam corresponding to this example is hard to draw, thus we do omit it.

5. Note, that each removed  $\overline{n_i}$  node was dual to the corresponding removed  $n'_i$  node. Thus for each positive open half-link that is left in  $\Gamma$  because of removing  $n'_i$  there is a negative open half-link in  $\overline{\mathcal{G}_{in}}$  left there because of removing  $\overline{n_i}$ . Moreover there is a map between these half-links induced by the duality map  $\phi_{\{n^*_i,n'_i\}}$  that we picked in point 4 of the previous procedure (in fact the map  $\phi_{\{n^*_i,n'_i\}}$  relates the half-link incident to the nodes, however for each temporary open half-link there is precisely one half-link incident to the removed node, so the generalisation of the duality map is natural). Thus let us connect these open half-links according to the map and obtain  $\mathcal{G}_{out}$ .

The coloring of the out-graph is in natural way induced by the coloring of  $\mathcal{G}_{in}$  and of  $\Gamma$ .

This procedure is very similar to the one used in subsection 3.2.3.4 to find the boundary graph of a graph diagram, however here instead of the graph of a diagram we use the initial graph. For graphical illustration of this procedure see figure 3.17.

#### 3.4.3 Glueing of diagrams

Consider two graphs  $\mathcal{G}$  and  $\mathcal{G}'$  with decomposition into connected components given by  $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_n\}$  and  $\mathcal{G}' = \{\Gamma'_1, \ldots, \Gamma'_m\}$ . We say that  $\mathcal{G}$  and  $\mathcal{G}'$  have a compatible connected component iff there is a pair of connected graphs  $(\Gamma_i, \Gamma'_j)$ , where  $\Gamma_i$  is a connected component of  $\mathcal{G}$ ,  $\Gamma'_j$  is a connected component of  $\mathcal{G}'$  and  $\Gamma_i = \overline{\Gamma'_j}$ . We say, that  $\mathcal{G}$  and  $\mathcal{G}'$  have k compatible connected components iff there are k pairs  $(\Gamma_{i_1}, \Gamma'_{j_1}), \ldots, (\Gamma_{i_k}, \Gamma'_{j_k})$  satisfying the above condition (each connected component  $\Gamma_{i_a}, \Gamma'_{j_b}$  may appear in these pairs at most once).

Given a spin-network Hilbert spaces  $\mathcal{H}_{\mathcal{G},\vec{j}}$  and  $\mathcal{H}_{\mathcal{G}',\vec{j}'}$  supported on the graphs  $\mathcal{G}$  and  $\mathcal{G}'$  respectively we say, that they have k compatible connected component, if the graphs  $\mathcal{G}$  and  $\mathcal{G}'$  have k compatible connected components, and for each pair  $\left(\Gamma_{i_a}, \Gamma'_{j_a}\right)$  spins of



Figure 3.17: Graphical illustration of the procedure of finding boundary of a oneinteraction diagram, presented for the diagram of figure 3.16a.

the corresponding links match.

1

Let us now consider two diagrams:

$$\mathcal{D}_1 = (\mathcal{G}_1, (\mathcal{R}_{\text{node},1}, \mathcal{R}_{\text{link},1}); C_1) \quad \text{and} \quad \mathcal{D}_2 = (\mathcal{G}_2, (\mathcal{R}_{\text{node},2}, \mathcal{R}_{\text{link},2}); C_2)$$
(3.33)

such that their boundary graph Hilbert spaces has k compatible connected components. We can glue the diagrams along these components. The resulting diagram  $\mathcal{D}_{1+2}$  is defined up to a spin-network automorphism-induced duality maps  $\phi_a : \Gamma_{i_a} \to \Gamma'_{j_a}$  for each pair of compatible connected components (see subsection 3.1.2.3). Let us thus pick k spin-network duality maps  $\phi_1, \ldots, \phi_k$  and describe the glued diagram.

- 1. The graph of  $\mathcal{D}_{1+2}$  is simply the disjoint sum of graphs of the ingredients:  $\mathcal{G}_{1+2} = \mathcal{G}_1 \sqcup \mathcal{G}_2.$
- 2. The node relation  $\mathcal{R}_{node,1+2}$  is the sum of the node relations  $\mathcal{R}_{node,1}$  and  $\mathcal{R}_{node,2}$ , extended by the gluing node-relation.
  - The gluing node relation  $\mathcal{R}_{node}^{\text{gluing}}$  is defined as follows: each node  $n_I^{(B)}$  of the boundary graph  $\partial \mathcal{D}_1$  (or  $\partial \mathcal{D}_2$ ) has its corresponding node  $n_I$  in the graph  $\mathcal{G}_1$  (or  $\mathcal{G}_2$  respectively), being unrelated with any other node. Suppose, that  $n_I \in \mathcal{G}_1$  is an unrelated node such that  $n_I^{(B)}$  belongs to a component  $\Gamma_{i_a}$  that we are gluing along. Then there is a node  $n_J^{(B)} = \phi_a\left(n_I^{(B)}\right)$  of the component  $\Gamma'_{j_a}$  and its corresponding unrelated node  $n_J \in \mathcal{G}_2$ . Thus we put a pair  $(n_I, n_J)$  into the relation  $\mathcal{R}_{node}^{\text{gluing}}$ .

Having defined the  $\mathcal{R}_{node}^{gluing}$  we can formally define  $\mathcal{R}_{node,1+2}$  as

$$\mathcal{R}_{\text{node},1+2} = \mathcal{R}_{\text{node},1} \cup \mathcal{R}_{\text{node},2} \cup \mathcal{R}_{\text{node}}^{\text{gluing}}$$
(3.34)

which means that two nodes of  $\mathcal{G}_{1+2}$  are in node relation iff they are either in  $\mathcal{R}_{node,1}$  relation, or in  $\mathcal{R}_{node,2}$  relation, or in  $\mathcal{R}_{node}^{gluing}$  relation.

3. The link relation is defined analogously:

$$\mathcal{R}_{\text{link},1+2} = \mathcal{R}_{\text{link},1} \cup \mathcal{R}_{\text{link},2} \cup \mathcal{R}_{\text{link}}^{\text{gluing}}$$
(3.35)

where the  $\mathcal{R}_{\text{link}}^{\text{gluing}}$  is defined as follows. Given a pair of nodes  $(n_1, n_2)$  in  $\mathcal{R}_{\text{node}}^{\text{gluing}}$  relation, there is a duality map between them induced by the graph morphism  $\phi_a$  of the components they belong to (see subsection 3.1.2.1). This duality map induces the  $\mathcal{R}_{\text{link}}^{(n_1,n_2)}$  being a part of the family  $\mathcal{R}_{\text{link}}^{\text{gluing}}$ .

4. The coloring of links is completely induced by the colorings  $C_1$  and  $C_2$  since no new links appeared. The same works for the coloring of graphs by contractors.

The coloring of nodes needs more care. It is also induced at these nodes, that do not belong to any pair of  $\mathcal{R}_{node}^{\text{gluing}}$  relation. For the pairs of nodes that do belong to  $\mathcal{R}_{node}^{\text{gluing}}$  we need to do some extra adjustment. Namely if  $(n_1, n_2) \in \mathcal{R}_{node}^{\text{link}}$ , we color both the nodes by the operator  $\widehat{P_{(n_1,n_2)}} := \widehat{P_{n_1}} \circ \widehat{P_{n_2}}$ . This composition is permitted, because the node Hilbert spaces of  $n_1$  and  $n_2$  are dual (see subsection 3.1.2.3 and Appendix A.3.2). Note, that if the diagram is colored in one of the schemes presented in subsection 3.3.3, the operators  $\widehat{P_{n_1}}$  and  $\widehat{P_{n_2}}$  are projections, thus the operator  $\widehat{P_{(n_1,n_2)}} = \widehat{P_{n_1}} = \widehat{P_{n_2}}$ .

The transition amplitude of a composed diagram is the composition of the transition amplitudes of the diagrams being composed. To prove it consider a decomposition of the graph  $\partial \mathcal{D}_1$  into  $\partial \mathcal{D}_1^{\text{free}} \sqcup \partial \mathcal{D}_1^{\text{glued}}$ , where  $\partial \mathcal{D}_1^{\text{glued}} := \Gamma_{i_1} \sqcup \Gamma_{i_2} \sqcup \cdots \sqcup \Gamma_{i_k}$  is the sum of the components compatible with  $\partial \mathcal{D}_2$ . We can decompose  $\partial \mathcal{D}_2$  analogously into  $\partial \mathcal{D}_2^{\text{free}} \sqcup \partial \mathcal{D}_2^{\text{glued}}$  where  $\partial \mathcal{D}_2^{\text{glued}} = \left(\partial \mathcal{D}_1^{\text{glued}}\right)^*$ . Then, according to subsection 1.2.2.3, we can treat the transition amplitude  $\langle A_{\mathcal{D}_1} | \in \left(\mathcal{H}_{\partial \mathcal{D}_1^{\text{free}}} \otimes \mathcal{H}_{\partial \mathcal{D}_1^{\text{glued}}}\right)^*$  as an operator  $\widehat{A_{\mathcal{D}_1}} : \mathcal{H}_{\partial \mathcal{D}_1^{\text{free}}} \to \mathcal{H}^*_{\partial \mathcal{D}_1^{\text{glued}}}$ , and similarly  $\langle A_{\mathcal{D}_2} | \in \left(\mathcal{H}_{\partial \mathcal{D}_2^{\text{free}}} \otimes \mathcal{H}_{\partial \mathcal{D}_2^{\text{glued}}}\right)^*$  as an operator  $\widehat{A_{\mathcal{D}_2}} : \mathcal{H}_{\partial \mathcal{D}_2^{\text{glued}}} \to \mathcal{H}^*_{\partial \mathcal{D}_2^{\text{free}}}$ . Now since  $\partial \mathcal{D}_2^{\text{glued}} = \left(\partial \mathcal{D}_1^{\text{glued}}\right)^*$ , the corresponding Hilbert spaces are also dual:  $\mathcal{H}_{\partial \mathcal{D}_2^{\text{glued}}} = \left(\mathcal{H}_{\partial \mathcal{D}_1^{\text{glued}}}\right)^*$ , so the image of  $\widehat{A_{\mathcal{D}_1}}$  is in the domain of  $\widehat{A_{\mathcal{D}_2}}$ , so the operators can be composed.

The composition of these operators would be given by taking the tensor product of them and then contracting all the indices, that refer to half-links of  $\mathcal{G}_{1/2}^{\text{glued}}$ :

$$\widehat{A_{\mathcal{D}_1}} \circ \widehat{A_{\mathcal{D}_2}} = [A_{\mathcal{D}_1} \otimes A_{\mathcal{D}_2}] \lrcorner \prod_{\ell^{\epsilon} \in \mathcal{L}_{\mathcal{G}_1^{\text{glued}}}} \delta_{n_{\ell^{\epsilon}}}^{m_{\ell^{\epsilon}}}$$
(3.36)

where  $m_{\ell^{\epsilon}}$  is the index referring to a half-link of  $\partial \mathcal{D}_{1}^{\text{glued}}$  and  $n_{\ell^{\epsilon}}$  is referring to a half-link of  $\partial \mathcal{D}_{2}^{\text{glued}}$  iff  $\epsilon = +$ , and opposite if  $\epsilon = -$ . But this contraction is precisely what is done, when the operator  $\widehat{P_{(n_{1},n_{2})}} := \widehat{P_{n_{1}}} \circ \widehat{P_{n_{2}}}$  is calculated. So we already know, that the unweighted amplitude operator is the composition of unweighed amplitude operators of the ingredients:

$$\widehat{A_{\mathcal{D}_{1+2}}^{\text{unweighted}}} = A_{\mathcal{D}_1}^{\text{unweighted}} \circ A_{\mathcal{D}_2}^{\text{unweighted}}$$
(3.37)

To prove, that the weighted amplitude operator matches, we have to check the face and link amplitude factors. Again only the amplitudes referring to  $\partial \mathcal{D}_{1/2}^{\text{glued}}$  matters, because all the other terms in the weight amplitude are simply multiplied. In the diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$  each link  $\ell$  of  $\partial \mathcal{D}_{1/2}^{\text{glued}}$  corresponds to an open equivalence class of the face relations (one in each diagram, i.e.  $f_{\ell}^{(1)}$  of  $\mathcal{R}_{\text{face},1}$  and  $f_{\ell}^{(2)}$  of  $\mathcal{R}_{\text{face},2}$ ). Thus the corresponding weight factors are  $A_{f_{\ell}^{(1)}} \cdot A_{\ell}$  for  $\ell \in \mathbf{L}_{\partial \mathcal{D}_1^{\text{glued}}}$  and  $A_{f_{\ell}^{(2)}} \cdot A_{\ell^*}$  for  $\ell^* \in \mathbf{L}_{\partial \mathcal{D}_2^{\text{glued}}}$ , so the overall

factor is

$$A^{(\ell),\text{unglued}} = A_{f_{\ell}^{(1)}} \cdot A_{\ell} \cdot A_{f_{\ell}^{(2)}} \cdot A_{\ell^*} = (2j+1) \frac{1}{\sqrt{2j+1}} (2j+1) \frac{1}{\sqrt{2j+1}} = 2j+1$$
(3.38)

In the diagram  $\mathcal{D}_{1+2}$  each link  $\ell \in \partial \mathcal{D}_1^{\text{glued}}$  corresponds to a closed equivalence class of the face relation. Indeed, the positive boundary member of each equivalence class  $f_{\ell}^{(1)}$  is glued with the negative boundary member of  $f_{\ell}^{(2)}$  and vice versa - so the half-links form a cyclic series, and the face  $f_{\ell}$  is internal. Thus the corresponding weight factor is

$$A^{(\ell),\text{glued}} = A_{f_{\ell}} = 2j + 1 = A^{(\ell),\text{unglued}}$$
(3.39)

So the weight factors are the same in the composed operator  $\widehat{A_{D_1}} \circ \widehat{A_{D_2}}$  and in the operator  $\widehat{A_{D_{1+2}}}$ . Thus at the end of the day we have

$$\widehat{A_{\mathcal{D}_{1+2}}} = \widehat{A_{\mathcal{D}_1}} \circ \widehat{A_{\mathcal{D}_2}} \tag{3.40}$$

 $Quod\ erat\ demonstrandum.$ 

Two remarks on the glueing procedure should be done. First: the glueing procedure can be easily generalised to the case, where the boundary graphs  $\partial \mathcal{D}_1$  and  $\partial \mathcal{D}_2$  do not have to have compatible connected components, but only a number of compatible nodes. The procedure is analogous, but the property (3.40) does not have to be satisfied any more. Second: the very special case is gluing the trivial diagram  $\mathcal{D}_{\Gamma}^{\text{Tr}}$  because it's operator is the identity operator. These remarks are discussed in subsection 3.4.3.1 and subsection 3.4.3.2 respectively.

#### 3.4.3.1 Generalisation of the gluing procedure

Consider two diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$  with the boundary graphs  $\partial \mathcal{D}_1$  and  $\partial \mathcal{D}_2$  respectively. Let us pick k pairs of pairwise dual nodes  $(n_i, n'_i)_{i=1,...,k}$  and the duality maps  $\{\phi_i\}_{i=1,...,k}$  such that  $n_i \in \mathbf{N}_{\partial \mathcal{D}_1}$  and  $n'_i \in \mathbf{N}_{\partial \mathcal{D}_1}$  and each duality map is compatible with the spinnetwork structure of the graphs (i.e. each  $\phi_i$  preserves the coloring of the links with spin-labels). Then we can glue the two diagrams along these pairs of nodes, and the resulting diagram  $\mathcal{D}_{1+2}$  is defined as follows:

- 1. The graph of the diagram  $\mathcal{D}_{1+2}$  is just the sum of graphs of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  as in the step 1 of the previous procedure.
- 2. The node and link relations are defined by the same extension of ingredient relations, as it was presented in the steps 2 and 3 of the previous procedure. However now the  $\mathcal{R}_{node}^{gluing}$  connects the nodes of the selected pairs  $(n_i, n'_i)_{i=1,...,k}$  which do not have to form any subgraph of any of graphs, and the  $\mathcal{R}_{face}^{gluing}$  is indicated by the picked duality maps $\{\phi_i\}_{i=1,...,k}$ , instead of graph automorphisms.
- 3. The coloring of  $\mathcal{D}_{1+2}$  is defined analogously to the one defined in the step 4 of previous procedure: the glued nodes are colored by  $\widehat{P_{(n_i,n'_i)}} := \widehat{P_{n_i}} \circ \widehat{P_{n'_i}}$ , the coloring of all the other cells is precisely the same as the coloring of  $\mathcal{D}_1$  or  $\mathcal{D}_2$ .

To study the transition amplitude of a result of generalised glueing procedure first note, that each spin-network Hilbert space can be decomposed into product of node Hilbert space (see (3.3)):

$$\mathcal{H}_{\mathcal{G}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \mathcal{H}_n^{\mathrm{Inv}} \tag{3.41}$$

thus if we named two sets of glued nodes  $\mathbf{N}_1 := \{n_i\}_{i=1,\dots,k}$  and  $\mathbf{N}_2 := \{n'_i\}_{i=1,\dots,k}$ , we can say, that

$$\mathcal{H}_{\partial \mathcal{D}_1} = \bigotimes_{n \in \mathbf{N}_{\partial \mathcal{D}_1} \setminus \mathbf{N}_1} \mathcal{H}_n^{\mathrm{Inv}} \otimes \bigotimes_{n \in \mathbf{N}_1} \mathcal{H}_n^{\mathrm{Inv}} =: \mathcal{H}_1^* \otimes \mathcal{H}^{\mathrm{glued}}$$
(3.42)

and

$$\mathcal{H}_{\partial \mathcal{D}_2} = \bigotimes_{n \in \mathbf{N}_{\partial \mathcal{D}_2} \setminus \mathbf{N}_2} \mathcal{H}_n^{\mathrm{Inv}} \otimes \bigotimes_{n \in \mathbf{N}_2} \mathcal{H}_n^{\mathrm{Inv}} =: \mathcal{H}_2 \otimes \left(\mathcal{H}^{\mathrm{glued}}\right)^*$$
(3.43)

so  $\widehat{A_{\mathcal{D}_1}} : \mathcal{H}_1 \to \mathcal{H}^{\text{glued}}$  and  $\widehat{A_{\mathcal{D}_2}} : \mathcal{H}^{\text{glued}} \to \mathcal{H}_2$ , so composing these two amplitudes is admissible. And in fact the unweighted amplitude of a composed diagram is the composition of unweighted amplitudes:

$$\widehat{A_{\mathcal{D}_{1+2}}^{\text{unweighted}}} = A_{\mathcal{D}_1}^{\widehat{\text{unweighted}}} \circ A_{\mathcal{D}_2}^{\widehat{\text{unweighted}}}$$
(3.44)

However when we consider the weight factors, the following problem arise: in the previous procedure we were always sure, that either all the boundary link were glued to another boundary link, or it was left unglued. Thus in the resulting diagram each face containing a link of glued subgraphs was an internal face being a composition of two external faces of the ingredient diagrams. That's why the identity (3.39) holds. Conversely in the generalised procedure it may happen that only one half-link of a boundary link is glued, and thus the resulting diagram contains an external face being a composition of two external faces of the ingredient diagrams (see figure 3.18a). Even more complicated scenarios may happen: a face of the resulting diagram may be composed of many more than two faces of the ingredient diagram (see figure 3.18b). Thus the weight factors of  $A_{\mathcal{D}_{1+2}}$  are related to the weight factors of  $A_{\mathcal{D}_1}$  and  $A_{\mathcal{D}_2}$  in a nontrivial, glueing dependent way. We shell call this difference of weight factor the gluing weight factor and denote as  $A^{\text{gluing}}$ , thus

$$\widehat{A_{\mathcal{D}_{1+2}}} = A^{\text{gluing}} \cdot \widehat{A_{\mathcal{D}_1}} \circ \widehat{A_{\mathcal{D}_2}}$$
(3.45)

The gluing weight factor is a product over pairs of glued nodes of node gluing weight factor  $A^{\text{gluing}} = \prod_{i=1}^{k} A^{\text{gluing}}_{(n_i,n'_i)}$ , which are products of terms related to each glued pair of half-links  $A^{\text{gluing}}_{(n_i,n_i')} = \prod_{\ell^{\epsilon} \in \mathcal{L}_{n_i}} A^{\text{gluing}}_{(\ell^{\epsilon},\ell'-\epsilon)}$ . The gluing weight factor of any pair  $(\ell^{\epsilon}, \ell'^{-\epsilon})$  is in general  $A^{\text{gluing}}_{(\ell^{\epsilon},\ell'-\epsilon)} = \frac{1}{A_{\ell}\cdot A_{f_{\ell}}}$  - because it reduces by one both number of boundary links and number of internal faces. The only exception is when two half-links of the same boundary link are glued (two examples can be found at figure 3.18c and figure 3.18d) - then the gluing weight factor is  $A^{\text{gluing}}_{(\ell^{\epsilon},\ell'-\epsilon)} = \frac{1}{A_{\ell}}$  - because number of faces does not change. The gluing weight factor should be calculated node by node, i.e. when calculating  $A^{\text{gluing}}_{(n_i,n'_i)}$  one asks whether two glued half-links are of the same boundary link of the boundary graph of diagram after the pairs of nodes  $(n_1, n'_1), \ldots, (n_{i-1}, n'_{i-1})$  had been already glued.

The generalised gluing procedure in fact is simply extending of the node and link relations. An example of the generalised gluing procedure is each one-interaction diagram (see subsection 3.4.2) being the trivial diagram (see subsection 3.4.1) glued to a one-vertex diagram (i.e. a diagram with graph with precisely one connected component - see subsection 3.2.2.1). We will also use it a lot in the next chapter while defining the algorithm to find all diagrams with a fixed boundary (see section §4.2).



Figure 3.18: Examples of different configurations of gluing weight factor. For simplicity assume that each half-link at  $n_I$  and  $n'_{II}$  is positive and each half-link at  $n'_I$  and  $n_{II}$ is negative. (a) - The diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are glued via the nodes  $n_I$  and  $n'_I$ . The links  $\ell_1$  and  $\ell'_1$  were external faces of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively. After gluing they are form a single external face of  $\mathcal{D}_{1+2}$ . (b) - The diagrams were glued also via  $n_{II}$  and  $n'_{II}$ . The duality map  $\phi_{n_{II} \rightarrow n'_{II}}$  was chosen in such a way, that now the series of links  $\{\ell'_5, \ell_1, \ell'_1, \ell_2, \ell'_2, \ell_3, \ell'_3, \ell_4\}$  for a single external face of  $\mathcal{D}_{1+2}$ . Before the gluings each of these links belonged to separate external face of  $\mathcal{D}_1$  or  $\mathcal{D}_2$ . (c) - Simple example of closing the face: the duality map  $\phi_{n_{II} \to n'_{II}}$  was chosen in such a way, that  $\{\ell_1, \ell'_1\}$  is an internal face of  $\mathcal{D}_{1+2}$ . The gluing weight factor for connecting  $n_{II}$  and  $n'_{II}$  is  $A_{(\ell_1^-, \ell_1^{++})}^{\text{gluing}} = \frac{1}{A_{\ell_1}}$ . (d) - More complicated example of closing the face: Before gluing  $n_{II}$  with  $n'_{II}$  there were had three external faces  $\{\ell_1, \ell'_1\}, \{\ell_2, \ell'_2\}$  and  $\{\ell_3, \ell'_3\}$ , which now became one internal face  $\{\ell_1, \ell'_1, \ell_2, \ell'_2, \ell_3, \ell'_3\}$ . The gluing weight factor is  $\frac{1}{A_{\ell_1}} = \frac{1}{A_{\ell_2}} = \frac{1}{A_{\ell_3}}$ .



Figure 3.19: Equivalent chains of diagrams  $\mathcal{D}_{\theta}$  of  $\theta$ -like graphs.

#### 3.4.3.2 Gluing of the trivial diagram.

Second remark refers to the properties of the trivial diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$ . As we have proven in subsection 3.4.1, the transition amplitude of the trivial diagram is precisely the identity operator. Thus given a diagram  $\mathcal{D} = (\mathcal{G}, \mathcal{R}, C)$  we can glue it with the trivial diagram  $\mathcal{D}_{\Gamma}^{\mathrm{Tr}}$ of any connected component  $\Gamma \in \mathcal{G}$  without changing neither the transition amplitude nor the boundary structure of the diagram. Moreover the 2-complex of a diagram  $\mathcal{D}$  is equivalent to the 2-complex of the diagram  $\mathcal{D}$  glued with  $\mathcal{D}_{\Gamma}^{\mathrm{Tr}}$ . Thus we will consider these two diagrams equivalent.

Moreover, compiling the concept of trivial diagrams with the generalised gluing procedure, consider a diagram  $\mathcal{D}_{\theta_n}$  being a simple  $\theta$ -like graph (of some node n), with void relations, and colored by the  $A^{\text{Tr}}$  contractor. Given an arbitrary diagram  $\mathcal{D}$  such that its boundary has a node  $n^*$  dual to n one can glue this diagram to  $\mathcal{D}_{\theta_n}$  with no effects to neither the transition amplitude nor the topology.

One can go even one step further. Consider a series of  $\mathcal{D}_{\theta_n}$  diagrams, glued together into a chain (see figure 3.19). The input of such chain into a transition amplitude of any graph is precisely the same as the input of one  $\theta$ -like graph - what can be proven analogously to the calculation presented in subsection 3.4.1. The bivalent nodes in this chain has no influence on the spin-network structure of the boundary, thus they can be removed harmlessly. The only meaning of them is when they are related in a nontrivial way to other nodes of the diagram. Then they effect the topology of the 2-complex of the diagram.

#### 3.4.4 Disconnected diagrams

Consider now a diagram  $\mathcal{D}$  such that its graph  $\mathcal{G}$  consists of two disconnected parts  $\mathcal{G}_1$ and  $\mathcal{G}_2$  (each of them may have arbitrary number of connected components), such that in the node relation there is no pair containing one node from  $\mathcal{G}_1$  and one node from  $\mathcal{G}_2$ . We say, that such diagram is *disconnected* (or that the graph  $\mathcal{G}$  of  $\mathcal{D}$  is *diagram-disconnected* - see also definition 5.8 in subsection 5.1.1). An operator spin-network diagram based on a disconnected graph diagram can be treated as two independent quantum processes  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Indeed, its diagram operator factorizes into a tensor product of two operators, one related to  $\mathcal{G}_1$  and the other to  $\mathcal{G}_2$ . One can see it easily by considering the formula (3.14). Let us recall it here:

$$\langle A_{\mathcal{D}} | = \left( \bigotimes_{\Gamma \in \mathcal{G}} \langle A_{\Gamma} | \right) \lrcorner_{\mathcal{D}} \left( \bigotimes_{[n] \in \mathcal{R}_{edge}} \widehat{P_n} \right) \cdot \left( \prod_{f \in \mathcal{R}_{face}} A_f \right) \cdot \left( \prod_{\ell \in \partial \mathcal{D}} A_\ell \right)$$
(3.46)

The contractor part of the formula is a product over connected components of  $\mathcal{G}$ , thus it can be naturally split into

$$\bigotimes_{\Gamma \in \mathcal{G}} \langle A_{\Gamma} | = \left( \bigotimes_{\Gamma \in \mathcal{G}_1} \langle A_{\Gamma} | \right) \otimes \left( \bigotimes_{\Gamma \in \mathcal{G}_2} \langle A_{\Gamma} | \right)$$
(3.47)

The projector part is a product over equivalence classes of edge relation. But since there is no pairs in  $\mathcal{R}_{node}$  that mix nodes of  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , this product again split into

$$\bigotimes_{[n]\in\mathcal{R}_{edge}}\widehat{P_n} = \left(\bigotimes_{[n]\in\mathcal{R}_{edge}:n\in\mathbf{N}_{\mathcal{G}_1}}\widehat{P_n}\right) \otimes \left(\bigotimes_{[n]\in\mathcal{R}_{edge}:n\in\mathbf{N}_{\mathcal{G}_2}}\widehat{P_n}\right)$$
(3.48)

The same happens for face amplitudes - since there are no pairs of nodes linking  $\mathcal{G}_1$  with  $\mathcal{G}_2$ , there must not be any equivalence classes of face relation containing links from both  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , thus

$$\prod_{f \in \mathcal{R}_{\text{face}}} A_f = \left(\prod_{f \in \mathcal{R}_{\text{face}}: \ell_f \in \mathbf{L}_{\mathcal{G}_1}} A_f\right) \otimes \left(\prod_{f \in \mathcal{R}_{\text{face}}: \ell_f \in \mathbf{L}_{\mathcal{G}_2}} A_f\right)$$
(3.49)

and for the same reason there are no boundary links that come from one half-link of  $\mathcal{G}_1$ and one half-link of  $\mathcal{G}_2$ , so

$$\prod_{\ell \in \partial \mathcal{D}} A_{\ell} = \left(\prod_{\ell \in \partial \mathcal{D}: \ell^{\pm} \in \mathbf{B}_{\mathcal{G}_1}} A_{\ell}\right) \otimes \left(\prod_{\ell \in \partial \mathcal{D}: \ell^{\pm} \in \mathbf{B}_{\mathcal{G}_2}} A_{\ell}\right)$$
(3.50)

The diagram driven contraction  $_{\mathcal{D}}$  connects only the indices that refer to the half-links, that are somehow connected within the diagram (i.e. either they are the same link or there is a link-relation), thus it does not mix the terms referring to  $\mathcal{G}_1$  with the terms referring to  $\mathcal{G}_2$ , thus the overall amplitude can be written as

$$\langle A_{\mathcal{D}} | = \langle A_{\mathcal{D}_1} | \otimes \langle A_{\mathcal{D}_2} | \tag{3.51}$$

# 3.5 Conclusions and discussion

We developed a framework in which one can describe the structure of a spin-foam in terms of 2-dimentional drawings, that we named graph diagrams. This framework is capacious, it allows many schemes of coloring a foam by SU(2)-tensors, among which there are standard BF model and both Euclidean and Lorentzian EPRL models (see subsection 3.3.3), but there are many more. In fact our framework allows even more

general approach, i.e. to consider spin-foam models for an arbitrary gauge group G -what is discussed in subsection 3.5.2.

It is worth to notice, that our approach do not change the calculation of a transition amplitude of a foam. Every step of the diagram-driven contraction (introduced in subsection 3.3.2.1) was designed in such a way, that the final formulae agrees with the standard BF or EPRL computation (if appropriate choice of contractors was done).

Our framework however has some input on the calculation of total transition amplitude being the sum over all quantum processes between the *in* and *out* states. The graph diagrams precise the list of processes, that should be taken into account in such summing - see subsection 3.5.1.

Next chapter describes, how to use operator spin-network diagrams when addressing concrete questions. We provide there a detailed algorithm to find and order all diagrams with some properties (i.e. with fixed boundary). We also show, how they can be used to simplify the process of reading the formula of the transition amplitude out of a diagram - by adopting Feynman-like rules of spin-foams [37] to Feynman-like rules of diagrams.

#### 3.5.1 The class of 2-complexes

Our framework provide an answer to the question raised in section §2.1: what class of 2-complexes is appropriate in defining the spin-foam models for Loop Quantum Gravity?

Let us recall the question with more details. The spin-foam models claim, that the transition amplitude of a boundary state  $|\psi\rangle$  is

$$A_{\psi} = \sum_{\kappa:\partial\kappa=\psi} A_{\kappa} \tag{3.52}$$

The question is: the spin-foams  $\kappa$  in the above formula are elements of what class? Which of them are considered equivalent, and thus counted as one and the same? Are there any 2-complex that we consider (for some reason) inadmissible?

Our answer is: the class of the 2-complexes should be all the 2-complexes that can be obtained out of a graph-diagram by the procedure described in subsection 3.2.2. Moreover, two 2-complexes that differ only by some removable cells colored in BF manner are considered the same diagram.

The answer above although being strict, is not very convenient. What one would expect is a set of assertions listing the properties that the admissible 2-complex must or must not satisfy. We have such list (see below), however we do not know whether it is complete.

1. The boundary vertices are always incident to precisely one internal edge.

Whenever two or more internal edges meet, an interaction takes place. The interaction vertices are always internal vertices of the 2-complex (which is a straightforward conclusion of constructing the one-vertex foams in subsection 3.2.2.1). An example of inadmissible situation together with explanation, how to correct it, is shown at figure 3.20

2. Each internal vertex's structure can be represented by a *connected* graph.

This is an obvious conclusion of the fact, that in subsection 3.2.2.1 we were creating the one-vertex foams only for connected components of the graph  $\mathcal{G}$ .

To be more precise - in our framework we admit the vertices with structure defined by a disconnected graph. One can obtain it by gluing together the vertices represented by the connected component of it. However the input to the transition



Figure 3.20: Admissible boundary vertices. (a) - The forbidden situation, where three internal edges meet at the boundary. (b) - Possible representation of similar process in our framework.

amplitude coming from such a vertex is precisely the product of the inputs of vertices of the connected components. Thus we simply treat a foam containing such vertices as equivalent to the foam with the connected components separated - see figure 3.21.

3. All removable cells that are colored in BF scheme are irrelevant in calculation of the transition amplitude. Thus we treat each foams that differ only by a number of such cells as equivalent - see figure 3.22.

In the points 2 and 3 of the above enumeration we claimed some classes of spin-foams being equivalent. One of the reasons we did it was the fact, that the transition amplitude of them was always equal. Obviously this was not the only reason. The other one was a common sense. It might happen, that there are many spin-foams giving the same transition amplitude for the same boundary conditions, however representing qualitatively different processes. This decision was our choice, based on our understanding what should be considered physical in the spin-foam models. One could provide a selfconsistent framework without these assumptions. Of course the final distinction should be done by comparing transition amplitudes computed in both approaches with some experimentally measurable quantities.

#### **3.5.2** Generalisation to groups other then SU(2)

The framework presented in this chapter can be easily generalised to the theories of dynamical connections of an arbitrary group, not only SU(2). This of course require redefining the rules of coloring. The general rules, for an arbitrary group G, should be as follows:

• Links of the graph of a diagram should be colored by unitary representations  $\rho_{\ell}$  of the group G (perhaps the framework might be generalised also for the coloring by non-unitary representations, the author however did not studied this issue).



Figure 3.21: Vertices with structure described by a disconnected graph. The transition amplitude of the fragment of a spin-foam presented at (a) is precisely the same, as the transition amplitude of the one presented at (b). We treat them as equivalent.



Figure 3.22: The two spin-foams  $\kappa_1$  and  $\kappa_2$  differ only by a removable edge  $V_1V_2$  and two removable vertices  $V_1$  and  $V_2$ . The vertices are colored by  $\langle A^{\rm Tr} |$  and the edge is colored by  $\widehat{P^{\rm Inv}}$ , thus it is the *BF* coloring. They spin-foams  $\kappa_1$  and  $\kappa_2$  are equivalent, which means that when summing the transition amplitudes, we will always count only one of them.

- Nodes of the graph of a diagram should be colored by operators  $\widehat{P}_n : \mathcal{H}_n \to \mathcal{H}_n$ acting on the *node Hilbert spaces*  $\mathcal{H}_n^G := \left(\bigotimes_{\ell^+ \in \mathcal{L}_n^+} \mathcal{H}_{\rho_\ell}\right) \otimes \left(\bigotimes_{\ell^- \in \mathcal{L}_n^-} \mathcal{H}_{\rho_\ell}^*\right)$ . However for the author it seams convenient to use everywhere  $\mathcal{H}_n^{G, \text{Inv}}$  being the *G*-invariant subspace of  $\mathcal{H}_n^G$ .
- Connected components of the graph of a diagram should be colored by contractors  $\langle A_{\Gamma} | \in (\mathcal{H}_{\Gamma}^G)^*$  where  $\mathcal{H}_{\Gamma}^G$  is the Hilbert space of *G*-spin-networks with the support on the graph  $\Gamma$ . A *G*-spin-network is a gauge-invariant, square-integrable function on *L* copies of the group *G*, where the gauge action is determined by the structure of the graph  $\Gamma$  (i.e. the gauge transformations act in nodes and act only on the variables related to the links incident to the node see also Appendix A.3.3.3). Thus  $\mathcal{H}_{\Gamma}^G = L^2 \left( \frac{G^L}{G^N} \right)$ , where *L* and *N* are numbers of links and nodes of  $\Gamma$ ., and the gauge action is determined by the graph structure.
- Face and link weight factors are some functions depending on the representation  $\rho_f$ and  $\rho_\ell$ . They might be related to the dimension of the representation, but they do not have to.

By comparing the above rules with the subsection 3.3.1 one can clearly see, that the coloring scheme presented there is precisely an application of this general rules to the SU(2) group.

# Chapter 4

# Boundary Formalism for OSD and its application in Dipole Cosmology

In chapter 3 we have introduced the formalism of Operator Spin-network Diagrams (OSDs) as an alternative language for Spin-Foams theories. A spin-foam - and equivalently a graph diagram - can be treated as a quantum process between the *in* and *out* states determined by its boundary, as it was discussed in subsection 1.2.2.3. Considering any quantum transition amplitude one has to take into account all possible elementary processes have some property - like for example a specific boundary condition. An important issue arises: how to find and characterise all such processes? As we have mentioned in section §3.5, the operator spin-network diagrams provides a convenient tool to find all spin-foams (i.e. all OSDs) that have a fixed, given boundary graph. This chapter is devoted to this problem.

This chapter is organised as follows. First in we describe a decomposition of an OSD into very elementary building blocks. The graphs and relations of half-links we were using in the previous chapter are already a simplification when compared to 2-complexes, however, here we will find that it is still not the elementary level - each graph is in fact a set of nodes connected in some way, and the structure of each node is encoded in a squid what will be described in detail in section \$4.1. Then, in section \$4.2, we will describe the main part of this chapter, namely the algorithm of finding all diagrams (and thus all spinfoams) with a fixed boundary. The algorithm's input is the boundary graph. We read out of the graph, which elements of the diagram are necessary to have such boundary, then we add an arbitrary number of extra building blocks and connect them in an arbitrary way to obtain one of the result diagrams. In fact for each boundary graph there are infinitely many diagrams, however, the procedure order them in a well organised series. One of the additional results of our procedure and of the diagrammatic representation of quantum processes is a simple way to read out the formulae for the transition amplitude out of a diagram. These *Feynman-EPRL* rules, adopted from the Feynman rules for spin-foams [37], are presented in section §4.3. The tools presented in this chapter are applied to the model of Dipole Cosmology (described in [55, 56, 57, 58], recalled in section §2.2). The section §4.4 presents the application of our algorithm to find all the diagrams in the first order. The section §4.5 presents, how to use the Feynman-EPRL rules to calculate their input to the transition amplitude.

This chapter is based mainly on [64]. The paper is a common work of the Author and of Marcin Kisielowski. However, the results presented in sections 4.1, 4.2 and 4.3 were mainly effects of the Author's research. Application of these results to the example of Dipole Cosmology model was done mainly by Marcin Kisielowski.

## 4.1 Elementary building blocks of OSD

There are two main representation of the spin-network states: the holonomy representation and the invariant tensors representation. The first one focus on the links, along which the integrals of the underlying spin-connection are taken (see Appendix A.3.3). In the later one main role is played by the nodes the graph - where the tensors seat. In this chapter we will use only the tensor representation of the spin-network states, because it simplifies identifying the elementary building blocks of states.

The spin-network Hilbert space of a graph  $\mathcal{G}$  with fixed spin-labelling  $\vec{j}$  can be decomposed into

$$\mathcal{H}_{\mathcal{G},\vec{j}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \mathcal{H}_{n}^{\mathrm{Inv}} \quad \text{where } \mathcal{H}_{n}^{\mathrm{Inv}} = \mathrm{Inv} \left[ \bigotimes_{\ell^{\epsilon} \in \mathcal{L}_{n}} \mathcal{H}_{j_{\ell}}^{\epsilon} \right]$$
(4.1)

One can see that what is crucial for the structure of the Hilbert space is the structure of the nodes, not the particular way of connection between them. Of course adjacency relations matter when an evaluation of a spin-network state  $|\psi\rangle \in \mathcal{H}_{\mathcal{G},\vec{j}}$  on a certain field configuration  $\{u_\ell\}_{\ell \in \mathbf{L}_{\mathcal{G}}}$  is taken:

$$\psi\left(u_{\ell}\right) = \left\|\left\langle u_{\ell}\right| \left|\psi\right\rangle\right\| := \left(\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} D^{j_{\ell}} \left(u_{\ell}\right)_{m_{\ell^{+}}}^{n_{\ell^{-}}}\right) \cdot \left[\psi\right]_{\overrightarrow{n_{\ell^{-}}}}^{\overrightarrow{m_{\ell^{+}}}}$$
(4.2)

where the way, in which the indices are contracted is determined by the links. However, the tensor structure of the Hilbert space  $\mathcal{H}_{\mathcal{G},\vec{j}}$  is indifferent on neighbour relations of the nodes. The only thing that matters is the tensor structure of the node Hilbert spaces. But we have already introduced an object that captures the structure of a node: a squid (see subsection 3.1.2.1 and Appendix A.1.1.4).

In the end, the full information contained in a spin-network state can be encoded in a set of colored squids and an instruction, how to connect their legs. To obtain an operator spin-network diagram we need to enhance this structure by a set of duality maps between some pairs of heads of squids, which will induce the node and link relations.

Next three subsections concretise the intuitions presented in above paragraph and illustrate them by appropriate figures. In subsection 4.1.1 we provide a strict definition of an *oriented squid set* (OSS), in subsection 4.1.2 we show, how to construct all possible OSDs from a given OSS, and in subsection 4.1.3 we discuss in what way the coloring of squids influence all the construction.

#### 4.1.1 Squids and oriented squid sets

The notion of *squids* was introduced in subsection 3.1.2.1 and it was described in details in Appendix A.1.1.4. A squid is a graph with one node (*head*) in the middle and several links (*legs*) incident with it - thus its shape is reminiscent of a squid or an octopus (that's the origin of the name). The precise definition is as follows:

#### **Definition 4.1.** Oriented squid

An oriented squid is a graph  $\lambda = (\mathbf{N}_{\lambda}, \mathbf{L}_{\lambda})$ , where  $\mathbf{N}_{\lambda} = \{n\} \cup \{x_1, \ldots, x_k\}$  and  $\mathbf{L}_{\lambda} = \{\ell_1, \ldots, \ell_k\}$  and each link  $\ell_i$  (a leg) connects the node n (the head) with the node  $x_i$  (a leg-node). A link (leg) is called outgoing or positive, iff  $s(\ell_i) = n \wedge t(\ell_i) = x_i$ . A link (leg) is called incoming or negative in the opposite case, namely  $s(\ell_j) = x_j \wedge t(\ell_j) = n$ . The number k is called the valency of the squid. The numbers of positive/negative legs  $k^{+/-}$  are called positive/negative valency respectively. An example of a squid can be found at figure 4.1a.

**Remark 4.1.** Squids are always open graphs. In fact they are the only open graphs we ever consider in this section. Whenever in what follows we mention a *graph* - we mean *closed*.

The squids are *oriented*, because the orientation of their legs matters. Such definition of squids were used in [64] (in contrary to [53], where all the legs of all squids were by definition oriented positive - thus actually their orientation had no meaning and the squids were unoriented). The word *oriented* in terms of both squids and squid-sets is used here to keep consistency with [64] notation. However, we often drop this word, because both in this and in previous chapter we use almost only the oriented squids (whenever we consider the unoriented ones, we explicitly mention it).

Let us introduce a notion of oriented squid sets. A general oriented squid set S is simply a set of oriented squids:  $S = \{\lambda_1, \ldots, \lambda_N\}$  and does not contain any extra information - see figure 4.1b. However, it is convenient to define a squid set of a graph  $\mathcal{G}$ .

#### **Definition 4.2.** Oriented squid set of a graph

An oriented squid set of a graph  $\mathcal{G}$  is a collection of squids: one per each node of  $\mathcal{G}$ , capturing the structure of this node. Namely  $\mathcal{S}_{\mathcal{G}} := \{\lambda_n : n \in \mathbf{N}_{\mathcal{G}}\}$ . Graphically one can express it by erasing a point from interior of each link of  $\mathcal{G}$ , and then "shaking" the whole think, so that the disconnected parts of each link "go apart" - see figure 4.1c.

Note, that two different graphs may have the same squid set. This was a hint for us to treat the squid, not the graph as an elementary building block in our constructions.

Not every squid set S is a squid set of any graph. This happens if and only if the total number of positive legs of all squids of a squid set equals to the total number of negative legs. In such case such procedure is obviously not unique (except some very special cases). This gives us a way to describe graphs: to get a graph one can provide its squid set an a prescription of connecting its legs (which we will call the closing of a squid set) - see figure 4.2.

#### 4.1.2 Constructing an OSD from an oriented squid set

Let us now describe, how to find all graph diagrams sharing a given oriented squid set S. We have noted already, that a graph can be constructed out of a squid-set by indicating, which legs should be connected. To obtain a graph diagram, one needs to introduce furthermore a node relation and a family of link relations. In order to do it, note that all the possible nodes to be related are in fact heads of the squids of the squid set. Thus one needs to pick some pairs of dual squids and introduce the duality maps (see subsection A.1.1.4) between them. This leads us to the following algorithm of constructing all graph diagrams:

- 1. Squids: fix a squid set S (see figure 4.3). Ensure it may be closed into a graph (namely the total numbers of incoming and outgoing legs are equal).
- 2. Node relation: choose N pairs of dual squids  $\{\lambda_i, \lambda'_i\}_{i=1,...,N}$  nothing, that a squid can appear at most once among the pairs. The relation  $\mathcal{R}_{node}$  is such, that heads of the chosen squids  $\lambda_i$  and  $\lambda'_i$  are related to each other, and heads of other squids are not related at all (see figure 4.4).
- 3. Link relation: for each pair of squids chosen in previous step pick a duality map  $\phi_i : \lambda_i \to \lambda'_i$ . The duality map induces the node relation, as it was described in subsection 3.2.1.2.



Figure 4.1: Oriented squid set's basics. (a) - An oriented squid with  $k^+ = 4$  outgoing (positive) and  $k^- = 2$  incoming (negative) legs. (b) - An example of an oriented squid set. (c) - A graph (on the lhs) and its oriented squid set (on the rhs) consisting of four three-legged squids.



Figure 4.2: Three examples of closing of an oriented squid set. (a) - An oriented squid set (lhs) and a unique graph obtained by gluings the legs (rhs). (b) - An oriented squid set whose legs cannot be glued to give a closed oriented graph. (c) - An oriented squid set (lhs) and three different oriented graphs (rhs), each of which can be obtained by different closing prescription.



Figure 4.3: The step 1 of the squid-set algorithm: An oriented squid set  $\mathcal{S}$ .



Figure 4.4: The step 2 of the squid-set algorithm: Three pairs of squids have been chosen to be related by the node relation.



Figure 4.5: The step 4 of the squid-set algorithm: One of the graph diagrams that can be constructed from S (the link relation is omitted).

- 4. Closing the graph: connect the leg-node of each outgoing leg with the leg-node of exactly one incoming leg of either the same of a different squid (see figure 4.5).
- 5. Iterating: repeat each of above steps, such that all possible choices are done precisely once (see for example figure 4.6). For example:
  - (a) In step 4 perform all possible options of connecting the graph
  - (b) In step 3 pick all possible combinations of duality maps, and hence link relations.
  - (c) In step 2 pick all possible combinations of dual pairs.

Note, that step 4 commutes with 2 and 3. We choose to put it at the end to emphasise the role of squid set over the particular choice of graph. However, it can be done as well before choosing the pairs to be node-related (as it was presented in chapter 3), or after choosing the node relation, but before the link relation (as it is expressed at figure 4.6).

The steps 2 and 3 obviously do *not* commute - the link relation is strictly dependent to the node relation and cannot be determined before picking related nodes.

We will call the above procedure the squid-set algorithm. Each graph diagram obtained in the squid set algorithm becomes an OSD after coloring by SU(2)-tensors - as



Figure 4.6: The step 5 of the squid-set algorithm: Three graph diagrams obtained by different choices of link relation for diagram of figure 4.5. Diagrams (a) and (b) can be mapped one to another by an automorphism of the right graph. The diagram (c) is essentially different from (a) and (b), because they cannot be mapped on it by any graph automorphism. In total, there are 16 different graph diagrams for this choice of the node relation and graph-closing (which reduces to 4 diagrams, when applying the graph automorphisms), and many more for arbitrary choice of a node relation and graph closings.

it was described in subsection 3.3.1. To obtain all diagrams one needs to iterate over all possible coloring, where *possible* means consistent with the diagram structure. It might be however convenient to determine the coloring already at the level of squids, so at the step 1. Such approach causes some constraint on the above procedure, which are discussed in the next subsection.

#### 4.1.3 Constraints on oriented squid sets caused by coloring

In previous subsection we constructed a graph diagram, and then considered only these colorings, that are consistent with its structure. Of course the inversed approach is also valid: one can consider from the very beginning a *colored oriented squid set* and for each step of the squid-set algorithm allow only such connections, that the consistency of the coloring is held. Such variation of the procedure will be called the *colored-squid-set algorithm*.

A squid captures the structure of graph's node, thus a colored squid captures the structure of a spin-network's node. We will consider squids colored in the following way: each leg  $\ell$  of a squid is colored by a spin  $j_{\ell}$  indicating a unitary SU(2)-representation. The head n of a squid is colored by an operator  $\widehat{P}_n : \mathcal{H}_n^{\text{Inv}} \to \mathcal{H}_n^{\text{Inv}}$ , where  $\mathcal{H}_n^{\text{Inv}}$  is the invariant part of the node Hilbert space  $\mathcal{H}_n := \bigotimes_{\ell} \mathcal{H}_{j_{\ell}}^{\epsilon}$  where  $\epsilon$  stands for \* (i.e. the dual representation space) for the negative leg and stands for nothing (i.e. the representation space) for the positive leg (see subsection 3.1.2.3).

The coloring of the squids allows to construct an Operator Spin-network Diagram, not just a graph diagram out of the squids. However, it cause certain constraints on the construction. To be precise let us list them:

- Closing the graph: when considering colored squids, one can connect only legs of the same coloring in the step 4 of the squid-set algorithm. This induce an extra constraint on the oriented colored squid set to be closable: now it is not enough to have the same total number of incoming and outgoing legs. One has to ensure, that there is the same number of positive and negative legs colored by each spin.
- Node relations: When in the step 2 of the squid-set algorithm one chooses the pairs to be in the node relation one has to ensure, that not only the squids are dual, but also the node Hilbert spaces are dual. This requirement allows to compose the operators attached to the related heads, thus in the resulting OSD each related pair of squids  $\{\lambda, \lambda'\}$  is labelled by the operator  $\widehat{P_{\{n,n'\}}} = \widehat{P_n} \circ \widehat{P_{n'}}$ .
- Link relations: the duality maps chosen at step 3 of the squid-set algorithm must be consistent with the coloring by mapping each leg onto the leg colored by the same spin (i.e. they must be the node-Hilbert spaces duality maps).
- The procedure based on the oriented squid sets do not create any restrictions on the coloring by the contractors (besides the requirement that they have to be in a contractor Hilbert space indicated by the graphs structure). The only difference with respect to the previous chapter is that here at the beginning of the procedure we do not know, how many connected components the graph of our diagram will have.

#### 4.1.4 Summary of the squid sets

The approach of *(colored-)squid-set algorithm* presented in this section allows to control the complexity of the diagram by the following measures: the number of internal edges
(namely the number of pairs of squids related by the node relation), the number of disconnected components of the graph (which gives the number of internal vertices of the spin-foam) and the complexity of each component (i.e. the the number of legs of each squid - the more legs, the more complicated graph it produces).

However, from the point of view of the physical application, what is more important is the boundary part of the diagram. Here it is build out of all the squids that were not chosen to be paired at the step 2 of the squid-set algorithm. In each particular case the squid set can be decomposed into the boundary and the internal part, but this decomposition varies during the procedure. Thus in order to keep the boundary structure of considered diagrams under control, we developed an improved algorithm, presented in the next section.

# 4.2 The algorithm finding all diagrams with a given boundary

As we have mentioned in previous section, the (colored-)squid-set algorithm is a tool to find all diagrams described by a certain squid set, however, in the physical applications it is rather the boundary, not the squid set, that we are given. In this section we will study the problem of finding all diagrams with a given fixed boundary.

Note, that the drawback of the squid-set algorithm was that we had no control on the boundary of our diagram. Nevertheless, the procedure of finding all diagrams with a given squid set might still be useful in problem considered here. The boundary structure of the diagram depend only on these squids, that are not related with any others. Thus we should keep control on the squids that are necessary to reconstructed the given boundary, and enhance our squid-set by all possible *pairs* of squids, that *must* be pairwise related by the node relation (what ensures that they will not influence the boundary). The subsection 4.2.1 explains our concept in detail.

There are two possible approaches to construct an operator spin-network diagrams using the graph diagrams obtained in our procedure: one can first construct all possible graph diagrams and then for each diagram consider its colorings, or one can color the input data (namely the boundary graph) at first, and then consider only the diagrams that are compatible with this particular coloring. The subsection 4.2.2 presents the first approach, the subsection 4.2.3 explains, how to adopt the algorithm presented in subsection 4.2.2 to the second approach.

## 4.2.1 The concept

In subsection 3.4.1 we have introduced a graph diagram corresponding to the *static spin-foam* of an arbitrary graph  $\mathcal{G}$ , describing the trivial evolution of the graph. Such graph diagram was denoted by  $\mathcal{D}_{\mathcal{G}}^{\text{Tr}}$  and is called the *static graph diagram of*  $\mathcal{G}$ . The boundary of  $\mathcal{D}_{\mathcal{G}}^{\text{Tr}}$  is the disjoint union  $\mathcal{G} \sqcup \mathcal{G}^*$ . The static graph diagram colored in the *BF*-manner represents the identity operator on the boundary graph's Hilbert space.

All the diagrams constructed by the improved algorithm, called the *fixed-boundary* algorithm (introduced below), will have the boundary  $\mathcal{G}$  given as the input, and they will have a subdiagram being the trivial diagram of the boundary  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$ . Given a boundary graph  $\mathcal{G}$ , our construction provide all the diagrams  $\mathcal{D}$  with the boundary  $\partial \mathcal{D} = \mathcal{G}$ , modulo the trivial subdiagram. In other words, in terms of spin-foams, we construct all the spinfoams which boundary is the graph  $\mathcal{G}$ , such that the neighbourhood of the boundary is a cylinder  $\mathcal{G} \times [0, 1]$ . The key trick behind our construction is the following observation: given a graph  $\mathcal{G}$  of the squid set  $\mathcal{S}_{\mathcal{G}}$  and an arbitrary graph diagram  $\mathcal{D}_{int}$  called an *interaction diagram*, whose boundary graph  $\partial \mathcal{D}_{int}$  has the same squid set:

$$\mathcal{S}_{\partial \mathcal{D}_{\text{int}}} = \mathcal{S}_{\mathcal{G}} \tag{4.3}$$

we can combine the diagram  $\mathcal{D}_{int}$  with the trivial diagram  $\mathcal{D}_{\mathcal{G}}^{Tr}$  into a new graph diagram  $\mathcal{D}_{\mathcal{G}}^{Tr} \# \mathcal{D}_{int}$  such that the graph  $\mathcal{G}$  is the boundary of the combined diagram. We achieve that by gluing each node of the graph  $\mathcal{G}^*$  of the boundary of  $\mathcal{D}_{\mathcal{G}}^{Tr}$  with one of the nodes of the boundary of  $\mathcal{D}_{int}$  according to the generalised gluing procedure described in subsection 3.4.3.1. It is possible thanks to the matching of the squid-sets (4.3): the squid set of  $\partial \mathcal{D}_{\mathcal{G}}^{Tr}$  is  $\mathcal{S}_{\mathcal{G}} \cup \mathcal{S}_{\mathcal{G}^*}$  and thus each squid of the boundary of  $\mathcal{D}_{int}$  is for sure dual to one of the squids of  $\mathcal{S}_{\mathcal{G}^*}$ . The only squids left unrelated are those of  $\mathcal{S}_{\mathcal{G}}$  and they form a graph  $\mathcal{G}$  being the boundary of the resulting diagram.

The identification of the squid sets  $S_{\mathcal{G}^*}$  and  $S_{\partial \mathcal{D}_{int}}$  is defined module symmetries of  $S_{\mathcal{G}^*}$  (exchanging identical squids, different duality maps). Of course there are more such symmetries, then simply the automorphisms of  $\mathcal{G}$ , what enriches the set of results of our procedure.

Below we implement this idea in detail.

## 4.2.2 The algorithm

The input of the fixed-boundary algorithm is an unoriented boundary graph  $|\mathcal{G}|$ . We consider an unoriented graph, because given two oriented graphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$  such that  $|\mathcal{G}_1| = |\mathcal{G}_2|$  there is always a unique unitary map identifying  $\mathcal{H}_{\mathcal{G}_1}$  and  $\mathcal{H}_{\mathcal{G}_2}$ . Thus from the physical point of view these two Hilbert spaces represent the same sets of states (see subsection 1.1.2.4, in fact two diagrams obtained by our procedure that differ only by some reorientations of links are considered one and the same diagram - see subsection 3.4.4). Nevertheless a specific orientation of the boundary graph may cause some interaction diagrams forbidden. However, since we want to find *all* processes with the fixed boundary, we need to consider all possible orientations.

The fixed-boundary algorithm goes as follows: given a boundary unoriented graph  $|\mathcal{G}|$  we pick one of its possible orientations  $\mathcal{G}$  and construct the trivial diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$ , then construct all possible interaction diagrams  $\mathcal{D}_{\mathrm{int}}$  compatible with  $\mathcal{G}$ , and finally glue them together in all possible ways. The first step was described in detail in subsection 3.4.1, thus let us just recall it at figure 4.7. The last step was discussed in subsection 3.4.3.1, thus we also just illustrate it at figure 4.8. The middle step, namely constructing all interaction diagrams compatible with  $\mathcal{G}$ , uses the squid-set algorithm (see subsection 4.1.2), with certain constrains imposed on the squid-sets considered. All the procedure is described below step by step.

- 1.  $\mathcal{G}$ : choose an orientation of each link of  $|\mathcal{G}|$ . The result is an oriented graph (figure 4.9). Keep in mind the static diagram of this graph, that will be recalled in the step 4.
- 2.  $S_{\text{int}}$ : Take the squid set  $S_{\mathcal{G}}$  of the graph  $\mathcal{G}$  and add to it N pairs of squids  $\lambda_1, \lambda'_1, \ldots, \lambda_N, \lambda'_N$ , such that for each pair the squid  $\lambda_i$  is dual to  $\lambda'_i$ . Denote the resulting squid set as  $S_{\text{int}}$ . Within each pair  $(\lambda_i, \lambda'_i)$  the heads of the squids are related by the node relation (figure 4.10). No other node relation will be introduced (compare this step with the steps 1 and 2 of the squid-set algorithm).



Figure 4.7: A static diagram. (a) - A given graph  $\mathcal{G}$  consisting of two connected components. (b) - The corresponding trivial graph diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  build of  $\theta$ -graphs and suitable defined node and link relations (the link relation is omitted). (c) - The scheme of building a trivial foam from the diagram. (d) - The boundary graph  $\partial \mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  of the diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  is the disjoint union of  $\mathcal{G}$  and  $\mathcal{G}^*$ .



Figure 4.8: The static diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  of figure 4.7 glued to an interaction diagram  $\mathcal{D}_{\mathrm{int}}^{\mathrm{Tr}}$ . The result is the diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}} \# \mathcal{D}_{\mathrm{int}}$  (link relations are omitted).



Figure 4.9: The step 1 of the fixed-boundary algorithm: choosing the orientation.



Figure 4.10: The step 2 of the fixed-boundary algorithm: (a) -  $S_{\mathcal{G}}$ , which becomes  $S_{\text{int}}$  in case of N = 0 pairs of squids added. (b) -  $S_{\text{int}}$  obtained by adding N = 2 pairs of squids paired by a node relation.



Figure 4.11: The step 3 of the fixed-boundary algorithm: glueing of a graph diagram  $\mathcal{D}_{int}$  (one of several possibilities) from the squid set  $\mathcal{S}_{int}$  presented at figures figure 4.10a and figure 4.10b respectively. The dotted lines mark the glueing of the legs of the squids. The link relation is not drawn, to keep the clarity of the figure.

- 3.  $\mathcal{D}_{int}$ : take the squid set  $\mathcal{S}_{int}$  with the node relation as defined in the previous step and apply to it the steps 3 and 4 of the squid-set algorithm (see subsection 4.1.2), namely choose the link relation at each pair of related squids and define gluing of the legs of the squids, such that they can be connected into closed graphs. The resulting diagram we will denote by  $\mathcal{D}_{int}$  (see figure 4.11).
- 4.  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}} \# \mathcal{D}_{\mathrm{int}}$ : take the static graph diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  of the boundary graph  $\mathcal{G}$  and glue it to the interaction diagram  $\mathcal{D}_{\mathrm{int}}$ , as it was explained in subsection 4.2.1 (see figure 4.12). In order to perform the glueing, pick an automorphism of the oriented squid set  $\mathcal{S}_{\mathcal{G}} \subset \mathcal{S}_{\mathrm{int}}$  covering the unrelated nodes of  $\mathcal{D}_{\mathrm{int}}$ . This automorphism will characterise the  $\mathcal{R}_{\mathrm{node}}^{\mathrm{gluing}}$  and  $\mathcal{R}_{\mathrm{link}}^{\mathrm{gluing}}$  relations.
- 5. Coloring: define an arbitrary coloring of the diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}} \# \mathcal{D}_{\mathrm{int}}$ , which turns it into an operator spin-network diagram (see figure 4.13). An alternative approach is possible, namely one can pick the coloring at the very beginning or the procedure and then in the step 3 perform gluing and relating of the edges consistently with the coloring chosen, i.e. using the colored-squid-set algorithm instead of the squid-set algorithm (see subsection 4.1.3).
- 6. Consider all possible: orientations of  $|\mathcal{G}|$  in the step 1, N-tuples of pairs of squids added to  $\mathcal{S}_{\mathcal{G}}$  in the step 2, ways of connecting the legs of  $\mathcal{S}_{int}$ , link relations for each



Figure 4.12: Step 4 of the fixed-boundary algorithm: the graph diagram  $\mathcal{D}_{int}$  of figures 4.11a and 4.11b respectively, combined with the trivial diagram  $\mathcal{D}_{\mathcal{G}}^{Tr}$  of the graph  $\mathcal{G}$  of figure 4.9 into the final graph diagram  $\mathcal{D}_{int} \# \mathcal{D}_{\mathcal{G}}^{Tr}$ .



Figure 4.13: The graph diagram of figure figure 4.12a colored in two possible ways. The coloring of the boundary trivial diagram by operators and contractors is omitted for transparency of the figure (each pair of nodes in this part of the diagram is colored by appropriate  $1_j$ , each graph is colored by  $A_{\theta}^{\text{Tr}}$ ). The coloring of (a) represents diagram with trivial evolution on the vertical edges. The coloring of (b) represents diagram with the EPRL norm calculated for each vertical edge.

 $\lambda_i, \lambda'_i$  in the step 3, ways of gluing the trivial diagram in the step 4 and colorings in the step 5.

The construction presented above allows us to control the level of complexity of resulting diagram by the level of complexity of the diagram  $\mathcal{D}_{int}$ . The complexity can be measured by the number of pairs of the nodes related by the node relation, i.e. the number of internal edges. The simplest case is zero internal edges, i.e. the squid set

$$S_{\rm int}^{(N=0)} = S_{\mathcal{G}} \tag{4.4}$$

A general example of the interaction graph diagram  $\mathcal{D}_{int}$  in  $N^{th}$  order is build of a graph  $\tilde{\mathcal{G}}_{int}$  and a node relation  $\mathcal{R}_{node}^{int}$  consisting of exactly N pairs of nodes. Increasing the number, we increase the complexity of the diagram  $\mathcal{D}_{\mathcal{G}}^{Tr} \# \mathcal{D}_{int}$ .

**Remark 4.2.** One remark on the orientations should be done. In fact in order to find all diagrams with the boundary fixed by an unoriented graph, it may be convenient to consider unoriented squids in the interaction squid set  $S_{int}$ . We choose to use the oriented ones to keep the control on the gluing of the boundary static diagram to the interaction diagram. One could choose the other approach: one could construct all the interaction diagram using unoriented squids (then closing the legs simplifies - one can connect each two legs), and then when the orientation of the boundary is chosen - use only these interaction diagrams, that can be oriented consistently with the boundary. Anyway, even using this approach the unoriented squid may be only the ones that comes from  $S_G$ . The added pairs of the squids must be oriented, because we must ensure that they are dual within each pair - to be connectible with the node relation.

## 4.2.3 Constraints on the coloring

Consider a boundary state  $|\psi\rangle \in \mathcal{H}_{\mathcal{G},\vec{j}}$  with a fixed spins. Obviously any diagram with boundary spins different than the ones indicated by  $\vec{j}$  gives zero contribution to the transition amplitude of such state. Thus it is reasonable to consider an algorithm for which the input graph will have fixed spins - we will call it the *colored-fixed-boundary algorithm*.

In such approach we slightly change the procedure presented in previous subsection:

- In the step 2 the squids of the squid set  $S_{\mathcal{G}}$  have already chosen their coloring. This of course influence the coloring of squids of  $S_{int}$  that come from  $S_{\mathcal{G}}$ .
- When adding the pairs of squids in the step 2, within each pair the two colored squids must be dual to each other (namely node Hilbert spaces of their heads must by dual).

Moreover, in order to obtain any nontrivial interaction with the internal edges, at least two legs of the added squids must be colored by spins that agree with spins of the boundary. Otherwise the only possible graph diagrams obtained in the next steps are such, that all internal edges form a disjoint subdiagram. Such diagram is an isolated bubble (for more details see chapter 5) and can be treated analogously to the vacuum self-energy in quantum field theory (see figure 4.14).

• In the step 3 one has to follow appropriate steps of the colored-squid-set algorithm, not the squid-set algorithm, i.e. only the legs labelled by the same spin can be connected when defining both the link relations and the graph course.



Figure 4.14: A vacuum self-energy example. (a) - A disjoint diagram. (b) - The foam corresponding to the diagram.

- Similarly in the step 4 the coloring limits a lot the possible ways to attach the trivial diagram namely the automorphism that determine the gluing must preserve the coloring of the links.
- In the step 5 one has to pick the coloring of the diagram by the contractors. The coloring of the links was already chosen and is fixed. The coloring of the nodes is usually fixed by the considered scheme of the coloring. In general from the point of view of the coloring there are three possible classes of nodes:
  - the boundary nodes of the diagram: they are only the nodes of the static diagram, that refer to the  $\mathcal{G}$  part of its boundary. They are colored as it is indicated in figure 3.15 by the operators  $\widehat{P_n^{\text{Inv}}}$ .
  - the pairs of nodes such that one node in the pair comes from  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  and the other comes from  $\mathcal{D}_{\mathrm{int}}$  then before gluing the node n coming from the static diagram had been colored by  $\widehat{P_n^{\mathrm{Inv}}}$ , the node n' coming from the interaction diagram had been colored by  $\widehat{P_{n'}}$  determined by the coloring scheme one had chosen, and in the final diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}} \# \mathcal{D}_{\mathrm{int}}$  the pair of the nodes is colored by the composition of the operators:  $\widehat{P_{(n,n')}} = \widehat{P_n^{\mathrm{Inv}}} \circ \widehat{P_{n'}}$
  - the coloring of pairs of nodes coming from heads of squids added at the step 2 to form the interaction squid is fully determined by the coloring scheme one had chosen.
- When iterating over all possibilities of diagram's content (in the step 6) one has to limit the iteration to such configurations, that the above rules of consistency hold (namely the coloring of links match at every step of the construction).

The (colored)-fixed-boundary algorithm will be used in section §4.4 to find all diagrams with 0 internal edges in the Dipole Cosmology model.

## 4.3 The Feynman - EPRL rules

The operator spin-network diagrams encode not only the topological structure of spinfoam models. They also provide a tool to read out transition amplitudes of the processes. They do not change the calculations, nevertheless it is convenient to represent the complicated formulae in a graphical way. In this sense they play the role of Feynman diagrams of the theory - they encode the formulae of transition amplitudes in drawings.

In this section we explain procedure of decoding the formula for the transition amplitude out of a diagram.

## 4.3.1 The components of Feynman-EPRL rules

Before going to the Feynman-EPRL rules, let us describe the components of the formulae: the Livine-Speziale semicoherent states, the EPRL map and the Generalized Wigner Matrices.

## 4.3.1.1 The Livine-Speziale semicoherent states

The LS semicoherent states were introduces in [63]. They form an overcomplete basis in the Hilbert space of spin-network functions. They are interpreted as states with fixed areas of faces dual to the spin-network links, that are peaked at certain orientations of vectors normal to these faces.

In order to describe them let us recall, that given a unit vector

$$\vec{u}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4.5}$$

one can define an SU(2)-element u such that  $u\sigma_3 u^{-1} = \vec{u} \cdot \vec{\sigma}$  (where  $\sigma_i$  are Pauli matrices). It is given by

$$u(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$
(4.6)

(of course it is defined up to a z-rotation, but we choose it to have real diagonal elements). Let us also introduce a shortcut notation for SU(2)-elements in *j*-representation

$$SU(2) \ni h \mapsto |h\rangle_j := D^{(j)}(h) |j\rangle_j = \sum_{m=-j}^j D^{(j)}(h)_j^m |m\rangle_j$$
(4.7)

and in the dual representation:

$$\langle h|_j := |h\rangle_j^{\dagger} \tag{4.8}$$

The Livine-Speziale semicoherent states are determined as follows: given a graph  $\mathcal{G}$ , for each link  $\ell \in \mathbf{L}_{\mathcal{G}}$  pick a triple: a half-integer spin  $j_{\ell}$  and two unit vectors  $\vec{u}_{\ell^+}$  and  $\vec{u}_{\ell^-}$  referring to the half-links of the edge. As we have already shown, the vectors can be translated to the SU(2)-elements  $u_{\ell^{\pm}}$ . Now using the above notation, the non-gaugeinvariant LS semicoherent state is defined as the following spin-network function

$$\psi_{j_{\ell}, u_{\ell^{+}}, u_{\ell^{-}}}^{\text{NGI}} (U_{\ell}) := \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \langle u_{\ell^{+}} | U_{\ell} | u_{\ell^{-}} \rangle_{j_{\ell}}$$
(4.9)

and the gauge-invariant states are

$$\psi_{j_{\ell},u_{\ell^{+}},u_{\ell^{-}}}^{\mathrm{GI}}\left(U_{\ell}\right) := \int_{SU(2)^{N}} \mathrm{d}h_{n} \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \left\langle u_{\ell^{+}} \right| h_{s(\ell)}^{-1} U_{\ell} h_{t(\ell)} \left| u_{\ell^{-}} \right\rangle_{j_{\ell}}$$
(4.10)

In Dirac notation we will denote the LS states as  $\psi_{j_{\ell},u_{\ell^+},u_{\ell^-}}^{\text{GI}} := |j_{\ell},u_{\ell^+},u_{\ell^-}\rangle$  (and  $\psi_{j_{\ell},u_{\ell^+},u_{\ell^-}}^{\text{NGI}} := |j_{\ell},u_{\ell^+},u_{\ell^-}\rangle^{\text{NGI}}$ ). The gauge-invariant LS states form an overcomplete basis of spin-network functions. The scalar product between the non-gauge-invariant states is

$$\langle j_{\ell}, u_{\ell^{+}}, u_{\ell^{-}} |^{\mathrm{NGI}} \left| j_{\ell}', u_{\ell^{+}}', u_{\ell^{-}}' \right\rangle^{\mathrm{NGI}} = \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \delta_{j_{\ell} j_{\ell}'} \left\langle u_{\ell^{+}}' \right| \left| u_{\ell^{+}} \right\rangle_{j_{\ell}} \left\langle u_{\ell^{-}} \right| \left| u_{\ell^{-}}' \right\rangle_{j_{\ell}}$$
(4.11)

and between gauge-invariant it is

$$\langle j_{\ell}, u_{\ell^{+}}, u_{\ell^{-}} | \left| j_{\ell}', u_{\ell^{+}}', u_{\ell^{-}}' \right\rangle = \int_{SU(2)^{N}} \mathrm{d}h_{n} \prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \delta_{j_{\ell}j_{\ell}'} \left\langle u_{\ell^{+}}' \right| h_{s(\ell)}^{-1} |u_{\ell^{+}}\rangle_{j_{\ell}} \left\langle u_{\ell^{-}} \right| h_{t(\ell)} \left| u_{\ell^{-}}' \right\rangle_{j_{\ell}}$$
(4.12)

which we will denote shortly as  $\prod_{\ell \in \mathbf{L}_{\mathcal{G}}} \delta_{j_{\ell}j'_{\ell}} \langle u'_{\ell^+} | |u_{\ell^+} \rangle_{j_{\ell}}^{\mathrm{GI}} \langle u_{\ell^-} | | u'_{\ell^-} \rangle_{j_{\ell}}^{\mathrm{GI}}$ .

## 4.3.1.2 The EPRL map

The Feynman-EPRL rules are valid for the EPRL spin-foam models. The EPRL map was discussed in subsection 1.2.2.1, here we will just recall it.

The EPRL map Y maps the vectors of representations of SU(2) group to the vectors of representations of G group, where G depends on the particular model. In the Lorentzian model the group  $G = SL(2, \mathbb{C})$  and the map is given by

$$Y : |m\rangle_j \mapsto |(\gamma j, j); j, m\rangle \tag{4.13}$$

where  $\gamma$  is the Barbero-Imirzi parameter, and  $(\gamma j, j)$  is the (p, k)-pair that labels the primary series of  $SL(2, \mathbb{C})$  unitary representations. In the Euclidean model the group G = SO(4) and the EPRL map acts as

$$Y : |m\rangle_{j} \mapsto C_{m}^{m_{+},m_{-}} |m_{+}\rangle_{j_{+}} \otimes |m_{-}\rangle_{j_{-}} \quad \text{for} \quad j_{\pm} = \frac{|1 \pm \gamma|}{2}j$$
(4.14)

where  $C_{m_+m_-}^m$  are the Clebsh-Gordan coefficients.

## 4.3.1.3 Generalized Wigner matrices

The Wigner matrix elements  $D^{(j)}(h)_n^m$  are building blocks of unitary representations of SU(2):

$$SU(2) \ni h \mapsto D^{(j)}(h)_m^n := \langle n|_j h | m \rangle_j \tag{4.15}$$

Thanks to the EPRL map we have a representation-like functions of the G-group, acting on the SU(2) representation spaces. It is convenient to introduce a similar to Wigner's notation for these functions - it will simplify a lot the formulae in what follows. We call them generalized Wigner matrices and define them as by the formula:

$$G \ni g \mapsto \tilde{D}^{(j)}(g)_m^n := \langle n|_j Y^{\dagger} g Y | m \rangle_j$$

$$(4.16)$$

They are not true representations, because they do not satisfy the composition principle:

$$\tilde{D}^{(j)}(g)_{m'}^{m} \tilde{D}^{(j)}(g')_{m''}^{m'} \neq \tilde{D}^{(j)}(g \cdot g')_{m''}^{m}$$
(4.17)

## 4.3.2 The rules

## 4.3.2.1 Labelling

Knowing the assumption, namely the coloring consistent with the EPRL scheme, we can relabel our diagram by the terms that will appear in the formula for the amplitude. We do it according to the following scheme:

- 1. Each link  $\ell \in \theta_n$  at the boundary static diagram  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  label by  $|u_{\ell,n}\rangle_{j_{\ell}}$  they are the *external legs* of the diagram. For this procedure we neglect the fact that the links in  $\mathcal{D}_{\mathcal{G}}^{\mathrm{Tr}}$  are split into two parts and we treat each such pair as a one link.
- 2. Each link  $\ell'$ , that is in the face relation with one of the boundary links  $\ell$ , label by the spin  $j_{\ell'} = j_{\ell}$ .
- 3. Each link  $\ell_f$ , that *is not* in the face relation with any boundary link, label by an arbitrary spin  $j_f$  the same for all equivalence class  $[\ell_f]$ .
- 4. Each node I of each graph colored by  $A^{\text{EPRL}}$  label by a G-group element  $g_I$  (where  $G = SL(2, \mathbb{C})$  in the Lorentzian case and G = SO(4) in the Euclidean case).

Example of this relabelling can be found at figure 4.15a.

### 4.3.2.2 External faces

The terms for the external legs of the diagram, namely the external faces, are obtained as follows:

- 1. Pick a node n in the boundary graph and a link  $\ell$  outgoing from it.
- 2. Go to  $\theta_n$  in the boundary static diagram and pick the link corresponding to  $\ell$ . Write for it

$$\langle u_{\ell,n}|_{j_{\ell}}\cdots$$

3. Follow the link relation of  $\ell$ . Whenever it leads to a node I of a graph colored by  $A^{\text{EPRL}}$ , write  $Y^{\dagger}g_{I}^{-1}$ :

$$\langle u_{\ell,n}|_{j_{\ell}} Y^{\dagger} g_I^{-1} \cdots$$

4. Then, whenever leaving a graph colored by  $A^{\text{EPRL}}$  at node II, write  $g_{II}Y$ :

$$\langle u_{\ell,n}|_{j_{\ell}} Y^{\dagger} g_I^{-1} g_{II} Y \cdots$$

5. Repeat it for each graph passed - if it is colored by  $A^{\text{EPRL}}$ . But if the graph passed is colored by  $A^{\text{Tr}}$  - write the identity (or just do not write anything):

$$\langle u_{\ell,n}|_{j_{\ell}} Y^{\dagger} g_I^{-1} g_{II} Y \mathbb{1} Y^{\dagger} g_V^{-1} g_{VI} Y \cdots$$

6. When reaching again the boundary static diagram, end the formula with  $|u_{\ell',n'}\rangle_{j_{\ell'}}$ , where  $\ell'$  is the appropriate boundary link (it does not need to be  $\ell$ ):

$$W_{\ell}^{\text{ext}} := \left\langle u_{\ell,n} \right|_{j_{\ell}} Y^{\dagger} g_{I}^{-1} g_{II} Y \, \mathbb{1} \, Y^{\dagger} g_{V}^{-1} g_{VI} Y \, Y^{\dagger} g_{VII}^{-1} g_{VIII} Y \left| u_{\ell',n'} \right\rangle_{j_{\ell'}} \tag{4.18}$$

thanks to the consistency conditions (see subsection 3.3.1), we are sure that  $j_{\ell} = j_{\ell'}$ .



Figure 4.15: An illustration to section 4.3.2. (a) - A labelling, as in 4.3.2.1. To make the figure more legible, only those elements of the labelling and the link relations are written that are used in the following examples. (b) - The elements of the diagram that contribute the term  $\langle u_1|_{j_1} Y^{\dagger} g_I^{-1} g_{II} Y \mathbbm 1 Y^{\dagger} g_V^{-1} g_{VI} Y |u_{1'}\rangle_{j_1}$  coming from an external face. (c) - The elements of the diagram that contribute to another external term, this time including only  $\theta$ -graphs:  $\langle u_2|_{j_2} |u_{2'}\rangle_{j_2}$ . (d) - The elements of the diagram that contribute to the internal face's term:  $\sum_{j_f} \operatorname{Tr}_{j_f} [Y^{\dagger} g_{VIII}^{-1} g_{II} Y \mathbbm 1 Y^{\dagger} g_V^{-1} g_{VII} Y]$ .

One can find two examples on figures figure 4.15b and figure 4.15c.

Recalling our notation introduced in subsection 4.3.1.3, all the elements of the formula (4.18) are in fact the generalised Wigner matrices subsection 4.3.1.3 in the  $j_{\ell}$  representation. Thus treat (4.18) as a shortcut notation of

$$W_{\ell}^{\text{ext}} := D^{(j_{\ell})} (u_{\ell})_{m}^{j_{\ell}} \tilde{D}^{(j_{\ell})} \left(g_{I}^{-1}g_{II}\right)_{m'}^{m} \delta_{m''}^{m'} \cdots$$

$$\cdots \tilde{D}^{(j_{\ell})} \left(g_{V}^{-1}g_{VI}\right)_{m'''}^{m''} \tilde{D}^{(j_{\ell})} \left(g_{VII}^{-1}g_{VIII}\right)_{m'''}^{m'''} D^{(j_{\ell})} (u_{\ell'})_{j_{\ell}}^{m''''}$$

$$(4.19)$$

Repeat this procedure for all links in the boundary graphs.

Each term  $W_{\ell}^{\text{ext}}$  must be multiplied by the weight factor (see subsection 3.3.2.2). In the simplest case, when the sequence ends at the same link, at which it started (i.e.  $\ell' = \ell$ ), this term takes into account, that our term  $W_{\ell}^{\text{ext}}$  represents one face and one boundary link, thus its weight factor is  $A_{\ell}^{\text{ext}} = A_{f_{\ell}} \cdot A_{\ell} = \sqrt{2j_{\ell} + 1}$ . However, if the link closing considered series is not the starting one, then the face we consider consist of several generalised faces with separate boundaries, and we have a series of boundary links  $\{\ell_1, \ldots, \ell_k\}$  (let us denote is as an equivalence class  $[\ell_i] := \{\ell_1, \ldots, \ell_k\}$  for  $i = 1, \ldots, k$ ), and one has to take the boundary amplitudes of each of them into account. The rule is than the following: for each link in the series the weight factor is  $A_{\ell}^{\text{ext}} = A_{\ell} = \frac{1}{\sqrt{2j_{\ell}+1}}$ , however, for the last link in the series we multiply it also by the face amplitude, so that  $A_{\ell_k}^{\text{ext}} = A_{f_{\ell_k}} \cdot A_{\ell_k} = \sqrt{2j_{\ell_k} + 1}$ . Thus after all we have

$$W_{\text{external}} = \prod_{\ell \in \partial \mathcal{D}} A_{\ell}^{\text{ext}} W_{\ell}^{\text{ext}}$$
(4.20)

which simplifies to product over boundary links of link amplitude and product over equivalence classes of face amplitude:

$$W_{\text{external}} = \left(\prod_{\ell \in \partial \mathcal{D}} A_{\ell}\right) \cdot \left(\prod_{[\ell] \subset \partial \mathcal{D}} A_{f_{\ell}}\right) \cdot W_{\ell}^{\text{ext}}$$
(4.21)

#### 4.3.2.3 Internal faces

For the remaining links in the graph diagram do as follows.

- 1. Pick a link  $\ell_f$  that has not been used yet and write  $\langle m|_{j_f}$  for it (using a half-integer  $m \in \{-j_f, -j_f + 1, \dots, j_f\}$ ).
- 2. Repeat the steps 3-5 of the procedure for external faces (see subsection 4.18).
- 3. When reaching the link  $\ell$  again, close the formula with  $|m\rangle_{j_f}$  (since the link we have chosen was a member of closed equivalence class of the face relation, for sure we will reach it again).
- 4. Sum over m:

$$W_{f}^{\text{int}} := \sum_{m=-j_{j}}^{j_{f}} \langle m|_{j_{f}} Y^{\dagger} g_{I}^{-1} g_{II} Y \cdots Y^{\dagger} g_{III}^{-1} g_{IV} Y |m\rangle_{j_{f}}$$
(4.22)

The resulting term is in fact the SU(2) trace of the generalized Wigner matrices:

$$W_{f}^{\text{int}} = \text{Tr}^{(j_{f})} \left[ \tilde{D}^{(j_{f})} \left( g_{I}^{-1} g_{II} \right)_{m'}^{m} \cdots \tilde{D}^{(j_{f})} \left( g_{III}^{-1} g_{IV} \right)_{m'''}^{m''} \right]$$
(4.23)

An example can be found in figure 4.15d.

Repeat this procedure until all links of the diagram were visited.

As in case of the external faces, the terms  $W_f^{\text{int}}$  must be multiplied by the weight factor (see subsection 3.3.2.2). In their case the weight factor is simply  $A_f^{\text{int}} = A_f = 2j_f + 1$ . Then multiply all the terms together:

$$W^{\text{internal}} = \prod_{f-\text{internal}} A_f W_f^{\text{int}}$$
(4.24)

## 4.3.2.4 Total amplitude

In order to obtain the final transition amplitude of the diagram, one has to multiply the terms  $W^{\text{internal}}$  and  $W^{\text{external}}$ , and then sum and integrate all the variables that are not determined by the boundary data, thus:

• Sum with respect to all possible combinations of  $j_f$ s (only  $W^{\text{internal}}$ , because  $W^{\text{external}}$  does not depend on  $j_f$ s):

$$\sum_{\{j_f\}} W^{\text{internal}}$$

• For each node I of the graphs colored by the  $A^{\text{EPRL}}$  contractor integrate over the group G with respect to the Haar measure  $dg_I$ :

$$\int_{G^N} \prod_{I \in \mathbf{N}_{\mathcal{S}_{\text{int}}^{\text{EPRL}}}} \mathrm{d}g_I \ W^{\text{external}} \cdot \sum_{\{j_f\}} W^{\text{internal}}$$

Note, that for each graph the last integration is trivial and simply multiplies the result by the volume of the group G. In the Euclidean case this volume is equal to 1 thanks to normalisation of the Haar measure. However, in the Lorentzian case the integration goes over a non compact group and the volume is infinite. Thus, to avoid this infinity, remember to drop one integration for each graph (see [50, 46]).

The resulting amplitude is therefore:

$$A_{\mathcal{D}} = \int_{G^{N}} \prod_{I \in \mathbf{N}_{\mathcal{S}_{int}^{EPRL}}} \mathrm{d}g_{I}$$

$$\sum_{\{j_{f}\}} \prod_{f} A_{f}(j_{f}) \operatorname{Tr}^{(j_{f})} \left[ Y^{\dagger}g_{I}^{-1}g_{II}Y \cdots Y^{\dagger}g_{III}^{-1}g_{IV}Y \right] \qquad (4.25)$$

$$\prod_{\ell} A_{\ell}(j_{\ell}) \langle u_{\ell,n}|_{j_{\ell}} Y^{\dagger}g_{I}^{-1}g_{II}Y \cdots Y^{\dagger}g_{III}^{-1}g_{IV}Y | u_{\ell',n'} \rangle_{j_{\ell}}$$

## 4.4 Application of the algorithm in Dipole Cosmology

In this section we apply the fixed-boundary algorithm to the case of Dipole Cosmology model (see subsection 2.2.1, and [55, 56, 57, 58, 59]).

The Dipole Cosmology model tries to test the behaviour of the spin-foam transition amplitudes in the limit of the homogeneous and isotropic boundary states. The boundary state in this model is a coherent state of two 3-sphears, each of which is represented by



Figure 4.16: The dipole graph  $\left|\mathcal{G}^{\text{dipole}}\right| := \left|\theta_{\text{in}}^{(4)}\right| \sqcup \left|\theta_{\text{out}}^{(4)}\right|.$ 

two glued tetrahedra. As a spin-network it is represented by a graph being a disjoint sum of two 4-link  $\theta$ -graphs (see figure 4.16), called Dipole Graph  $|\mathcal{G}^{\text{dipole}}| := |\theta_{\text{in}}^{(4)}| \sqcup |\theta_{\text{out}}^{(4)}|$ .

The transition amplitude in DC model was calculated under several approximations. One of them was the so-called vertex expansion: at the first order one considers only the contribution of the spin-foams with precisely 1 interaction vertex. The original DC paper [55] considered only one of such foams, in [59] some more examples were found and studied.

In our paper [64] we proposed a different expansion: the edge- expansion. It is easier to control within our framework, but moreover it seams to be more physically justified. Having the number of internal edges unlimited even in the first step of the vertex expansion one would have to consider an infinite series of spin-foams with loop edges starting and ending at this vertex (see figure 4.16). On the other hand having a fixed number of internal edges one can easily restrict oneself to a defined number of vertices. In our framework vertices are represented by the connected components of the interaction graph, thus each one can simply restrict the results to only these ones, that has a certain number of connected components. In our paper - and in this section - we find all the diagrams with boundary being the dipole graph, in the first order of the edge-expansion.

The process of finding all the interaction graphs was dome mainly by Marcin Kisielowski, thus it is just recalled here, without getting deep into details.

## 4.4.1 Adaptation of the fixed-boundary algorithm

As we have already mentioned, we are going to use the fixed-boundary algorithm presented in subsection 4.2.2 to the dipole boundary graph  $|\mathcal{G}^{\text{dipole}}|$ . We will do it only in the first order of the edge-expansion, thus for no internal edges - which will mean no pairs of squids added to the interaction squid set.

The individual steps of the procedure are described below:

1.  $\mathcal{G}$ : choose an orientation of each link of each two graphs  $|\theta^{(4)}|$  (figure 4.17a).



Figure 4.17: Construction of the interaction graph diagram. (a) - Step 1: choose an orientation of each link of  $\mathcal{G}^{\text{dipole}}$ . (b) - Step 2: construct the squid set  $\mathcal{S}_{\text{int}}$ . (c) - Step 3: construct an interaction graph  $\mathcal{G}_{\text{int}}$  being in fact the interaction diagram  $\mathcal{D}_{\text{int}}$  - one of possible examples.

- 2.  $S_{\text{int}} = S_{\mathcal{G}}$ : for the interaction squid set take the squid set  $S_{\mathcal{G}^{\text{dipole}}}$  of the graph  $\mathcal{G}^{\text{dipole}}$ . That means that in point 2 of the fixed-boundary algorithm we add N = 0 pairs of squids. The interaction squid-set consists of four 4-valent squids, oriented according to the previous step (figure 4.17b). No link relation has to be set, because the node relation is void.
- 3.  $\mathcal{G}_{int}$ : close the squid-set  $\mathcal{S}_{\mathcal{G}}$  by connecting each incoming leg with one outgoing leg of the squids. In the next subsection we will show all the graphs that can be constructed in this way (figure 4.17c).
- 4.  $\mathcal{D}_{\mathcal{G}^{\text{dipole}}} \# \mathcal{D}_{\text{int}}$ : take the static diagram  $\mathcal{D}_{\mathcal{G}^{\text{dipole}}}$  of the graph  $\mathcal{G}^{\text{dipole}}$  (figure 4.18a) and attach it to the interaction diagram (figure 4.18b).
- 5. Coloring: Color the interaction part of the diagram in the EPRL scheme, color the static diagram in the BF scheme. The spins on the boundary links choose arbitrary (but consistently with the diagram).
- 6. Consider all possible: orientations of the graph  $\mathcal{G}^{\text{dipole}}$  in the step 1, closing of the interaction graph in the step 3, ways of attaching the static diagram in the step 4 (the last iteration will be discussed in subsection 4.4.3).

## 4.4.2 All possible interaction graphs

The orientation of links in the fixed-boundary algorithm is chosen only to keep control on the possible ways of connecting it to the static diagram of the boundary. In fact in order to find all graphs that share an *unoriented* squid set, one can use the unoriented squids, and afterwards, when attaching the boundary static diagram, for each orientation of the boundary choose only these diagrams, that can be oriented consistently with it (see remark 4.2 in subsection 4.2.2). Let us do it in this case.



Figure 4.18: Construction of the full graph diagram. (a) - The static diagram of the dipole graph (the dotted lines denote the link relations). (b) - A graph diagram  $\mathcal{D}_{int}$  attached to the static diagram gives the final diagram  $\mathcal{D}_{\mathcal{G}^{dipole}} \# \mathcal{D}_{int}$ 

All the interaction graphs that are allowed by the above construction can be characterised as follows:

- Each graph  $|\mathcal{G}_{int}|$  has precisely four nodes.
- Each node of  $|\mathcal{G}_{int}|$  is precisely *four-valent*.

Each such graph can be characterised by so called *adjacency matrix*: the  $N \times N$ -matrix (N is the number of nodes of the graph), such that  $A_{ij} = x$  iff there are precisely x link connecting the node i and j (in case of i = j the loop links are counted twice). This is the *unoriented adjacency matrix*, there is also the oriented version of it, but we will not use it here. The above conditions put some constraints on this matrix, namely: it is a  $4 \times 4$  matrix, and the sum of elements in each row and each column equals precisely 4. There are also constraints on  $A_{ij}$  coming from the fact that it is adjacency matrix: namely it has to be symmetric:  $A_{ij} = A_{ji}$ , and the diagonal elements must be even number.

Each matrix satisfying the above conditions describe one graph that can be an interaction graph of our problem. However permutations of nodes may lead to different matrices that represent the same graph. Marcin Kisielowski developed a method to find and parametrise all such matrices, and thus all interaction graphs. It was described with details in [64], section 4.2. Here we will just recall the result of his procedure: all the interaction graphs are represented at figure 4.19.

In our case of 4-node graph one could find all these graphs without using the sophisticated method of Marcin Kisielowski, however one could easily miss one of them. And if the squid set we considered were bigger or the squids had more legs - the number of possible combinations would likely exceed human possibilities and thus the strict procedure is needed.



Figure 4.19



Figure 4.19: (Continued.) The list of all possible interaction graphs in the first order of edge-expansion. The graphs are unoriented - given a specific orientation of the boundary one has to use only those that can be oriented consistently with it. Note, that spin-foams considered in [59] have the interaction graphs of type 18, 19 or 20.



Figure 4.20: An example of node-exchange ambiguity.

## 4.4.3 Gluing the static diagram

There are 20 various interaction graphs, however there are much more graph diagrams that can be constructed out of them. This is because of many possible ways to attach the static diagram:

- When defining the  $\mathcal{R}_{node}^{gluing}$  it may happen, that two nodes of  $\mathcal{G}$  has the same structure. Then the two nodes of  $\mathcal{G}_{int}$  can be exchanged. The two resulting diagrams are usually different. An example is shown at figure 4.20
- When defining the  $\mathcal{R}_{link}^{gluing}$ , for each node one can permute the links (of course preserving their orientation) and obtain a different diagrams. An example is shown at figure 4.21. However, it may happen, that two different permutations of links give the same diagram (for example see figure 4.21c).

Furthermore, when gluing a colored static boundary diagram, one realises, that some interaction diagrams may give immediately zero transition amplitude. We will call such interaction graphs *incompatible* with the certain coloring of the boundary. Let us explain it on an example.

Consider a boundary graph colored an oriented like on figure 4.22, and the interaction graph of type 12. Let us choose the  $\mathcal{R}_{node}^{gluing}$  and  $\mathcal{R}_{link}^{gluing}$  as on figure 4.21a. Note, that thanks to the consistency conditions (see subsection 4.2.3) the transition amplitude is zero *unless*  $j_2 = j_3$  and  $j_1 = j_5$ . The two loops in the interaction graph ensure, that the amplitude is non-zero *only if* there are pairs of equal spins among  $j_1, \ldots, j_8$ .

A similar analysis can be done for other graphs. One can see, that the constraints on the spins arise if there is a loop link (then the constraint relates the spins of one connected part of the boundary), or if there is a node connected at least two other nodes (then the constraint relates the spins of *in* and *out* part of the boundary). It leads to an interesting conclusion: for all graphs other then 20 there are always such constraints, so it is the only interaction graph, that can give non-zero amplitude for a generic boundary state. Nevertheless, it also can be attached in a way that produces constraints - see figure 4.23.



Figure 4.21: Examples of link-exchange ambiguity. The diagrams (a) and (b) are different. The diagram (c) has different link-relation, than (a), but an automorphism of the interaction graph may map it back into (a).



Figure 4.22: An example of coloring of  $\mathcal{G}^{\text{dipole}}$  by spin-labels.

# 4.5 The transition amplitudes in 1st order Dipole Cosmology

In this section we will calculate the transition amplitudes of some diagrams that contribute to the Dipole Cosmology model in our approximation (i.e. when restricting oneself to the diagrams found in section §4.4). We will show, that in large j limit the amplitude related to interaction graphs containing at least one loop-link is negligible when compared to the original DC amplitude. Our calculation will be done using the Feynman-EPRL rules developed in section §4.3.

The boundary state used in the Dipole Cosmology model is a semicoherent state  $|\Psi_{H_{\ell}}\rangle$  defined by a wave function

$$\Psi_{H_{\ell}}(U_{\ell}) = \int_{SU(2)^{N}} \mathrm{d}h_{n} \prod_{\ell} K_{t} \left( h_{s(\ell)}^{-1} U_{\ell} h_{t(\ell)} H_{\ell}^{-1} \right)$$
(4.26)

where  $K_t$  is the heat kernel function on the SU(2) group, t labels the spread of the heat kernel, and  $H_{\ell} \in SL(2, \mathbb{C})$  encode the phase space point, on which the coherent state is peaked (see[55, 56]). In case of DC the parameters  $H_{\ell}$  has special form, determined by the requirement of homogeneity and isotropy of the geometry we want to describe:

$$H_{\ell}\left(z_{\rm in/out}\right) := u_{\ell} e^{-iz_{\rm in/out}\sigma_3} u_{\ell}^{-1} \tag{4.27}$$

where  $z_{in/out} = \alpha \dot{a}_{in/out} + i\beta a_{in/out}^2$  (for some constants  $\alpha$  and  $\beta$  [55, 57]) and a is the scale factor of the state we describe (for more details see subsection 2.2.1). Of course one can express these states as a superposition of LS semi-coherent states. The coefficients are

$$\langle j_{\ell}, u_{\ell^+}, u_{\ell^-} | | \Psi_{H_{\ell}} \rangle \sim \prod_{\ell} e^{-\frac{(j_{\ell} - j_{\ell}^0)^2}{2\Delta j_{\ell}}} \langle u_{\ell} | | u_{\ell^+} \rangle_{j_{\ell}}^{\text{GI}} \langle u_{\ell^-} | | u_{\ell} \rangle_{j_{\ell}}^{\text{GI}}$$
(4.28)



Figure 4.23: An example of a constraint on spin-labels produced by the choice of linkrelation when gluing the interaction graph of type 20 to the boundary static diagram. Here the link relations enforces  $j_5 = j_6$ .



Figure 4.24: Two diagrams that are considered in section §4.5. To make the figure more legible the spin labels and the link relations are omitted. (a) - The interaction graph 16 from figure 4.19. Each link  $\ell$  carry a spin label  $j_{\ell}$ . For the link relations see figure 4.25. (b) - The interaction graph 20 from figure 4.19. Each link of the *in* part of the static part of the diagram carry the same spin label j and each link of the *out* part of the static part of the diagram carry the same spin label j'. The link relation is the simplest possible: each two links with the same  $u_{\ell}$  are connected through the interaction graph.

for  $j_{\ell}^0 = \frac{2t\hbar - iz}{4t\hbar}$  and  $\Delta j_{\ell} = \frac{1}{4t\hbar}$ . The scalar products  $\langle u_{\ell} | |u_{\ell^+} \rangle_{j_{\ell}}^{\text{GI}}$  and  $\langle u_{\ell^-} | |u_{\ell} \rangle_{j_{\ell}}^{\text{GI}}$  ensure that the vectors on both ends of each link are the same:  $u_{\ell^+} = u_{\ell^-} = u_{\ell}$ , thus we will consider only the LS states of the form  $|j_{\ell}, u_{\ell}, u_{\ell} \rangle$ .

We will do our calculations using the interaction graph of type 16 of figure 4.19. The full diagram is shown at figure 4.24a. We will compare the amplitude of this diagram with the amplitude of the original DC calculation, which refers to the interaction graph of type 20, presented at figure 4.24b. Although we use only one interaction graph, our result depend only on one feature of this graph, namely the loop-link, and there is a straightforward generalisation of our calculation to to all interaction graphs with that feature.

We start (in subsection 4.5.1) with recalling the calculation of the original transition amplitude of the DC model and pointing out some features that we are interested in. Then, in subsection 4.5.2, we calculate the amplitude of a diagram having a loop-link in the interaction graph. Finally (in subsection 4.5.3) we compare these two amplitudes and discuss, what do we mean by negligible.

## 4.5.1 Original Dipole Cosmology amplitude

Let us first calculate the transition amplitude of the diagram presented at figure 4.24b. It was computed first in [57], and it was recalled in subsection 2.2.1. It was shown that the

main contribution to this amplitude comes from the terms in which  $j_{\ell} = j$  for  $\ell = 1, \ldots, 4$ and  $j_{\ell} = j'$  for  $\ell = 5, \ldots, 8$ . Since here we use this diagram only as a reference point for the second calculation, it is enough to use only this types of boundary states.

Using the Feynman-EPRL rules of section §4.3 we can read the transition amplitude of this diagram as:

$$A_{20} = [A_{\ell}(j)]^{4} \left[ \int_{G} dg_{N} dg_{S} \prod_{\ell=1}^{4} \langle u_{\ell} |_{j} Y^{\dagger} g_{N}^{-1} g_{S} Y | u_{\ell} \rangle_{j} \right] \cdot \left[ A_{\ell}(j') \right]^{4} \left[ \int_{G} dg_{N'} dg_{S'} \prod_{\ell=5}^{8} \langle u_{\ell} |_{j'} Y^{\dagger} g_{N'}^{-1} g_{S'} Y | u_{\ell} \rangle_{j'} \right]$$

$$= \left[ \int_{G} dg \prod_{\ell=1}^{4} \langle u_{\ell} |_{j} Y^{\dagger} g Y | u_{\ell} \rangle_{j} \right] \left[ \int_{G} dg' \prod_{\ell=1}^{4} \langle u_{\ell} |_{j} Y^{\dagger} g' Y | u_{\ell} \rangle_{j} \right]$$

$$\cdot (2j+1)^{2} (2j'+1)^{2} \left[ \int_{G} dg'' \Big]^{2}$$
(4.29)

The term  $\int_G dg''$  comes from the invariance of the Haar measure under multiplication and inversion. In the Euclidean case we can calculate it explicitly: it is equal 1 thanks to the normalisation of the Haar measure. In the Lorentzian case such integral would give an infinite factor, but we regularise the integrals by dropping one integration per each connected component - and thus we simply drop both these terms (see subsection 1.2.3.2). Thus both in Euclidean and Lorentzian case the unweighted amplitude is

$$A_{20}^{\text{unweighted}} = \left[ \int_{G} \mathrm{d}g \prod_{\ell=1}^{4} \left\langle u_{\ell} \right|_{j} Y^{\dagger} g Y \left| u_{\ell} \right\rangle_{j} \right] \left[ \int_{G} \mathrm{d}g' \prod_{\ell=1}^{4} \left\langle u_{\ell} \right|_{j} Y^{\dagger} g' Y \left| u_{\ell} \right\rangle_{j} \right]$$
(4.30)

and the weight factor is

$$w(j,j') = (2j+1)^2 (2j'+1)^2$$
(4.31)

and it is a polynomial in j and j'.

In the Euclidean case each term  $\int_{SO(4)} dg \prod \langle u_\ell |_j Y^{\dagger}gY | u_\ell \rangle_j$  is the coherent regular tetrahedron of LS, that was calculated in [63] to be equal  $N_0 j^{-3}$  in the large jlimit. In the Lorentzian case the terms  $\int_{SL(2,\mathbb{C})} dg \prod \langle u_\ell |_j Y^{\dagger}gY | u_\ell \rangle_j$  are the matrix elements of the Lorentzian Polyhedra Propagator (studied in [83], described in chapter 6), which in this case behave similarly as  $N_0 j^{-3}$ . Thus in both Euclidean and Lorentzian case the unweighted transition amplitude decays polynomially. The overall amplitude  $A_{20} = w(j, j') A_{20}^{\text{unweighted}}$  also decays polynomially - to be more precise:

$$A_{20} = O\left(j^{-1} \cdot j^{\prime - 1}\right) \tag{4.32}$$

### 4.5.2 Amplitude for a vertex containing a loop

Let us now calculate the the terms of transition amplitude of the diagram present at figure 4.24a. We do it using the Feynman-EPRL rules.

Thanks to very simple structure of the diagram, each (unweighted) term coming from external faces has a form

$$\langle u_{\ell}|_{j_{\ell}} Y^{\dagger} g_{I}^{-1} g_{II} Y | u_{\ell'} \rangle_{j_{\ell'}}$$

$$\tag{4.33}$$

Some examples are presented at figure 4.25.



Figure 4.25: Examples of terms appearing in the amplitude  $A_{16}$ : (a):  $\langle u_1 |_{j_1} Y^{\dagger} g_N^{-1} g_S Y | u_1 \rangle_{j_1}$ , (b):  $\langle u_2 |_{j_2} Y^{\dagger} g_N^{-1} g_S Y | u_3 \rangle_{j_3}$  (c):  $\langle u_4 |_{j_4} Y^{\dagger} g_N^{-1} g_S Y | u_5 \rangle_{j_5}$ , (d):  $\langle u_6 |_{j_6} Y^{\dagger} g_N^{-1} g_S Y | u_7 \rangle_{j_7}$ 

We are not going to calculate all these terms. We will focus on the last example, presented at figure 4.25d. In this case the group elements  $g_I$  and  $g_{II}$  are the same element  $g_{N'}$ , and thus

$$\langle u_6|_{j_6} Y^{\dagger} g_{N'}^{-1} g_{N'} Y | u_7 \rangle_{j_7} = \langle u_6|_{j_6} Y^{\dagger} Y | u_7 \rangle_{j_7} = \langle u_6| | u_7 \rangle_{j_6} \delta_{j_6, j_7}$$
(4.34)

Obviously the spins  $j_6$  and  $j_7$  must match. Assuming  $j_6 = j_7 = j$  we can calculate this term. First let us expand the  $|u\rangle_j$  notation (see equation (4.7)) and use the properties of Wigner matrices to express it in terms of matrix elements of  $u_6$  and  $u_7$ :

$$\langle u_6 | | u_7 \rangle_j = \langle j | u_6^{-1} u_7 | j \rangle_j = \left( \left\langle \frac{1}{2} | u_6^{-1} u_7 | \frac{1}{2} \right\rangle_{\frac{1}{2}} \right)^{2j}$$
 (4.35)

Now using the definition of  $u_{\ell}$  elements we can express it all in terms of angles:

$$\left\langle \frac{1}{2} \middle| u_6^{-1} u_7 \left| \frac{1}{2} \right\rangle_{\frac{1}{2}} = \cos \frac{\theta_6}{2} \cos \frac{\theta_7}{2} + \sin \frac{\theta_6}{2} \sin \frac{\theta_7}{2} e^{-i(\phi_6 - \phi_7)} =: \alpha_{6-7}$$
(4.36)

Using some trigonometric identities we can estimate the modulus of  $\alpha_{6-7}$ :

$$|\alpha_{6-7}|^2 = \frac{1}{2} \left( 1 + \cos\theta_6 \cos\theta_7 + \sin\theta_6 \sin\theta_7 \cos(\phi_6 - \phi_7) \right)$$
  
$$\leq \frac{1}{2} \left( 1 + \cos\theta_6 \cos\theta_7 + \sin\theta_6 \sin\theta_7 \right)$$
(4.37)

$$= \frac{1}{2} \left( 1 + \cos\left(\theta_6 - \theta_7\right) \right) \le 1 \tag{4.38}$$

where the inequality equation (4.37) is strong for all pairs such that  $\phi_6 \neq \phi_7$  (because  $\phi \in [0, 2\pi[)$ ) and the inequality equation (4.38) is strong for all pairs such that  $\theta_6 \neq \theta_7$  (because  $\theta \in [0, \pi]$ ). Thus we have

$$|\alpha_{6-7}| < 1 \text{ for } \vec{u_6} \neq \vec{u_7} \text{ and } |\alpha_{6-7}| = 1 \text{ for } \vec{u_6} = \vec{u_7}$$
 (4.39)

The unweighted contribution coming from the external leg presented at figure 4.25d to the amplitude  $A_{16}$  is

$$\langle u_6|_{j_6} Y^{\dagger} g_{N'}^{-1} g_{N'} Y | u_7 \rangle_{j_7} = \langle u_6| | u_7 \rangle_{j_6} \,\delta_{j_6, j_7} = \delta_{j_6, j_7} \left(\alpha_{6-7}\right)^{2j_6} \tag{4.40}$$

Unless  $\vec{u_6} = \vec{u_7}$  this term decays exponentially with  $j_6$ .

L

Note, that the calculation above is valid whenever there is a loop-link in the interaction diagram (to be more precise: whenever there is a loop-link at one of the nodes of interaction diagram, that is attached to the boundary static diagram - but in our case there are no other nodes). Indeed, the cancellation at equation (4.34) happens because both the starting and ending node of the considered link are the same node, and thus the corresponding group element is the same. Thus the result equation (4.40) and equation (4.39) is general - there is always a multiplicative exponentially decaying term in the amplitude of an interaction diagram containing a loop attached to the boundary static diagram.

All the other contributions are at most polynomially growing. The weight amplitudes  $A_{\ell}$  are bounded from above by  $\sqrt{2j_{\ell}+1}$ . The other terms form an evaluation of a spinnetwork with the loop cut, thus it can be estimated. In the Euclidean case it is bounded by the norms of elements of the integrand:

$$\left| \int_{SO(4)^4} \mathrm{d}g_n \prod_{\ell,\ell'} \langle u_\ell | Y^{\dagger} g_n^{-1} g_{n'} Y^{\dagger} | u_{\ell'} \rangle \right| \leq \int_{SO(4)^4} \mathrm{d}g_n \prod_{\ell,\ell'} \| \langle u_\ell | \| \cdot \left\| Y^{\dagger} \right\| \cdot \left\| g_n^{-1} g_{n'} \right\| \cdot \left\| Y^{\dagger} \right\| \cdot \| u_{\ell'} \rangle |$$

$$(4.41)$$

where by ||g|| we mean the norm of the tensor product of representations  $\left\|\bigotimes_{\ell} D^{(j_{\ell}^+, j_{\ell}^-)}(g)\right\|$ . Each of the terms in (4.41) has the norm equal 1, thus the total expression is bound by  $\int_{SO(4)^4} dg_n = V_{SO(4)}^4 = 1$  (because of the normalisation of Haar measure). In the Lorentzian case the graph is 2-edge connected, thus one has to regularise the expression by removing appropriate number of integrations over  $SL(2, \mathbb{C})$  to obtain a formula like for a 3-edge connected graph (in this particular case one have to drop 2 integrations), and the amplitude for a 3-edge connected graph is bounded as a function of *j*-s (see [84, 50, 46]). Thus in the end the whole expression apart from the  $\alpha_{6-7}$ -term grows at most polynomially in *j*. In the large *j* regime such asymptotic behaviour is dominated by the  $(\alpha_{6-7})^{2j_6}$ term, and thus

$$A_{16} = O\left(\left(\alpha_{6-7}\right)^{2j_6}\right) \quad \text{unless} \quad \vec{u_6} = \vec{u_7}$$
 (4.42)

Note that this property of the amplitudes with loop-link does not depend on the signature of our model - the calculation of exponentially decaying term are independent on the signature, and the other facts used above are true for both Euclidean and Lorentzian models. Thus our result is signature independent.

## 4.5.3 Comparison of these two amplitudes and discussion

As we have shown in subsection 4.5.2, the amplitude  $A_{16}$  (and in fact amplitudes of all interaction diagrams with a loop link) decays exponentially in the large j regime if the vectors  $\vec{u_6}$  and  $\vec{u_7}$  differ. This is however the generic case. Consider a boundary state where  $\vec{u_6} = \vec{u_7}$ . In terms of LS semicoherent states the vectors corresponds to directions perpendicular to the faces of the polyhedra, pointing outside the polyhedra. Thus  $\vec{u_6} = \vec{u_7}$ means that two faces of the tetrahedron are in fact a single face, artificially divided into two. A tetrahedron with such property is degenerate (for example it has volume equal 0). Thus let us restrict our consideration to only these tetrahedra, where  $\vec{u_6} \neq \vec{u_7}$  (or more general: where  $\vec{u_i} \neq \vec{u_j}$  for  $i \neq j$  if both  $i, j = 1, \ldots, 4$  or  $i, j = 5, \ldots, 8$  thus it may happen that  $\vec{u_1} = \vec{u_5}$  etc.).

Having this restriction let us assume, that we scale the spins in  $A_{16}$  uniformly to the large j regime. This assumptions prevents from the situation, where the boundary state represents a universe with hi volume, but the spins  $j_6$  and  $j_7$  are far smaller then the other spins. Under these conditions it is obvious that  $A_{16} \ll A_{20}$  in large j regime - this is because the exponent function  $O(\alpha^j)$  for  $\alpha < 1$  and  $j \gg 1$  decays faster than any inversed polynomial  $O(j^{-k})$ . The same happen for all  $A_i$  where  $i = 1, \ldots, 16$ . Thus all the graphs of type  $1, \ldots, 16$  presented at figure 4.19 are negligible when compared to  $A_{20}$  in the large j regime, quod erat demonstrandum.

The amplitudes of diagrams containing graphs 18, 19, 20 of figure 4.19 was calculated in [59]. The amplitude of diagram containing graph 17 of figure 4.19 has not been computed yet.

# Chapter 5

# Identification of bubble subdiagram

As it was discussed in the literature [65, 66, 67, 68, 69, 70] and recalled in section §2.3, bubbles are these parts of spin-foams that are responsible for divergent amplitudes. So far some study was done, but concerning only spin-foams dual to triangulations (for example [69, 65, 70]). Here we expand the study to all 2-complexes admissible by the Operator Spin-network Diagram framework (see section §3.5).

Bubbles are spin-foam analogues of Feynman diagram's loops. A spin-foam containing a bubble in most cases admits infinitely many colorings giving nonvanishing amplitude for fixed boundary conditions (see subsection 5.2.4). Since the transition amplitude for a boundary state is defined as a sum over all internal configurations, the above property of bubbles make them natural candidates for sources of possible infinite amplitudes of the theory. Thus in order to understand the theory well one has to be able to identify bubbles and characterise their properties.

One of the advantages of Operator Spin-network Diagram framework is that they provide an easy way to identify a subdiagram equivalent to a spin-foam being an isolated bubble. A very similar algorithm allows to count the rank of the bubble - namely the number describing, how many sub-bubbles it contains. This chapter is devoted to this issue.

This chapter is organized as follows. Firstly, in section §5.1, we define bubbles in terms of Operator Spin-network Diagrams. We also introduce some naming conventions that simplify further sections and we discuss the relation between our definition and the ones used in literature. The section §5.2 presents the algorithms mentioned above together with several examples of diagrams to which we apply the algorithms and explaining some classes of bubbles that may appear and calculate amplitudes related to some of these examples.

## 5.1 Definitions

In section §3.2 we introduced and studied the framework of graph diagrams. They were designed to represent spin-foam calculations, thus some restrictive assumptions were made for their components.

On the contrary, this chapter is devoted finding parts of graph diagrams satisfying certain conditions, what requires a strict procedure of identifying and classifying objects obtained by removing parts of graph diagrams. These objects will be called *generalized graph diagrams*. In contrast, the graph diagrams as they were defined in section §3.2 will



Figure 5.1: Generalized graph diagrams. (a) - An example of an ordinary graph diagram  $\mathcal{D}$ . (b) - The one node 1-valent node and two links added, one link erased and link-relation *spoiled* - so that it is not always a duality relation - becomes a generalized graph diagram  $\mathcal{D}'$ .

be called *ordinary graph diagrams*. Strict definitions of generalized graph diagrams and operations on them are given in subsection 5.1.1.

Having defined generalized graph diagrams we use them to the issue of bubbles, what is described in subsection 5.1.2. We introduce a definition of a bubble in terms of graph diagram framework, we discuss its relation with definitions used in the literature and we introduce a notion of rank of a bubble, that helps to characterize them.

## 5.1.1 Generalized graph diagrams

#### **Definition 5.1.** Generalized graph diagram

A generalized graph diagram is a pair  $\mathcal{D} = (\mathcal{G}, \mathcal{R})$  of a graph  $\mathcal{G}$  and a family of relations  $\mathcal{R} = (\mathcal{R}_{node}, \mathcal{R}_{link})$ , where  $\mathcal{G}$  is a general graph (not necessarily closed) and the relations  $\mathcal{R}_{node}$  and  $\mathcal{R}_{link}$  are similar to the node- and link-relations of a graph diagram (see definition 3.1 in subsection 3.2.1.1) with the following conditions loosen:

- The  $\mathcal{R}_{node}$  relation may relate pairs of nodes that are not dual.
- For each pair  $(n, n') \in \mathcal{R}_{node}$  there is a relation  $\mathcal{R}_{link}^{(n,n')} \subset \mathcal{L}_n \times \mathcal{L}_{n'}$  in the family  $\mathcal{R}_{link}$ , however, the relation  $\mathcal{R}_{link}^{(n,n')}$  does not have to be induced by a duality map between  $\mathcal{L}_n$  and  $\mathcal{L}_{n'}$ . Nevertheless, the relation  $\mathcal{R}_{link}^{(n,n')}$  must be induced by a duality map  $\phi : \tilde{\mathcal{L}}_n \to \tilde{\mathcal{L}}_{n'}$ , where  $\tilde{\mathcal{L}}_n \subset \mathcal{L}_n$  and  $\tilde{\mathcal{L}}_{n'} \subset \mathcal{L}_{n'}$ . Obviously it may happen, that  $\tilde{\mathcal{L}}_n = \mathcal{L}_n$  and  $\tilde{\mathcal{L}}_{n'} = \mathcal{L}_{n'}$  then the relation is a duality relation, as in the case of ordinary graph diagrams. It may also happen, that  $\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}_{n'} = \emptyset$  then the relation  $\mathcal{R}_{link}^{(n,n')}$  is the empty relation.

Obviously an ordinary graph diagram is a generalized graph diagram, but not the opposite.

The easiest way to obtain a generalized graph diagram is to take a graph diagram and add or remove some links and nodes to it, without paying attention to the consistency of link and node relations - see figure 5.1.



Figure 5.2: The boundary graph of the the generalized graph diagram  $\mathcal{D}'$  of figure 5.1b. (a) - The boundary  $\partial \mathcal{D}'$ . (b) - The same graph, but deformed to make the drawing more legible.

The notions of face- and edge-relations generalize straightforward to generalized graph diagrams.

It is useful to redefine the notion of a boundary concerning generalized graph diagrams.

#### **Definition 5.2.** Boundary of a generalized graph diagram

We say, that a half-link of a generalized graph diagram is a boundary half-link iff it is not related by any of  $\mathcal{R}_{link}^{(n,n')}$ -relations.

We say, that a node of a generalized graph diagram is a boundary node iff at least one of half-links incident to it is a boundary half-link.

The boundary graph  $\partial \mathcal{D}$  of a generalized graph diagram  $\mathcal{D}$  is a graph build of all the boundary nodes and boundary half-links of  $\mathcal{D}$  connected as follows:

- obviously each boundary half-link is incident to the node of the boundary graph, to which it was incident in the diagram.
- two boundary half-links form a link iff they belong to the same equivalence class of the face relation.

Thanks to definition of the face relation for each positive boundary half-link there is a negative boundary half-link belonging to the same (open) equivalence class of the face relation, thus there are no half-links left open in the boundary graph. However, there might be open links in the boundary graph, i.e. links incident to 1-valent nodes.

An example can be found at figure 5.2.

In case of ordinary graph diagram the above definition overlaps with the definition introduced in subsection 3.2.3.4. Indeed, in case of ordinary graph diagrams a half-link incident to a node is a boundary half-link iff this node is not related with any other node by the node relation. In such case all the other half-links incident to this node are also boundary.

For generalized graph diagram one can define operations of erasing links and nodes in a way preserving the generalized graph diagram category (which was not the case for ordinary graph diagrams).



Figure 5.3: Erasing links of generalized graph diagrams. (a) - A diagram  $\mathcal{D}$  which links we will erase. (b) - Erasing one link: the diagram  $\mathcal{D} \setminus \ell_1$ . (c) - Erasing several links: the diagram  $\mathcal{D} \setminus \{\ell_1, \ell_2, \ell_3\}$ . (d) - Outer links. The link  $\ell_1$  is simply outer. The link  $\ell_3$  is outer, but not simply-outer: one has to erase  $\ell_1$  and  $\ell_2$  to make it simply-outer. The link  $\ell_4$  is inner.

## Definition 5.3. Erasing a link

Given a generalized graph diagram  $\mathcal{D} = (\mathcal{G}, \mathcal{R})$  and a link  $\ell \in \mathbf{L}_{\mathcal{G}}$  we define a graph diagram with erased link (which we shell denote as  $\mathcal{D} \setminus \ell$ ) as follows:

- The graph  $\mathcal{G} \setminus \ell$  is the graph with the set of nodes  $\mathbf{N}_{\mathcal{G} \setminus \ell} = \mathbf{N}_{\mathcal{G}}$  and the set of links  $\mathbf{L}_{\mathcal{G} \setminus \ell} = \mathbf{L}_{\mathcal{G}} \setminus \{\ell\}$ . All the incidence relations in  $\mathcal{G} \setminus \ell$  are induced by the incidence relations in  $\mathcal{G}$ .
- The node relation  $\mathcal{R}_{node}$  of the diagram  $\mathcal{D} \setminus \ell$  is the same as in  $\mathcal{D}$ .
- The link relations change as follows: if a half-link l̃<sup>ϵ</sup> was related to one of half-links of l in the diagram D, then it becomes a boundary half-link in the diagram D \ l. In other words each pair containing l<sup>+</sup> of l<sup>-</sup> is removed from any R<sup>(n,n')</sup><sub>link</sub> relation.

Since all the operations presented above are subtractions of sets, the result does not depend on the order in which they are performed. This leads to a conclusion, that erasing many links can be done simultaneously. For erasing several links  $\{\ell_1, \ldots, \ell_N\}$  we introduce the notation  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\} := (\cdots (\mathcal{D} \setminus \ell_1) \setminus \ldots) \setminus \ell_N$ .

An example of erasing links can be found at figure 5.3.

For later convenience let us introduce a notion of outer links. The definition is recursive, thus first let us introduce the base of the recursion:

#### **Definition 5.4.** Simply outer link

A link  $\ell$  of a generalized graph diagram is called simply outer iff at least one of the following is true:

- at least one of half-links of  $\ell$  is a boundary half-link,
- $\ell$  is incident to a 1-valent node.

Erasing the links may cause some links to become simply outer. This leads us to next definition:

## **Definition 5.5.** Outer link

A link  $\ell$  of a generalized graph diagram  $\mathcal{D}$  is called outer iff it is simply outer or there exists a series of links  $(\ell_1, \ldots, \ell_N)$  such that

- the link  $\ell_1$  is a simply outer link of  $\mathcal{D}$ ,
- for each i = 2, ..., N the link  $\ell_i$  is a simply outer link of  $\mathcal{D} \setminus \{\ell_1, ..., \ell_{i-1}\},\$
- the link  $\ell$  is a simply outer link of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ .

Note, that although erasing the links is order-independent, the ordering of the series  $(\ell_1, \ldots, \ell_N)$  matters. Indeed, the diagram  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$  does not depend on the order of erasings, however, the intermediate steps differ for different orderings and the property of being simply outer is required to be satisfied through all intermediate steps.

Obviously if  $\ell$  is outer, there may be more than one series satisfying the above conditions.

Illustration of outer links can be found at figure 5.3d

Erasing nodes is more complicated. In order to preserve the generalized graph diagrams category one has to deal with the links incident to the nodes one erases and there are several ambiguous ways to do it. From the point of view concerning our purpose (namely dealing with bubble subdiagrams) it is convenient to consider only the following approach: in order to erase a node one has to first erase all the links incident to it and then erase the 0-valent node using the following procedure.

## Definition 5.6. Erasing a 0-valent node

Given a generalized graph diagram  $\mathcal{D} = (\mathcal{G}, \mathcal{R})$  and a 0-valent node  $n \in \mathbf{N}_{\mathcal{G}}$  we define a graph diagram with erased node (which we shell denote as  $\mathcal{D} \setminus n$ ) as follows:

- The graph G \ n is the graph with the set of nodes N<sub>G\ℓ</sub> = N<sub>G</sub> \ {n} and the set of links L<sub>G\ℓ</sub> = L<sub>G</sub>. All the incidence relations in G \ n are induced by the incidence relations in G.
- If the node n was related with another node n' in the diagram  $\mathcal{D}$ , then the node relation  $\mathcal{R}_{node\setminus n}$  of the diagram  $\mathcal{D} \setminus n$  equals  $\mathcal{R}_{node} \setminus \{(n,n')\}$ , otherwise the node relation  $\mathcal{R}_{node\setminus n}$  of the diagram  $\mathcal{D} \setminus n$  is the same as in  $\mathcal{D}$ .
- Even if the node n was related with another node n' in  $\mathcal{D}$ , the only possible link relation  $\mathcal{R}_{link}^{(n,n')}$  was the void relation, because n was by assumption 0-valent. We remove it in such case. In any case, the equivalence classes of  $\mathcal{R}_{link}$  in the diagram  $\mathcal{D} \setminus n$  are the same, as in  $\mathcal{D}$ .

Again removing several 0-valent nodes is independent and can be done simultaneously, we will denote it as  $\mathcal{D} \setminus \{n_1, \ldots, n_N\} := (\cdots (\mathcal{D} \setminus n_1) \setminus \ldots) \setminus n_N$ .

Let us now define and characterize subdiagrams of generalized graph diagrams:

### **Definition 5.7.** Subdiagrams

A generalized graph diagram  $\mathcal{D}'$  is called a generalized subdiagram of  $\mathcal{D}$  iff it can be obtained by erasing some links and then erasing some 0-valent nodes of  $\mathcal{D}$ .

A generalized subdiagram  $\mathcal{D}'$  of  $\mathcal{D}$  that is an ordinary graph diagram is called a proper subdiagram of  $\mathcal{D}$ .

A proper subdiagram  $\mathcal{D}'$  of an ordinary graph diagram  $\mathcal{D}$  such that the graph  $\mathcal{G}'$  of  $\mathcal{D}'$ is a subgraph of the graph  $\mathcal{G}$  of  $\mathcal{D}$  obtained by erasing only all connected components of  $\mathcal{G}$ is called a simple subdiagram of  $\mathcal{D}$ .

We will also need to identify connected parts of (generalized-) graph diagrams. For our purpose it is convenient to consider connectivity in terms of the edges of the 2-complex related to an ordinary graph diagram (what can be easily generalized to generalized graph diagrams). This intuition is concretized by the following definition:

#### **Definition 5.8.** Diagram-connectivity

Given a (generalized-) graph diagram  $\mathcal{D} = (\mathcal{G}, \mathcal{R})$  we say, that each connected component  $\Gamma_I$  of  $\mathcal{G}$  is diagram-connected. Moreover, each two connected components  $\Gamma_I, \Gamma_{II}$  are diagram-connected, if there is a node relation relating a node  $n \in \mathbf{N}_{\Gamma_I}$  with  $n' \in \mathbf{N}_{\Gamma_{II}}$ . Moreover we want diagram-connectivity to be transitive, so if  $\Gamma_I$  is diagram connected with  $\Gamma_{II}$ , and  $\Gamma_{II}$  is diagram connected with  $\Gamma_{III}$ , then  $\Gamma_I$  is diagram-connected with  $\Gamma_{III}$ .

A diagram  $\mathcal{D}$  is diagram-connected if each connected component of its graph is diagramconnected with each other connected component. A diagram-connected part of a graph diagram  $\mathcal{D}$  is a maximal subdiagram of  $\mathcal{D}$ , which is diagram-connected.

Two elements (nodes or links) of a graph diagram  $\mathcal{D}$  are diagram-connected iff they are in the same diagram-connected part of  $\mathcal{D}$ .

If two elements (nodes, links or connected components) are not diagram-connected, they are called diagram-disconnected.

For a generalized diagram its diagram-connected parts are its generalized subdiagrams (they can be proper, but they do not have to). For an ordinary graph diagram each diagram-connected part is a simple subdiagram.

In case of graphs one has to always emphasise whether considering topological connectivity of diagram-connectivity. However, in the category of (generalized-) graph diagrams there is no notion of connectivity other then diagram-connectivity. Thus whenever it will not cause a confusion we drop the prefix *diagram*-.

As it was discussed in subsection 3.4.4, a diagram-disconnected diagram has transition amplitudes that factorizes into a product of transition amplitudes of its diagramconnected parts.

## 5.1.2 Bubble-part of graph diagram

In the spin-foam community the word *bubble* was used mainly in context of 2-complexes dual to a triangulation of a 4-manifold. The bubble was defined as a set of faces of the foam that form a closed surface [77, 69, 65] (the surface does not mean smooth surface, since faces of spin-foams are not smooth manifolds, it means a topological surface). This definition was not used strictly - it was applied also to for example the melonic bubble
[79, 78] or *self-energy-bubble* [69], which are not topological surfaces, but rather closed (i.e. boundaryless) 2-CW-complexes. This leads us to formulating our definition, which though being more strict, covers the traditional definition.

In what follows a bubble in context of spin-foams will be called a spin-foam bubble or a topological bubble. The name *bubble* without any adjective will be reserved for representation of the ones in terms of (generalized-) graph diagrams.

### 5.1.2.1 Bubble in graph-diagram framework

First let us define an isolated bubble.

#### **Definition 5.9.** An isolated bubble

An isolated bubble is a nonempty graph diagram with no boundary.

Let us justify the above definition by showing its relation to the traditional one. As we have showed in chapter 3, graph-diagrams define a class of 2-CW-complexes, that can be used to define a spin-foam theory. Thus an isolated bubble has its interpretation in terms of 2-CW-complexes and vice versa.

Let us firstly consider a 2-CW-complex  $\kappa_1$  forming a closed surface. We assume that  $\kappa_1$  is a 2-CW-complex of some spin-foam, thus by assumption there is a graph diagram representing it (see subsection 3.5.1). Since  $\kappa_1$  has no boundary, the graph diagram  $\mathcal{D}_{\kappa_1}$  also has no boundary. Since  $\kappa_1$  form a surface, it must have at least one face, thus  $\mathcal{D}_{\kappa_1}$  must have at least one link, so it is nonempty. Thus  $\mathcal{D}_{\kappa_1}$  is an isolated bubble.

As we have already recalled, the notion *bubble* was used also in case of more general foams, i.e. for subcomplexes that are not topological manifolds (like for example the *melonic* bubble). Nevertheless, they were still nonempty boundaryless 2-complexes. Let  $\kappa_2$  be such complex. Then again, by assumption, there must be a graph-diagram  $\mathcal{D}_{\kappa_2}$  representing this 2-complex. The diagram  $\mathcal{D}_{\kappa_2}$  must be non-empty and must not have a boundary - so it must be an isolated bubble. Consider now the opposite situation: let  $\mathcal{D}_3$  be an isolated bubble and  $\kappa_{\mathcal{D}_3}$  be the 2-complex it represents. Since  $\mathcal{D}_3$  is boundaryless, the 2-complex  $\kappa_{\mathcal{D}_3}$  must not have boundary, and since  $\mathcal{D}_3$  is nonempty,  $\kappa_{\mathcal{D}_3}$  contains at least one face, so  $\kappa_{\mathcal{D}_3}$  is a bubble in the traditional means.

Although the above considerations may seam tautological, they are not. The traditional definition was formulated in context of 2-complexes dual to triangulations of manifolds. The class of 2-complexes we introduced in chapter 3 is more general and thus we must have assured that the traditional definition of a bubble could be translated to this new context.

In case of diagrams with boundary, the full diagram cannot be a bubble, but it can contain a bubble:

### **Definition 5.10.** A diagram containing a bubble

If a graph diagram contains a proper subdiagram being an isolated bubble, we say that it is a diagram containing a bubble or simply it is a diagram with a bubble. The subdiagram is called the bubble of the diagram.

The assumption, that the isolated bubble part is a proper subdiagram, assures, that within the class of admissible 2-complexes there must be one corresponding to this bubble part of considered diagram.

Thanks to the correspondence between graph-diagrams and 2-complexes, our definition applies also to spin-foam bubbles: **Definition 5.11.** A spin-foam containing a bubble

If a 2-complex of a spin-foam can be represented by a graph diagram with a bubble, we say that this spin-foam contains a bubble.

### 5.1.2.2 Characterisation of bubbles

Let us characterise some basic properties of possible bubbles. We characterise only isolated bubbles. A diagram with a bubble is characterised by the isolated bubble it contains and the information about the way the isolated bubble is embedded it.

### **Definition 5.12.** Connected bubbles

An isolated bubble that is diagram-connected is called a connected bubble.

In what follows we consider only connected bubbles. Obviously each diagramconnected part of an isolated bubble is an isolated bubble itself. Disconnected bubbles are disjoint products of connected bubbles. In terms of spin-foam amplitudes the contribution of disconnected parts of a diagram simply multiply (see subsection 3.4.4). Although it may happen, that a connected diagram with a bubble contains a disconnected isolated bubble, it is the issue of embedding bubbles in diagrams, unrelated to internal structure of the bubbles.

Let us define the rank of a bubble:

### **Definition 5.13.** Rank of a bubble:

Given an isolated bubble  $\mathcal{D}$  the rank of the bubble  $\mathcal{D}$  is the minimal number of links that must be erased from  $\mathcal{D}$  to obtain a bubbleless diagram.

Obviously the rank of a disconnected bubble is the sum of ranks of its connected parts.

The rank of a bubble measures the quantity that can be intuitively called as the number of surfaces in the bubble. In fact, the above definition translates to the spin-foam bubbles as the minimal number of faces one has to remove in order to make the foam bubbleless (in other words: it is the minimal number of faces of a CW-complex homotopic to the bubble).

Consider for a moment an isolated bubble  $\mathcal{D}$  and a diagram  $\mathcal{D}'$  containing a bubble such that it's bubble part is precisely  $\mathcal{D}$ . Note, that if one erases these links of  $\mathcal{D}'$ , which would cause  $\mathcal{D}$  to become bubbleless, the diagram  $\mathcal{D}$  also becomes bubbleless. Indeed, erasing links cannot increase the number of inner links of a diagram, so no new bubble may appear. But the old bubble cannot preserve such operation, because the links we erased by definition cause all the bubble part of  $\mathcal{D}'$  to become outer. Thus we conclude, that rank is a property not only of isolated bubbles, but also of the whole diagram (and the definition is straightforward, namely the rank of a diagram is the minimal number of links that must be erased from it to obtain a bubbleless diagram, which equals to the rank of its maximal subdiagram being an isolated bubble). The topological interpretation also generalizes to whole diagram: given a 2-CW-complex of an ordinary graph diagram  $\mathcal{D}'$  the minimal number of faces of a general CW-complex<sup>1</sup> homotopic to it equals to the rank of the diagram  $\mathcal{D}'$ , because each 2-comples of a bubbleless diagram is homotopic to a 1-complex [85].

There is a topological invariant that measures a similar quantity, namely the 2nd Betti number. It measures the number of holes enclosed by a 2-dimensional complex. It was already used to characterise bubbles of spin-foams [70]. Nevertheless, they do not

<sup>&</sup>lt;sup>1</sup>See remark A.2.

equal to the rank of the bubble - for example a projective plane is a bubble of rank 1 (see subsection 5.2.3.3), while its 2nd Betti number equals 0.

### 5.2 The algorithm finding bubble subdiagram

This section presents two algorithms that the Author formulated. The first algorithm, presented in subsection 5.2.1, is a tool to find a subdiagram of an operator spin-network diagram being a pure bubble together with a proof of correctness. The second algorithm, presented in subsection 5.2.2, is a slight modification of the previous algorithm, that allows to count the order of the bubble found using the previous algorithm.

The algorithms are described in a condensed manner, without illustrations. Examples of applications of the algorithms can be found in subsection 5.2.3.

Finally, in subsection 5.2.4, we study some basic harmonic-analytical properties of bubbles.

### 5.2.1 Does a diagram have a bubble?

Given an ordinary graph diagram  $\mathcal{D}$  we will find its bubble. The general concept of the algorithm is based on the following fact:

### **Theorem 5.1.** The bubble of a graph diagram

Given a (generalized-) graph diagram  $\mathcal{D}$  with a bubble, the bubble of  $\mathcal{D}$  is the subdiagram build of all inner link of  $\mathcal{D}$ .

### Proof

First let us proof a lemma saying, that if  $\mathcal{D}'$  is a closed proper subdiagram of  $\mathcal{D}$  and  $\ell \notin \mathcal{G}_{\mathcal{D}'}$ , then  $\mathcal{D}'$  is a closed proper subdiagram of  $\mathcal{D} \setminus \ell$ . Indeed, erasing  $\ell$  must not spoil any of properties of  $\mathcal{D}'$ . If there were a link of  $\mathcal{D}'$  related within  $\mathcal{D}$  with  $\ell$  via  $\mathcal{R}_{\text{link}}$  relation,  $\mathcal{D}'$  would not be closed. If there is a node of  $\mathcal{D}'$  incident to  $\ell$ , it must be incident to at leas 2 other links, both in  $\mathcal{D}'$ , otherwise  $\mathcal{D}'$  would not be proper. Thus erasing  $\ell$  neither opens any equivalence class of face-relation in  $\mathcal{D}'$  nor reduce the valency of any nodes of  $\mathcal{D}'$  to 1, so  $\mathcal{D}'$  is a closed proper subdiagram of  $\mathcal{D} \setminus \ell$ .

Now we will prove one direction of implication, namely that no outer link of a diagram  $\mathcal{D}$  is a member of the bubble of  $\mathcal{D}$ .

From the above lemma we conclude, that if  $\ell$  does not belong to any closed proper subdiagram of  $\mathcal{D}$ , each closed proper subdiagram of  $\mathcal{D}$  is a closed proper subdiagram of  $\mathcal{D} \setminus \ell$ .

Consider thus a link  $\ell$  being an outer link of  $\mathcal{D}$ . We will show that there is no proper boundaryless subdiagram of  $\mathcal{D}$  that contains  $\ell$ .

If  $\ell$  is outer, there are 2-possibilities: either it is simply outer or it is not.

If  $\ell$  is simply outer, by definition it may not be a member of a proper closed subdiagram. If  $\ell$  has a boundary half-link, each subdiagram containing  $\ell$  must have a boundary, thus it cannot be closed. If  $\ell$  is incident to a 1-valent node, each subdiagram containing  $\ell$  must have a 1-valent node, thus it is not proper. So each simply outer link  $\ell$  must not be a member of a proper closed subdiagram of  $\mathcal{D}$ .

If  $\ell$  is not simply outer, there is a sequence of links  $(\ell_1, \ldots, \ell_N)$  satisfying the property of definition 5.5 (i.e. for each  $i = 1, \ldots, N$  the link  $\ell_i$  is a simply outer link of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_{i-1}\}$  and  $\ell$  is a simply outer link of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ ). Since  $\ell_1$  is simply outer, it must not be a member of any proper closed subdiagram of  $\mathcal{D}$ , thus (as it was

concluded from the lemma) each closed proper subdiagram of  $\mathcal{D}$  is a closed proper subdiagram of  $\mathcal{D} \setminus \ell_1$ . But  $\ell_2$  is a simply outer link of  $\mathcal{D} \setminus \ell_1$ , thus each closed proper subdiagram of  $\mathcal{D} \setminus \ell_1$  is a closed proper subdiagram of  $\mathcal{D} \setminus \{\ell_1, \ell_2\}$ , and of course it is a closed proper subdiagram of  $\mathcal{D}$ . Iterating though the whole series we conclude, that each closed proper subdiagram of  $\mathcal{D}$  is a closed proper subdiagram of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ . But  $\ell$  is a simply outer link of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ , so it must not belong to any closed proper subdiagram of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ , so it must not belong to any closed proper subdiagram of  $\mathcal{D} \setminus \{\ell_1, \ldots, \ell_N\}$ , so it must not belong to any closed proper subdiagram of  $\mathcal{D}$ .

We are left with the proof that if a link  $\ell$  is inner, it belongs to a proper closed subdiagram of  $\mathcal{D}$ .

Let  $\ell \in \mathbf{L}_{\Gamma}$  for one of connected components  $\Gamma$  of the graph  $\mathcal{G}_{\mathcal{D}}$ . Since  $\Gamma$  is closed (what follows from the assumption, that  $\mathcal{D}$  is an ordinary graph diagram), there must be an unoriented cycle of links in  $\Gamma$  containing  $\ell$ . If  $\ell$  is inner, there must be an unoriented cycle of *inner links* containing  $\ell$  within  $\Gamma$ . Indeed, assume, that each unoriented cycle containing  $\ell$  has at least one outer link, and then erase all outer links of  $\mathcal{D}$  - so that only the inner links are left in  $\Gamma$ . Because of the assumption, there are no unoriented cycles containing  $\ell$  in  $\Gamma$  - but this is possible only in an open graph. But if  $\Gamma$  is open, at least one of its links is simply outer. But if it became simply outer after removing some outer links, it was outer by definition, what is in the contrary with our assumption, that we erased all outer links. Thus a subgraph  $\Gamma'$  of  $\Gamma$  build of only inner links is a closed graph.

Note now, that if  $\ell$  is inner, then both of its ends must not be boundary links. Indeed, since  $\mathcal{D}$  is an ordinary graph diagram, if one of ends of  $\ell$  were boundary nodes  $\ell$  would be a boundary, thus simply outer link. Moreover,  $\ell$  must not be a member of an open equivalence class of the face relation. If it were, the links in the link relation would form a series of links ending with a boundary, thus simply outer link. Moreover, if  $\ell$  is inner, than none of the links in the equivalence class of face relation to which  $\ell$  belongs may be outer. Indeed, consider an internal equivalence class of the face relation such that it contains a link  $\ell'$  being for some an outer link. Let's erase all the links that cause  $\ell'$ to become simply outer and then erase  $\ell'$ . Then the equivalence class becomes an open equivalence class, thus all its members are also outer. Thus if  $\ell$  is an inner link, its whole face-relation equivalence class consists also of only inner links.

Consider now a subdiagram  $\mathcal{D}'$  of  $\mathcal{D}$  build of an inner link  $\ell$  together with the whole inner subgraph  $\Gamma'$  of its connected component  $\Gamma$ , extended by whole equivalence classes of all the links in  $\Gamma'$ , and then extended of all inner subgraphs of all the links added in previous step, and then extended by whole equivalence classes of the links added in the previous step, and so on, until no new links are added. This subdiagram consists of only closed graphs, because each inner subgraph of a connected component of the graph is closed. This subdiagram has no boundary nodes (and thus it has no boundary), as we have proven in previous paragraph. Thus it is an isolated bubble.

Quod erat demonstrandum

The theorem was proven for generalized graph diagrams, because even starting from an ordinary graph diagram removing links quickly leads to generalized graph diagrams.

The proof of the theorem is in fact the base for our algorithm, which is presented below:

### **Proposition 5.1.** Find-bubble algorithm

Consider a graph diagram  $\mathcal{D}$ . In order to find the bubble part of  $\mathcal{D}$  do as follows:

- 1. Erase each simply-outer link of  $\mathcal{D}$ .
- 2. If after doing step 1 some links become simply-outer go back to step 1 and erase them.

- 3. If no new simply-outer links appeared erase all 0-valent nodes.
- If all the diagram was erased it means it had no bubble. If not what is left is the bubble part of D, in the form of an isolated bubble.

At first the only simply-outer links to be erased are the links with boundary half-links. However, starting from the second iteration some links incident to 1-valent nodes may start to appear.

### 5.2.2 What is the rank of the bubble?

Consider now an isolated bubble (possibly the one obtained as a result of find-bubble algorithm). We present here a way to find its rank.

If a bubble is disconnected, one should apply the following algorithm to each connected part and sum the results.

### **Proposition 5.2.** Rank-of-bubble algorithm

Let  $\mathcal{D}$  be a connected isolated bubble. In order to find its rank do as follows: Start with rank r = 0.

- 1. Pick a link  $\ell$  of  $\mathcal{D}$  and erase it, increase r by 1.
- Apply the find-bubble algorithm to the diagram D \ ℓ, obtaining another diagram D.
- If D
   is disconnected rewind the procedure to the step 1 and pick another link l', such that it was not one of the links erased in the step 2.
- If D
   is an empty diagram the number r is the rank of the original bubble. Otherwise
   repeat the procedure starting form D
   .

The step 3 prevents from the situation, when the 2-complex representing the diagram  $\mathcal{D}$  has a form of two (or more) isolated bubbles connected by a set of faces homeomorphic to a disc (or several disconnected discs). If a link corresponding to one of the faces of this disc were chosen, the result of find-bubble algorithm would be these disconnected bubbles. However, erasing the links corresponding to faces of the component bubbles will lead at some point to the situation, when the links corresponding to the disc connecting the bubbles become outer links without increasing the rank counter. And since the rank counter is the minimal number of links erased, one must not count the links of the connecting disc. An illustration of such situation can be found in the example presented in subsection 5.2.3.5.

### 5.2.3 Examples

Here we provide several examples of applications of the algorithms presented above. We start with three examples of bubbleless diagrams in subsection 5.2.3.1. Then we introduce several realisations of a bubble with topology of a sphere (see subsection 5.2.3.2). In subsection 5.2.3.3 we study other basic possibilities of bubbles with rank 1, i.e. real projective plane, torus and Klein-bottle. In subsection 5.2.3.4 we present an example of a diagram containing a bubble considered in [78] (called *melonic* bubble). Finally in subsection 5.2.3.5 we show, how do our algorithms work in case of multiple bubbles, both when they form a disconnected isolated bubble and if they form a bubble connected by contractible faces (i.e. when the step 3 of the rank-of-bubble algorithm applies). For the

simple bubbles of subsections 5.2.3.2-5.2.3.3 we also add the calculation of BF transition amplitudes of isolated bubbles.

The examples are described mostly by the figures. For most of bubbles there are extra figures presenting construction of spin-foams corresponding to them (except from those, which are non-embeddable in 3d). The simply outer links are highlighted red (sometimes links which are outer, but not simply outer, are also highlighted).

The diagrams presented at the figures are results of erasing of links, thus they are mostly generalized graph diagrams. This means, that it might happen, that two nodes in the diagram are related by the  $\mathcal{R}_{node}$  relation, but no links incident to them are related by  $\mathcal{R}_{link}$  relation. However, in most situations this is not the case, and if it is, it is emphasised explicitly in the description of the figure. Usually when there is a pair of of nodes related by the node relation and no link relation is drawn, it means that the link relation is a duality relation, however its precise form is irrelevant for the step we present at this particular figure and it would make the figure illegible.

Due to limit the number of figures we usually omit the figures presenting few last final steps of each procedure (if they are obvious). Usually we stop at the step, when all the remaining links are simply-outer.

### 5.2.3.1 Bubbleless diagrams

Let us start with three examples of bubbleless diagrams.

First let us consider the simplest nontrivial diagram, that was used as an example in subsection 3.4.2. It is obviously bubbleless and thus it is a good example to show, how our algorithm behaves in such situation. The figure figure 5.4 present the steps of find-bubble algorithm.

As the next let us take the diagram with the same boundary, presented in section §4.3 at figure 4.15a. It appears to be also bubbleless, however it requires more steps to see it. It is analysed at figure 5.5. First two steps of the procedure effect only the boundary static diagram, that is attached to the interaction part of the diagram. In fact the boundary static part of the diagram, being a topological cylinder over the boundary, never contains a bubble and is always removed by the first two steps of the find-bubble algorithm.

As the third example let us take the diagram that we recalled at the beginning of this chapter at figure 5.1a. It is also bubbleless, what can be showed in three steps, presented at figure 5.6.

### 5.2.3.2 Spherical bubbles

Let us now consider few simple realisations of diagrams with bubbles. In each example we choose possibly simple diagrams (usually with all outer links being simply-outer) in order to keep attention on the pattern of bubble subdiagrams rather than the way they are embedded in full diagrams.

First let us consider a diagram containing a *dumpling bubble*, presented at figure 5.7. The name comes from the shape of the bubble obtained in this procedure. The dumpling bubble topologically is a sphere, what is shown at figure 5.8.

One can simply calculate the BF transition amplitude of such bubble.

Given a spin-label j of the face, the nodes are labelled by a projection operator  $\hat{P}$ : Inv  $(\mathcal{H}_j \otimes \mathcal{H}_j) \to$  Inv  $(\mathcal{H}_{j^*} \otimes \mathcal{H}_{j^*})$ . But the space Inv  $(\mathcal{H}_j \otimes \mathcal{H}_j)$  is 1-dimensional, spanned by  $\epsilon^{m_\ell m_{\ell'}}$ , thus the projection operator is  $\hat{P} = \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'}+} \epsilon_{n_{\ell}-n_{\ell'}-}$ . The *BF* contractor is  $A^{\text{Tr}} = \delta^{n_{\ell'}-}_{m_{\ell'}+} \delta^{n_{\ell'}-}_{m_{\ell'}+}$ . The face weight amplitude is  $A_f = 2j + 1$  per each face,



Figure 5.4: An example of bubbleless diagram. (a) - Step 0: The diagram that is studied. (b) - Step 1: All the simply-outer links are colored red. (c) - Step 2: Some (in fact all) of the remaining links became simply-outer. (d) - Step 3: After repeating the Step 1 no links are left. All the remaining nodes are 0-valent, thus we remove them and we go to the Step 4, where we are left with the empty diagram. Thus there is no bubble.



Figure 5.5: The second example of a bubbleless diagram. Until (e) the link-relations are irrelevant, and thus they are omitted at the figures (it is enough to know, that whenever two nodes are connected by the node-relation, there is a duality relation between their links). (a) - The diagram we consider. (b) - The first iteration - the simply outer links of the original diagram. (c) - The simply outer links of the diagram obtained after the first iteration. (d) - Since this moment the boundary static part of the diagram will not be relevant, thus focus on the interaction part. The third iteration of the simply outer links. (e) - The first moment when the precise form of the link-relation matters: it indicates, which links of the remaining  $\theta$ -graph are simply outer (because they were related to the links we already removed), and which one is not. (f) - The last iteration - only one link remains and it is simply outer.



Figure 5.6: Another example of bubbleless diagram. (a) - The simply outer links of the diagram  $\mathcal{D}$  of figure 5.1a are highlighted. (b) - Removing all the simply outer links cause some other links become simply-outer. (c) - The last link remain. But it is also simply outer, because it is incident to 1-valent nodes. Thus we must erase it and then, left with only 0-valent nodes, we conclude that there is no bubble in the diagram  $\mathcal{D}$ .



Figure 5.7: A diagram containing a *dumpling bubble*. (a) - The diagram. (b) - All simply outer links of the diagram are highlighted. (c) - The subdiagram being an isolated bubble - i.e. the dumpling bubble.



Figure 5.8: Construction of a 2-complex corresponding to the dumpling bubble. (a) - The 1-vertex foam obtained from the dumpling bubble diagram. The blue square points are the middles of links, that appear when dividing a graph into a squid-graph (see subsection 3.2.2) (b) - The foam closed according to the relations becomes a sphere. The boundary links are glued like a dumpling - and that's the origin of the name.

and there are 2 equivalence classes of the face relation. There are obviously no boundary links, so the amplitude related to them is 1. Thus the amplitude (for fixed j) is

$$A_{\text{dumpling}}(j) = (2j+1)^2 \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'}+} \epsilon_{n_{\ell}-n_{\ell'}-} \delta^{n_{\ell}-}_{m_{\ell}+} \delta^{n_{\ell'}-}_{m_{\ell'}+} = (2j+1) \epsilon^{AB} \epsilon_{AB} = (2j+1)^2$$
(5.1)

Thus the total transition amplitude  $A_{\text{dumpling}} = \sum_{j=0}^{\infty} A_{\text{dumpling}}(j)$  is infinite. If one regularise it by putting a maximum spin cut-off  $\Lambda$ , one gets

$$A_{\text{dumpling}}^{\Lambda} := \sum_{j=0}^{\Lambda} A_{\text{dumpling}}(j) = O\left(\Lambda^{3}\right)$$
(5.2)

The sum goes every half-integer, so it is convenient to change the summing index to the dimension d of the spin-j representation:

$$A_{\text{dumpling}}^{\Lambda} = \sum_{d=0}^{2\Lambda+1} d^2 = \frac{1}{6} \left( 2\Lambda + 1 \right) \left( 2\Lambda + 2 \right) \left( 4\Lambda + 3 \right) = \frac{2}{3} \Lambda^3 + O\left( \Lambda^2 \right)$$
(5.3)

The Lorentzian EPRL amplitude of this bubble is not well defined, because the graph is only 2-edge connected, not 3-edge connected (see [84]). One can regularise it by removing all the redundant  $SL(2, \mathbb{C})$ -integrals - but there is one redundant integral for each 2-valent node, and this graph contain only 2-valent nodes, so one has to remove all  $SL(2, \mathbb{C})$ -integrals and the amplitude becomes the same as the *BF*-amplitude. This is the case for all bubbles with only 2-valent nodes.

Of course there are more realisations of spherical bubbles. Another example is a double-cone bubble, presented at figure 5.9.

The transition amplitude of a double-cone bubble is the same as the one of the dumpling bubble. Indeed, now the projection operator is  $\hat{P}$ : Inv  $(\mathcal{H}_j \otimes \mathcal{H}_{j^{\star}}) \to$  Inv  $(\mathcal{H}_j \otimes \mathcal{H}_{j^{\star}})$ , and its matrix elements in the spin basis are  $\hat{P} = \frac{1}{2j+1} \delta_{n_{\ell'}}^{m_{\ell'}} \delta_{n_{\ell'}}^{m_{\ell'}}$ . The *BF* contractors for each graph are  $A^{\text{Tr}} = \delta_{m_{\ell+}}^{n_{\ell}}$  and  $A'^{\text{Tr}} = \delta_{m_{\ell'+}}^{n_{\ell'}}$  respectively. There is only one equivalence class of the face relation, so the face amplitude contributes only once. Thus

$$A_{2-\text{cone}}(j) = (2j+1) \frac{1}{2j+1} \delta_{n_{\ell^{-}}}^{m_{\ell^{+}}} \delta_{m_{\ell^{+}}}^{m_{\ell^{+}}} \delta_{m_{\ell^{+}}}^{m_{\ell^{-}}} = \delta_{A}^{A} \delta_{B}^{B} = (2j+1)^{2}$$
(5.4)



Figure 5.9: A diagram containing a *double-cone bubble*. (a) - The diagram. (b) - All simply outer links of the diagram are highlighted. (c) - The subdiagram being the isolated bubble - i.e. the double-cone bubble.



Figure 5.10: Construction of a 2-complex corresponding to the double-cone bubble. (a) - Two 1-vertex foams obtained from the double-cone bubble diagram. (b) - The foams glued along their boundaries become a sphere.



Figure 5.11: Another diagram containing a spherical bubble. (a) - The diagram. (b) - Again all outer links of this diagram are simply outer, they are highlighted red. (c) - The isolated spherical bubble. (d) - 1-vertex foams of this isolated bubble are very similar to the double cone bubble. (e) - The 2-complex of considered bubble is a 2-sphear.

so, similarly, the total (regularized) amplitude is

$$A_{2-\text{cone}}^{\Lambda} = \frac{2}{3}\Lambda^3 + O\left(\Lambda^2\right) \tag{5.5}$$

One can also obtain spherical bubbles concerning more then one pair of related nodes, as it is presented at figure 5.11.

The common feature of all spherical bubbles is that the bubble subdiagram contains only 2-valent nodes. This leads to a conclusion that their rank equals 1. Indeed, erasing any link of a closed connected diagram containing only 2-valent nodes cause that each outer link becomes outer. Because of connectivity there must be a sequence of links either adjacent or link-related between any two links in the diagram; erasing a link at one end of such sequence makes the next link simply outer, and then the sequence becomes the sequence of simply-outer links required by the definition of being an outer link.

However, not only spherical bubbles has this property. Other examples with this property are described in the next subsection.



Figure 5.12: A diagram containing a real-projective-plane bubble. (a) - The diagram - being the diagram of figure 5.7a with the link relation slightly modified. (b) - The simply outer links of the diagram remain the same as in the dumpling bubble case. (c) - The subdiagram being an isolated bubble with the topology of real projective plane.

### 5.2.3.3 Other bubbles of rank 1

### Real projective plane bubble

A real projective plane is the next to the simplest example of a bubble. It was introduced in subsection 3.2.3.3 (at figure 3.10), but it can be also realised by a simple modification of the dumpling bubble from previous subsection, what is presented at figure 5.12. Since the projective plane is not embeddable in  $\mathbb{R}^3$ , we do not even try to draw the foam representing it.

Let us calculate the transition amplitude of the projective-plane bubble. All the components of the amplitude are the same as in case of the dumpling bubble (see subsection 5.2.3.2), however there two subtleties. First: there is only *one* equivalence class of the face relation, thus the factor (2j + 1) contributes only once. Second: the order of the indices of the projector operator is switched, so  $\hat{P} = \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'}+} \epsilon_{n_{\ell'}-n_{\ell-}}$ . This leads to the amplitude

$$A_{\mathbb{RP}(2)}(j) = (2j+1)\hat{P} = \frac{1}{2j+1}\epsilon^{m_{\ell}+m_{\ell'}+}\epsilon_{n_{\ell'}-n_{\ell'}-}\delta^{n_{\ell'}-}_{m_{\ell'}+}\delta^{n_{\ell'}-}_{m_{\ell'}+} = \epsilon^{AB}\epsilon_{BA} = -(2j+1)$$
(5.6)

thus the total amplitude is

$$A_{\mathbb{RP}(2)}^{\Lambda} = \sum_{d=0}^{2\Lambda+1} -d = -\frac{(2\Lambda+1)(2\Lambda+2)}{2} = -2\Lambda^2 + O(\Lambda)$$
(5.7)

Let us explain more the switch of the indices, which caused the minus sign. For a moment let us consider a diagram of the real projective plane bubble without gluing of the nodes. Then each node has its separate projection operator with free indices: the upper node has  $\hat{P}^+ = \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'}+} \epsilon_{A_{\ell}+A_{\ell'}+}$  and the lower node has  $\hat{P}^- = \frac{1}{2j+1} \epsilon^{B_{\ell}-B_{\ell'}-} \epsilon_{n_{\ell}-n_{\ell'}-}$  (of course there is also a boundary-link term  $\frac{1}{(\sqrt{2j+1})^2}$  and one more face-weight term (2j+1), but they together give the factor of 1, so we neglect them). As it was described in subsection 3.4.3, relating these two nodes results with appropriate contraction of the



(c)

Figure 5.13: Another realisation of a diagram containing a real projective plane bubble. (a) - The diagram. (b) - All outer links of this diagram are simply outer, they are highlighted red. (c) - The real projective plane subdiagram.

free indices (and appropriate adjustment of the weight factors, but as we mentioned, it does not matter in this case). The contraction is determined by the link relation. Since the half-link  $\ell^+$  is related to  $\ell'^-$  and the half-link  $\ell'^+$  is related to  $\ell^-$ , the contraction is indicated by  $\delta^{A_{\ell^+}}_{B_{\ell^-}} \delta^{A_{\ell'^+}}_{B_{\ell^-}}$ , what gives

$$\hat{P} = \hat{P}^{+} \circ \hat{P}^{-} = \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'+}} \epsilon_{A_{\ell}+A_{\ell'+}} \frac{1}{2j+1} \epsilon^{B_{\ell}-B_{\ell'-}} \epsilon_{n_{\ell}-n_{\ell'-}} \delta^{A_{\ell}+}_{B_{\ell'-}} \delta^{A_{\ell'+}}_{B_{\ell'-}} \\
= \frac{1}{2j+1} \epsilon^{m_{\ell}+m_{\ell'+}} \epsilon_{n_{\ell}-n_{\ell'-}} \cdot \frac{1}{2j+1} \left[ \epsilon_{A_{\ell}+A_{\ell'+}} \delta^{A_{\ell+}}_{B_{\ell'-}} \delta^{A_{\ell'+}}_{B_{\ell'-}} \epsilon^{B_{\ell}-B_{\ell'-}} \right]$$
(5.8)

The term in bracket is precisely  $\epsilon_{AB}\epsilon^{BA} = -(2j+1)$ , so the overall factor is -1, what we incorporated in the switch of the indices of  $\hat{P}$ .

An example of a diagram containing the real projective plane bubble as it was introduced in subsection 3.2.3.3 is presented at figure 5.13

### Toroidal bubble

Let us introduce another bubble diagram with only 2-valent nodes. It is presented at figure 5.14. The topology of a 2-complex corresponding to it is a torus - see figure 5.15.



Figure 5.14: A diagram containing a toroidal bubble. (a) - The diagram. (b) - All outer links of this diagram are simply outer, they are highlighted red. (c) - The isolated bubble subdiagram. One of possible orientations of links is indicated.



Figure 5.15: A spin-foam of the isolated diagram of a toroidal bubble. (a) - The 1-vertex foam. (b) - A pair of nodes glued. (c) - The second pair of nodes glued resulting in the topology of a torus.



Figure 5.16: Klein-bottle bubble. (a) - The diagram of an isolated bubble forming the Klein bottle. (b) - The intermediate step of construction of a spin-foam representing such bubble (note the link relation different than the one at figure 5.15b). We do not draw the final step, because the Klein bottle is not embeddable in 3d and such drawing would be illegible.

The *BF*-amplitude of such bubble can be calculated similarly to the amplitude of a spherical bubble. Let us orient and name links as it is indicated at figure 5.14c. Each pair of nodes is colored by the projection operator with appropriate indices, i.e. the vertical one is  $\hat{P}_{\text{vert}} = \frac{1}{2j+1} \epsilon^{m_{\ell_1}+m_{\ell_4}+} \epsilon_{n_{\ell_2}-n_{\ell_3}}$  and the horizontal one is  $\hat{P}_{\text{hor}} = \frac{1}{2j+1} \delta_{n_{\ell_1}-}^{m_{\ell_1}+m_{\ell_4}+} \delta_{n_{\ell_2}-n_{\ell_3}-}^{n_{\ell_3}-n_{\ell_3}-}$ . The contractor is  $A^{\text{Tr}} = \delta_{m_{\ell_1}+1}^{n_{\ell_1}-n_{\ell_2}-n_{\ell_3}-n_{\ell_3}-n_{\ell_4}-} \delta_{m_{\ell_4}+1}^{n_{\ell_4}-n_{\ell_3}-n_{\ell_3}-n_{\ell_4}-}$ . There is only one equivalence class of face relation, so the weight factor is (2j+1). Thus

$$A_{\text{torus}}(j) = (2j+1) \frac{1}{2j+1} \epsilon^{m_{\ell_{1}}+m_{\ell_{4}}+} \epsilon_{n_{\ell_{2}}-n_{\ell_{3}}-} \cdot \frac{1}{2j+1} \delta^{m_{\ell_{2}}+}_{n_{\ell_{1}}-} \delta^{m_{\ell_{1}}-}_{n_{\ell_{1}}-} \delta^{n_{\ell_{2}}-}_{m_{\ell_{1}}+} \delta^{n_{\ell_{2}}-}_{m_{\ell_{1}}+} \delta^{n_{\ell_{2}}-}_{m_{\ell_{1}}+} \delta^{n_{\ell_{2}}-}_{m_{\ell_{3}}+} \delta^{n_{\ell_{3}}-}_{m_{\ell_{4}}+} \delta^{n_{\ell_{4}}-}_{m_{\ell_{4}}+} \delta^{n_{\ell_{4}}-}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}-}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}+}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}+}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}+}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}-}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}-}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}+}_{m_{\ell_{4}+}+} \delta^{n_{\ell_{4}+}+}_{m_{\ell_{4}+}+}_$$

so the regularized amplitude is

$$A_{\rm torus}^{\Lambda} = 2\Lambda + 1 \tag{5.10}$$

### Klein-bottle bubble

A little modification of the link relation of the diagram figure 5.14a leads to the bubble with the topology of Klein bottle - see figure 5.16.

The *BF*-amplitude of Klein-bottle bubble is very similar to the one of the toroidal bubble. The only difference is the switch of the indices in the  $\hat{P}_{\text{vert}}$  operator - in the same way, as we did it in case of the projective plane bubble. Thus the amplitude differs only by a sign and equals

$$A_{\text{Klein bottle}}^{\Lambda} = -\left(2\Lambda + 1\right) \tag{5.11}$$

### 5.2.3.4 Melonic bubble

One of important bubbles considered so far in the literature is the so called *melonic* bubble. It was considered in [69] and studied with details in [78, 79]. A simple example

of a diagram containing such bubble is presented at figure 5.17 - it is a melonic bubble sandwiched between two static diagrams.

On figure 5.17d there is the fragment of diagram representing the edge with a melonic bubble. This is precisely the pattern one has to look for when searching for an edge with a melonic bubble. The outer links of this diagram are in fact the external faces of such edge. Note, that such diagram can be obtained by taking a  $\theta$ -like-graph build of the outer links, cutting it in half and inserting an isolated bubble into them. The isolated melonic bubble (with the link relation specified) is presented at figure 5.17e.

Let us now study the rank of such bubble using the *rank-of-bubble algorithm*. The steps of this algorithm are presented and explained at figure 5.18. As a result we get the rank of melonic bubble equal to 3, which agrees with the intuition: the melonic bubble has topology of three spheres glued by fragments of their surfaces (see figure 5.19).

### 5.2.3.5 Multiple bubbles

**Two disjoint bubbles** Let us now consider a diagram containing two disjoint bubbles. In the example presented at figure 5.20 the bubble subdiagrams becomes disconnected although the starting diagram was connected. Each of obtained diagrams is a spherical bubble, thus one can easily calculate the rank of the bubbles.

Two bubbles connected by a face Now let us consider a diagram with a bubble being two isolated bubbles connected by a contractible face. An example of such diagram is presented at figure 5.17e. The find-bubble algorithm ends up with a connected subdiagram, unlike in previous example. Applying the rank-of-bubble algorithm (see figure 5.22) one faces the situation mentioned in subsection 5.2.2 in the step 3 of the algorithm - namely erasing one of the links and applying the find-bubble algorithm results with a disconnected subdiagram (see figure 5.22c). Thus one has to go back and choose another link (see figure 5.22d). If we did not rewind the procedure at figure 5.22c, the result of it would have been 3 (i.e. 1 plus the rank of the bubble figure 5.22c, which is obviously 2). And of course removing three faces of such bubble makes it contractible to a 1-complex. But the rank is the minimal number of faces one has to remove in order to obtain something contractible to a 1-complex, and this minimal number is 2 - what is shown at figures 5.22d-5.22g.



Figure 5.17: The melonic bubble sandwiched between two static diagrams. Due to make figures legible the link relations are omitted, so far it is important that for each pair of nodes related by the node relation all the links incident to one node are in 1-to-1 correspondence with the links of the other node. (a) - The diagram. (b) and (c) - Erasing the simply-outer links of the static boundary subdiagrams. (d) - The subdiagram representing the melonic bubble, including the outer faces of the edge on which the bubble appears. The link relation is not drawn, but it is given by the following conditions: for each related pair of nodes the outer links are in link relation with each other and the link relation on the inner links is indicated by figure 5.17e. (e) - The melonic bubble with the link relations omitted.



Figure 5.18: Rank-of-bubble algorithm applied to the melonic bubble. (a) - Step 1: first link chosen to be removed. The counter r = 1. (b) - Step 2: erasing the link in previous step caused one link to become outer. (c) - The diagram obtained after (b) was neither disconnected nor empty, thus we pass through the steps 3 and 4 and we repeat the step 1: another link chosen to be erased. Now the counter r = 2. (d) - The links that became outer after the previous step. (e) - Again step 1: we choose another link to be erased. Now the counter r = 3. (f) - Step 2: now all remaining links became outer. Erasing them leads us to the empty diagram, thus as we proceed to step 4 with the counter r = 3, we conclude that the rank of the melonic bubble is 3.



Figure 5.19: Construction of a 2-complex corresponding to the melonic bubble. (a) - Two 1-vertex foams obtained from the melonic bubble diagram. (b) - The foams glued along their boundaries.



Figure 5.20: A diagram containing two disjoint bubbles. (a) - The diagram, the link relations are omitted. (b) - The simply-outer links of diagram are highlighted red. (c) - We assume that the link relation was such that now at each node the green links are related. The red links are the outer links. (d) - The isolated bubble subdiagram. It consist of two disjoint diagrams: one being the double-cone, and the second being either sphere (as at figure 5.11) or a projective plane (as at figure 5.13) - depending on the link relation.



Figure 5.21: A diagram containing and isolated bubble consisting of two bubbles connected by a contractible face. (a) - The diagram. (b) - The simply-outer edges of the diagram are highlighted red. (c) - The result of the find-bubble algorithm, i.e. the subdiagram being an isolated bubble.



Figure 5.22: Rank-of-bubble algorithm applied to two bubbles connected by a contractible face. (a) - Step 1: first link chosen to be removed. The counter r = 1. (b) - Step 2: erasing the link in previous step caused one link to become outer. (c) - Step 3: proceeding the find-bubble algorithm leads to a disjoint diagram - thus we have to go back to step 1. (d) - Step 1 again: choosing another link to erase. We cannot choose neither  $\ell_1$  nor  $\ell_2$ , because both of them were chosen in a previous attempt. The counter r still equals 1. (e) - Step 2: again erasing the outer links. The links  $\ell_1$  and  $\ell_2$  are now erased because of being outer. (f) - Step 4: the diagram obtained in previous steps is connected, thus again go to step 1, where we choose one of the links to be erased and increase the counter r to 2. (g) - Step 2: Only one link left and it is an outer link, thus we remove it, obtaining the empty diagram. We proceed to step 4 with the counter r = 2, being the total rank of the bubble.

### 5.2.4 Bubbles in terms of *SU*(2)-labelled diagrams.

Let us briefly analyse some properties of bubbles from the point of view of colored graph diagrams, namely Operator Spin-Network Diagrams.

Our analysis is based on coloring of the diagram by SU(2)-tensors (where the EPRL contractors are considered as SU(2)-tensors, because thanks to the EPRL-map they have SU(2)-index structure - see Appendix A.4.3). We do not know, whether our results can be generalized to colorings by tensors of other groups, however we state the necessary condition to do so.

First let us observe a property of isolated bubbles:

Claim 5.1. SU(2)-colorings of an isolated bubbles

An isolated bubble admits infinitely many colorings by SU(2) labels giving non-zero amplitude.

### Proof

Let us pick an integer spin j and assume that each link of a bubble is colored by this spin. Then for each node the node Hilbert space will be of the form  $\mathcal{H}_n^{\text{Inv}} = \text{Inv}\left(\bigotimes_{\ell_i^{\epsilon} \in \mathcal{L}_n} \mathcal{H}_j^{\epsilon_i}\right)$ . Such Hilbert space is never 0-dimensional. Note, that

$$\operatorname{Inv}\left(\mathcal{H}_{1}\otimes\cdots\otimes\mathcal{H}_{k}\right)\supset\operatorname{Inv}\left(\mathcal{H}_{1}\otimes\mathcal{H}_{2}\right)\otimes\operatorname{Inv}\left(\mathcal{H}_{3}\otimes\cdots\otimes\mathcal{H}_{k}\right)$$
(5.12)

For k even, we can find the subspace of the node Hilbert space

$$\mathcal{H}_{n}^{\mathrm{Inv}} \supset \mathrm{Inv}\left(\mathcal{H}_{1} \otimes \mathcal{H}_{2}\right) \otimes \cdots \otimes \mathrm{Inv}\left(\mathcal{H}_{k-1} \otimes \mathcal{H}_{k}\right)$$
(5.13)

If k is odd, the last term in such division has three elements:

$$\mathcal{H}_{n}^{\mathrm{Inv}} \supset \mathrm{Inv}\left(\mathcal{H}_{1} \otimes \mathcal{H}_{2}\right) \otimes \cdots \otimes \mathrm{Inv}\left(\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-1} \otimes \mathcal{H}_{k}\right)$$
(5.14)

Since all the spins are equal, for each k, l we have either  $\mathcal{H}_k = \mathcal{H}_l$  or  $\mathcal{H}_k = \mathcal{H}_l^*$ . In case of 2-spin terms there is always a nontrivial invariant tensor: either  $\epsilon^{mn}$  or  $\delta_n^m$ . For 3-spin space there is a non-trivial invariant tensor of  $\operatorname{Inv}\left(\mathcal{H}_j \otimes \mathcal{H}_j \otimes \mathcal{H}_j^*\right)$  iff j is an integer (not half-integer), and it is a Clebsh-Gordan coefficient  $C_l^{mn}$ . Thus we have proven that there is always a nontrivial invariant tensor  $|\iota_n\rangle \in \mathcal{H}_n^{\operatorname{Inv}}$  for all links incident to n having the same integer spin j.

Consider now a following coloring of an isolated bubble: let all links be colored by the same integer spin j and each pair of nodes be colored by an operator  $\hat{P}_n := |\iota_n\rangle \langle \iota_n|$ . The transition amplitude of such diagram is (see section §3.3)

$$A = \prod_{\text{faces}} A_f \cdot \bigotimes_{\Gamma} A_{\Gamma}^{\text{Tr}} \lrcorner_{\mathcal{D}} \bigotimes_{n} \hat{P}_n$$
(5.15)

It factorises into a product over connected components of the graph:

$$A = \prod_{\text{faces}} A_f \cdot \prod_{\Gamma} A_{\Gamma}^{\text{Tr}} \lrcorner \bigotimes_{n \in \mathbf{N}_{\Gamma}} |\iota_n\rangle = \prod_{\Gamma} A_{\Gamma}$$
(5.16)

Inserting the form of BF-contractors we get

$$A_{\Gamma} = N \cdot \prod_{\ell \in \mathbf{L}_{\Gamma}} \delta_{m_{\ell^+}}^{n_{\ell^-}} \cdot \prod_n [\iota_n]_{n_{\ell'^-}}^{m_{\ell^+}}$$
(5.17)

where N is a number coming from normalisation of the projection operators  $\hat{P}_n$ .

We do not want to calculate the particular value of the amplitude. We only want to show it is non-vanishing. So far we showed it is a product of the terms  $A_{\Gamma}$ , now we will show, that each  $A_{\Gamma}$  is non-vanishing. Let us insert the particular form of the  $\iota_n$ contractors. We obtain a product of  $\delta_B^A$ ,  $\epsilon^{CD}$ ,  $\epsilon_{EF}$  and Clebsh-Gordan coefficients  $C_I^{GH}$ , with all indices contracted, so that:

$$A_{\Gamma} = N \cdot \delta_B^A \cdots \epsilon^{CD} \cdots \epsilon_{EF} \cdots C_I^{GH}$$
(5.18)

There are as many Clebsh-Gordan terms in the product, as there are odd-valent nodes in  $\Gamma$ . Thanks to the Handshake Lemma it must be an even number. The delta- end epsilonterms cause that the indices of the Clebsh-Gordan coefficients get contracted in some way. Contractions of Clebsh-Gordan coefficients leads to more delta- and epsilon- terms (normalised by factors dependent on j), thus in the end we obtain  $A_{\Gamma}$  equal to trace of unity operator multiplied by some combinatoric factor. Thus each  $A_{\Gamma}$  is non-zero so all the amplitude is also non-zero.

Quod erat demonstrandum

This leads us to another claim:

Claim 5.2. A diagram containing a bubble has infinitely many SU(2) - BF-colorings giving non-zero contribution to the amplitude for a boundary state being a constant function.

Proof

A constant function is represented by a coloring in which each boundary link has a spin  $j_{\ell} = 0$ .

Consider a following coloring of a graph diagram: let each outer link be colored by a spin  $j_{\ell_{\text{outer}}} = 0$  and each inner link be colored by a spin  $j_{\ell_{\text{innerr}}} = j \in \mathbb{Z}_+$ . The transition amplitude of a diagram colored this way is non-zero, as it was proven above. Moreover, a diagram colored this way contributes to the transition amplitude of a boundary state given by a constant function, because all the boundary links are outer links so they are colored by  $j_{\ell} = 0$ . Since there are infinitely many ways to choose the spin j coloring the inner links, the claim is proven.

One can easily interpret the notion of rank of a bubble in the above terms.

**Claim 5.3.** The rank of a bubble (of or a diagram) is the minimal number of spins that must be set *by hand* to zero in order to spoil the above properties of diagrams containing a bubble.

**Remark 5.1.** In order to generalize the results of this subsection for groups G other then SU(2), one has to ensure one property of the representation theory on G, that we used above. Let us introduce the intertwiners of pairs of representations of G:

$$[\delta^{\rho}]_{B}^{A} \in \operatorname{Inv}\left(\mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho}^{\star}\right) \quad [\epsilon^{\rho}]^{AB} \in \operatorname{Inv}\left(\mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho}\right) \quad [\epsilon^{\star\rho}]_{AB} \in \operatorname{Inv}\left(\mathcal{H}_{\rho}^{\star} \otimes \mathcal{H}_{\rho}^{\star}\right) \quad (5.19)$$

then the group G must have infinitely many inequivalent irreducible representations  $\rho$  such that there is at least one nontrivial tensor  $[T^{\rho}]_{C}^{AB} \in \text{Inv} (\mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho} \otimes \mathcal{H}_{\rho}^{\star})$  such that each contraction of a product of  $[T^{\rho}]$ s with  $[\delta^{\rho}]$ ,  $[\epsilon^{\rho}]$  and  $[\epsilon^{\star\rho}]$  does not vanish. This property of G ensures, that an isolated bubble admits infinitely many inequivalent colorings, that gives non-vanishing amplitude, which is a key of further considerations. In case of SU(2) the  $[T^{\rho}]$ -tensors are the Clebsh-Gordan coefficients, which always exists for three equal integer spins and which have well known properties. Perhaps there is a weaker condition, that allows to generalize our result for even wider class of groups.

### Chapter 6

## Lorentzian Polyhedra Propagator

When doing a direct spin-foam calculation one meets the terms of the form

$$\int_{SL(2,\mathbb{C})^N} \mathrm{d}g_n \prod_{\ell} \langle u_\ell | Y^{\dagger} g_n^{-1} g_{n'} Y | u_{\ell'} \rangle_{j_{\ell}}$$
(6.1)

(see section §4.3). The simplest case of such terms appear when interaction vertices of structure given by  $\theta$ -graphs are considered, like in Dipole Cosmology model (see section §4.5) - then the group elements  $g_n$  and  $g_{n'}$  form always the same pair, thus one of them can be shifted thanks to the invariance of the Haar measure, so that one gets

$$\int_{SL(2,\mathbb{C})^2} \mathrm{d}g \mathrm{d}g' \prod_{\ell} \langle u_{\ell} | Y^{\dagger} g Y | u_{\ell'} \rangle_{j_{\ell}}$$
(6.2)

where the integral over dg' can be dropped thanks to the regularisation (see [84, 50]), and one ends up with

$$\int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{\ell} \langle u_{\ell} | Y^{\dagger} g Y | u_{\ell'} \rangle_{j_{\ell}}$$
(6.3)

The result is a matrix element of some operator, that we decided to investigate.

Such operator appear whenever a spin-foam has an edge split by a vertex colored by a Lorentzian EPRL contractor. Such edge can be interpreted as a self-interacting (because of the vertex) propagator of the objects at the ends of the edge. Since this objects are spin-network nodes, that has interpretation of quantum polyhedra (see subsection 1.1.2.4), we named the operator *Lorentzian Polyhedra Propagator* (LPP).

The same operator appears also in calculation of [78], where the radiative correction of a simple bubble to a spin-foam edge is calculated. There the result is proportional to  $\mathbb{T}^2$ .

The analysis of  $\mathbb{T}$  requires estimation of integrals of matrix elements of unitary representations of  $SL(2,\mathbb{C})$ -group. A brief introduction to harmonic analysis on  $SL(2,\mathbb{C})$ based on [86] can be found in Appendix A.4. The estimation of the integrals was done by applying the saddle point approximation method, generalised to integrands that do not have the explicit form required in the classical version of the theorem.

This chapter is organised as follows. First, in section §6.1, we provide the formal definition of LPP and we study some basic properties of it. Then, in section §6.2, we present the key tool used in calculations, namely the generalised saddle point approximation method of estimating integrals. The section §6.3 shows how to apply each step of section §6.2 to the integrand that we have in LPP. Finally, in section §6.4, we discuss the

implications of obtained form of LPP on Dipole Cosmology model and on renormalization of transition amplitude of an edge in spin-foam theories.

In order to keep the main text of the chapter clear and fluent, some strictly technical details were skipped in sections 6.2 and 6.3 and moved to section §6.5. These are mostly details concerning intermediate steps of derivations, which, though not mathematically trivial, are not of key importance conceptually.

This chapter is Author's own work. The results presented here are also described in a paper of the Author [83].

### 6.1 Definition of Lorentzian Polyhedra Propagator

Let us start with the formal definition of the Lorentzian Polyhedra Propagator. All the conventions about SU(2)- and  $SL(2, \mathbb{C})$ -tensors can be found in Appendix A.3 and Appendix A.4 respectively.

For simplicity we do all the calculations for  $\mathbb{T}$  acting on a tensor space  $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ , however, there is a straightforward generalisation to  $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N} \otimes \mathcal{H}^*_{j_{N+1}} \otimes \cdots \otimes \mathcal{H}^*_{j_{N+M}}$ .

### 6.1.1 Definition

Given a node Hilbert space (see Appendix A.3.3.2)  $\mathcal{H}_n = \mathcal{H}_{\vec{j}} = \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$  for *N*-tuple of spins  $j_j, \ldots, j_N$  let us define an operator  $\mathbb{T} : \mathcal{H}_{\vec{j}} \to \mathcal{H}_{\vec{j}}$  by the formula

$$\mathbb{T} := \int_{SL(2,\mathbb{C})} \mathrm{d}g \ Y^{\dagger} g Y \tag{6.4}$$

where Y is the Lorentzian EPRL map (see Appendix A.4.3) in the  $\vec{j}$ -representation. In the spin-z basis  $|\vec{m}\rangle_{\vec{j}} = |m_1, \ldots, m_N\rangle_{\vec{j}}$  the matrix elements of  $\mathbb{T}$  are

$$\mathbb{T}_{m'_{1}\cdots m'_{N}}^{m_{1}\cdots m_{N}} := \langle \vec{m} | \,\mathbb{T} \, | \vec{m} \rangle_{\vec{j}} = \int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{i=1}^{N} D^{(\gamma j_{i},j_{i})}(g)_{j_{i}m'_{i}}^{j_{i}m_{i}} \tag{6.5}$$

This operator is in fact build of the matrix-elements of the regularized Lorentzian EPRL contractor of a  $\theta$ -graph. Such graph can be interpreted as a part of an Operator Spin-network Diagram and it leads to a spin-foam with an edge split by a vertex with the Lorentzian EPRL amplitude (see figure 6.1)

Let us now study the basic properties of the  $\mathbb{T}$  operator.

### 6.1.2 Domain and range

Recall that each element  $g \in SL(2, \mathbb{C})$  can be decomposed into  $g = k \cdot u$ , where  $u \in SU(2)$ and  $k \in H^3$  (see Appendix A.4.1). Using the fact that SU(2)-elements commute with Y-map, the  $\mathbb{T}$  can be rewritten as

$$\mathbb{T} = \int_{H^3 \times SU(2)} \mathrm{d}k \mathrm{d}u \ Y^{\dagger} k \cdot u \ Y = \int_{H^3} \mathrm{d}k \ Y^{\dagger} k \ Y \ \int_{SU(2)} \mathrm{d}u \ u \tag{6.6}$$

The same derivation works for inversed decomposition  $g = \tilde{u} \cdot \tilde{k}$ :

$$\mathbb{T} = \int_{SU(2)} \mathrm{d}u \, u \, \int_{H^3} \mathrm{d}k \, Y^{\dagger} k \, Y \tag{6.7}$$



Figure 6.1: The  $\mathbb{T}$ -operator in context of spin-foam models. A fragment of a spin-foam representing  $\mathbb{T}$ : the edge with a vertex, the structure of the vertex is precisely  $\theta$ -graph labelled by a Lorentzian EPRL contractor.

Since in any unitary representation  $\int_{SU(2)} du \, u = \widehat{P^{\text{Inv}}}$ , we have  $\mathbb{T} = \hat{A} \cdot \widehat{P^{\text{Inv}}} = \widehat{P^{\text{Inv}}} \cdot \hat{A}$ , thus  $\mathbb{T} = \hat{A} \cdot \widehat{P^{\text{Inv}}} \hat{A}$  (6.8)

for  $\hat{A} = \int_{H^3} dk \ Y^{\dagger} k Y$ . This implies that  $\mathbb{T}|_{\left(\mathcal{H}_{\vec{j}}^{\text{Inv}}\right)^{\perp}} = 0$  and it is enough to study  $\mathbb{T} : \mathcal{H}_{\vec{j}}^{\text{Inv}} \to \mathcal{H}_{\vec{j}}^{\text{Inv}}$ .

Given two invariant tensors  $\iota, \iota' \in \mathcal{H}_{\vec{i}}^{\mathrm{Inv}}$  we have

$$\mathbb{T}_{\iota'}^{\iota} := \langle \iota | \mathbb{T} | \iota' \rangle_{\vec{j}} = \int_{SL(2,\mathbb{C})} \mathrm{d}g \, \langle \iota | Y^{\dagger}g \, Y | \iota' \rangle_{\vec{j}}$$
(6.9)

Let us introduce a function  $\Phi_{\iota'}^{\iota}(g) := \langle \iota | Y^{\dagger}g Y | \iota' \rangle_{\vec{i}}$ , so that

$$\mathbb{T}_{\iota'}^{\iota} = \int_{SL(2,\mathbb{C})} \mathrm{d}g \,\Phi_{\iota'}^{\iota}(g) \tag{6.10}$$

In what follows we will study the integrand  $\Phi_{\iota'}^{\iota}(g)$ .

### 6.1.3 Symmetries

The EPRL map distinguishes one of SU(2) subgroups of  $SL(2, \mathbb{C})$ , which commutes with the Y operator (i.e.  $SL(2, \mathbb{C}) \subset SU(2) \ni u \to uY |x\rangle_j = Y u |x\rangle_j$  - see Appendix A.4.3). Such choice divides the invariant fields  $X \in \mathfrak{sl}(2, \mathbb{C})$  into generators of this distinguished SU(2)-subgroup, which we shall call rotation generators  $L^i$ , and their orthogonal completion, namely boost generators  $K^i$ . The group elements that can be generated by a linear combination of only rotation generators  $e^{i\vec{\omega}\cdot\vec{L}}$  and only boost generators  $e^{i\vec{\eta}\cdot\vec{K}}$  will be called *pure rotations* and pure boosts respectively. One can check, that the integrand  $\Phi_{\iota'}^{\iota}(g)$  is invariant under pure rotations. Indeed, given  $u \in SU(2)$  we have consider the right action of SU(2) on  $SL(2, \mathbb{C})$ :

$$\Phi_{\iota'}^{\iota}(g \cdot u) = \langle \iota | Y^{\dagger}g \cdot u Y | \iota' \rangle = \langle \iota | Y^{\dagger}g Y u | \iota' \rangle$$
(6.11)

and since  $\iota'$  is an invariant tensor, we have  $u |\iota'\rangle = |\iota'\rangle$ , thus

$$\Phi_{\iota'}^{\iota}(g \cdot u) = \Phi_{\iota'}^{\iota}(g) \tag{6.12}$$

Analogous derivation goes for the left action, so:

$$\Phi_{\iota'}^{\iota}\left(u\cdot g\right) = \Phi_{\iota'}^{\iota}\left(g\right) \tag{6.13}$$

Thus, given a decomposition  $g = k \cdot u$  for  $k \in H^3$  and  $u \in SU(2)$  the integrand depend only on the pure boost:

$$\Phi_{\iota'}^{\iota}(k \cdot u) = \Phi_{\iota'}^{\iota}(k) = \Phi_{\iota'}^{\iota}(\tilde{u} \cdot k)$$
(6.14)

Let us now consider dependence of the integrand on the direction of the boost. Each pure boost can be expressed as  $k(\vec{\eta}) = e^{i\vec{\eta}\cdot\vec{K}}$  for  $\vec{\eta}$  called the boost vector. Each boost vector can be obtained by an appropriate pure rotation (called  $u_{\vec{\eta}}$ ) of the z-versor, such that  $\frac{\vec{\eta}\cdot\vec{K}}{\eta} = u_{\vec{\eta}}^{\dagger}K^3u_{\vec{\eta}}$  (where  $\eta := |\vec{\eta}|$ ). Thus each boost can be expressed as

$$k(\vec{\eta}) = e^{i\vec{\eta}\cdot\vec{K}} = e^{i\eta u_{\vec{\eta}}^{\dagger}K^{3}u_{\vec{\eta}}} = u_{\vec{\eta}}^{\dagger}e^{i\eta K^{3}}u_{\vec{\eta}}$$
(6.15)

Combining (6.14) and (6.15) we get

$$\Phi_{\iota'}^{\iota}\left(k\left(\vec{\eta}\right)\right) = \Phi_{\iota'}^{\iota}\left(e^{i\eta K^{3}}\right) \tag{6.16}$$

so the integrand depend only on the length of the boost vector of the pure-boost-part of its argument. Let us denote this dependence as

$$\Phi_{\iota'}^{\iota}(\eta) := \Phi_{\iota'}^{\iota}\left(e^{i\eta K^3}\right) = \left\langle \iota \right| e^{i\eta K^3} \left| \iota' \right\rangle \tag{6.17}$$

### 6.1.3.1 Integral measure

In order to obtain  $\mathbb{T}_{t'}^{\iota}$  the integrand  $\Phi_{t'}^{\iota}(g)$  is integrated over the  $SL(2, \mathbb{C})$  group with respect to the Haar measure. Since  $SL(2, \mathbb{C})$  is not compact, the measure is not defined uniquely. All the calculation can be done for measure  $dg = c \cdot dg$  for a positive constant c, and appropriate rescaling of the result must be done. We pick the one used in [86] (see also Appendix A.4.2.3), which ensures, that the measure induced on the SU(2)-subgroup in the defining representation is normalized. Using the decomposition  $g = u \cdot k$  the measure dg factorises into dudk. The integral over SU(2) is trivial (see subsection 6.1.3) and gives 1 (thanks to normalisation of Haar measure on a compact group), thus we are left with

$$\mathbb{T}_{\iota'}^{\iota} = \int_{H^3} \mathrm{d}k(\vec{\eta}) \,\Phi_{\iota'}^{\iota}(\eta) \tag{6.18}$$

According to [86] the measure dk is

$$dk(\vec{\eta}) = \frac{1}{(4\pi)^2} d\phi(\vec{\eta}) \sin \theta(\vec{\eta}) d\theta(\vec{\eta}) (\sinh |\vec{\eta}|)^2 d|\vec{\eta}|$$
(6.19)

where  $\phi(\vec{\eta})$  and  $\theta(\vec{\eta})$  are the spherical angles of the direction of the boost vector.

Let us now introduce a measure function that will simplify the expression. Let

$$\mu\left(\vec{x}\right) := \left(\frac{\sinh\left|\vec{x}\right|}{4\pi\left|\vec{x}\right|}\right)^2 \tag{6.20}$$

Obviously  $dk(\vec{\eta}) = \mu(\vec{\eta}) d^3\vec{\eta}$ . Plugging it into (6.18) and (6.19) we get

$$\mathbb{T}_{\iota'}^{\iota} = \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{\eta} \, \mu\left(|\vec{\eta}|\right) \Phi_{\iota'}^{\iota}\left(|\vec{\eta}|\right) \tag{6.21}$$

and this is the integral we will study in what follows.

### 6.1.4 Result

Before doing the calculation let us state the result.

We do not have the strict form of the  $\mathbb{T}$  operator. The calculation is done in the first order in  $\frac{1}{J}$ -expansion, where J is the maximum of the spins  $j_1, \ldots, j_N$ . The result is

$$\mathbb{T} = \left(\frac{1}{4\pi}\right)^2 \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \left[\frac{3}{(1+\gamma^2)\sum_{i=1}^N x_i}\right]^{\frac{3}{2}} \mathbb{1}_{\vec{j}} + O\left(J^{-\frac{5}{2}}\right)$$
(6.22)

where  $\gamma$  is the Barbero-Imirzi parameter and  $x_i := \frac{j_i}{J}$  are numbers, that belong to the interval [0, 1], but cannot be considers small (at least one of them equals 1, and the sum must be greater or equal 2).

# 6.2 Scheme of calculation - Generalised Saddle Point Approximation method

The operator  $\mathbb{T}$  has a form of an integral over the  $SL(2,\mathbb{C})$  group. We will not do this integral explicitly. We will find the leading order of the  $\mathbb{T}$  operator using the saddle point approximation (SPA) method [87].

The SPA theorem gives a tool to estimate integrals of the form

$$I(\Lambda) = \int \mathrm{d}^N x \, g(x) e^{-\Lambda f(x)} \tag{6.23}$$

for large values of  $\Lambda$  as a power series in  $\frac{1}{\Lambda}$ . The leading order is estimated by a Gaussianlike integral around a critical point  $x_0$  of the exponent function f(x):

$$I(\Lambda) = \int \mathrm{d}^{N} x \, g(x) e^{-\Lambda f(x)} = \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left(\left|\frac{\partial^{2} f}{\partial x^{2}}\right|_{x_{0}}\right)^{-\frac{1}{2}} g(x_{0}) e^{-\Lambda f(x_{0})} \left(1 + O\left(\Lambda^{-1}\right)\right)$$

$$\tag{6.24}$$

where  $\left|\frac{\partial^2 f}{\partial x^2}\right|$  is the determinant of the Hessian matrix of the function f(x). If the function f has more than one critical point  $\{x_1, \ldots, x_k\}$ , than the argument  $x_0$  that appear in the formula (6.24) is the maximal critical point, i.e. such a point  $x_0 \in \{x_1, \ldots, x_k\}$  that  $\Re(-f(x_0))$  is maximal (if there is more then one maximal critical point, than we sum over them).

Several assumptions must be satisfied for the formula (6.24) to be valid. First of all the function f(x) must be smooth and twice differentiable at the point  $x_0$ . Moreover

the integrand  $g(x)e^{-\Lambda f(x)}$  must vanish outside a compact region  $\Omega \subset \mathbb{R}^N$ . (or decay sufficiently fast with  $|\vec{x}| \to \infty$  – see subsection 6.2.2).

The integrand  $\Phi_{\iota'}(g)$  for the Lorentzian EPRL amplitude does *not* satisfy assumptions of the SPA theorem. It neither have form (6.23) nor is defined on a compact region. However, we prove appropriate generalisation of the SPA theorem. The subsection 6.2.1 generalises the theorem for the functions that do not have the form (6.23), but satisfy some other necessary conditions. The subsection 6.2.2 shows, how to generalize the theorem for non-compact domain of integration.

Three further subsections discuss two more technical issues of the integration. The subsection 6.2.3 shows, how does the calculation simplifies when the integrand is spherically symmetric. The subsection 6.2.4 explain, how does the result change when an integrand is multiplied by a measure function. Lastly the subsection 6.2.5 explains, how to avoid looking for all critical point of a considered function.

The section §6.3 will be devoted to detailed study of our integrand  $\Phi^{\iota}_{\iota'}(g)$  showing step by step, how to apply the generalised SPA method to it.

### 6.2.1 The Theorem

When considering the Euclidean EPRL model, the transition amplitudes factorises into matrix elements of SU(2)-representations (see subsection 1.2.3.2), which can be expressed in terms of combinations of spin  $\frac{1}{2}$  matrix elements (see Appendix A.86):

$$\langle u | h | u' \rangle_j = \langle \uparrow | u^{-1} h \, u' | \uparrow \rangle^{2j} \tag{6.25}$$

thus for large spins  $j_i$  the largeness parameter J naturally appear in the exponent and the integrals have the form (6.23) and one can apply the SPA theorem directly. For the Lorentzian model this is not the case. We have to then extend the SPA theorem to the integrands with more general form.

Consider a smooth function  $\Phi : \mathbb{R}^+ \times \Omega \to \mathbb{C}$ . Let us define

$$\chi(\Lambda, x) := \ln\left(\Phi\left(\Lambda, x\right)\right) \tag{6.26}$$

and let us assume, that  $\chi$  has a simple pole of degree 1 in the first argument at  $\Lambda \to \infty$ . The saddle point approximation theorem can be generalized so that it can be applied to  $\Phi$  being the integrand:

**Theorem 6.1.** Given a smooth function  $\chi : \mathbb{R}^+ \times \Omega \ni (\Lambda, x) \mapsto \chi(\Lambda, x) \in \mathbb{C}$ , (where the compact set  $\Omega \subset \mathbb{R}^n$  contains a neighbourhood of 0) with a simple pole of degree 1 at  $\Lambda \to \infty$  and assuming, that

$$\forall_{x \neq 0, \Lambda \in \mathbb{R}^+} \qquad \Re\left(\chi\left(\Lambda, x\right)\right) < 0 \tag{6.27}$$

$$\forall_{\Lambda \in \mathbb{R}^+} \lim_{x \to \partial \Omega} \qquad \Re \left( \chi \left( \Lambda, x \right) \right) = -\infty \tag{6.28}$$

$$\forall_{\Lambda \in \mathbb{R}^+} \qquad \chi\left(\Lambda, 0\right) = 0 \tag{6.29}$$

$$\forall_{i=1,\dots,N} \lim_{\Lambda \to \infty} \left. \frac{1}{\Lambda} \left. \frac{\partial \chi \left( \Lambda, x \right)}{\partial x^{i}} \right|_{x=0} = 0 \tag{6.30}$$

$$\forall_{i,j=1,\dots,N} \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \Re \left( \frac{\partial^2 \chi \left(\Lambda, x\right)}{\partial x^i \partial x^j} \Big|_{x=0} \right) = -C_i \delta_{ij} < 0$$
(6.31)

one has

$$\int_{\Omega} e^{\chi(\Lambda, x)} \mathrm{d}x = \frac{1}{\sqrt{\Lambda^n}} I(\Lambda)$$
(6.32)

for a smooth function  $I\left(\frac{1}{\Lambda}\right): [0,\infty[\to\mathbb{C}, \text{ such that } I\left(0\right) = \sqrt{(2\pi)^n} \frac{1}{\sqrt{-\det(\partial_x^2\chi_{-1}(0))}}.$ 

The proof of the theorem can be found in Appendix B.2.

Having the above theorem let us define the exponent part of  $\Phi$  as

$$\phi(x) := \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \log \left( \Phi(\Lambda, x) \right) \tag{6.33}$$

Let  $A := \{x \in \Omega : \nabla \phi(x) = 0\}$ . Let the maximal critical point of  $\phi(x)$ , i.e. the point  $x_{\max} \in A$  such that  $\Re(\phi(x))$  is maximal among all the points  $x \in A$ . The integral of the function  $\Phi(\Lambda, x)$  over  $\Omega$  can be then estimated by

$$\int_{\Omega} \mathrm{d}^{N} x \, \Phi(\Lambda, x) = \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left(\left.\frac{\partial^{2} \phi}{\partial x^{2}}\right|_{x_{\mathrm{max}}}\right)^{-\frac{1}{2}} \Phi(\Lambda, x_{\mathrm{max}}) \left(1 + O\left(\Lambda^{-1}\right)\right) \tag{6.34}$$

### 6.2.2 Noncompact integration range

The SPA theorem requires the integration range  $\Omega$  to be compact and the the integrand to vanish at  $\partial \Omega$ . Nevertheless, under certain assumptions, one can generalise this theorem to noncompact integration ranges.

Let us consider a noncompact  $\Omega$  and assume that exists  $\Lambda_0$  such that for all  $\Lambda > \Lambda_0$ the following is true: for each  $\epsilon > 0$  exists a compact region  $R_{\epsilon} \subset \Omega$ , such that

$$\int_{\Omega \setminus R_{\epsilon}} \mathrm{d}^{N} x \, \left| \Phi(\Lambda, x) \right| < \epsilon \tag{6.35}$$

and that  $R_{\epsilon} \subset R_{\epsilon'}$  for  $\epsilon > \epsilon'$ . Then let us introduce for each  $\epsilon$  another compact region  $\tilde{R}_{\epsilon}$ , such that  $R_{\epsilon} \subsetneq \tilde{R}_{\epsilon} \subset \Omega$  and a smooth function  $\chi_{\epsilon}(x)$  such that

$$\chi_{\epsilon}(x) = \begin{cases} 1 & x \in R_{\epsilon} \\ 0 \le \chi_{\epsilon}(x) \le 1 & x \in \tilde{R}_{\epsilon} \setminus R_{\epsilon} \\ 0 & x \in \Omega \setminus \tilde{R}_{\epsilon} \end{cases}$$
(6.36)

Obviously  $\int_{\Omega \setminus R_{\epsilon}} \mathrm{d}^N x \, |\chi_{\epsilon}(x) \Phi(\Lambda, x)| < \epsilon.$ 

Let  $I(\Lambda) := \int_{\Omega} d^N x \Phi(\Lambda, x)$  and  $I_{\epsilon}(\Lambda) := \int_{\Omega} d^N x \chi_{\epsilon}(x) \Phi(\Lambda, x)$ . Obviously for all  $\Lambda > \Lambda_0$  we have  $|I(\Lambda) - I_{\epsilon}(\Lambda)| < 2\epsilon$ , thus function  $I_{\epsilon}(\Lambda)$  converges to  $I(\Lambda)$  uniformly with respect to  $\Lambda$ . But each integral  $I_{\epsilon}(\Lambda)$  is in fact an integral over a compact region  $\tilde{R}_{\epsilon}$ , so it can be calculated using the SPA method:

$$I_{\epsilon}(\Lambda) = \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left( \left|\frac{\partial^2 \phi}{\partial x^2}\right|_{x_{\epsilon}} \right)^{-\frac{1}{2}} \Phi(\Lambda, x_{\epsilon}) \left(1 + O\left(\Lambda^{-1}\right)\right)$$
(6.37)

where  $x_{\epsilon}$  is the maximal critical point of  $\phi$  in the region  $R_{\epsilon}$ . For  $\epsilon$  sufficiently small, for example for  $\epsilon < \frac{1}{\Lambda} \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left(\left|\frac{\partial^2 \phi}{\partial x^2}\right|_{x_0}\right)^{-\frac{1}{2}} \Phi(\Lambda, x_0)$ , the region  $R_{\epsilon}$  must contain the maximal critical point of  $\phi$ , so  $x_{\epsilon} = x_{\max}$ , thus the leading term does not depend on  $\epsilon$ . So the leading term of the function  $I(\Lambda)$  is the limit at  $\epsilon \to 0$  of the leading terms of  $I_{\epsilon}(\Lambda)$ . Quod erat demonstrandum.

### 6.2.3 Case of spherically symmetric integrals

Consider now a spherically symmetric integrand  $\Phi(\Lambda, \vec{x}) = \Phi(\Lambda, r)$ , for  $r = |\vec{x}|$ , with the maximal critical point at r = 0 (being in the interior of the region  $\Omega$ ). Then obviously

 $\phi(\vec{x})$  is also a spherically symmetric function. This simplifies a lot calculation of the Hessian matrix of the exponent part of the integrand.

The Hessian matrix of a spherically symmetric function  $\phi$  is

$$\frac{\partial^2 \phi}{\partial x^i \partial x^j} = \frac{\partial^2 r}{\partial x^i \partial x^j} \frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} \frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2}$$
(6.38)

If r = 0 is the critical point, the differential  $\frac{d\phi}{dr}\Big|_{r=0} = 0$ , thus the first term vanish.

Further simplification comes from the fact, that

$$\frac{1}{2}\frac{\partial^2 r^2}{\partial x^i \partial x^j} = \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} + r\frac{\partial^2 r}{\partial x^i \partial x^j} = \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} \quad \text{at } r = 0$$
(6.39)

and since  $r^2 = \sum (x^i)^2$ , we have

$$\frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} = \delta_{ij} \quad \text{at } r = 0$$
(6.40)

Thus the Hessian matrix is

$$\frac{\partial^2 \phi}{\partial x^i \partial x^j} \bigg|_{\vec{x}=0} = \left. \frac{\mathrm{d}^2 \phi}{\mathrm{d} r^2} \right|_{r=0} \delta_{ij} \tag{6.41}$$

And the Hessian determinant:

$$\left| \frac{\partial^2 \phi}{\partial x^i \partial x^j} \right|_{\vec{x}=0} = \left| \frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2} \right|_{r=0}^N \tag{6.42}$$

### 6.2.4 Multiplying of the integrand by a $\Lambda$ -independent function

Consider now an integral of the form

$$\tilde{I}(\Lambda) := \int_{\Omega} \mathrm{d}^{N} x \, \mu(\vec{x}) \Phi(\Lambda, \vec{x}) \tag{6.43}$$

where  $\mu(\vec{x})$  is a nonvanishing function and  $\Phi(\Lambda, \vec{x})$  satisfies the assumption of generalised SPA. When using the SPA method, the integral  $\tilde{I}(\Lambda)$  can be easily related to  $I(\Lambda) := \int_{\Omega} d^N x \Phi(\Lambda, x)$ . Indeed, note, that presence of  $\mu(\vec{x})$  does not effect the exponent part of the integrand:

$$\tilde{\phi}(\vec{x}) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln \mu(\vec{x}) + \frac{1}{\Lambda} \ln \Phi(\Lambda, \vec{x}) = 0 + \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln \Phi(\Lambda, \vec{x}) = \phi(\vec{x})$$
(6.44)

thus the maximal critical points of the exponent of the integrand does not know about presence of  $\mu(\vec{x})$ . So using the SPA we get

$$\tilde{I}(\Lambda) = \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left(\left.\frac{\partial^2 \phi^n}{\partial x^2}\right|_{x_{\max}}\right)^{-\frac{1}{2}} \mu(x_{\max}) \Phi(\Lambda, x_{\max}) \left(1 + O\left(\Lambda^{-1}\right)\right) = \mu(x_{\max}) I(\Lambda)$$
(6.45)

### 6.2.5 The role or the maximal critical point

Consider now an integrand  $\Phi(\Lambda, x)$  such that

$$|\Phi(\Lambda, x)| \le e^{-\Lambda\alpha(x)} |\Phi(\Lambda, x_0)| \tag{6.46}$$

where  $x_0$  is a critical point of the exponent part  $\phi(x)$  and  $\forall_{x \neq x_0} \alpha(x) > 0$ . Then  $x_0$  is the maximal critical point of  $\phi(x)$ . Indeed, let us estimate the real part of  $\phi(x)$  for any other point:

$$\Re (\phi (x)) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln |\Phi (\Lambda, x)|$$

$$< \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln \left( e^{-\Lambda \alpha(x)} |\Phi (\Lambda, x_0)| \right)$$

$$= -\alpha (x) + \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln |\Phi (\Lambda, x_0)| = -\alpha (x) + \Re (\phi (x_0))$$
(6.47)

and since (by assumption)  $\alpha(x) > 0$ , obviously  $\Re(\phi(x)) < \Re(\phi(x_0))$ .

This leads to a convenient conclusion: if one finds a point, such that (6.46) is satisfied, the derivation (6.47) shows, that it provides maximum real value of the exponent part of the integrand  $\phi(x)$ . It is thus enough to check, if it is a critical point of the exponent part of the integrand. If it is, there is no need to look for any other critical point. If it is not, the theorem cannot be applied (the integrand does not behave Gaussian-like around the maximum point).

One can easily estimate the contribution from Gaussian-like integral  $I_1(\Lambda)$  around any other critical point  $x_1$  of  $\phi(x)$ 

$$I_{1}(\Lambda) = \left(\frac{2\pi}{\Lambda}\right)^{\frac{N}{2}} \left( \left| \frac{\partial^{2} \phi^{n_{1}}}{\partial x^{2}} \right|_{x_{1}} \right)^{-\frac{1}{2}} \Phi(x_{1},\Lambda) \left( 1 + O\left(\Lambda^{-1}\right) \right)$$
$$= C(x_{1}) \Lambda^{-\frac{N}{2}} \Phi(x_{1},\Lambda) \left( 1 + O\left(\Lambda^{-1}\right) \right)$$
(6.48)

Let  $I_0(\Lambda) = C(x_0) \Lambda^{-\frac{N}{2}} \Phi(x_0, \Lambda) (1 + O(\Lambda^{-1}))$ . Then the modulus of  $I_1(\Lambda)$  is

$$|I_{1}(\Lambda)| = |C(x_{1})| \Lambda^{-\frac{N}{2}} |\Phi(x_{1},\Lambda)| (1 + O(\Lambda^{-1}))$$

$$\leq e^{-\Lambda\alpha(x_{1})} |C(x_{1})| \Lambda^{-\frac{N}{2}} |\Phi(x_{0},\Lambda)| (1 + O(\Lambda^{-1}))$$

$$= e^{-\Lambda\alpha(x_{1})} \frac{|C(x_{1})|}{|C(x_{0})|} |I_{0}(\Lambda)| (1 + O(\Lambda^{-1}))$$

$$= |I_{0}(\Lambda)| \cdot O(e^{-\Lambda\alpha(x_{0})})$$
(6.49)

so  $I_1(\Lambda)$  is exponentially suppressed.

### 6.3 Details of calculations

In previous section we developed the tools needed to integrate the matrix elements  $\Phi_{\iota'}^{\iota}(g)$  over  $SL(2,\mathbb{C})$ . In this section we study properties of this particular integrand, that are needed to apply the above methods.

As we have already mentioned (see subsection 6.1.3.1), the integral (6.10) can be easily expressed as an integral over  $\mathbb{R}^3$ :

$$\mathbb{T}_{\iota'}^{\iota} = \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{\eta} \,\mu\left(\vec{\eta}\right) \Phi_{\iota'}^{\iota}\left(\vec{\eta}\right) \tag{6.50}$$

where  $\iota, \iota' \in \text{inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$ . The integrand is in fact a spherically symmetric function dependent only on  $\eta := |\vec{\eta}|$ :

$$\Phi(\eta) := \mu(\eta) \Phi_{\iota'}^{\iota}(\eta) \tag{6.51}$$

Our largeness parameter is  $J = \max \{j_i\}_{i=1,\dots,N}$ . For later convenience we also introduce the numbers  $x_i := \frac{j_i}{J}$ , which although being  $0 \le x_i \le 1$ , cannot be considered small parameters.

This section is organised as follows. Firstly, in subsection 6.2.4, we identify the parts dependent and independent on the largeness parameter in the integrand. We also fix some notations used later on. Then, in subsection 6.3.2, we express the functions appearing in the integrand in terms of hypergeometric functions. In subsection 6.3.3 we identify the maximal critical point of the exponent part of the integrand. The subsection 6.3.4 shows that despite noncompactness of the integration region we can apply the SPA method. The subsection 6.3.5. Finally, the subsection 6.3.6 we collect all the results of this section into the final form of the leading order of the  $\mathbb{T}$  operator.

### 6.3.1 Decomposition into measure and exponent part

Recalling (6.51) the integrand is  $\tilde{\Phi}(\eta) = \mu(\eta) \Phi_{\mu}^{\iota}(\eta)$  with

$$\mu(\eta) = \left(\frac{\sinh \eta}{4\pi\eta}\right)^2 \quad \Phi^{\iota}_{\iota'}(\eta) = \langle \iota | Y^{\dagger} e^{i\eta K^3} Y | \iota' \rangle \tag{6.52}$$

Obviously the function  $\mu(\eta)$  does not depend on the largeness parameter J. Thus, using subsection 6.2.4, it is enough to find the critical points of  $\Phi_{\iota'}^{\iota}(\eta)$ . From now on we will call the function  $\tilde{\Phi}(\eta)$  the *full integrand*, the function  $\Phi_{\iota'}^{\iota}(\eta)$  the *integrand*, and the function  $\mu(\eta)$  the measure part of the integrand or simply the measure.

In this section we will often go back to the  $|m\rangle_j$  basis, thus let us introduce appropriate notation:

$$\Phi_{\iota'}^{\iota}(\eta) = \sum_{\vec{m}\vec{m'}} \langle \iota | |\vec{m}\rangle_{\vec{j}} \left\langle \vec{m'} \right| \left| \iota' \right\rangle_{\vec{j}} \Phi_{\vec{m'}}^{\vec{m}}(\eta)$$
(6.53)

where

$$\Phi_{\vec{m'}}^{\vec{m}}(\eta) := \prod_{i=1}^{N} \langle m_i | Y^{\dagger} e^{i\eta K^3} Y \left| m'_i \right\rangle_{j_i}$$

$$(6.54)$$

Since  $[K^3, L^3] = 0$ , each term  $\langle m_i | Y^{\dagger} e^{i\eta K^3} Y | m'_i \rangle_{j_i}$  is proportional to  $\delta_{m_i, m'_i}$ , thus we can define the function

$$f_m^{(j)}(\eta) := \langle m | Y^{\dagger} e^{i\eta K^3} Y | m \rangle_j \tag{6.55}$$

such that

$$\Phi_{\vec{m'}}^{\vec{m}}(\eta) = \prod_{i=1}^{N} \delta_{m_i, m'_i} f_{m_i}^{(j_i)}(\eta)$$
(6.56)

Thanks to (6.56) it is now obvious, that in the basis  $|\vec{m}\rangle_{\vec{j}}$  all the nondiagonal matrix elements are precisely zero.

The exponent part of the integrand is

$$\phi_{\iota'}^{\iota}(\eta) = \lim_{J \to \infty} \phi_{\iota'}^{\iota}(\eta, J) := \lim_{J \to \infty} \frac{1}{J} \ln \left[ \sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' e^{\sum_{i=1}^{N} \ln f_{m_i}^{(j_i)}(\eta)} \right]$$
(6.57)

Because of the sum under the logarithm it is quite inconvenient object, thus most calculations we be done for the exponent part of the function  $\Phi_{\vec{m'}}^{\vec{m}}(\eta)$ , which is

$$\phi_{\vec{m}}(\eta) = \lim_{J \to \infty} \phi_{\vec{m}}(\eta, J) := \lim_{J \to \infty} \frac{1}{J} \sum_{i=1}^{N} \ln f_{m_i}^{(j_i)}(\eta)$$
(6.58)

Note, that

$$\phi_{\iota'}^{\iota}(\eta, J) = \frac{1}{J} \ln \left[ \sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota_{\vec{m}}^{\prime} e^{J\phi_{\vec{m}}(\eta, J)} \right]$$
(6.59)

### 6.3.2 Hypergeometric representation

Let us now focus on the function  $f_m^{(j)}(\eta)$ . According to [86] it can be written as

$$f_m^{(j)}(\eta) = (2j+1) {\binom{2j}{j+m}} e^{-(j+m+1)\eta} e^{i\gamma j\eta}$$

$$\cdot \int_0^1 \mathrm{d}t \, t^{j+m} \, (1-t)^{j-m} \left[1 - \left(1 - e^{-2\eta}\right) t\right]^{i\gamma j - (j+1)}$$
(6.60)

(see also Appendix A.4.3). Recall now the integral definition of the Gauss's hypergeometric function [88]:

$${}_{2}F_{1}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \mathrm{d}t \, t^{b-1} \, (1-t)^{c-b-1} \, (1-zt)^{-a} \tag{6.61}$$

Comparison of formulas (6.60) and (6.61) gives a conclusion, that

$$f_m^{(j)}(\eta) = e^{-(j+m+1)\eta} e^{ij\gamma\eta} {}_2F_1\left(j+1-i\gamma j, \, j+1+m; \, 2j+2; \, 1-e^{-2\eta}\right)$$
(6.62)

More detailed study of properties of functions  $f_m^{(j)}(\eta)$  can be found in subsection 6.5.1.

### 6.3.3 Maximal critical point

To apply the SPA method we need to find the maximal critical point of  $\phi_{t'}^{\iota}(\eta)$ . We will find it as follows: first we will identify the only point such that if it were a critical point, it would be relevant (i.e. was there any other critical point, it's contribution would be exponentially suppressed - see subsection 6.2.5), then we will check that the point we found is actually a critical point.

### 6.3.3.1 Identification of the potentially maximal critical point.

The natural candidate for the maximal critical point is  $\eta = 0$ . Since  $f_m^{(j)}(\eta)\Big|_{\eta=0} = 1$  (see (6.119)), it is obvious to find the value of the integrand

$$\Phi_{\vec{m}}^{\vec{m}}(0) = 1 \quad \text{so} \quad \Phi_{\iota'}^{\iota} = \langle \iota | | \iota' \rangle \tag{6.63}$$

Now let us show, that  $\eta = 0$  is the only critical point that counts, i.e. was there any other critical point  $\eta_1$ , The value of  $\Phi_{\vec{m}}^{\vec{m}}(\eta_1)$  would be exponentially suppressed in large J limit (see subsection 6.2.5). Indeed, one can estimate the modulus of  $f_m^{(j)}(\eta)$  by

$$\left| f_m^{(j)}(\eta) \right| \le \left( e^{1 - 2\eta - e^{-2\eta}} \right)^{\frac{(j+1)^2 - m^2}{4(2j+3)}} \tag{6.64}$$

(see subsection 6.5.1.4) thus

$$\left|\Phi_{\vec{m}}^{\vec{m}}(\eta_{1})\right| \leq \left(e^{1-2\eta_{1}-e^{-2\eta_{1}}}\right)^{\sum_{i=1}^{N}\frac{(j_{i}+1)^{2}-m_{i}^{2}}{4(2j_{i}+3)}} = \left[C(\eta_{1})\right]^{\sum_{i=1}^{N}\frac{(j_{i}+1)^{2}-m_{i}^{2}}{4(2j_{i}+3)}} \tag{6.65}$$

Now since

$$\left|\Phi_{\iota'}^{\iota}(\eta_{1})\right| = \left|\sum_{\vec{m}} \Phi_{\vec{m}}^{\vec{m}}(\eta_{1})\overline{\iota_{\vec{m}}}\iota_{\vec{m}}'\right| \le \left|\sum_{\vec{m}} \left|\Phi_{\vec{m}}^{\vec{m}}(\eta_{1})\right|\overline{\iota_{\vec{m}}}\iota_{\vec{m}}'\right|$$
(6.66)

we have

$$|\Phi_{\iota'}^{\iota}(\eta_1)| \le \left| \sum_{\vec{m}} \left[ C(\eta_1) \right]^{\sum_{i=1}^{N} \frac{(j_i+1)^2 - m_i^2}{4(2j_i+3)}} \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' \right|$$
(6.67)

Now using the fact, that the states  $|\iota\rangle$  and  $|\iota'\rangle$  are SU(2)-gauge invariant we can apply the lemma of subsection 6.5.2 to replace  $\sum_{i=1}^{N} m_i^2$  by  $\sum_{i=1}^{N} \frac{j_i(j_i+1)}{3}$  in the above formula and obtain

$$\left|\Phi_{\iota'}^{\iota}(\eta_{1})\right| \leq \left|\left\langle\iota\right|\left|\iota'\right\rangle\right| \left[C(\eta_{1})\right]^{\frac{1}{12}\sum_{i=1}^{N} j_{i}+1} = \left|\left\langle\iota\right|\left|\iota'\right\rangle\right| \left[C(\eta_{1})\right]^{\frac{1}{12}J\sum_{i=1}^{N} x_{i}+\frac{1}{J}}$$
(6.68)

(see (6.175)). Adopting the above formula to the form of the condition (6.46):

$$|\Phi_{\iota'}^{\iota}(\eta_1)| \le |\Phi_{\iota'}^{\iota}(0)| e^{-J\tilde{\alpha}(\eta_1, J)}$$
(6.69)

where

$$\tilde{\alpha}(\eta, J) = -\frac{\sum_{i=1}^{N} x_i + \frac{1}{J}}{12} \ln [C(\eta)]$$
(6.70)

Since  $\forall_{\eta_1>0}C(\eta_1) < 1$  (see subsection 6.5.3.1), the logarithm is always negative and thus  $\tilde{\alpha}(\eta, J)$  is always positive. Although the obtained function  $\tilde{\alpha}$  depends on J, one can find a weaker estimation independent on J, namely  $\alpha(\eta) = -\frac{\sum_{i=1}^{N} x_i}{12} \ln [C(\eta)]$ . Obviously  $\forall_{J>0}\alpha(\eta) < \tilde{\alpha}(\eta, J)$ , and thus

$$\forall_{\eta>0} |\Phi_{\iota'}^{\iota}(\eta)| \le |\Phi_{\iota'}^{\iota}(0)| e^{-J\alpha(\eta)}$$
(6.71)

so  $\eta = 0$  is the point maximising the real part of exponent part of the integrand.

### 6.3.3.2 Smoothness check

Having proven that the maximal critical point of  $\phi_{\iota'}^{\iota}$  is at  $\eta = 0$  (if any), let us check, whether the exponent part of the integrand is smooth at this point, i.e. if  $\frac{\mathrm{d}\phi_{\iota'}^{\iota}}{\mathrm{d}\eta}\Big|_{\eta=0} = 0$ ? Using the property (6.59) we have

$$\frac{\mathrm{d}\phi_{\iota'}}{\mathrm{d}\eta}\Big|_{\eta=0} = \frac{\frac{1}{J}\sum_{\vec{m}}\overline{\iota_{\vec{m}}}\iota'_{\vec{m}}Je^{J\phi_{\vec{m}}(\eta)}\frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta}}{\Phi_{\iota'}^{\iota}(\eta)}\Big|_{\eta=0} = \sum_{\vec{m}}\frac{\overline{\iota_{\vec{m}}}\iota'_{\vec{m}}}{\langle\iota|\,|\iota'\rangle}\frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta}\Big|_{\eta=0}$$
(6.72)

We will show, that  $\frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta}\Big|_{\eta=0} = 0$  for all  $\vec{m}$ .

Let us then calculate:

$$\frac{\partial}{\partial \eta}\phi_{\vec{m}}(\eta,J) = \frac{1}{J}\sum_{i=1}^{N} \frac{\frac{\mathrm{d}f_{m_i}^{(j_i)}}{\mathrm{d}\eta}}{f_{m_i}^{(j_i)}(\eta)}$$
(6.73)
Details of calculations can be found in subsection 6.5.1.2, the result is

$$\frac{\partial}{\partial \eta}\phi_{\vec{m}}(\eta,J) = \frac{1}{J} \left[ \sum_{i=1}^{N} -\delta_i + 2\frac{a_i b_i}{c_i} e^{-2\eta} \frac{{}_2F_1\left(a_i+1, b_i+1; c_i+1; 1-e^{-2\eta}\right)}{{}_2F_1\left(a_i, b_i; c_i; 1-e^{-2\eta}\right)} \right]$$
(6.74)

with  $\delta_i = \delta := j_i + 1 + m_i - i\gamma j_i$ ,  $a_i = 1 + j_i - i\gamma j_i$ ,  $b_i = 1 + j_i + m_i$ ,  $c_i = 2j_i + 2$ . For  $\eta = 0$  all the hypergeometric functions value 1, and we are left with

$$\left. \frac{\partial}{\partial \eta} \phi_{\vec{m}}(\eta, J) \right|_{\eta=0} = -\frac{1}{J} \sum_{i=1}^{N} \delta_i - 2 \frac{a_i b_i}{c_i} \tag{6.75}$$

using simple algebra (similarly to (6.132)) we can simplify it to:

$$\left. \frac{\partial}{\partial \eta} \phi_{\vec{m}}(\eta, J) \right|_{\eta=0} = -i\gamma \frac{\sum_{i=1}^{N} m_i}{J} + O\left(J^{-1}\right) \tag{6.76}$$

Thanks to SU(2) invariance of the states we act on, the sum  $\sum_{i=0}^{N} m_i = 0$ . The second term vanish when the limit  $J \to \infty$  is taken. Thus for all  $\vec{m}$ 

$$\left. \frac{\mathrm{d}}{\mathrm{d}\eta} \phi_{\vec{m}}(\eta) \right|_{\eta=0} = 0 \tag{6.77}$$

 $\mathbf{SO}$ 

$$\left. \frac{\mathrm{d}\phi_{\iota'}^{\iota}}{\mathrm{d}\eta} \right|_{\eta=0} = 0 \tag{6.78}$$

so the exponent part of the integrand is appropriately smooth at  $\eta = 0$  and thus  $\eta = 0$ is the maximal critical point.

#### 6.3.4 Asymptotics

As we have already shown in (6.68), the modulus of the integrand is bounded by

$$|\Phi_{\iota'}^{\iota}(\eta)| \le \left(e^{1-2\eta-e^{-2\eta}}\right)^{\frac{J}{12}\sum_{i=1}^{N}x_i+\frac{1}{J}}$$
(6.79)

Thus for arbitrary small  $\epsilon$  one can find such an  $\eta_\epsilon$  that

$$\forall_{J>1} \quad \left| \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \, \mu(\eta) \Phi_{\iota'}^{\iota}(\eta) \right| < \epsilon \tag{6.80}$$

Indeed, this integral can be estimated by

$$\begin{aligned} \left| \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \, \mu(\eta) \Phi_{\iota'}^{\iota}(\eta) \right| &\leq \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \, \left| \mu(\eta) \Phi_{\iota'}^{\iota}(\eta) \right| \\ &\leq \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \, \left( \frac{\sinh \eta}{4\pi\eta} \right)^2 \left( e^{1-2\eta-e^{-2\eta}} \right)^{\frac{J}{12}\sum_{i=1}^{N} \left( x_i + \frac{1}{J} \right)} \quad (6.81) \\ &\leq \frac{1}{2} \left( \frac{e}{4\pi\eta_{\epsilon}} \right)^2 \left( e - e^{1-e^{-2\eta_{\epsilon}}} - e^{1-2\eta_{\epsilon}-e^{-2\eta_{\epsilon}}} \right) \end{aligned}$$

where the last inequality holds for  $\frac{J}{12}\sum_{i=1}^{N} (x_i + \frac{1}{J}) > 3$  (see subsection 6.5.3.2) - what is for sure true for J > 36, and since we consider the large J limit, this condition holds in the limit. As it is shown in subsection 6.5.3.2, the inequality

$$\frac{1}{2} \left( \frac{e}{4\pi\eta_{\epsilon}} \right)^2 \left( e - e^{1 - e^{-2\eta_{\epsilon}}} - e^{1 - 2\eta_{\epsilon} - e^{-2\eta_{\epsilon}}} \right) < \epsilon \tag{6.82}$$

has always a solution. Thus the assumptions of the lemma of subsection 6.2.2 are satisfied, so our integrand  $\Phi_{t'}^{\iota}$  has proper asymptotic behaviour and the SPA method can be applied.

#### 6.3.5Hessian matrix

Since the maximal critical point is  $\eta = 0$ , we need to know the Hessian determinant  $\left| \frac{\partial^2 \phi_{\iota'}^{\iota}}{\partial \eta^2} \right|$ at this point (see (6.34)). Thanks to the spherical symmetry it equals  $\begin{vmatrix} \frac{\partial^2 \phi_{\iota'}^{\iota}}{\partial \bar{\eta}^2} \end{vmatrix}_{\eta=0}^{2} = \left| \frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2} \right|_{\eta=0}^{3} \text{ (see subsection 6.2.3).}$ To calculate the second derivative of  $\phi_{\iota'}^{\iota}(\eta)$  we use the equation (6.72):

$$\frac{\mathrm{d}^{2}\phi_{\iota'}^{\iota}}{\mathrm{d}\eta^{2}} = \frac{\mathrm{d}}{\mathrm{d}\eta} \frac{\sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota_{\vec{m}}^{\prime} e^{J\phi_{\vec{m}}(\eta)} \frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta}}{\Phi_{\iota'}^{\iota}(\eta)} \\
= \sum_{\vec{m}} \frac{\overline{\iota_{\vec{m}}} \iota_{\vec{m}}^{\prime} e^{J\phi_{\vec{m}}(\eta)}}{\Phi_{\iota'}^{\iota}(\eta)} \frac{\mathrm{d}^{2}\phi_{\vec{m}}}{\mathrm{d}\eta^{2}} - \frac{\mathrm{d}\Phi_{\iota'}^{\iota}}{\mathrm{d}\eta} \sum_{\vec{m}} \frac{\overline{\iota_{\vec{m}}} \iota_{\vec{m}}^{\prime} e^{J\phi_{\vec{m}}(\eta)}}{\left[\Phi_{\iota'}^{\iota}(\eta)\right]^{2}} \frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta} \quad (6.83)$$

The second term in above formula vanishes at  $\eta = 0$ . Indeed, in subsection 6.3.3.2 we have checked, that  $\forall_{\vec{m}} \frac{\mathrm{d}\phi_{\vec{m}}}{\mathrm{d}\eta}\Big|_{\eta=0} = 0$  up to  $O(J^{-1})$  corrections. Thus noting, that  $\Phi_{\iota'}^{\iota}(0) = \langle \iota | | \iota' \rangle$  and  $\phi_{\vec{m}}(0) = 0$ , we have

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = \frac{1}{\langle \iota | |\iota' \rangle} \sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} \left. \frac{\mathrm{d}^2 \phi_{\vec{m}}}{\mathrm{d}\eta^2} \right|_{\eta=0} \tag{6.84}$$

Let us then analyse  $\frac{d^2\phi_{\vec{m}}}{d\eta^2}\Big|_{\eta=0}$ . Obviously it decomposes into a sum

$$\frac{\mathrm{d}^2 \phi_{\vec{m}}}{\mathrm{d}\eta^2} \bigg|_{\eta=0} = \sum_{i=1}^N \frac{1}{J} \left. \frac{\mathrm{d}^2 \ln\left[ f_{m_i}^{(j_i)} \right]}{\mathrm{d}\eta^2} \right|_{\eta=0}$$
(6.85)

Using the calculations of subsection 6.5.1.3 we get (up to  $\frac{1}{J}$  corrections)

$$\frac{\mathrm{d}^2 \ln\left[f_{m_i}^{(j_i)}\right]}{\mathrm{d}\eta^2}\bigg|_{\eta=0} = -\left(1+\gamma^2\right)\frac{(j_i+1)^2 - m_i^2}{(2j_i+3)}\left(1+O\left(J^{-1}\right)\right)$$
(6.86)

so (neglecting the  $O(J^{-1})$  terms)

$$\frac{\mathrm{d}^2 \phi_{\vec{m}}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\frac{1+\gamma^2}{J} \sum_{i=1}^N \frac{(j_i+1)^2 - m_i^2}{(2j_i+3)}$$
(6.87)

Now recalling (6.84) we have

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = \frac{1}{\langle \iota | \, |\iota' \rangle} \sum_{\vec{m}} \left[ \overline{\iota_{\vec{m}}} \iota_{\vec{m}}^{\prime} \left( -\frac{1+\gamma^2}{J} \right) \sum_{i=1}^{N} \frac{(j_i+1)^2 - m_i^2}{(2j_i+3)} \right]$$
(6.88)

Thanks to SU(2) symmetry we may use the lemma of subsection 6.5.2 and substitute each term  $m_i^2$  by  $\frac{j_i(j_i+1)}{3}$  (see (6.176)). Then we get

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\frac{1+\gamma^2}{J} \frac{\sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota'_{\vec{m}}}{\langle \iota | \, |\iota' \rangle} \sum_{i=1}^N \frac{(j_i+1)}{3} \tag{6.89}$$

The *ι*-dependent factor cancels out, because  $\sum_{\vec{m}} \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} = \langle \iota | | \iota' \rangle$ :

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\frac{1+\gamma^2}{3} \left(\sum_{i=1}^N x_i + \frac{1}{J}\right)$$
(6.90)

Neglecting the  $\frac{1}{J}$ -order terms simplify the formula a lot and gives

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2} \Big|_{\eta=0} = -\frac{1+\gamma^2}{3} \sum_{i=1}^N x_i \tag{6.91}$$

So the Hessian determinant is

$$\left|\frac{\partial^2 \phi_{\iota'}^{\iota}}{\partial \vec{\eta}^2}\right|_{\eta=0} = \left[\frac{1+\gamma^2}{3} \sum_{i=1}^N x_i\right]^3 \tag{6.92}$$

#### 6.3.6 Results and overall factors

Let us summarise all the calculations done above. The Lorentzian Polyhedra Propagator  $\mathbb T$  is given by the integral

$$\mathbb{T}_{\iota'}^{\iota} = \int_{\mathbb{R}^3} d^3 \vec{\eta} \, \mu(\eta) \Phi_{\iota'}^{\iota}(\eta)$$
(6.93)

For  $J \gg 1$  it can by approximated by the value of an integrand at  $\eta = 0$  and the Hessian matrix of the exponent part of the integrand at this point:

$$\mathbb{T}_{\iota'}^{\iota} = \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \left|\frac{\partial^2 \phi}{\partial \bar{\eta}^2}\right|_{\eta=0}^{-\frac{1}{2}} \mu(0) \Phi_{\iota'}^{\iota}(0) \left(1 + O\left(J^{-1}\right)\right)$$
(6.94)

Noting that  $\mu(0) = \left(\frac{1}{4\pi}\right)^2$ ,  $\Phi_{\iota'}^{\iota}(0) = \sum_{\vec{m},\vec{m'}} \overline{\iota_{\vec{m}}} \iota_{\vec{m'}} \delta_{\vec{m},\vec{m'}} = \langle \iota | | \iota' \rangle$ , taking the value of  $\left| \frac{\partial^2 \phi}{\partial \vec{\eta}^2} \right|_{\eta=0}$  from (6.92) and neglecting  $\frac{1}{J}$  corrections we get

$$\mathbb{T}_{\iota'}^{\iota} = \left(\frac{1}{4\pi}\right)^2 \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \left[\frac{3}{(1+\gamma^2)\sum_{i=1}^N x_i}\right]^{\frac{3}{2}} \langle \iota | \left|\iota'\right\rangle \tag{6.95}$$

This means that the leading order of  $\mathbb{T}$  is proportional to the identity:

$$\mathbb{T} = \left(\frac{1}{4\pi}\right)^2 \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \left[\frac{3}{(1+\gamma^2)\sum_{i=1}^N x_i}\right]^{\frac{3}{2}} \mathbb{1}_{\vec{j}}$$
(6.96)

The proportionality factor will be denoted

$$\alpha(x_i) := \left(\frac{1}{4\pi}\right)^2 \left[\frac{6\pi}{(1+\gamma^2)\sum_{i=1}^N x_i}\right]^{\frac{3}{2}}$$
(6.97)

so that

$$\mathbb{T} = \alpha\left(x_i\right) J^{-\frac{3}{2}} \mathbb{1}_{\vec{j}} \tag{6.98}$$

One can easily check, that  $\|\mathbb{T}\| < 1$ . Indeed, the factor  $\alpha(x_i) < \frac{0.52}{\left((1+\gamma^2)\sum_{i=1}^N x_i\right)^{3/2}}$  is less then 1, because  $1 + \gamma^2 \ge 1$  and  $\sum_{i=1}^N x_1 \ge 2$ . Moreover, by assumption  $J \gg 1$ .

The higher order terms of  $\mathbb{T}$  have not been studied yet, but we have already found (see (6.56)), that in the spin-z basis  $|\vec{m}\rangle_{\vec{i}}$  only the diagonal terms are nonzero:

$$\mathbb{T}_{\vec{m'}}^{\vec{m}} = \left(\frac{1}{4\pi}\right)^2 \left( \left[\frac{6\pi}{J\left(1+\gamma^2\right)\sum_{i=1}^N x_i}\right]^{\frac{3}{2}} + \left(\frac{1}{J}\right)^{\frac{5}{2}} T_{\vec{m}} \right) \delta_{\vec{m'}}^{\vec{m}}$$
(6.99)

#### 6.4 Application of Lorentzian Polyhedra Propagator

The operator  $\mathbb{T}$  obtained above can be applied in some explicit calculations of spin-foam theory. The very simple form of its leading order, i.e. fact, that it is proportional to identity, simplifies some calculations a lot. In this section we will list few of applications that we considered.

First in subsection 6.4.1 we discuss the applications in the transition amplitude of the Dipole Cosmology model. In subsection 6.4.2 we discuss the input of  $\mathbb{T}$  operator on the radiative correction caused by the *melonic* bubble (studied by Aldo Riello in [78], recalled in subsection 2.3.2.1). Finally, in subsection 6.4.3, we make an attempt to study the issue of renormalization of edge-amplitude in spin-foam models.

#### 6.4.1 Dipole Cosmology

As it was calculated in [57, 58] and recalled in section §2.2 and subsection 4.5.1, the transition amplitude of the Dipole Cosmology model is given by

$$W(z) = \sum_{\{j_{\ell}\}} \prod_{\ell=1}^{4} (2j_{\ell}+1) e^{-2t\hbar j_{\ell}(j_{\ell}+1)-izj_{\ell}} \int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{\ell=1}^{4} \langle j_{\ell} | u_{\vec{n_{\ell}}}^{\dagger} Y^{\dagger} g Y u_{\vec{n_{\ell}}} | j_{\ell} \rangle_{j_{\ell}}$$

$$= \sum_{\{j_{\ell}\}} \prod_{\ell=1}^{4} (2j_{\ell}+1) e^{-2t\hbar j_{\ell}(j_{\ell}+1)-izj_{\ell}} \langle \iota | \mathbb{T} | \iota' \rangle$$
(6.100)
with  $\iota^{(\prime)} := \int_{SU(2)} \mathrm{d}u \prod_{\ell=1}^{4} u \cdot u_{\vec{n_{\ell}}} | j_{\ell} \rangle$ 

Our calculations influence only the term  $\langle \iota | \mathbb{T} | \iota' \rangle$ . Authors of [57, 58] assume that this term behaves like  $\frac{\tilde{N}_0}{j_0^3}$  (with  $j_0 = \frac{\Im z}{4t\hbar}$  and  $\tilde{N}_0$  being a constant for a given graph). The direct formula for the Lorentzian Polyhedra propagator shows, that it is

$$\langle \iota | \mathbb{T} | \iota' \rangle = \alpha (x_i) \cdot J^{-\frac{3}{2}} \langle \iota | | \iota' \rangle$$
 (6.101)

where the factor

$$\alpha(x_i) = \alpha_0 \left(\frac{4}{\sum_{i=1}^4 x_i}\right)^{\frac{3}{2}} =: \alpha_0 \xi(x_i)$$
(6.102)

depend on ratios of spins  $x_i = \frac{j_i}{J}$ , but it does not scale with J. The coherent SU(2)intertwiner have norm squared scaling as  $j^{-\frac{3}{2}}$  (see [63]), thus the overall scaling of  $\langle \iota | \mathbb{T} | \iota' \rangle$ is  $N \cdot j^{-3}$ , as expected. So taking into account the direct calculation of the Lorentzian
Polyhedra Propagator, the Dipole Cosmology transition amplitude reproduce the original
DC formula of [55, 57, 58]:

$$W(z) = \sum_{\{j\ell\}} \frac{N(x_i)}{j_0^3} \prod_{\ell=1}^4 (2j_\ell + 1) e^{-2t\hbar j_\ell (j_\ell + 1) - i\lambda v_0 j_\ell^{\frac{3}{2}} - izj_\ell}$$
(6.103)



Figure 6.2: A 2-vertex contribution to Dipole Cosmology transition amplitude. (a) - The graph diagram representing the interaction vertices of the process. (b) - The spin-foam of the process.

with a minor correction: the graph-shape dependent factor N now depends also on the ratios of spins.

Using the same summation technique, as in [55], (i.e. summing by integrating the Gaussian integrals, see also subsection 2.2.1) one obtains a similar result

$$W(z) = \frac{N(x_i)}{j_0^3} \left( 2j_0 \sqrt{\frac{\pi}{t}} e^{-\frac{z^2}{8t\hbar}} \right)^4 = \mathcal{N}(x_i) z e^{-\frac{z^2}{2t\hbar}}$$
(6.104)

where (likewise in (6.102)) the normalisation factor  $\mathcal{N}(x_i) = \mathcal{N}_0 \xi(x_i)$  for  $\mathcal{N}_0$  being the normalisation factor of the original SC The original DC transition amplitude (see (2.23)) can be obtained by setting all parameters  $x_i = 1$ .

It is important to notice, that the factor  $\xi(x_i)$  varies from  $\xi(x_i) = 1$  for all spins being equals to  $\xi(x_i) = 2\sqrt{2}$  for a degenerate configuration (for example  $j_1 = j_2$  and  $j_3 = j_4 = 0$ ). Thus it does not influence the exponential behaviour of the total amplitude.

**Simple 2-vertex Dipole Cosmology** The fact that the T-operator is proportional to identity allows to calculate the contribution of one of 2-vertex terms to Dipole Cosmology transition amplitude.

Consider a diagram presented at figure 6.2a. It is equivalent to the spin-foam presented at figure 6.2b, with eight rectangular faces and one internal edge connecting two interaction vertices. Recalling the form of the boundary state (subsection 2.2.1.2) and

the Feynman-EPRL rules (section §4.3), the transition amplitude of such process is:

$$W^{(2v)}(z_{\rm in}, z_{\rm out}) = \sum_{\{j_\ell\}} \prod_{\ell} (2j_\ell + 1) e^{-2t\hbar j_\ell (j_\ell + 1) - iz_\ell j_\ell} \cdot \int_{SL(2,\mathbb{C})^{6-2}} (dg_N dg_{N'} dg_{N''} dg_S dg_{S'} dg_{S''})'' \qquad (6.105) \cdot \prod_{\ell=1}^4 \langle j_\ell | u_{\vec{n_\ell}}^{\dagger} Y^{\dagger} g_N^{-1} g_{N''} Y Y^{\dagger} g_{S''}^{-1} g_S u_{\vec{n_\ell}}^{-1} | j_\ell \rangle_{j_\ell} \cdot \prod_{\ell=5}^8 \langle j_\ell | u_{\vec{n_\ell}}^{\dagger} Y^{\dagger} g_{N'}^{-1} g_{N''} Y Y^{\dagger} g_{S''}^{-1} g_{S'} u_{\vec{n_\ell}}^{-1} | j_\ell \rangle_{j_\ell}$$

where by  $\int_{SL(2,\mathbb{C})^{6-2}} (dg_N dg_{N'} dg_{N''} dg_S dg_{S'} dg_{S''})^{''}$  we mean, that two of the integrals are dropped (thanks to the property recalled in subsection 1.2.3.3), and  $z_{\ell} = z_{\text{in}}$  for  $\ell = 1, \ldots, 4$  and  $z_{\ell} = z_{\text{out}}$  for  $\ell = 5, \ldots, 8$  (similarly to subsection 2.2.1.4 we omit the face amplitudes). Note, that the only terms that prevent the formula from the factorization on *in* and *out* parts are the  $g_{N''}$  and  $g_{S''}$  group elements. Thus, using the regularization (see [46, 50] or subsection 1.2.3.3), we can choose to drop these two group elements, obtaining

$$W^{(2v)}(z_{\rm in}, z_{\rm out}) = W^{(2v)}(z_{\rm in}) W^{(2v)}(z_{\rm out})$$
(6.106)

for

$$W^{(2v)}(z_{in}) = \sum_{\{j_{\ell}\}} \prod_{\ell} (2j_{\ell}+1) e^{-2t\hbar j_{\ell}(j_{\ell}+1)-izj_{\ell}}$$

$$\cdot \int_{SL(2,\mathbb{C})^{2}} \mathrm{d}g_{N} \mathrm{d}g_{S} \prod_{\ell=1}^{4} \langle j_{\ell} | u_{\vec{n_{\ell}}}^{\dagger} Y^{\dagger} g_{N}^{-1} Y Y^{\dagger} g_{S} u_{\vec{n_{\ell}}} | j_{\ell} \rangle_{j_{\ell}}$$
(6.107)

Comparing it with (6.100) we can write, that

$$W^{(2\mathbf{v})}(z) = \sum_{\{j_{\ell}\}} \prod_{\ell=1}^{4} \left(2j_{\ell}+1\right) e^{-2t\hbar j_{\ell}(j_{\ell}+1)-i\lambda v_{0}j_{\ell}^{\frac{3}{2}}-izj_{\ell}} \left\langle\iota\right| \mathbb{T}^{2}\left|\iota'\right\rangle$$
(6.108)

and recalling (6.98) we get

$$\langle \iota | \mathbb{T}^2 | \iota' \rangle = \alpha^2 (x_i) \cdot J^{-3} \langle \iota | | \iota' \rangle = \alpha (x_i) J^{-\frac{3}{2}} \langle \iota | \mathbb{T} | \iota' \rangle$$
(6.109)

 $\mathbf{SO}$ 

$$W^{(2\mathbf{v})}(z) = \alpha(x_i) j_0^{-\frac{3}{2}} W(z)$$
(6.110)

and we conclude, that the correction coming from this particular 2-vertex contribution to Dipole Cosmology transition amplitude is proportional to the original amplitude and is negligible in large j limit.

#### 6.4.2 Bubble Divergence

As it was proven in [78] and recalled in subsection 2.3.2.1, the radiative correction to the spin-foam edge coming from the *melonic* bubble (see figure 6.3) is proportional to the  $\mathbb{T}^2$  operator. To be precise, given a cutoff  $\Lambda$  on the internal spins of the bubble, the leading



Figure 6.3: Since the  $\mathbb{T}$  operator represents a single EPRL vertex on an edge, and the identity operator represents a single edge, the *melonic* bubble regularised in [78] can be regularised one step further, as it is shown above.

order of the transition amplitudes are proportional to matrix elements of the following operator

$$W^{\Lambda}_{\mathcal{M}} \sim W^{\Lambda} := \ln\left(\Lambda\right) \cdot \mathbb{T}^2 \tag{6.111}$$

Our calculation shows, that in large J limit (with J being the maximal spin of the *external* faces of the bubble) up to  $\frac{1}{J}$  corrections this radiatively-corrected edge operator is proportional to the identity with the constant dependent on  $\Lambda$  and J:

$$W_{\iota,\iota'}^{\Lambda} = \ln\Lambda \cdot \left(\frac{1}{J}\right)^{3} \cdot \frac{27}{32\pi} \left[\frac{1}{(1+\gamma^{2})\sum_{i=1}^{N} x_{i}}\right]^{3} \delta_{\iota,\iota'}$$
(6.112)

The proportionality factor we will call  $a(J, \Lambda)$ :

$$a(J,\Lambda) := \ln\Lambda \cdot \left(\frac{1}{J}\right)^3 \cdot \frac{27}{32\pi} \left[\frac{1}{(1+\gamma^2)\sum_{i=1}^N x_i}\right]^3 \tag{6.113}$$

we will use it in the next subsection in the edge renormalization problem.

Note that since  $\Lambda$  is the maximum spin appearing in the bubble causing infinity, whereas J is the maximum spin appearing on the external faces, so they cannot be identified. To make the formula (6.112) correct we have to assume  $1 \ll J \ll \Lambda$ .

#### 6.4.3 Edge renormalization

Consider now a series of spin-foams  $(\kappa_n)_{n=0,1,\dots}$  such that they differ only by a number of vertices on one selected edge e, i.e.  $\kappa_n$  has precisely n EPRL vertices on e (see figure 6.4). Then the spin-foam operator related to  $\kappa_n$  is

$$W_{\kappa_n} = W_{\kappa_0} \cdot \mathbb{T}^n \tag{6.114}$$

As we have mentioned in subsection 6.3.6,  $||\mathbb{T}|| < 1$ . Thus we can sum the series of  $W_{\kappa_n}$  to

$$W_{\kappa_n}^R = \sum_{n=0}^{\infty} W_{\kappa_n} = W_{\kappa_0} \frac{1}{1 - \mathbb{T}} = W_{\kappa_0} \frac{1}{1 - \alpha (x_i) J^{-3/2}}$$
(6.115)

The same procedure can be done for a series of spin-foams  $\tilde{\kappa}_n$  that differ by a number of *melonic* bubbles on a 4-valent edge (see figure 6.5). Then, using the results of previous subsection (and assuming  $W^{\Lambda}_{\mathcal{M}} = C \cdot W^{\Lambda}$  for some constant C) we have

$$W_{\tilde{\kappa}_n} = W_{\tilde{\kappa}_0} \cdot \left(C \cdot \ln \Lambda \cdot \mathbb{T}^2\right)^n = W_{\tilde{\kappa}_0} \cdot \left(C \cdot a\left(J,\Lambda\right)\right)^n \tag{6.116}$$



Figure 6.4: A series of spin-foams  $(\kappa_n)_{n=0,1,\dots}$ . For each foam only a fragment containing the considered edge *e* is presented.



Figure 6.5: A series of spin-foams  $(\kappa_n)_{n=0,1,\dots}$ . For each foam only a fragment containing the considered edge e is presented.

We can write a formal sum of the power series:

$$W_{\tilde{\kappa}_n}^{R,\text{bubble}} = \sum_{n=0}^{\infty} W_{\tilde{\kappa}_n} \, " = W_{\tilde{\kappa}_0} \frac{1}{1 - C \cdot a \left(J, \Lambda\right)} "$$
(6.117)

which equals to the actual sum if the series is convergent, thus if  $C \cdot a(J, \Lambda) < 1$ .

Let us now study the factor  $a(J, \Lambda)$ . Thanks to the logarithmic dependence on the cut-off it takes relatively small values even for huge values of the cut-off. Assume for a moment, that the maximum internal spin  $\Lambda$  is the inverse cosmological constant expressed in Plan units, i.e. that  $\Lambda = 10^{120}$  (such choice is justifies for example in [78]), Then we get an upper bound for a, namely

$$a(J,\Lambda) < \frac{9.28}{J^3}$$
 (6.118)

thus for each reasonable value of the proportionality constant C there is a scale J such that  $C \cdot a(J,\Lambda)|_{\Lambda=10^{120}} < 1$  and the sum (6.117) converges. If C = 1, this condition is satisfied even for  $J \ge 2\frac{1}{2}$  (note, that our approximations were made in large J limit, so they do not apply for  $J \le 2$  and it is possible, that the sum (6.117) with C = 1 converges for all Js).

Note, that the formula (6.115) is valid for edges with arbitrary valency. In contrast, the derivation concerning bubbles is based on the calculation of [78], that was done only for a 4-edge bubble on a 4-valent edge. In order to derive a formula similar to (6.117) for other types of edges and bubbles one has to calculate first the divergent parts of these bubbles.

#### 6.5 Technical issues

In this section there one can intermediate steps of the calculations done above (mainly in section §6.3).

In subsection 6.5.1 we study the properties of the functions  $f_m^{(j)}(\eta)$  (introduced in subsection 6.3.1). Subsection 6.5.2 states and proves the theorem about functions dependent on squared magnetic momentum number used to simplify lots of formulae. Finally the subsection 6.5.3 contains proves of some estimations done in the main text of the chapter.

### 6.5.1 Some properties of $f_m^{(j)}(\eta)$ function

This subsection is devoted to the function  $f_m^{(j)}(\eta)$ . First we study some basic symmetries of  $f_m^{(j)}(\eta)$ . Then we calculate it's derivatives. Finally we prove it decays exponentially in  $\eta$ .

Before doing so let recall some definitions and fix some notations.

The  $f_m^{(j)}(\eta)$  functions are very closely related to Gauss hypergeometric function. Let us recall the definition of  $_2F_1$  in both series and integral representation [88]:

$${}_{2}F_{1}(a, b; c; z): = \sum_{k=0}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \mathrm{d}t \, t^{b-1} \, (1-t)^{c-b-1} \, (1-zt)^{-a}$$
(6.119)

where

$$x^{\overline{k}} := \prod_{n=0}^{k-1} (x+n)$$
 (6.120)

Note that the if all arguments a, b, c > 0 and  $0 \le z < 1$ , then the series is sum of real positive numbers, thus in this case  ${}_{2}F_{1}(a, b; c; z) \ge 0$ .

The function  $f_m^{(j)}(\eta)$ :

$$f_m^{(j)}(\eta) = e^{-(j+1+m-i\gamma j)\eta} {}_2F_1\left(j+1-ij\gamma, \, j+1+m; \, 2j+2; \, 1-e^{-2\eta}\right)$$
(6.121)

Let us define another function,  $g_m^{(j)}(\eta)$ , by setting  $\gamma = 0$  in the above formula:

$$g_m^{(j)}(\eta) = e^{-(j+1+m)\eta} {}_2F_1\left(j+1, \, j+1+m; \, 2j+2; \, 1-e^{-2\eta}\right) \tag{6.122}$$

One can easily see, that  $g_m^{(j)}(\eta) \ge 0$ . Indeed, all the parameters of  $_2F_1$  function are positive and  $z(\eta) = 1 - e^{-2\eta} < 1$ .

In this subsection it is convenient to consider  $f_m^{(j)}$  and  $g_m^{(j)}$  as functions of  $z(\eta)$ :

$$f_m^{(j)}(\eta) = (1-z)^{\frac{(j+1+m-i\gamma)}{2}} {}_2F_1(j+1-ij\gamma, j+1+m; 2j+2; z)$$
(6.123)

We will also use a following shortcut notation:

$$J := j + 1 \quad G := \gamma j \tag{6.124}$$

$$a := J - iG$$
  $b := J + m$   $c := 2J$   $\delta := J + m - iG$  (6.125)

thus

$$f_m^{(j)}(\eta) = (1-z)^{\frac{\delta}{2}} {}_2F_1(a, b; c; z)$$
(6.126)

in case of  $g_m^{(j)}(\eta)$  it is convenient to use also  $\tilde{a} := J$ , so

$$g_m^{(j)}(\eta) = (1-z)^{\frac{b}{2}} {}_2F_1(\tilde{a}, b; c; z)$$
(6.127)

## **6.5.1.1** Mirror symmetry: $f_m^{(j)}(z) = \overline{f_{-m}^{(j)}(z)}$

Let us use the symmetry of  $_2F_1$  function:

$$_{2}F_{1}(a, b; c; z) = (1 - z)^{c - a - b} {}_{2}F_{1}(c - a, c - b; c; z)$$
 (6.128)

and the mirror symmetry  $_2F_1(\overline{a}, \overline{b}; \overline{c}; \overline{z}) = \overline{_2F_1(a, b; c; z)}$  to relate  $f_m^{(j)}(z)$  with  $f_{-m}^{(j)}(z)$ . In our case  $c - a = J + iG = \overline{a}$  and c - b = J - m, and c - a - b = -m + iG, thus for  $0 \le z < 1$  we have

$$\begin{aligned}
f_m^{(j)}(z) &= (1-z)^{\frac{\delta}{2}} {}_2F_1 \left(J - iG, J + m; 2J; z\right) \\
&= (1-z)^{\frac{J+m-iG}{2}} (1-z)^{-m-iG} {}_2F_1 \left(J + iG, J - m; 2J; z\right) \\
&= \frac{(1-z)^{\frac{J-m+iG}{2}} {}_2F_1 \left(J + iG, J - m; 2J; z\right)}{(1-z)^{\frac{J-m-iG}{2}} {}_2F_1 \left(J - iG, J - m; 2J; z\right)} = \overline{f_{-m}^{(j)}(z)}
\end{aligned}$$
(6.129)

Obviously the same calculation show that  $g_m^{(j)}(z) = g_{-m}^{(j)}(z)$ .

#### 6.5.1.2 First derivative of $f_m^{(j)}(z(\eta))$

Let us now calculate the first derivative of  $f_m^{(j)}$  and  $g_m^{(j)}$  with respect to  $\eta$ . Using  $\partial_{z} _2F_1(a, b; c; z) = \frac{ab}{c} _2F_1(a+1, b+1; c+1; z)$  we get:

$$\frac{\mathrm{d}f_{m}^{(j)}(z(\eta))}{\mathrm{d}\eta} = -\delta e^{-\delta\eta} {}_{2}F_{1}(a, b; c; z)$$

$$+e^{-\delta\eta} \frac{\mathrm{d}z(\eta)}{\mathrm{d}\eta} \frac{ab}{c} {}_{2}F_{1}(a+1, b+1; c+1; z)$$
(6.130)

using (6.125) and  $z = 1 - e^{-2\eta}$  we get  $\frac{dz}{d\eta} = 2e^{-2\eta} = 2(1-z), \frac{ab}{c} = \frac{(J-iG)(J+m)}{2J} = \frac{\delta}{2} - \frac{iGm}{2J}$  so

$$\frac{\mathrm{d}f_{m}^{(j)}(z(\eta))}{\mathrm{d}\eta} = -\delta e^{-\delta\eta} {}_{2}F_{1}(a, b; c; z)$$

$$+ e^{-(\delta+2)\eta} \left(\delta - \frac{iGm}{J}\right) {}_{2}F_{1}(a+1, b+1; c+1; z)$$
(6.131)

At  $\eta = 0$  we use z(0) = 0 and  $_2F_1(a, b; c; 0) = 1$  to obtain

$$\frac{\mathrm{d}f_m^{(j)}(\eta)}{\mathrm{d}\eta}\bigg|_{\eta=0} = -\delta + \delta - \frac{iGm}{J} = -\frac{iGm}{J} = -i\frac{\gamma jm}{j+1} = -im\left(1+O\left(j^{-1}\right)\right) \quad (6.132)$$

To find the derivative of  $g_m^{(j)}(\eta)$  we just set G = 0 in the above formulae and get

$$\frac{\mathrm{d}g_m^{(j)}(z(\eta))}{\mathrm{d}\eta} = -\delta e^{-\delta\eta} {}_2F_1(a, b; c; z) + \delta e^{-(\delta+2)\eta} {}_2F_1(a+1, b+1; c+1; z) = -\delta e^{-\delta\eta} \left[ {}_2F_1(a, b; c; z) - (1-z) {}_2F_1(a+1, b+1; c+1; z) \right]$$
(6.133)

We can simplify the hypergeometric term using the series expansion:

$$_{2}F_{1}(a, b; c; z) - (1 - z) _{2}F_{1}(a + 1, b + 1; c + 1; z) =$$

$$= 1 + \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} - 1 - \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k}} (b+1)^{\overline{k}}}{(c+1)^{\overline{k}} k!} z^{k} + \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k}-1} (b+1)^{\overline{k}-1}}{(c+1)^{\overline{k}-1} (k-1)!} z^{k}$$

$$= \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} \left[ 1 - \frac{(a+k)(b+k)c}{ab(c+k)} + \frac{ck}{ab} \right]$$

$$= \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} \frac{abc + abk - abc - akc - bkc - ck^{2} + c^{2}k + ck^{2}}{ab(c+k)}$$

$$= \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} \frac{(c-a)(c-b)k}{ab(c+k)}$$

$$= z \frac{(c-a)(c-b)}{c(c+1)} \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k}-1} (b+1)^{\overline{k}-1}}{(c+2)^{\overline{k}-1} (k-1)!} z^{k-1}$$

$$= z \frac{(c-a)(c-b)}{c(c+1)} \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k}} (b+1)^{\overline{k}}}{(c+2)^{\overline{k}} k!} z^{k}$$

$$= z \frac{(c-a)(c-b)}{c(c+1)} \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k}} (b+1)^{\overline{k}}}{(c+2)^{\overline{k}} k!} z^{k}$$

 $\mathbf{so}$ 

$$\frac{\mathrm{d}g_m^{(j)}(z(\eta))}{\mathrm{d}\eta} = -\delta e^{-\delta\eta} z(\eta) \frac{(c-a)(c-b)}{c(c+1)} {}_2F_1(a+1, b+1; c+2; z(\eta))$$
(6.135)

what we will need in (6.5.1.4).

# **6.5.1.3** Second derivative of $f_m^{(j)}(z(\eta))$ and $\ln \left[ f_m^{(j)}(z(\eta)) \right]$ at $\eta = 0$

In subsection 6.3.5 we need to know  $\frac{d^2}{d\eta^2} \ln \left[ f_m^{(j)}(z(\eta)) \right]$  at  $\eta = 0$ .

Let us first calculate  $\frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2}$ . By differentiating the equation (6.131) and recalling, that  $\left(\delta - \frac{iGm}{J}\right) = 2\frac{ab}{c}$  we get

$$\frac{\mathrm{d}^{2} f_{m}^{(j)}(z(\eta))}{\mathrm{d}\eta^{2}} = -\delta \left[ -\delta e^{-\delta\eta} {}_{2}F_{1}\left(a, b; c; z(\eta)\right) + 2\frac{ab}{c} e^{-(\delta+2)\eta} {}_{2}F_{1}\left(a+1, b+1; c+1; z(\eta)\right) \right] 
- \left(\delta+2\right) 2\frac{ab}{c} e^{-(\delta+2)\eta} {}_{2}F_{1}\left(a+1, b+1; c+1; z(\eta)\right) 
+ 2\frac{ab}{c} \frac{(a+1)(b+1)}{(c+1)} \frac{\mathrm{d}z(\eta)}{\mathrm{d}\eta} e^{-(\delta+2)\eta} {}_{2}F_{1}\left(a+2, b+2; c+2; z(\eta)\right)$$
(6.136)

Fortunately we do not need to know the second derivative for all  $\eta$ , it is enough to find its value at  $\eta = 0$ , where z(0) = 0 and  $_2F_1(a, b; c; 0) = 1$ , so the formula simplifies:

$$\frac{\mathrm{d}^2 f_m^{(j)}(z(\eta))}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\delta \left[-\delta + 2\frac{ab}{c}\right] - (\delta+2) \, 2\frac{ab}{c} + 4\frac{ab}{c}\frac{(a+1)(b+1)}{(c+1)} \tag{6.137}$$

Now using (6.125) one get

$$\frac{\mathrm{d}^2 f_m^{(j)}(z(\eta))}{\mathrm{d}\eta^2}\bigg|_{\eta=0} = 4\left[\frac{\delta^2}{4} - \frac{ab\delta}{c} - \frac{ab}{c} + \frac{ab}{c}\frac{(a+1)(b+1)}{(c+1)}\right]$$
(6.138)

We are interested in the second differential of  $\ln \left[ f_m^{(j)}(z(\eta)) \right]$ . Using the identity

$$\frac{\mathrm{d}^2 \ln f(x)}{\mathrm{d}x^2} = \frac{f''(x)f(x) - [f'(x)]^2}{\left[f(x)\right]^2} \tag{6.139}$$

and the fact, that  $f_m^{(j)}(z(0)) = 1$  we have

$$\frac{\mathrm{d}^{2} \ln\left[f_{m}^{(j)}\left(z(\eta)\right)\right]}{\mathrm{d}\eta^{2}}\bigg|_{\eta=0} = \frac{\mathrm{d}^{2}f_{m}^{(j)}\left(z(\eta)\right)}{\mathrm{d}\eta^{2}}\bigg|_{\eta=0} - \left(\frac{\mathrm{d}f_{m}^{(j)}\left(z(\eta)\right)}{\mathrm{d}\eta}\bigg|_{\eta=0}\right)^{2}$$
(6.140)

putting here the expressions (6.131) and (6.138) we get

$$\frac{d^{2} \ln \left[ f_{m}^{(j)} \left( z(\eta) \right) \right]}{d\eta^{2}} \bigg|_{\eta=0} = 4 \left[ \frac{\delta^{2}}{4} - \frac{ab\delta}{c} - \frac{ab}{c} + \frac{ab}{c} \frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)} \right] - \left[ -\delta + 2\frac{ab}{c} \right]^{2} \\
= \delta^{2} - 4\frac{ab\delta}{c} - 4\frac{ab}{c} + 4\frac{ab}{c} \frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)} - \delta^{2} + 4\frac{ab\delta}{c} - 4\left(\frac{ab}{c}\right)^{2} \\
= 4\frac{ab}{c} \left[ -1 - \frac{ab}{c} + \frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)} \right] \\
= 4\frac{ab}{c} \cdot \frac{-c^{2} - c - abc - ab + abc + ac + bc + c}{c\left(c+1\right)} \qquad (6.141) \\
= -4\frac{ab}{c} \frac{\left(c-a\right)\left(c-b\right)}{c\left(c+1\right)} \\
= -4\frac{a\left(c-a\right) \cdot b\left(c-b\right)}{c^{2}\left(c+1\right)}$$

Now using (6.125) we have

$$\frac{\mathrm{d}^2 \ln\left[f_m^{(j)}\left(z(\eta)\right)\right]}{\mathrm{d}\eta^2}\bigg|_{\eta=0} = -4\frac{\left(J^2 + G^2\right)\left(J^2 - m^2\right)}{4J^2\left(2J+1\right)} = -\left(1 + \frac{G^2}{J^2}\right)\frac{J^2 - m^2}{\left(2J+1\right)} \quad (6.142)$$

and using (6.124)

$$\frac{\mathrm{d}^2 \ln\left[f_m^{(j)}(z(\eta))\right]}{\mathrm{d}\eta^2}\bigg|_{\eta=0} = -\left(1 + \left(\frac{\gamma j}{j+1}\right)^2\right)\frac{(j+1)^2 - m^2}{(2j+3)}$$
(6.143)

Since  $\frac{\gamma j}{j+1} = \gamma \left( 1 + O\left(j^{-1}\right) \right)$ , the leading order of  $\left. \frac{\mathrm{d}^2 \ln \left[ f_m^{(j)}(z(\eta)) \right]}{\mathrm{d}\eta^2} \right|_{\eta=0}$  is

$$\frac{\mathrm{d}^2 \ln\left[f_m^{(j)}\left(z(\eta)\right)\right]}{\mathrm{d}\eta^2}\bigg|_{\eta=0} = -\left(1+\gamma^2\right)\frac{\left(j+1\right)^2 - m^2}{\left(2j+3\right)}\left(1+O\left(j^{-1}\right)\right) \tag{6.144}$$

The term dependent on m can be further simplified - see subsection 6.5.2.

#### 6.5.1.4 Estimations

It is easy to see, that  $||f_m^{(j)}(\eta)|| \leq g_m^{(j)}(\eta)$ . Indeed, taking the integral representation of  $_2F_1$  we see, that

$$\begin{aligned} \left| f_{m}^{(j)}\left(z(\eta)\right) \right| &= \left| e^{-(j+1+m-i\gamma j)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \int_{0}^{1} \mathrm{d}t \, \frac{t^{j+m} \left(1-t\right)^{j-m}}{\left[1-\left(1-e^{-2\eta}\right)t\right]^{(j+1)-i\gamma j}} \right| \\ &= e^{-(j+1+m)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \left| \int_{0}^{1} \mathrm{d}t \, \frac{t^{j+m} \left(1-t\right)^{j-m}}{\left[1-\left(1-e^{-2\eta}\right)t\right]^{(j+1)-i\gamma j}} \right| \\ &\leq e^{-(j+1+m)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \int_{0}^{1} \mathrm{d}t \, \left| \frac{t^{j+m} \left(1-t\right)^{j-m}}{\left[1-\left(1-e^{-2\eta}\right)t\right]^{(j+1)-i\gamma j}} \right| \\ &= e^{-(j+1+m)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \tag{6.145} \\ &\quad \cdot \int_{0}^{1} \mathrm{d}t \, t^{j+m} \left(1-t\right)^{j-m} \left[1-\left(1-e^{-2\eta}\right)t\right]^{-(j+1)} \\ &= e^{-(j+1+m)\eta} \, _{2}F_{1}\left(j+1, \, j+1+m; \, 2j+2; \, 1-2^{-2\eta}\right) \\ &= g_{m}^{(j)}\left(z(\eta)\right) \end{aligned}$$

Let us now estimate  $g_m^{(j)}(\eta)$  from above and from below.

To estimate  $g_m^{(j)}$  we will estimate it's derivative. We will consider bounds of  $\frac{\frac{\mathrm{d}g_m^{(j)}}{\mathrm{d}\eta}}{g_m^{(j)}}$ :

$$\frac{\frac{\mathrm{d}g_m^{(j)}}{\mathrm{d}\eta}\left(z(\eta)\right)}{g_m^{(j)}} = -\frac{(J+m)(J-m)}{2(2J+1)} \frac{z_2 F_1\left(J+1, J+m+1; 2\left(J+1\right); z\right)}{2F_1\left(J, J+m; 2J; z\right)} \tag{6.146}$$

To simplify the formulae let us introduce a constant  $\omega := \frac{(J+m)(J-m)}{2(2J+1)} = \frac{b(c-b)}{2(c+1)}$ .

**Estimation of**  $g_m^{(j)}(z)$  from above One can easily see, that for nonnegative a, b, c, for  $0 \le z < 1$  and for c = 2a and  $b \le c$  the following inequality holds

$$_{2}F_{1}(a, b; c; z) \leq _{2}F_{1}(a+1, b+1; c+2; z)$$
(6.147)

Indeed, using the series representation we can compare each term, obtaining:

$$\frac{1}{2a} = 1 \quad (k = 0) \\
\frac{ab}{2a} < \frac{(a+1)(b+1)}{2a+2} \quad (k = 1) \\
\frac{ab}{2a} < \frac{(a+k)(b+k)}{2a+2} \quad (k > 2)$$
(6.148)

where the last inequality is equivalent to

$$0 < k^{2} (4a + 2 - b) + k (4a^{2} + 2a + b)$$
(6.149)

which is satisfied for our assumptions. Thus  $\frac{z_2F_1(J+1, J+m+1; 2J+2; z)}{{}_2F_1(J, J+m; 2J; z)} \ge z$ , so

$$\frac{\frac{\mathrm{d}g_m^{(j)}}{\mathrm{d}\eta}\left(z(\eta)\right)}{g_m^{(j)}} \le -\frac{(J+m)(J-m)}{2(2J+1)}z(\eta) = -\omega\left(1-e^{-2\eta}\right) \tag{6.150}$$

Integrating the above equations one gets

$$\ln\left(g_m^{(j)}\right)\Big|_0^\eta \le \frac{\omega}{2} - \omega\eta - \frac{\omega}{2}e^{-2\eta} \tag{6.151}$$

so we have the upper bound:

$$g_m^{(j)}(z(\eta)) \le \left[e^{1-e^{-2\eta}-2\eta}\right]^{\frac{(J+m)(J-m)}{4(2J+1)}} \tag{6.152}$$

and replacing J by j + 1 (see (6.124))

$$\underline{g_m^{(j)}(z(\eta))} \le \left[e^{1-e^{-2\eta}-2\eta}\right]^{\frac{(j+1+m)(j+1-m)}{4(2j+3)}}$$
(6.153)

**Estimation of**  $g_m^{(j)}(z)$  from below To find the lower bound of (6.146) let us show, that under even weaker assumptions (i.e.  $a, b, c \ge 0$  and  $0 \le z < 1$ ) the following is true:

$$\exists_{\alpha>0} \alpha_2 F_1(a, b; c; z) \ge z_2 F_1(a+1, b+1; c+2; z)$$
(6.154)

Let us introduce an auxiliary quantity  $A_{\alpha} := \alpha {}_{2}F_{1}(a, b; c; z) - z {}_{2}F_{1}(a + 1, b + 1; c + 2; z)$ and use the series expansion of the hypergeometric functions:

$$A_{\alpha} = \alpha + \alpha \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} - \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k-1}} (b+1)^{\overline{k-1}}}{(c+2)^{\overline{k-1}} (k-1)!} z^{k}$$

$$= \alpha + \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} \left( \alpha - \frac{c(c+1)k}{ab(c+k)} \right)$$

$$= \alpha + \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^{k} \left( \frac{\alpha abc + \alpha abk - c(c+1)k}{ab(c+k)} \right)$$

$$= \alpha + \alpha \sum_{k=1}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{(c+1)^{\overline{k}} k!} z^{k} + \frac{(\alpha ab - c(c+1))}{c(c+1)} \sum_{k=1}^{\infty} \frac{(a+1)^{\overline{k-1}} (b+1)^{\overline{k-1}}}{(c+2)^{\overline{k-1}} (k-1)!} z^{k}$$

$$= \alpha _{2}F_{1}(a, b; c+1; z) + \frac{z}{c(c+1)} (\alpha ab - c(c+1)) _{2}F_{1}(a+1, b+1; c+2; z)$$

For  $\alpha \ge \alpha_0 := \frac{c(c+1)}{ab}$  all the elements of the above formula are nonnegative, thus we have  $A_{\alpha} \ge 0|_{\alpha \ge \alpha_0}$ , which proves the lemma. Thus

$$\frac{z_2 F_1(a+1, b+1; c+2; z)}{_2 F_1(a, b; c; z)} \le \alpha_0$$
(6.156)

 $\mathbf{SO}$ 

$$\frac{\frac{\mathrm{d}g_m^{(j)}}{\mathrm{d}\eta}(z(\eta))}{g_m^{(j)}} \ge -\omega\alpha_0 = -\frac{b(c-b)}{2(c+1)}\frac{c(c+1)}{ab} = -\frac{c}{2a}(c-b)$$
(6.157)

In our case c = 2a, so

$$\frac{\frac{\mathrm{d}g_m^{(j)}}{\mathrm{d}\eta}\left(z(\eta)\right)}{g_m^{(j)}} \ge -\left(c-b\right) = -\left(j+1-m\right) \tag{6.158}$$

Integrating above formula one gets

$$g_m^{(j)}(z(\eta)) \ge e^{-\eta(j+1-m)}$$
 (6.159)

So finally we have

$$\underbrace{e^{-\eta(j+1-m)} \le g_m^{(j)}\left(z(\eta)\right) \le \left[e^{1-e^{-2\eta}-2\eta}\right]^{\frac{(j+1+m)(j+1-m)}{4(2j+3)}}}_{(6.160)}$$

#### 6.5.2 Squared magnetic momentum number

Several times in the calculations above the squared magnetic number  $m^2$  appears. It seem to break SU(2) invariance, however when considering the invariant states  $|\iota\rangle \in \text{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$ , one can express such components in terms of gauge invariant quantities, what we will prove below.

At the beginning let us remind, that given an invariant state  $|\iota\rangle$  we can decompose it in the magnetic momentum basis

$$|\iota\rangle = \sum_{m_1,\dots,m_N} \iota_{m_1\dots m_N} |m_1\rangle_{j_1} \otimes \dots \otimes |m_N\rangle_{j_N} =: \sum_{\vec{m}} \iota_{\vec{m}} |\vec{m}\rangle_{\vec{j}}$$
(6.161)

note, that

$$\iota_{\vec{m}} := \langle \vec{m} | \, | \iota \rangle_{\vec{i}} \tag{6.162}$$

We will learn how to compute the formulae of the form

$$\sum_{\vec{m}} f\left(m_a^2, j_b\right) \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' \tag{6.163}$$

for a real analytic function f.

#### 6.5.2.1 Single squared magnetic momentum number

First let us consider an expression  $\sum_{\vec{m}} m_i^2 \overline{\iota_{\vec{m}}} \iota'_{\vec{m}}$  for a single index *i*. By definition we have  $m_i^2 |\vec{m}\rangle_{\vec{j}} = \widehat{L_{z,(i)}}^2 |\vec{m}\rangle_{\vec{j}}$ . Now using (6.162) we get

$$\sum_{\vec{m}} m_i^2 \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} = \sum_{\vec{m}} \langle \iota | |\vec{m}\rangle_{\vec{j}} \langle \vec{m} | |\iota'\rangle_{\vec{j}} m_i^2$$

$$= \sum_{\vec{m}} \langle \iota | m_i^2 |\vec{m}\rangle_{\vec{j}} \langle \vec{m} | |\iota'\rangle_{\vec{j}}$$

$$= \sum_{\vec{m}} \langle \iota | \widehat{L_{z,(i)}}^2 |\vec{m}\rangle_{\vec{j}} \langle \vec{m} | |\iota'\rangle_{\vec{j}}$$
(6.164)

Now since  $\sum_{\vec{m}} |\vec{m}\rangle \langle \vec{m} |$  is the identity operator, we have

$$\sum_{\vec{m}} m_i^2 \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} = \langle \iota | \widehat{L_{z,(i)}}^2 | \iota' \rangle$$
(6.165)

Now thanks to SU(2) invariance we have

$$\langle \iota | \widehat{L_{z,(i)}}^{2} | \iota' \rangle = \langle \iota | \widehat{L_{x,(i)}}^{2} | \iota' \rangle = \langle \iota | \widehat{L_{y,(i)}}^{2} | \iota' \rangle$$

$$= \frac{1}{3} \langle \iota | \left[ \widehat{L_{x,(i)}}^{2} + \widehat{L_{y,(i)}}^{2} + \widehat{L_{z,(i)}}^{2} \right] | \iota' \rangle$$

$$= \frac{1}{3} \langle \iota | \widehat{L_{(i)}^{2}} | \iota' \rangle$$

$$(6.166)$$

The operator  $\widehat{L_{(i)}^2}$  is an invariant with eigenvalue  $j_i(j_i+1)$ , thus after all we have

$$\sum_{\vec{m}} m_i^2 \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = \frac{j_i(j_i+1)}{3} \left\langle \iota \right| \left| \iota' \right\rangle \tag{6.167}$$

#### 6.5.2.2 Real function of $m_i^2$

Consider now a real analytic function  $f(m_i^2)$  instead of  $m_i^2$ , i.e. let us calculate  $\sum_{\vec{m}} f(m_i^2) \overline{\iota_{\vec{m}}} \iota'_{\vec{m}}$ . We will start with a polynomial:  $(m_i^2)^n$ . Since  $\widehat{L_{z,(i)}}^2$  is a positive, selfadjoint operator, we can repeat the above procedure and obtain

$$\sum_{\vec{m}} \left(m_i^2\right)^n \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = \langle \iota | \left(\widehat{L_{z,(i)}}^2\right)^n | \iota' \rangle$$
(6.168)

Now we can insert an orthonormal intertwiner basis between each two  $\widehat{L_{z,(i)}}^2$  operators:

$$\sum_{\vec{m}} (m_i^2)^n \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} = \sum_{\iota_1 \dots \iota_{n-1}} \langle \iota | \widehat{L_{z,(i)}}^2 | \iota_1 \rangle \langle \iota_1 | \dots | \iota_{n-1} \rangle \langle \iota_{n-1} | \widehat{L_{z,(i)}}^2 | \iota' \rangle$$
(6.169)

Each expression  $\langle \iota_I | \widehat{L_{z,(i)}}^2 | \iota_{I+1} \rangle$  equals  $\frac{1}{3} \langle \iota_I | \widehat{L_{(i)}^2} | \iota_{I+1} \rangle$  (see (6.166)) giving the eigenvalue  $\frac{j_i(j_i+1)}{3}$ , thus we have

$$\sum_{\vec{m}} \left(m_i^2\right)^n \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = \left[\frac{j_i(j_i+1)}{3}\right]^n \left\langle \iota \right| \left|\iota'\right\rangle \tag{6.170}$$

For a real analytic function f more general than the polynomial we expand f in a power series and follow the above steps for each power of  $m_i^2$ , obtaining

$$\sum_{\vec{m}} f\left(m_i^2\right) \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = f\left(\frac{j_i(j_i+1)}{3}\right) \langle \iota | \left| \iota' \right\rangle$$
(6.171)

#### 6.5.2.3 Function of many $m_i^2$ s

Consider now a real function  $f(m_1^2, \ldots, m_N^2)$ . Since the operators  $\widehat{L_{z,(i)}}$  commute for different *i*, for each term  $m_i^2$  we can do the procedure of subsection 6.5.2.2 separately, obtaining

$$\sum_{\vec{m}} f\left(m_1^2, \dots, m_N^2\right) \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = f\left(\frac{j_1(j_1+1)}{3}, \dots, \frac{j_N(j_N+1)}{3}\right) \langle \iota | \left| \iota' \right\rangle$$
(6.172)

#### 6.5.2.4 Function of many $m_i^2$ s and many $j_i$ s

At least let us consider a real function  $f(m_1^2, \ldots, m_N^2, j_1, \ldots, j_N) =: f(\overrightarrow{m^2}, \overrightarrow{j})$ . Note, that given an expression  $\sum_{\vec{m}} f(\overrightarrow{m^2}, \overrightarrow{j}) \overline{\iota_{\vec{m}}} \iota'_{\vec{m}}$ , the  $j_i$ -dependent parts do not differ when  $\vec{m}$  changes. We can thus follow the procedure of subsection 6.5.2.3, treating all  $j_i$ dependences as parameters of the function, and obtain the result:

$$\sum_{\vec{m}} f\left(\overrightarrow{m^2}, \vec{j}\right) \overline{\iota_{\vec{m}}} \iota'_{\vec{m}} = f\left(\frac{\overline{j\left(j+1\right)}}{3}, \vec{j}\right) \left\langle \iota \right| \left|\iota'\right\rangle \tag{6.173}$$

#### 6.5.2.5 Application in the text

In the main text of the article the above lemma is used twice: when the large J behaviour of the function  $\Phi_{t'}^{\iota}(\eta)$  is considered subsection 6.3.3, and when the Hessian matrix of  $\phi_{t'}^{\iota}(\eta)$  is calculated subsection 6.3.5.

In the first case we estimate  $\left|\Phi_{\iota'}^{\iota}(\eta_1)\right| = \left|\sum_{\vec{m}} \Phi_{\vec{m}}^{\vec{m}}(\eta_1) \overline{\iota_{\vec{m}}} \iota_{\vec{m}}'\right| \leq \left|\sum_{\vec{m}} \left|\Phi_{\vec{m}}^{\vec{m}}(\eta_1)\right| \overline{\iota_{\vec{m}}} \iota_{\vec{m}}'\right|$ , knowing that  $\left|\Phi_{\vec{m}}^{\vec{m}}(\eta_1)\right| \leq \left[C(\eta_1)\right]^{\sum_{i=1}^{N} \frac{(j_i+1)^2 - m_i^2}{4(2j_i+3)}}$ , where the factor  $C(\eta_1)$  does not depend on  $\vec{m}$ . We have thus

$$|\Phi_{\iota'}^{\iota}(\eta_1)| \le \left| \sum_{\vec{m}} \left[ C(\eta_1) \right]^{\sum_{i=1}^{N} \frac{(j_i+1)^2 - m_i^2}{4(2j_i+3)}} \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' \right|$$
(6.174)

and using our lemma we can substitute each appearance of  $m_i^2$  by  $\frac{j_i(j_i+1)}{3}$ , obtaining

$$\left|\Phi_{\iota'}^{\iota}(\eta_{1})\right| \leq \left[C(\eta_{1})\right]^{\sum_{i=1}^{N} \frac{(j_{i}+1)^{2} - \frac{j_{i}(j_{i}+1)}{3}}{4(2j_{i}+3)}} \left|\langle\iota|\left|\iota'\right\rangle\right| = \left[C(\eta_{1})\right]^{\frac{1}{12}\sum_{i=1}^{N} j_{i}+1} \left|\langle\iota|\left|\iota'\right\rangle\right| \quad (6.175)$$

In the second case we calculate  $\frac{\mathrm{d}^2 \phi_{\iota'}^i}{\mathrm{d}\eta^2}\Big|_{\eta=0} = \frac{1}{\langle \iota | \iota' \rangle} \sum_{\vec{m}} \frac{\mathrm{d}^2 \phi_{\vec{m}}}{\mathrm{d}\eta^2}\Big|_{\eta=0} \overline{\iota_{\vec{m}}} \iota'_{\vec{m}}$ , knowing that  $\frac{\mathrm{d}^2 \phi_{\vec{m}}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\frac{1+\gamma^2}{J} \sum_{i=1}^N \frac{(j_i+1)^2 - m_i^2}{(2j_i+3)}$ . Again using our lemma allows to substitute each appearance of  $m_i^2$  by  $\frac{j_i(j_i+1)}{3}$ , obtaining

$$\frac{\mathrm{d}^2 \phi_{\iota'}^{\iota}}{\mathrm{d}\eta^2}\Big|_{\eta=0} = -\frac{1+\gamma^2}{J} \sum_{i=1}^N \frac{(j_i+1)^2 - \frac{j_i(j_i+1)}{3}}{(2j_i+3)} \frac{\langle \iota | \, |\iota' \rangle}{\langle \iota | \, |\iota' \rangle} = -\frac{1+\gamma^2}{3J} \sum_{i=1}^N (j_i+1) \quad (6.176)$$

#### 6.5.3 **Proofs of estimations**

Twice in the text of this chapter some inequalities were stated without a proof. They were

- The prove of the inequality  $\forall_{\eta>0}C(\eta) < 1$  for  $C(\eta) := e^{1-2\eta-e^{-2\eta}}$  in subsection 6.3.3.1
- The prove of the estimation of the integral  $\int_{\eta_{\epsilon}}^{\infty} d\eta \left(\frac{\sinh \eta}{4\pi\eta}\right)^2 \left(e^{1-2\eta-e^{-2\eta}}\right)^{\frac{J}{12}\sum_{i=1}^{N} \left(x_i+\frac{1}{J}\right)}$  used in (6.81) in subsection 6.3.4.

Since they are not obvious, but they are rather mathematical analysis exercises, we skipped proves of them in the main text, providing them here.

#### **6.5.3.1** The inequality $\forall_{\eta>0}C(\eta) < 1$

Let us show the following inequality

$$\forall_{\eta \ge 0} \quad \left(e^{1-2\eta-e^{-2\eta}}\right) \le 1 \tag{6.177}$$

where the equality holds only for  $\eta = 0$ .

Obviously the case  $\eta = 0$  is satisfied:

$$e^{1-0-e^0} = e^0 = 1 \tag{6.178}$$

Since both sides of the inequality are nonnegative, we can take the logarithm of the inequality

$$1 - 2\eta - e^{-2\eta} \stackrel{?}{<} 0 \tag{6.179}$$

We can differentiate both sides of the inequality with respect to  $\eta$ :

$$-2 + 2e^{-2\eta} < 0 \tag{6.180}$$

which is obviously true for  $\eta > 0$ . Thus for  $\eta > 0$ 

$$1 - 2\eta - e^{-2\eta} = 1 + \int_0^\eta \left( -2 + 2e^{-2\tilde{\eta}} \right) \mathrm{d}\tilde{\eta} < 1 + \int_0^\eta \mathrm{Od}\tilde{\eta} = 1$$
 (6.181)

Quod erat demonstrandum.

#### 6.5.3.2 The estimation in the equation (6.68)

We want to estimate the integral

$$I_{\eta_{\epsilon}} := \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \left(\frac{\sinh \eta}{4\pi\eta}\right)^2 \left(e^{1-2\eta-e^{-2\eta}}\right)^{\kappa} \tag{6.182}$$

for  $\kappa > 3$  and  $\eta_{\epsilon} > 0$ .

First let us note, that  $\frac{1}{\eta} \leq \frac{1}{\eta_{\epsilon}}$  for all  $\eta$  in the integration range, thus

$$I_{\eta_{\epsilon}} \le \left(\frac{1}{4\pi\eta_{\epsilon}}\right)^2 \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \,(\sinh\eta)^2 \left(e^{1-2\eta-e^{-2\eta}}\right)^{\kappa} \le \cdots \tag{6.183}$$

Now note, that  $(\sinh \eta)^2 \leq e^{2\eta}$ , so

$$\dots \leq \left(\frac{1}{4\pi\eta_{\epsilon}}\right)^2 \int_{\eta_{\epsilon}}^{\infty} \mathrm{d}\eta \ e^{2\eta} \left(e^{1-2\eta-e^{-2\eta}}\right)^{\kappa} = \dots \tag{6.184}$$

Now let us change the integration variables to  $x = e^{-2\eta}$ :

$$\dots = \left(\frac{1}{4\pi\eta_{\epsilon}}\right)^{2} \frac{1}{2} \int_{0}^{e^{-2\eta_{\eta}} = :x_{\eta}} \mathrm{d}x \frac{1}{x^{2}} \left(xe^{1-x}\right)^{\kappa} \le \dots$$
(6.185)

all the integration rage is the subset of  $x \leq 1$ , so  $e^{1-x} \leq e$ , thus we can remove the denominator:

$$\dots \leq \frac{1}{2} \left(\frac{e}{4\pi\eta_{\epsilon}}\right)^2 \int_0^{x_{\epsilon}} \mathrm{d}x \left(xe^{1-x}\right)^{\kappa-2} \leq \dots \tag{6.186}$$

One can easily prove, that  $\forall_{x \in [0,1]} x \leq e^{x-1}$ . Indeed, for x = 1 we have 1 = 1, and the derivative of left-hand side is x' = 1 is bigger than the derivative of the right-hand side  $(e^{x-1})' = e^{x-1}$ . Knowing that, and since  $\kappa > 3$ , we can estimate  $(xe^{1-x})^{\kappa-2} \leq xe^{1-x}$ , and thus

$$\dots \leq \frac{1}{2} \left(\frac{e}{4\pi\eta_{\epsilon}}\right)^2 \int_0^{x_{\epsilon}} \mathrm{d}x \ x e^{1-x} = \dots \tag{6.187}$$

which can be integrated by parts:

$$\dots = \frac{1}{2} \left( \frac{e}{4\pi \eta_{\epsilon}} \right)^2 \left[ -(x+1) e^{1-x} \right]_0^{x_{\epsilon}} = \dots$$
 (6.188)

Now putting back  $x_{\epsilon} = e^{-2\eta_{\epsilon}}$  we get

$$I_{\eta_{\epsilon}} \leq \frac{1}{2} \left( \frac{e}{4\pi\eta_{\epsilon}} \right)^2 \left( e - e^{1 - e^{-2\eta_{\epsilon}}} - e^{1 - 2\eta_{\epsilon} - e^{-2\eta_{\epsilon}}} \right) = \tilde{I}_{\eta_{\epsilon}}$$
(6.189)

**Proof of existence of**  $\eta_{\epsilon}$  Now let us proof, that for each  $\epsilon > 0$  there is  $\eta_{\epsilon}$  such that  $I_{\eta_{\epsilon}} \leq \epsilon$ . We will do it by showing (using Darboux theorem), that the equation  $\tilde{I}_{\eta_{\epsilon}} = \epsilon$  has a solution for each  $\epsilon > 0$ .

Let us check the limits of  $\tilde{I}_{\eta_{\epsilon}}$ . For  $\eta_{\epsilon} \to 0$  we have

$$\lim_{\eta_{\epsilon}\to 0} \tilde{I}_{\eta_{\epsilon}} = \frac{1}{2} \left(\frac{e}{4\pi}\right)^2 (e-2) \left(\frac{1}{\eta_{\epsilon}}\right)^2 = +\infty$$
(6.190)

For  $\eta_{\epsilon} \to +\infty$  we have

$$\lim_{\eta_{\epsilon} \to +\infty} \tilde{I}_{\eta_{\epsilon}} = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left[ \frac{1}{+\infty} \left( e - e^{1-0} - e^{-\infty} \right) \right] = 0$$
(6.191)

Thus  $I_{\eta_{\epsilon}}$  runs through all real positive numbers, so for each  $\epsilon > 0$  there is  $\eta_{\epsilon}$  being the solution to the equation  $\tilde{I}_{\eta_{\epsilon}} = \epsilon$ , and since  $I_{\eta_{\epsilon}} \leq \tilde{I}_{\eta_{\epsilon}}$  (what we have shown in (6.189)), it is obvious now, that

$$\forall_{\epsilon>0} \exists_{\eta_{\epsilon}} I_{\eta_{\epsilon}} \le \epsilon \tag{6.192}$$

# Part III Discussion and outlook

## Chapter 7

## Open issues and further directions

The research presented in previous, although being complete, does not exhaust the subject. Various aspects of each issue we have studied may be investigated deeper. Some results we obtained opens new directions of research. This chapter presents some of the open questions we see and find being of most importance. Of course our list may be extended.

This chapter is organized as follows. In section §7.1 we discuss the problems related with the topological structure underlying Spin-foam models. In section §7.2 we present possible directions in Dipole Cosmology model. In section §7.3 we point out the major problems related to bubble divergences and renormalization of Spin-foam theories. Finally in section §7.4 we discuss several possible applications of Operator Spin-network Diagrams framework.

#### 7.1 Open questions about 2-complexes in spin-foams

Our answer to the question about the appropriate class of 2-complex for Spin-foam models to make them compatible with Loop Quantum Gravity kinematics (posed in section §2.1) was given by the claim, that one should use all the 2-complex that can be constructed out of a graph diagram (see section §3.5). We see three most important further questions one should ask, i.e.

- Better characterisation of these class, possibly in terms of topologically invariant properties of the 2-complexes.
- A way to impose the cylindrical consistency on these 2-comples, possibly in terms of equivalence classes of graph diagrams.
- A rule of calculating combinatoric weight factors to judge, how many times an equivalent diagram appear in inequivalent ways in an expansion of a quantum process.

#### **Topological characterization**

Our definition of the class of 2-complexes appropriate for Spin-foam models was a constructive definition, i.e. we gave a rule to construct all 2-complexes that one should consider. Such definition, although being complete, is not very useful if one is given a given 2-complex and needs to know, whether it is a member of the class. In such case one would rather need a theorem stating a set of sufficient and necessary conditions for a 2-complex to be in our class. Since the only limitation on the members of our class was of topological origin (i.e. to be a result of a sequence of topological operations), these conditions must be formulated in terms of topologically invariant properties of the complex (i.e. the properties, that are invariant under a homeomorphism). We have found already two such conditions for a 2-complex  $\kappa$ :

- $\kappa$  must be a surjective 2-complex (see Appendix A.1.2) since by definition only such complexes appear in our construction and this property is never spoiled during the gluing procedure.
- the boundary  $\partial \kappa$  must have a neighbourhood in  $\kappa$  homeomorphic with  $\partial \kappa \times [0, 1]$  (see subsection 3.5.1).

These conditions are necessity conditions. However, we have no guarantee that they are sufficient conditions.

#### Cylindrical consistency

The space of spin-network states is designed in such a way, that two states which differ in a certain way (i.e. by a number of 2-valent nodes or a number of links colored by the spin j = 0), are considered equivalent [16, 89], because they represent the same physical state. This requirement is called cylindrical consistency.

Such construction suggests, that a similar requirement should be imposed on the spinfoams representing the interactions between them. Indeed, given two pairs of states, that are cylindrically equivalent but represented by topologically different graphs, one should expect, that quantum transition amplitudes for both pairs are the same, not depending on the topological representation. Thus one should be able to identify the quantum processes, that are physically equivalent.

Some preliminary attempts to identify the rules of spin-foam cylindrical consistency in terms of Operator Spin-network Diagrams was already done by us and our conclusions were presented in subsection 3.5.1. Further research is being continued by the others (for example [90]).

#### **Combinatoric factors**

The issue of cylindrical consistency leads to another puzzle: how many times a given quantum process should be counted to the total amplitude? To be more precise, consider a series of all diagrams (equivalent to spin-foams) connecting a given *in* state with an *out* state (for example obtained as a result of the boundary algorithm - see section §4.2). There might be some elements of the series which although being different diagrams, represent an equivalent physical process, and thus should be counted as one diagram. There might be also a diagram that correspond to more than one physical process, each of which gives identical contribution the transition amplitude and thus it should be counted more then once.

This question should be answered by a prescription to find a (perhaps automorphism group dependent) factor, that should multiply each diagram appearing in the expansion. These factor would be analogous of the combinatoric factors multiplying Feynman diagrams in ordinary Quantum Field Theory.

#### 7.2 Further development in Dipole Cosmology model

The Dipole Cosmology model, recalled in section §2.2, provides itself a wide class of open problems. In sections 4.4-4.5 we addressed only two of them, i.e. we found all diagrams in certain (in fact very low) order of expansion and we estimated the contribution of some of them. In our opinion the most important further directions here are:

- Finding and classifying diagrams with more vertices and more internal edges.
- Calculating (or at least estimating) the transition amplitude of each diagram found.
- Examining the influence of gluing ambiguities.
- Proposing a prescription to isolate the contribution from the bubble part of a diagram from the bubbleless amplitude.

#### Higher orders

We have found the diagrams that contribute to transition amplitude of Dipole Cosmology model in the very first order of the expansion, i.e. for precisely 0 internal edges (with all possible number of internal vertices, i.e. from 1 to 4). As we have noted in subsection 3.5.1, the number of vertices is not of key importance, especially if the vertices are not connected by any edge. Thus the next step would be to find all diagrams in 1-edge expansion.

One of such vertices was already considered and in subsection 6.4.1, several other can be easily constructed from the examples of bubble diagrams in chapter 5. Allowing bubbles leads to infinite number of diagrams at each step of expansion, since even with one edge one can introduce a bubble of arbitrary rank. Restricting to a fixed rank of a possible bubble (even for rank equal 0) the number of possible diagrams is finite, but huge. Thus one has to not only find them, but also classify them.

Some preliminary attempts were made to organize the diagrams appearing at first order of edge expansion, together with a proposal of treating some of them as *unphysical* [91]. Lots of further work needs to be done.

#### Calculations

Of course finding the diagrams is not enough to study a model. One has to also calculate their transition amplitudes, or at least (if an exact calculation is out of reach) estimate them. So far the diagrams corresponding to three possible interaction graphs (i.e. the graphs numbers 18-20 of figure 4.19) were calculated (in [55, 56, 57, 58, 59]). In section §4.5 and in [64] we have shown that the contribution coming from sixteen another interaction graphs (i.e. the graphs number 1-16 of figure 4.19) can be neglected in certain limit. Out of all 20 possible interaction graphs in the lowest order of expansion there is still needs to be calculated (however, some attempts to estimate it are already in being studied [92]).

Concerning diagrams with one internal edge, only one of them was already studied (see subsection 6.4.1). Any estimation of contribution of other diagrams is still an open issue.

#### Various gluings

In subsection 4.4.3 we have presented various possibilities of attaching the interaction vertex to the boundary state. So far most of calculations in Dipole Cosmology was done

with the assumption, that the gluing is done in the most natural way, i.e. there is the maximal possible number of faces in the spin-foam. We expect, that the contribution from other gluings is suppressed in the large-j-limit, because in such case it is likely to obtain the terms enforcing the boundary state to be peaked on the degenerate tetrahedron, however, the direct proof is not ready yet.

#### Renormalization

Considering all possible diagrams leads spin-foams containing bubbles already at the first order of edge expansion. The intuition from ordinary Quantum Field Theory suggests, that the contribution of the bubble diagrams might be absorbed by the bubble-free terms. Such situation is desired, since at each step of the expansion the number of bubble-free diagrams is finite.

Of course this intuition needs a strict framework. The Dipole Cosmology model, thanks to its simplicity, would be a great candidate to test such framework when it is designed.

#### 7.3 Open problems concerning bubbles

The detailed study of bubble divergences is necessary to understand the relation between transition amplitudes of a single spin-foam and the physical transition amplitude between two states.

Within this topic we see the following important problems related to the issues addressed in this thesis:

- Further classification of bubble, related with the Lorentzian EPRL transition amplitude of them.
- Finding an extension of the boundary algorithm to keep under control the bubble that appear in the resulting diagram.
- Calculation of higher order contribution of the Lorentzian Polyhedra Propagator operator.
- Introduction of a procedure allowing to absorb bubbles by changing the coloring of bubble-less diagrams.

#### Classification of bubbles by EPRL amplitude

In subsection 5.2.3 we have calculated SU(2)-BF -amplitudes of some of the examples of bubbles. These bubbles were of rank 1, which caused that the Lorentzian EPRL amplitude of isolated bubbles like them is ill defined (due to problems with applying the regularization of subsection 1.2.3.3 to 2-link connected graphs). Nevertheless, one can consider the Lorentzian EPRL amplitude of a diagram containing such bubble and try to identify the part that comes from the bubble. For bubbles of rank higher than 1 the Lorentzian EPRL amplitude in general should be well defined (see [84]). It is thus important to study and classify the bubbles by their transition amplitude. Such classification would let to identify the bubbles of most importance from the point of view of divergences.

One of the problems that one faces trying to calculate any more complicated bubble is the presence of the matrix elements  $\langle m' | Y^{\dagger}g'gY | m \rangle_{i}$  for  $g, g' \in SL(2, \mathbb{C})$ , where the composition of group elements cannot be omitted by any shift of integrals. These matrix elements are far more complicated than  $\langle m' | Y^{\dagger}gY | m \rangle_j$ , which we studied in chapter 6. We expect, that the object one has to deal with is the matrix element of the form  $\langle j', m' |_{(\gamma j, j)} gY | m \rangle_j$  for general j and j'. It can be reduced to the matrix elements of pure z-boost  $\langle j', m |_{(\gamma j, j)} k_{\eta}Y | m \rangle_j$  which is less complicated than the most general formula (A.128), but more complicated than  $\langle m | Y^{\dagger}k_{\eta}Y | m \rangle_j$  (see (A.165)).

#### Controlling the bubbles in diagrams we construct

The fixed-boundary algorithm (presented in 4.2) allows to construct all diagrams with certain properties, namely with a fixed boundary and fixed number of internal edges. From the point of view of renormalization of spin-foam models it is important to be able to impose another constraint on the diagrams, namely to control the number and type of bubbles in the diagram.

There is an obvious generalization of the algorithm allowing to obtain all diagrams with fixed boundary, number of edges, that contain at least the chosen bubble. Let  $\mathcal{D}_{bub}$ be the diagram of the bubble we want to have. The generalization is as follows: after the step 2 of the fixed-boundary algorithm (see subsection 4.2.2) one should add the diagram  $\mathcal{D}_{bub}$  (with the node and link relation already fixed and the half-links already connected into links). In order to let  $\mathcal{D}_{bub}$  be connected with the rest of the diagram one has to add some open half-links to the nodes of  $\mathcal{D}_{bub}$  (otherwise one would obtain a diagram with a disconnected isolated bubble). Due to keep the diagram consistent one has to add half-links in pairs, i.e. whenever one adds a half-link to a node n of  $\mathcal{D}_{bub}$ , one has to add a dual link to the node n' being in relation with n (since  $\mathcal{D}_{bub}$  is a bubble - there is always n' for each n). At this step one do not close the half-links added to  $\mathcal{D}_{bub}$  - they will be used to connect the bubble with the rest of the diagram. Then, in the step 3, one closes the half-links of the oriented squid set and the open half-links added to  $\mathcal{D}_{bub}$  and then one continue the procedure.

The draw-back of the generalization presented above is that there is no guarantee, that  $\mathcal{D}_{bub}$  is the only bubble in the obtained diagram. The modification of this procedure that gives such guarantee is not known yet (but the research is in progress [90]).

#### Higher order of Lorentzian Polyhedra Propagator

The Lorentzian Polyhedra Propagator  $\mathbb{T}$  calculated in chapter 6 appears to be an important operator, not only when considering bubbles. So far we found the leading order of its matrix elements, obtaining an operator proportional to identity at each  $\mathcal{H}_j$  space (with the proportionality factor dependent on j). This is however only the first approximation. Finding subleading orders of  $\mathbb{T}$  would help to control this approximation.

The most desired result would be to find a good approximation of  $\mathbb{T}$  by the heat kernel function on SU(2) for spread depending on the cut-off  $\Lambda$  and the scale J - in such case a simple physical interpretation of its action in terms of spreading the wave function would be natural.

#### Renormalization

As we have already mentioned in section §7.2, it is important to find a way to replace the bubble part of a diagram by a bubble-less diagrams with modified coloring. Note, that each bubble can be contracted to one complicated vertex, with the amplitude dependent on the data on external elements and the cut-off put on the internal spins (and on the "type of the bubble"). This amplitude obviously is not an ordinary EPRL amplitude. It may be thus expressed as an artificial contractor of some graph. The heuristic procedure needs however an exact calculations of these artificial contractors together with rules to remove the regularizations.

## 7.4 What else can we use Operator Spin-network Diagrams for?

The Operator Spin-network Diagrams framework introduced in chapter 3 was originally introduced to answer the question about the proper class of 2-complexes in Spin-foam models (see section §2.1). However, they appeared to be a much more powerful tool. They allowed to formulate the fixed-boudnary algorithm (see section §4.2) to find all spin-foams with a fixed boundary. They help to simplify the procedure of reading the transition amplitude of a spin-foam thanks to Feynman-EPRL rules (see section §4.3). Moreover, they were a starting point to the design the tools we used to study bubbles in chapter 5.

Despite the applications we presented, there are many others. In our opinion the most important ones are the following

- Detailed study of other boundary problems then Dipole Cosmology model.
- A possibility to test other (then EPRL) schemes of calculating the transition amplitude.
- A convenient framework to modify the theory in order to incorporate matter fields.

#### Other boundary problems

Although the Spin-foam models were designed to be a tool to study the boundary problems in Loop Quantum Gravity (by calculating the analogue of *S*-matrix elements), in fact not many problems have been in ever calculated. The Dipole Cosmology model [55, 56, 57, 58, 59, 64] and the Graviton Propagator [93, 94, 95, 96] are the most known examples of calculations of full transition amplitude. The melonic bubble is a study of partial transition amplitude.

We find three issues responsible for such little number of problems being well studied. First and the most important is the complicated form of functions that appear in the formulae of transition amplitudes. The Operator Spin-network Diagrams framework does not effect this issue. The second one was a difficulty in finding the boundary states with a clear physical interpretation - which was partially solved by a recent development of coherent states in Loop Quantum Gravity [63, 97, 60]. However, since a generic coherent state may be represented by a very complicated spin-network state, its evolution would be a complicated spin-foam, which leads us to the third problem: a difficulty in characterizing the possible spin-foams for a given boundary problem.

The third issue is what Operator Spin-network Diagrams are a very well suited for. The boundary algorithm (section §4.2) should be applied not only to Dipole Cosmology model. One can use it to find the higher-order corrections to the graviton propagator. One can use it to find and order spin-foams for any boundary problem. Moreover, thanks to the adaptation of Feynman-EPRL rules of [37] to Operator Spin-network Diagrams (see section §4.3), one can use the Diagrams to decompose the difficult calculation into manageable subproblems. Furthermore (as we have mentioned in section §7.3), our asymptotic analysis of  $\langle m | Y^{\dagger} k_{\eta} Y | m \rangle_{j}$  terms is the first step to analyse the terms  $\langle j', m |_{(\gamma j, j)} k_{\eta} Y | m \rangle_{j}$ , which are the basic building blocks of any transition amplitude in the EPRL Spin-foam model.

#### Various transition amplitudes

The Operator Spin-network Diagrams in a natural way isolates the elements of the model, which may vary when choosing a different scheme of calculating transition amplitudes. One can easily change the vertex-contractor, one can change the scheme of choosing the edge-projectors. In fact we have already introduced a model which is more general, than the ordinary EPRL, i.e. we considered spin-foams with some of edges and vertices colored in BF and the other in EPRL manner.

Using this feature of Operator Spin-network Diagrams (and having calculated a boundary problem) one can easily test other then EPRL models of spin-foam transition amplitude by comparing the physical predictions obtained for various models.

#### Introducing matter fields

The convenience of changing the prescription to calculate the transition amplitude in Operator Spin-network Diagrams framework has another advantage. It makes a good opportunity to extend the model by introducing matter fields.

The kinematic structure of Loop Quantum Gravity with matter was studied in [1, chapter 12]. It leads to spin-network states enhanced by an extra coloring, with possibly more general gauge group. As we have mentioned in subsection 3.5.2, there is an easy way to generalize the Operator Spin-network Diagrams to any gauge group. An analogous generalisation would work for spin-networks enhanced by scalar or spinor fields. The only difficulty is to find an appropriate dynamics, which is encoded in three objects: the face amplitude, the edge projector and the vertex contractor.

There are two ways to find this dynamics. One can derive a discretisation of the action of gravity and matter, and then adopt it to Spin-foam formalism (and thus to Operator Spin-network Diagrams) ore one can postulate some kind of vertex amplitude and then verify its semiclassical limit. Operator Spin-foam models are suitable for both approaches.

Part IV Appendixes

# Appendix A Mathematical introduction

This appendix is devoted to a brief review of mathematical concepts being the base for the topics presented in this thesis. The issues presented here are organised in 4 sections. In Appendix A.1 we present the theory of CW-complexes emphasising graphs and 2-CW-complexes. In Appendix A.2 we discuss the issues of differential geometry. In Appendix A.3 and Appendix A.4 we present the representation theory on SU(2) and  $SL(2, \mathbb{C})$  groups respectively.

This appendix should not be treated as a detailed study of the subjects mentioned above. Most of notions recalled here are commonly known mathematical objects. However, since various conventions of notations, naming and even of some details in definitions are used, here we fix these conventions and state explicitly the ambiguous details. Moreover some more sophisticated objects are introduced.

The sections are organised in collections of definitions. Each definition contains an exact definition of the object, but also some helper definitions and naming conventions, sometimes also it may contain some basic facts abut the defined object. In some cases the definitions are followed by examples, usually when the object was assessed by the Author to be nontrivial. Only several theorems and claims are stated explicitly, with a proof.

This appendix is based on classical textbooks and lecture notes. Each section in its introduction has references to the textbooks on which it is based. However, the conventions used by the Author are only based on the textbooks and in some cases they differ markedly. Moreover some definitions recalled are not present in cited textbooks these are either Author's own definitions, or definitions of the objects that Author judges that they are trivial. Some notions, like the *topological space* etc., are also used without defining them - they are also considered trivial.

#### A.1 Topology of CW-complexes

Graphs (both abstract and embedded) are key topological structure underlying Loop Quantum Gravity. They are 1-dimmensional case of CW-complexes, however, they can be considered without all tools from the full theory. First subsection of this section is devoted to them.

A trace of a time-evolution of a graph is a 2-dimmensional structure, namely 2-CWcomplex, and it is a topological structure on which the Spin-Foam models are build. In the second subsection of this section we define general *n*-dimensional CW-complexes and operations on them. Most of the facts recalled for graphs are either the common knowledge at the level of graduate student of physics (and thus the we find pointless to refer to a particular textbook) or thy are Author's own definitions (thus they cannot be found in the textbooks yet). The subsection about CW-complexes was based on the chapter 0 of [82]. Some of the new definitions introduced by the Author can be found in [53] and [64].

#### A.1.1 Graphs

In this subsection we define graphs and some operations on them. We introduce a few useful notions that help to describe the structure of nodes of the graphs (half-links, squids) and we discuss the group of automorphisms of graphs. Moreover we define the special class of graphs, called  $\theta$ -like graphs, that are very often used in the thesis, and we define the graphs with enhanced structure, called *squid-graphs*, that simplify some operations and proofs in chapter 3 and chapter 4.

#### A.1.1.1 Basic definitions

#### **Definition A.1.** Graph

A graph  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$  is a set of nodes  $\mathbf{N} = \{n_1, \dots, n_N\}$  and a set of links  $\mathbf{L} = \{\ell_1, \dots, \ell_L\}.$ 

Unless specified else, all graphs are oriented, which means that each link has its source node  $n = s(\ell)$  and its target node  $n' = t(\ell)$ . Some links might be loops, i.e. such links, that  $s(\ell) = t(\ell)$ .

The unoriented graph is denoted by  $|\mathcal{G}|$  and is a class of all graphs, that differ only by some changes of orientation of their links.

In general graphs are not required to be connected. However for each graph one can define its decomposition into connected components. We will write  $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_K\}$  if  $\mathcal{G} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_K$  and each of graphs  $\Gamma_i$  is connected itself. General graphs will be denoted by  $\mathcal{G}$  letter, connected graphs will be denoted by  $\Gamma$ .

A closed graph is a graph in which each node has at least two links incident to it. Unless specified else, all graphs are assumed to be closed.

Let us define a half-link of a graph.

#### **Definition A.2.** Half-link

A half-link  $\ell^{\epsilon}$  is a pair of a link  $\ell \in \mathbf{L}$  and a sign  $\epsilon \in \{+, -\}$ . A half-link is incident to only one node, called its boundary  $n = \partial \ell^{\epsilon}$ . The boundary of positive half-link  $\ell^{+}$  is the source node of  $\ell$ , the boundary of the negative half-link  $\ell^{-}$  is the target node of  $\ell$ :

$$\partial \ell^+ = s(\ell) \qquad \partial \ell^- = t(\ell)$$
 (A.1)

It is convenient to define the set of all half-links incident to n:

$$\mathcal{L}_n := \{ l^\epsilon : \partial \ell^\epsilon = n \}$$
(A.2)

Obviously if a link  $\ell$  is a loop, it appears in the set  $\mathcal{L}_n$  twice: once as a positive, and once as a negative half-link. It is also natural to define the sets of positive and negative half-links at n:

$$\mathcal{L}_{n}^{+/-} := \left\{ l^{+/-} : \partial \ell^{+/-} = n \right\}$$
(A.3)

The set of all half-links of the graph is  $\mathcal{L}_{\mathcal{G}} = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n$ . An example is shown at figure A.1b.



Figure A.1: Graphs. (a) - A graph  $\mathcal{G} = \{\Gamma_1, \Gamma_2\}$ . The source of the link  $\ell_1$  is  $s(\ell_1) = n_I$ , the target is  $t(\ell_1) = n_{II}$ . (b) - Half-links of the node  $n_{III}$  are highlighted (positive by red, negative by blue). (c) - The nodes  $n_{IV}$  and  $n_V$  are dual. The dotted line describe a duality map between them.

The notion of half-links makes the following definition very simple:

#### **Definition A.3.** Duality of nodes

Let n and n' be nodes of a graph. We say they are dual iff the number of positive half-links of n equals to number of negative half-links of n' and vice versa: number of negative half-links of n equals to number of positive half-links of n'.

It is also convenient to define a *duality map*:

#### **Definition A.4.** Duality map

Given two nodes n and n' consider the sets of half-links incident to them:  $\mathcal{L}_n$  and  $\mathcal{L}_{n'}$  respectively. We say that a map  $\phi_{n,n'} : \mathcal{L}_n \to \mathcal{L}_{n'}$  is a duality map between n and n' iff it is bijective and it maps  $\mathcal{L}_n^+$  onto  $\mathcal{L}_{n'}^-$  and  $\mathcal{L}_n^-$  onto  $\mathcal{L}_{n'}^+$ .

Obviously if n and n' are dual, there might be many duality maps between them. However the following lemma is true: if there is at least one duality map between n and n', the nodes are dual.

Illustration of notion of duality is presented at figure A.1c.

#### A.1.1.2 Maps of graphs

We will consider two types of maps between graphs: *morphisms*, i.e. maps that preserve the structure of the graphs, and the maps that does not have to preserve the graph's structure - we will call them *half-link-maps*.

#### Morphisms of graphs

#### **Definition A.5.** Morphism of graphs

Given two graphs  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$  and  $\mathcal{G}' = (\mathbf{N}', \mathbf{L}')$  a morphism of graphs  $\phi : \mathcal{G} \to \mathcal{G}'$  is a pair of maps  $\phi^{\mathbf{N}} : \mathbf{N} \to \mathbf{N}'$  and  $\phi^{\mathbf{L}} : \mathbf{L} \to \mathbf{L}'$  such that for each  $\ell \in \mathbf{L}$  if  $n = s(\ell)$  and  $n' = t(\ell)$ , we have  $\phi^{\mathbf{N}}(n) = s(\phi^{\mathbf{L}}(\ell))$  and  $\phi^{\mathbf{N}}(n') = t(\phi^{\mathbf{L}}(\ell))$ . A morphism  $\phi = (\phi^{\mathbf{N}}, \phi^{\mathbf{L}})$  is called node-surjective if  $\phi^{\mathbf{N}}$  is a surjective map, and it is

A morphism  $\phi = (\phi^{\mathbf{N}}, \phi^{\mathbf{L}})$  is called node-surjective if  $\phi^{\mathbf{N}}$  is a surjective map, and it is called link-surjective if  $\phi^{\mathbf{L}}$  is a surjective map. Similarly it is called node- or link-injective if  $\phi^{\mathbf{N}}$  or  $\phi^{\mathbf{L}}$  are injective maps respectively.

A morphism that is both node- and link-surjective is a graph-epimorphism, and is called graph-surjection. A morphism that is both node- and link-injective is a graph-monomorphism and is called graph-injection.

A map  $\phi: \mathcal{G} \to \mathcal{G}', \phi = (\phi^{\mathbf{N}}, \phi^{\mathbf{L}})$  such that for each  $\ell \in \mathbf{L}$  if  $n = s(\ell)$  and  $n' = t(\ell)$ , we have **either**  $\phi^{\mathbf{N}}(n) = s(\phi^{\mathbf{L}}(\ell))$  and  $\phi^{\mathbf{N}}(n') = t(\phi^{\mathbf{L}}(\ell))$  or  $\phi^{\mathbf{N}}(n) = s(\phi^{\mathbf{L}}(\ell))$  and  $\phi^{\mathbf{N}}(n') = t(\phi^{\mathbf{L}}(\ell))$  is called an unoriented-morphism of graphs. The classification of morphisms presented above extends straightforwardly to unoriented-morphisms.

The special case of morphisms are the automorphisms:

#### **Definition A.6.** Automorphism of a graph

Given a graph  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$  a morphism  $\phi : \mathcal{G} \to \mathcal{G}$  that is both graph-surjection and graph-injection is called an automorphism of  $\mathcal{G}$ . All the automorphisms of  $\mathcal{G}$  form a group Aut  $(\mathcal{G})$ .

Usually we are more interested in unoriented automorphisms of  $\mathcal{G}$ 

#### **Definition A.7.** Unoriented automorphism of a graph

Given two graphs  $\mathcal{G}$  and  $\mathcal{G}'$  such that  $|\mathcal{G}| = |\mathcal{G}'|$  a morphism  $\phi : \mathcal{G} \to \mathcal{G}'$  that is both graph-surjection and graph-injection is called an unoriented automorphism of  $\mathcal{G}$  or an automorphism of  $|\mathcal{G}|$ . All the unoriented automorphisms of  $\mathcal{G}$  form a group Aut  $(|\mathcal{G}|)$ .

There is a normal subgroup of  $\operatorname{Aut}(|\mathcal{G}|) \triangleright \operatorname{Reo}(\mathcal{G})$  of all unoriented automorphisms of  $\mathcal{G}$  such that both  $\phi^{\mathbf{N}}$  and  $\phi^{\mathbf{L}}$  are the identity maps. The elements of  $\operatorname{Reo}(\mathcal{G})$  are called reorientations of  $\mathcal{G}$ . Each reorientation can be characterised a subset of links for which it inverts orientations (in fact). One can show, that  $\operatorname{Aut}(|\mathcal{G}|)/\operatorname{Reo}(\mathcal{G}) = \operatorname{Aut}(\mathcal{G})$ .

There is a special element of  $\text{Reo}(\mathcal{G}) \ni \epsilon$  called the inversion of the graph. It is defined by inverting all the links of  $\mathcal{G}$ . The inverted graph is denoted as  $\mathcal{G}^* := \epsilon(\mathcal{G})$ .

It is worth to notice, that in case of oriented automorphisms each pair of bijections  $\phi^{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$  and  $\phi^{\mathbf{L}} : \mathbf{L} \to \mathbf{L}$  that is consistent with the graph structure of  $\mathcal{G}$  determines fully the automorphism. It is not true in case of unoriented automorphism - one need an extra information of the structure of  $\mathcal{G}'$  (which is in fact encoded in the Reo ( $\mathcal{G}$ )-part of the automorphism).

Whenever considering colored graph one can introduce color-preserving morphisms by introducing requirement, that for each object (node, link) the image with respect to the color-preserving morphism is colored by the same value, as the argument. Obviously color-preserving automorphisms form a subgroup of all automorphisms.

Half-link maps Consider now maps that do not preserve the structure of the graph, but they do preserve the structure of each node. We will call them the half-link maps.

#### **Definition A.8.** Half-link map

Given two graphs  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$  and  $\mathcal{G}' = (\mathbf{N}', \mathbf{L}')$  a half-link map between them  $\phi : \mathcal{G} \to \mathcal{G}'$  is a map  $\phi^{\mathbf{N}} : \mathbf{N} \to \mathbf{N}'$  and a map  $\phi^{\mathcal{L}} : \mathcal{L}_{\mathcal{G}} \to \mathcal{L}_{\mathcal{G}'}$  such that for each half-link  $\ell^{\epsilon} \in \mathcal{L}_{\mathcal{G}}$  if  $n = \partial \ell^{\epsilon}$  then  $\phi^{\mathbf{N}}(n) = \partial (\phi^{\mathcal{L}}(\ell^{\epsilon}))$ .

Obviously each morphism of graphs define half-link map, but not the opposite - a half-link map may change the graph structure.

Likewise the morphisms, half-link maps may be node-surjective, half-link-surjective, node-injective, half-link-injective, graph-surjective and graph-injective.

A half-link map is orientation-preserving iff  $\phi^{\mathcal{L}}(\mathcal{L}_{\mathcal{G}}^+) \subset \mathcal{L}_{\mathcal{G}'}^+$  and  $\phi^{\mathcal{L}}(\mathcal{L}_{\mathcal{G}}^-) \subset \mathcal{L}_{\mathcal{G}'}^-$ . A half-link map is orientation-consistent if in the image of  $\phi^{\mathcal{L}}$  the number of positive half-links is equal to the number of negative half-links. It might happen that a half-link map
is orientation preserving but not orientation-consistent or orientation-consistent, but not orientation-preserving. However each half-link-injective map and each half-link-surjective map must be orientation-consistent.

If  $\mathcal{G} = \mathcal{G}'$  we may define half-link-automorphisms:

# **Definition A.9.** Half-link-automorphism

A half-link map  $\phi : \mathcal{G} \to \mathcal{G}$  that is both graph-surjective and graph-injective is called a half-link-automorphism. Obviously a graph-automorphism induces a half-link automorphism, but there might be half-link map that violates the graph-structure. Half-link automorphisms form a group  $\operatorname{Aut}^{\pm}(\mathcal{G})$ .

Obviously each half-link automorphism is orientation-consistent. They may be also orientation-preserving (at least the identity map is).

A special case of half-link-automorphisms are so called duality-maps of the graph  $\mathcal{G}$ . A half-link-automorphism is a duality-map of the graph if the for each node  $n \in \mathbf{N}$  the map  $\phi^{\mathcal{L}}$  restricted to  $\mathcal{L}_n$  is the duality map of the nodes n and  $\phi^{\mathbf{N}}(n)$ .

The half-link maps, especially the duality maps, are used mainly in the gluing of 2-complexes and gluing of graph-diagrams.

#### A.1.1.3 $\theta$ -like graphs

Let us define a class of graphs of great importance in the construction of boundary formalism for the Operator Spin-network Diagrams, namely the  $\theta$ -like graphs. First let us define a  $\theta$ -graph:

# **Definition A.10.** $\theta$ -graph

A  $\theta$ -graph is a graph  $\Gamma_{\theta} = (\mathbf{N}, \mathbf{L})$  such that  $\mathbf{N} = \{n, n'\}$  is a 2-element set, and  $\mathbf{L} = \{\ell_1, \ldots, \ell_L\}$  is at least 2-element set, and for each link  $\ell_i$  we have either  $s(\ell_i) = n$  and  $t(\ell_i) = n'$  or the opposite, i.e.  $s(\ell_i) = n'$  and  $t(\ell_i) = n$ . The number L is called the valency of the  $\theta$ -graph. The nodes of the  $\theta$ -graph are called the poles. An example can be found at figure A.2a.

The name of the  $\theta$ -graphs comes from the fact, that the letter  $\theta$  is a 3-valent  $\theta$ -graph. The basic  $\theta$ -graphs are not used a lot in this thesis. More common are simple  $\theta$ -like graphs:

#### **Definition A.11.** Simple $\theta$ -like graph

A simple  $\theta$ -like graph is a graph  $\Gamma_{\theta} = (\mathbf{N}, \mathbf{L})$  such that  $\mathbf{N} = \mathbf{N}^p \cup \mathbf{X}$  where  $\mathbf{N}^p = \{n, n'\}$  and  $\mathbf{X} = \{x_1, \dots, x_L\}$  and  $\mathbf{L} = \mathbf{L}^n \cup \mathbf{L}^{n'}$  where  $\mathbf{L}^n = \{\ell_1^n, \dots, \ell_L^n\}$  and  $\mathbf{L}^{n'} = \{\ell_1^{n'}, \dots, \ell_L^{n'}\}$  (for  $L \geq 2$ , of course L is the same number everywhere), and the following conditions for the links are satisfied:

- for each  $\ell_i^n \in \mathbf{L}^n$  we have either  $s(\ell_i^n) = n$  and  $t(\ell_i^n) = x_i$  or the opposite, i.e.  $s(\ell_i^n) = x_i$  and  $t(\ell_i^n) = n$
- for each  $\ell_i^{n'} \in \mathbf{L}^{n'}$  we have either  $s\left(\ell_i^{n'}\right) = n'$  and  $t\left(\ell_i^{n'}\right) = x_i$  or the opposite, i.e.  $s\left(\ell_i^{n'}\right) = x_i$  and  $t\left(\ell_i^{n'}\right) = n'$

Obviously each node  $x_i$  is bivalent.

We say, that a link  $\ell_i^n$  (or  $\ell_j^{n'}$ ) is positive if it is outgoing from n (or n' respectively). The nodes n and n' are called the poles, the nodes  $x_i$  are called the link-nodes.

A simple  $\theta$ -like graph is called simply oriented iff each node  $x_i$  has one incoming and one outgoing link. Otherwise it is a generally oriented simple  $\theta$ -like graph (see figure A.2b).

One can easily see that a simple  $\theta$ -like graph is a  $\theta$ -graph with one node  $x_i$  inserted at each link  $\ell_i$  (and thus each link divided into two links  $\ell_i^n$  and  $\ell_i^{n'}$ ). One can generalise this by placing an arbitrary number of nodes at each link:

#### **Definition A.12.** General $\theta$ -like graph

A L-valent general  $\theta$ -like graph (for  $L \geq 2$ ) is a graph  $\Gamma = (\mathbf{N}, \mathbf{L})$  such that  $\mathbf{N} = \mathbf{N}^p \cup \mathbf{X}_1 \cup \cdots \cup \mathbf{X}_L$ , where  $\mathbf{N}^p = \{n, n'\}$  and each of  $(\mathbf{X}^i)_{i=1,...,L}$  sets is  $\mathbf{X}_i = \{x_i^1, \ldots, x_i^{k_i}\}$  (the numbers  $k_i$  might be 0, and then the corresponding  $\mathbf{X}_i$  is the empty set) and  $\mathbf{L} = \mathbf{L}^i \cup \ldots \cup \mathbf{L}^L$  where each of  $(\mathbf{L}^i)_{i=1,...,L}$  sets is  $\mathbf{L}_i = \{\ell_i^0, \ldots, \ell_i^{k_i}\}$ (they cannot be empty), and and the following conditions for the links are satisfied:

- for each l<sub>i</sub><sup>0</sup> we have we have either s (l<sub>i</sub><sup>0</sup>) = n and t(l<sub>i</sub><sup>0</sup>) = x<sub>i</sub><sup>1</sup> or the opposite,
   i.e. s (l<sub>i</sub><sup>0</sup>) = x<sub>i</sub><sup>1</sup> and t(l<sub>i</sub><sup>0</sup>) = n
- for each  $\ell_i^{k_i}$  we have we have either  $s\left(\ell_i^{k_i}\right) = n'$  and  $t\left(\ell_i^{k_i}\right) = x_i^{k_i}$  or the opposite, i.e.  $s\left(\ell_i^{k_i}\right) = x_i^{k_i}$  and  $t\left(\ell_i^{k_i}\right) = n'$
- for each  $\ell_i^j$  where  $j = 1, \dots, k_i 1$  we have we have either  $s\left(\ell_i^j\right) = x_i^j$  and  $t\left(\ell_i^j\right) = x_i^{j+1}$  or the opposite, i.e.  $s\left(\ell_i^j\right) = x_i^{j+1}$  and  $t\left(\ell_i^j\right) = x_i^j$

Obviously each node  $x_i^j$  is bivalent.

A general  $\theta$ -like graph is called simply oriented iff each node  $x_i$  has one incoming and one outgoing link. Otherwise it is a generally oriented  $\theta$ -like graph. An example can be found at figure A.2c.

In the main part of the thesis we sometimes use the same name  $\theta$ -like graphs for both simple- and general  $\theta$ -like graphs.

# A.1.1.4 Squid-graphs

If one combine the notions of half-links and duality of nodes with the notion of graphs, a new convenient object arises: the squid graphs. They were first introduced in [53] and improved in [64].

### Definition A.13. Squid

A squid is an open graph  $\lambda = (\mathbf{N}_{\lambda}, \mathbf{L}_{\lambda})$  with  $\mathbf{N}_{\lambda} = \{n\} \cup \{x_1, \ldots, x_k\}$  and  $\mathbf{L}_{\lambda} = \{\ell_1, \ldots, \ell_k\}$  such that each link  $\ell_i$  connects the node n and the node  $x_i$ . Example of a squid is on figure A.3a.

The node n is called a head of the squid. The links  $\ell_i$  are called legs of the squid, the nodes  $x_i$  are called leg-nodes.

A leg  $\ell_i$  is called outgoing iff  $s(\ell_i) = n \wedge t(\ell_i) = x_i$ . A leg is called incoming iff  $s(\ell_i) = x_i \wedge t(\ell_i) = n$ .

Valence of a squid is number of its legs k. One can also introduce the positive valence  $k^+$  being number of outgoing legs, and negative valence  $k^-$  being the number of incoming legs.



Figure A.2: Theta-like graphs. (a) - A  $\theta$ -graph. (b) - A simple  $\theta$ -like graph. (c) - A general  $\theta$ -like graph.

Given a graph, for each node one can introduce a squid capturing structure of the node. Indeed, a sufficiently small neighbourhood of each node has form of a squid. Duality of nodes can be naturally generalised to duality of squids. If two nodes are dual, then the squids corresponding to them are also dual. The set of squids related to all nodes of a graph  $\mathcal{G}$  we shall call a *squid-set* of this graph and denote as  $S_{\mathcal{G}}$ .

Note, that every half-link of a graph can be uniquely identified with a leg of one of squids in the squid-set of this graph.

It appears that it is convenient to introduce a so called squid graph  $\gamma$  of a graph  $\mathcal{G}$ , being a graph enhanced with an extra structure:

# Definition A.14. Squid-graph

A squid-graph of a graph  $\mathcal{G}$  is a pair of a graph and a set of squids  $(\mathcal{G}^{(s)}, S)$ , where

- $\mathcal{G}^{(s)}$  (called the split graph) is a graph obtained by splitting each link  $\ell_i$  of  $\mathcal{G}$  into two links  $\ell_i^s$  and  $\ell_i^t$  by putting an extra node  $x_{\ell_i}$  in the interior of each link (the link  $\ell_i^s$  starts at  $s(\ell_i)$  and ends at  $x_{\ell_i}$ , the link  $\ell_i^t$  starts at  $x_{\ell_i}$  and ends at  $t(\ell_i)$ ).
- S is the squid-set of the graph  $\mathcal{G}$ , i.e. for each node n of the graph  $\mathcal{G}$  there is one squid  $\lambda_n \in S$ , and the legs of  $\lambda_n$  are links of  $\mathcal{G}^{(s)}$  incident to n (in fact they are halves of links of  $\mathcal{G}$ )

A procedure of obtaining a squid-graph of a graph is illustrated on figure A.3.

**Remark A.1.** Later we will need to know the precise graph structure of  $\mathcal{G}^{(s)}$ . Let  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$ , then

$$\mathcal{G}^{(s)} = (\mathbf{N} \cup \mathbf{X}, \mathbf{L}^+ \cup \mathbf{L}^-) \tag{A.4}$$

where  $\mathbf{X} = \{x_{\ell_i} \in \ell_i : \ell_i \in \mathbf{L}\}$  is the set of middle points of the links of  $\mathcal{G}$ , and the sets  $\mathbf{L}^{+/-}$  contain the links of  $\mathcal{G}^{(s)}$  representing the positive/negative half-links of  $\mathcal{G}$  respectively, i.e.  $\mathbf{L}^+ = \{\ell_i^s : \ell_i \in \mathbf{L}\}$  and  $\mathbf{L}^- = \{\ell_i^t : \ell_i \in \mathbf{L}\}$ .



Figure A.3: Squid-graphs. (a) - An example of 6-valent squid, with  $k^+ = 4$  and  $k^- = 2$ . (b) - A graph  $\Gamma$  that we will turn into a squid-graph. (c) - The split graph of the graph  $\Gamma^{(s)}$ . One of the squids, i.e. the squid  $\lambda_n$  of the node n of  $\Gamma$ , is emphasised.

# A.1.2 CW-complexes

The 2-complexes are widely used in the thesis, as the topological structure underlying the spin-foam models. They are the special case of more general concept of CW-complexes with maximal dimension of cells set to 2. We shell use the definition of CW-complexes consistent with [82]. It can be also found in the appendix of [53].

# A.1.2.1 Basic definitions

# **Definition A.15.** CW-complex

The n-dimensional CW-complex is a collection of sets of k-dimensional cells  $C_k$  where k = 0, 1, ..., n together with set of maps  $f_j$  for j = 1, 2, ..., n and relations  $\sim_l$  defined as follows.

- Each cell  $c \in C_k$  is a k-dimensional disc with a boundary. For each set of cells we define its boundary as  $\partial C_k = \bigsqcup_{c \in C_k} \partial c$ .
- Each relation  $\sim_l$  is defined on the set  $\partial C_l$ .
- The relation  $\sim_0$  is the empty relation.
- Each function function  $f_m$  is a continuous, map

$$f_m: \partial \mathcal{C}_m \to \mathcal{C}_{m-1}/\sim_{m-1} \tag{A.5}$$

• For m > 0 the relations are defined by the functions  $f_m$ :

$$x \sim_m y \Leftrightarrow f_m(x) = f_m(y)$$
 (A.6)

The functions  $f_m$  are called the boundary functions. They define the way in which higher dimension cells are glued onto the lower-dimensional skeleton. The relations says,

which points are glued together. Note, that since  $\sim_0$  is empty (indeed,  $\sim_0$  is a relation on  $\partial C_0 = \emptyset$ ), it can be omitted in the construction. Then any other relation is inductively constructed from the functions  $f_m$ . Thus what is essential in the construction of a CW-complex are the boundary functions, not the relations (however, they are useful in geometrical interpretation). So we will denote a CW-complex as

$$\kappa = (\mathcal{C}_n, \dots, \mathcal{C}_0; f_n, \dots, f_1) \tag{A.7}$$

The above definition is too wide to our purpose. Almost everywhere in the thesis we require CW-complexes to not posses any cells of non-maximal dimensions, that are not subcells of higher dimension cells. In order to ensure it let us thus introduce an auxiliary definition:

#### **Definition A.16.** Surjective CW-complex

A CW-complex such that each of its boundary functions  $f_m$  is a surjective map  $f_m : \partial \mathcal{C}_m \to \mathcal{C}_{m-1} / \sim_{m-1} is$  called a surjective CW-complex.

The following claim is obvious:

Claim A.1. In a *n*-dimensional surjective CW-complex each *k*-cell (for k < n) lies on a boundary of at least one (k + 1)-cell.

thus the surjective CW-complexes are a good class of CW-complexes for our further considerations.

**Remark A.2.** Whenever in this thesis we refer to a *CW-complex*, we mean a *surjective CW-complex*. If we want to refer to a non-necessarily surjective CW-complex, we call it a *general CW-complex*.

As a topological space the CW-complex is  $X_{\kappa} := \mathcal{C}_n / \sim_n$ . This leads to a notion of cellular decomposition:

#### **Definition A.17.** Cellular decomposition

Given a topological space X and a CW-complex  $\kappa$  such that  $X_{\kappa}$  is homeomorphic with X we call  $\kappa$  the cellular decomposition of X.

We say, that the cellular decomposition is n-dimensional iff it is given by a surjective n-CW-complex.

CW-complex contains more information than just its topological space. This additional information is contained in a set of *skeletons*:

#### Definition A.18. Skeleton

For  $k \leq n$  a k-dimensional skeleton  $\kappa^{(k)}$  of an n-dimensional CW-complex  $\kappa = (\mathcal{C}_n, \ldots, \mathcal{C}_0; f_n, \ldots, f_1)$  is a subcomplex of  $\kappa$  defined by omitting all objects of dimension higher then k:

$$\kappa^{(k)} = (\mathcal{C}_k, \dots, \mathcal{C}_0; f_k, \dots, f_1)$$
(A.8)

The skeletons allows us to define strong equivalence of CW-complexes:

### **Definition A.19.** Complexomorphism

A complexomorphism is a map between CW-complexes preserving the CW-complex structure.

Given two complexes  $\kappa_1$  and  $\kappa_2$  and a map  $\phi : X_{\kappa_1} \to X_{\kappa_2}$  we say, that  $\phi$  is a complexomorphism iff  $\phi$  is a homeomorphism of  $X_{\kappa_1}$  and  $X_{\kappa_2}$  and for each k-skeleton of  $\kappa$  the truncated map  $\phi|_{X_{\kappa_2}^{(k)}}$  is a homeomorphism of the skeletons  $X_{\kappa_1^{(k)}}$  and  $X_{\kappa_2^{(k)}}$ .

We say that two CW-complexes are *strongly equivalent* iff there is a complexomorphism between them.

In contrary, the *weak equivalence* does not require equivalence of skeletons:

# Definition A.20. Weak equivalence of CW-complexes

We say that two CW-complexes  $\kappa_1$  and  $\kappa_2$  are weekly equivalent iff their topological spaces  $X_{\kappa_1}$  and  $X_{\kappa_2}$  are homeomorphic (i.e. they are cellular decompositions of the same topological space).

In practice weak equivalence means that the CW-complexes differ by a number of removable cells - what shell be clarified in Appendix A.1.2.3.

Given a CW-complex, its cells naturally inherits the structure of CW-complex:

# **Definition A.21.** Induced complex

Given a k-cell c of a n-CW-complex  $\kappa$  the induced complex of c is the k-CW-complex  $\kappa_c$  being the intersection of the k-skeleton of  $\kappa$  with c:

$$\kappa_c := \kappa^{(k)} \cap c \tag{A.9}$$

To be precise let us spell out the ingredients of  $\kappa_c$ :

$$\kappa_c := \left( \tilde{\mathcal{C}}_k, \dots, \tilde{\mathcal{C}}_0; \ f_k|_{\partial \tilde{\mathcal{C}}_k}, \dots, \ f_1|_{\partial \tilde{\mathcal{C}}_1} \right)$$
(A.10)

where  $\tilde{\mathcal{C}}_k := \{c\}$ , and for l < k we have  $\tilde{\mathcal{C}}_l := \mathcal{C}_l \cap f_{l+1}\left(\partial \tilde{\mathcal{C}}_{l+1}\right)$ .

In case of 2-CW-complexes the 2-cells will be called *faces*  $f \in \mathbf{F}$ , the 1-cells will be called *edges*  $e \in \mathbf{E}$ , and the 0-cells will be called *vertices*  $v \in \mathbf{V}$ , and thus we will introduce a notation

$$\kappa = (\mathbf{F}, \mathbf{E}, \mathbf{V}; f_2, f_1) \tag{A.11}$$

Graphs are 1-CW-complexes (although usually it is inconvenient to use the full CWcomplex notation to describe them). As we noted already in Appendix A.1.1, the 1-cells of graphs are called *links* and the 0-cells of graphs are called *nodes*.

# A.1.2.2 Relations of cells

Now let us introduce some notions to describe the relative position of cells in CWcomplexes.

#### **Definition A.22.** Incidence

Given a cell  $c \in C_k$  we say, that a bigger cell  $c' \in C_{k+1}$  is incident to c iff  $c \subset f_{k+1}(\partial c')$ . Moreover if we have a cell  $c'' \in C_{k+l}$  such that there is a sequence of cells  $c_1 \in C_{k+1}, \ldots, c_{l-1} \in C_{k+l-1}$  such that each  $c_i$  is incident to  $c_{i-1}$ , and  $c_1$  is incident to c, and c'' is incident to  $c_{k+l-1}$ , then c'' is also called incident to c.

We want incidence to be a symmetric relation, thus if c is incident to c', then by definition c' is incident to c.

If  $c \in C_k$  and  $c' \in C_l$  are incident, and l < k, we say that c' is one of the boundary cells of the cell c - this is because c' has its counterimage with respect to  $f_{l+1} \circ f_{l+2} \circ \cdots \circ f_k$  intersecting with  $\partial c$ , i.e.  $(f_k^{-1} \circ f_{k-1}^{-1} \circ \cdots \circ f_{l+1}^{-1}(c')) \cap \partial c \neq \emptyset$ . We denote it shortly as  $c' \subset \partial c$ .

A higher dimensional cell c may be glued onto a lower dimensional cell c' several times. This takes reflection is the number of copies of counterimage of c' in  $\partial c$ . If a projection from  $(f_k^{-1} \circ f_{k-1}^{-1} \circ \cdots \circ f_{l+1}^{-1}(c'))$  to c' is a bijective map, we say that c' is single-incident to c, otherwise for c' incident to c we say that c' is multi-incident to c.



Figure A.4: Adjacency. The cells  $f_1$  and  $f_2$  are adjacent, because they share a 1-cell  $e_I$ . The cells  $f_1$  and  $f_3$  are not adjacent although they share a 0-cell V.

Incidence is a relation between cells of different dimension. In case of cells of the same dimension we say about adjacency:

#### **Definition A.23.** Adjacency

Two k-cells  $c, c' \in C_k$  are called adjacent iff there is a (k-1)-cell c'' such, that c'' is incident to c and c'' is incident to c'.

Thus for example two faces need to share an edge to be called adjacent. If they share only a vertex - they have nontrivial intersection, but they are not adjacent in our terminology (see figure A.4).

Let us now define the boundary of a CW-complex.

Definition A.24. Boundary cell of a CW-complex

- A (n-1)-cell of a n-CW-comples κ is called a boundary cell of κ iff it is singleincident to precisely one n-cell of κ
- A cell of dimension lower then n-1 is called a boundary cell of  $\kappa$  iff it is incident to at least one boundary (n-1)-cell.

A (n-1)-subcomplex of  $\kappa$  build of all the boundary cells is called the boundary complex of  $\kappa$  and denoted by  $\partial \kappa$ .

A CW-complex with the empty boundary is called closed.

The non-boundary cells of  $\kappa$  are called interior cells of  $\kappa$ .

Obviously the boundary of a 2-CW-complex is a graph.

It is important not to confuse two meanings of the notion *boundary*. The boundary cells of the interior cells does not have to be the boundary cells of the whole CW-complex. For each *n*-dimensional CW-complex  $\kappa$  the following two claims are true:

- If c is a boundary cell of at least one cell c' such that c' is a boundary cell of  $\kappa$ , then c is a boundary cell of  $\kappa$ .
- If c is a boundary cell of  $\kappa$ , then it is a boundary cell of at least one cell c' being the boundary cell of  $\kappa$ , **unless** c is a (n-1)-dimensional cell (then there is no c' boundary cell of  $\kappa$  such that  $c \subset \partial c'$ ).

# A.1.2.3 Removing cells

There are some cases, when removing cells do not change the topology of  $X_{\kappa}$ . In particular it happens, when the cell has the following property:

#### **Definition A.25.** Removable cell

A k-cell  $c \in C_k$  is called removable iff it is incident to **precisely two** (k + 1)-cells  $c', c'' \in C_{k+1}$  and none of the boundary cells  $c''' \subset \partial c$  is a removable cell (the definition is recursive, however, the recursion stops for 0-cells, which has no boundary).

In this case the complex with removed cell c is denoted by  $\kappa \setminus c$  and defined as follows:

- The sets of cells  $C_l$  for  $l \neq k$  and  $l \neq k+1$  are left unchanged. The set of k-cells equals  $C_k^{\text{new}} = C_k \setminus \{c\}$ . The set of (k+1)-cells has a new cell  $\tilde{c}$  instead of c' and c'', i.e.  $C_{k+1}^{\text{new}} = \{\tilde{c}\} \cup C_{k+1} \setminus \{c', c''\}$ .
- The new cell  $\tilde{c}$  is the cell c' glued to the cell c'' along the cell c. I.e. it is

$$\tilde{c} = \left(c' \cup c''\right) / \left(f_{k+1}|_{f_{k+1}^{-1}(c)}\right)$$
(A.12)

In other words: It is the sum of the sets representing c' and c'', but for each pair of points, that the map  $f_{k+1}$  maps on the same point of the cell c, the points are identified.

- The boundary maps for all the cells apart from c' and c'' do not change.
- We need to define the boundary map  $f_{k+1}^{\text{new}}$  for the new cell  $\tilde{c}$ . Since  $\partial \tilde{c} \subset \partial c' \cup \partial c''$ , the new boundary map is just the truncation of the old one:

$$f_{k+1}^{\text{new}}\Big|_{\partial \tilde{c}} = f_{k+1}\Big|_{\partial \tilde{c}} \tag{A.13}$$

It may happen, that after removing a removable cell, some cells become removable. Thus let us introduce one more definition:

### Definition A.26. Pre-removable cell

A k-cell  $c \in C_k$  is called pre-removable iff there exists a series of removal operations, after which it becomes removable (see figure A.5c for a graphical explanation).

The pre-removable cells are the cells that does not have any crucial input to the topology of  $X_{\kappa}$ , i.e. if there is a removable cell c in  $\kappa$ , then there is a complex  $\kappa'$  weekly equivalent to  $\kappa$  without the cell c. In the main text of the thesis we will often refer to both removable and pre-removable cells as to removable cells, because usually we will to all

the removings of the cells at once. However, since sometimes we use colored 2-complexes and the coloring of some parts of the skeleton is relevant, there are some situations, when the cells that are topologically removable will not be removed in order to not loose the coloring data stored at them. In other words in case of colored 2-complexes we say that the cell is removable if it is topologically pre-removable and if the coloring data stored at it has the default value that can be unambiguously deduced from the data stored at the other cells (where *default* usually means *default with respect to the chosen coloring scheme*).

# A.1.2.4 Orientation

One can easily enhance n-CW-complexes with an extra structure - orientation.

### **Definition A.27.** Oriented n-CW-complex

An oriented n-CW-complex is a n-CW-complex with an orientation assigned to each n-cell of the complex.

The orientation of lower dimensional cells is not defined a priori. One can consider orientation of k-cell c with respect a (k + 1)-cell c' to which c is incident - then the orientation of c is induced by the orientation of c'.

The boundary (n-1)-cells have uniquely determined orientation, because they are incident to precisely one n-cell. Thus the boundary complex of an oriented CW-complex is also an oriented CW-complex.

We say, that orientation of two adjacent k-cells  $c_1$ ,  $c_2$  agree, iff for each (k-1)-cell c' they share the orientation of c' induced by  $c_1$  is opposite than the orientation of c' induced by  $c_2$  (see figure A.5).

Given a *n*-cell *c* of an oriented *n*-CW-complex  $\kappa$ , the induced complex  $\kappa_c$  of the cell *c* is an oriented *n*-CW-complex with the orientation of *c* determined by its orientation in  $\kappa$ . Given a (n-1)-cell *c'* of  $\kappa$  being a boundary cell, the induced complex  $\kappa_{c'}$  is also oriented. Given any other *k*-cell *c''* for k < n there is in general no natural way to determine the orientation of *c''* in  $\kappa_{c''}$ , thus  $\kappa_{c''}$  is in general an unoriented CW-complex.

In case of oriented CW-complexes it is harder to remove cells. The updated definition is as follows:

#### **Definition A.28.** Removable cell of an oriented CW-complex

A k-cell c of an oriented CW-complex is removable iff it is incident to precisely two (k+1)-cells  $c', c'' \in \mathcal{C}_{k+1}$  and none of the boundary cells  $c''' \subset \partial c$  is a removable cell, and the orientation of c' and c'' agree.

Since the orientation of the cells with non-maximal dimension is defined only with respect to one of n-cells, the last condition must be satisfied for each possible choice of the cells determining the orientation.

The procedure of removing such cell is the same, as in definition A.25. Moreover, if the removed cell is of co-dimension 1, we have to determine the orientation of the new cell  $\tilde{c}$ . Since the orientation of c' and c'' agrees, we say the orientation of  $\tilde{c}$  to be in agreement with the orientation of c' and c'' (see figure A.5c).

# A.1.2.5 Special cases of CW-complexes

There are two special cases of CW-complexes of high importance for the rest of this theses: Squid-2-CW-complexes and  $\Delta$ -complexes. We define them below.



Figure A.5: Orientation of cells. (a) - A simple oriented 2-CW-complex. Orientations of faces are given by arrows. (b) - Induced orientation. The edge  $e_1$  can be oriented with respect to  $f_1$  (see the upper arrow) and with respect to  $f_2$  (the lower arrow). The edge  $e_2$  can be oriented with respect to  $f_2$  (the left arrow) and with respect to  $f_3$  (the right arrow). The boundary edges are oriented uniquely. (c) - Consistent orientation. The faces  $f_2$  and  $f_3$  are oriented consistently, because the orientations induced by them on  $e_2$  are opposite. Thus  $e_2$  is a removable cell. The edge  $e_1$  is not removable. The vertex V was not removable when the edge  $e_2$  was present, but now it is removable, thus the vertex V was pre-removable.

### Squid-2-CW-complexes

It may happen, that the boundary of a 2-CW-complex has form of a split graph  $\mathcal{G}^{(s)}$  of a squid-graph  $\gamma = (\mathcal{G}^{(s)}, S)$ . In such case we can introduce an extra structure on the boundary of this 2-complex:

# Definition A.29. Squid-2-CW-complex

A Squid-2-CW-complex is a pair  $(\kappa, S)$ , such that  $(\partial \kappa, S)$  is a squid-graph.

# $\Delta$ -complexes

In order to define  $\Delta$ -complexes let us first define a k-simplex:

# **Definition A.30.** *k*-simplex

A k-simplex is the subset of  $\mathbb{R}^k$  defined as

$$\Delta^{k} := \left\{ x \in \mathbb{R}^{k} : \left( \forall_{i=1,\dots,k} x^{i} \ge 0 \right) \land \left( \sum_{i=1}^{k} x^{i} \le 1 \right) \right\}$$
(A.14)

The boundary of a k-simplex is  $\partial \Delta^k = \bigcup_{I=0}^k \Delta_I^{k-1}$  where for  $I = 1, \ldots, k$  each set  $\Delta_I^{k-1}$  is defined as  $\Delta_I^{k-1} := \{x \in \Delta^k : x^I = 0\}$  and for I = 0 we have  $\Delta_0^{k-1} := \{x \in \Delta^k : (\sum_{i=1}^k x^i = 1)\}$ . Each of the sets  $\Delta_I^{k-1}$  can be affine transformed into a (k-1)-simplex, thus there is a natural decomposition of  $\partial \Delta^k$  into (k+1) simplexes  $\Delta^{k-1}$ . This decomposition continues recursively until k = 0, where 0-simplex is a point and has no boundary. This lets us introduce a structure of a k-CW-complex on each k-simplex.

# **Definition A.31.** $\Delta$ -complex

A n- $\Delta$ -complex  $\kappa^{\Delta}$  is a n-CW-complex such, that each k-dimensional cell of  $\kappa^{\Delta}$  is a k-simplex with the k-CW-complex structure introduced as above, and each of the boundary maps  $f_k$  is consistent with the k-simplex structures (i.e. given a k-cell  $c = \Delta^k$  each boundary cell of c introduced by k-simplex structure is mapped onto precisely one boundary cell of c with respect to  $\kappa^{\Delta}$ ).

A cellular decomposition into a  $\Delta$ -complex  $\kappa^{\Delta}$  is called triangulation.

# A.1.2.6 Duality of CW-complexes

In spin-foam theories one needs a notion of a CW-complex dual to a cellular decomposition of a topological space. Although it is hard to define such duality in general, there is no need to do so, because in this thesis it is used only in two cases: a 1-complex dual to a 3-dimensional cellular decomposition and a 2-complex dual to a 4-dimensional cellular decomposition.

Let us define it here. The definitions are very technical. We will start with defining 1and 2-complexes dual to 3- and 4-dimensional triangulations respectively, since in case of triangulations it is relatively easy to visualise all the procedure and to find the relations between properties of the cellular decomposition and resulting dual complex. Then we will discuss, how to apply the same procedure to more general cellular decompositions.

## Graph dual to 3-dimensional cellular decomposition

Consider a triangulation  ${}^{3}\Delta$  of a 3-dimensional topological space  $\Sigma$ . One can define a graph dual to the triangulation  $\mathcal{G} = {}^{3}\Delta^{\star}$  in the following way. Let  ${}^{3}\Delta = (\mathcal{C}^{3}, \mathcal{C}^{2}, \mathcal{C}^{1}, \mathcal{C}^{0}, f_{3}, f_{2}, f_{1})$ , then the graph  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$ , where  $\mathbf{N} = \mathbf{N}_{\text{int}} \cup \mathbf{N}_{\text{bound}}$  such that

- There is a 1-to-1 correspondence between 3-cells of  ${}^{3}\Delta$  and internal nodes of  $\mathcal{G}$ , i.e. for each tetrahedron  $\theta \in \mathcal{C}^{3}$  there is one and only one node  $n_{\theta} \in \mathbf{N}_{\text{int}}$ , and there are no other nodes in  $\mathbf{N}_{\text{int}}$ .
- There is a 1-to-1 correspondence between 2-cells of  ${}^{3}\Delta$  and links of  $\mathcal{G}$ , i.e. for each triangle  $t \in \mathcal{C}^{2}$  there is one and only one link  $\ell_{t} \in \mathbf{L}$ , and there are no other links.
- There is a 1-to-1 correspondence between boundary 2-cells of  ${}^{3}\Delta$  and boundary nodes of  $\mathcal{G}$ , i.e. for each triangle  $t \in \mathcal{C}^2 \cap \partial^{3}\Delta$  there is one and only one node  $n_t \in \mathbf{N}_{\text{bound}}$  and there are no other nodes in  $\mathbf{N}_{\text{bound}}$ .
- Whenever two tetrahedra  $\theta$ ,  $\theta'$  of  ${}^{3}\Delta$  are adjacent (i.e. share a triangle t), the link  $\ell_t$  of  $\mathcal{G}$  connects the nodes  $n_{\theta}$  and  $n_{\theta'}$  (one can pick an arbitrary orientation).
- For each boundary triangle t of the tetrahedra  $\theta$  of  ${}^{3}\Delta$  the link  $\ell_{t}$  of  $\mathcal{G}$  connects  $n_{t}$  and  $n_{\theta}$  (again one can pick an arbitrary orientation)

Each graph obtained in such a way have two interesting properties:

- Each internal node  $n \in \mathbf{N}_{int}$  is precisely 4-valent.
- Each boundary node  $n \in \mathbf{N}_{\text{bound}}$  is precisely 1-valent (thus if the manifold  $\Sigma$  and respectively the triangulation  ${}^{3}\Delta$  is boundaryless, the graph  $\mathcal{G}$  is closed).

Obviously the graph  $\mathcal{G}$  does not capture all the information encoded in the triangulation it is insensitive on gluing of 1- and 0-cells of  ${}^{3}\Delta$ . However it captures enough information to construct (after appropriate coloring) a diffeomorphism-invariant spin-network state on  $\Sigma$  (see Appendix A.3.3).

The above procedure can be easily generalized to the cellular decomposition that is not a triangulation. Indeed, none of the steps of the procedure depends on the shape of cells. The only point where we used the fact, that  ${}^{3}\Delta$  is a triangulation was reading the valency of the nodes  $n \in \mathbf{N}_{int}$  - it is 4 because each tetrahedron has precisely 4 faces. In case of a general cellular decomposition this number may be arbitrary, thus a graph dual to an arbitrary cellular decomposition may have nodes of arbitrary valence.

### 2-complex dual to 4-dimensional triangulation

The above procedure can be easily generalised 1-dimension up. Indeed, let  $\mathcal{M}$  be a 4-dimensional topological space and let  ${}^{4}\Delta = (\mathcal{C}^{4}, \mathcal{C}^{3}, \mathcal{C}^{2}, \mathcal{C}^{1}, \mathcal{C}^{0}, f_{4}, f_{3}, f_{2}, f_{1})$  be a triangulation of  $\mathcal{M}$  (i.e.  $X_{4\Delta} = \mathcal{M}$ ). Then the 2-complex dual to the triangulation  $\kappa = {}^{4}\Delta^{\star}$  is defined as follows. Let  $\kappa = (\mathbf{F}, \mathbf{E}, \mathbf{V}; f_{2}, f_{1})$  where  $\mathbf{E} = \mathbf{E}_{int} \cup \mathbf{L}$  and  $\mathbf{V} = \mathbf{V}_{int} \cup \mathbf{N}$ , such that:

- There is a 1-to-1 correspondence between 4-cells of  ${}^{4}\Delta$  and internal vertices of  $\kappa$ , i.e. for each 4-simplex  $s \in C^{4}$  there is one and only one vertex  $v_{s} \in \mathbf{V}_{int}$ , and there are no other vertices in  $\mathbf{V}_{int}$ .
- There is a 1-to-1 correspondence between 3-cells of  ${}^{4}\Delta$  and internal edges of  $\kappa$ , i.e. for each tetrahedron  $\theta \in C^{3}$  there is one and only one edge  $e_{\theta} \in \mathbf{E}_{int}$ , and there are no other edges in  $\mathbf{E}_{int}$ .
- There is a 1-to-1 correspondence between 2-cells of  ${}^{4}\Delta$  and faces of  $\kappa$ , i.e. for each triangle  $t \in C^{2}$  there is one and only one face  $f_{t} \in \mathbf{F}$ , and there are no other faces.
- The boundary 3- $\Delta$ -complex  $\partial^4 \Delta$  is a triangulation of the boundary manifold  $\Sigma = \partial \mathcal{M}$ . Applying the above procedure one obtain a graph  $\mathcal{G} = (\partial^4 \Delta)^*$ . This graph defines the boundary edges **L** and boundary vertices **N** of the 2-complex  $\kappa$ , together with incidency relations on them.
- Whenever two 4-simplices s, s' of  ${}^{4}\Delta$  are adjacent (i.e. share a tetrahedron  $\theta$ ), the edge  $e_{\theta}$  of  $\kappa$  connects the vertices  $v_{s}$  and  $v_{s'}$ .
- For each boundary tetrahedron  $\theta$  of a 4-simplex s of  ${}^{4}\Delta$  the edge  $e_{\theta}$  of  $\kappa$  connects the node  $n_{\theta}$  and  $v_{s}$ .
- The faces  $f \in \mathbf{F}$  are glued onto the skeleton obtained above as follows.
  - Let  $t \in C^2$  be an triangle of  ${}^{4}\Delta$ . It is incident to a number of 4-simplices  $s_1, \ldots, s_k$ . If t is multi-incident to  $s_i$ , the 4-simplex  $s_i$  must appear multiple times in the series  $s_1, \ldots, s_k$ .
  - Let us order the 4-simplices in such a way, that  $s_i$  and  $s_{i+1}$  are adjacent and they share a tetrahedron  $\theta_i$ , that is incident to t.
    - \* If t is an internal face of  ${}^{4}\Delta$ , we also want  $s_k$  and  $s_1$  to be adjacent and to share a tetrahedron  $\theta_k$  incident to t.
  - Consider a series of edges  $e_{\theta_1}, \ldots, e_{\theta_{k-1}}$ , each one oriented from  $s_i$  to  $s_{i+1}$

- \* If t is an internal face of  ${}^{4}\Delta$ , we add the edge  $e_{\theta_{k}}$  oriented from  $s_{k}$  to  $s_{1}$  at the end of the series.
- \* If t is a boundary face of  ${}^{4}\Delta$ , we add the edge  $\ell_{t}$  at the end of the series, we orient it from  $n_{\theta_{k}}$  to  $n_{\theta_{1}}$  (because the tetrahedra  $\theta_{1}$  and  $\theta_{k}$  must be boundary tetrahedra).
- The boundary of  $f_t$  is a circle. The series  $e_{\theta_1}, \ldots, e_{\theta_k}$  (or  $e_{\theta_1}, \ldots, e_{\theta_{k-1}}, \ell_t$ ) form a closed path  $\gamma$ . We glue the face onto the 1-sceleton of  $\kappa$  by gluing the circle  $\partial f_t$  along the path  $\gamma$ . The orientation of  $\gamma$  induces the orientation of  $f_t$ (but of course it can be reversed)
- A 2-complex dual to a triangulation of a 4-manifold has the following properties:
- Each internal vertex  $v \in \mathbf{V}_{int}$  is precisely 5-valent.
- Each boundary vertex  $n \in \mathbf{N}$  is also precisely 5-valent, but precisely one of the edges incident of it is internal, and the other four are the boundary links.
- Each internal edge  $e \in \mathbf{E}_{int}$  is incident to precisely 4 faces (when counting multiply the faces, that are multi-incident).
- Each boundary edge  $\ell \in \mathbf{L}$  is incident to precisely 1 face.

Again a 2-complex  $\kappa = {}^{4}\Delta^{\star}$  does not capture all the information encoded in the triangulation, however, it is enough to introduce the spin-foam structure on it (see subsection 1.2.2).

Again the above construction does not depend on the fact, that  ${}^{4}\Delta$  is a triangulation this fact is necessary only when calculating the valency of vertices and edges. Thus given an arbitrary 4-dimensional cellular decomposition one can apply the same procedure obtaining a dual 2-complex without the above constrains on the valency of vertices and edges.

# A.1.2.7 Gluing of CW-complexes

Let us now introduce a procedure of gluing CW-complexes.

# Definition A.32. Gluing a CW-complex

Consider a n-CW-complex  $\kappa = (\mathcal{C}_n, \ldots, \mathcal{C}_0; f_n, \ldots, f_1)$  and a pair of k-cells  $\alpha = (c_A, c_B)$  together with a map  $f^{\alpha} : c_A \to c_B$  that is a complexomorphism of the induced complexes  $\kappa_{c_A}$  and  $\kappa_{c_B}$ . We can glue  $\kappa$  along  $f^{\alpha}$  obtaining the new n-CW-complex, that we shall denote by  $\kappa/f^{\alpha}$  or simply by  $\kappa/\alpha$ . The resulting complex is determined as follows. Let  $\kappa/f^{\alpha} = (\tilde{\mathcal{C}}_n, \ldots, \tilde{\mathcal{C}}_0; \tilde{f}_n, \ldots, \tilde{f}_1)$ .

- The sets  $\tilde{\mathcal{C}}_k$  for  $k > \dim(c_A)$  of  $\kappa/_{f^{\alpha}}$  are the same as  $\mathcal{C}_k$  of  $\kappa$ .
- On the sets  $C_k$  for  $k \leq \dim(c_A)$  the map  $f^{\alpha}$  introduces a relation defined as follows: the cells c and c' are in the relation iff c is incident to  $c_A$  (or  $c = c_A$ ) and  $f^{\alpha} : c \mapsto c'$ . In such case the map  $f^{\alpha}$  defines a homeomorphism of c and c'. Let us extend this relation to a minimal equivalence relation on  $C_k$  and call this equivalence relation  $\sim_{\alpha,k}$ , then  $\tilde{C}_k = C_k / \sim_{\alpha,k}$ .
- The maps  $\tilde{f}_k$  are defined as follows. Let  $\pi_{\alpha,k} : \mathcal{C}_k \to \tilde{\mathcal{C}}_k$  be the projection on the equivalence classes of  $\sim_{\alpha,k}$ . Then  $\tilde{f}_k = \pi_{\alpha,k-1} \circ f_k \circ \pi_{\alpha,k}^{-1}$ .

In order to make sure, that the definition is consistent, one has to check, if the maps  $\tilde{f}_k$  are well defined, i.e. whether they not depend on the choice of the cell in the image of  $\pi_{\alpha,k}^{-1}$ . This issue appear only when considering cell with more-than-one-element equivalence classes with respect to  $\sim_{\alpha,k}$ , i.e. the cells of  $k < \dim(c_A)$ , that are incident with  $c_A$  or  $c_B$ . Thus let us pick such k-cell c that is incident to  $c_A$  and let us act by  $\tilde{f}_k$  on the equivalence class  $[\partial c]_{\sim_{\alpha,k}}$ . We get

$$\tilde{f}_k(\partial c) = \pi_{\alpha,k-1} \left( f_k \left( \pi_{\alpha,k}^{-1} \left( [\partial c]_{\sim_{\alpha,k}} \right) \right) \right)$$
(A.15)

Since c is incident to  $c_A$ , we the inversed projection  $\pi_{\alpha,k}^{-1}$  may pick (at least) two possible representatives of  $[\partial c]_{\sim_{\alpha,k}}$ , namely  $\partial c$  or  $f^{\alpha}(\partial c)$ . For  $\partial c$  the result is

$$\tilde{f}_k \left(\partial c\right)^{(1)} = \pi_{\alpha,k-1} \left( f_k \left(\partial c\right) \right) \tag{A.16}$$

for  $f^{\alpha}\left(\partial c\right)$  the result is

$$\tilde{f}_k \left(\partial c\right)^{(2)} = \pi_{\alpha,k-1} \left( f_k \left( f^\alpha \left(\partial c \right) \right) \right) \tag{A.17}$$

now since  $f^{\alpha}$  is a complexomorphism, we have  $f_k(f^{\alpha}(\partial c)) = f^{\alpha}(f_k(\partial c))$  (together with a homeomorphism of corresponding cells). But this means, that the images  $f_k(\partial c)$  and  $f_k(f^{\alpha}(\partial c))$  lie in the same equivalence class of  $\sim_{\alpha,k-1}$ , thus the projection  $\pi_{\alpha,k-1}$  maps them onto the same object, so  $\tilde{f}_k(\partial c)^{(1)} = \tilde{f}_k(\partial c)^{(2)}$ . If one chose c incident to  $c_B$ , the same derivation would work with  $f^{\alpha}$  replaced by  $(f^{\alpha})^{-1}$ , thus the maps  $\tilde{f}_k$  are well defined.

Quod erat demonstrandum.

One can prove, that in 2-CW-complex gluing along pairs of edges commute, i.e. given a 2-complex  $\kappa$  and to pairs  $\alpha = (e_1, e_2)$  and  $\beta = (e_3, e_4)$  for  $e_1, \ldots, e_4 \in \mathbf{E}_{\kappa}$  the 2complexes  $(\kappa/\alpha)/\beta$  and  $(\kappa/\beta)/\alpha$  are complexomorphic. The proof is presented in Appendix B.1.

# A.2 Differential geometry

In this section we provide a short introduction to differential geometry. First in A.2.1 we recall some basing notions of differential topology, like a manifold, tangent space, tensor bundles etc. Then in A.2.2 we introduce Riemann and Cartan structures on manifolds, namely: metrics, connections, covariant derivatives and holonomies. All the definitions are based on [98].

As it is done in all sections of this appendix, this section is rather a recall and overview then a detailed study of differential geometry. Only the most important definitions are recalled, and most of facts are stated without a proof.

# A.2.1 Differential topology

A differential topology is a tool to study differential properties of functions on topological manifolds. The issues discussed here are often called the differential geometry, however, we reserve the name *geometry* to the part of the field studied in the second subsection, where we can define angles, volumes and twists.

First we define the topological manifolds without and with boundary and some basic notions related to them. Then we introduce fibre bundles: vector and tensor bundles, tangent bundles and cotangent bundles, together with vector fields, tensor fields and differential forms. Finally we recall the special class of manifolds enhanced with a group structure, namely the Lie groups, and we define the gauge action of a Lie group on fibre bundles.

# A.2.1.1 Manifolds

We will define various categories of manifolds: topological manifolds (called also  $C^{0}$ manifolds), differential manifolds of class  $C^{k}$  (called also  $C^{k}$ -manifolds) and smooth manifolds (called also  $C^{\infty}$ -manifolds). The categories differ only by the class of functions considered admissible in various cases. Wherever in this section we say about a function  $f: \mathbb{R}^{n} \to \mathbb{R}^{m}$  appropriately smooth, we mean of class  $C^{k}$  for k corresponding to considered category (where k = 0 for topological manifolds and  $k = \infty$  for smooth manifolds).

#### **Definition A.33.** Topological, differentiable and smooth manifold, submanifold

A n-dimensional topological manifold (also called  $C^0$ -manifold) is a pair  $(M, \mathcal{A})$  of a topological separable Hausdorff space M and an atlas  $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$  being a set of pairs of open sets  $U_\alpha \subset M$  and homeomorphisms  $\phi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{R}^n$ , such that the sets  $U_\alpha$  cover M (i.e.  $\bigcup_\alpha U_\alpha = M$ ). Each homeomorphism  $\phi_\alpha$  is called a coordinate chart, and its inverse  $\phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha)}$  is called a map or a parametrisation. Two atlases  $\mathcal{A}$  and  $\mathcal{A}'$  are subatlases of  $\mathcal{A}''$  (i.e.  $\mathcal{A} \subset \mathcal{A}''$  and  $\mathcal{A}' \subset \mathcal{A}''$ ).

A topological manifold is called  $C^k$ -differentiable manifold (or smooth manifold) if the maps of the atlas satisfy an extra condition, namely for each two intersecting sets  $U_{\alpha}$ ,  $U_{\beta}$  the change of coordinates map  $\Phi_{\alpha\beta} := \phi_{\alpha} \circ \phi_{\beta}^{-1}$  (defined on  $\phi_{\beta} (U_{\alpha} \cap U_{\beta}) \subset \mathbb{R}^n$ ) is a diffeomorphism of the sets  $\phi_{\beta} (U_{\alpha} \cap U_{\beta})$  and  $\phi_{\alpha} (U_{\alpha} \cap U_{\beta})$  of class  $C^k$  (or  $C^{\infty}$  in case of a smooth manifold). In the differentiable case two atlases  $\mathcal{A}$  and  $\mathcal{A}'$  are considered equivalent iff there is an atlas  $\mathcal{A}''$  of the same  $C^k$ -class, such that both  $\mathcal{A}$  and  $\mathcal{A}'$  are subatlases of  $\mathcal{A}''$ .

A m-dimensional submanifold (m < n) of a manifold is a pair of a subset  $\Sigma \subset M$  and the atlas  $\mathcal{A}^{\Sigma} := \mathcal{A}|_{\Sigma}$  (i.e. the atlas obtained by restriction of all coordinate charts to their intersection with  $\Sigma$ ), such that for each coordinate chart in  $\mathcal{A}^{\Sigma}$  the image  $\phi_{\alpha}(U_{\alpha} \cap \Sigma)$ can be mapped by a  $C^k$  bijective map to a subset of  $\mathbb{R}^m$  (of course in case of topological manifolds k = 0, and in case of smooth ones  $k = \infty$ ). We say, that the manifold structure of  $\Sigma$  is induced by M.

Submanifold can be considered both as an independent manifold  $\tilde{\Sigma}$ , or as a submanifold  $\Sigma \subset M$ , called an embedded manifold; there is a map, that maps each point of  $\tilde{\Sigma}$  onto corresponding point of  $\Sigma \subset M$ , called the embedding map and denoted by  $i: \tilde{\Sigma} \hookrightarrow M$ . Whenever we want to emphasise the fact of a submanifold being embedded into its supermanifold we use the notation  $\Sigma \hookrightarrow M$ .

In most cases we will not need to use the atlases of manifolds explicitly, thus we will refer to the topological space M of the manifold as to the manifold.

Each coordinate chart  $\phi_{\alpha}$  defines a set of coordinate functions  $\{x^i\}_{i=1,...,\dim M}$  defined on  $U_{\alpha}$ . They act as  $x^i(p) := (\phi_{\alpha}(p))^i$ . However, whenever the exact choice of the coordinate chart is not crucial, we will use a shortcut notation as  $x^i(p) = p^i$ . The parametrisations will be denoted as  $p(x^i)$ . In most cases we will also not need to know, what parametrisation are we using, thus whenever this choice is irrelevant, we will simply not mention it. For example given a function f on M and a coordinate chart, there is a  $(\mathbb{R}^n \to \mathbb{R})$ -function induced by the parametrisation, namely  $f(p(x^i))$  - if the choice of coordinates is obvious or irrelevant, we will simply denote it as  $f(x^i)$ .

## **Definition A.34.** Manifold with a boundary

A manifold with a boundary is defined as a manifold, however the coordinate charts are allowed to be homeomorphisms  $\phi_{\alpha} : U_{\alpha} \to \phi(U_{\alpha}) \subset \mathbb{R}^{n}_{\geq 0}$ , where  $\mathbb{R}^{n}_{\geq 0} : \{x \in \mathbb{R}^{n} : x^{1} \geq 0\}$ (and likewise in case of changes of coordinates; where for  $C^{k}$ - and smooth-manifolds the differentials on the boundary are considered only with respect to the directions tangent to the boundary). The boundary of a manifold with a boundary is a sum of the points, that has  $x^{1}$ -coordinate equal to 0 in at least one coordinate chart:

$$\partial M = \left\{ p \in m : \exists_{\alpha: U_{\alpha} \ni p} x^{1} \left( \phi_{\alpha} \left( p \right) \right) = 0 \right\}$$
(A.18)

The boundary of a manifold has the manifold structure induced from M (it is a submanifold of M).

One can consider also manifolds with corners of degree l - then the coordinate charts are assumed to map on  $\{x \in \mathbb{R}^n : x^1 \ge 0, \dots, x^l \ge 0\}$ .

Let us introduce a notion of region:

### Definition A.35. Region

A k-dimensional region  $\mathcal{R}^{(k)}$  of n-dimensional manifold M is a k-dimensional connected submanifold of M, possibly with a boundary. We do not use the name region when k = 0 - the only 0-dimensional regions would be points, and we call them simply points.

Let us pay some more attention to two special cases of manifolds: the paths and the surfaces, i.e. 1- and 2-dimensional manifolds.

### Paths

Given a manifold M, a path is a continuous map  $\gamma : I \to M$  where  $\mathbb{R} \supset I$  is a connected interval. We say, that  $\gamma$  is a *simple path* if I is a closed interval (then one can rescale the parametrisation such that  $\gamma : [0, 1] \to M$ ). One can also consider *open paths*  $\gamma : [0, 1] \to M$  and *infinite paths*  $\gamma : \mathbb{R} \to M$ . Given a manifold of class  $\mathcal{C}^k$  we say, that the path is of class  $\mathcal{C}^{k'}$   $(k' \leq k)$ , iff in any coordinate chart the coordinates of the path are  $\mathcal{C}^{k'}$  functions.

A path  $\gamma$  define a 1-dimensional submanifold of M iff it is injective. The submanifold being an image of a path will be called a link and denoted by  $\ell$ .

A simple link is an image of a simple path. It is a manifold with a boundary consisting two distinct points  $\partial \ell := \{\ell(0), \ell(1)\}$ . The boundary points are called *source*  $s(\ell) := \ell(0)$  and *target*  $t(\ell) := \ell(1)$ . The path with the parametrisation inverted will be denoted as  $\ell^{-1}$  (as a manifold  $\ell$  and  $\ell^{-1}$  are the same object, however the distinction between  $\ell$  and  $\ell^{-1}$  is needed in case of embedded directed graphs).

An open link is an image of an open path. It has only one boundary point - the source:  $\partial \ell = \{\ell(0)\}.$ 

It may happen, that a simple path is injective on ]0, 1[, but  $\gamma(0) = \gamma(1)$ . The image of such path will be called a *loop*. Loops are always topological submanifolds of M, however, the start/end point may be a corner, and thus it may violate the differential structure (i.e. even if a path is of class  $C^{k'}$ , the loop might be a submanifold of class lower then k'). Loops will be also often denoted by  $\ell$ .

#### Surfaces

A surface is a connected 2-dimmentional manifold. We will usually denote them by f or S.

A simple surface (a simple surface with a boundary) is a 2-dimmentional manifold, that is homeomorphic to a disc (a disc with a boundary respectively). The boundary of a simple surface with a boundary is a loop.

In case of embedded surfaces it also may happen, that the induced differential structure is different, than the one presented above (likewise in case of paths).

# A.2.1.2 Fibre bundles, vector and tensor bundles

This section is valid for each class of differentiability, i.e. we assume all the manifolds are of class  $C^k$  (for  $k \ge 1$ ) and whenever we consider a map between manifolds (or a function from manifold to  $\mathbb{R}^n$ ), it is assumed to be  $C^k$ .

#### **Definition A.36.** Fibre bundle

Given two manifolds M and X the fibre bundle is a manifold B and a map  $\pi : B \to M$ such that for each point  $p \in M$  the counterimage  $\pi^{-1}(p)$  is diffeomorphic with X, and for each point  $p \in M$  there is a neighbourhood  $U \ni p$  such that there exists a diffeomorphism  $\phi_U : \pi^{-1}(U) \to U \times X$ .

The manifold M is called the base of the bundle, the manifold X is called the typical fibre, each diffeomorphism  $\phi_U$  is called local trivialisation of the bundle, the map  $\pi$  is called the projection map, and the manifold B is usually called simply the bundle of the bulk of the bundle. A fibre bundle is denoted by  $\left(B \xrightarrow{}{} M\right)$ , or simply by  $(B \rightarrow M)$ , if the projection map is obvious or irrelevant. If one wants to emphasise the typical fibre, one can write  $(B \rightarrow M)_X$ .

An obvious example of a fibre bundle is the trivial bundle  $B = M \times X$  with the natural projection  $\pi : (p, x) \to p$ .

Nontrivial bundles appear when the base manifold is not homeomorphic to a region. Consider base manifold being the circle  $M = S^1$  and the typical fibre being also a circle  $X = S^1$ . The trivial bundle is the torus  $B = S^1 \times S^1 = \mathbf{T}^2$ . Let us now parametrise the base manifold by  $x \in [0, 2\pi]$  and the fibre by  $\alpha \in [0, 2\pi]$  (obviously 0 and  $2\pi$  are glued) and consider the following local trivialisation: on  $x \in [0, 2\pi]$  there is a natural local trivialisation of onto the product  $]0, 2\pi[ \times S^1$ , but in a neighbourhood of  $x = 0 \equiv 2\pi$  the fibres are inversed (i.e.  $\lim_{x\to 0^+} (x, \alpha) \equiv \lim_{x\to 2\pi^-} (x, 2\pi - \alpha)$ ). The resulting manifold is the Klein bottle, and it is a  $S^1$  fibre bundle over  $S^1$ .

#### **Definition A.37.** Vector bundle

Assume, that the typical fibre is a vector space V. Then the bundle  $\left(B \xrightarrow{\pi} M\right)_V$  is called the Fibre bundle.

Fibre bundles provide a convenient definition of vector fields on manifolds.

## Definition A.38. Vector field

Given an open set  $U \subset M$  of the base of a vector bundle  $(B \to M)$ , the section of the bundle B is a (appropriately smooth) map  $\omega : U \to B$ , such that  $\forall_{p \in U} \pi (\omega (p)) = p$ . Sections have a structure of vector space induced by the structure of fibres: if  $\omega_{1,2}(p) = (x, v_{1,2}(p))$ , then  $(\omega_1 + \omega_2)(p) = (p, v_1(p) + v_2(p))$ .

The sections are often called vector fields when applied to physics. They are also called tensor fields if the space V has an extra structure of a tensor product  $V = V_1 \otimes \cdots \otimes V_n^*$ . Moreover, given two bundles  $(B_1 \to M)_{V_1}$  and  $(B_2 \to M)_{V_2}$  and two sections  $\omega_1$  of  $B_1$ and  $\omega_2$  of  $B_2$  one can define the tensor product  $\omega_1 \otimes \omega_2$  as the section of the third bundle  $(B_3 \to M)_{V_1 \times V_2}$ , such that  $(\omega_1 \otimes \omega_2)(p) := (p, \omega_1(p) \otimes \omega_2(p))$ . We will call the bundle  $B_3$  the tensor product of  $B_1$  and  $B_2$ . The set of all sections (fields) of a bundle  $(B \to M)_V$  is denoted as  $\Xi(B)$  (or  $\Xi(B)_M$ or  $\Xi(B)_{M,V}$  if one wants to emphasise the structure of the bundle), set of all sections over a certain open set  $U \subset M$  is denoted as  $\Xi(B)_U$  or  $\Xi(B)_{U,V}$ .

The most basic example of vector bundles and vector fields are tangent and cotangent bundle, containing tangent vectors and one-forms. There are many equivalent definitions of the tangent vectors. Let us provide the one given in [98]:

### Definition A.39. Tangent space at a point, tangent bundle

Given a manifold M and a point  $p \in M$  consider a set of all functionals X on the set of functions on M, satisfying the following conditions

- linearity: X(f+g) = X(f) + X(g)
- Leibniz rule: X(fg) = f(p)X(g) + g(p)X(f)
- if f is a constant function, then X(f) = 0

The set of all such functionals form a vector space of dimension equal to the dimension of M. We call it the tangent space to M at p, and denote as  $T_pM$ . One can attach a tangent space to each point of M obtaining the tangent bundle TM (obviously the fibre of this bundle is isomorphic to  $\mathbb{R}^{\dim M}$ ). Sections of TM are called tangent fields to M(or tangent vector fields on M, the value of a tangent field in a particular point is called a tangent vector to M). Set of all tangent fields is denoted as  $\mathcal{X}(M) := \Xi(TM)$ .

Given a coordinate chart  $(x^i)$  on M there is a natural basis of tangent vectors, given by  $\partial_i := \frac{\partial}{\partial x^i}$ . A complete set of basis vectors is called frame. Vector fields will be denoted as  $X(p) = X^i(p) \partial_i$ , or in abstract index notation as  $X^i(p)$ .

An example of a tangent vector to M is the velocity vector of a path: given a path  $\gamma : ]-\epsilon, \epsilon[ \to M \text{ such that } \gamma(0) = p \text{ and a function } f \text{ defined on a neighbourhood of } p$  the tangent vector  $X_p^{\gamma}$  acts on the function f as

$$X_{p}^{\gamma}(f) := \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} f\left(\gamma\left(t\right)\right) \tag{A.19}$$

Each tangent vector is the velocity vector of some path (however, different paths may have the same velocity vector), we will sometimes denote it shortly as  $X_p = \frac{d}{dt}\Big|_{t=0} \gamma(t)$  or  $X_p = \dot{\gamma}(t)$ 

Tangent fields can be considered as functions on M valued in a vector space V isomorphic with  $T_p M$ .

Having the tangent fields we can introduce tangent multivector fields as tensor products of tangent fields. The space tangent multivectors will be denoted as  $\mathcal{X}^{\otimes k}(M) := \Xi(TM \otimes \cdots \otimes TM)$ , the space of symmetric tangent multivectors will be denoted as  $\mathcal{X}^{\odot k}(M) := \Xi(TM \odot \cdots \odot TM)$ , the space of antisymmetric tangent multivectors will be denoted as  $\mathcal{X}^{\wedge k}(M) := \Xi(TM \wedge \cdots \wedge TM)$ , or as  $\Lambda^{k}(M)$ .

Given a tangent field one can always locally define family of diffeomorphisms called the flow of the tangent field.

# **Definition A.40.** The flow of the tangent field

Let X be a tangent field on M. The flow of X is the family of diffeomorphisms  $\phi_X^t$ (for  $t \in ]-\epsilon, \epsilon[$ ), such that for each point p the path  $\gamma_p(t) := \phi_X^t(p)$  satisfy the condition

$$\forall_{t\in ]-\epsilon,\epsilon[} \quad X_{\phi_X^t(p)} = \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \gamma\left(t+s\right) \tag{A.20}$$

In other words the flow of the tangent field X is defined by the set of curves tangent to X at each point and parametrised in such a way, that the velocity vector is precisely X.

The diffeomorphisms of the flow of a tangent fields obviously effects the tensor fields on a manifold M. The infinitesimal change with respect to the flow is measured by the Lie derivative:

### **Definition A.41.** Lie derivative

Given a tensor field  $Y \in \Xi(B)_{M,V}$  and a tangent field  $X \in \mathcal{X}(M)$  the Lie derivative of Y along X is a tensor field  $\partial_X Y$  defined as follows

$$\left(\partial_X Y\right)(p) := \lim_{\epsilon \to 0} \frac{T\phi_X^{-\epsilon}\left(Y\left(\phi_X^{\epsilon}\left(p\right)\right)\right) - T\left(p\right)}{\epsilon}$$
(A.21)

If  $Y \in \mathcal{X}(M)$ , the Lie derivative is denoted as  $\partial_X Y := [X, Y]$  and indeed it is the vector field defined by the commutator of the tangent fields, i.e.

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$
(A.22)

The Lie derivative commutes with tensor product and with contraction. It also acts by the duality on the fields dual to the tangent fields, that are introduced below.

#### **Definition A.42.** Cotangent bundle and differential forms

Consider a linear functional dual at each point  $p \in M$  to the tangent fields, i.e.  $\omega$ such that  $\forall_{p \in M} \omega(p) : T_p M \to \mathbb{R}$  is a linear map. Assuming, that given an appropriately smooth field X the action of  $\omega$  on X is a smooth function on M (i.e.  $f(p) := \omega(p) \sqcup X(p)$ is of appropriate differentiability class), such functional is called a differential form or a one-form on M. The set of all one-form on M form a fibre bundle, called the cotangent bundle over M, denoted by  $T^*M$ .

Consider an antisymmetric tensor of product of k copies of cotangent bundle, we will denote it as  $\Lambda^{\star k}(M)$  and it is obviously a tensor bundle over M. The sections of such bundle are antisymmetric tensor products of one-forms. Each such section is a functional on k tangent fields. These sections are called k-forms on M, and the space of all kforms is denoted as  $\Omega^{k}(M) = \Xi(\Lambda^{\star k}(M))$ . Obviously  $\Omega^{1}(M) = \Xi(T^{\star}M)$ , one can also say, that the space of all differentiable functions on M is the space of 0-forms, and thus  $\mathcal{C}^{\infty}(M) = \Omega^{0}(M)$ .

Obviously given a coordinate chart  $(x^i)$  on M one can at each point (where the chart is defined) introduce a basis dual to  $\partial_i$  - denoted as  $dx^i$ . One-forms will be denoted as  $\omega(p) = \omega_i(p) dx^i$ , and k-forms as  $\omega(p) = \omega_{i_1 \cdots i_k}(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ , or in abstract index notation as  $\omega_i(p)$  and  $\omega_{i_1 \cdots i_k}(p)$  respectively.

It will be also convenient to consider forms giving values more general, then simply real numbers:

**Definition A.43.** V-valued functions, V-valued k-forms and V-valued tangent vector fields and tangent multifields, (p, q)-tensors

Consider a vector space V and a manifold M. One can construct a vector bundle  $(B \to M)_V$  of V-valued functions on M (for example  $M \times V$ ). We will denote them like ordinary functions:  $v : M \to V, v : p \mapsto v(p)$ , however remembering, that  $v(p) \in V$ . Given a basis  $\{e_I\}$  of V we have  $v(p) = e_I v^I(p)$ , sometimes we will use the abstract index notation, writing simply  $v^I(p)$ .

Consider now a bundle  $B \otimes \Lambda^{\star k}(M)$ . The sections of these bundle are V-valued kforms on M. We denote them as ordinary forms  $\omega : M \to \Lambda^{k}(T_{p}^{\star}M) \otimes V, \omega : p \mapsto \omega(p)$ , however remembering, that  $\omega(p) \in \Lambda^k(T_p^*M) \otimes V$ . Given a basis  $\{e_I\}$  of V and a coordinate chart  $(x^i)$  on M we write  $\omega(p) = \omega_{i_1\cdots i_k}^I(p) e_I \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k}$  (of course we will also use the abstract index notation). The space of V-valued k-forms will be denoted as  $V \otimes \Omega^k(M) := \Xi(B \otimes \Lambda^{*k}(M))$ .

In a similar way one can define a V-valued vector field E, as a section of a bundle  $B \otimes TM$ . In a coordinate chart  $(x_i)$  the components of a E are  $E(p) = E^{I,i} e_I \otimes \partial_i$ . Similarly one can define multi-vector fields, as  $\Xi(B \otimes \mathcal{X}^{\wedge k}(M))$ .

There is a special nomenclature for sections of bundles  $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$  - they are called tensors fields p-covariant q-contravariant, or simply (p,q)-tensor fields. Given a  $(B \to M)_V$  bundle we have also V-valued (p,q)-tensors, being sections of  $B \otimes (TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$ . Some of these tensor products may be symmetrised or antisymmetrized, thus we have for example p-covariant symmetric q-contravariant, antisymmetric in last two indices tensor fields being the sections of  $(TM)^{\odot p} \otimes (T^*M)^{\otimes (q-2)} \otimes (T^*M)^{\wedge 2}$ , etc.

Given a diffeomorphism of a base manifold, one can consider the way it influence the tangent and cotangent fields.

### **Definition A.44.** Tangent and cotangent map

Let  $\phi: M \to M$  be a diffeomorphism. The tangent map  $T\phi$  is a collection of linear maps  $T_p\phi: T_pM \to T_{\phi(p)}M$  acting as follows: given a vector  $X_p^{\gamma}$  defined by the path  $\gamma$  it is mapped to the vector defined by the path  $\phi(\gamma)$ :

$$T_p\phi: X_p^{\gamma} \mapsto X_{\phi(p)}^{\phi(\gamma)} \tag{A.23}$$

The cotangent map  $T^*\phi$  is defined by the algebraic duality: given a form  $\alpha$  and a vector X we have  $[T_p^*\phi(\alpha)] \lrcorner X_{\phi^{-1}(p)} = \alpha \lrcorner [T_{\phi^{-1}(p)}\phi(X_{\phi^{-1}(p)})].$ 

The tangent and cotangent maps can be naturally generalised to multivectors and higher order forms.

# A.2.1.3 Lie groups

A special class of manifolds are the Lie Groups:

#### Definition A.45. Lie group

A Lie group is a group G with a manifold structure, such that all the group operations (i.e. multiplication and inversion) are smooth maps.

Two Lie groups are used most in our work: SU(2) and  $SL(2, \mathbb{C})$ .

The SU(2)-group is the group of special unitary matrices of dimension 2:

$$SU(2) := \left\{ u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2} : \det(u) = 1 \land u^{\dagger} = u^{-1} \right\}$$
(A.24)

It is a 3-dimensional manifold, homeomorphic to the 3-sphere  $S^3$ . It covers the group of 3-dimensional rotations:  $SO(3) = SU(2)/\mathbb{Z}_2$ . It is described in more detail in Appendix A.3. The elements of SU(2) will be denoted by h or by u, v, U, V.

The  $SL(2,\mathbb{C})$ -group is the group of  $2 \times 2$  matrices with determinant equal to 1:

$$SL(2,\mathbb{C}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2} : \det(u) = 1 \right\}$$
 (A.25)

It is a 6-dimensional, noncompact manifold. It is the universal covering of the proper orthochronous subgroup of the Lorentz group:  $SL(2, \mathbb{C}) \to SO(1, 3)^+$ . It is described in more detail in Appendix A.4, the detailed study can be found in the textbook [86]. The elements of  $SL(2, \mathbb{C})$  will be denoted by g.

Lie groups are groups and thus they have linear representations in vector spaces. A representation of a Lie group on a vector space being a fibre of a vector bundle leads us to the notion of gauge action and gauge transformations:

#### **Definition A.46.** Gauge Transformation

Consider a fibre bundle  $(B \to M)_V$ , a Lie group G and a representation  $\rho : G \to \operatorname{Aut}(V)$ . Consider a G-valued field  $g(\cdot)$  on M (i.e. a smooth function  $g: M \to G$ ). A gauge transformation defined by the field  $g(\cdot)$  is a map on the set of sections  $g(\cdot) : \Xi(B) \to \Xi(B)$  (i.e. a map on V-valued fields on M - see Appendix A.2.1.2) defined as follows. Let us pick a field  $X \in \Xi(B)$  and choose a local trivialisation, so that the field looks like a function  $X: M \to V$ . Then the gauge transformation maps  $g(\cdot): X \mapsto X^g$ , where the new field  $X^g(x) := \rho(g(x)) X(x)$ .

The equivalence classes of gauge transformations are called gauge orbits. Obviously two fields that vanish in different points lay on different gauge orbits. Moreover in most cases the representation maps G to  $\rho(G)$  that is a proper subgroup of Aut(V), thus there are more than one gauge orbit per point of M.

The tensor fields with gauge action of a group G will be called G-tensor fields or G-tensors. Of course it might happen, that we want to introduce an action of G-group on a bundle, that is *not* a linear representation of G on the fibre (see for example Appendix A.2.2.2) - then even though the fibre bundle has some tensor structure, we will not call it's sections G-tensors.

Groups obviously acts on themselves:

#### **Definition A.47.** Action of a group on itself

Given an element  $g \in G$  we let us define a map  $\phi_g^L : G \to G$  given by  $\phi_g^L : g' \mapsto g \cdot g'$ . Thanks to smoothness and invertibility of multiplication the map  $\phi_g^L$  is a diffeomorphism. We call it left action of G on itself. Similarly we define the right action of G on itself as  $\phi_g^L : g' \mapsto g' \cdot g$ .

Having the left and right action we can define invariant fields on a group:

#### **Definition A.48.** Left- and right-invariant fields

Consider a tangent field  $X_e \in T_eG$ , where  $e \in G$  is the unity element. Using the family of left and right action of G on itself we can extend each such field to each point  $g \in G$  as:

$$X_g^L := T\phi_g^L X_e \qquad X_g^R := T\phi_g^R X_e \tag{A.26}$$

The fields  $X^L$  and  $X^R$  are called left- and right-invariant fields on G, respectively. Sometimes they are called left- (right-) invariant extensions of a vector  $X_e$ .

The invariant fields are used to define the Lie algebra of a Lie group:

### Definition A.49. Lie algebra

Given a group G its Lie algebra  $\mathfrak{g}$  is the vector space isomorphic to the tangent space to G at the unity element

$$\mathfrak{g} := \frac{1}{i} T_e G \tag{A.27}$$

The Lie algebra product is given by the Lie derivative of the left invariant fields: let  $A_X, A_Y \in \mathfrak{g}$  be given by  $A_X := \frac{1}{i}X$  and  $A_Y := \frac{1}{i}Y$  for some tangent vectors  $X, Y \in T_eG$ , let  $X^L$  and  $Y^L$  are the left-invariant extensions of X and Y. The the Lie product of  $A_X$  and  $A_Y$  equals:

$$[A_X, A_Y]^{\mathfrak{g}} := i \left[ X^L, Y^L \right] \tag{A.28}$$

where  $[\cdot, \cdot]$  is the commutator of tangent fields. Given a representation of the group  $\rho: G \to \operatorname{Aut}(V)$  there is always a representation of the Lie algebra determined by the tangent map to the representation map:  $T\rho: \mathfrak{g} \to \operatorname{End}(V)$ .

Concerning generators of Lie groups there are two possible conventions. Given a Lie group G, its Lie algebra may be defined either as the vector space  $\mathfrak{g}^A := T_e G$  or  $\mathfrak{g}^H := \frac{1}{i}T_e G$  (Lie algebra is a real vector space), so that in the defining representation the elements  $g \in G$  (in some neighbourhood of the unity) are given by

$$g = e^{\alpha X^A} = e^{i\alpha X^H} \tag{A.29}$$

for  $\alpha \in \mathbb{R}$  and both  $X^A$  and  $X^H$  being appropriate endomorphisms of the carrier space of the defining representation of G. The algebra multiplication  $\circ$  in both  $\mathfrak{g}^A$  and  $\mathfrak{g}^H$  is induced by the adjoint action of G and it is

$$X^{A} \circ_{A} Y^{A} := \begin{bmatrix} X^{A}, Y^{A} \end{bmatrix} \quad \text{and} \quad X^{H} \circ_{H} Y^{H} := i \begin{bmatrix} X^{H}, Y^{H} \end{bmatrix}$$
(A.30)

where [X, Y] := XY - YX is the ordinary matrix commutator. It is easy to see, that  $\mathfrak{g}^A$  and  $\mathfrak{g}^H$  are isomorphic algebras. Indeed,  $\forall_{X^A \in \mathfrak{g}^A} \frac{1}{i} X^A \in \mathfrak{g}^H$  and the algebra multiplication transforms as

$$\frac{1}{i}X^A \circ_H \frac{1}{i}Y^A = i\left[\frac{1}{i}X^A, \frac{1}{i}Y^A\right] = \frac{1}{i}\left[X^A, Y^A\right] = \frac{1}{i}\left(X^A \circ_A Y^A\right)$$
(A.31)

Consider now a unitary representation  $\rho$  of G. The algebras  $\mathfrak{g}^A$  and  $\mathfrak{g}^H$  are represented by the tangent representation  $T\rho$ . The algebra  $\mathfrak{g}^A$  is mapped onto the space of antihermitian generators of  $\rho(G)$ , while the algebra  $\mathfrak{g}^H$  is mapped onto the space of hermitian generators of  $\rho(G)$ , thus let us call  $\mathfrak{g}^A$  the anti-hermitian Lie algebra of G and  $\mathfrak{g}^H$  the hermitian Lie algebra of G.

As we have shown above, the hermitian and anti-hermitian Lie algebras of G are equivalent. However, the matrix elements of their members differ by the factor of i, which influence some formulae, thus it is important to fix this convention.

**Remark A.3.** The hermitian convention is more convenient for the functional analysis (for example in this convention one has the well known action of rotation-generator in Wigner representation, given in (A.75)), and since the main use of Lie algebras in the thesis are in the functional analysis context, we decide to use the hermitian generators in most cases. It is the opposite choice to the one used for example in [6]. On the other hand, the anti-hermitian convention is more convenient when considering connections, because it allows to introduce a covariant derivative operator (see Appendix A.2.2.2) which is an anti-hermitian operator, similarly to ordinary derivative operator. Thus whenever we say  $\mathfrak{g}$ -valued connection, we mean  $\mathfrak{g}^A$ -valued connection. Elsewhere we either use  $\mathfrak{g}^H$  or we state the use of  $\mathfrak{g}^A$  explicitly by keeping the superscript A (thus whenever - except the phrase  $\mathfrak{g}$ -valued connection - we use  $\mathfrak{g}$  without a superscript, we mean  $\mathfrak{g}^H$ ).

# A.2.2 Geometry

We start with defining the (pseudo-)Riemannian metric on a manifold, together with basic geometric notions. Then we introduce affine connections on fibre bundles. We relate them with the Riemannian structure obtaining the Levi-Civita connection. Finally we study holonomies of connections and we prove the formula of gauge transformation of a holonomy.

# A.2.2.1 Riemannian geometry

Consider a manifold M and a tensor field  $g: M \to T_p^* M \odot T_p^* M$  (i.e. g is a section of  $T^*M \odot T^*M$ ). We say, that g is nondegenerated, if in each coordinate chart the matrix  $g_{ij}$  is non degenerated. Similarly we say, that g is positive, if in each coordinate chart it has only positive eigenvalues.

# Definition A.50. (pseudo-)Riemannian manifold

A Riemannian manifold is a pair (M,g) of a manifold M and a field  $g: M \to T_p^* M \odot T_p^* M$ , such that at each point p the field g is positive and nondegenerated. The field g is called the metric field. A manifold is called pseudo-Riemannian iff the field g is nondegenerated, but id does not have to be positive. At each point  $p \in M$  the directions  $X \in T_p M$  such that  $g_p(X, X) = 0$  are called null directions. The metric field defines a (pseudo-)scalar product at each space  $T_p M$ .

Obviously given a submanifold  $\Sigma \hookrightarrow M$  of a (pseudo-)Riemannian manifold there is a induced field  $q := g|_{\Sigma}$ . If (M, g) is a Riemannian manifold,  $(\Sigma, q)$  is also a Riemannian manifold. If (M, g) is a pseudo-Riemannian manifold it may happen, that the induced metric q is Riemannian, pseudo-Riemannian, or degenerate (and this property may vary in various points of  $\Sigma$ ).

The matrix elements of a metric tensor has a natural interpretation as angles:

#### **Definition A.51.** Angles in Riemannian manifold

Given two tangent vectors  $X, Y \in T_pM$ , where (M, g) is a Riemannian manifold, the angle  $\alpha_{XY}$  between X and Y is defined by the formula

$$\cos \alpha_{XY} := \frac{g_p(X,Y)}{\sqrt{g_p(X,X)g_p(Y,Y)}} \tag{A.32}$$

This definition generalises in a natural way to the angle between two intersecting curves: given  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1(0) = \gamma_2(0) = p$ , the angle between them at p is the angle between the their velocity vectors.

The metric tensor allows to define other geometric quantities, i.e. k-volume:

#### **Definition A.52.** *k*-volume

Given a k-dimensional region  $\mathcal{R}^{(k)} \hookrightarrow M$  in a Riemannian manifold M (k may be equal to dim M) the k-volume form is the k-differential form  $dV^{(k)}$  on  $\mathcal{R}^{(k)}$  define in a coordinates  $(x^i)_{i=1,\dots,k}$  at  $\mathcal{R}^{(k)}$  as

$$\mathrm{d}V^k := \sqrt{\mathrm{det}\,(q_{ij})} \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k \tag{A.33}$$

Although defined in a specific choice of coordinates, the k-volume form does not depend on the coordinate chart. For k = 1 the volume form is called length form, for k = 2 it is called area form, for k = 3 it is called volume form, for higher k it is called k-volume form. The total k-volume of a region  $\mathcal{R}^{(k)}$  is given by

$$V\left(\mathcal{R}^{(k)}\right) := \int_{\mathcal{R}^{(k)}} \mathrm{d}V^{(k)} \tag{A.34}$$

(and it is called length, area, volume, and k-volume respectively) and is additive and invariant under a change of coordinates.

# A.2.2.2 Affine connections

Let us define a connection on a vector bundle  $(B \to M)_V$ :

#### **Definition A.53.** Connection on a vector bundle

Consider a vector bundle  $(B \to M)_V$  and a tangent bundle TM. A connection on B is a map

$$\nabla : \Xi (TM) \times \Xi (B)_{M,V} \to \Xi (B)_{M,V}$$
$$\nabla : (X,Y) \mapsto \nabla_X Y$$
(A.35)

satisfying the following three conditions:

• It is linear in the second argument, i.e.

$$\nabla_X \left( aY + bY' \right) = a \nabla_X Y + b \nabla_X Y' \quad \text{for } a, b \in \mathbb{C}$$
(A.36)

• It is functionally linear in first argument, i.e.:

$$\nabla_{fX+gX'}Y = f\nabla_X Y + g\nabla_{X'}Y \quad \text{for } f, g, \in \mathcal{C}^{\infty}(M)$$
(A.37)

• It satisfies the Leibniz rule when multiplying the second argument by a function:

$$\nabla_X f Y = f \nabla_X Y + X (f) \cdot Y \quad \text{for } f \in \mathcal{C}^\infty (M)$$
(A.38)

In other words a connection is a directed differential operator on the fields of a bundle B, where the direction is defined by a tangent field on M.

A very trivial example of a connection is the *Lie derivative* (see definition A.41). For connections other than Lie derivative the operation (A.35) is usually called *covariant derivative* with respect to the connection  $\nabla$ .

Since a connection is functionally linear in the tangent field, one can associate a one form with each connection.

#### **Definition A.54.** Connection one-form

Given a connection  $\nabla$  on  $(B \to M)_V$ , its connection one-form  $\omega$  is a one-form valued in End  $(V) = V \otimes V^*$  defined as

$$\nabla_X Y =: \partial_X Y + \omega(X) Y \tag{A.39}$$

Consider now a coordinate chart  $(x^i)$  on M and a basis  $e_I$  on the vector space V. The connection one-form can be expressed in this basis as

$$\omega = \omega_J^I = \omega_{J,k}^I \mathrm{d}x^k \tag{A.40}$$

and the derivative along the connection is given by

$$\nabla_i Y^I = \partial_i Y^I + \omega^I_{J,i} Y^J \tag{A.41}$$

A connection one-form determines fully the connection, thus we will say connection  $\omega$  instead of connection one-form  $\omega$  of the connection  $\nabla$ . Having a connection  $\omega$  we denote the covariant derivative by  $\stackrel{(\omega)}{\nabla}$ .

Assume now, that the tensor field Y is a G-tensor for some gauge group G.

#### **Definition A.55.** Gauge covariant connection

A connection is called G-covariant, iff it transforms under gauge transformations in such a way, that the covariant derivative does not change (it is gauge invariant), i.e.

$$\nabla_X Y = \nabla^g_{X^g} Y^g \tag{A.42}$$

this leads to conditions on the action of gauge-transformation on connection one-forms. A simple calculation leads to the formula

$$\omega^{g} = \rho\left(g^{-1}\right)\omega\rho\left(g\right) + \rho\left(g^{-1}\right)d\left[\rho\left(g\right)\right]$$
(A.43)

where  $\rho$  is the representation of G on the bundle  $(B \to M)_V$ . The action is non-linear, and that's why the connection one-form is not considered a G-tensor no matter that it is a section of the bundle  $(\tilde{B} \to M)_{V^* \otimes V}$ .

The equivalence classes of the gauge action (A.43) on the connections are called the gauge invariant connections.

Consider now connections defined on a bundle  $(B \to M)_V$  such that V is the defining representation of the group G. The gauge action of the group simplifies to

$$\omega^g = g^{-1}\omega g + g^{-1}\mathrm{d}g \tag{A.44}$$

Moreover, note that the Lie algebra  $\mathfrak{g}$  of the group G is the subspace of  $V^* \otimes V$ . Let us then consider only these connections, that takes values in  $\mathfrak{g}$ . Such connection can be very easily generalised to other G-bundles:

# Definition A.56. G-connection

A connection  $\omega$  is called G-connection for the group G, if it is defined by a one-form that takes values in  $\mathfrak{g}^A = \mathfrak{ig}$ , which is G-covariant.

A G-connection with the base manifold M defines a G-invariant connection on each bundle of G-tensors over M. It acts on the tensors via the representation  $T\rho$  of the Lie algebra  $\mathfrak{g}^A$  tangent to appropriate representation  $\rho$  of the gauge group G:

$$\stackrel{(\omega)}{\nabla}_X Y := \partial_X Y + T\rho(\omega(X))Y \tag{A.45}$$

Covariant derivatives with respect to different fields in general do not commute. The measure of non-commutativity of a connection is the Curvature form:

#### **Definition A.57.** Curvature form of a connection

Given a connection  $\omega$  the curvature form is a Aut (V)-valued 2-form on M, which action is given by

$$\Omega(X_1, X_2) Y := \overset{(\omega)}{\nabla}_{X_1} \overset{(\omega)}{\nabla}_{X_2} Y - \overset{(\omega)}{\nabla}_{X_2} \overset{(\omega)}{\nabla}_{X_1} Y - \overset{(\omega)}{\nabla}_{[X_1, X_2]} Y$$
(A.46)

One can prove, that if  $\omega$  is a G-covariant connection, its curvature form  $\Omega$  is a G-tensorvalued 2-form. For G-connections the following equation holds

$$\Omega = \mathrm{d}\omega + \frac{1}{2} \left[\omega, \omega\right] \tag{A.47}$$

A special class of connections are the connections on the tangent bundle to the manifold. Although they are connections on a vector bundle, we will call them *connections defined on manifolds*:

#### **Definition A.58.** Connection on a manifold

A connection on a manifold M is a  $GL(n, \mathbb{R})$ -connection on the bundle TM (where  $n = \dim M$ ). The connection one-form of such connection is denoted by  $\Gamma$  instead of  $\omega$ . Given a coordinate chart  $(x^i)$  on M and related frame  $(\partial_i)$  and coframe  $(dx^i)$  the matrix elements of  $\Gamma$  are given by

$$\Gamma_{j,i}^{k} = \mathrm{d}x^{k} \lrcorner \left[ \stackrel{(\Gamma)}{\nabla}_{i} \partial_{j} \right]$$
(A.48)

and they are called Christofel symbols of second kind.

Consider a coordinate transformation  $\phi : (x^i) \mapsto (y^a)$ . Because of  $GL(n, \mathbb{R})$ -covariance the Christofel symbols transform as

$$\Gamma_{b,a}^{c} = \Gamma_{j,i}^{k} \frac{\partial x^{i}}{\partial y^{a}} \frac{\partial x^{j}}{\partial y^{b}} \frac{\partial y^{c}}{\partial x^{k}} + \frac{\partial y^{c}}{\partial x^{u}} \frac{\partial^{2} x^{u}}{\partial y^{a} \partial y^{b}}$$
(A.49)

Since  $\Gamma$  is a  $GL(n, \mathbb{R})$ -connection, it can be naturally generalised to any bundle of multivectors or k-forms on M.

Although the connection form has one upper and two lower indices, it is not a (1, 2)-tensor field, because the transformation rules (A.49) are not linear. Moreover, the connection acts on two tangent fields, but they are not treated equally. The antisymmetry of such connection is measured by Torsion tensor:

Definition A.59. Torsion of a connection on a manifold

Given a connection  $\Gamma$  on M the torsion of this connection is the field  $\stackrel{(1)}{T}$  defined as follows:

$$\overset{(\Gamma)}{T}(X,Y) := \overset{(\Gamma)}{\nabla}_X Y - \overset{(\Gamma)}{\nabla}_Y X - [X,Y]$$
(A.50)

One can prove, that for each connection on M the torsion is a (1,2)-tensor field.

An example of a connection on a manifold is the Levi-Civita connection. It is defined as follows:

#### Definition A.60. Levi-Civita connection

Given a (pseudo-)Riemannian manifold (M, g) the Levi-Civita connection is the SO (g)connection  $\stackrel{(L-C)}{\nabla}$  defined by the following property:

- It is torsionless:  $\overset{(L-C)}{T} = 0$
- Covariant derivative of the metric tensor vanishes:  $\stackrel{(L-C)}{\nabla}_i g = 0$

One can prove, that there is a unique connection satisfying these both conditions for a fixed (pseudo-)metric field. The Christofel symbols of Levi-Civita connections can be expressed by derivatives of the metric field:

$$\Gamma_{j,k}^{i} = g^{il} \frac{1}{2} \left( \frac{\partial g_{lk}}{\partial x^{k}} + \frac{\partial g_{lj}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{l}} \right)$$
(A.51)

# A.2.2.3 Parallel transport and holonomies

Connections gives a way to connect the fibre spaces of a fibre bundle in separate points of the base manifold. This is done using the holonomies. In order to define holonomies let us first introduce selfparallelness with respect to a connection.

### **Definition A.61.** Self-parallel field

Given a connection  $\omega$  on a bundle  $(B \to M)_V$  we say, that the field  $Y \in \Xi(B)$  is self-parallel with respect to  $\omega$  along a path  $\gamma$ : ]-1,1[  $\to M$  iff for each  $t \in$  ]-1,1[ we have  $\nabla_{X^{\gamma}}Y_{\gamma(t)} = 0.$ 

Consider now a single tensor  $Y_p \in V$ . One can extend it in a self-parallel way along any path passing through p via the parallel transport

# **Definition A.62.** Parallel transport of a tensor

Given a connection  $\omega$  on a bundle  $\left(B \xrightarrow{\pi} M\right)_V$ , a tensor  $Y_p \in \pi^{-1}(p)$  and a path  $\gamma : ]-1, 1[ \to M$  we say, that the parallel transport with respect to  $\omega$  of  $Y_p$  along the path  $\gamma$  is the function  $Y : ]-1, 1[ \to \pi^{-1}(\gamma(]-1,1[))$  such that for each  $t \in ]-1, 1[$  we have  $\stackrel{(\omega)}{\nabla}_{X^{\gamma}}Y(t) = 0$  (where obviously  $Y(t) = Y_{\gamma(t)} \in \pi^{-1}(\gamma(t))$ , but we will use the notation without the subscript).

The condition for parallel transport can be expressed in terms of the connection form

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \omega|_{\gamma(t)}Y(t) = 0 \tag{A.52}$$

(where  $\omega|_{\gamma(t)} := \omega \lrcorner X_{\gamma(t)}^{\gamma}$ ), which in coordinates gives the so called *geodesic equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}Y^{I}(t) + \omega^{I}_{J,i}\left(X^{\gamma}\right)^{i}Y^{J} = 0 \tag{A.53}$$

Since one can parallel transport each tensor, one can transport all the fibre, which is a vector space. Thanks to the linearity of the covariant derivative, the geodesic equation is a linear differential equation on a vector space:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -\left.\omega\right|_{\gamma(t)}U(t) \tag{A.54}$$

This equation can be integrated to a linear operator on a vector space.

# Definition A.63. Path ordered exponential

The unique solution of the equation (A.54) with the initial condition U(0) = 1 is called the path ordered exponential, and denoted as  $\mathcal{P}\exp\left(\int_0^t -\omega|_{\gamma}\right)$  where  $\omega|_{\gamma}$  is a one-form on the interval [0,t] defined as  $\omega|_{\gamma}(s) := \omega|_{\gamma(s)} ds$ .

Given a compact path  $\gamma: [0,1] \to M$  the path ordered exponential along the full path will be called the *holonomy*:

### **Definition A.64.** Holonomy

A holonomy of a connection  $\omega$  along a path  $\gamma : [0,1] \to M$  is  $U_{\gamma}[\omega] := \mathcal{P} \exp\left(\int_{0}^{1} - \omega|_{\gamma}\right)$ . Sometimes we will also use the shortcut notation  $U_{\ell}[\omega] = \mathcal{P} \exp\left(\int_{\ell} -\omega\right)$ .

Consider now a holonomy on a bundle of G-tensors. The following statement is true:

Claim A.2. Gauge transformation of a holonomy of a connection

Given a G-covariant connection  $\omega$  that transforms according to (A.43), the holonomies of  $\omega$  transform as

$$g(\cdot): U_{\ell}[\omega] \mapsto U_{\ell}[\omega^{g}] = \rho\left[g^{-1}\left(\ell\left(1\right)\right)\right]\left(U_{\ell}[\omega]\right)\rho\left[g\left(\ell\left(0\right)\right)\right]$$
(A.55)

Likewise, given the G-connection  $\omega$ , that transforms according to (A.44), the holonomies of  $\omega$  transform as

$$g(\cdot): U_{\ell}[\omega] \mapsto U_{\ell}[\omega^g] = g^{-1}(\ell(1))(U_{\ell}[\omega])g(\ell(0))$$
(A.56)

Proof

It is enough to prove the claim for G-connection. The proof for general G-covariant connections goes analogously.

Consider both sites of the equation (A.56) as functions of the parametrisation of the curve. The left-hand site is

$$U_{\ell}\left[\omega^{g}\right](t) := \mathcal{P}\exp\left(\int_{0}^{t} - \omega^{g}|_{\ell}\right)$$
(A.57)

The right-hand site is

$$U_{\ell}^{g}\left[\omega\right]\left(t\right) := g^{-1}\left(\ell\left(t\right)\right) \mathcal{P}\exp\left(\int_{0}^{t} - \left.\omega\right|_{\ell}\right) g\left(\ell\left(0\right)\right) \tag{A.58}$$

Let us differentiate both  $U_{\ell}[\omega^g](t)$  and  $U_{\ell}^g[\omega](t)$  with respect to the t parameter. We get for the left hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{\ell}\left[\omega^{g}\right]\left(t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\exp\left(\int_{0}^{t} - \left.\omega^{g}\right|_{\ell}\right) = -\left.\omega^{g}\right|_{\ell(t)}\mathcal{P}\exp\left(\int_{0}^{t} - \left.\omega^{g}\right|_{\ell}\right) \tag{A.59}$$

and for the right hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t} U_{\ell}^{g} \left[\omega\right](t) = \frac{\mathrm{d}}{\mathrm{d}t} \left[g^{-1}\left(\ell\left(t\right)\right) \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g\left(\ell\left(0\right)\right)\right] \\
= \frac{\mathrm{d}g^{-1}\left(\ell\left(t\right)\right)}{\mathrm{d}t} \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g\left(\ell\left(0\right)\right) \\
+ g^{-1}\left(\ell\left(t\right)\right) \left(-\omega|_{\ell\left(t\right)}\right) \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g\left(\ell\left(0\right)\right)$$
(A.60)

Inserting the identity  $\mathbb{1} = g(\ell(t)) g^{-1}(\ell(t))$  before both  $\mathcal{P} \exp\left(\int_0^t - \omega|_\ell\right)$  we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} U_{\ell}^{g} [\omega] (t) = \frac{\mathrm{d}g^{-1} (\ell(t))}{\mathrm{d}t} g(\ell(t)) g^{-1} (\ell(t)) \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g(\ell(0)) 
-g^{-1} (\ell(t)) \omega|_{\ell(t)} g(\ell(t)) g^{-1} (\ell(t)) \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g(\ell(0)) 
= \left(\frac{\mathrm{d}g^{-1} (\ell(t))}{\mathrm{d}t} g(\ell(t)) - g^{-1} (\ell(t)) \omega|_{\ell(t)} g(\ell(t)) g^{-1} (\ell(t))\right) 
\cdot \left[g^{-1} (\ell(t)) \mathcal{P} \exp\left(\int_{0}^{t} - \omega|_{\ell}\right) g(\ell(0))\right]$$
(A.61)

Note, that  $\frac{\mathrm{d}g^{-1}(\ell(t))}{\mathrm{d}t}g(\ell(t)) = -g^{-1}(\ell(t))\frac{\mathrm{d}g(\ell(t))}{\mathrm{d}t}$  (indeed, since  $0 = \mathrm{d}\mathbb{1}$  we have  $0 = \mathrm{d}\left(g^{-1} \cdot g\right) = (\mathrm{d}g^{-1})g + g^{-1}\mathrm{d}g$ ), and that the last term in (A.61) is precisely  $U_{\ell}^{g}[\omega](t)$ . Thus we get

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{\ell}^{g}\left[\omega\right]\left(t\right) = \left(\frac{\mathrm{d}g^{-1}\left(\ell\left(t\right)\right)}{\mathrm{d}t}g\left(\ell\left(t\right)\right) - g^{-1}\left(\ell\left(t\right)\right)\omega|_{\ell\left(t\right)}g\left(\ell\left(t\right)\right)g^{-1}\left(\ell\left(t\right)\right)\right)U_{\ell}^{g}\left[\omega\right]\left(t\right)$$
(A.62)

but the factor in the bracket is precisely  $\omega^{g}|_{\ell(t)}$  (see (A.44)), so

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{\ell}^{g}\left[\omega\right]\left(t\right) = \left.\omega^{g}\right|_{\ell\left(t\right)}U_{\ell}^{g}\left[\omega\right]\left(t\right) \tag{A.63}$$

So both  $U_{\ell}[\omega^g](t)$  and  $U_{\ell}^g[\omega](t)$  satisfy the same differential equation. Since the initial condition also match (for  $U_{\ell}[\omega^g]$  we have  $U_{\ell}[\omega^g](0) = 1$  by definition of  $\mathcal{P} \exp(\cdot)$ , for  $U_{\ell}^g[\omega]$  we have  $U_{\ell}^g[\omega](0) = g^{-1}(\ell(0)) \mathbb{1}g(\ell(0)) = 1$ ), whole functions match, and thus the holonomy of the transformed connection  $U_{\ell}[\omega^g]$  equals the transformed holonomy  $U_{\ell}^g[\omega]$ .

Quod erat demonstrandum.

**Remark A.4.** The above notions were introduced in terms of connections valued in  $\mathfrak{g}^A$ . They can be easily translated to  $\mathfrak{g}^H$  by noticing, that the canonical isomorphism between  $\mathfrak{g}^A$  and  $\mathfrak{g}^H$  is  $\mathfrak{g}^A \ni X^A \mapsto \frac{1}{i}X^A =: X^H \in \mathfrak{g}^H$ . Thus given a  $\mathfrak{g}^A$ -valued connection  $\omega$  there is a  $\mathfrak{g}^H$ -valued connection  $\tilde{\omega} = \frac{1}{i}\omega$ . A geodesic equation for  $\mathfrak{g}^H$ -valued connection is

$$\frac{\mathrm{d}}{\mathrm{d}t}Y^{I}(t) + i\tilde{\omega}_{J,i}^{I}(X^{\gamma})^{i}Y^{J} = 0 \qquad (A.64)$$

and the path-ordered exponential is the unique solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -i\,\tilde{\omega}|_{\gamma(t)}\,U(t) \tag{A.65}$$

and we can write it as  $\mathcal{P} \exp\left(i\int_0^t - \tilde{\omega}|_{\gamma}\right)$ .

# A.3 Harmonic analyses on SU(2)

Both Spin-foams and Operator Spin-network Diagrams are topological objects colored by some tensors. Most of these tensors are elements of Hilbert spaces build of representation spaces of SU(2)-group. Let us thus provide here a short introduction to the harmonic analysis on SU(2) and the algebra of SU(2)-tensors.

We start with fixing conventions about denoting SU(2)-elements. Then we introduce spin- and tensor-representations of SU(2). Finally we define spin network functions.

The SU(2) is a widely known group. The detailed study of the representation theory on SU(2) can be found for example in [99] and [86, chap. 2]. The detailed introduction to spin network functions can be found in for example in [100].

# A.3.1 Conventions

# A.3.1.1 Angular momentum operators

The hermitian generators of SU(2) will be denoted by  $\hat{L}_x$ ,  $\hat{L}_y$  and  $\hat{L}_z$ . In the defining representation they are given by halves of the hermitian Pauli matrices

$$\hat{L}_x = \frac{\sigma_1}{2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \hat{L}_y = \frac{\sigma_2}{2} = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad \hat{L}_z = \frac{\sigma_3}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
(A.66)

and they have the following commutation relations:

$$i\left[\hat{L}_{i},\hat{L}_{j}\right] = -\epsilon_{ij}{}^{k}\hat{L}_{k} \tag{A.67}$$

# A.3.1.2 Parametrisation of SU(2)

Elements of SU(2)-group will be denoted by letters u, v or h. A general  $u \in SU(2)$  element can be parametrised by 3 angles:

$$u = u\left(\theta, \alpha, \beta\right) = \begin{pmatrix} e^{i\frac{\alpha-\beta}{2}}\cos\frac{\theta}{2} & e^{i\frac{\alpha+\beta}{2}}\sin\frac{\theta}{2} \\ -e^{-i\frac{\alpha+\beta}{2}}\sin\frac{\theta}{2} & e^{-i\frac{\alpha-\beta}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$
(A.68)

the ranges of the angles are

$$\theta \in [-\pi, \pi]$$
  $\alpha \in ]-2\pi, 2\pi]$   $\beta \in [0, 2\pi[$  (A.69)

Although the range of  $\alpha$  may be surprising, it must be so in order to admit all possible combinations of phases.

The elements of a U(1)-subgroup of rotations along the z-axis are denoted by

$$u_{\alpha} := u\left(0, \alpha, 0\right) = \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0\\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix} = \exp\left(i\alpha\hat{L}_{z}\right)$$
(A.70)

Note, that here also  $\alpha \in [-2\pi, 2\pi]$ , and that  $u(0, 2\pi, 0) = -1$ . The elements  $u_{\alpha}$  act on general SU(2)-elements as follows:

$$u_{\alpha}u\left(\theta,\alpha',\beta\right) = u\left(\theta,\alpha+\alpha',\beta\right) \tag{A.71}$$

It is convenient to derive matrix elements of a group element which corresponds to the rotation of the z-axis to a given direction  $\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ , being the element  $u_{\vec{n}} \in SU(2)$  such that

$$u_{\vec{n}}^{\dagger}\sigma_3 u_{\vec{n}} = \vec{n} \cdot \vec{\sigma} \tag{A.72}$$

Obviously  $u_{\vec{n}}$  is defined up to left multiplication by a rotation around z-axis, i.e. if  $u_{\vec{n}}$  satisfies (A.72), then  $u_{\alpha}u_{\vec{n}}$  also does. In calculations we will often use the  $u_{\vec{n}}$  defined in such a way, that it's diagonal elements are real. In such convention the matrix elements are

$$u_{\vec{n}} = \begin{pmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \\ -e^{-i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = u\left(\theta,\phi,\phi\right)$$
(A.73)

and the group element  $u_{\vec{n}}$  of a vector  $\vec{n}$  is sometimes denoted as n.

### A.3.2 Representations of SU(2)

In general a SU(2)-tensor is an object on which we can act with a linear representation of the SU(2)-group. The simplest case is SU(2)-vector, i.e. elements of spin-*j* Hilbert space:

# **Definition A.65.** Spin-j Hilbert space

Given a non-negative half-integer number  $j \in \frac{\mathbb{Z}_+}{2} \cup \{0\}$  the spin-j Hilbert  $\mathcal{H}_j$  space is the carrier vector space of (2j + 1)-dimensional unitary irreducible representation of SU(2)-group. Its elements will be denoted by ket notation:  $|x\rangle_j$ . The action of SU(2) on  $\mathcal{H}_j$  will be denoted by

$$SU(2) \times \mathcal{H}_j \ni (u, x) \mapsto u |x\rangle_i \in \mathcal{H}_j$$
 (A.74)

The spin-j Hilbert space has a basis that diagonalise the  $\hat{L}_z$  operator, i.e.  $|m\rangle_i$  for  $m \in \{-j, -j+1, \dots, j-1, j\}$ . The action of the generator of the z-rotation  $\hat{L}_z$  is given by:

$$\hat{L}_{z} \left| m \right\rangle_{j} = m \left| m \right\rangle_{j} \tag{A.75}$$

Unless explicitly stated otherwise we use the summation convention over spin-indices:

$$|x\rangle_j = x^m |m\rangle_j := \sum_m x^m |m\rangle_j \tag{A.76}$$

There is of course the basis diagonalising  $\hat{L}_x$  and  $\hat{L}_y$ , we denote them as  $|m_x\rangle_i$  and  $|m_y\rangle_i$  respectively. The z-angular momentum basis can be also denoted with z-subscript:  $|m_z\rangle_i$ ; however whenever there is no subscript, we assume that m refers to  $\hat{L}_z$ -basis.

The spin-j Hilbert spaces are eigenspaces of the Casimir operator  $\hat{L^2}$  with the eigenvalue j(j+1).

There are two special cases of spin-j Hilbert spaces. First is the scalar space  $\mathcal{H}_0 = \mathbb{C}$ . In most cases we will simply omit its vectors and treat them as numbers. Second one is the defining representation  $\mathcal{H}_{\frac{1}{2}} = \mathbb{C}^2$ . We will denote its basis as  $|\uparrow\rangle := \left|\frac{1}{2}\right\rangle_{\frac{1}{2}}$  and  $|\downarrow\rangle := \left|-\frac{1}{2}\right\rangle_{\frac{1}{2}}$ (of course there are also  $|\uparrow_x\rangle$ ,  $|\downarrow_x\rangle$  and  $|\uparrow_y\rangle$ ,  $|\downarrow_y\rangle$ ). The higher spin representations can be seen as symmetrized tensor products of the defining representation, i.e.

$$\mathcal{H}_{j} = \underbrace{\mathcal{H}_{\frac{1}{2}} \odot \cdots \odot \mathcal{H}_{\frac{1}{2}}}_{2j \text{ times}} \qquad |m\rangle_{j} = N_{m}^{j} \underbrace{|\downarrow\rangle \odot \cdots \odot |\downarrow\rangle}_{(j+m) \text{ times}} \odot \underbrace{|\downarrow\rangle \odot \cdots \odot |\downarrow\rangle}_{(j-m) \text{ times}} \qquad (A.77)$$

where  $N_m^j := {\binom{2j}{j+m}}^{\frac{1}{2}}$  is a normalisation factor (see [86]). The dual space  $\mathcal{H}_j^*$  is the space of linear functionals on  $\mathcal{H}_j$ , its elements are denoted either by kets  $|y\rangle_{j^*}$  or by bras:  $\langle y|_j$ , or by lower indices:  $\langle y|_j = \langle n|_j y_n$ .

The group SU(2) acts on  $\mathcal{H}_j$  via Wigner matrices:

# **Definition A.66.** Wigner matrices

Wigner matrices are matrix representations of action of SU(2) on  $\mathcal{H}_i$  spaces. They are defined as

$$SU(2) \ni u \mapsto D^{j}(u)_{m}^{n} := \langle n | u | m \rangle_{j} \in \mathbb{C}$$
 (A.78)

thus action of SU(2) on  $\mathcal{H}_i$  is given by the formula

$$SU(2) \times \mathcal{H}_j \ni (u, x) \mapsto u |x\rangle_j = D^j (u)_m^n x^m |n\rangle_j \in \mathcal{H}_j$$
(A.79)

and on  $\mathcal{H}_{i}^{*}$  the group SU(2) acts via the hermitian conjugation of Wigner matrices

$$SU(2) \times \mathcal{H}_j^* \ni (u, y) \mapsto \langle y | u = \overline{D^j(u)_m^n} y_n \langle m |_j$$
 (A.80)

Using the (A.77) one can get an exact formula for matrix elements of Wigner matrices (see [86]):

$$D^{j}(u)_{m}^{n} = {\binom{2j}{j+m}}^{\frac{1}{2}} {\binom{2j}{j+n}}^{-\frac{1}{2}}$$

$$+ \sum_{k} {\binom{j+m}{k}} {\binom{j-m}{j+n-k}} (u_{+}^{+})^{k} (u_{-}^{+})^{j+n-k} (u_{-}^{-})^{j+m-k} (u_{-}^{-})^{k-m-n}$$
(A.81)

where  $u_{+}^{+} = \langle \uparrow | u | \uparrow \rangle$ ,  $u_{-}^{+} = \langle \uparrow | u | \downarrow \rangle$ ,  $u_{+}^{-} = \langle \downarrow | u | \uparrow \rangle$  and  $u_{-}^{-} = \langle \downarrow | u | \downarrow \rangle$ . Imposing the unitarity and unit-determinant conditions on  $u_{\pm}^{\pm}$  leads to further simplifications of this formula.

Thanks to the Peter-Weyl theorem the Wigner Matrices matrix elements form an orthogonal set of square-integrable functions on SU(2). The orthogonality relation is

$$\int_{SU(2)} \mathrm{d}u \overline{D^{j}(u)_{n}^{m}} D^{j'}(u)_{n'}^{m'} = \delta^{j,j'} \delta^{mm'} \delta_{nn'} \frac{1}{2j+1}$$
(A.82)

Given several spins one can introduce a product representation space:

#### **Definition A.67.** Product representation

Consider a tensor product of k spin-j Hilbert spaces  $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_k}$ . We will denote them shortly as  $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_k} = \mathcal{H}_{j_1} \otimes \cdots \otimes j_k = \mathcal{H}_{\vec{j}}$ . The basis of this spaces will be denoted as  $|m_1, \ldots m_k\rangle_{j_1 \otimes \cdots \otimes j_k}$  or shortly:  $|\vec{m}\rangle_{\vec{j}}$ . This space has a natural action of SU(2) given by:

$$SU(2) \times \mathcal{H}_{\vec{j}} \ni (u,T) \mapsto D^{j_1}(u)_{m_1}^{n_1} \cdots D^{j_k}(u)_{m_k}^{n_k} T^{m_1 \cdots m_k} \left| \vec{n} \right\rangle_{\vec{j}} \in \mathcal{H}_{\vec{j}}$$
(A.83)

It can be simply generalised to tensors with both upper and lower indices: for  $T_{m_{k+1}\cdots m_{k+l}}^{m_1\cdots m_k} \in \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_k} \otimes \mathcal{H}_{j_{k+1}}^* \otimes \cdots \otimes \mathcal{H}_{j_{k+l}}^*$  we have

$$SU(2) \times \mathcal{H}_{\vec{j}} \ni (u, T) \mapsto D^{j_1}(u)_{m_1}^{n_1} \cdots D^{j_k}(u)_{m_k}^{n_k} \\ \cdot \overline{D^{j_{k+1}}(u)_{n_{k+1}}^{m_{k+1}} \cdots D^{j_{k+l}}(u)_{n_{k+l}}^{m_{k+l}}} \\ \cdot T^{m_1 \cdots m_k}_{m_{k+1} \cdots m_{k+l}} |\vec{n}\rangle_{\vec{j}} \in \mathcal{H}_{\vec{j}}$$
(A.84)

we will often use a shortcut notation:

$$D^{\vec{j}}(u)_{\vec{m}}^{\vec{n}}T^{\vec{m}} := D^{j_1}(u)_{m_1}^{n_1}\cdots D^{j_k}(u)_{m_k}^{n_k} \cdot \overline{D^{j_{k+1}}(u)_{n_{k+1}}^{m_{k+1}}\cdots D^{j_{k+l}}(u)_{n_{k+l}}^{m_{k+l}}} \cdot T^{m_1\cdots m_k}_{m_{k+1}\cdots m_{k+l}}$$
(A.85)

Among the SU(2)-tensors there are tensors, that does not change under the action of the SU(2)-group. They are called *invariant tensors*:

# **Definition A.68.** SU(2)-invariant tensors

A tensor  $T^{\vec{m}} \in \mathcal{H}_{\vec{i}}$  is SU(2)-invariant iff

$$\forall_{u \in SU(2)} D^{\vec{j}} \left( u \right)_{\vec{m}}^{\vec{n}} T^{\vec{m}} = T^{\vec{n}}$$
(A.86)

The space of SU(2)-invariant tensors will be denoted by  $Inv\left(\mathcal{H}_{\vec{j}}\right)$ . The projection onto  $Inv\left(\mathcal{H}_{\vec{j}}\right)$  is

$$\widehat{P_{\text{Inv}}} := \int_{SU(2)} \mathrm{d}u \ D^{\vec{j}}(u)_{\vec{m}}^{\vec{n}} \tag{A.87}$$

Sometimes we will expand the formula (A.87) in the spin-z-basis:

$$\widehat{P_{\text{Inv}}} = \int_{SU(2)} \mathrm{d}u \ D^{j_1}(u)_{m_1}^{n_1} \cdots D^{j_k}(u)_{m_k}^{n_k} \cdot \overline{D^{j_{k+1}}(u)_{n_{k+1}}^{m_{k+1}} \cdots D^{j_{k+l}}(u)_{n_{k+l}}^{m_{k+l}}} \tag{A.88}$$

or simplify to a formal expression

$$\widehat{P_{\text{Inv}}} = \int_{SU(2)} \mathrm{d}u \ u \tag{A.89}$$

The invariant tensors are often denoted by  $\iota$  instead of T, and in ket-notation as  $|\iota\rangle_{\vec{i}}$ 

The necessary condition for a tensor  $\iota \in \mathcal{H}_{\vec{j}}$  to be invariant is  $\hat{L}_z |\iota\rangle_{\vec{j}} = 0$  (as well as  $\hat{L}_x |\iota\rangle_{\vec{j}} = \hat{L}_y |\iota\rangle_{\vec{j}} = 0$ ). This put a constraint on the multi-index  $\vec{m}$  - since  $|\vec{m}\rangle_{\vec{j}}$  are eigenstates of  $\hat{L}_z$  with the eigenvalue  $\sum \epsilon_i m_i$  (where  $\epsilon_i = -1$  if the spin  $j_i$  is present in  $\mathcal{H}_{\vec{j}}$ as a dual representation, namely  $\mathcal{H}_{j_i}^*$ , and  $\epsilon_i = +1$  otherwise), the invariant tensors has non-zero only the components  $\iota^{\vec{m}} |\vec{m}\rangle_{\vec{j}}$  for  $\vec{m}$  such that  $\sum \epsilon_i m_i = 0$ . Thanks to rotations of the generators one can prove that some tensor spaces may contain no nontrivial invariant tensors. In order to have a non-trivial SU(2)-invariant subspace, the following condition must be satisfied:

$$\forall_{i=1,\dots,k} j_i \le \sum_{a \ne i} j_a \tag{A.90}$$

called generalised triangle inequality.

For invariant tensors one can also prove a convenient identity that were used in the main text of the theses:

# **Theorem A.1.** On functions of invariant tensors

Given two invariant tensors  $\iota, \iota' \in \mathcal{H}_{\vec{j}}$  (where  $\vec{j} = (j_1, \ldots, j_n) |\iota\rangle_{\vec{j}} = \sum_{\vec{m}} \iota_{\vec{m}} |\vec{m}\rangle_{\vec{j}}$  and  $|\iota'\rangle_{\vec{j}} = \sum_{\vec{m}} \iota'_{\vec{m}} |\vec{m}\rangle_{\vec{j}}$ ) and a function  $f : \mathbb{R}^{2n} \to \mathbb{C}$  being a uniform limit of a series of polynomials we have

$$\sum_{\vec{m}} f\left(m_1^2, \dots, m_n^2; j_1, \dots, j_n\right) \overline{\iota_{\vec{m}}} \iota_{\vec{m}}' = f\left(\frac{m_1\left(m_1+1\right)}{3}, \dots, \frac{m_n\left(m_n+1\right)}{3}; j_1, \dots, j_n\right) \langle \iota | \left| \iota' \right\rangle_{\vec{j}}$$
(A.91)

The proof of this theorem can be found in subsection 6.5.2.

# A.3.3 Spin-networks

Let us now introduce Spin-network states. The detailed definitions and deeper study of spin-networks can be found in [100, 19]. Here we introduce a usable definitions, without discussing their origins.

Spin network states are functions on configurations of a gauge field on a graph. Thus first let us define the gauge configuration on a graph. Then we define the spin-network functions for the SU(2)-group and express them in two pictures - as a function on a group and as a tensor product of invariant tensors. Finally we discuss how to generalise this picture to other groups.

# A.3.3.1 Gauge field on a graph

Consider a manifold M with a SU(2)-connection defined by a  $\mathfrak{su}(2)$ -valued SU(2)covariant one-form  $\omega$  (see Appendix A.2.2.2). Recalling the Appendix A.2.2.3, given a
path  $\ell : [0,1] \to M$  one can define a holonomy of  $\omega$  along  $\ell$  as  $U_{\ell}[\omega] := \mathcal{P} \exp\left(i \int_{\ell} - \omega|_{\ell}\right)$ ,
that transforms under the gauge transformation  $h(\cdot)$  as

$$h(\cdot): U_{\ell}[\omega] \mapsto h(t(\ell))^{-1} U_{\ell}[\omega] h(s(\ell))$$
(A.92)

Consider now a graph  $\mathcal{G}$  embedded in  $\mathcal{M}$ . One can define a holonomy of the field  $\omega$  along the graph  $\mathcal{G}$  as a functional  $\operatorname{Hol}_{\mathcal{G}} : \Omega^1(\mathcal{M}) \otimes \mathfrak{su}(2) \to SU(2)^L$  defined by the formula:

$$\operatorname{Hol}_{\mathcal{G}}: A \mapsto \{U_{\ell}[\omega]\}_{\ell \in \mathbf{L}_{\mathcal{G}}} := \left\{ \mathcal{P} \exp\left(i \int_{\ell} -\omega\right) \right\}_{\ell \in \mathbf{L}_{\mathcal{G}}}$$
(A.93)

One can check, that for  $\omega$  being SU(2)-connection the holonomy of  $\omega$  along the graph  $\mathcal{G}$  transforms under a gauge transformations according to the formula:

$$h(\cdot) : \left\{ U_{\ell}[A] \right\}_{\ell \in \mathbf{L}_{\mathcal{G}}} \mapsto \left\{ U_{\ell}\left[A^{h}\right] \right\}_{\ell \in \mathbf{L}_{\mathcal{G}}} = \left\{ h\left(t(\ell)\right)^{-1} U_{\ell}[A] h\left(s(\ell)\right) \right\}_{\ell \in \mathbf{L}_{\mathcal{G}}}$$
(A.94)

Consider now a collection of SU(2)-elements  $\{U_\ell\}_{\ell \in \mathbf{L}_{\mathcal{G}}}$  and action of N copies of SU(2)-group on them (N is the power of the set of nodes  $\mathbf{N}_{\mathcal{G}}$ ), similar to (A.94), namely

$$SU(2)^N \ni \{h_n\}_{n \in \mathbf{N}_{\mathcal{G}}} : \{U_\ell\}_{\ell \in \mathbf{L}_{\mathcal{G}}} \mapsto \left\{h_{s(\ell)}^{-1} U_\ell h_{t(\ell)}\right\}_{\ell \in \mathbf{L}_{\mathcal{G}}}$$
(A.95)

Let us call  $\{U_\ell\}_{\ell\in\mathbf{L}_{\mathcal{G}}}$  a SU(2)-holonomy on the graph  $\mathcal{G}$ , and the action of  $SU(2)^N$  will be called the SU(2)-gauge action. A SU(2)-holonomy can be defined for a graph embedded in a manifold (then it might be a holonomy of some connection field on this manifold), as well as for an abstract, non-embedded graph. The equivalence classes of SU(2)holonomies on the graph  $\mathcal{G}$  of the gauge action of  $SU(2)^N$  are called gauge-equivalent configurations of SU(2)-holonomies. The set of all gauge-equivalent configurations will be denoted by  $SU(2)^L/_{\mathcal{G}}SU(2)^N$ , where  $\mathcal{G}$  indicates the precise action of the gauge transformations (of course even if  $\mathcal{G}$  and  $\mathcal{G}'$  have the same number of links and nodes, the spaces  $SU(2)^L/_{\mathcal{G}}SU(2)^N$  and  $SU(2)^L/_{\mathcal{G}'}SU(2)^N$  are in general different).

#### A.3.3.2 Spin-network functions

Let us define a general spin-network function.

# Definition A.69. A spin-network function

A spin-network function is a function  $\mathcal{N}_{\mathcal{G}j_{\ell}j_nv_n\iota_n}: SU(2)^L \to \mathbb{C}$  determined by a graph  $\mathcal{G} = (\mathbf{N}, \mathbf{L})$ , colored as follows:

- each link of the graph  $\ell \in \mathbf{L}$  is colored by a spin  $j_{\ell}$
- each node of the graph  $n \in \mathbf{N}$  is colored by a triple: a spin  $j_n$ , a vector  $v_n \in \mathcal{H}_{j_n}$ and an invariant tensor  $\iota_n \in \operatorname{Inv} \left[ \mathcal{H}_{j_n}^{\star} \otimes \left( \bigotimes_{\ell^+ \in \mathcal{L}_n^+} \mathcal{H}_{j_\ell} \right) \otimes \left( \bigotimes_{\ell^- \in \mathcal{L}_n^-} \mathcal{H}_{j_\ell}^{\star} \right) \right]$

Given a set of SU(2)-elements  $\{U_\ell\}_{\ell \in \mathbf{L}}$  the evaluation of the spin-network function is given by

$$\mathcal{N}_{\mathcal{G}j_{\ell}j_{n}v_{n}\iota_{n}}\left(\{U_{\ell}\}_{\ell\in\mathbf{L}}\right) = \prod_{\ell\in\mathbf{L}}\sqrt{2j_{\ell}+1} \ D^{j_{\ell}}\left(U_{\ell}\right)_{m_{\ell}+}^{m_{\ell}-} \cdot \prod_{n\in\mathbf{N}} \iota_{n} m_{\ell_{1}}^{m_{\ell_{1}}+\cdots m_{\ell_{k}}} u_{n} m_{j_{n}} \cdot v_{n}^{m_{j_{n}}} \left(A.96\right)$$

where k is the number of positive half-links and l is the number of negative half-links at the corresponding node.

Spin-network functions span a basis of the Hilbert space  $L^2(SU(2)^{\mathbf{L}})$  (see [19]).

Consider a graph embedded in a manifold  $\mathcal{M}$  on which one defined a connection oneform A. Then a spin-network state defines a functional on the connection one-form by the formula

$$\mathcal{N}_{\mathcal{G}j_{\ell}j_{n}v_{n}\iota_{n}}\left[A\right] := \mathcal{N}_{\mathcal{G}j_{\ell}j_{n}v_{n}\iota_{n}}\left(\left\{U_{\ell} = \mathcal{P}\exp\left(i\int_{\ell}-A\right)\right\}_{\ell\in\mathbf{L}}\right)$$
(A.97)

Since the gauge transformations act on connections (as it was defined in Appendix A.2.2.2 and recalled in Appendix A.3.3.1), they also effects the value of a spin-network functional on a connection. In order to have a functional insensitive on the gauge choice we define the gauge-invariant spin-network functions:

**Definition A.70.** Gauge-invariant spin-network functions

A spin-network function  $\mathcal{N}_{\mathcal{G}j_{\ell}j_n v_n \iota_n}$  is a gauge-invariant spin-network function iff all the spins  $j_n = 0$ . Thus the vectors  $v_n$  are redundant (since they only multiply the tensors  $\iota_n$  by numbers) and thus the gauge-invariant spin-network functions can be described in a simpler way as  $\mathcal{N}_{\mathcal{G}j_{\ell}\iota_n}$ , where  $\iota_n \in \operatorname{Inv}\left[\left(\bigotimes_{\ell^+ \in \mathcal{L}_n^+} \mathcal{H}_{j_\ell}\right) \otimes \left(\bigotimes_{\ell^- \in \mathcal{L}_n^-} \mathcal{H}_{j_\ell}^*\right)\right]$ , and evaluated as

$$\mathcal{N}_{\mathcal{G}j_{\ell}\iota_{n}}\left(\{U_{\ell}\}_{\ell\in\mathbf{L}}\right) = \prod_{\ell\in\mathbf{L}}\sqrt{2j_{\ell}+1}D^{j_{\ell}}\left(U_{\ell}\right)_{m_{\ell}+}^{m_{\ell}-} \cdot \prod_{n\in\mathbf{N}}\iota_{n}\frac{m_{\ell_{1}}+\cdots+m_{\ell_{k}}}{u_{n}m_{\ell_{1}}-\cdots+m_{\ell_{l}}}$$
(A.98)

where k is the number of positive half-links and l is the number of negative half-links at the corresponding node. We may say, that a gauge-invariant spin-network function is a function  $\mathcal{N}_{\mathcal{G}j_{\ell\ell}n}: SU(2)^{\mathbf{L}}/_{\mathcal{G}}SU(2)^{\mathbf{N}} \to \mathbb{C}$ .

# Structure of spin-network Hilbert spaces

Let us now study the structure of the Hilbert space of all spin-network functions. In order to do this it is convenient to name the Hilbert spaces of the form

$$\mathcal{H}_{n} := \left(\bigotimes_{\ell^{+} \in \mathcal{L}_{n}^{+}} \mathcal{H}_{j_{\ell}}\right) \otimes \left(\bigotimes_{\ell^{-} \in \mathcal{L}_{n}^{-}} \mathcal{H}_{j_{\ell}}^{\star}\right)$$
(A.99)

as the node Hilbert spaces and

$$\mathcal{H}_{n}^{\mathrm{Inv}} := \mathrm{Inv}\left[\left(\bigotimes_{\ell^{+} \in \mathcal{L}_{n}^{+}} \mathcal{H}_{j_{\ell}}\right) \otimes \left(\bigotimes_{\ell^{-} \in \mathcal{L}_{n}^{-}} \mathcal{H}_{j_{\ell}}^{\star}\right)\right]$$
(A.100)

the *invariant node Hilbert spaces* (such Hilbert spaces appeared already in definitions A.69 and A.70 at each node). We will use this notation in what follows.

Consider now all gauge-equivalent configurations of SU(2)-holonomies on a graph  $\mathcal{G}$ , i.e. all equivalence classes of gauge action on the space of SU(2)-holonomies on  $\mathcal{G}$  (see Appendix A.3.3.1). These configurations form a compact topological space  $SU(2)^L/_{\mathcal{G}}SU(2)^N$ . Let us call the the space of all functions square-integrable with respect to L copies of Haar measure on that space as  $\mathcal{H}_{\mathcal{G}}$ , i.e.  $\mathcal{H}_{\mathcal{G}} := L^2 \left( SU(2)^L/_{\mathcal{G}}SU(2)^N ; (d\mu_{\text{Haar}})^L \right)$ . Gaugeinvariant spin-network functions are a dense subset of  $\mathcal{H}_{\mathcal{G}}$ . They can be used to describe the structure of  $\mathcal{H}_{\mathcal{G}}$  [6, 100].

Each coloring of links of  $\mathcal{G}$  by spins  $\{j_\ell\}$  labels a separate orthogonal component, thus:

$$\mathcal{H}_{\mathcal{G}} = \bigoplus_{\{j_\ell\}_{\ell \in \mathbf{L}_{\mathcal{G}}}} \mathcal{H}_{\mathcal{G},\{j_\ell\}}$$
(A.101)

Given a fixed spin-labels note, that in a gauge-invariant spin-network each node of n labelled by a tensor  $\iota_n \in \mathcal{H}_n$ . Thus the space of all gauge-invariant spin-networks with fixed spin-labels contains a copy of  $\mathcal{H}_n$  for each node. Since the choice of the invariant tensors is independent for different nodes, the node Hilbert spaces are tensor-multiplied. Since there is no more freedom for gauge-invariant spin-network functions, we have

$$\mathcal{H}_{\mathcal{G},\{j_\ell\}} = \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \mathcal{H}_n^{\mathrm{Inv}}$$
(A.102)

Thus the Hilbert space of all gauge invariant spin-network functions of a given graph is

$$\mathcal{H}_{\mathcal{G}} = \bigoplus_{\{j_\ell\}_{\ell \in \mathbf{L}_{\mathcal{G}}}} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \mathcal{H}_n^{\mathrm{Inv}}$$
(A.103)

Given an orthonormal basis of each of the intertwiner spaces  $\mathcal{H}_n$  (denoted by  $(\iota_n^I)_{I=1,\dots,\dim \mathcal{H}_n}$ ) we have the orthonormal basis of the space of spin-network functions:

$$\mathcal{H}_{\mathcal{G}} = \overline{\operatorname{Span}\left(\mathcal{N}_{\mathcal{G}\{j_{\ell}\}\{\iota_{n}^{I}\}}\right)} \text{ all possible } \{j_{\ell}\}$$

$$\operatorname{all possible } \{\iota_{n}^{I}\} \text{ for given } \{j_{\ell}\}$$

$$(A.104)$$

The Hilbert space  $\mathcal{H}_{\mathcal{G}}$  is often called the graph Hilbert space or the space of spinnetworks on  $\mathcal{G}$ . The space  $\mathcal{H}_{\mathcal{G},\{j_\ell\}}$  is called fixed-spin Hilbert space on  $\mathcal{G}$ ; it is often denoted shortly as  $\mathcal{H}_{\mathcal{G},\vec{j}}$ . The spaces  $\mathcal{H}_n$  and  $\mathcal{H}_n^{\text{Inv}}$  were already named the node Hilbert space and the invariant node Hilbert space. The spaces  $\mathcal{H}_{\mathcal{G}}$  and  $\mathcal{H}_{\mathcal{G},\{j_\ell\}}$  are gauge invariant by definition.

Spin-network functions are often used as vectors in a Hilbert space, not as functions on a group. Because of this we often refer to them as to *spin-networks*, not *spin-network-functions*.

#### A.3.3.3 Spin-network-like states for other groups

The framework of spin-network functions can be easily generalised to any compact Lie group G. The generalisation is defined as follows.

A spin-network-like function is determined by the following modification of the definition A.69 : Each spin  $j_{\ell}$  and  $j_v$  is substituted by a irreducible unitary representation  $\rho_{\ell}$  or  $\rho_v$  of *G*-group respectively. The Wigner matrices are replaced by appropriate matrix elements of corresponding representations. The factors  $\sqrt{2j_{\ell}+1}$  are replaced by the dimension of the appropriate representation  $\sqrt{d_{\rho_{\ell}}}$ . The invariant tensors  $\iota_n$  are now invariant tensors of the appropriate representations of *G*-group, the vectors  $v_n$  are elements of the representation carrier Hilbert space  $\mathcal{H}_{\rho_v}$ . Thus the evaluation of the spin-network function is

$$\mathcal{N}_{\mathcal{G}\rho_{\ell}\rho_{n}v_{n}\iota_{n}}\left(\{g_{\ell}\}_{\ell\in\mathbf{L}}\right) = \prod_{\ell\in\mathbf{L}}\sqrt{d_{\rho_{\ell}}}\rho_{\ell}\left(g_{\ell}\right)^{a_{\ell^{-}}}_{b_{\ell^{+}}} \cdot \prod_{n\in\mathbf{N}}\iota_{n}a^{b_{\ell^{+}_{1}}\cdots b_{\ell^{+}_{k}}}_{a_{\ell^{-}_{1}}\cdots a^{-}_{\ell^{-}_{k}}c_{j_{n}}} \cdot v_{n}^{c_{j_{n}}} \qquad (A.105)$$

The same substitutions are straightforward through all the formulae in Appendix A.3.3.2. Thus in the end the space of gauge-invariant G-spin-network-like functions is decomposes into

$$\mathcal{H}_{\mathcal{G}}^{G} = \bigoplus_{\{\rho_{\ell}\}_{\ell \in \mathbf{L}_{G}}} \bigotimes_{n \in \mathbf{N}_{\mathcal{G}}} \mathcal{H}_{n}^{G, \operatorname{Inv}}$$
(A.106)

with  $\mathcal{H}_n^{G,\operatorname{Inv}} := \operatorname{Inv}\left[\left(\bigotimes_{\ell^+ \in \mathcal{L}_n^+} \mathcal{H}_{\rho_\ell}\right) \otimes \left(\bigotimes_{\ell^- \in \mathcal{L}_n^-} \mathcal{H}_{\rho_\ell}^{\star}\right)\right]$  being the node Hilbert spaces of *G*-group.

Spin-network-like states can be also defined for a noncompact Lie group G, however, the Peter-Weyl theorem is not valid for non-compact groups and the matrix elements of irreducible representations of G are not any more square-integrable functions on G - they are distributions.

# A.4 The $SL(2,\mathbb{C})$ group

The  $SL(2, \mathbb{C})$  group, being the universal cover of proper orthochronous Lorentz group  $SO(1,3)^+$ , is the group of key importance in the dynamics of General Relativity. In this section we recall some basic properties of this group and fix notation conventions used
used in this thesis. This section is based mostly on the textbook [86], and partially on [101], however conventions used here are more similar to [37].

In subsection A.4.1 we focus on algebraic properties of the group, we show it's relation with the Lorentz group and show the decomposition of  $SL(2, \mathbb{C})$  into rotations and boosts. In subsection A.4.2.3 present the brief recall of representation theory of the  $SL(2, \mathbb{C})$ group. In subsection A.4.3 we discuss some technical issues related to the EPRL map.

# A.4.1 Algebra of $SL(2,\mathbb{C})$ elements

In the defining representation the  $SL(2,\mathbb{C})$  group is the set of  $2 \times 2$  complex matrices with the determinant equal to 1:

$$SL(2,\mathbb{C}) \ni g = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix}$$
 :  $g_1^1 g_2^2 - g_2^1 g_1^2 = 1$  (A.107)

In this subsection we fix our convention about relating  $SL(2, \mathbb{C})$  elements with Lorentz transformations, then we show, how is the SU(2)-subgroup immersed in  $SL(2, \mathbb{C})$ , and finally we present some formulae allowing to decompose a  $SL(2, \mathbb{C})$ -element into *pure* rotation and *pure boost*.

#### A.4.1.1 Lorentz representation

Consider a 4-vector x in the Minkowski space. One can assign to it a hermitian  $2 \times 2$  complex matrix  $X := x^{\mu} \sigma_{\mu}$ , where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(A.108)

Note, that det  $(X) = \eta_{\mu\nu} x^{\mu} x^{\nu}$ . Indeed:

$$\det (X) = \det \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$
(A.109)

Given an element  $g \in SL(2, \mathbb{C})$  one can define its action on x as follows. Let  $Y = g^{\dagger}Xg$ , where  $Y = y^{\mu}\sigma_{\mu}$ . We say that  $\Lambda_g x = y$ . It is easy to check, that this representation preserved the scalar product  $\eta$ . Indeed:

$$\eta_{\mu\nu}y^{\mu}y^{\nu} = \det\left(Y\right) = \det\left(g^{\dagger}Xg\right) \underset{\det(g)=1}{=} \det\left(X\right) = \eta_{\mu\nu}x^{\mu}x^{\nu}$$
(A.110)

thus  $\Lambda_q$  is a Lorentz transformation.

Since  $SL(2,\mathbb{C})$  is connected, it covers only a connected component of SO(1,3) containing the unity, namely  $SO(1,3)^+$ . Moreover, one can easily check, that  $\Lambda_{-g} = \Lambda_g$ , thus each  $SO(1,3)^+$  element is represented by a pair of  $SL(2,\mathbb{C})$  elements.

#### A.4.1.2 Little group

Given any hermitian matrix X one can identify a subgroup  $H < SL(2, \mathbb{C})$  preserving this matrix. This subgroup is called the little group of the vector  $x^{\mu}$  (where  $X = x^{\mu}\sigma_{\mu}$ ).

If the matrix X is positive-definite, the vector  $x^{\mu}$  is timelike and future-pointing. Indeed, the determinant of a positive-defined matrix is positive, thus  $x^{\mu}x^{\nu}\eta_{\mu\nu} = \det(X) > 0$ , so the vector is timelike. Now consider a matrix element  $\langle \uparrow | X | \uparrow \rangle = x^0 + x^3$ . If  $x^{\mu}$  were past-pointing, we would have  $x^0 < 0$ . Since  $x^{\mu}$  is timelike, obviously  $|x^0| > |x^3|$ , thus  $x^0 + x^3 < 0$ , so X is would not be positive-definite.

If X is positive-definite, it defines a hermitian scalar product on  $\mathbb{C}^2$ . A group that preserves a hermitian scalar product is the group of unitary matrices U(2). Its intersection with  $SL(2,\mathbb{C})$  are the unitary matrices with unit determinant, i.e. SU(2). Thus given a timelike future-pointing vector  $x^{\mu} \in \mathbb{R}^4$  or given a hermitian positive-definite matrix  $X \in \mathbb{C}^{2 \times 2}$  we can pick one of SU(2)-subgroups of  $SL(2,\mathbb{C})$ .

The Lorentz transformations that preserve a timelike future-pointing vector  $x^{\mu}$  are the rotations of the 3-space orthogonal to  $x^{\mu}$ . Thus the little group of proper orthochronous Lorentz group SO(1,3) of  $x^{\mu}$  is SO(3), being doubly covered by SU(2).

Each particular choice of SU(2)-subgroup of  $SL(2, \mathbb{C})$  can be interpreted as a choice of time direction in the space-time. All the elements of Lorentz invariant theory are invariant under such change of coordinates. Thus in most cases we chose the SU(2)subgroup preserving the natural hermitian scalar product in the defining space  $\mathbb{C}^2$  of  $SL(2, \mathbb{C})$ , which corresponds to the choice of timelike vector x = (1, 0, 0, 0).

### A.4.1.3 Decomposition into boost and rotation

Having chosen the little subgroup  $SU(2) < SL(2, \mathbb{C})$  one can provide a decomposition of  $g \in SL(2, \mathbb{C})$  into the unitary and non-unitary part. In this subsection we pick the SU(2) subgroup corresponding to the vector x = (1, 0, 0, 0).

One can perform the polar decomposition of g. Let  $H = g^{\dagger}g$ . Obviously H is a positive hermitian matrix (since for each  $|\psi\rangle$  we have  $\langle\psi|H|\psi\rangle = ||g\psi|| \ge 0$ ), thus it has a unique positive hermitian square root k. The matrix  $u := gk^{-1}$  is unitary, thus for each g we have a unique decomposition

$$g = uk \tag{A.111}$$

for unitary u and hermitian k. Obviously both u and k can be treated as  $SL(2, \mathbb{C})$ elements (with appropriately trivial polar decomposition). One can perform another decomposition (by choosing  $H' = gg^{\dagger}$ ) into g = k'u' for unitary u' and hermitian k', where in general  $u' \neq u$  and  $k' \neq k$ .

The unitary subgroup of  $SL(2,\mathbb{C})$  is SU(2), thus we have  $u \in SU(2)$  and  $k \in SL(2,\mathbb{C})/SU(2) := H^3$ . The choice of the little group  $SU(2) < SL(2,\mathbb{C})$  corresponds to the choice of the hermitian scalar product with respect to which we consider u and k to be unitary and hermitian respectively. Although  $H^3$  is the quotient space  $SL(2,\mathbb{C})/SU(2)$ , there is a natural embedding  $H^3 \hookrightarrow SL(2,\mathbb{C})$ . For each element of the quotient space there is a unique hermitian representative. Thus we will usually consider  $H^3$  as a subset of  $SL(2,\mathbb{C})$ .

In order to characterise the quotient space  $H^3$  let us consider a general hermitian  $k = k^{\dagger} \in SL(2, \mathbb{C})$ . In the defining representation it can be written as

$$k(\eta, \theta, \phi) = \begin{pmatrix} \cosh \frac{\eta}{2} + \cos \theta \sinh \frac{\eta}{2} & e^{i\phi} \sin \theta \sinh \frac{\eta}{2} \\ e^{-i\phi} \sin \theta \sinh \frac{\eta}{2} & \cosh \frac{\eta}{2} - \cos \theta \sinh \frac{\eta}{2} \end{pmatrix}$$
(A.112)

It is easy to check that given a unit vector  $\vec{n}_{\theta,\phi}$  the element  $k(\eta,\theta,\phi)$  is

$$k(\eta,\theta,\phi) = u_{\vec{n}_{\theta,\phi}}^{\dagger} \begin{pmatrix} e^{\frac{\eta}{2}} & 0\\ 0 & e^{-\frac{\eta}{2}} \end{pmatrix} u_{\vec{n}_{\theta,\phi}} = u_{\vec{n}_{\theta,\phi}}^{\dagger} \exp\left(i\eta\hat{K}_z\right) u_{\vec{n}_{\theta,\phi}}$$
(A.113)

for  $u_{\vec{n}_{\theta,\phi}}$  given by (A.73) and  $\hat{K}_z = \frac{\sigma_3}{2i}$ . Introducing two other anti-hermitian Pauli matrices  $\hat{K}_x = \frac{\sigma_1}{2i}$  and  $\hat{K}_y = \frac{\sigma_2}{2i}$ , recalling (A.72) and noting, that for unitary u we have

 $u^{-1} \exp(A) u = \exp(u^{-1}Au)$ , we can express  $k(\eta, \theta\phi)$  as

$$k(\eta, \theta, \phi) = \exp\left(i\eta \vec{n}_{\theta, \phi} \cdot \vec{K}\right) = \exp\left(i\vec{\eta} \cdot \vec{K}\right) =: k(\vec{\eta})$$
(A.114)

Thus the hermitian elements  $k \in SL(2, \mathbb{C})$  can be labelled by a vector  $\vec{\eta} = \eta \vec{n}_{\theta,\phi} \in \mathbb{R}^3$ . We will often identify  $H^3$  with  $\mathbb{R}^3$ , however,  $H^3$  has a measure induced from  $SL(2, \mathbb{C})$ 's Haar measure, being different from the natural measure on  $\mathbb{R}^3$ .

The elements of  $SU(2) < SL(2, \mathbb{C})$  are called *pure rotations*, because in the Lorentz representation they correspond to rotations of the space  $x^{\perp} \subset \mathbb{R}^4$ . The elements of  $H^3 \subset SL(2, \mathbb{C})$  are called pure boosts, because in the Lorentz representation the element  $k(\vec{\eta})$  boost the vector x in the direction  $\eta$  (with the velocity monotonic with  $|\vec{\eta}|$ ). The generators  $\hat{L}_i$  (introduced in Appendix A.3) and  $\hat{K}_j$  are called *rotation generators* and *boost generators* respectively. Their commutation relations are following:

$$i\left[\hat{L}_{i},\hat{L}_{j}\right] = -\epsilon_{ij}{}^{k}\hat{L}_{k} \qquad i\left[\hat{L}_{i},\hat{K}_{j}\right] = -\epsilon_{ij}{}^{k}\hat{K}_{k} \qquad i\left[\hat{K}_{i},\hat{K}_{j}\right] = \epsilon_{ij}{}^{k}\hat{L}_{k} \qquad (A.115)$$

Combining (A.113) with (A.111) one can obtain another decomposition of a general  $SL(2, \mathbb{C})$  element:

$$g = u^{-1}k_{\eta}v \tag{A.116}$$

for  $u, v \in SL(2, \mathbb{C})$  and  $k_{\eta} := \exp\left(i\eta \hat{K}_z\right)$ . This decomposition is not unique, because the z-rotations  $u_{\phi}$  commute with z-boosts, so  $g = u^{-1}k_{\eta}v = u^{-1}u_{\phi}^{-1}k_{\eta}u_{\phi}v$ . Nevertheless, this decomposition is very useful in direct calculations.

# A.4.2 Harmonic analysis on $SL(2, \mathbb{C})$

A very good and detailed study of harmonic analysis on  $SL(2, \mathbb{C})$ -group can be found in [86]. Here we summarize the aspects of the theory, which are used in our thesis.

We start with finite dimensional representation of the  $SL(2, \mathbb{C})$ -group (i.e. the spinor representations), which are non-unitary. Then we describe the principal series of unitary representations of  $SL(2, \mathbb{C})$  (without going into derivations, but simply by characterising these properties and these matrix elements, that are necessary in our calculations). Finally we recall some formula for Haar measure on  $SL(2, \mathbb{C})$ .

#### A.4.2.1 Spinor representations of $SL(2,\mathbb{C})$

The generators, when treated as complex matrices, are proportional to corresponding generators of rotations. This leads to a conclusion, that  $SL(2, \mathbb{C})$ -group can be obtained by a complexification of SU(2). Indeed, consider the formula (A.68) for  $u(\theta, \alpha, \beta)$ . Matrix elements are combinations of trigonometric and exponent functions of the angles. The determinant of  $u(\theta, \alpha, \beta)$  is

$$\det\left(u\left(\theta,\alpha,\beta\right)\right) = e^{i\frac{\alpha-\beta}{2}}e^{-i\frac{\alpha-\beta}{2}}\cos^2\frac{\theta}{2} + e^{i\frac{\alpha+\beta}{2}}e^{i\frac{\alpha+\beta}{2}}\sin^2\frac{\theta}{2}$$
(A.117)

It is equal to 1 thanks properties of exponent and trigonometric functions, that hold also for complex arguments. However, the unitarity of  $u(\theta, \alpha, \beta)$  is ensured by the reality of the arguments. Thus by complexifying the angles one obtains a general  $2 \times 2$  complex matrix with determinant equal to 1, i.e. a general  $SL(2, \mathbb{C})$  element.

Given a general  $SL(2, \mathbb{C}) \ni g(z_1, z_2, z_3)$  consider a Wigner matrix  $D^{(j)}(\cdot)$ . Its matrix elements are analytic functions of matrix elements of its argument. Since  $g(z_1, z_2, z_3)$  is complexified  $u(\theta, \alpha, \beta)$ , one can calculate the Wigner matrix of it, obtaining  $D^{(j)}(g)$ .

The spin-j Wigner matrices of  $SL(2, \mathbb{C})$  elements are invertible operators acting on  $\mathcal{H}_j$ . However, they are not unitary, except from the SU(2) subgroup of  $SL(2, \mathbb{C})$ . They form finite-dimensional irreducible non-unitary representations of  $SL(2, \mathbb{C})$ .

#### A.4.2.2 Unitary representations of $SL(2,\mathbb{C})$

All unitary representations of  $SL(2, \mathbb{C})$  are infinite dimensional (except the trivial one). Let us consider here the principal series of unitary representations of  $SL(2, \mathbb{C})$ . They are labelled by a pair of a real, positive number p and a non-negative half-integer k.

The carrier space of each of (p, k)-representations is a subspace of the space of squareintegrable functions on a sphere  $\mathcal{H}^{(p,k)} \subset L^2(S^2)$ . The precise action of  $SL(2, \mathbb{C})$  on these functions is crucial for the derivation and can be found in [86, chapter 3.]. However, it is irrelevant for our considerations, thus we shall treat it as given.

Let us introduce a convenient basis in (p, k). Fixing a SU(2) subgroup of  $SL(2, \mathbb{C})$  decomposes  $\mathcal{H}^{(p,k)}$  into subspaces invariant under the action of SU(2). On each of such subspace the little group acts via spin-*j* representation, for a different spin:

$$\mathcal{H}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j^{(p,k)} \tag{A.118}$$

where the sum goes through j = k + n for a nonnegative integer n (i.e. there is no  $j = k + \frac{1}{2}$  etc.).<sup>1</sup> Fixing the z-direction (i.e. choosing a little group U(1) of SU(2)) splits each  $\mathcal{H}_{j}^{(p,k)}$  into eigenstates of  $\hat{L}_{z}$ , so that there is a basis in  $\mathcal{H}^{(p,k)}$ :

$$\mathcal{H}^{(p,k)} = \boxed{\begin{array}{c} \operatorname{Span} & \left(|j,m\rangle_{(p,k)}\right) \\ j \in k + \mathbb{Z} \\ m \in \{-j, -j+1, \dots, j\} \end{array}}$$
(A.119)

the states  $|j,m\rangle_{(p,k)}$  will be sometimes denoted as  $|(p,k);j,m\rangle$ . In this basis the matrix elements of g will be denoted in one of the following ways:

$$D^{(p,k)}(g)_{j_2,m}^{j_1,n} = \langle j_1, n | g | j_2, m \rangle_{(p,k)} = \langle (p,k); j_1, n | g | (p,k); j_2, m \rangle$$
(A.120)

Of course  $D^{(p,k)}(g)_{j_{2},m}^{j_{1},n}$  is an infinite dimensional unitary matrix, even for  $g \in SL(2,\mathbb{C})$  non-unitary in the defining representation.

As we have mentioned, given an element  $u \in SU(2) < SL(2, \mathbb{C})$ , it acts on  $|j, m\rangle_{(p,k)}$  via spin-j Wigner matrices, so

$$\langle j_1, n | u | j_2, m \rangle_{(p,k)} = \delta_{j_1, j_2} D^{(j_1)} (u)_m^n$$
 (A.121)

Thanks to the formula (A.113) it is now enough to find the matrix elements of  $k_{\eta} = \exp\left(\eta \hat{K}_z\right)$ . Since  $\hat{K}_z$  commutes with  $\hat{L}_z$ ,  $D^{(p,k)}(k_{\eta})_{j_2,m}^{j_1,n}$  is diagonal in m, n:

$$D^{(p,k)}(k_{\eta})_{j_{2},m}^{j_{1},n} = \delta_{m}^{n} D^{(p,k)}(k_{\eta})_{j_{2},m}^{j_{1},m}$$
(A.122)

However, it is not diagonal in  $j_1, j_2$ . For general  $j_1$  and  $j_2$  it is given by the following integral:

$$D^{(p,k)}(k_{\eta})_{j_{2},m}^{j_{1},m} = \sqrt{(2j_{1}+1)(2j_{2}+1)} \int_{0}^{1} \mathrm{d}t \frac{d_{k,m}^{j_{1}}(2t-1)d_{k,m}^{j_{2}}(2t_{d}-1)}{[te^{-\eta}+(1-t)e^{\eta}]^{1-ip}}$$
(A.123)

<sup>&</sup>lt;sup>1</sup> The absence of subspaces with spin lower then k follows from certain assumptions on the regularity of the functions  $f \in L^2(S^2)$ , that are chosen to build  $\mathcal{H}^{(p,k)}$ .

where

$$t_d = \frac{te^{-\eta}}{te^{-\eta} + (1-t)e^{\eta}}$$
(A.124)

and  $d_{k,q}^{j}(\cos \theta) = D^{(j)}(u(\theta, 0, 0))_{m}^{k}$ , i.e.

$$d_{k,m}^{j}(\cos\theta) = {\binom{2j}{j+m}}^{\frac{1}{2}} {\binom{2j}{j+k}}^{-\frac{1}{2}}$$

$$(A.125)$$

$$\cdot \sum_{n} {\binom{j+m}{n}} {\binom{j-m}{j+k-n}} (-1)^{j+k-n} \left(\cos\frac{\theta}{2}\right)^{2n-m-k} \left(\sin\frac{\theta}{2}\right)^{2j-2n+m+k}$$

(see (A.81)). Putting  $\cos \theta = 2t - 1$  for  $t \in [0, 1]$  one gets  $\cos^2 \frac{\theta}{2} = t$  and  $\sin^2 \frac{\theta}{2} = 1 - t$ , thus the formula simplify

$$d_{k,m}^{j}(2t-1) = {\binom{2j}{j+m}}^{\frac{1}{2}} {\binom{2j}{j+k}}^{-\frac{1}{2}}$$

$$\cdot \sum_{n} {\binom{j+m}{n}} {\binom{j-m}{j+k-n}} (-1)^{j+k-n} (t)^{n-\frac{m+k}{2}} (1-t)^{j-n+\frac{m+k}{2}}$$
(A.126)

and thus

$$d_{k,m}^{j}(2t_{d}-1) = \left[\frac{1}{te^{-\eta} + (1-t)e^{\eta}}\right]^{j} {\binom{2j}{j+m}}^{\frac{1}{2}} {\binom{2j}{j+k}}^{-\frac{1}{2}} \cdot \sum_{n} {\binom{j+m}{n}} {\binom{j-m}{j+k-n}} (-1)^{j+k-n} \cdot (t)^{n-\frac{m+k}{2}} (1-t)^{j-n+\frac{m+k}{2}} e^{\eta(j+m+k-2n)}$$
(A.127)

Combining (A.123) with (A.126) and (A.127) we get

$$D^{(p,k)}(k_{\eta})_{j_{2},m}^{j_{1},m} = (2j_{1}+1)^{\frac{1}{2}}(2j_{2}+1)^{\frac{1}{2}}\binom{2j_{1}}{j_{1}+m}^{\frac{1}{2}}\binom{2j_{1}}{j_{1}+k}^{-\frac{1}{2}}\binom{2j_{2}}{j_{2}+m}^{\frac{1}{2}}\binom{2j_{2}}{j_{2}+k}^{-\frac{1}{2}} \cdot (-1)^{j_{1}+j_{2}+2k} e^{\eta(j_{2}+m+k)} \cdot \int_{0}^{1} dt \frac{(t)^{-(m+k)}(1-t)^{j_{1}+j_{2}+(m+k)}}{[te^{-\eta}+(1-t)e^{\eta}]^{1+j_{2}-ip}}$$
(A.128)  
  $\cdot \sum_{n_{1},n_{2}}\binom{j_{1}+m}{n_{1}}\binom{j_{1}-m}{j_{1}+k-n_{1}}\binom{j_{2}+m}{n_{2}}\binom{j_{2}-m}{j_{2}+k-n_{2}} \cdot (-1)^{n_{1}+n_{2}}(t)^{n_{1}+n_{2}}(1-t)^{-(n_{1}+n_{2})} e^{-2\eta n_{2}}$ 

(since k is half-integer, we cannot reduce  $(-1)^{2k}$ ). For each value of the  $n_1$  and  $n_2$  the integral over dt in (A.128) gives a Hypergeometric Function with some parameters, depending on  $p, k, j_1, j_2, m, n_1$  and  $n_2$ .

## A.4.2.3 Measures on $SL(2,\mathbb{C})$

In order to perform integrals over  $SL(2,\mathbb{C})$  group one has to introduce an invariant measure. We use the measure derived in [86]. Since  $SL(2,\mathbb{C})$  is non-compact, any

invariant measure is defined up to a multiplicative constant, which we fix consistently with the conventions presented there.

Given a parametrisation of  $SL(2,\mathbb{C})$  by a complex vector  $g^{\mu} \in \mathbb{C}^4$ 

$$g\left(g^{\mu}\right) = g^{0}\mathbf{1} + \vec{g}\cdot\vec{\sigma} \tag{A.129}$$

the measure is

$$dg = C_{SL(2,\mathbb{C})} \delta \left( g^{\mu} g^{\nu} \eta_{\mu\nu} - 1 \right) \prod_{\mu=0}^{3} Dg^{\mu}$$
(A.130)

for an arbitrary constant  $C_{SL(2,\mathbb{C})}$ , where for z = x + iy we have Dz = dxdy and  $\delta(z) = \delta(x) \delta(y)$ . Such a choice was introduced in [86, p.13].

One can introduce an analogous parametrisation of SU(2) by a real vector  $u^{\mu} \in \mathbb{R}^4$ :

$$u\left(u^{\mu}\right) = u^{0}\mathbb{1} + i\vec{u}\cdot\vec{\sigma} \tag{A.131}$$

In this parametrisation the Haar measure on SU(2) is

$$du = C_{SU(2)}\delta\left(\sum_{\nu=0}^{3} (u^{\nu})^2 - 1\right)\prod_{\mu=0}^{3} du^{\mu}$$
(A.132)

for an arbitrary constant  $C_{SU(2)}$ . In [86] Rühl postulates to set  $C_{SL(2,\mathbb{C})} = C_{SU(2)}^2$ , while the constant  $C_{SU(2)}$  can be fixed by the normalisation requirement  $\int_{SU(2)} du = 1$ . Such condition leads to

$$C_{SU(2)} = \pi^{-2} \tag{A.133}$$

thus, according to our convention

$$C_{SL(2,\mathbb{C})} = \pi^{-4} \tag{A.134}$$

We are especially interested in the measure expressed in terms of polar decomposition of g into  $g(u, \vec{\eta}) = u \cdot k(\vec{\eta})$ . In [86, appendix A] one can find, that given a decomposition  $g = u_1 k_{\eta} u_2$  the measure is

$$\mathrm{d}g = \frac{1}{4\pi} \mathrm{d}u_1 \mathrm{d}u_2 \sinh^2 \eta \mathrm{d}\eta \tag{A.135}$$

One can shift the decomposition into  $g = u_1 u_2^{-1} k_\eta u_2$  without change of the measure, thanks to invariance of du on the shift. Let us now focus on du. In terms of the parametrisation introduced in Appendix A.3.1.2 it is

$$du(\theta,\phi,\psi) = \frac{1}{(4\pi)^2} \sin\theta d\theta d\phi d\psi \qquad (A.136)$$

Note, that given  $u_2(\theta, \psi, \phi)$  one can decompose it into  $u_2(\theta, \psi, \phi) = e^{i(\psi-\phi)L_z}u_2(\theta, \phi, \phi)$ . The term  $e^{i\alpha L_z}$  commute with  $k_\eta$  (see Appendix A.4.1.3), thus we can transform the term  $k(\vec{\eta}) := u_2^{-1}(\theta, \phi, \psi) k_\eta u_2(\theta, \phi, \psi)$  as follows

$$k(\vec{\eta}) := u_2^{-1}(\theta, \phi, \psi) k_{\eta} u_2(\theta, \phi, \psi) = u_2^{-1}(\theta, \phi, \phi) e^{-i(\psi-\phi)L_z} k_{\eta} e^{i(\psi-\phi)L_z} u_2(\theta, \phi, \phi) = u_2^{-1}(\theta, \phi, \phi) e^{-i(\psi-\phi)L_z} e^{i(\psi-\phi)L_z} k_{\eta} u_2(\theta, \phi, \phi) = u_2^{-1}(\theta, \phi, \phi) k_{\eta} u_2(\theta, \phi, \phi)$$
(A.137)

obtaining the term independent on  $\psi$ . Thus we can perform the integral over  $\psi$  (which ranges over  $[-2\pi, 2\pi]$  - see (A.69)) reducing the measure term  $du_2$  to

$$du_2(\theta,\phi,\phi) = \frac{1}{4\pi}\sin\theta d\theta d\phi \qquad (A.138)$$

being proportional to the area element of a 2-sphere.

Collecting all this together we have

$$dg = \frac{1}{(4\pi)^2} \sin\theta d\theta d\phi \sinh^2\eta d\eta du_2$$
(A.139)

Let us now focus on the part corresponding to  $k(\vec{\eta})$ , i.e.

$$dk\left(\vec{\eta}\right) = \frac{1}{\left(4\pi\right)^2} \sin\theta d\theta d\phi \sinh^2\eta d\eta \qquad (A.140)$$

Note, that the natural volume element on  $\mathbb{R}^3$  is  $d^3\vec{\eta} = \eta^2 \sin\theta d\theta d\phi d\eta$ . This means, that  $dk(\vec{\eta})$  is proportional to  $d^3\vec{\eta}$ :

$$dk(\vec{\eta}) = \frac{1}{(4\pi)^2} \frac{\sinh^2 \eta}{\eta^2} d^3 \vec{\eta}$$
(A.141)

so that finally we can rewrite dg in the form that was used in chapter 6:

$$dg(\vec{\eta}, u) = \left(\frac{\sinh \eta}{4\pi\eta}\right)^2 d^3\vec{\eta} du \qquad (A.142)$$

#### A.4.3 EPRL map

In this section we introduce the Lorentzian EPRL map. We start with a recall of explanation of its construction. Then we discuss basic properties of the Y-map. Finally we apply the simplifications induced by the EPRL map to the formula (A.128) for matrix elements of a z-boost.

The Lorentzian EPRL map was first introduced in [41], then it was widely studied. Most of formulae are based on the version presented in [37].

#### A.4.3.1 Explanation

The EPRL map was designed to be weak solution to the simplicity constraint, i.e. to find the states  $|\psi\rangle_{(p,k)} \in \mathcal{K}^{(p,k)} \subset \mathcal{H}^{(p,k)}$  such that

$$\left\langle \psi \right| \vec{K} + \gamma \vec{L} \left| \psi' \right\rangle_{(p,k)} = 0 \tag{A.143}$$

In order to do that it is convenient to recall the matrix elements of the generators  $\vec{L}$  and  $\vec{K}$ . Let us decompose  $\vec{L}$  and  $\vec{K}$  into  $L_z$ ,  $K_z$  and  $L_{\pm} := L_x \pm iL_y$  and  $K_{\pm} := K_x \pm iK_y$ .<sup>2</sup> Obviously (A.143) is equivalent to

$$\begin{cases} \langle \psi | K_z + \gamma L_z | \psi' \rangle_{(p,k)} &= 0\\ \langle \psi | K_+ + \gamma L_+ | \psi' \rangle_{(p,k)} &= 0\\ \langle \psi | K_- + \gamma L_- | \psi' \rangle_{(p,k)} &= 0 \end{cases}$$
(A.144)

<sup>&</sup>lt;sup>2</sup>Note, that although in the defining representation  $K_i = \frac{1}{i}L_i$ , in the unitary representation this is not the case, because in the unitary representation both  $L_i$  and  $K_i$  are hermitian operators.

In the representation (p, k) the matrix elements of L and K are following (since all the vectors  $|j, m\rangle$  are in (p, k)-representation, to make the formulae more transparent we omit the subscript):

$$L_{z} |j,m\rangle = m |j,m\rangle$$

$$L_{\pm} |j,m\rangle = \sqrt{(j \mp m) (j + 1 \pm m)} |j,m \pm 1\rangle$$

$$K_{z} |j,m\rangle = -\alpha (j) \sqrt{j^{2} - m^{2}} |j - 1,m\rangle - \beta (j) m |j,m\rangle$$

$$+\alpha (j + 1) \sqrt{(j + 1)^{2} - m^{2}} |j + 1,m\rangle$$
(A.145)
$$K_{\pm} |j,m\rangle = -\alpha (j) \sqrt{(j \mp m) (j - 1 \mp m)} |j - 1,m \pm 1\rangle$$

$$-\beta (j) \sqrt{(j \mp m) (j + 1 \pm m)} |j,m \pm 1\rangle$$

$$-\alpha (j + 1) \sqrt{(j + 1 \pm m) (j + 2 \pm m)} |j + 1,m \pm 1\rangle$$

for

$$\alpha(j) = \frac{i}{j} \sqrt{\frac{(j^2 - k^2)(j^2 + p^2)}{4j^2 - 1}} \quad \beta(j) = \frac{kp}{j(j+1)}$$
(A.146)

(see [47]).

We postulate (the detailed argument is not a subject of our thesis, it can be found for example in [41]), that the the space of weak solutions of the constraint (A.143) is present iff  $p = \gamma (k + 1)$  and it is $\mathcal{K}^{(p,k)} = \mathcal{H}_k^{(p,k)}$ . In order to check it, let us calculate matrix elements of (A.144) between elements  $|k, m\rangle_{(p,k)}$  and  $|k, m'\rangle_{(p,k)}$  for  $p = \gamma (k + 1)$ :

$$\langle k, m' | K_z + \gamma L_z | k, m \rangle = -\beta (k) m \langle k, m' | | k, m \rangle + \gamma m \langle k, m' | | k, m \rangle$$
  
=  $m \delta_{m,m'} (\gamma - \beta (k))$  (A.147)

$$\langle k, m' | K_{\pm} + \gamma L_{\pm} | k, m \rangle = -\beta (k) \sqrt{(j \mp m) (j + 1 \pm m)} \langle k, m' | | k, m \pm 1 \rangle$$

$$+ \gamma \sqrt{(j \mp m) (j + 1 \pm m)} \langle k, m' | | k, m \pm 1 \rangle$$

$$= \sqrt{(j \mp m) (j + 1 \pm m)} \delta_{m \pm 1, m'} (\gamma - \beta (k))$$
(A.148)

and

$$(\gamma - \beta(k))|_{p=\gamma(k+1)} = \gamma - \frac{k\gamma(k+1)}{k(k+1)} = 0$$
 (A.149)

The spaces  $\mathcal{K}^{(p,k)}$  are in fact parametrised by one half-integer number k. Consider now a direct sum of all these spaces

$$\mathcal{K} = \bigoplus_{k \in \{0\} \cup \frac{\mathbb{Z}_+}{2}} \mathcal{K}^{(p(k),k)} = \bigoplus_{k \in \{0\} \cup \frac{\mathbb{Z}_+}{2}} \mathcal{H}_k^{(\gamma(k+1),k)}$$
(A.150)

Note, that  $\mathcal{K}$  is isomorphic to sum of all carrier spaces of irreducible representations of SU(2)

$$\mathcal{H}^{SU(2)} = \bigoplus_{k \in \{0\} \cup \frac{\mathbb{Z}_+}{2}} \mathcal{H}_k \tag{A.151}$$

Let us now introduce a map from  $\mathcal{H}^{SU(2)}$  to  $\mathcal{K}$ , called the EPRL map:

$$Y_{\gamma} : \mathcal{H}^{SU(2)} \to \mathcal{K}$$
  

$$Y_{\gamma} : |m\rangle_{k} \mapsto |k, m\rangle_{(\gamma(k+1), k)}$$
(A.152)

This map has a natural generalisation to any tensor product of any of  $\mathcal{H}_j$ -spaces.

Sometimes it is convenient to use so called approximated EPRL map:

$$Y_{\gamma} : |m\rangle_k \mapsto |k, m\rangle_{(\gamma k, k)} \tag{A.153}$$

It's image does not satisfy exactly the simplicity constraint (A.143), but the difference is of the order of  $\frac{1}{j}$ . In the case of semicalssical calculations this difference is usually negligible, and the corresponding formulae are simpler. More discussion of this issue can be found in [47]. In chapter 6 we use only  $\tilde{Y}_{\gamma}$  without stating it explicitly.

We will often omit the  $\gamma$ -subscript, when it is not necessary.

#### A.4.3.2 Basic properties

First of all note, that the EPRL map is defined because of a choice of a little subgroup SU(2) of  $SL(2, \mathbb{C})$ . Indeed, in its definition there is an asymmetry between K and L generators, which means, that there must by a way to identify rotation and boost part of  $SL(2, \mathbb{C})$ .

Although the elements of  $\mathcal{K}$  have  $SL(2, \mathbb{C})$ -indices, they can be treated as SU(2)tensors. Indeed, the map Y (or  $\tilde{Y}$ ) is a bijection on its image, and thus the conjugate  $Y^{\dagger}: \mathcal{K} \to \mathcal{H}^{SU(2)}$  is its inverse, so  $Y^{\dagger}Y = \mathbf{1}$  and  $YY^{\dagger} = P_{\mathcal{K}}$ .
The easiest way to translate the objects lying in  $\mathcal{H}^{SL(2,\mathbb{C})}$  into SU(2)-tensors using

The easiest way to translate the objects lying in  $\mathcal{H}^{SL(2,\mathbb{C})}$  into SU(2)-tensors using the Y map is to introduce Y as a collection of tensors  $Y_k \in \mathcal{H}^{(\gamma(k+1),k)} \otimes \mathcal{H}_k$  The matrix elements of  $Y_k$  are following:

$$[Y_k]_m^{j,n} = \delta_k^j \delta_m^n \tag{A.154}$$

One can insert them whenever needed and then simply use the summation convention.

Thanks to the EPRL map one can introduce a new action of  $SL(2, \mathbb{C})$  on the SU(2) tensors. Let us focus on one space  $\mathcal{H}_j$ . Given an element g we can calculate

$$\langle n | Y^{\dagger}gY | m \rangle_j := \langle j, n | g | j, m \rangle_{(p(j),j)} = D^{(p(j),j)} (g)_{j,m}^{j,n} =: \tilde{D}^j (g)_m^n$$
(A.155)

We call the matrices  $\tilde{D}^{j}(g)_{m}^{n}$  the generalized Wigner matrices. However, they are not a representation of  $SL(2,\mathbb{C})$ , because they do not satisfy the composition principle:

$$\tilde{D}^{j}(g)_{m'}^{n}\tilde{D}^{j}(g')_{m}^{m'} = \langle n|Y^{\dagger}gYY^{\dagger}g'Y|m\rangle_{j} 
= \langle n|Y^{\dagger}gP_{\mathcal{K}}g'Y|m\rangle_{j} 
\neq \langle n|Y^{\dagger}gg'Y|m\rangle_{j} = \tilde{D}^{j}(gg')_{m'}^{n}$$
(A.156)

Nevertheless, the thanks to (A.121), the chosen subgroup SU(2) of  $SL(2, \mathbb{C})$  acts unitarily via generalized Wigner matrices. Let  $u \in SU(2) < SL(2, \mathbb{C})$ , then

$$\tilde{D}^{j}(u)_{m}^{n} = \langle n | Y^{\dagger}uY | m \rangle_{j} = \delta_{j_{2}}^{j} \delta_{m''}^{n} \delta_{j_{1}}^{j_{2}} D^{(j_{1})}(u)_{m'}^{m''} \delta_{j}^{j_{1}} \delta_{m}^{m'} = D^{(j)}(u)_{m}^{n} = \langle n | u | m \rangle_{j}$$
(A.157)

## A.4.3.3 Matrix elements of boosts in EPRL map

The EPRL map reduced a lot the number of free parameters in the formula (A.128). This makes further calculation manageable. To simplify some formula even further we will use here the approximated EPRL map  $\tilde{Y}$ .

Let us calculate a matrix element  $\langle m | \tilde{Y}^{\dagger} k_{\eta} \tilde{Y} | m \rangle_{j} = D^{(\gamma j, j)} (k_{\eta})_{j,m}^{j,m}$ . Recalling (A.128) it is given as

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = (2j+1)^{\frac{1}{2}} (2j+1)^{\frac{1}{2}} {2j \choose j+m}^{\frac{1}{2}} {2j \choose j+j}^{-\frac{1}{2}} {2j \choose j+m}^{\frac{1}{2}} {2j \choose j+j}^{-\frac{1}{2}} \cdot (-1)^{j+j+2j} \cdot e^{\eta(j+m+j)} \cdot \int_{0}^{1} dt \ \frac{(t)^{-(m+j)} (1-t)^{j+j+(m+j)}}{[te^{-\eta} + (1-t) e^{\eta}]^{1+j-i\gamma j}} \qquad (A.158)$$
$$\cdot \sum_{n_{1},n_{2}} {j+m \choose n_{1}} {j-m \choose j+j-n_{1}} {j+m \choose n_{2}} {j-m \choose j+j-n_{2}} \cdot (-1)^{n_{1}+n_{2}} (1-t)^{-(n_{1}+n_{2})} e^{-2\eta n_{2}}$$

simplifying some of the numeric factors one gets

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = (2j+1) \binom{2j}{j+m} \cdot e^{\eta(2j+m)} \\ \cdot \int_{0}^{1} dt \frac{(t)^{-(m+j)}(1-t)^{3j+m}}{[te^{-\eta}+(1-t)e^{\eta}]^{1+j-i\gamma j}}$$
(A.159)  
$$\cdot \sum_{n_{1},n_{2}} \binom{j+m}{n_{1}} \binom{j-m}{2j-n_{1}} \binom{j+m}{n_{2}} \binom{j-m}{2j-n_{2}} \\ \cdot (-1)^{n_{1}+n_{2}}(t)^{n_{1}+n_{2}}(1-t)^{-(n_{1}+n_{2})} e^{-2\eta n_{2}}$$

Now consider the binomial factors in the sum. In order  $\binom{j+m}{n_1}$  not to vanish we need  $n_1 \leq j+m$ . In order  $\binom{j-m}{2j-n_1}$  not to vanish we need  $2j - n_1 \leq j - m$ , so  $n_1 \geq j + m$ . Thus the only  $n_1$  giving nonzero contribution is  $n_1 = j + m$ . The same happens to  $n_2$ . The binomial coefficients become equal to 1, the sign factor  $(-1)^{n_1+n_2} = (-1)^{2(j+m)} = 1$  because j + m must be an integer, thus we end up with

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = (2j+1) \binom{2j}{j+m} \cdot e^{\eta(2j+m)} \\ \cdot \int_{0}^{1} dt \ \frac{(t)^{-(m+j)} (1-t)^{3j+m}}{[te^{-\eta} + (1-t) e^{\eta}]^{1+j-i\gamma j}} \\ \cdot (t)^{2(j+m)} (1-t)^{-2(j+m)} \ e^{-2\eta(j+m)}$$
(A.160)

collecting similar terms we get

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = (2j+1) \binom{2j}{j+m} \cdot e^{-m\eta}$$

$$\cdot \int_{0}^{1} dt \quad \frac{(t)^{j+m} (1-t)^{j-m}}{[te^{-\eta} + (1-t) e^{\eta}]^{1+j-i\gamma j}}$$
(A.161)

One can factor out the term  $e^{\eta}$  from the denominator of the integral, obtaining

$$\left[te^{-\eta} + (1-t)e^{\eta}\right]^{1+j-i\gamma j} = e^{(j+1-i\gamma j)\eta} \left[1 - t\left(1 - e^{-2\eta}\right)\right]^{1+j-i\gamma j}$$
(A.162)

which leads to

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = (2j+1) \binom{2j}{j+m} \cdot e^{-(j+1+m)\eta} e^{i\gamma j\eta}$$

$$\cdot \int_{0}^{1} dt \ (t)^{j+m} (1-t)^{j-m} \left[1 - t \left(1 - e^{-2\eta}\right)\right]^{i\gamma j - (j+1)}$$
(A.163)

Recalling the integral definition of the Gauss's hypergeometric function [88]:

$${}_{2}F_{1}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \mathrm{d}t \, t^{b-1} \left(1-t\right)^{c-b-1} \left(1-zt\right)^{-a}$$
(A.164)

we get

$$D^{(\gamma j,j)}(k_{\eta})_{j,m}^{j,m} = e^{-(j+m+1)\eta} e^{ij\gamma\eta} {}_{2}F_{1}\left(j+1-i\gamma j, j+1+m; 2j+2; 1-e^{-2\eta}\right)$$
(A.165)

# Appendix B

# Lemmas, theorems, calculations

In this appendix we gathered some strictly technical proofs or calculations, that we decided to remove from the core part of the thesis to make the text more fluent.

In section §B.1 we present the theorem, that was proven in the appendix to [53], stating, that gluing 2-complexes along pairs of edges does not depend on the order in which we perform the gluing.

In section §B.2 we present the generalized Saddle Point Approximation theorem, that is of key importance in the asymptotic approximation done in chapter 6 and in [83].

# B.1 Commutativity of gluing 2-complex

In this section we will use the following notation for the elements of sets: when writing sets by explicit list of their elements (for example  $A = \{a, b, c\}$ ) we allow the elements to appear more then once, for example  $B = \{a, b, c, b\}$ . We will treat the multiple elements as one element, so that

$$\{a, b, c, a, b, b\} = \{a, b, c\}$$
(B.1)

This is only the notation trick, so that we may write a set not knowing, whether its elements overlap, for example

$$\{a, b\} \cup \{x, y\} = \{a, b, x, y\}$$
(B.2)

and when we eventually realize, that a = x, we simply write

$$\{a,b\} \cup \{x,y\} = \{a,b,x,y\} = \{a,b,y\} = \{x,b,y\}$$
(B.3)

Let us also introduce a notion of a set divided by a pair of its elements:

**Definition B.1.** A set divided by a pair

Let A be a set and  $x, y \in A$ . We define A divided by a pair (x, y) as

$$A/(x,y) := (A \setminus \{x,y\}) \cup \{\{x,y\}\}$$
(B.4)

The elements A/(x, y) are denoted by  $[w]_{(x,y)}$  for  $w \in A$ , where  $[w]_{(x,y)} = w$  if  $w \notin \{x, y\}$  and  $[w]_{(x,y)} = \{x, y\}$  for  $w \in \{x, y\}$  (the subscript (x, y) is often omitted, if it does not cause confusion).

Now let us prove a lemma about sets divided by pairs:

Lemma B.1. Commutativity of dividing by pairs

Let  $\alpha = (x, y)$  and  $\beta = (w, z)$  for  $x, y, w, z \in A$ . There is a canonical bijection between  $(A/\alpha)/\beta$  and  $(A/\beta)/\alpha$ , i.e. we have  $(A/\alpha)/\beta \equiv (A/\beta)/\alpha$ .

## Proof

Using the definition we have

$$(A/\alpha)/\beta = \left(\left((A \setminus \{x, y\}) \cup \{[x]_{\alpha}\}\right) \setminus \{[w]_{\alpha}, [z]_{\alpha}\}\right) \cup \left\{\left[[w]_{\alpha}\right]_{\beta}\right\}$$
(B.5)

using the set identities we can rewrite it as

$$(A/\alpha)/\beta = (A \setminus \{x, y\}) \setminus \{[w]_{\alpha}, [z]_{\alpha}\} \cup (\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \{[[w]_{\alpha}]_{\beta}\}$$
(B.6)

Let us focus on the first term. We can transform it to

$$(A \setminus \{x, y\}) \setminus \{[w]_{\alpha}, [z]_{\alpha}\} = A \setminus \{x, y, [w]_{\alpha}, [z]_{\alpha}\}$$
(B.7)

Now if  $w, z \notin \{x, y\}$ , the equivalence classes are  $[w]_{\alpha} = w$  and  $[z]_{\alpha} = z$  respectively, so

$$(A \setminus \{x, y\}) \setminus \{[w]_{\alpha}, [z]_{\alpha}\} = A \setminus \{x, y, w, z\}$$
(B.8)

on the other hand if  $w \in \{x, y\}$ , then  $[w]_{\alpha} = \{x, y\}$ . But then  $[w]_{\alpha} \notin A$ , so

$$A \setminus \{x, y, [w]_{\alpha}, [z]_{\alpha}\} = A \setminus \{x, y, [z]_{\alpha}\}$$
(B.9)

Moreover, thanks to our notation  $\{x, y, z\} = \{x, y, w, z\}$ , because w = x or w = y. The same argument works for  $z \in \{x, y\}$ . Thus either case

$$(A \setminus \{x, y\}) \setminus \{[w]_{\alpha}, [z]_{\alpha}\} = A \setminus \{x, y, w, z\}$$
(B.10)

The overall set  $(A/\alpha)/\beta$  is thus

$$(A/\alpha)/\beta = (A \setminus \{x, y, w, z\}) \cup (\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \left\{[[w]_{\alpha}]_{\beta}\right\}$$
(B.11)

while the set  $(A/\beta)/\alpha$  is

$$(A/\alpha)/\beta = (A \setminus \{x, y, w, z\}) \cup \left(\left\{[w]_{\beta}\right\} \setminus \left\{[x]_{\beta}, [y]_{\beta}\right\}\right) \cup \left\{\left[[x]_{\beta}\right]_{\alpha}\right\}$$
(B.12)

Let us now focus on the term  $(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \{[[w]_{\alpha}]_{\beta}\}$ . There are three possibilities:  $w, z \notin \{x, y\}, w, z \in \{x, y\}$  and only one of w, z is an element of  $\{x, y\}$ . We will deal with them separately.

1.  $w, z \notin \{x, y\}$ : Then  $[w]_{\alpha} = w$  and  $[z]_{\alpha} = z$  and  $(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) = \{[x]_{\alpha}\} = \{\{x, y\}\}$ . Moreover  $[[w]_{\alpha}]_{\beta} = \{w, z\}$ . Thus we have

$$(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \left\{[[w]_{\alpha}]_{\beta}\right\} = \{\{x, y\}, \{w, z\}\}$$
(B.13)

2.  $w, z \in \{x, y\}$ : Then  $[w]_{\alpha} = [z]_{\alpha} = [x]_{\alpha}$ , so  $(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) = \emptyset$ . Moreover  $[[w]_{\alpha}]_{\beta} = [w]_{\alpha}$ , because it is an equivalence class of a trivial relation, thus the set

$$(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \{[[w]_{\alpha}]_{\beta}\} = \{[w]_{\alpha}\} = \{[x]_{\alpha}\} = \{\{x, y\}\}$$
(B.14)

3.  $w \in \{x, y\}$ ,  $z \notin \{x, y\}$ : then  $[w]_{\alpha} = [x]_{\alpha}$ , so  $(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) = \emptyset$ . Moreover  $[[w]_{\alpha}]_{\beta} = \{\{x, y\}, z\}$ , thus the set

$$(\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \{[[w]_{\alpha}]_{\beta}\} = \{\{\{x, y\}, z\}\}$$
(B.15)

Let us now compare the sets  $(A/\alpha)/\beta$  and  $(A/\beta)/\alpha$ .

$$(A/\alpha)/\beta = (A \setminus \{x, y, w, z\}) \cup (\{[x]_{\alpha}\} \setminus \{[w]_{\alpha}, [z]_{\alpha}\}) \cup \left\{[[w]_{\alpha}]_{\beta}\right\}$$
(B.16)

while the set  $(A/\beta)/\alpha$  is

$$(A/\alpha)/\beta = (A \setminus \{x, y, w, z\}) \cup \left(\left\{[w]_{\beta}\right\} \setminus \left\{[x]_{\beta}, [y]_{\beta}\right\}\right) \cup \left\{\left[[x]_{\beta}\right]_{\alpha}\right\}$$
(B.17)

In the first case it is  $(A \setminus \{x, y, w, z\}) \cup \{\{x, y\}, \{w, z\}\}$ , and it is symmetric with respect to interchange of the pairs (x, y) and (w, z), so  $(A/\alpha)/\beta = (A/\beta)/\alpha$ .

In the second and third case the equivalence of  $(A/\alpha)/\beta$  and  $(A/\beta)/\alpha$  less obvious. The set  $(A/\alpha)/\beta$  is

$$(A/\alpha)/\beta = (A \setminus \{x, y, w, z\}) \cup B$$
(B.18)

for  $B = B_1 := \{\{x, y\}\}$  or  $B = B_2 = \{\{\{x, y\}, z\}\}$ . Let us now write explicitly  $(A/\beta)/\alpha$ . There are two possibilities: either both  $x, y \in \{w, z\}$ , or only one of them, say  $x \in \{w, z\}$  and  $y \notin \{w, z\}$ , so we have

$$(A/\beta)/\alpha = (A \setminus \{x, y, w, z\}) \cup C \tag{B.19}$$

for  $C = C_1 = \{\{w, z\}\}$  or  $C = C_2 = \{\{\{w, z\}, y\}\}$ . Each of the sets  $B_1, B_2, C_1$  and  $C_2$  is one-element, thus there is a natural bijection between them. This natural bijection determines the natural bijection between  $(A/\alpha)/\beta$  and  $(A/\beta)/\alpha$ .

To make the proof complete, let us explicitly write the bijection  $\phi : (A/\alpha)/\beta \to (A/\beta)/\alpha$ . The map  $\phi|_{A \setminus \{x,y,w,z\}}$  is the identity map. On the glued part there are two possibilities:

- The map  $\phi|_{((A/\alpha)/\beta)\setminus(A\setminus\{x,y,w,z\})}$  is the identity map on  $\left\{[x]_{\alpha}, [w]_{\beta}\right\}$  if  $\{x,y\} \cap \{w,z\} = \emptyset$ .
- In the opposite case (i.e. if  $\{x, y\} \cap \{w, z\} \neq \emptyset$ ), we have  $\phi : [[w]_{\alpha}]_{\beta} \mapsto [[x]_{\beta}]_{\alpha}$ .

After this introduction let us state and prove the theorem:

## Theorem B.1. On commutativity of gluing 2-complex along edges

Let  $\kappa = (\mathbf{F}, \mathbf{E}, \mathbf{V}; f_2, f_1)$  be a 2-comples. Let  $e_1, e_2, e_3, e_4 \in \mathbf{E}_{\kappa}$  be four edges (it may happen, that  $e_3 = e_2$ ). Let  $f_{\alpha} : e_1 \to e_2$  and  $f_{\beta} : e_3 \to e_4$  be two homeomorphic maps of cells. The 2-complexes  $\kappa_1 := (\kappa/f_{\alpha})/f_{\beta}$  and  $\kappa_2 := (\kappa/f_{\beta})/f_{\alpha}$  are equivalent.

#### Proof

To prove the theorem we have to show the equivalence of each element of the structure of  $\kappa_1 = \left(\mathbf{F}_1, \mathbf{E}_1, \mathbf{V}_1; f_2^{(1)}, f_1^{(1)}\right)$  and  $\kappa_2 = \left(\mathbf{F}_2, \mathbf{E}_2, \mathbf{V}_2; f_2^{(2)}, f_1^{(2)}\right)$ .

The sets of faces are trivially the same, because the gluing of edges does not effect faces.

The sets of edges are also the same. Indeed, let  $\alpha^1 = (e_1 e_2)$  and  $\beta^1 = (e_3, e_4)$ . Then the  $\mathbf{E}_1 = (\mathbf{E}/\alpha^1)/\beta^1$  and  $\mathbf{E}_2 = (\mathbf{E}/\beta^1)/\alpha^1$ . Using lemma B.1 there is a natural bijection  $\phi^E : \mathbf{E}_1 \to \mathbf{E}_2$ .

To identify the sets  $\mathbf{V}_1$  and  $\mathbf{V}_2$  let us name the end-points of each of the edges  $e_1, \ldots, e_4$  by  $v_i^{\epsilon}$  for  $\epsilon \in \{+, -\}$ , such that the map  $f_{\alpha}$  maps  $v_1^{\epsilon}$  to  $v_2^{\epsilon}$  and the map  $f_{\beta}$  maps  $v_3^{\epsilon}$  to  $v_4^{\epsilon}$ . Let us name the pairs  $\alpha^{0\epsilon} := (v_1^{\epsilon}, v_2^{\epsilon})$  and  $\beta^{0\epsilon} := (v_3^{\epsilon}, v_4^{\epsilon})$ . Under such notation the set of vertices of  $\kappa/f_{\alpha}$  is  $(\mathbf{V}/\alpha^{0+})/\alpha^{0-}$ , which is equivalent to  $(\mathbf{V}/\alpha^{0-})/\alpha^{0+}$  (thanks to lemma B.1). Thus the set  $\mathbf{V}_1 = (((\mathbf{V}/\alpha^{0+})/\alpha^{0-})/\alpha^{1+})/\alpha^{1-}$  and  $\mathbf{V}_2 = (((\mathbf{V}/\alpha^{1+})/\alpha^{1-})/\alpha^{0+})/\alpha^{0-}$ . Again, thanks to lemma B.1, we can change the order of divisions and obtain the natural bijection  $\phi^V : \mathbf{V}_1 \to \mathbf{V}_2$ .

Let us now consider the action of the boundary maps  $f_2$  and  $f_1$  and check, whether they are consistent with the bijections  $\phi^E$  and  $\phi^V$ .

In order to consider the the map  $f_2$  let us consider the projections  $\pi_1^{(1)} : \mathbf{E} \to \mathbf{E}^1$  and  $\pi_1^{(2)} : \mathbf{E} \to \mathbf{E}^2$ . The map  $f_2^{(1)} := \pi_1^{(1)} \circ f_2$ , and similar  $f_2^{(2)} := \pi_1^{(2)} \circ f_2$ . The exact formula of the projections can be read out of the proof of lemma B.1.

The exact formula of the projections can be read out of the proof of lemma B.1. Obviously  $\pi_1^{(1)}\Big|_{\mathbf{E}\setminus\{e_1,e_2,e_3,e_4\}} = \pi_1^{(2)}\Big|_{\mathbf{E}\setminus\{e_1,e_2,e_3,e_4\}} = \mathbb{1}$ . If  $\{e_1,e_2\} \cap \{e_3,e_4\} \neq \emptyset$ , all the edges are glued to one edge, and thus we have  $\pi_1^{(1)}: \{e_1,e_2,e_3,e_4\} \rightarrow \{[[e_3]]\}$  and  $\pi_1^{(2)}: \{e_1,e_2,e_3,e_4\} \rightarrow \{[[e_1]]\}$ , otherwise (i.e. if  $\{e_1,e_2\} \cap \{e_3,e_4\} = \emptyset$ ) both projections map  $\{e_1,e_2\}$  onto  $\{[e_1]\}$  and  $\{e_3,e_4\}$  onto  $\{[e_3]\}$ .

Consistency of 2-complex structure require  $\phi^E \circ f_2^{(1)} = f_2^{(2)}$ . In fact it is enough to check, whether  $\phi^E \circ \pi_1^{(1)} = \pi_1^{(2)}$ . At  $\mathbf{E} \setminus \{e_1, e_2, e_3, e_4\}$  the bijection  $\phi^E$  and both projections are the identity maps, it is enough to check it at  $\{e_1, e_2, e_3, e_4\}$ . If  $\{e_1, e_2\} \cap \{e_3, e_4\} = \emptyset$  the bijection  $\phi^E$  is the identity map. If  $\{e_1, e_2\} \cap \{e_3, e_4\} \neq \emptyset$ , we have  $\pi_1^{(1)}(e_i) = [[e_3]]$ , but  $\phi^E([[e_3]]) = [[e_1]] = \pi_1^{(2)}(e_i)$ , which proves the consistency of the  $f_2$  maps.

In order to consider the map  $f_1$  let us consider the projections  $\pi_0^{(1)} : \mathbf{V} \to \mathbf{V}^1$  and  $\pi_0^{(2)} : \mathbf{V} \to \mathbf{V}^2$ . The map  $f_1^{(1)} := \pi_0^{(1)} \circ f_1 \circ \left(\pi_1^{(1)}\right)^{-1}$ , and similarly  $f_1^{(2)} := \pi_0^{(2)} \circ f_1 \circ \left(\pi_1^{(2)}\right)^{-1}$ . The consistency of 2-complex structure at the level of  $f_1$  function requires  $\phi^V \circ f_1^{(1)} \circ (\phi^E)^{-1} = f_1^{(2)}$ , so

$$\phi^{V} \circ \pi_{0}^{(1)} \circ f_{1} \circ \left(\pi_{1}^{(1)}\right)^{-1} \circ \left(\phi^{E}\right)^{-1} \stackrel{?}{=} \pi_{0}^{(2)} \circ f_{1} \circ \left(\pi_{1}^{(2)}\right)^{-1}$$
(B.20)

First let us show, that  $\phi^V \circ \pi_0^{(1)} = \pi_0^{(2)}$ . Note, that to obtain  $\mathbf{V}_2$  from  $\mathbf{V}_1$  one has to do four changes of order of quotients. Let us introduce four bijection maps:

$$\phi_{a}^{V}: \mathbf{V}_{1} = \left(\left(\left(\mathbf{V}/\alpha^{0+}\right)/\alpha^{0-}\right)/\alpha^{1+}\right)/\alpha^{1-} \rightarrow \left(\left(\left(\mathbf{V}/\alpha^{0+}\right)/\alpha^{1+}\right)/\alpha^{0-}\right)/\alpha^{1-} =: \mathbf{V}_{a}\right) \\
\phi_{b}^{V}: \mathbf{V}_{a} = \left(\left(\left(\mathbf{V}/\alpha^{0+}\right)/\alpha^{1+}\right)/\alpha^{0-}\right)/\alpha^{1-} \rightarrow \left(\left(\left(\mathbf{V}/\alpha^{0+}\right)/\alpha^{1+}\right)/\alpha^{0-}\right)/\alpha^{0-} =: \mathbf{V}_{b}\right) \\
\phi_{c}^{V}: \mathbf{V}_{b} = \left(\left(\left(\mathbf{V}/\alpha^{0+}\right)/\alpha^{1+}\right)/\alpha^{1-}\right)/\alpha^{0-} \rightarrow \left(\left(\left(\mathbf{V}/\alpha^{1+}\right)/\alpha^{0+}\right)/\alpha^{1-}\right)/\alpha^{0-} =: \mathbf{V}_{c}\right) \\
\phi_{d}^{V}: \mathbf{V}_{c} = \left(\left(\left(\mathbf{V}/\alpha^{1+}\right)/\alpha^{0+}\right)/\alpha^{1-}\right)/\alpha^{0-} \rightarrow \left(\left(\left(\mathbf{V}/\alpha^{1+}\right)/\alpha^{0+}\right)/\alpha^{0-}\right) =: \mathbf{V}_{2}\right) \\$$
(B.21)

Obviously  $\phi^V = \phi_d^V \circ \phi_c^V \circ \phi_b^V \circ \phi_a^V$ . Moreover let us introduce three projection maps:  $\pi_0^{(i)} : \mathbf{V} \to \mathbf{V}_i$  for i = a, b, c. By the same argument, as we used in case of  $\phi^E$  we see, that  $\pi_a = \phi_a^V \pi_0^{(1)}, \pi_b = \phi_b^V \pi_a, \pi_c = \phi_c^V \pi_b$  and  $\pi_0^{(2)} = \phi_d^V \pi_c$ . By composing these identities we obtain  $\pi_0^{(2)} = \phi_d^V \circ \phi_c^V \circ \phi_b^V \circ \phi_a^V \pi_0^{(1)}$ , so

$$\pi_0^{(2)} = \phi^V \pi_0^{(1)} \tag{B.22}$$

Now let us check the consistency of the counterimage of  $(\pi_1^{(i)})^{-1}$ . As it was stated above, if  $\{e_1, e_2\} \cap \{e_3, e_4\} = \emptyset$ , both projections  $\pi_1^{(1)}$  and  $\pi_1^{(2)}$  are equal, thus their inverse  $(\pi_1^{(1)})^{-1} = (\pi_1^{(2)})^{-1}$  give the same counterimages for each argument. On the other hand if  $\{e_1, e_2\} \cap \{e_3, e_4\} \neq \emptyset$  the projections are different, but the counterimages of each argument also agrees - indeed,

$$\left(\pi_1^{(1)}\right)^{-1}\left([[e_3]]\right) = \{e_1, e_2, e_3, e_4\} = \left(\pi_1^{(2)}\right)^{-1}\left([[e_1]]\right) = \left(\pi_1^{(2)}\right)^{-1}\left(\phi^E\left([[e_1]]\right)\right) \quad (B.23)$$

Applying (B.22) and (B.23) to (B.20) we obtain the identity, so indeed  $\phi^V \circ f_1^{(1)} \circ (\phi^E)^{-1} = f_1^{(2)}$ , which ends the proof. *Quod erat demonstrandum* 

# B.2 Proof of generalized SPA

The statement of the theorem and the sketch of the proof is taken from [102]. The extension of the proof to more general integrands was performed by the Author.

Let us recall the Saddle Point Approximation theorem in the original formulation.

**Theorem B.2.** Saddle Point Approximation (original formulation):

Given two smooth functions  $f, g : [a, b] \to \mathbb{R}$  and assuming, that f has a global minimum at a unique point  $c \in [a, b]$  such that f''(c) > 0, one has

$$\int_{a}^{b} g(x) e^{-f(x)/\hbar} \mathrm{d}x = \sqrt{\hbar} e^{-f(c)/\hbar} I(\hbar)$$
(B.24)

for a smooth function  $I(\hbar) : [0, \infty[$ , such that  $I(0) = \sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}}$ .

One can easily re-express this theorem in such a way, that the function g disappears (i.e. for  $\tilde{f}(x) = f(x) \ln g(x)$ ). One can assume c = 0. One can assume f(c) = 0 (by multiplying both sides of the formula by  $e^{f(c)/\hbar}$ ). One should also assume  $\Re(f(x)) \to \infty$ for  $x \to a$  and  $x \to b$  - in order to avoid boundary terms in the integral. It is also convenient to change the sign of f and consider f having maxima instead of minima etc. Moreover, there is no good reason to restrict only to real functions, however one has to slightly modify the assumptions about f.

Thus we have another theorem:

#### Theorem B.3. Simplified form of Saddle Point Approximation

Given a smooth function  $f : [a, b] \to \mathbb{C}$  and assuming, that  $\Re(f)$  has a global maximum at  $0 \in [a, b]$  such that f''(0) < 0 and f(0) = 0, and f'(0) = 0, and  $\forall_{x \neq 0} \Re(f(x)) < 0$ , and  $\Re(f(x)) \to -\infty$  for  $x \to a$  and  $x \to b$ , one has

$$\int_{a}^{b} e^{f(x)/\hbar} \mathrm{d}x = \sqrt{\hbar} I(\hbar)$$
(B.25)

for a smooth function  $I(\hbar) : [0, \infty[$ , such that  $I(0) = \sqrt{2\pi} \frac{1}{\sqrt{-f''(0)}}$ .

However, we are interested in integrating functions  $\Phi(x, \hbar)$  of more general form than  $e^{f(x)/\hbar}$ . Consider an integrand of the form  $e^{\chi(x,\hbar)}$  where the function  $\chi(x,\hbar)$  has a simple pole of degree 1 at  $\hbar = 0$ , i.e. it has a Laurent series

$$\chi(x,\hbar) = \frac{\chi_{-1}(x)}{\hbar} + \sum_{\alpha=0}^{\infty} \chi_{\alpha}(x) \hbar^{\alpha}$$
(B.26)

An intuition suggests, that the contribution from the terms other than  $\chi_{-1}$  will be negligible. The intuition is right, what we shall prove in the following theorem:

#### **Theorem B.4.** Generalized Saddle Point Approximation

Given a smooth function  $\chi : [a,b] \times ]0,1[ \ni (x,\hbar) \mapsto \chi (x,\hbar) \in \mathbb{C}$ , (where  $0 \in [a,b]$ ) with a simple pole of degree 1 at  $\hbar = 0$  and assuming, that

$$\forall_{x \neq 0, \hbar \in [0,1[} \qquad \Re\left(\chi\left(x, \hbar\right)\right) < 0 \tag{B.27}$$

$$\forall_{\hbar\in]0,1[} \lim_{x\to a \text{ and } x\to b} \qquad \Re\left(\chi\left(x,\hbar\right)\right) = -\infty \tag{B.28}$$

$$\forall_{\hbar\in]0,1[} \qquad \chi\left(0,\hbar\right) = 0 \tag{B.29}$$

$$\lim_{\hbar \to 0} \quad \hbar \left. \frac{\partial \chi \left( x, \hbar \right)}{\partial x} \right|_{x=0} = 0 \tag{B.30}$$

$$\lim_{\hbar \to 0} \qquad \hbar \Re \left( \left. \frac{\partial^2 \chi \left( x, \hbar \right)}{\partial x^2} \right|_{x=0} \right) = -C < 0 \tag{B.31}$$

one has

$$\int_{a}^{b} e^{\chi(x,\hbar)} \mathrm{d}x = \sqrt{\hbar} I(\hbar) \tag{B.32}$$

for a smooth function  $I(\hbar): [0,\infty[ \to \mathbb{C}, \text{ such that } I(0) = \sqrt{2\pi} \frac{1}{\sqrt{-\chi''_{-1}(0)}}$ .

#### Proof

The function  $I(\hbar)$  can be defined by the equation (B.32). What we need to do is to find it's value at  $\hbar = 0$  and prove its smoothness.

Let us start with a sketch of the proof. It goes in three steps of estimations, i.e. we find three classes of functions  $I_1(\hbar)$  (a one-element class),  $I_2^n(\hbar)$  for  $n \in \mathbb{Z}_+ \cup \{0\}$  and  $I_3^n(\hbar)$  again for  $n \in \mathbb{Z}_+ \cup \{0\}$  (we shall define these classes later), such that:

$$\forall_{N \in \mathbb{Z}_{+}} \lim_{\hbar \to 0} \frac{|I(\hbar) - I_{1}(\hbar)|}{\hbar^{N}} = 0$$
(B.33)

$$\forall_{N \in \mathbb{Z}_{+}} \lim_{\hbar \to 0} \frac{\left| I_{1}\left(\hbar\right) - I_{2}^{2N}\left(\hbar\right) \right|}{\hbar^{N+\epsilon}} = 0$$
(B.34)

$$\forall_{N,\tilde{N}\in\mathbb{Z}_{+}}\lim_{\hbar\to0}\frac{\left|I_{2}^{N}\left(\hbar\right)-I_{3}^{N}\left(\hbar\right)\right|}{\hbar^{\tilde{N}}}=0$$
(B.35)

From (B.33)-(B.35) we see, that

$$\forall_{N \in \mathbb{Z}_{+}} \exists_{k \in \mathbb{Z}_{+}} \left| I\left(\hbar\right) - I_{3}^{k}\left(\hbar\right) \right| = O\left(\hbar^{N}\right)$$
(B.36)

so the series of functions  $I_3^n(\hbar)$  can be used to estimate the original integral. These functions will be chosen in such a way, that at each order it is easy to evaluate the integral and obtain the corresponding term of expansion.

Having the sketch, let us go to the proof.

Let  $\epsilon \in \left[0, \frac{1}{2}\right]$  (it will never be considered a small number).

The function  $I_1(\hbar)$  is defined by the following integral:

$$I_1(\hbar) := \frac{1}{\sqrt{\hbar}} \int_{-\hbar^{\frac{1}{2}-\epsilon}}^{\hbar^{\frac{1}{2}-\epsilon}} e^{\chi(x,\hbar)} \mathrm{d}x \tag{B.37}$$

note, that for  $\hbar \ll 1$  we have  $\hbar^{\frac{1}{2}-\epsilon} \ll 1$ . Having defined  $I_1(\hbar)$  let us prove (B.33).

Proof of (B.33):

To simplify the notation, let us name the integration ranges:  $[a, b] =: \Omega$ and  $\left[-\hbar^{\frac{1}{2}-\epsilon}, \hbar^{\frac{1}{2}-\epsilon}\right] =: \mathcal{O}_{\hbar,\epsilon}.$ 

The difference  $I(\hbar) - I_1(\hbar)$  is given by the integral

$$I(\hbar) - I_1(\hbar) = \frac{1}{\sqrt{\hbar}} \int_{\Omega \setminus \mathcal{O}_{\hbar,\epsilon}} e^{\chi(x,\hbar)} \mathrm{d}x$$
(B.38)

we can estimate its modulus by

$$|I(\hbar) - I_1(\hbar)| \le \frac{1}{\sqrt{\hbar}} |\Omega \setminus \mathcal{O}_{\hbar,\epsilon}| \sup_{\Omega \setminus \mathcal{O}_{\hbar,\epsilon}} \left| e^{\chi(x,\hbar)} \right|$$
(B.39)

The measure of the set  $|\Omega \setminus \mathcal{O}_{\hbar,\epsilon}| < |\Omega| = b - a$ . The supremum is given by the supremum of the real part of the exponent, so that

$$|I(\hbar) - I_1(\hbar)| < \frac{1}{\sqrt{\hbar}} (b-a) e^{\sup_{\Omega \setminus \mathcal{O}_{\hbar,\epsilon}}(\Re(\chi(x,\hbar)))}$$
(B.40)

Since  $\chi(0,\hbar)$  is the global maximum of the real part of  $\chi(x,\hbar)$ , for  $\hbar$  sufficiently small the supremum will be at one of the borders of  $\mathcal{O}_{\hbar,\epsilon}$ . Without loose of generality we may assume that it is for  $x = \hbar^{\frac{1}{2}-\epsilon}$ , obtaining

$$|I(\hbar) - I_1(\hbar)| < \frac{1}{\sqrt{\hbar}} (b-a) e^{\Re\left(\chi\left(\hbar^{\frac{1}{2}-\epsilon},\hbar\right)\right)}$$
(B.41)

Since  $\hbar^{\frac{1}{2}-\epsilon} \ll 1$ , we may apply the Taylor expansion to  $\chi$  in x around x = 0. Notting the Laurent series (B.26) of  $\chi$ , we get

$$\chi(x,\hbar) = \sum_{\alpha=-1}^{\infty} \sum_{n=0}^{\infty} \frac{\chi_{\alpha}^{(n)}(0)}{n!} x^n \hbar^{\alpha}$$
(B.42)

Now applying the assumptions on the function  $\chi$  we get rid of some terms in (B.42). First of all, since  $\chi(0,\hbar) = 0$  for all  $\hbar$  (see (B.29)), we have  $\forall_{\alpha}\chi_{\alpha}(0) = 0$ . Moreover, (B.30) puts an extra constraint on  $\chi_{-1}$ , i.e.  $\chi'_{-1}(0) = 0$ . Thus

$$\chi(x,\hbar) = \sum_{n=2}^{\infty} \frac{\chi_{-1}^{(n)}(0)}{n!} \frac{x^n}{\hbar} + \sum_{\alpha=0}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_{\alpha}^{(n)}(0)}{n!} x^n \hbar^{\alpha}$$
(B.43)

The assumption (B.31) gives another constraint on the real part of second derivative of  $\chi_{-1}$ , i.e.  $\Re \left( \chi''_{-1}(0) \right) = -C$ . Thus

$$\Re\left(\chi\left(x,\hbar\right)\right) = -\frac{C}{2}\frac{x^{2}}{\hbar} + \sum_{n=3}^{\infty}\frac{\Re\left(\chi_{-1}^{(n)}\left(0\right)\right)}{n!}\frac{x^{n}}{\hbar} + \sum_{\alpha=0}^{\infty}\sum_{n=1}^{\infty}\frac{\Re\left(\chi_{\alpha}^{(n)}\left(0\right)\right)}{n!}x^{n}\hbar^{\alpha} \qquad (B.44)$$

For  $x = \hbar^{\frac{1}{2}-\epsilon}$  the real part of the exponent becomes

$$\Re\left(\chi\left(\hbar^{\frac{1}{2}-\epsilon},\hbar\right)\right) = -\frac{C}{2}\hbar^{-2\epsilon} + \sum_{n=3}^{\infty} \frac{\Re\left(\chi^{(n)}_{-1}(0)\right)}{n!} \hbar^{\frac{n}{2}-1-n\epsilon} + \sum_{\alpha=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Re\left(\chi^{(n)}_{\alpha}(0)\right)}{n!} \hbar^{\frac{n}{2}-n\epsilon+\alpha}$$
$$= -\frac{C}{2}\hbar^{-2\epsilon} \left[1 - \frac{2}{C}\zeta\left(\hbar\right)\right]$$
(B.45)

with

$$\zeta(\hbar) := \sum_{n=3}^{\infty} \frac{\Re\left(\chi_{-1}^{(n)}(0)\right)}{n!} \hbar^{\frac{n}{2}-1-n\epsilon+2\epsilon} + \sum_{\alpha=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Re\left(\chi_{\alpha}^{(n)}(0)\right)}{n!} \hbar^{\frac{n}{2}-n\epsilon+\alpha+2\epsilon} \tag{B.46}$$

Now note, that  $\lim_{\hbar\to 0} \zeta(\hbar) = 0$ . Indeed, in the first series in each term  $\hbar$  has the exponent  $\frac{n}{2} - 1 - n\epsilon - 2\epsilon = (\frac{1}{2} - \epsilon)(n-2)$ , and since by  $n \ge 3$  and (by assumption)  $\frac{1}{2} > \epsilon$ , it is positive. In the second series in each term  $\hbar$  has the exponent  $n(\frac{1}{2} - \epsilon) + \alpha + 2\epsilon$  being the sum of two positive quantities and one non-negative quantity, so it is also always positive. Thus for  $\hbar$  small enough we may be sure that  $|\zeta(\hbar)| < \frac{C}{4}$ . Under such assumption we have  $|1 - \frac{2}{C}\zeta(\hbar)| > \frac{1}{2}$ , so

$$\Re\left(\chi\left(x,\hbar\right)\right) < -\frac{C}{4}\hbar^{-2\epsilon} \tag{B.47}$$

and thus

$$|I(\hbar) - I_1(\hbar)| < \frac{1}{\sqrt{\hbar}} (b-a) e^{-\frac{C}{4}\hbar^{-2\epsilon}}$$
(B.48)

so it is bounded by an exponent function of  $\frac{1}{\hbar}$  to a positive power, which indeed goes to 0 faster than any polynomial of  $\hbar$ , which ends the proof of (B.33).

Having proven (B.33) let us define the family  $I_2^n(\hbar)$  and proceed to the proof of (B.34). In order to define  $I_2^n(\hbar)$  let us first change the integration variable in (B.37) to  $y := \frac{x}{\sqrt{\hbar}}$ , so that

$$I_{1}(\hbar) = \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} e^{\chi \left(y\sqrt{\hbar},\hbar\right)} \mathrm{d}y \tag{B.49}$$

Note, that now the boarders of the integration range  $\hbar^{-\epsilon} \gg 1$ , but the argument of the function f is still small, i.e.  $y\sqrt{\hbar} \ll 1$ . The family  $I_2^n(\hbar)$  is defined as

$$I_2^n(\hbar) = \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} \left[ \left( e^{\chi(y\sqrt{\hbar},\hbar)} \right)_{n^{\text{th}} \text{ order of Taylor expantion in } \sqrt{\hbar} \right] \mathrm{d}y \qquad (B.50)$$

i.e. let us introduce  $h := \sqrt{\hbar}$  and a function  $\Phi(y,h) := e^{\chi(yh,h^2)}$ , then we have

$$\Phi(y,h) = \sum_{n=0}^{\infty} \left. \frac{\partial^n \Phi}{\partial h^n} \right|_{(y,0)} \frac{h^n}{n!} \tag{B.51}$$

according to Taylor's theorem we can introduce a family of functions  $\Phi^{n}(y,h)$  such that

$$\Phi(y,h) = \Phi^{n}(y,h) + R^{n}(y,h)$$
(B.52)

where  $R^{n}(y,h)$  is the reminder of  $n^{\text{th}}$  order, i.e

$$\forall_y \lim_{h \to 0} \frac{R^n(y,h)}{h^n} = 0 \tag{B.53}$$

The functions  $I_{2}^{n}(\hbar)$  can be rewritten in terms of h and  $\Phi(y,h)$ , obtaining

$$I_2^n(h) = \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi^n(y,h) \,\mathrm{d}y \tag{B.54}$$

Let us now proceed to the proof of (B.34)

$$\forall_{N \in \mathbb{Z}_{+}} \lim_{\hbar \to 0} \frac{\left| I_{1}\left(\hbar\right) - I_{2}^{2N}\left(\hbar\right) \right|}{\hbar^{N+\epsilon}} = 0$$
(B.55)

Proof of (B.34):

Note, that in terms of  $\Phi(y, h)$  the function  $I_1(h)$  is

$$I_1(h) = \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi(y,h) \,\mathrm{d}y \tag{B.56}$$

thus the difference  $I_1(h) - I_2^n(h)$  can be written as

$$I_{1}(h) - I_{2}^{n}(h) = \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} R^{n}(y,h) \,\mathrm{d}y$$
 (B.57)

Let us recall the Lagrange form of the reminder:

$$R^{n}(y,h) = \left. \frac{\partial^{n+1}\Phi}{\partial h^{n+1}} \right|_{y,\xi} \frac{h^{n+1}}{(n+1)!} \tag{B.58}$$

for some  $\xi \in [0, h]$ . We shall estimate the value of the reminder.

Since we are interested in  $\Phi$  as a function of h, in what follows we shall treat y as a parameter, i.e. we shall consider  $\Phi_y(h) := \Phi(y, h)$ 

First let us introduce the exponent function  $\phi_y(h) := \chi(yh, h^2)$ , so that  $\Phi_y(h) = e^{\phi_y(h)}$ . The derivatives of  $\Phi$  with respect to h are:

$$\begin{aligned}
\Phi'_{y}(h) &= \phi'_{y}(h) e^{\phi_{y}(h)} \\
\Phi''_{y}(h) &= \left[\phi''_{y}(h) + \left(\phi'_{y}(h)\right)^{2}\right] e^{\phi_{y}(h)} \\
\Phi'''_{y}(h) &= \left[\phi'''_{y}(h) + 2\phi''_{y}(h) \phi'_{y}(h) + \left(\phi'_{y}(h)\right)^{3}\right] e^{\phi_{y}(h)} \\
&\vdots &\vdots \\
\Phi^{(n)}_{y}(h) &= W^{n}\left(\phi'_{y}(h), \phi''_{y}(h), \dots, \phi^{(n)}_{y}(h)\right) e^{\phi_{y}(h)}
\end{aligned}$$
(B.59)

where  $W^n(x_1,\ldots,x_n)$  is a polynomial:

$$W^{n}(x_{1},...,x_{n}) = \sum w_{p_{1}p_{2}\cdots p_{n}} x_{1}^{p_{1}} x_{2}^{p_{2}} \cdots x_{n}^{p_{n}}$$
(B.60)

and the coefficients  $w_{p_1p_2\cdots p_n}$  are zero unless  $\sum_{i=1}^n i \cdot p_i = n$ , when they are non-negative (they can be derived from the derivative of a composition of functions).

Consider now the Taylor series of the exponent function  $\phi_y(y)$ . Since through all the domain the argument of  $\chi$  is small (indeed,  $yh < h^{-2\epsilon}h \ll 1$ , since  $h \ll 1$  and  $1-2\epsilon > 0$ ), one can apply the expansion (B.43). We obtain

$$\phi_y(h) = \frac{\chi_{-1}''(0)}{2}y^2 + y^2 \sum_{k=1}^{\infty} \frac{\chi_{-1}^{(k+2)}(0)}{(k+2)!} y^k h^k + \sum_{\alpha=0}^{\infty} \sum_{k=1}^{\infty} \frac{\chi_{\alpha}^{(k)}(0)}{k!} y^k h^{k+2\alpha}$$
(B.61)

Recalling the same argument, as in (B.43) we argue, that the tail goes to 0 with  $h \to 0$ . Thus, for sufficiently small h the modulus of the tail is less then  $\left|\frac{\Re(\chi''_{-1}(0))y^2}{4}\right|$ , so

$$\forall_{y \in [-h^{-2\epsilon}, h^{-2\epsilon}]} \Re\left(\phi_y\left(h\right)\right) < -\frac{C}{4}y^2 \tag{B.62}$$

Now let us study the derivatives of  $\phi_y(h)$ . By reorganizing the terms in (B.61) one can order them by the powers of h, so that

$$\phi_y(h) = \frac{\chi_{-1}''(0)}{2} y^2 + \sum_{n=0}^{\infty} h^n \sum_{\alpha=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{\chi_{\alpha-1}^{(n-2(\alpha-1))}(0) y^{n-2(\alpha-1)}}{(n-2(\alpha-1))!}$$
(B.63)

thus the  $n^{\text{th}}$  derivative over h reads as

$$\phi_{y}^{(n)}(h) = \sum_{k=n}^{\infty} h^{k-n} \frac{k!}{(k-n)!} \sum_{\alpha=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \frac{\chi_{\alpha-1}^{(k-2(\alpha-1))}(0) y^{k-2(\alpha-1)}}{(k-2(\alpha-1))!}$$
$$= \sum_{k=0}^{\infty} h^{k} \frac{(k+n)!}{n!} \sum_{\alpha=0}^{\left\lfloor \frac{k+n+1}{2} \right\rfloor} \frac{\chi_{\alpha-1}^{(k+n-2(\alpha-1))}(0) y^{k+n-2(\alpha-1)}}{(k+n-2(\alpha-1))!}$$
(B.64)

Again reorganizing the terms in (B.64) one can obtain a more convenient form, i.e

$$\phi_{y}^{(n)}(h) = \sum_{\alpha=-1}^{\lfloor \frac{n-1}{2} \rfloor} y^{n-2\alpha} \left( \frac{n! \chi_{\alpha}^{(n-2\alpha)}(0)}{(n-2\alpha)!} + \sum_{k=1}^{\infty} \frac{(k+n)! \chi_{\alpha}^{(k+n-2\alpha)}(0)}{k! (k+n-2\alpha)!} (hy)^{k} \right) + \sum_{\alpha=\lfloor \frac{n+1}{2} \rfloor}^{\infty} h^{2\alpha-n} \sum_{l=1}^{\infty} \frac{(2(\alpha-1)-l+2)! \chi_{\alpha}^{(l)}(0)}{l! (2\alpha-n+l)!} (hy)^{l}$$
(B.65)

By a similar argument we see, that for each  $\alpha$  both tails dependent on h vanishes with h (indeed, because  $hy \ll 1$  and  $h \ll 1$ ). Thus given an arbitrary constant  $\tilde{C}_n^{\alpha}$  for sufficiently small h the modulus of the sum of the tails can be estimated by  $\tilde{C}_n$  times the modulus of a h-independent term:<sup>1</sup>

$$\left|\phi_{y}^{(n)}(h)\right| < C_{n}^{\infty} + \sum_{\alpha=-1}^{\lfloor\frac{n-1}{2}\rfloor} \left|y^{n-2\alpha}\right| \frac{\left|\chi_{\alpha}^{(n-2\alpha)}(0)\right| n!}{(n-2\alpha)!} C_{n}^{\alpha} \quad \text{for } C_{n} = 1 + \tilde{C}_{n} \qquad (B.66)$$

where  $C_n^{\alpha} = 1 + \tilde{C}_n^{\alpha}$  and  $C_n^{\infty}$  is the total bound of all the terms for  $\alpha \geq \lfloor \frac{n+1}{2} \rfloor$  For simplicity we assume all  $C_n^{\alpha}$  and  $C_n^{\infty}$  to be less then 2. Thus we have an estimation for derivative of  $\phi_y(h)$  of each degree n:

$$\left|\phi_{y}^{(n)}\left(h\right)\right| < \mathcal{P}_{\chi}^{n}\left(y\right) \tag{B.67}$$

where  $\mathcal{P}_{\chi}^{n}$  is a polynomial in |y| with positive coefficients, *independent* on h.

Going back to the derivative of  $\Phi_y(h)$  (see (B.59)), thanks to (B.62) we can estimate its modulus by

$$\begin{aligned} \left| \Phi_{y}^{(n)}(h) \right| &\leq \left| W^{n} \left( \phi_{y}'(h), \phi_{y}''(h), \dots, \phi_{y}^{(n)}(h) \right) \right| \cdot \left| e^{\phi_{y}(h)} \right| \\ &\leq \left| W^{n} \left( \phi_{y}'(h), \phi_{y}''(h), \dots, \phi_{y}^{(n)}(h) \right) \right| \cdot e^{-\frac{C}{4}y^{2}} \end{aligned} \tag{B.68}$$

recalling the positiveness of the coefficient of each of the polynomials  $W^n$  we can estimate

$$|W^{n}(x_{1},...,x_{n})| \le W^{n}(|x_{1}|,...,|x_{n}|)$$
 (B.69)

combining (B.69) and (B.67) we obtain

$$\left| W^{n}\left(\phi_{y}^{\prime}\left(h\right),\ldots,\phi_{y}^{\left(n\right)}\left(h\right)\right) \right| \leq W^{n}\left(\mathcal{P}_{\chi}^{1}\left(y\right),\ldots,\mathcal{P}_{\chi}^{n}\left(y\right)\right) = \tilde{W}^{n}\left(\left|y\right|\right)$$
(B.70)

being another polynomial, with positive coefficients (one can check, that the minimal power of |y| is 0 and the maximal is 3n).

<sup>&</sup>lt;sup>1</sup>Note, that now we estimate the modulus, not the real part, thus we have to add the modulus of each term.

Now let us go back to the integral (B.57). Thanks to (B.58) we have

$$\int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} R^n(y,h) \,\mathrm{d}y = \frac{h^{n+1}}{(n+1)!} \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi_y^{(n+1)}(\xi) \,\mathrm{d}y \tag{B.71}$$

Since  $\xi \in [0, h]$ , we can use (B.68) and (B.70) to estimate the modulus of the integral

$$\left| \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} R^n(y,h) \,\mathrm{d}y \right| \le \frac{h^{n+1}}{(n+1)!} \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \left| \Phi_y^{(n+1)}(\xi) \right| \,\mathrm{d}y \le \frac{h^{n+1}}{(n+1)!} \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \tilde{W}^n(|y|) \, e^{-\frac{C}{4}y^2} \,\mathrm{d}y$$
(B.72)

Since  $\forall_{y \in \mathbb{R}}$  the polynomial  $\tilde{W}^n(|y|) > 0$ , we can estimate the integral from the above by extending it's limits to  $\pm \infty$ :

$$\int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \tilde{W}^n\left(|y|\right) e^{-\frac{C}{4}y^2} \mathrm{d}y \le \int_{-\infty}^{\infty} \tilde{W}^n\left(|y|\right) e^{-\frac{C}{4}y^2} \mathrm{d}y \tag{B.73}$$

For each polynomial the integral of the right-hand-side of (B.73) is equals to a number M depending only on C and the coefficients of the polynomial. Both C and the coefficients of the polynomial are numbers depending only on the derivatives of  $\chi_{\alpha}(x)$  at x = 0, thus

$$\int_{-\infty}^{\infty} \tilde{W}^n\left(|y|\right) e^{-\frac{C}{4}y^2} \mathrm{d}y = M\left(\chi_{\alpha}^{(n)}\left(0\right)\right) = M_{\chi} \tag{B.74}$$

Is important that  $M_{\chi}$  does not depend in any way on h.

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The final step of the proof of (B.34) is to plug (B.74) and (B.72) and (B.57) to (B.34), obtaining

$$\lim_{h \to 0} \frac{\left| I_1(h) - I_2^{2N}(h) \right|}{h^{2N+2\epsilon}} = \lim_{h \to 0} \frac{\left| \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} R^{2N}(y,h) \,\mathrm{d}y \right|}{h^{2N+2\epsilon}} \tag{B.75}$$

$$\leq \lim_{h \to 0} \frac{1}{h^{2N+2\epsilon}} \frac{h^{2N+1}}{(n+1)!} \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \tilde{W}^{2N}(|y|) e^{-\frac{C}{4}y^2} dy$$
(B.76)

$$\leq \frac{1}{(n+1)!} \lim_{h \to 0} h^{1-2\epsilon} M_{\chi}$$
 (B.77)

$$= \frac{M_{\chi}}{(n+1)!} \lim_{h \to 0} h^{1-2\epsilon}$$
(B.78)

$$= 0$$
 (B.79)

where from (B.75) to (B.76) we applied the estimation (B.72), from (B.76) to (B.77) we used the estimation (B.74) and (B.73), from (B.77) to (B.78) we used the fact, that  $M_{\chi}$  does not depend on h and from (B.78) to (B.79) we used the fact, that  $1 - 2\epsilon > 0$ , by definition of  $\epsilon$ .

This ends the proof of (B.34).

Having proven (B.34) let us define the family  $I_3^n(\hbar)$  and proceed to the proof of (B.35). We will do it in terms of  $h = \sqrt{\hbar}$  and the Taylor decomposition of the function  $\Phi(y,h)$  (see (B.52)). The function  $I_3^n(h)$  is defined as

$$I_3^n(h) := \int_{-\infty}^{\infty} \Phi^n(y,h) \,\mathrm{d}y \tag{B.80}$$

Recalling (B.50) we have

$$I_2^n(h) = \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi^n(y,h) \,\mathrm{d}y \tag{B.81}$$

so we can rephrase our claim (B.35) into

$$\lim_{h \to 0} \frac{\left| \int_{-\infty}^{\infty} \Phi^n\left(y,h\right) \mathrm{d}y - \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi^n\left(y,h\right) \mathrm{d}y \right|}{h^{2\tilde{N}}} = 0$$
(B.82)

Proof of (B.35):

Let us recall the Taylor series of the function  $\Phi(y,h) = \Phi_y(h)$ . The  $n^{\text{th}}$  order of the expansion is

$$\Phi^{n}(y,h) = \sum_{i=0}^{n} \frac{\Phi_{y}^{(i)}(0)}{i!} h^{i}$$
(B.83)

Recalling the form (B.59) of  $i^{\text{th}}$  derivative of  $\Phi_y(h)$  we get

$$\Phi_{y}^{(i)}(0) = W^{i}\left(\phi_{y}'(0), \phi_{y}''(0), \dots, \phi_{y}^{(i)}(0)\right) e^{\phi_{y}(0)}$$
(B.84)

From (B.65) we have

$$\phi_y^{(n)}(0) = \sum_{\alpha=-1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} y^{n-2\alpha} \frac{n! \chi_\alpha^{(n-2\alpha)}(0)}{(n-2\alpha)!} =: \tilde{\mathcal{P}}_\chi^n(y)$$
(B.85)

being a polynomial in y. Inserting the form (B.61) of  $\phi_y(h)$  and (B.85) of the derivatives of  $\phi_y(h)$  we get (now exactly, without approximation)

$$\Phi_{y}^{(i)}(0) = W^{i}\left(\tilde{\mathcal{P}}_{\chi}^{1}(y), \dots, \tilde{\mathcal{P}}_{\chi}^{i}(y)\right) e^{\frac{\chi''_{-1}(0)}{2}y^{2}}$$
(B.86)

thus the  $n^{\text{th}}$  order of the Taylor expansion  $\Phi^{n}(y,h)$  is

$$\Phi^{n}(y,h) = e^{\frac{\chi''_{-1}(0)}{2}y^{2}} \sum_{i} W^{i}\left(\tilde{\mathcal{P}}_{\chi}^{1}(y), \dots, \tilde{\mathcal{P}}_{\chi}^{i}(y)\right) \frac{h^{i}}{i!} = e^{\frac{\chi''_{-1}(0)}{2}y^{2}} \sum_{i} \mathcal{W}^{i}(y) \frac{h^{i}}{i!} \quad (B.87)$$

for some polynomials  $\mathcal{W}^{i}(y)$  (note, that they are different then  $\tilde{W}^{i}(y)$  of (B.70)). Consider now the nominator of the formula (B.82). It can be rewritten as

$$\int_{-\infty}^{\infty} \Phi^n(y,h) \,\mathrm{d}y - \int_{-h^{-2\epsilon}}^{h^{-2\epsilon}} \Phi^n(y,h) \,\mathrm{d}y = \int_{\mathbb{R}\setminus \left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} \mathrm{d}y \Phi^n(y,h) \tag{B.88}$$

thus the domain of integration is symmetric with respect to the change of the sign of y. Thus only the part of  $\Phi^n(y,h)$  symmetric in y contributes. The term  $e^{\frac{\chi''_{-1}(0)}{2}y^2}$  is symmetric in y, so it is enough to take into account only the part of each of the polynomials  $\mathcal{W}^i$  consisting of even powers of y, namely  $\mathcal{W}^i_{\text{sym}}$ . Note, that again the polynomials  $\mathcal{W}^i_{\text{sym}}(y)$  do not depend on h in any way, they are fully determined by derivatives of  $\chi_{\alpha}$  at 0. Thus the nominator of (B.82) is

$$I_{3}^{n}(h) - I_{2}^{n}(h) = \int_{\mathbb{R}\setminus \left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{\frac{\chi_{-1}^{\prime\prime}(0)}{2}y^{2}} \sum_{i=0}^{n} \mathcal{W}_{\text{sym}}^{i}(y) \frac{h^{i}}{i!}$$
(B.89)

let us now organize the integrand in (B.89) by the powers of y. Each term  $y^{2j}$  is multiplied by a polynomial in h (let us call it  $\mathcal{Q}_j^n(h)$ ). Each  $\mathcal{Q}_j^n(h)$  is a polynomial of degree less or equal then n. The least power of y in  $\mathcal{W}^i(y)$  is 0 and the highest is 3n for even n and 3n-1 for odd n. Let  $N := \lfloor \frac{3n}{2} \rfloor$ , then

$$I_{3}^{n}(h) - I_{2}^{n}(h) = \int_{\mathbb{R} \setminus \left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{\frac{\chi''_{-1}(0)}{2}y^{2}} \sum_{j=0}^{N} y^{2j} \mathcal{Q}_{j}^{n}(h) = \sum_{j=0}^{N} \mathcal{Q}_{j}^{n}(h) \int_{\mathbb{R} \setminus \left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{\frac{\chi''_{-1}(0)}{2}y^{2}} y^{2j}$$
(B.90)

We estimate the modulus  $|I_{3}^{n}(h) - I_{2}^{n}(h)|$ , thus

$$|I_{3}^{n}(h) - I_{2}^{n}(h)| \leq \sum_{j=0}^{N} \left| \mathcal{Q}_{j}^{n}(h) \right| \int_{\mathbb{R} \setminus \left[ -h^{-2\epsilon}, h^{-2\epsilon} \right]} dy \left| e^{\frac{\chi''_{-1}(0)}{2}y^{2}} y^{2j} \right|$$
(B.91)

recalling, that

$$\left| e^{\frac{\chi_{-1}''(0)}{2}y^2} \right| = e^{\frac{\Re\left(\chi_{-1}''(0)\right)}{2}y^2} = e^{-\frac{C}{2}y^2}$$
(B.92)

we have

$$|I_{3}^{n}(h) - I_{2}^{n}(h)| \leq \sum_{j=0}^{N} |\mathcal{Q}_{j}^{n}(h)| \int_{\mathbb{R}\setminus \left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{-\frac{C}{2}y^{2}} y^{2j}$$
(B.93)

Now we shall estimate the integrals

$$J_{j}(h) := \int_{\mathbb{R}\setminus\left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{-\frac{C}{2}y^{2}}y^{2j}$$
(B.94)

They are given by even momenta of Gaussian integrals with the spread  $\sigma := \sqrt{\frac{2}{C}}$ . Note, that since  $-2\epsilon < 0$ , the bounds of the integration range  $h^{-2\epsilon}$  may reach arbitrary multiple of the spread  $\sigma$ . Let us now consider a series

$$h_k := (k\sigma)^{-2\epsilon} \tag{B.95}$$

i.e. for each  $k \in \mathbb{Z}_+$  we have  $h_k^{-2\epsilon} = k\sigma$ . Obviously  $\lim_{k \to \infty} h_k = 0$ . We will show, that

$$\lim_{k \to \infty} \frac{|I_3^n(h_k) - I_2^n(h_k)|}{h_k^{2\tilde{N}}} = 0$$
(B.96)

Note, that

$$J_{j}(h_{k}) = \int_{-\infty}^{\infty} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2j} \mathrm{d}y - \int_{-k\sigma}^{k\sigma} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2j} \mathrm{d}y$$
(B.97)

Let us introduce a family of serieses of functions

$$\mathcal{I}_{n}^{k}(\sigma) := \frac{1}{\sqrt{\pi}} \int_{-k\sigma}^{k\sigma} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2n} \mathrm{d}y$$
(B.98)

with the limit

$$\mathcal{I}_{n}(\sigma) := \mathcal{I}_{n}^{\infty}(\sigma) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2n} \mathrm{d}y$$
(B.99)

One can find (see Appendix B.2.1.1) that

$$\mathcal{I}_n\left(\sigma\right) = a_n \sigma^{2n+1} \tag{B.100}$$

where the series

$$a_n := \frac{(2n+1)!!}{2^n (2n+1)} \quad \text{for } n \ge 0$$
 (B.101)

or equivalently

$$a_0 = 1$$
 and  $a_n = \frac{(2n-1)!!}{2^n}$  for  $n > 0$  (B.102)

where

$$(2n+1)!! := (2n+1)(2n-1)\cdots 3 \cdot 1$$
 (B.103)

The series  $a_n$  satisfies the recurrence relation:

$$a_{n+1} = \alpha_n a_n$$
 for  $\alpha_n = \frac{2n+1}{2}$  (B.104)

with the initial condition  $a_0 = 1$ .

The integrals  $\mathcal{I}_{n}^{k}\left(\sigma\right)$  are given by

$$\mathcal{I}_n^k(\sigma) = a_n^k \sigma^{2n+1} \tag{B.105}$$

where for each k the series  $a_n^k$  satisfies the recurrence relation:

$$a_{n+1}^k = \alpha_n a_n^k + \gamma^k \beta_n^k$$
 for  $\alpha_n = \frac{2n+1}{2}$  and  $\beta_n^k = k^{2n+1}$  and  $\gamma^k = -\frac{e^{-k^2}}{\sqrt{\pi}}$  (B.106)

with the initial condition  $a_0^k := \frac{1}{\sqrt{\pi}} \int_{-k\sigma}^{k\sigma} e^{-\frac{x^2}{\sigma^2}} dx$  (see (B.181) in Appendix B.2.1.1). The exact value of the initial condition cannot be obtained analytically, but it can be estimated by

$$\sqrt{1 - e^{-k^2}} < a_0^k < \sqrt{1 - e^{-2k^2}} \tag{B.107}$$

(see (B.169)). One can expand the recursion and express  $a_n^k$  by  $a_n$  and  $\alpha_n, \beta_n^k, \gamma^k$  - each known explicitly - obtaining

$$a_n^k = a_0^k \cdot a_n + a_n \gamma^k \beta_{n-1}^k \sum_{i=0}^{n-1} \frac{\beta_i^k}{\beta_{n-1}^k} \frac{1}{a_i} =: a_n - \delta a_n^k$$
(B.108)

(see (B.166)). Inserting the exact formulae for  $a_n$ ,  $\beta_n^k$  and  $\gamma^k$  and the estimation of  $a_0^k$ , one can estimate  $\delta a_n^k$ . For n = 0 the estimation is simply

$$\left|\delta a_0^k\right| = 1 - a_0^k < 1 - \sqrt{1 - e^{-k^2}} < e^{-k^2}$$
 (B.109)

For  $k \geq 1$  we have:

$$\delta a_n^k = a_n \left( 1 - a_0^k - \gamma^k \beta_{n-1}^k \sum_{i=0}^{n-1} \frac{\beta_i^k}{\beta_{n-1}^k} \frac{1}{a_i} \right) = a_n \left( 1 - a_0^k + \frac{e^{-k^2}}{\sqrt{\pi}} k^{2n-1} \sum_{i=0}^{n-1} \frac{k^{2i+1}}{k^{2n-1}} \frac{2^i (2i+1)}{(2i+1)!!} \right)$$
(B.110)

Obviously:

$$\left|\delta a_{n}^{k}\right| \leq a_{n} \left( \left|1 - a_{0}^{k}\right| + \frac{e^{-k^{2}}}{\sqrt{\pi}} k^{2n-1} \left|\sum_{i=0}^{n-1} \frac{1}{k^{2(n-1-i)}} \frac{2^{i} \left(2i+1\right)}{\left(2i+1\right)!!}\right| \right)$$
(B.111)

Let us estimate the sum in the tail. Obviously

$$\sum_{i=0}^{n-1} \frac{1}{k^{2(n-1-i)}} \frac{2^i (2i+1)}{(2i+1)!!} = \frac{1}{k^{2n-2}} + \sum_{i=1}^{n-1} \frac{1}{k^{2(n-1-i)}} \frac{2^i (2i+1)}{(2i+1)!!}$$
(B.112)

Now let us investigate the term  $\frac{2^i(2i+1)}{(2i+1)!!}$  for  $i \ge 1$ :

$$\frac{2^{i}(2i+1)}{(2i+1)!!} = \left(\prod_{K=0}^{i-1} 2\right) \cdot \left(\prod_{L=0}^{i} \frac{1}{2L+1}\right) \cdot (2i+1) \\
= \prod_{K=0}^{i-1} \frac{2}{2K+1} \\
= \frac{2}{(2\cdot 0+1)} \prod_{K=1}^{i-1} \frac{2}{2K+1}$$
(B.113)

now since obviously  $\frac{2}{2K+1} < \frac{2}{2K} = \frac{1}{K}$ , so

$$\frac{2^{i}(2i+1)}{(2i+1)!!} \le 2\prod_{K=1}^{i-1} \frac{1}{K} = \frac{2}{(i-1)!}$$
(B.114)

Now since  $k \ge 1$ , for all *i* we have  $\frac{1}{k^{2(n-1-i)}} \le 1$ . Moreover the sum of non-negative terms up to n-1 is less than the sum up to  $\infty$ , thus

$$\left|\sum_{i=0}^{n-1} \frac{1}{k^{2(n-1-i)}} \frac{(2i+1)}{2^i (2i+1)!!}\right| < \left|1 + \sum_{i=1}^{\infty} \frac{2}{(i-1)!}\right| = 1 + 2\sum_{i=0}^{\infty} \frac{1}{i!} = 1 + 2 \cdot e \qquad (B.115)$$

Applying (B.115) together with (B.107) and (B.109) to (B.111) we got

$$\left|\delta a_n^k\right| \le a_n \left( e^{-k^2} + \frac{e^{-k^2}}{\sqrt{\pi}} k^{2n-1} \cdot (1+2 \cdot e) \right) = a_n e^{-k^2} \left( 1 + k^{2n-1} \frac{1+2e}{\sqrt{\pi}} \right)$$
(B.116)

Going back to  $J_{j}(h_{k})$  note, that

$$J_{j}(h_{k}) = \sqrt{\pi} \left( \mathcal{I}_{j}(\sigma) - \mathcal{I}_{j}^{k}(\sigma) \right) = \sqrt{\pi} \sigma^{2j+1} \delta a_{j}^{k}$$
(B.117)

Thanks to (B.116) we have

$$|J_j(h_k)| \le e^{-k^2} a_j \left(\sqrt{\pi} + 8k^{2j-1}\right) \sigma^{2j+1}$$
(B.118)

Going back to (B.93) we can apply (B.118) and obtain

$$|I_3^n(h_k) - I_2^n(h_k)| \le e^{-k^2} \sum_{j=0}^N \left| \mathcal{Q}_j^n(h_k) \right| a_j \left( \sqrt{\pi} + 8k^{2j-1} \right) \sigma^{2j+1}$$
(B.119)

for  $\sigma = \sqrt{\frac{2}{C}}$  independent on n, j or k.

Let us finally consider the limit  $\lim_{k\to\infty} \frac{|I_3^n(h_k)-I_2^n(h_k)|}{h_k^{2\tilde{N}}}$ . Recalling (B.95) we have  $h_k := (k\sigma)^{-2\epsilon}$ , thus the denominator becomes a power function of k, i.e.  $h_k^{2\tilde{N}} = k^{-4\epsilon\tilde{N}}\sigma^{-4\epsilon\tilde{N}}$ , thus the limit is

$$\lim_{k \to \infty} \frac{|I_3^n(h_k) - I_2^n(h_k)|}{h_k^{2\tilde{N}}} \le \lim_{k \to \infty} e^{-k^2} \sum_{j=0}^N |\mathcal{Q}_j^n(h_k)| a_j \left(\sqrt{\pi} + 8k^{2j-1}\right) k^{4\epsilon \tilde{N}} \sigma^{2j+1+4\epsilon \tilde{N}}$$
(B.120)

Each polynomial  $Q_j^n(h_k)$  goes to a constant  $\mathcal{Q}_j^n(0)$  as  $h_k$  goes to 0. Note, that

$$\lim_{k \to \infty} e^{-k^2} k^{\alpha} = 0 \tag{B.121}$$

for any positive  $\alpha$ . Thus the overall limit is

$$\lim_{k \to \infty} \frac{|I_3^n(h_k) - I_2^n(h_k)|}{h_k^{2\tilde{N}}} = 0$$
(B.122)

for each  $\tilde{N}$ .

This ends the proof of (B.35).

Now to find the leading order of the integral (B.32) it is enough to check  $I_3^0(\hbar)$ . It is given by assuming  $\hbar = 0$  in (B.80). We have thus

$$I_3^0(\hbar) = \int_{-\infty}^{+\infty} e^{\chi_{-1}'(0)y^2} dy = \frac{\sqrt{2\pi}}{\sqrt{-\chi_{-1}''(0)}}$$
(B.123)

 $\mathbf{SO}$ 

$$\int_{a}^{b} e^{\chi(x,\hbar)} \mathrm{d}x = \sqrt{\hbar} \frac{\sqrt{2\pi}}{\sqrt{-\chi_{-1}''(0)}} \left(1 + O\left(\hbar^{\epsilon}\right)\right) \tag{B.124}$$

Moreover, it is easy to show a stronger approximation. From (B.33)-(B.35) we have, that  $|I(\hbar) - I_3^2(\hbar)| = O(\hbar)$ . Consider thus the integral  $I_3^2(\hbar)$ :

$$I_{3}^{2}(\hbar) = \int_{-\infty}^{+\infty} e^{\chi_{-1}'(0)y^{2}} \left( 1 + \sqrt{\hbar}\phi_{y}'(0) + \frac{\hbar}{2} \left[ \phi_{y}''(0) + \left(\phi_{y}'(0)\right)^{2} \right] \right) \mathrm{d}y$$
(B.125)

Recalling (B.85) one can check, that  $\phi'_{y}(0)$  is an odd polynomial of y, thus it vanishes when integrated with  $e^{\chi''_{-1}(0)y^2}$  over a symmetric range. The integral

$$\int_{-\infty}^{+\infty} e^{\chi_{-1}'(0)y^2} \left[ \phi_y''(0) + \left( \phi_y'(0) \right)^2 \right] =: A$$
(B.126)

gives a constant factor A (independent on  $\hbar$ ), so

$$I_{3}^{2}(\hbar) = I_{3}^{0}(\hbar) + \hbar \cdot A = I_{3}^{0}(\hbar) + O(\hbar)$$
(B.127)

and thus finally

$$\int_{a}^{b} e^{\chi(x,\hbar)} dx = \sqrt{\hbar} \frac{\sqrt{2\pi}}{\sqrt{-\chi_{-1}''(0)}} (1 + O(\hbar))$$
(B.128)

Quod erat demonstrandum.

The multidimensional generalisation of the theorem is straightforward. Indeed, let us do the following modification of the proof:

- We use a compact set  $\Omega \subset \mathbb{R}^N$ , such that  $\Omega$  is a neighbourhood of  $\{0\}$  instead of the interval [a, b]. Moreover, let us choose the coordinates in  $\mathbb{R}^N$ , such that the matrix  $\frac{\partial^2 \chi}{\partial x^i \partial x^j}\Big|_{x=0}$  is diagonal (one can do this ensuring that the Jacobian  $J := \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = 1$  for all  $x \in \mathbb{R}^N$ ).
- We use the set  $\mathcal{O}_{\hbar,\epsilon} := \left\{ x \in \mathbb{R}^N : |x| < \hbar^{\frac{1}{2}-\epsilon} \right\}$  instead of  $\left[ -\hbar^{\frac{1}{2}-\epsilon}, \hbar^{\frac{1}{2}-\epsilon} \right]$ , and  $\tilde{\mathcal{O}}_{\hbar,\epsilon} := \left\{ x \in \mathbb{R}^N : |x| < \hbar^{-\epsilon} \right\}$  instead of  $\left[ -\hbar^{-\epsilon}, \hbar^{-\epsilon} \right]$  respectively.
- We adjust the assumptions, i.e.

$$\forall_{x \neq 0, \hbar \in ]0,1[} \qquad \Re\left(\chi\left(x, \hbar\right)\right) < 0 \tag{B.129}$$

$$\forall_{\hbar\in]0,1[}\lim_{x\to\partial\Omega}\qquad \Re\left(\chi\left(x,\hbar\right)\right)=-\infty\tag{B.130}$$

$$\forall_{\hbar\in]0,1[} \qquad \chi\left(0,\hbar\right) = 0 \tag{B.131}$$

$$\left| i_{i=1,\dots,N} \lim_{\hbar \to 0} \quad \hbar \left| \frac{\partial \chi \left( x, \hbar \right)}{\partial x^{i}} \right|_{x=0} = 0$$
 (B.132)

$$\forall_{i,j=1,\dots,N} \lim_{\hbar \to 0} \qquad \hbar \Re \left( \left. \frac{\partial^2 \chi \left( x, \hbar \right)}{\partial x^i \partial x^j} \right|_{x=0} \right) = -C_i \delta_{ij} < 0 \tag{B.133}$$

• The final formula reads

$$\int_{\Omega} e^{\chi(x,\hbar)} \mathrm{d}^{N} x = \left(\sqrt{\hbar}\right)^{N} I(\hbar)$$
(B.134)

for a smooth function 
$$I(\hbar) : [0, \infty[ \to \mathbb{C}, \text{ such that } I(0) = \left(\sqrt{2\pi}\right)^N \frac{1}{\sqrt{\det\left(-\frac{\partial^2 \chi_{-1}}{\partial x^i \partial x^j}\Big|_0\right)}}$$

Then all the steps of the proof can be translated to the multidimensional case. Whenever in the above proof one uses the Taylor series of the functions  $\chi_{\alpha}(x)$ , here one uses the multidimensional Taylor series. Whenever one uses the smallness of  $x \in \left[-\hbar^{\frac{1}{2}-\epsilon}, \hbar^{\frac{1}{2}-\epsilon}\right]$ , here one has it for all coordinates  $x^i$  for  $x \in \mathcal{O}_{\hbar,\epsilon}$ . Whenever in the above proof one derives, that a function is polynomial of |y| or of y, here one obtains a polynomial of many coordinates  $x^i$  or  $|x^i|$ .

The only nontrivial generalization is the estimation of the integrals

$$J_{j}(h) := \int_{\mathbb{R}\setminus\left[-h^{-2\epsilon}, h^{-2\epsilon}\right]} dy e^{-\frac{C}{2}y^{2}}y^{2j}$$
(B.135)

in the proof of (B.35). In the multidimensional case one obtains a Gaussian integral over  $\mathbb{R}^N \setminus \tilde{\mathcal{O}}_{\hbar,\epsilon}$  of a polynomial of multiple variables  $y^i$ , with positive coefficients. One can estimate such polynomial by a polynomial in r = |y|, because obviously  $r > y^i$ . One can also estimate the Gaussian function  $e^{-\sum_i \frac{C_i}{2}(y^i)^2} \leq e^{-\frac{Cr^2}{2}}$  for  $C = \min(C_i)$ . Passing to the spherical coordinates one obtains integrals

$$\tilde{J}_m(h) := \int_{|y| > h^{-2\epsilon}} |y|^{m-(N-1)} e^{-\frac{Cr^2}{2}} \mathrm{d}^N y = S^{(N-1)} \int_{h^{-2\epsilon}}^{\infty} r^m e^{-\frac{Cr^2}{2}} \mathrm{d}r$$
(B.136)

where  $S^{(N-1)}$  is the (N-1)-volume of the unit (N-1)-sphere, and m can be both odd and even. Again one introduces  $\sigma := \sqrt{\frac{2}{C}}$  and the series  $h_k := (k\sigma)^{-2\epsilon}$  and consider the limit with  $k \to \infty$ . The case of even m is handled by the derivation in the above proof, because  $\tilde{J}_{2j}(h_k) = \frac{S^{(N-1)}}{2} J_j(h_k)$ , so it vanish at  $k \to \infty$ . Here let us study the case of odd m.

Let us introduce a series of integrals

$$\tilde{\mathcal{I}}_{j}^{k}\left(\sigma\right) := \int_{0}^{k\sigma} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2j+1} \mathrm{d}y \tag{B.137}$$

and the limit

$$\tilde{\mathcal{I}}_{j}^{\infty}\left(\sigma\right) := \int_{0}^{\infty} e^{-\frac{y^{2}}{\sigma^{2}}} y^{2j+1} \mathrm{d}y \tag{B.138}$$

Similarly to the calculations in the proof above, we have

$$\tilde{\mathcal{I}}_{j}^{\infty}(\sigma) = \frac{\sigma^{2j+2}}{2}\tilde{a}_{j} \quad \text{and} \quad \tilde{\mathcal{I}}_{j}^{k}(\sigma) = \frac{\sigma^{2j+2}}{2}\tilde{a}_{j}^{k}$$
(B.139)

where by appropriate changes of variables one gets

$$\tilde{a}_{j} = \int_{0}^{\infty} e^{-u} u^{j} du \quad \text{and} \quad \tilde{a}_{j}^{k} = \int_{0}^{k^{2}} e^{-u} u^{j} du$$
(B.140)

The integrals  $\tilde{a}_j$  can be easily calculated, giving

$$\tilde{a}_j = j! \tag{B.141}$$

The integrals  $\tilde{a}_{i}^{k}$  satisfy a recursive relation, similar to (B.106), i.e.

$$\tilde{a}_{j+1}^k = \tilde{\alpha}_j \tilde{a}_j^k + \tilde{\gamma}^k \tilde{\beta}_j^k \quad \text{for} \quad \tilde{\alpha}_j = j+1 \quad \text{and} \quad \tilde{\beta}_j^k = k^{2j+2} \quad \text{and} \quad \tilde{\gamma}^k = -e^{-k^2} \tag{B.142}$$

with the initial condition

$$\tilde{a}_0^k = 1 - e^{-k^2} = 1 + \tilde{\gamma}^k \tag{B.143}$$

(see Appendix B.2.1.2). By the same argument, as in the above proof, one obtains the difference  $\delta \tilde{a}_j^k := \tilde{a}_n - \tilde{a}_n^k$  to be given by the formula

$$\delta \tilde{a}_j^k = \tilde{a}_j \left( 1 - \tilde{a}_0^k - \tilde{a}_0^k \tilde{\gamma}^k \tilde{\beta}_{j-1}^k \sum_{i=0}^{j-1} \frac{\tilde{\beta}_i^k}{\tilde{\beta}_{j-1}^k} \frac{1}{\tilde{a}_i} \right)$$
(B.144)

applying the exact form of  $\tilde{a}_j$ ,  $\tilde{\beta}_j^k$  and  $\tilde{\gamma}^k$  one gets

$$\delta \tilde{a}_{j}^{k} = j! e^{-k^{2}} \left( 1 + \left( 1 - e^{-k^{2}} \right) k^{2j} \sum_{i=0}^{j-1} \frac{1}{k^{2(j-i-1)}} \frac{1}{i!} \right)$$
(B.145)

Again we can estimate  $\left(1 - e^{-k^2}\right) < 1$  and  $\sum_{i=0}^{j-1} \frac{1}{k^{2(j-i-1)}} \frac{1}{i!} \le \sum_{i=0}^{j-1} \frac{1}{i!} < \sum_{i=0}^{\infty} \frac{1}{i!} = e$ , so

$$\left|\delta \tilde{a}_{j}^{k}\right| \leq j! e^{-k^{2}} \left(1 + e \cdot k^{2j}\right) \tag{B.146}$$

so  $\tilde{J}_{2j+1}(h_k) = \frac{S^{(N-1)}}{2} \left( \tilde{\mathcal{I}}_j^{\infty}(\sigma) - \tilde{\mathcal{I}}_j^k(\sigma) \right)$  is bound by

$$\left|\tilde{J}_{2j+1}(h_k)\right| < \frac{S^{(N-1)}}{2} \frac{\sigma^{2j+2}}{2} \left|\delta \tilde{a}_j^k\right| \underset{k \to \infty}{\longrightarrow} 0 \tag{B.147}$$

The final calculation of  $I_3^0(\hbar)$  factorises into N independent Gaussian integrals, which leads to the result

$$I(0) = \left(\sqrt{2\pi}\right)^{N} \frac{1}{\sqrt{\det\left(-\frac{\partial^{2}\chi_{-1}}{\partial x^{i}\partial x^{j}}\Big|_{x=0}\right)}}$$
(B.148)

and ends the multidimensional generalisation of the proof.

The following subsection present the calculation of the Gaussian integrals.

## **B.2.1** Gaussian integrals over finite intervals

Let us derive the recursive relations for integrals  $\mathcal{I}_{n}^{k}(\sigma)$  and  $\tilde{\mathcal{I}}_{n}^{k}(\sigma)$  used in above proof.

### B.2.1.1 Gaussian integral with even polynomial

Consider a family of integrals

$$\mathcal{I}_{n}^{k}(\sigma) := \frac{1}{\sqrt{\pi}} \int_{-k\sigma}^{k\sigma} e^{-\frac{x^{2}}{\sigma^{2}}} x^{2n} \mathrm{d}x$$
(B.149)

with the limit

$$\mathcal{I}_{n}(\sigma) := \mathcal{I}_{n}^{\infty}(\sigma) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^{2}}{\sigma^{2}}} x^{2n} \mathrm{d}x$$
(B.150)

#### Integral over all real line

First let us consider  $\mathcal{I}_{n}(\sigma)$ . By changing variables to  $u := \frac{x}{\sigma}$  we get

$$\mathcal{I}_n\left(\sigma\right) = \sigma^{2n+1} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} u^{2n} \mathrm{d}u =: \sigma^{2n+1} a_n \tag{B.151}$$

For the series  $a_n$  we can find a recurrence relation by integrating by parts:

$$a_{n} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^{2}} u^{2n} du$$

$$= \frac{1}{\sqrt{\pi}} \left[ \frac{u^{2n+1}e^{-u^{2}}}{2n+1} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-2u) e^{-u^{2}} \frac{u^{2n+1}}{2n+1} du \right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{2}{2n+1} \int_{-\infty}^{\infty} u^{2n+2} e^{-u^{2}} du$$

$$= \frac{2}{2n+1} a_{n+1}$$
(B.152)

thus the recurrence relation is

$$a_{n+1} = \frac{2n+1}{2}a_n \tag{B.153}$$

The initial condition is

$$a_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$
 (B.154)

(we kept the factor  $\frac{1}{\sqrt{\pi}}$  in order to have this simple initial condition). Let us introduce the series

$$\alpha_n := \frac{2n+1}{2} \tag{B.155}$$

so that

$$a_{n+1} = \alpha_n a_n \tag{B.156}$$

Now it is easy to solve the recursion:

$$a_n = a_0 \prod_{i=0}^{n-1} \alpha_i \tag{B.157}$$

The product can be done explicitly:

$$a_n = \frac{(2n-1)!!}{2^n}$$
 for  $n > 0$  and  $a_0 = 1$  (B.158)

where

$$(2N+1)!! := \prod_{k=0}^{N} (2k+1)$$
(B.159)

# Integral over a multiplicity of the spread of the Gaussian

Now let us consider  $\mathcal{I}_{n}^{k}(\sigma)$ . Again the change of variables gives us a series

$$\mathcal{I}_{n}^{k}(\sigma) = \sigma^{2n+1} \frac{1}{\sqrt{\pi}} \int_{k}^{k} e^{-u^{2}} u^{2n} \mathrm{d}u =: \sigma^{2n+1} a_{n}^{k}$$
(B.160)

and integration by parts gives us a recurrence relation for  $a_n^k$ :

$$a_{n}^{k} = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} e^{-u^{2}} u^{2n} du$$

$$= \frac{1}{\sqrt{\pi}} \left[ \frac{u^{2n+1}e^{-u^{2}}}{2n+1} \Big|_{-k}^{k} - \int_{-k}^{k} (-2u) e^{-u^{2}} \frac{u^{2n+1}}{2n+1} du \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[ \frac{2}{2n+1} k^{2n+1} e^{-k^{2}} + \frac{2}{2n+1} \int_{-k}^{k} u^{2n+2} e^{-u^{2}} du \right]$$

$$= \frac{2}{2n+1} \left[ \frac{1}{\sqrt{\pi}} k^{2n+1} e^{-k^{2}} + a_{n+1}^{k} \right]$$
(B.161)

giving the recurrence relation

$$a_{n+1}^k = \frac{2n+1}{2}a_n^k - k^{2n+1}\frac{e^{-k^2}}{\sqrt{\pi}}$$
(B.162)

Let us now introduce two more serieses

$$\beta_n^k := k^{2n+1} \qquad \gamma^k := -\frac{e^{-k^2}}{\sqrt{\pi}}$$
 (B.163)

so that the recurrence relation becomes

$$a_{n+1}^k = \alpha_n a_n^k + \beta_n^k \gamma^k \quad \text{for } n \ge 1$$
(B.164)

with  $\alpha_n$  the same, as in case of  $a_n$ .

One can expand the recurrence relation (B.164) by inserting it in place of  $a_n^k$ :

$$a_{n}^{k} = \alpha_{n-1}a_{n-1}^{k} + \beta_{n-1}^{k}\gamma^{k}$$

$$= \alpha_{n-1}\alpha_{n-2}a_{n-2}^{k} + \alpha_{n-1}\beta_{n-2}^{k}\gamma^{k} + \beta_{n-1}^{k}\gamma^{k}$$

$$= \alpha_{n-1}\alpha_{n-2}\alpha_{n-3}a_{n-3}^{k} + \alpha_{n-1}\alpha_{n-2}\beta_{n-3}^{k}\gamma^{k} + \alpha_{n-1}\beta_{n-2}^{k}\gamma^{k} + \beta_{n-1}^{k}\gamma^{k}$$

$$\vdots :$$

$$= \left(\prod_{i=0}^{n-1}\alpha_{i}\right) \cdot a_{0}^{k} + \gamma^{k}\sum_{i=0}^{n-1}\beta_{i}^{k}\prod_{j=i+1}^{n-1}\alpha_{j}$$
(B.165)

now since  $a_n = \prod_{i=0}^{n-1} \alpha_i$ , we have  $\prod_{j=i+1}^{n-1} \alpha_j = \frac{\prod_{j=0}^{n-1} \alpha_j}{\prod_{j=0}^i \alpha_j} = \frac{a_n}{a_{i+1}}$ 

$$a_n^k = a_n \cdot a_0^k + a_n \cdot \gamma^k \sum_{i=0} \frac{\beta_i^k}{a_{i+1}} = a_n \cdot a_0^k + a_n \cdot \beta_{n-1}^k \cdot \gamma^k \sum_{i=0}^{n-1} \frac{\beta_i^k}{\beta_{n-1}^k} \frac{1}{a_i}$$
(B.166)

The initial condition  $a_0^k$  for (B.164) cannot be found directly, but it can be estimated. Let us consider  $(a_0^k)^2$ :

$$\left(a_{0}^{k}\right)^{2} = \frac{1}{\sqrt{\pi}} \int_{-k}^{k} e^{-x^{2}} dx \frac{1}{\sqrt{\pi}} \int_{-k}^{k} e^{-y^{2}} dy$$

$$= \frac{1}{\pi} \iint_{\max(|x|,|y|) < k} e^{-(x^{2}+y^{2})} dx dy$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} d\phi \int_{0}^{r(\phi)} e^{-r^{2}} r dr$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \int_{0}^{r^{2}(\phi)} e^{-u} du$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left(1 - e^{-r^{2}(\phi)}\right) d\phi$$
(B.167)

where  $r(\phi)$  is describes the borders of the square. The exact integration is impossible in terms of elementary functions, however one can estimate it. Note, that for each  $\phi$  we have  $k^2 \leq r^2(\phi) \leq 2k^2$ . Thus we have

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left( 1 - e^{-\max\left(r^{2}(\phi)\right)} \right) \mathrm{d}\phi \quad < \left(a_{0}^{k}\right)^{2} < \frac{1}{2\pi} \int_{0}^{2\pi} \left( 1 - e^{-\min\left(r^{2}(\phi)\right)} \right) \mathrm{d}\phi$$
$$\frac{1}{2\pi} \int_{0}^{2\pi} \left( 1 - e^{-2k^{2}} \right) \mathrm{d}\phi \quad < \left(a_{0}^{k}\right)^{2} < \frac{1}{2\pi} \int_{0}^{2\pi} \left( 1 - e^{-k^{2}} \right) \mathrm{d}\phi \qquad (B.168)$$
$$1 - e^{-2k^{2}} \quad < \left(a_{0}^{k}\right)^{2} < 1 - e^{-k^{2}}$$

 $\mathbf{SO}$ 

$$\sqrt{1 - e^{-2k^2}} < a_0^k < \sqrt{1 - e^{-k^2}} \tag{B.169}$$

### B.2.1.2 Gaussian integral with odd polynomial

Let us now consider another family of integrals

$$\tilde{\mathcal{I}}_{n}^{k}\left(\sigma\right) := \int_{0}^{k\sigma} e^{-\frac{x^{2}}{\sigma^{2}}} x^{2n+1} \mathrm{d}x \tag{B.170}$$

with the limit

$$\tilde{\mathcal{I}}_{n}\left(\sigma\right) := \int_{0}^{\infty} e^{-\frac{x^{2}}{\sigma^{2}}x^{2n+1}} \mathrm{d}x \tag{B.171}$$

## Integral over all positive real half-line

First let us consider  $\tilde{\mathcal{I}}_n(\sigma)$ . By changing variables to  $u := \frac{x^2}{\sigma^2}$  we get

$$\tilde{\mathcal{I}}_n(\sigma) = \frac{\sigma^{2n+2}}{2} \int_0^\infty e^{-u} u^n \mathrm{d}u =: \frac{\sigma^{2n+2}}{2} \tilde{a}_n \tag{B.172}$$

Integrating by parts we get a recurrence relation for  $\tilde{a}_n$ :

$$\tilde{a}_n = -e^{-u}u^n \Big|_0^\infty - \int_0^\infty -e^{-u} \cdot n \cdot u^{n-1} du = n\tilde{a}_{n-1} \quad \text{for } n \ge 1 \quad (B.173)$$

Again introducing a series

$$\tilde{\alpha}_n := n+1 \tag{B.174}$$

we have

$$\tilde{a}_n = \tilde{\alpha}_{n-1} a_{n-1} \tag{B.175}$$

with the initial condition

$$\tilde{a}_0 = \int_0^\infty e^{-u} \mathrm{d}u = 1 \tag{B.176}$$

which gives

$$\tilde{a}_n = \tilde{a}_0 \prod_{i=0}^{n-1} \alpha_i = n!$$
(B.177)

# Integral over a multiplicity of the spread of the Gaussian

Now let us consider  $\tilde{\mathcal{I}}_{n}^{k}(\sigma)$ . Again the change of variables gives us a series

$$\tilde{\mathcal{I}}_{n}^{k}\left(\sigma\right) = \frac{\sigma^{2n+2}}{2} \int_{0}^{k^{2}} e^{-u} u^{n} \mathrm{d}u =: \frac{\sigma^{2n+2}}{2} \tilde{a}_{n}^{k}$$
(B.178)

and integration by parts gives us a recurrence relation for  $\tilde{a}_n^k$ :

$$\tilde{a}_{n}^{k} = \int_{0}^{k^{2}} e^{-u} u^{n} du$$

$$= -e^{-u} u^{n} \Big|_{0}^{k^{2}} - \int_{0}^{k^{2}} -e^{-u} \cdot n \cdot u^{n-1} du$$

$$= -e^{-k^{2}} k^{2n} + n \cdot \tilde{a}_{n-1}^{k}$$

$$= ((n-1)+1) \tilde{a}_{n-1}^{k} - e^{-k^{2}} k^{2(n-1)+2}$$
(B.179)

Introducing two more serieses

$$\tilde{\beta}_n^k := k^{2n+2} \qquad \gamma^k := -e^{-k^2}$$
(B.180)

we get the recurrence relation

$$\tilde{a}_{n}^{k} = \tilde{\alpha}_{n-1}\tilde{a}_{n-1}^{k} + \tilde{\beta}_{n}^{k}\tilde{\gamma}^{k} \quad \text{for } n \ge 1$$
(B.181)

similar to the (B.164).

Applying the same technique we used in case of  $a_n^k$  we can expand the recursion obtaining

$$\tilde{a}_n^k = \tilde{a}_n \cdot \tilde{a}_0^k + \tilde{a}_n \cdot \tilde{\beta}_{n-1}^k \cdot \tilde{\gamma}^k \sum_{i=0}^{n-1} \frac{\tilde{\beta}_i^k}{\tilde{\beta}_{n-1}^k} \frac{1}{\tilde{a}_i}$$
(B.182)

Here the initial condition can be calculated explicitly

$$\tilde{a}_0^k = \int_0^{k^2} e^{-u} du = -e^{-u} \Big|_0^{k^2} = 1 - e^{-k^2}$$
(B.183)

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