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Heterotic Orbifolds

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Maximilian Felix Fischer

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Heterotic Orbifolds

Maximilian Felix Fischer

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Vorsitzender:		Univ.-Prof. Dr. Lothar Oberauer
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Michael Ratz
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	3.	Priv.-Doz. Dr. Stefan Groot Nibbelink Ludwig-Maximilians-Universität München

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In Erinnerung an

Helen Holtzmann

(1989^{*} – $2011^{\dagger}$)

Diese Arbeit ist dir gewidmet, meine Kriegerin – du warst ein fürchterlich richtiges Wesen in einer schrecklich falschen Welt. Ein Platz in meinem Herzen ist dir auf alle Ewigkeit gewiss.

Abstract

String theory is the leading candidate for a unified description of gravity and gauge interactions. Among the possible string constructions, heterotic string theory stands out as it can naturally yield $\mathcal{N} = 1$ supersymmetry in four dimensions. In this dissertation we study some geometrical aspects of the heterotic string utilising a method called orbifold compactification. We briefly revisit the construction of the heterotic string as well as the geometric definitions leading to orbifolds. Then we heavily exploit crystallographic methods to classify all possible symmetric toroidal orbifolds with Abelian and non-Abelian point groups which lead to $\mathcal{N} = 1$ or higher supersymmetry in four dimensions and give derivations of their fundamental groups and two independent ways of obtaining their orbifold Hodge numbers. A complete tabulation of all results can be found in the appendix.

Zusammenfassung

Stringtheorie stellt eine herausragende Möglichkeit dar, Gravitation und Eichwechselwirkungen einheitlich zu beschreiben. Die heterotischen Stringkonstruktionen sind dabei von besonderem Interesse, da sie auf natürliche Weise Theorien mit $\mathcal{N} = 1$ Supersymmetrie in vier Raumzeitdimensionen produzieren können. In dieser Arbeit widmen wir uns einigen geometrischen Aspekten heterotischer Stringtheorien, die auf sogenannten Orbifaltigkeiten kompaktifiziert werden. Wir geben eine kurze Einführung des heterotischen Strings und definieren Orbifaltigkeiten und verwandte Konzepte. Im Hauptteil dieser Arbeit nutzen wir kristallographische Methoden um alle symmetrischen toroidalen Orbifaltigkeiten mit abelschen und nichtabelschen Punktgruppen, die $\mathcal{N} \geq 1$ Supersymmetrie in vier Dimensionen erhalten, zu klassifizieren und geben deren Fundamentalgruppen und Hodgezahlen an. Im Anhang ist eine vollständige tabellarische Auflistung der Ergebnisse zu finden.

Contents

Chapter 1

Introduction

The underpinning principle of the universe seems to be symmetry. However, this was not apparently clear at the inception of physics as an exact science. But since the publication of Albert Einsteins theory of general relativity [1] which relies on the principle of locality and Emmy Noether's famous theorem [2] which links conserved observables directly to symmetries of the phase space, symmetries have always been rising in importance within physics. This development culminated in the merging of special relativity with quantum mechanics to quantum field theory (QFT) [3] in the early sixties. In QFT, physical properties of particles and their trajectories can be deduced directly from the symmetries of spacetime and the particles themselves. The latter, so-called gauge symmetries, are—in the standard model (SM) continuous—symmetry groups whose representations govern the transformation properties of bosons and fermions. The SM, with gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ has turned out to be one of the most accurate theories of physics that has been verified with unprecedented precision which culminated in the discovery of the Higgs Boson in 2012 [4]. But it is still clear that this model can not be the ultimate description of the universe we live in. This is due to a series of problems, some aesthetic, some practical and some fundamental. Several modifications and extensions to the SM have been proposed which address some of these problems; however, no extent of meddling will ever be able to solve the most fundamental problem of the SM, which is its incompatibility with general relativity: every attempt to quantise gravity in a gauge field theory is doomed to be non-renormalisable and thus can only yield an effective description at best [5].

The most promising answer to this conundrum is string theory, which was first proposed in 1970 (cf. [6]) as a description of strong interactions. In string theory, the fundamental entities are no longer point-like particles, but one-dimensional extended objects – strings. This leads to some astounding implications, one of which enforces the presence of a consistent description of gravity within it, guaranteeing the absence of one of the biggest problems of the SM. However, string theory comes with its own set of intrinsic problems. One of them, the prediction of extra dimensions and how to deal with them will be the focus of this thesis. To tackle this problem, we will make heavy use of symmetries of compact spaces and are therefore in good continuation of a long history of symmetries in physics.

The main result of this thesis is a complete classification of all 520 toroidal orbifold geometries with genus one which can be used to symmetrically compactify heterotic string theory and which allow for supersymmetry in the resulting four-dimensional low-energy theory. Some of these were known before, but most of them are new. We can also prove that our list is complete. If no realistic model can be found within it, then symmetric orbifold constructions could be ruled out at all. Note however, that due to the vast possibilities in how to chose the gauge embedding, there are still many models to be scanned and a complete search is still not an easy task.

Outline

This work is organised as follows: in Chapter two we will briefly review heterotic string theory and give a first outlook into orbifold compactifications. Chapter three will then be fully committed to orbifolds: we will first give a very general definition of orbifolds and will then subsequently reduce this definition to the case of toroidal genus one orbifolds. Afterwards we will introduce the space group as the object, which solely defines the orbifold, explore its constituents, most prominently the lattice and the point group and relate it to subgroups of the unimodular group $\mathrm{GL}(n, \mathbb{Z})$. Following that, our task will be to classify all possible orbifolds which give rise to $\mathcal{N} \geq 1$ supersymmetry in four dimensions. We will do this by introducing crystallographic language, namely the notions of $\mathbb{Q}$-, $\mathbb{Z}$- and affine classes of unimodular groups. We will then utilise the computer program Carat [7] to do the classification of geometries. We will also present two different ways of deciding whether a given space admits SUSY in four dimensions. Computations of the Hodge numbers and fundamental groups of the orbifold geometries obtained with this scheme will conclude the chapter. In Chapter 4 we will focus on one specific geometry and explore its implications for gauge coupling unification at high energies. Finally, Chapter 5 will conclude this thesis. In the appendices, we will give a short review of orbifolds in two dimensions—spawned by the famous wallpaper groups—a complete tabulation of all 520 SUSY-preserving geometries, Abelian and non-Abelian, we found, together with some of their topological data, and lastly reproduce a listing of the Mathematica package which enabled us to do explicit calculations with Clifford algebras.

Some parts of this thesis have been published in the following research papers.

- M. Fischer, M. Ratz, J. Torrado, and P. K. Vaudrevange, "Classification of symmetric toroidal orbifolds," *JHEP*, vol. 1301, p. 084, 2013

- M. Fischer, S. Ramos-Sánchez, and P. K. S. Vaudrevange, "Heterotic non-Abelian orbifolds," *JHEP*, vol. 1307, p. 080, 2013

Chapter 2

Heterotic String Theory

In this chapter we will give a very brief introduction to heterotic string theory and the orbifold compactification process. This chapter is, however, not meant to be a comprehensive introduction to the vast field that string theory is: we will merely repeat the most important results for convenience and in order to set the notation and provide little to no proof or motivation, which can be found in many standard textbooks, some of which are advertised in the following paragraph. In doing this, we will anticipate some of the results of the following chapters without going into too much tedious detail, or to be precise, we will use tori, point groups and orbifolds, where needed, without giving rigorous definitions yet, in order to focus on the stringy part of the discussion.

Complete introductions, not only in heterotic, but all of string theory can be found for instance in the monographs by Green, Schwarz and Witten [10] or Becker, Becker and Schwarz [11]. The reader who is more interested in the historic development and the methods first used to understand string physics might find the classic standard textbook by Polchinski [12, 13] particularly interesting. We will, however, adopt the more modern language and conventions used in the comprehensive textbook by Ibáñez and Uranga [14] and most closely follow this book throughout this chapter. For the background in supersymmetry required to understand superstring theory, the exceptional primer by Martin [15] is a very good starting point. Because the contents of this chapter are standard textbook knowledge, only few references will be given, where they are of special interest.

2.1 The world sheet

The parameter space of the free heterotic string is described by a two-dimensional orientable manifold with the topology of a cylinder, parametrised by t, denoting the eigentime of the string and σ, describing the compact internal string coordinate. On this, a two-dimensional supersymmetric conformal field theory (CFT) is declared

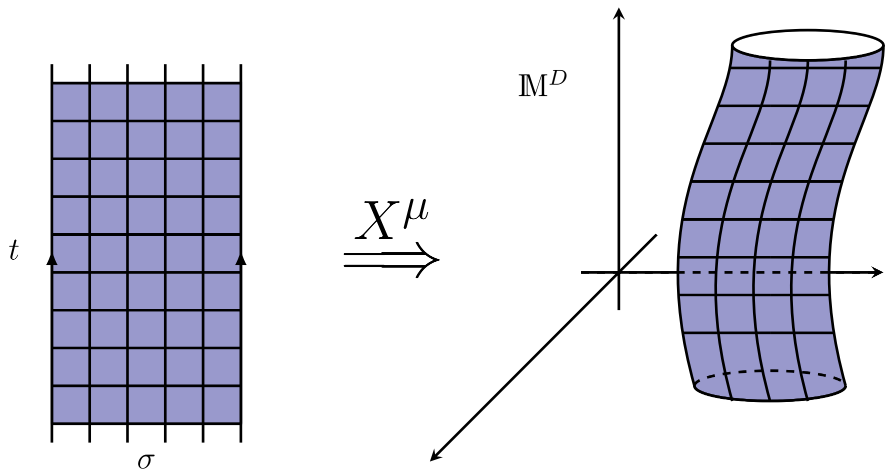


Figure 2.1: The string X^μ embeds the world sheet into the target space. In our case, the string is closed, which means opposite ends are identified, and thus the embedding traces out a cylindrical submanifold of the target space.

whose bosonic fields, denoted with $X^i(t,\sigma)$, describe the string. They take values in the so-called target space, which for type I and II superstring theories would be just spacetime, but in the heterotic formulation also contains gauge degrees of freedom. Since the world sheet has cylindrical topology, we are only dealing with closed strings which comply with a boundary condition

$$X(t,\sigma+\ell) \;=\; X(t,\sigma)\ , \tag{2.1}$$

where ℓ is the length of the string, cf. Figure 2.1. Their motion is governed by wave equations whose solutions are superpositions of left- and right-moving waves. We are therefore able to split the string function into these left- and right-moving parts

$$X(t,\sigma) \;=\; X_L(t+\sigma) + X_R(t-\sigma)\ . \tag{2.2}$$

Now, because the left- and right-movers are almost completely independent from each other, we can split them into distinct entities with differing properties. To be precise, we define 26 bosonic left-moving strings

$$X_L^\mu(t+\sigma), \qquad \text{and} \qquad X_L^I(t+\sigma)\ , \tag{2.3}$$

with $\mu \in \{0,\dots,9\}$ and $I \in \{10,\dots,25\}$[1] as well as ten right-moving bosonic and fermionic fields

$$X_R^\mu(t-\sigma), \qquad \text{and} \qquad \Psi_R^\mu(t-\sigma)\ , \tag{2.4}$$

[1]Note that in the literature I will often be defined to go from 1 to 16, which would, if one was to be very pedantic, cause a double definition of the lower $X^{\mu/I}$.

where we already made use of the correct results for the critical dimension from Section 2.2.

2.2 Light-cone quantisation

Of the various possible quantisation schemes for the string, we chose quantisation in the so-called light-cone gauge, which may be a comparatively "dirty" approach in the sense that the theory then no longer exhibits manifest Lorentz invariance, which we will need to restore later on, but it is also a fast way that gauge-fixes the conformal symmetries and thus avoids the deployment of the full Virasoro algebra machinery needed in BRST quantisation, since the Virasoro constraints can now be solved explicitly. Let the longitudinal light-cone coordinates be

$$X^{\pm} \;=\; \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right) \;, \tag{2.5}$$

and let from now on denote μ the remaining transverse spacetime coordinates 2, ..., 9. This leaves only a SO(8) Lorentz symmetry of the original $\mathrm{SO}(9,1)$ one of the target space manifest.

Equation (2.1) implies the X to be periodic. Therefore they can be Fourier expanded into periodic modes. Performing this so-called oscillator expansion for the bosonic and fermionic fields in the light-cone gauge yields

$$X^I_L(t+\sigma) \;=\; \frac{P^I}{\sqrt{2\alpha' p^+}}(t+\sigma)+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^I_n}{n}\mathrm{e}^{-2\pi\mathrm{i}n(t+\sigma)/\ell} \;, \tag{2.6a}$$

$$X^\mu_L(t+\sigma) \;=\; \frac{x^\mu}{2}+\frac{p_\mu}{2p^+}(t+\sigma)+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\mathrm{e}^{-2\pi\mathrm{i}n(t+\sigma)/\ell} \;, \tag{2.6b}$$

$$X^\mu_R(t-\sigma) \;=\; \frac{x^\mu}{2}+\frac{p_\mu}{2p^+}(t-\sigma)+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}\mathrm{e}^{-2\pi\mathrm{i}n(t-\sigma)/\ell} \;, \tag{2.6c}$$

$$\Psi^\mu_R(t-\sigma) \;=\; \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}}\tilde{\psi}^\mu_{r+\nu}\mathrm{e}^{-2\pi\mathrm{i}(r+\nu)(t-\sigma)/\ell} \;. \tag{2.6d}$$

Here, α' is the only dimensional parameter of string theory, the famous Regge-slope, which defines the string tension $T=(2\pi\alpha')^{-1}$, x^μ and p_μ are operators which define the centre of mass position and momentum of the string, and

$$p^+ \;=\; -\frac{\partial L}{\partial(\partial_t x^-)} \;=\; \frac{\ell}{2\pi\alpha'} \tag{2.7}$$

is the x^+ momentum. Equation (2.7) actually fixes the string length $\ell = 2\pi\alpha' p^+$ to be proportional to p^+. Furthermore, $\nu \in \{0, 1/2\}$ defines the boundary condition for the closed fermionic string to be either Ramond (R) or Neveu-Schwarz (NS)[2]. Note that the momenta P^I in Equation (2.6a) are made dimensionless by the $\sqrt{2\alpha'}$ factor. They are quantised on a discrete 16-dimensional lattice which will be specified below. Finally, the α, $\tilde{\alpha}$ and ψ are Fock space operators which create and annihilate bosonic and fermionic string states respectively. They, as well as the momentum and position operators, fulfil the usual commutation and anti-commutation relations

$$[x^\mu,\ p_\nu] = \mathrm{i}\delta^\mu_\nu, \qquad [\alpha^\mu_m,\ \alpha^\nu_n] = m\delta_{\mu\nu}\delta_{m+n,0}\ , \tag{2.8}$$
$$\{\psi^\mu_m,\ \psi^\nu_n\} = \delta^{\mu\nu}\delta_{m+n,0}, \qquad [\tilde{\alpha}^\mu_m,\ \tilde{\alpha}^\nu_n] = m\delta_{\mu\nu}\delta_{m+n,0}\ , \tag{2.9}$$

where for the fermionic operators ψ^μ_m, the indices m and n can be in $\mathbb{Z}$ or $\mathbb{Z} + 1/2$ for the R and NS sectors respectively.

The oscillator expansions in Equation (2.6) lead to the following left- and right-moving Hamiltonians:

$$H_L = \frac{1}{4p^+}\sum_\mu p_\mu^2 + \frac{1}{2\alpha' p^+}\sum_I \left(P^I\right)^2 + \frac{1}{\alpha' p^+}\left(N_B - 1\right)\ , \tag{2.10a}$$
$$H_R = \frac{1}{4p^+}\sum_\mu p_\mu^2 + \frac{1}{\alpha' p^+}\left(\tilde{N}_B + \tilde{N}_F + \tilde{E}_0\right)\ . \tag{2.10b}$$

Here, $\tilde{E}_0 = 2\nu(1-\nu)$ is the fermionic zero-point energy, derived from a ζ-function regularisation of the normal-ordered sum of all oscillator ground states[3] at the critical dimension $D = 10$ (cf. below), whereas the various oscillator number operators are declared as

$$\tilde{N}_B = \sum_\mu \sum_{n\in\mathbb{N}_0} \tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\mu_n\ , \qquad \tilde{N}_F = \sum_{r\in\mathbb{N}_0}(r+\nu)\psi^\mu_{-r-\nu}\psi^\mu_{r+\nu}\ , \tag{2.11}$$

and

$$N_B = \sum_\mu \sum_{n\in\mathbb{N}_0} \alpha^\mu_{-n}\alpha^\mu_n + \sum_I \sum_{n\in\mathbb{N}_0} \alpha^I_{-n}\alpha^I_n\ . \tag{2.12}$$

[2] A thorough treatment of the partition function would show that both possibilities need to be included at equal parts to achieve modular invariance.

[3] This sum does, obviously, diverge. However, it can be regularised, which is equivalent to introducing a counter-term in the Lagrangian of the theory.

Mass equations

With the Hamiltonians and oscillator number operators in place, we find the masses of left- and right-moving string states to be

$$\frac{\alpha' M_R^2}{2} = \tilde{N}_B + \tilde{N}_F - 2\nu(1-\nu) \ , \tag{2.13}$$

$$\frac{\alpha' M_L^2}{2} = N_B + \frac{1}{2}\sum_I \left(P^I\right)^2 - 1 \ . \tag{2.14}$$

Full physical states are built by tensoring one left- and one right-moving state together. However, this can not be done arbitrarily and is subject to the following condition: since the physical spectrum should be invariant of choice of origin $\sigma = 0$ for the string coordinate, the σ-momentum operator P_σ has to vanish. But since that operator is proportional to the difference of left- and right-moving mass squares

$$P_\sigma \propto M_L^2 - M_R^2 \ , \tag{2.15}$$

the only allowed states are tensor products of left- and right-movers with equal masses

$$M_L^2 = M_R^2 \ . \tag{2.16}$$

This is called the level-matching condition.

Critical dimension

It is a remarkable feature of string theory that the total number D of spacetime dimensions is not a free parameter as in quantum field theory, but a prediction, obtained from very few physical assumptions. Probably the most sound way to obtain this so-called critical dimension is to demand cancellation of the so-called conformal anomaly, first shown in the famous no-ghost theorem by Goddard and Thorn [16]. However, we will, again following [14], give a more simple argument based on the light-cone quantisation. If our resulting theory is to be Lorentz invariant, polarisation states of particles ought to transform in the little group of $\mathrm{SO}(D-1,1)$. We do not get Lorentz invariance for free, since we lost it when gauge-fixing the light-cone, so we need to restore it here. For massless particles, this little group is $\mathrm{SO}(D-2)$, while for massive ones it is $\mathrm{SO}(D-1)$. Thus, when computing the spectrum of the string theory with arbitrary dimension D, the condition that states transforming in $\mathrm{SO}(D-2)$ have to be massless arises. In superstring theory, theses states have mass squares

$$\frac{\alpha' M^2}{2} = \frac{1}{2} - \frac{D-2}{16} \ , \tag{2.17}$$

following the regularisation of zero-point energies. This immediately leads to the familiar result $D = 10$. The corresponding result for the non-supersymmetric bosonic string reads

$$\frac{\alpha' M^2}{2} \;=\; 2 - \frac{D-2}{12}\;, \tag{2.18}$$

yielding $D = 26$. In heterotic string theory, this is the dimension of the left-mover.

The gauge lattice

When computing string amplitudes using a path integral, one has to be careful not to over-count equivalent diagrams. When considering closed strings, the one-loop vacuum amplitude, also known as partition function, has the topology of a two-torus – this can be depicted easily as a σ-closed string also closing on itself in the t-coordinate. To obtain a physically meaningful expression for this amplitude, the partition function thus needs to be invariant under transformations which map that torus to itself. In order to identify the correct class of transformations, we introduce the complex world sheet coordinate $z = \sigma + \mathrm{i}t$. Then the torus is defined by the identifications $z \equiv z + \ell$ and $z \equiv z + \tau\ell$ (cf. Section 3.2.2 for more details about tori), where τ, also called complex structure in this context, is a complex number solely defining the geometry of the torus. It can be shown that all possible transformations of this torus which leave the partition function invariant, i. e. which map to congruent tori are generated by compositions of the maps $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. These are the generators of the famous modular group $\mathrm{PSL}(2,\mathbb{Z})$. This group acts on τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\;, \tag{2.19}$$

with $a,\ b,\ c,\ d \in \mathbb{Z}$ and $ad - bc = 1$. As can be seen from this, the parameters can be arranged into a 2×2 integer matrix with unit determinant, where elements defined by $(a\ , b\ , c\ , d\ )$ and $(-a\ , -b\ , -c\ , -d)$ give rise to the same transformation, thus justifying the name $\mathrm{PSL}(2,\mathbb{Z})$. A possible fundamental domain of τ under the modular group is defined by

$$-\,1/2 \leq \Re\tau < 1/2\;, \quad \text{and} \quad |\tau| \geq 1\;, \tag{2.20}$$

and depicted in Figure 2.2. Although the precise form of the partition function is not of utmost importance here, the factor describing the only left-moving parts X_L^I is given by

$$Z_{X^I}(\tau) \;=\; \eta(\tau)^{-16} \sum_{P\in\Lambda_{16}} q^{P^2/2}\;, \tag{2.21}$$

where η is the Dedekind η-function, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and Λ_{16} is a 16-dimensional lattice. This lattice is, however, subject to two very strong constraints from invariance of Z

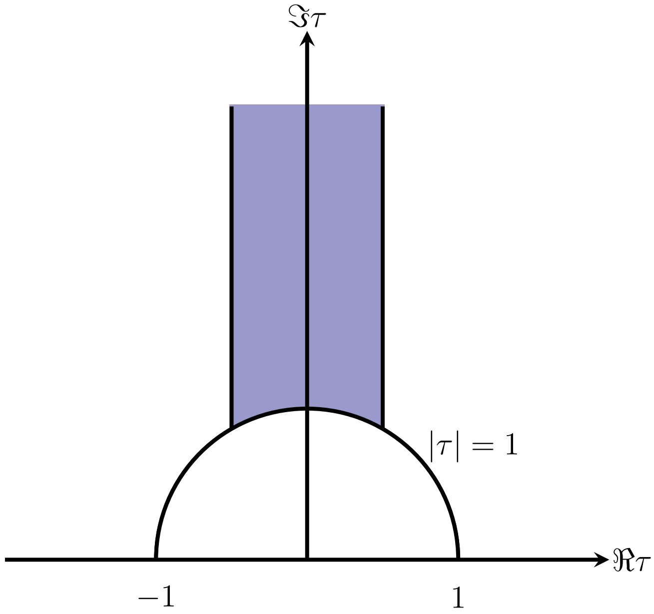


Figure 2.2: Fundamental domain of τ under the modular group.

under the modular group, one stemming from each generator of $\mathrm{PSL}(2,\mathbb{Z})$. First, invariance under $\tau \mapsto \tau + 1$ requires

$$\sum_{P\in\Lambda_{16}} \mathrm{e}^{2\pi\mathrm{i}(\tau+1)P^2/2} = \sum_{P\in\Lambda_{16}} \mathrm{e}^{2\pi\mathrm{i}\tau P^2/2} , \tag{2.22}$$

which enforces $P^2 \in 2\mathbb{Z}$. Such a lattice is called even. Secondly, invariance under $\tau \mapsto -1/\tau$ yields, after a Poisson resummation,

$$Z_{X^I}(-1/\tau) = (-\mathrm{i}\tau)^{-8}\eta(\tau)^{-16} \sum_{P\in\Lambda_{16}} \mathrm{e}^{2\pi\mathrm{i}(-1/\tau)P^2/2} \tag{2.23}$$

$$= \eta(\tau)^{-16}\frac{1}{|\Lambda_{16}^*/\Lambda_{16}|} \sum_{P'\in\Lambda_{16}^*} \mathrm{e}^{-2\pi\mathrm{i}\tau P'^2/2} . \tag{2.24}$$

Hence the lattice needs to be self-dual $\Lambda_{16}^* = \Lambda_{16}$.

It is a remarkable fact, that in sixteen Euclidean dimensions only two lattices which are both even and self-dual exist. Therefore, only two possibilities to compactify the gauge degrees of freedom present themselves, thus creating the two—instead of many if the lattices weren't so constrained—heterotic string theories in ten dimensions. These two lattices are $\Lambda_{\mathrm{SO}(32)}$, the root lattice of the Lie algebra Spin(32), divided by $\mathbb{Z}_2$[4], defined by

$$(n_1, \ldots, n_{16}) \qquad \text{and} \qquad \left(n_1 + \frac{1}{2}, \ldots, n_{16} + \frac{1}{2}\right) , \tag{2.25}$$

with $\forall i \in \{1, \ldots, 16\} \,:\, n_i \in \mathbb{Z}$ and $\sum_i n_i \in 2\mathbb{Z}$ and $\Lambda_{\mathrm{E}_8 \times \mathrm{E}_8}$, the direct sum of two E_8 root lattices, each given by

$$(n_1, \ldots, n_8) \qquad \text{and} \qquad \left(n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}\right) , \tag{2.26}$$

again with $\forall i \in \{1, \ldots, 8\} \,:\, n_i \in \mathbb{Z}$ and $\sum_i n_i \in 2\mathbb{Z}$.

Spectrum

To obtain a complete spectrum of ten-dimensional heterotic string theory, one forms the tensor products of left- and right-moving states honouring the level-matching condition (2.16) $M_L^2 = M_R^2$. Note that in order for the partition function to be modular invariant, the right-mover contributes states from the R as well as the NS sector. This leads to a projection condition, the so-called GSO projection, which removes all tachyons from the theory. The resulting massless spectrum can be tabulated as follows:

Sector	$\vert \cdot \rangle_L \otimes \vert \cdot \rangle_R$	SO(8) representation	target space field name
NS	$\mathbf{8}_V \otimes \mathbf{8}_V$	$\mathbf{1} + \mathbf{28}_V + \mathbf{35}_V$	$\phi,\ B_{MN},\ G_{MN}$
R	$\mathbf{8}_V \otimes \mathbf{8}_C$	$\mathbf{8}_S + \mathbf{56}_S$	$\lambda_\alpha,\ \psi_{M\alpha}$
NS	$\alpha^I_{-1}\mathbf{\Omega} \otimes \mathbf{8}_V$	$\mathbf{8}_V$	$A_M^{(I)}$
NS	$\vert P\rangle \otimes \mathbf{8}_V$	$\mathbf{8}_V$	$A_M^{(P)}$
R	$\alpha^I_{-1}\mathbf{\Omega} \otimes \mathbf{8}_C$	$\mathbf{8}_C$	$\lambda_{\dot\alpha}^{(I)}$
R	$\vert P\rangle \otimes \mathbf{8}_C$	$\mathbf{8}_C$	$\lambda_{\dot\alpha}^{(P)}$

Here, $\mathbf{\Omega}$ depicts the vacuum state of the string Fock space and SO(8) is the transverse Lorentz group of spacetime. The named fields are, in more detail, the scalar dilaton ϕ, the anti-symmetric tensor field B_{MN}, the graviton G_{MN}, the gravitino $\psi_{M\alpha}$,

[4]This is not the root lattice of SO(32) even if the algebras are isomorphic. The connotation is however oftentimes made in the literature in a slight abuse of notation and this thesis will be no exception.

the dilatino λ_α as well as gauge bosons and gauginos A_M and $\lambda_{\dot\alpha}$. Of these, the bosons $A_M^{(P)}$ carry quantised momentum P_I in the internal coordinates of the 16-dimensional lattice, which correspond to roots of the Lie algebra SO(32) or $E_8 \times E_8$, therefore forming representations of the respective non-Abelian algebra. Hence, the two possible lattices for the extra bosonic dimensions define the resulting gauge symmetries of the spacetime theory.

2.3 Compactification

Despite its marvellous beauty and the fact that in can naturally produce theories with non-Abelian gauge symmetries, the heterotic string theory described so far still lacks some features before it can be used for (semi-)realistic model-building. First, of course, the number of effective spacetime dimensions needs to be reduced to four by some compactification procedure. And second, the gauge group, be it SO(32) or $E_8 \times E_8$ is too big to make contact with reality, so it needs to be broken down – whether directly to the standard model or some intermediary GUT group is a matter of preference. As it turns out, solving the first problem in a way which preserves chirality and $\mathcal{N} = 1$ supersymmetry in four dimensions automatically alleviates the second. However, no compactification alone is able to reduce the rank of the gauge group; therefore, in general, one is confronted with an unpleasantly large set of extra U(1) factors. This problem can only be mitigated by the introduction of non-trivial background fields in the gauge space, so-called Wilson lines, which are beyond the scope of this thesis.

To compactify, we view spacetime as a direct product

$$\mathbb{M}^{10} \;=\; \mathbb{M}^{1,3} \times \mathcal{C}^6 \qquad (2.27)$$

of classical Minkowski space $\mathbb{M}^{1,3}$ and a topologically compact six-dimensional space $\mathcal{C}^6$. Let us for the moment assume this compactification takes place on a six-dimensional flat torus $\mathcal{C}^6 = \mathbb{T}^6$. Then this compactification scheme, in its most general form, breaks the transverse Lorentz group SO(8) to

$$\mathrm{SO}(8) \;\longrightarrow\; \mathrm{SO}(2) \times \mathrm{SO}(6)\ , \qquad (2.28)$$

which is locally isomorphic to U(1) × SU(4) [17]. Here, the U(1) factor can be interpreted as four-dimensional helicity, whereas the SU(4) factor is best understood as an R-symmetry. We will not derive the computation of the number of preserved supersymmetry generators in full glory, but rather give a short plausibility argument. The gravitino $\psi_{M\alpha}$ as $\mathbf{56}_S$ of SO(8) gets broken into SU(4) representations with U(1) charges as follows:

$$\mathbf{56}_S \longrightarrow \mathbf{4}_{3/2} + \overline{\mathbf{4}}_{1/2} + \mathbf{4}_{-1/2} + \overline{\mathbf{4}}_{-3/2} + \overline{\mathbf{20}}_{1/2} + \mathbf{20}_{-1/2}\ . \qquad (2.29)$$

Here we recognise the representations $\mathbf{4}_{3/2} + \overline{\mathbf{4}}_{-3/2}$ as the two helicity states of four fermions with spin $3/2$ and identify them as the spacetime gravitinos. Hence we count $\mathcal{N} = 4$ unbroken supersymmetries in four dimensional spacetime. But since any theory with such a high amount of supersymmetry is necessarily non-chiral, some of these gravitinos will need to get projected out of the spectrum in order for model building to arrive at a realistic spectrum. This is usually done by using a non-flat compactification space, namely one whose holonomy group is in SU(3). This in turn lets only one supersymmetric spinor survive the projection and thus yields $\mathcal{N} = 1$ SUSY in four dimensions. Manifolds with this property are called Calabi-Yau. However, these objects are notoriously difficult to work with and their (pseudo-)metrics are, in general, not available in an analytic form. A second approach is to give up on the condition that $\mathcal{C}^6$ needs to be smooth everywhere and allow for non-differentiable points. These spaces, called orbifolds, will be the main focus of this thesis.

2.3.1 Orbifolds

In this section we very briefly describe the derivation of a string theory compactified on an orbifold. Since we have not yet defined the notion of an orbifold itself, we will not go into too much detail here. Some details about twisted and untwisted sectors will be given during the computation of the orbifold cohomologies in Section 3.10.2, while the exact nature of orbifolds will be studied in Section 3.2.

Let, for now, an orbifold be the quotient of a torus $\mathbb{T}^6$ by a finite group of rotations Q, which will be later called a point group. This is a simplification of the complete picture which will be laid out further in this thesis that helps with the understanding of the argument but which is not necessary for the computations to be carried out successfully, cf. [18]. In addition to the action of Q on the six compact dimensions of spacetime, we may also have it act on the gauge torus $\mathbb{T}^{16}$ by some embedding G,

$$G: \quad P \hookrightarrow \mathrm{Aut}(\mathbb{T}^{16}) \ , \tag{2.30}$$

which we will give the obvious name gauge embedding. The whole setting then has the structure

$$\frac{\mathbb{T}^6}{Q} \oplus \frac{\mathbb{T}^{16}}{G(Q)} \ . \tag{2.31}$$

Here, the quotient of a torus by a discrete group yields an object which in general is not smooth everywhere, but contains non-differentiable points. This happens when some points on the torus are mapped to themselves by the action of Q and thus end up being singular in the quotient. These are called fixed points and play an important role in the construction of an anomaly-free string theory.

We will in the following, for simplicity, assume that the image of G is Abelian and therefore consists of translations $X^I \mapsto X^I + V^I$, where the V^I will be referred to as shift vectors. Therefore, in the spectrum, the gauge bosons carrying I-momentum P will acquire a phase

$$\exp\left(2\pi \mathrm{i} P \cdot V\right) \tag{2.32}$$

which is required to be 1 in order for the corresponding states to be invariant of the orbifold projection – which is a requirement for the consistency of the theory. This projects out some P and therefore bosonic states from the spectrum, thus reducing the gauge group.

Twisted and untwisted sectors

Since every string in a heterotic theory needs to close not only in classical spacetime but also in the compact extra-dimensions, additional boundary conditions arise. In our chosen compactification scheme, where we start from a six-torus and then divide out a rotation group, we can distinguish three different kinds of closing strings, cf. Figure 2.3.

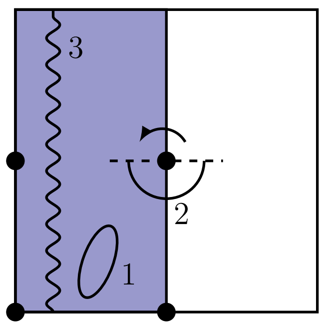


Figure 2.3: A string can close in three different ways on an orbifold. Depicted is the fundamental domain of a two-torus $\mathbb{T}^2$ as well as the fixed points of a $\mathbb{Z}_2$-rotation around the origin. The shaded area indicates the fundamental domain of the orbifold. String 1 closes already in flat space and is therefore called untwisted, while string 2 requires the $\mathbb{Z}_2$-twist to close and therefore falls in the twisted sector. Note that it is centred at its constructing fixed point and cannot move from it. Finally, string number 3 closes only after a torus lattice transformation and is therefore a winding mode string.

Untwisted strings are strings which already close in flat space. Hence, they can be studied as strings compactified on a six-torus and their spectrum is just a truncation of closed toroidal strings compatible with the orbifold projection.

Twisted strings do not close on $\mathbb{T}^6$, their endpoints only meet when applying the action of the symmetry group Q; to be precise, their boundary condition reads

$$X(t, \sigma + \ell) \;=\; \vartheta X(t, \sigma) + \lambda \,, \tag{2.33}$$

where $\vartheta \in Q$ and λ is an element of the torus lattice Λ^6. Such a string is necessarily located at a fixed point of the orbifold, defined by the group action and lattice translation (ϑ, λ) under which it is fixed; or to be more precise, the conjugacy class of the space group element (ϑ, λ), cf. also Section 3.2.4 for a more thorough explanation. The derivation of orbifold-invariant states of twisted strings is a bit more intricate, since the modular invariance conditions are more complex. In layman's terms, the invariance under modular transformations interacts with the boundary condition Equation (2.33) in non-trivial ways, cf. again Section 3.10.2. This may reduce the number of surviving states significantly, depending on the studied geometry. Also note that by virtue of its construction, the twisted string state is localised in spacetime at the corresponding fixed point and, in low excitation modes, can not move from it.

Winding strings are strings which do not close in flat space, but on the torus $\mathbb{T}^6$. They have, for generic choices of moduli, spatial extent at the order of the size of the compact space and therefore, by means of their inherent tension, can never be massless. Their contribution to the massless string spectrum therefore vanishes.[5]

As a final remark, we note that the twisted sector states are indeed necessary for the consistency of the theory, since, in general, the untwisted sector alone will be anomalous [14]. Furthermore, since modular invariance of the theory requires the presence of twisted sector states, this is an example of the interplay between modular invariance and anomaly cancellation.

2.4 $\mathbb{Z}_3$ orbifold

As a standard example (cf. for instance [14]), in this section we will demonstrate the construction of an explicit string model where the point group is $\mathbb{Z}_3$. Another, more complicated example will be studied in more detail in Chapter 4. In this most basic example, we choose the torus $\mathbb{T}^6$ to be the orthogonal product of three two-tori who are defined by a hexagonal lattice spanned by $e_1 = 1$ and $e_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$,

[5] Note that all massive particles of the low-energy theory need to come from massless string states, since string scale masses would be orders of magnitude bigger than even the mass of the top quark.

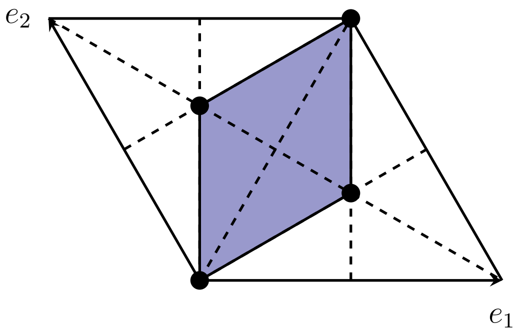


Figure 2.4: The $\mathbb{Z}_3$ orbifold is a direct product of three such spaces. Depicted are the basis vectors of the hexagonal lattice, together with the fundamental domain of the resulting torus. The dots mark fixed points of the orbifold action and the shaded area indicates the fundamental domain of the resulting orbifold.

where we identified $\mathbb{R}^2$ with $\mathbb{C}$. The point group is generated by ϑ which acts as a multiplication by $\alpha = \mathrm{e}^{2\pi \mathrm{i}/3}$ on every such plane. The gauge embedding is done by

$$V = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0, \ldots, 0 \right) . \tag{2.34}$$

This is called the standard embedding, where the action of the point group on the three complex planes that define compact spacetime mimics that on the gauge lattice, which we choose to be $\mathrm{E}_8 \times \mathrm{E}_8$. In the untwisted sector, the condition Equation (2.32) projects out all roots of the gauge group which do not fulfil $P \cdot V \in \mathbb{Z}$. The surviving non-zero roots of the first E_8 are therefore

$$(0, 0, 0, \underline{\pm 1, \pm 1, 0, 0, 0}) , \tag{2.35}$$

$$\pm \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2} \right) \quad \text{and} \tag{2.36}$$

$$(\underline{1, -1, 0}, 0, \ldots, 0) . \tag{2.37}$$

Here, the underline denotes permutation of entries and an even number of minus signs is demanded in the second line. These are the roots of E_6 and SU(3). The second E_8 group on the other hand does not get broken at all. Thus the complete gauge group of our simple example is given by

$$\mathrm{SU}(3) \times \mathrm{E}_6 \times \mathrm{E}_8 . \tag{2.38}$$

The spectrum can be computed in a similar fashion: by combining right- and left-moving states whose ϑ-eigenvalues multiply to 1 (for instance α^2-right-movers and α-left-movers), chiral states transforming as $(\mathbf{3}, \mathbf{27})$ of the first two gauge group factors can be constructed. The twisted spectrum on the other hand contains a 27-fold degeneracy from the 27 fixed points which, in the absence of Wilson lines, are equivalent. To find the states corresponding to the ϑ-twisted geometry, the sufficiently shifted versions of Equation (2.14),

$$\frac{\alpha' M_R^2}{2} = \frac{(r+v)^2}{2} - \frac{1}{6} = 0\,, \tag{2.39}$$

$$\frac{\alpha' M_L^2}{2} = \frac{(P+V)^2}{2} - \frac{2}{3} = 0\,, \tag{2.40}$$

where we already inserted the zero-point energy $E_0 = {}^1\!/_3$ and vanishing oscillator numbers for the ground states need to be solved. These lead, in a similar fashion to the computation above, to an additional $(\mathbf{1},\mathbf{27})$-plet per fixed point. Lastly, for a non-vanishing oscillator $N_B = {}^1\!/_3$, three additional $(\overline{\mathbf{3}},\mathbf{1})$-plets per fixed point are found. Therefore, in its entirety, the massless spectrum of the $\mathbb{Z}_3$ example orbifold is

$$3(\mathbf{3},\mathbf{27}) + 27(\mathbf{1},\mathbf{27}) + 81(\overline{\mathbf{3}},\mathbf{1}) \tag{2.41}$$

of $\mathrm{SU}(3) \times \mathrm{E}_6$.

As a concluding remark to this chapter, note that this spectrum and many more can be computed automatically using the C++ orbifolder [19] program, which is available on the Internet.

Chapter 3

Classification of Orbifolds

This chapter is dedicated to the classification of all toroidal and symmetric six-dimensional orbifolds which give rise to $\mathcal{N} \geq 1$ supersymmetry in four dimensions. We begin with a formal definition of orbifold spaces and restricting ourselves to special cases. Then we will utilise known results from crystallography to define the equivalence-relations we need for our classification. Examples will be given where they are in order. The last sections will deal with the computation of the surviving supersymmetry generators and the calculation of the orbifold Hodge numbers as well as their fundamental groups. All results obtained from the methods outlined in this chapter are tabulated in Appendix B. Parts of this chapter, along with results, have been published [8, 9, 20].

3.1 Definition

First we give the most general definition of an orbifold, which is due to Thurston [21], then we will state that the quotient of a manifold by a symmetry group is indeed an orbifold and henceforth abandon the general definition which is a tad too general for our purposes.

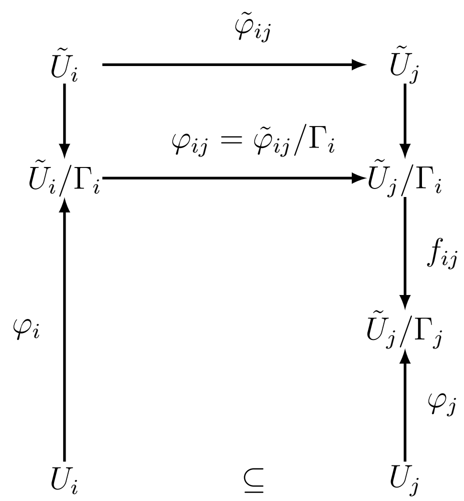
Definition 3.1 *An* orbifold $\mathcal{O}$ *is a topological Hausdorff space* X_0 *with the following structure data:* $\left\{U_i, \Gamma_i, \tilde{U}_i, \varphi_i\right\}_{i\in I}$, *such that:*

1. $\{U_i\}_{i\in I}$ *is an open covering of* X_O *which is closed under finite intersections,*

2. $\forall i \in I$, Γ_i *is a discrete group with an action on an open subset* $\tilde{U}_i \subseteq \mathbb{R}^n$,

3. *$\forall i \in I$, $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$ is a homeomorphism; $\tilde{U}_i/\Gamma_i$ means the set of equivalence classes one gets from identifying each point in U_i with its orbit under the action of Γ_i,*

4. *$\forall i, j \in I$ with $U_i \subseteq U_j$ there is an injective homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ such that the following diagram commutes.*



The open covering $\{U_i\}_{i \in I}$ is not intrinsic to the orbifold structure: Two coverings define the same orbifold structure when their union still satisfies the conditions given in the definition, analogous to the complex structures defining manifolds.

This definition of an orbifold was first given by Thurston [21]. Note that we loosened the original definition by allowing the Γ_i to be infinite discrete groups instead of finite.

Example 3.2 *A closed manifold is an orbifold, where each group Γ_i is the trivial group **0**, so that $\forall U : \quad \tilde{U} = U$.*

Theorem 3.3 *Let M be a manifold and Γ a properly discontinuously acting group on M. Then M/Γ has the structure of an orbifold.*

Proof: [21] Let π be the canonical projection from M to M/Γ. Let $\tilde{x} \in M/\Gamma = \pi(M)$. Pick any $x \in M$ with $\pi(x) = \tilde{x}$. Let then I_x be the isotropy group of x.

There exists a neighbourhood U_x of x which is invariant by I_x and disjoint from its translates by elements of Γ not in I_x. Then the projection $\varphi : U_x \longrightarrow U_x/I_x$ is a homeomorphism. For some open cover $\{U_x\}$, let V_x be the projected set U_x/I_x. Now take $\{V_x\}$ and adjoin all finite intersections. It remains to show that for each of these intersections a corresponding homeomorphism can be found. For any finite intersection $V_{x_1} \cap \ldots \cap V_{x_k}$ which is not empty, there exists a translated set

$$\mathcal{U} \;=\; \gamma_1 U_{x_1} \cap \ldots \cap \gamma_k U_{x_k} \neq \emptyset \tag{3.1}$$

with $\gamma_i \in \Gamma$ for all $i \in \{1, \ldots, k\}$. Now $\mathcal{U}$ may be taken to be $\pi^{-1}\left(V_{x_1} \cap \ldots \cap V_{x_k}\right)$ with the associated group $\gamma_1 I_{x_1} \gamma_1^{-1} \cap \ldots \cap \gamma_k I_{x_k} \gamma_k^{-1}$ acting on it. □

In the following, all orbifolds will have that structure, with M being $\mathbb{R}^6$ and Γ being a so-called space group S. Every manifold is an orbifold but the converse is not necessarily true. However, all orbifolds we will be concerned with are good, by virtue of their construction, i. e. they have coverings which are manifolds. These are often called blow-ups or resolutions in the literature [22].

3.2 Toroidal orbifolds

In this section, we will discuss the construction of toroidal orbifolds [23, 22] and review their building blocks, most prominently space groups, lattices and point groups.

A toroidal orbifold can be constructed by either (i) taking Euclidean space $\mathbb{R}^n$ and dividing out a discrete group S, the so-called space group or (ii) first dividing $\mathbb{R}^n$ by an n-dimensional lattice Λ, which will be defined in Section 3.2.2, which yields a torus $\mathbb{T}^n$ and subsequently dividing this torus by a discrete symmetry group G acting on it. Note that G, the so-called orbifolding group as defined in Section 3.2.4, is in general not equal to the point group introduced in Section 3.2.3. Hence, we can write the defining equation(s) of a toroidal orbifold as

$$\mathcal{O} \;=\; \mathbb{R}^n/S \;=\; \mathbb{T}^n/G\,. \tag{3.2}$$

In heterotic string theory, we will be mostly concerned with orbifolds of dimension $n = 6$, but since the definitions we will be using carry over smoothly to higher dimensions, we will keep n arbitrary for the time being. The following definitions are textbook knowledge in crystallography and follow [24] very closely.

3.2.1 The space group

A space group S is a discrete subgroup of the Euclidean group in $\mathbb{R}^n$ which contains n linearly independent translations. That is, S consists of rigid motions $g: \mathbb{R}^n \to \mathbb{R}^n$, i. e. rotations, inversions, translations and combinations thereof and is not "degenerate" in the sense that it contains translations in all directions of space. For $n = 3$, groups of this type have been studied in great detail by crystallographers since the 19th century, because they describe the symmetries of crystal structures.

Now let S be a space group. Any element $g \in S$ can be decomposed into a mapping ϑ that leaves at least one point invariant and a pure translation by a vector λ, i. e. $g = \lambda \circ \vartheta$. Therefore we can write any space group element as

$$g = (\vartheta, \lambda) , \tag{3.3}$$

and its action on an element $v \in \mathbb{R}^n$ as

$$v \overset{g}{\longmapsto} \vartheta v + \lambda . \tag{3.4}$$

It is convention to group actual representations of ϑ and λ together in an augmented matrix

$$g_{\text{aug}} = \left(\begin{array}{c|c} \vartheta & \lambda \\ \hline \mathbf{0} & 1 \end{array}\right) , \tag{3.5}$$

where ϑ is a $n \times n$ matrix, $\mathbf{0}$ a row vector of n zeros and λ a n-dimensional column vector. g_{aug} then acts on an augmented vector $v_{\text{aug}} = \left(\begin{array}{c} v \\ \hline 1 \end{array}\right)$ through simple matrix vector multiplication:

$$\left(\begin{array}{c|c} \vartheta & \lambda \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c} v \\ \hline 1 \end{array}\right) = \left(\begin{array}{c} \vartheta v + \lambda \\ \hline 1 \end{array}\right) . \tag{3.6}$$

Let $h = (\omega, \tau)$ be another element of the same space group. The group product of g and h is given by

$$h \circ g = (\omega\vartheta, \omega\lambda + \tau) , \tag{3.7}$$

which reads in augmented matrix notation

$$\left(\begin{array}{c|c} \omega & \tau \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \vartheta & \lambda \\ \hline \mathbf{0} & 1 \end{array}\right) = \left(\begin{array}{c|c} \omega\vartheta & \omega\lambda + \tau \\ \hline \mathbf{0} & 1 \end{array}\right) . \tag{3.8}$$

The inverse of g reads

$$g^{-1} = (\vartheta^{-1}, -\vartheta^{-1}\lambda) , \tag{3.9}$$

as can be easily verified by insertion into Equation (3.7). Here a remark about notation is in order: the order of the translational and rotational parts of g is swapped in mathematical literature, since the lattice (cf. the next section) is a normal subgroup of the space group and in mathematics, the normal subgroup element is canonically written to the left.

3.2.2 The lattice

Let S be a space group. The subset $\Lambda = \{(\mathrm{id}, \lambda)\} \subseteq S$ of all pure translations is called the lattice of the space group. We naturally identify it with its set of translation vectors

$$\Lambda \;=\; \{(\mathrm{id}, \lambda) \in S\} := \{\lambda \;:\; (\mathrm{id}, \lambda) \in S\} \;, \tag{3.10}$$

and will move freely between these definitions. Note that in general, for an element $g = (\vartheta, \lambda) \in S$ the vector λ needs not to be an element of the lattice. Elements of this form with $\lambda \notin \Lambda$ are called roto-translations, while elements $(\vartheta, n_i e_i)$ (summed over $i \in \{1, \ldots, n\}$) with e_i as below and $n_i \in \mathbb{Z}$ are called screwings.

Since every space group contains by definition n linearly independent translations, its lattice always contains a basis $\mathfrak{e} = \{e_i\}_{i\in\{1,\ldots,n\}} \subsetneq \Lambda$ of $\mathbb{R}^n$ whose $\mathbb{Z}$-span yields the whole lattice: $\mathrm{span}_{\mathbb{Z}}(\mathfrak{e}) = \{n_i e_i \;:\; n_i \in \mathbb{Z}\} = \Lambda$. Obviously, there is an infinite number of possible choices of basis. For instance, take two bases $\mathfrak{e} = \{e_1, \ldots, e_n\}$ and $\mathfrak{f} = \{f_1, \ldots, f_n\}$ of the same lattice Λ and let $B_{\mathfrak{e}}$ and $B_{\mathfrak{f}}$ be matrices whose columns are the respective basis vectors. Then the change of basis is described by the unimodular matrix $M = B_{\mathfrak{e}}^{-1} B_{\mathfrak{f}}$:

$$B_{\mathfrak{e}} M \;=\; B_{\mathfrak{f}} \;. \tag{3.11}$$

Hence, every vector of $\mathfrak{f}$ is an element of the integer span of $\mathfrak{e}$ and vice versa. Conversely, two bases $\mathfrak{e}$ and $\mathfrak{f}$ span the same lattice if and only if M (as defined above) is unimodular, i. e. if it is an element of $\mathrm{GL}(n, \mathbb{Z})$, the group of all $n \times n$ integer matrices with determinant ± 1.[1]

Every lattice Λ defines an equivalence relation on vectors from $\mathbb{R}^n$ when elements which differ by an element of Λ get identified:

$$v \approx w \;:\Leftrightarrow\; v - w \in \Lambda \;. \tag{3.12}$$

The fundamental domain of this relation is the unit cell of the lattice, with opposing hypersurfaces identified, i. e. a torus $\mathbb{T} := \mathbb{R}^n/\Lambda$, see Figure 3.1.

3.2.3 The point group

Let S be a space group and denote its elements of the form (ϑ, λ). Then the set P of all such ϑ is finite and closes under multiplication. Furthermore, since all ϑ

[1] This is equivalent to saying M is invertible over the integers.

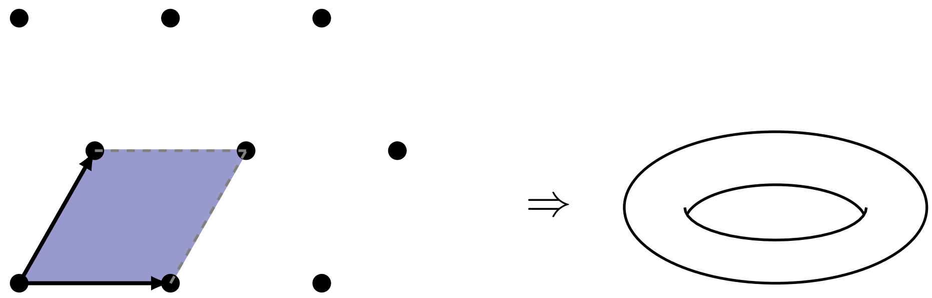


Figure 3.1: When $\mathbb{R}^n$ is compactified by identifying points which differ by a vector from the lattice on the left, the resulting space has the topology of a torus. The fundamental domain is the primitive unit cell of the lattice (shaded area).

are invertible (cf. below) and because the identity id is always an element of P[2], P is a group (cf. [24, p. 15]), which is called the point group of S. We will call its elements twists or rotations. Note that this is a slight abuse of notation, because P contains in general elements of $\mathrm{O}(n)$ and therefore may also include inversions and reflections.

The point group P of S maps the lattice Λ to itself. Consequently, when changing from Euclidean to (an arbitrary) lattice basis, point group elements can be represented by unimodular matrices $\vartheta \in \mathrm{GL}(n, \mathbb{Z})$. This description will later turn out to be advantageous for purposes of classification and computation, since it separates the internal algebraic structure of the point group from the underlying vector space $\mathbb{R}^n$. When an Euclidean representation is necessary, one can simply choose an explicit lattice basis $\mathfrak{e}$ and transform ϑ according to $B_\mathfrak{e} \vartheta B_\mathfrak{e}^{-1}$. When going from one lattice basis $\mathfrak{e}$ to another $\mathfrak{f}$ as in Equation (3.11), the twist transforms according to

$$\vartheta_\mathfrak{f} \;=\; M^{-1}\, \vartheta_\mathfrak{e}\, M\,. \tag{3.13}$$

Now, the versor product $(\vartheta, \mathbf{0}) \circ (\mathrm{id}, \lambda) \circ (\vartheta, \mathbf{0})^{-1}$ of a pure rotation with a pure translation always yields a pure translation. Hence the lattice is always a normal subgroup of the space group and therefore the space group S has the structure of a semi-direct product iff the point group P is a subgroup of it.[3] If this is indeed the case, S can be written as

$$S \;=\; P \ltimes \Lambda\,, \tag{3.14}$$

and the orbifold as

$$\mathcal{O} \;=\; \mathbb{R}^n/(P \ltimes \Lambda) \;=\; \mathbb{T}^n/P\,. \tag{3.15}$$

[2]This is because every space group contains a lattice.
[3]This statement also slightly abuses notation and means that $\forall \vartheta \in P \;:\; (\vartheta, \mathbf{0}) \in S$.

In general, however, this is not the case, due to the possible presence of roto-translations in the space group – elements of the form (ϑ, λ), where λ is not an element of the lattice. In this case, ϑ will always be accompanied by a non-trivial translation. This means that, in general, the point group P does not equal the orbifolding group G of Section 3.2.4.

In cases where the point group is Abelian, i. e. a direct sum of one or more $\mathbb{Z}_N$ factors, all of its elements can be brought into block-diagonal shape simultaneously, that is, an element $\vartheta \in P$ will, after an appropriate choice of basis is made, have the structure

$$\vartheta \;=\; \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix} . \tag{3.16}$$

Here, k is $n/2$ and the α_i are two-dimensional rotation matrices. For odd n an additional ± 1 entry on the diagonal will exist – however, since we are primarily concerned with $n = 6$, this is of little relevance to us. In this case, it is sometimes advantageous (cf. for instance [17]) to complexify the base space $\mathbb{R}^6 \cong \mathbb{C}^3$. Then, all elements ϑ of an Abelian point group P can be simultaneously written as diagonal matrices

$$\vartheta \;=\; \mathrm{diag}\left(\mathrm{e}^{2\pi \mathrm{i} v_1},\; \mathrm{e}^{2\pi \mathrm{i} v_2},\; \mathrm{e}^{2\pi \mathrm{i} v_3}\right) \;, \tag{3.17}$$

where the v_i are real numbers, which are sometimes grouped together to form the so-called twist vector

$$v \;=\; (v_1,\; v_2,\; v_3) \;. \tag{3.18}$$

3.2.4 The orbifolding group

Since, in general, a space group cannot be described as the semi-direct product of a lattice with a point group, due to the possibility of roto-translations, one can ask, how to properly augment a lattice to the full space group. The correct object will incorporate the necessary roto-translations, but can—because of this very fact—only close on the torus $\mathbb{R}^n/\Lambda$. It contains all elements of S with a translational part within the fundamental domain of $\mathbb{T}$, i. e. elements of the form $(\vartheta, n_i e_i)$ with $\forall i \in \{1, \ldots, n\} \;:\; 0 \leq n_i < 1$. This group is called the orbifolding group G.[4] In general, it is not equal to the point group. Now, the space group is generated by G and the lattice $S = \langle\{G, \Lambda\}\rangle$. Thus, the orbifold is defined by any of the following:

$$\mathcal{O} \;=\; \mathbb{R}^n/S \;=\; \mathbb{R}^n/\langle\{G, \Lambda\}\rangle \;=\; (\mathbb{R}^n/\Lambda)/G \;=\; \mathbb{T}^n/G \,. \tag{3.19}$$

[4] This group is not to be confused with the "orbifold group" which is a term, sometimes used to describe the full action of the space group on the six spacetime and sixteen gauge degrees of freedom.

Orbifolds can be manifolds (see e. g. Figure 3.2(b)), but in general, they are equipped with singularities of curvature, arising from fixed points of the space group which cannot be governed by a differential map, and hence violating the defining property of a manifold, (cf. e. g. Figure 3.2(a)).

3.2.5 Examples in two dimensions

This subsection provides two examples of space groups and resulting orbifolds with a point group isomorphic to $\mathbb{Z}_2$ in two dimensions.

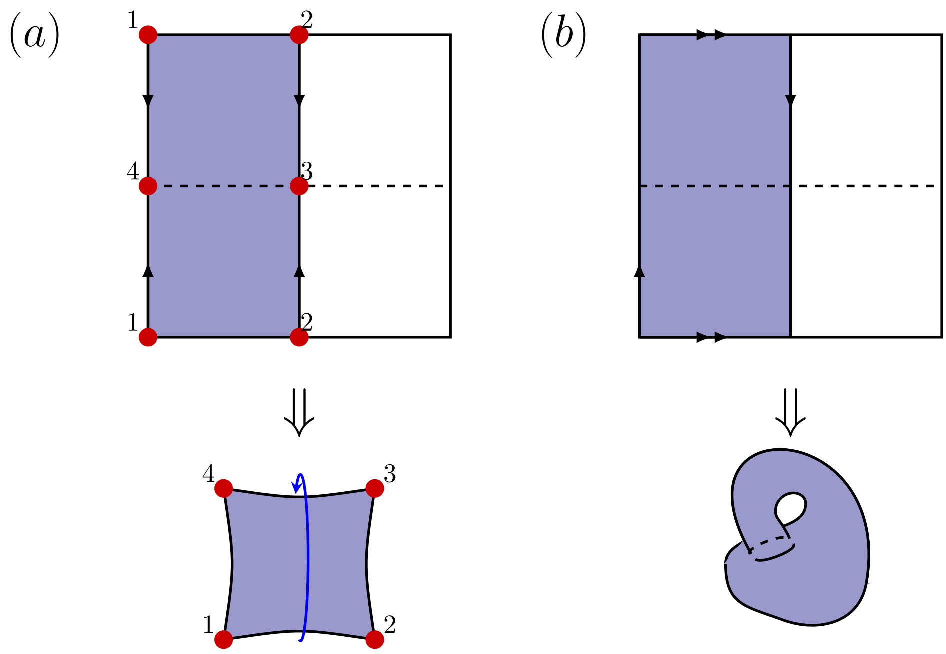


Figure 3.2: Two-dimensional examples from the fundamental domain of the torus: (a) “pillow” and (b) Klein bottle. The red dots mark fixed points and the shaded areas indicate the fundamental domains of the respective orbifolds.

The “pillow”

The space group S of our first example is generated by

$$\{(\mathrm{id}, e_1), (\mathrm{id}, e_2), (\vartheta, \mathbf{0})\} \ , \tag{3.20}$$

i. e. a lattice with basis $\mathfrak{e} = \{e_1, e_2\}$ and one point group element ϑ which acts as point inversion around the origin:

$$\vartheta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.21}$$

Note that ϑ is compatible with any two-dimensional lattice and since $\vartheta^2 = \mathrm{id}$, the point group is isomorphic to $\mathbb{Z}_2$.

The action of S on the torus, as seen in Figure 3.2(a), can be best pictured by dividing the torus into quadrants. Each quadrant then gets identified with the one diagonally adjacent to it, with the point labelled with 3 being the centre of mirroring. That means, the fundamental domain of the orbifold gets shrunk by half; this is expected, since the point group is of order two. Four points however map to themselves: they are fixed. One can now mentally overlap two of the quadrants in such a way that the identified edges meet. The resulting space, dubbed a “pillow”, has the topology of a 2-sphere with four fixed points. Because the orbifold metric is inherited from the torus on all other points, all curvature is localised at these points.

The Klein bottle

For a second example, we present the space group of a Klein bottle, cf. Figure 3.2(b). In this case, the lattice needs to be primitive rectangular, which means that the shortest basis consists of two orthogonal vectors e_1 and e_2. The full space group is generated by the lattice and a roto-translation g,

$$S = \langle (\mathrm{id}, e_1), (\mathrm{id}, e_2), g \rangle \qquad \text{with} \qquad g = \left(\vartheta, \tfrac{1}{2}e_1\right) , \tag{3.22}$$

where

$$(\vartheta, \tfrac{1}{2}e_1) = \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) , \tag{3.23}$$

making use of the augmented matrix notation introduced in Equation (3.5).

As in the previous example, the point group is isomorphic to $\mathbb{Z}_2$, since $\vartheta^2 = \mathrm{id}$, but the set $\{g, g^2\}$ only closes on the torus, since $g^2 = (\mathrm{id}, e_1)$. This space group also identifies diagonally adjacent quadrants on the fundamental domain of the torus, but its effect on the edges differs slightly from the previous example. Since g contains a translation, it leaves no points fixed, yielding a smooth (in this case non-orientable) manifold.

3.3 Equivalences of space groups

String theory imposes, a priori, no restrictions to the space group which is used in compactification: any choice leads to a consistent model – most of which will be unrealistic at first glance by, for instance, conserving too many or too few supersymmetry generators. However, a specific route that takes us to precisely one space group which then describes our universe, hat not yet been found. Thus a classification scheme for space groups, or to be more precise, their equivalence classes is required. This is a very old problem which has been studied in much detail in low dimensions in the past. The 17 up to isomorphism unique space groups in two dimensions, the so-called wallpaper groups, have been known for a very long time. A formal proof has been given in 1891 [25] by Fedorov. The three-dimensional case was studied in subsequent years, with Fedorov and Schönflies giving the correct 219 classes of space groups in 1895 (cf. [26]). The reason why this work was carried out so early in modern history is the significance two- and three-dimensional space groups have in crystallography. The four-dimensional case was later successfully carried out by Brown et. al in 1978 [24]. The for our purposes most relevant case of $n = 6$ dimensions was given by Plesken et. al in 2000 [27]. For higher dimensions, only parts of the full classification problem have been solved – a daunting task that gets quickly out of hand due to the rapidly growing numbers of equivalence classes of space groups, as we will see in this section.

Definitions

Every space group S fits in the following exact sequence [27]:

$$\mathbf{0} \longrightarrow \mathbb{Z}^n \longrightarrow S \longrightarrow P \longrightarrow \mathbf{1} \ , \tag{3.24}$$

where P is the point group and $\mathbb{Z}^n$ parametrises the lattice of the space group. Now since all point groups are, in lattice basis, finite matrix groups $P \leq \mathrm{GL}(n, \mathbb{Z})$, the task of classifying them up to isomorphism is basically to find all finite groups with a representation in $\mathrm{GL}(n, \mathbb{Z})$ [28]. We will make heavy use of the following definitions:

Definition 3.4 *Let S and S' be two space groups of the same degree n. Let P and P' be their point groups. They belong to the same ...*

1. *affine class, iff they are isomorphic, i. e. if there is an affine mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f^{-1}Sf = S'$.*

2. *$\mathbb{Z}$-class, iff P and P' are conjugate in* $\mathrm{GL}(n,\mathbb{Z})$*, i. e. if there is a matrix* $V \in \mathrm{GL}(n,\mathbb{Z})$ *such that* $V^{-1}PV = P'$.

3. *$\mathbb{Q}$-class, iff P and P' are conjugate in* $\mathrm{GL}(n,\mathbb{Q})$*, i e. if there is a matrix* $V \in \mathrm{GL}(n,\mathbb{Q})$ *such that* $V^{-1}PV = P'$.

The form space of P is

$$\mathcal{F}(P) \;=\; \left\{F \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \mid \forall p \in P \;:\; p^T F p = F\right\} \;. \tag{3.25}$$

Clearly, $\mathcal{F}(P)$ is never empty. The form space is the set of all Gram matrices of lattices compatible with the point group. In other words, it parametrises all possible lattices for the given point group. Given a base for the form space, a Gram matrix, which is a vector in said space, is thus defined by a set of real parameters, called moduli. The following subsections will explore these definitions in more detail.

3.3.1 Affine classes of space groups

As stated in Definition 3.4(1), two space groups S and S' are said to belong to the same affine class, when they are isomorphic. This implies the existence of an affine transformation $f = (A, t)$ with t being a translation and A being a linear mapping (allowing for rotations, inversions and rescalings), such that $f^{-1}Sf = S'$. Hence going from S to S' does not introduce new structure into the resulting orbifold, it just shifts, rotates and zooms the "viewpoint" from which it is seen. Physically speaking, new choices for the geometric moduli of the theory are made. Following that, we will be interested in only one representative for each affine class in the classification procedure. The question how to later fix the moduli parameters[5] at certain values thus remains unresolved at this step.

It can be shown that for every n, only a finite number of affine classes of space groups of degree n exist [24, p. 10]. A complete classification of orbifolds in any dimension is therefore possible, assuming sufficiently powerful computing hardware.

3.3.2 $\mathbb{Z}$-classes of space groups

One step up from affine classes comes the notion of $\mathbb{Z}$-classes. Following Definition 3.4(2), two space groups S and S' with point groups P and P' respectively,

[5]This is known in the literature as the moduli stabilisation problem.

belong to the same $\mathbb{Z}$-class, also known as arithmetic crystal class, iff there exists a unimodular matrix $V \in \mathrm{GL}(n, \mathbb{Z})$, such that

$$V^{-1} P V \;=\; P' \,. \tag{3.26}$$

Since P and P' are in accord with their actions on the lattices of their corresponding space groups, V acts on lattice vectors. V lying in $\mathrm{GL}(n, \mathbb{Z})$ (i. e. V and V^{-1} consist of integer entries only) now implies that lattice vectors get mapped to lattice vectors; hence, the lattice is invariant under V and therefore, space groups in the same $\mathbb{Z}$-class possess the same lattice, or to be more precise, they leave the same space of quadratic forms invariant, i. e. they share the same form space $\mathcal{F}$.

A remark about lattices

Here a remark about classification schemes of lattices seems in order. Since the seminal papers by Dixon et. al in 1985 and 1986 [23, 22] space group lattices have often been classified as root lattices of certain semi-simple Lie algebras, because of the ease with which one can identify the corresponding point group as a discrete subgroup of the holonomy group SU(3). However, root systems do not classify the symmetries of lattices (or to be more precise, form spaces) in a way suitable for space groups. Compared to the canonical way of classifying lattice symmetries using the means of Bravais types, root systems pose over- as well as under-counting problems:

Redundancies There exist different Lie algebras with distinct root systems, which span the same lattice nonetheless, cf. Figure 3.3.

Omissions On the other hand, Bravais types of lattices for which no corresponding Lie algebra exists start appearing in dimensions as low as three. There, e. g. the body-centred-cubic (bcc) or i-cubic lattice is not generated by any root system. This problem is immediately apparent when comparing the number of Bravais types of lattices in n dimensions with the number of simple Lie algebras with rank n; the former of which grows way faster and reaches already 826 for $n = 6$ [29].

In addition to these problems, the notion of root lattices also makes it somewhat difficult to identify all possible deformations of the lattice which leave the orbifold symmetries untouched. This is due to the fact that each root system defines exactly one lattice with fixed moduli, whereas a Bravais type of lattice is parametrised by its form space, which already inherently conveys all information about possible deformations.

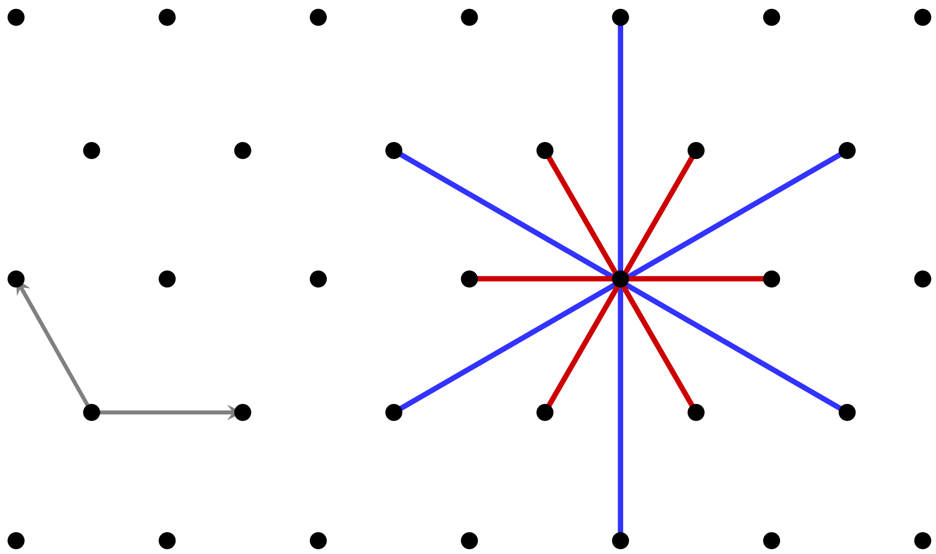


Figure 3.3: The hexagonal lattice. Red lines: the root system of SU(3). Red and blue lines: the G_2 root system. Grey lines: a possible basis for the lattice.

3.3.3 $\mathbb{Q}$-classes of space groups

Going upwards again from $\mathbb{Z}$-classes, we arrive at the notion of $\mathbb{Q}$-classes. According to Definition 3.4(3), two space groups S and S' with point groups P and P' belong to the same $\mathbb{Q}$-class, also known as geometric crystal class, when a matrix $V \in \mathrm{GL}(n, \mathbb{Q})$ exists such that

$$V^{-1} P V \;=\; P' \,. \tag{3.27}$$

Obviously, if two space groups belong to the same $\mathbb{Z}$-class they also belong to the same $\mathbb{Q}$-class. Clearly, the converse is not true. Going from one $\mathbb{Z}$-class to another in the same $\mathbb{Q}$-class thus means switching to another symmetry type of lattice. But since V is unimodular over $\mathbb{Q}$, the commutation relations and orders of the generators of the space group are conserved. This means that the point group of a space group stays a representation of the same finite group in $\mathrm{GL}(n, \mathbb{Z})$ within a given $\mathbb{Q}$-class. Therefore they share the same holonomy group and thus preserve the same amount of supersymmetry in four dimensions. Following from that, when searching for MSSM candidates, we will be able to select promising $\mathbb{Q}$-classes and split only these into $\mathbb{Z}$- and affine classes. The form spaces of space groups in the same $\mathbb{Q}$-class are also of the same dimension, which results in the same number of geometric moduli.

3.3.4 Some examples

In this subsection we will illustrate the definitions from above with some two-dimensional examples taken from Appendix A.

Space groups in the same affine class

Our first example covers two distinct, but isomorphic space groups, i. e. space groups in the same affine class. Let $\mathcal{O} = \mathbb{T}^2/\mathbb{Z}_2$ with the generating point group element $\vartheta = -\mathrm{id}$ as in Section 3.2.5. Since the negative of a lattice vector is always another lattice vector, ϑ imposes no restriction to the lattice – the form space allows for any symmetric non-degenerate matrix. Therefore we can complete the space group S by choosing two linearly independent vectors as a lattice basis, for instance

$$e_1 \;=\; \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 \;=\; \begin{pmatrix} r_2\cos(\alpha) \\ r_2\sin(\alpha) \end{pmatrix}\,, \tag{3.28}$$

with r_1, r_2 and α unequal zero.

This space group shares its affine class with one defined by any other valid lattice. Take for instance

$$\widetilde{e}_1 \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \widetilde{e}_2 \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{3.29}$$

For an explicit check, consider the affine mapping $f = (A, \mathbf{0})$, with

$$A \;=\; \begin{pmatrix} r_1 & r_2\cos(\alpha) \\ 0 & r_2\sin(\alpha) \end{pmatrix} \quad \text{and} \quad A^{-1} \;=\; \begin{pmatrix} \frac{1}{r_1} & -\frac{1}{r_1\tan(\alpha)} \\ 0 & \frac{1}{r_2\sin(\alpha)} \end{pmatrix}\,. \tag{3.30}$$

Now for an element $g = (\vartheta, n_i e_i)$, $n_i \in \mathbb{Z}$ from the space group, its action on an element $x \in \mathbb{R}^2$ readily yields

$$\left(f^{-1}\,g\,f\right)(x) \;=\; \left(f^{-1}\,g\right)(Ax) \;=\; f^{-1}(\vartheta Ax + n_i e_i) \tag{3.31a}$$
$$\;=\; \vartheta x + A^{-1}(n_i e_i) \tag{3.31b}$$
$$\;=\; \vartheta x + n_i\,\widetilde{e}_i \;=\; \widetilde{g}\,x\,, \tag{3.31c}$$

with $\widetilde{g}$ being an element of our second space group.

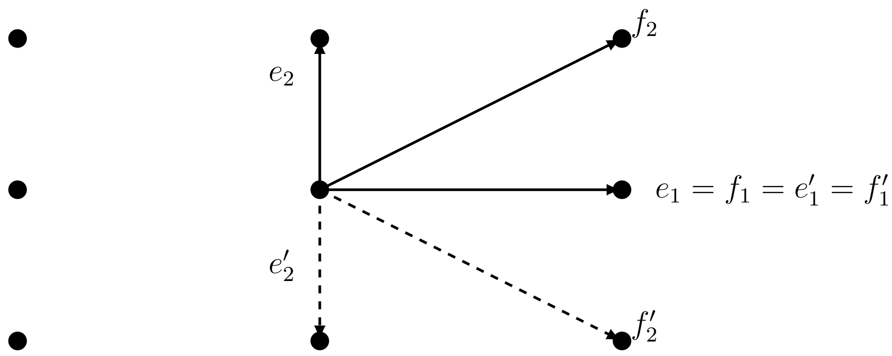


Figure 3.4: A primitive rectangular lattice with two bases $\mathfrak{e} = \{e_1, e_2\}$ and $\mathfrak{f} = \{f_1, f_2\}$. The primed vectors and dashed lines indicate the action of ϑ, which acts as a reflection at e_1.

Space groups in the same $\mathbb{Z}$-class

For our second example we consider the affine class $\mathbb{Z}_2$–II–1–1 from Appendix A. This space group is generated by the semi-direct product of a primitive rectangular lattice with the point group generated by ϑ, a reflection at one axis. Now choose two bases $\mathfrak{e}$ and $\mathfrak{f}$ for this lattice, as in Figure 3.4.

The two corresponding space groups are generated by these lattices and

$$\vartheta_{\mathfrak{e}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \vartheta_{\mathfrak{f}} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} , \tag{3.32}$$

respectively. $\vartheta_{\mathfrak{e}}$ and $\vartheta_{\mathfrak{f}}$ are, although not equal, $\mathbb{Z}$-conjugate to each other, as can be seen by direct computation of $U^{-1}\vartheta_{\mathfrak{e}}U = \vartheta_{\mathfrak{f}}$ with

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{3.33}$$

Hence, since $U \in \mathrm{GL}(2, \mathbb{Z})$ is just a change of lattice basis, $\vartheta_{\mathfrak{e}}$ and $\vartheta_{\mathfrak{f}}$ belong to the same $\mathbb{Z}$-class.

Space groups in the same $\mathbb{Q}$-class

Now consider two space groups generated by the $\mathbb{Z}_2$ point group elements

$$\vartheta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{3.34}$$

respectively. These are, again, taken from Appendix A and belong to the $\mathbb{Z}_2$-classes $\mathbb{Z}_2$–II–1 and $\mathbb{Z}_2$–II–2. They are distinct, as can be seen explicitly by giving the transformation matrix V that fulfils $V^{-1}\vartheta V = \omega$. One easily finds a whole set of matrices

$$V = \begin{pmatrix} x & x \\ y & -y \end{pmatrix} , \qquad \text{with} \qquad x, y \in \mathbb{Q} \tag{3.35}$$

with this property. However, for all values of x and y for which V becomes invertible, either V or V^{-1} gains non-integer entries and therefore $\mathrm{GL}(2,\mathbb{Q}) \ni V \notin \mathrm{GL}(2,\mathbb{Z})$ and hence the space groups generated by ϑ and ω fall in different $\mathbb{Z}$-classes within the same $\mathbb{Q}$-class. This means that they require geometrically different lattices. In this case, the first space group is compatible with any primitive rectangular lattice, while the second needs a centred rectangular lattice. Note that a square lattice would satisfy the consistency criterion for both $\mathbb{Z}$-classes – this is why in the classification of space groups, the lattice itself is a bad indicator for the symmetries inherent in the space group.

Additional translations

In the string orbifold literature, an alternative classification scheme gets used sometimes (cf. [30, 31]) for certain point groups. In this approach, one starts with a factorised lattice, that is the orthogonal sum of two-dimensional sublattices on which the point group generators act diagonally. This has the advantage of being able to directly determine the number of preserved supersymmetry generators in four-dimensional Minkowskian spacetime and an easy to implement check of the modular invariance of the partition function. To fully classify the possible $\mathbb{Z}$-classes, one introduces additional translational generators to the space group, hence increasing the number of lattice points (since each freely acting element of the space group is by definition an element of the lattice) and shrinking its unit cell. This is equivalent to switching to another $\mathbb{Z}$-class as we will illustrate with the following example.

Take the affine class $\mathbb{Z}_2$–II–1–1 from Appendix A, generated by a primitive rectangular lattice and

$$(\vartheta, \mathbf{0}) \;=\; \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) . \tag{3.36}$$

This is geometrically speaking a reflection at the second basis vector of the lattice. Now introduce the freely acting element

$$\tau \;=\; \left(\mathrm{id}, \frac{1}{2}(e_1 + e_2)\right) , \tag{3.37}$$

where $\mathfrak{e} = \{e_1, e_2\}$ is, as usual, the basis of the underlying lattice. This introduces a new lattice point in the unit cell of the original lattice and therefore shrinks it by half, cf. Figure 3.5. The new lattice (which contains the old one as a sublattice) has a shortest basis of τ and $e_1 - \tau$.

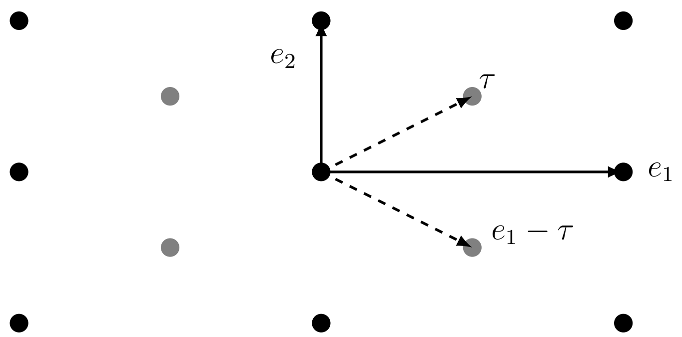


Figure 3.5: Introducing an additional translation $\tau = \frac{1}{2}(e_1+e_2)$ to a primitive rectangular lattice changes it to a centred rectangular lattice. The dark dots indicate the original lattice, the dark and bright dots the new lattice. Shortest bases are given as solid and dashed lines respectively.

The action of ϑ on this new shortest basis is to interchange τ and $e_1 - \tau$. Therefore, in this basis, the point group generator looks like

$$\vartheta' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.38}$$

and the conjugating element $M \in \mathrm{GL}(2, \mathbb{Q})$ with $M^{-1}\vartheta M = \vartheta'$ reads

$$M = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} . \tag{3.39}$$

Obviously, $M \notin \mathrm{GL}(2, \mathbb{Z})$. Thus we have switched to a different $\mathbb{Z}$-class, namely $\mathbb{Z}_2$–II–2.

This alternate way of classifying $\mathbb{Z}$-classes has the disadvantage of requiring explicit lattices to work on with all drawbacks (see Section 3.3.2) linked to it. Therefore we will not pursue this approach further. A complete classification of possible $\mathbb{Z}$-classes is guaranteed to cover all cases that could arise by introducing additional shifts in this way. Following from that we can—without loss of generality—assume all freely acting space group elements to be part of the original lattice and disregard any entanglements that would arise by the two-step approach from this subsection.

Space groups in different $\mathbb{Q}$-classes

For our last example, consider the $\mathbb{Q}$-classes $\mathbb{Z}_2$–I and $\mathbb{Z}_2$–II from Appendix A. The generators of their point groups are defined as an inversion at the origin and a reflection at one axis and read

$$\vartheta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.40}$$

We then find the transformation matrix V which would conjugate these two generators $\vartheta V = V\omega$ to be

$$V = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}, \tag{3.41}$$

with x and y being arbitrary elements of $\mathbb{Q}$. However, since the determinant of V equals zero for all values of x and y, V is not invertible and thus not an element of $\mathrm{GL}(2, \mathbb{Q})$ and therefore the point groups generated by ϑ and ω belong to different $\mathbb{Q}$-classes.

3.4 Classification strategy

As we have seen, to classify all possible crystal classes of space groups (and hence orbifolds), one faces the challenge of finding all $\mathbb{Z}$- and $\mathbb{Q}$-conjugacy classes of finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$. Luckily, by the famous theorem given by Zassenhaus in 1938, there is only a finite number of such classes for every $n \geq 1$, cf. for instance p. 563f of the great review of representation theory of finite groups in [32]. This theorem makes an automated approach feasible – which is a very desirable trait, considering the huge numbers of classes involved! For the classification as utilised in [8, 9], the software package Carat [7] (see also [27, 33]) was used. It comes with pre-computed representatives of each $\mathbb{Q}$-class[6] for dimensions up to six and features algorithms to split them into $\mathbb{Z}$- and affine classes.

To compute the full set of $\mathbb{Q}$-classes, the authors of Carat started with a set of irreducible maximal finite integral matrix groups, or short i. m. f. groups, which was obtained by looking for finite groups which have irreducible faithful representations of the necessary degree and then constructing the full set of subgroups of the i. m. f. groups [28]. Then, making additional use of some crystallographic invariants, these subgroups where tested for $\mathbb{Q}$-equivalence, yielding the full set of $\mathbb{Q}$-classes [27]. Following that, Carat splits $\mathbb{Q}$- into $\mathbb{Z}$-classes by computing the normalisers of the

[6] For a derivation thereof see [27, 28].

respective representations and the generation of lattices [33]. The last step, splitting $\mathbb{Z}$- into affine classes, makes use of the Zassenhaus algorithm.

In recent years, so-called asymmetric orbifolds [34], i. e. heterotic string theories where the right- and left-movers live on different orbifolds altogether, have seen increased interest, cf. for instance [35, 36, 37]. For these constructions, a classification of space groups in $n = 22$ dimensions would be a great step forward. And, in principal, the building blocks, namely the i. m. f. groups, are known [38]. However, Carat is currently not suited to the task, since its implementation details rely on the fact that the degree of the involved groups is at most six[7]. Also, for the big numbers involved in the process, Carat would need to be rewritten to utilise 64 bit integers internally[8].

3.5 Previous classifications

Prior to the classification which we present in this thesis and which was first published in [8], there were some incomplete attempts at a classification of orbifold symmetries which preserve $\mathcal{N} = 1$ supersymmetry. For instance, Bailin and Love [39] classified $\mathbb{Z}_N$ geometries using root lattices of semi-simple Lie algebras. However, as pointed out in Section 3.3.2, this method has its intricacies and consequently, they over-count the geometries as well as missing some of them.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds have been studied in much detail by different authors. For instance, Förste et. al [40] (see also [41]) based their study on Lie lattices, while omitting roto-translations, i. e. non-trivial affine classes. However, they also missed out four $\mathbb{Z}$-classes. Another approach, by Donagi and Wendland [30, 42] is complete, but over-counts one case.

The case of $\mathbb{Z}_N \times \mathbb{Z}_N$ point groups with $N = 3, 4, 6$ has been studied in [31], but also somewhat incomplete.

[7] This is true for at least the computation of the family symbol of a matrix group, which is needed for checking Bravais and, by extension, $\mathbb{Z}$-equivalence [33].
[8] W. Plesken, personal correspondence.

3.6 SUSY check

Now that we have the means of classifying all geometrically inequivalent space groups and thus, by division from the torus, orbifolds, we want to focus on those orbifolds which yield $\mathcal{N} = 1$ supersymmetry in four dimensions, since they are phenomenologically the most promising. The number of surviving supersymmetry generators in four dimensions is tied to the holonomy group of the compact space [43]. For instance, a six-dimensional torus, created by any space group with trivial point group, has trivial holonomy and therefore yields four covariantly constant spinors and therefore all $\mathcal{N} = 4$ supersymmetry generators survive in four dimensions. For orbifolds, the holonomy group is connected to the point group [22] and therefore to the $\mathbb{Q}$-class in question. Four-dimensional supersymmetry is preserved, if the point group is a subgroup of SU(3).

Subsequently, our approach to classification starts with a complete list of $\mathbb{Q}$-classes of degree $n = 6$ provided by Carat and proceeds to test their point groups for forming SU(3)-subgroups. In the following sections, we will present two different approaches to carry out this task. All candidates obtained in this manner are then split into $\mathbb{Z}$- and affine classes to create the full list of toroidal orbifold compactifications with $\mathcal{N} = 1$ supersymmetry in four dimensions. This list has been published in [8].

3.7 Spinor representation

In this section we will develop the means to directly translate a GL$(6, \mathbb{Z})$ point group element to its SU(4) representation. From there, one can easily read off if the whole point group falls into a SU(3) subgroup thereof, i. e. if it leaves one direction invariant.

In order to determine how many supersymmetric spinors survive the action of a point group, one could choose a basis in which all of the point group elements are in block-diagonal shape and the so-called twist vectors can be read off easily. For Abelian point groups, this is always possible. It can be done by choosing a transformation matrix which consists of (real) column vectors from the intersections of the eigenspaces of the point group elements. Since every point group element is in SO(6), eigenvectors come in complex conjugated pairs or—for eigenvalues ± 1—are purely real. If they are real, one part of the matrix can be made diagonal; if they are complex conjugated pairs, choosing the real and the imaginary part of one of them leads to a block-diagonal rotation matrix.

However, when one wants to consider non-Abelian point groups, this approach is no longer feasible (since the matrices can no longer be block-diagonalised at the same time) and it is mandatory to obtain the spinor representations of the point group elements. To do this, we require some additional machinery, which we will, for the sake of space, not fully develop here. Instead, we will give the main ideas of the construction we used. Similar constructions have been used before [44, 45] and we follow them closely, but generalise their method since we need it to properly handle so-called isoclinic rotations.

3.7.1 Rotations as reflections

By virtue of the theorem of Cartan-Dieudonné, every element of the orthogonal group of an Euclidean space $\mathbb{E}^n$ is a composition of at most n reflections at hyperplanes. Another way to state this is that every rotation in n-dimensional space is a composition of at most $\lfloor n/2 \rfloor$ simple rotations in orthogonal planes, where a simple rotation is a rotation within one two-dimensional subspace. In two dimensions, this means that every rotation is defined by just an angle, while in three dimensions, it is defined by one plane—which can be portrayed by its orthogonal vector, the rotation axis[9]—and one angle. Therefore, in six dimensions, every rotation and inversion can be decomposed into at most three simple rotations, each of them being the composition of two hyperplane reflections. The vectors orthogonal to these hyperplanes span the plane of rotation and they enclose half the angle of rotation. Therefore, each simple rotation has two possible representations: one which rotates around ϕ in the plane of rotation and one which rotates around $2\pi - \phi$ in the opposite direction. As rotations, these operations do the same thing, but in the spinor world they will not. Therefore, we are now challenged with two tasks: first, to find the correct hyperplane reflections for our orthogonal matrix at hand and second, to find a way of writing them down as an element of Spin(6). The first is achieved with a standard orthogonalisation scheme, while the second makes heavy use of explicit representations of Clifford algebras.

3.7.2 Decomposing a rotation

In this subsection, we will show how to decompose a (possibly isoclinic) rotation in six dimensions into a composition of hyperplane reflections. This has been done before, e. g. [44, 45], but these previous approaches were not interested in isoclinic

[9] Note that the concept of rotation axes does not translate well into spaces with dimensions differing from three.

rotations – therefore we have to adjust our algorithm accordingly. First, we need the theorem of Cartan-Dieudonné, as stated and proven in [46, p. 245]:

Definition 3.5 *Let E be a Euclidean space of finite dimension n. For any two subspaces F and G which are orthogonal $F = G^{\perp}$ and form a direct sum $E = F \oplus G$, the orthogonal symmetry with respect to F and parallel to G, or orthogonal reflection about F is the linear map $s: E \to E$, defined such that*

$$s(u) = 2p_F(u) - u \,, \tag{3.42}$$

for every $u \in E$. Here, p_F is the projection unto the subspace F. When G is a plane, i. e. two-dimensional, we call s a flip about F.

Theorem 3.6 *Let E be a Euclidean space of dimension $n \geq 3$. Every rotation $f \in \mathrm{SO}(E)$ is the composition of an even number of flips $f = f_{2k} \circ \cdots \circ f_1$, where $2k \leq n$. Furthermore, if $u \neq \mathbf{0}$ is invariant under f (i. e. $u \in \mathrm{kern}(f - \mathrm{id})$), we can pick the last flip f_{2k} such that $u \in F_{2k}^{\perp}$, where F_{2k} is the subspace of dimension $n-2$ determining f_{2k}.*

The proof's main idea is the equivalence between the composition of two hyperplane reflections, which make up the rotation, with the composition of two flips.

As the theorem already suggests, the trick is to orthogonalise the eigenspaces of f in order to obtain orthogonal planes, which is always possible. However, since one plane is constructed of two eigenvectors with complex conjugated eigenvalues which represent the sine and cosine of the angle of rotation, an ambiguity arises if two or three angles of rotation are identical. However, in this case every linear combination of the corresponding eigenvectors is again an eigenvector and hence the eigenspace is four- or even six-dimensional. Therefore, any pair of orthogonal vectors from this space will satisfy the requirement.

Once orthogonal planes of rotation are found, the rotation angles are easily obtained by picking one vector per plane and letting f act on it. The looked for hyperplane reflections are reflections at the hyperplanes defined by orthogonal vectors spanning the rotation plane which enclose an angle which is half the angle of rotation. However, as we will see below, we will not need these hyperplanes explicitly.

3.7.3 Clifford algebras

The concept of Clifford algebras is a very rich one with lots of mathematical structure and therefore, a complete construction is outside of the scope of this thesis; hence, we will only introduce the most important concepts for our approach. We closely follow the excellent review of the subject in [47] and recommend it to the readers interested in the details.

Definition 3.7 *Let V be a real finite-dimensional vector space, equipped with a symmetric, non-degenerate bilinear form $\varphi: V \times V \to \mathbb{R}$ and an associated quadratic form $\phi: v \mapsto \varphi(v,v)$. A Clifford algebra over (V, ϕ) is a real algebra $Cl(V, \phi)$, together with a linear mapping $\iota: V \to Cl(V, \phi)$, which satisfies $\forall v \in V: (\iota(v))^2 = \phi(v) \cdot \mathbf{1}$. In addition, the following universal property needs to hold: For every real algebra A and every linear map $f: V \to A$ with $\forall v \in V: (f(v))^2 = \phi(v) \cdot \mathbf{1}$ there is a unique homomorphism $h_f: Cl(V, \phi) \to A$ such that $f = h_f \circ \iota$.*

A few remarks are in order: First, recall that an algebra is a vector space with a well-defined multiplication law. Therefore, with $\mathbf{1}$ we denote the multiplicative one in this space. Also, since this definition is not constructive, one might wonder if Clifford algebras do exist and if they do, how they might be built. Here we observe that $T(V)/\mathfrak{A}$ where $T(V)$ is the Tensor algebra over V and $\mathfrak{A}$ is the ideal generated by elements of the form $v \otimes v - \phi(v) \cdot \mathbf{1}$ and ι is the canonical projection into the quotient $T(V)/\mathfrak{A}$ of the the canonical injection of V into $T(V)$ fulfils all requirements. Third, verify that with the above definitions

$$\varphi(u,v) \cdot \mathbf{1} \;=\; \frac{1}{2}\left(\iota(u)\iota(v) + \iota(v)\iota(u)\right) \;=\; \frac{1}{2}\left\{\iota(u), \iota(v)\right\}, \tag{3.43}$$

where the braces denote the anti-commutator.

Using definition 3.7, we now have the possibility to multiply vectors. Using the hyperplane-defining vectors which are ensured by theorem 3.6, we could now construct rotors, which are their products and act on vectors via versor product: Let M be a matrix describing a simple rotation, v_1 and v_2 the two such vectors found by the method described in the previous subsection and x an arbitrary vector from $\mathbb{E}^n$. Then,

$$\iota(Mx) \;=\; \iota(v_1)\iota(v_2)\iota(x)\left(\iota(v_1)\iota(v_2)\right)^{-1}. \tag{3.44}$$

A proof of this can be found in [48] (see also [46]).

The remaining task is now to find the structure of $Cl(V, \phi)$ and the injection ι. Observe that V always has a basis in which ϕ can be written as

$$\phi(x_1, \ldots, x_{p+q}) \;=\; x_1^2 + \cdots + x_p^2 - \left(x_{p+1}^2 + \cdots + x_{p+q}^2\right), \tag{3.45}$$

where (p,q) will be called the signature of ϕ. We will denote the Clifford algebra $Cl(V, \phi)$ as $Cl(p,q)$, since this describes it uniquely up to isomorphism. To finally construct the Clifford algebra $Cl(0,6)$, which is the one we are interested in, we exploit some beautiful recursion relations which allow us to construct this Clifford algebra as Kronecker products of only two building blocks.[10]

Theorem 3.8 *The following isomorphisms hold:*

$$Cl(0,n+2) \cong Cl(n,0) \otimes Cl(0,2), \tag{3.46}$$
$$Cl(n+2,0) \cong Cl(0,n) \otimes Cl(2,0). \tag{3.47}$$

Proof: See [47]. The main idea is to build generators $\{e_i \otimes f_1 f_2,\ \mathbf{1} \otimes f_1,\ \mathbf{1} \otimes f_2\}$ for $Cl(0,n+2)$, out of the generators e_i of $Cl(n,0)$ and f_i of $Cl(0,2)$ and prove the bijection property. The second isomorphism is completely analogous. □

Note that since we required V to be finite-dimensional, the tensor products can be understood as Kronecker products. Finally, verify that $\mathbb{R}^{2,2}$, generated by

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{3.48}$$

and $\mathbb{H}$, the space of quaternions, generated by

$$\left\{ \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \tag{3.49}$$

are representations of $Cl(2,0)$ and $Cl(0,2)$ respectively. Recall that despite the fact that the generators of $\mathbb{H}$ contain the imaginary unit i, the algebra still is real and therefore only multiplication with real scalars is meaningful.

3.7.4 The spin group and an accidental isomorphism

Every Clifford algebra $Cl(p,q)$ comes with an unique automorphism

$$\alpha: \ Cl(p,q) \to Cl(p,q)\ , \tag{3.50}$$

satisfying

$$\forall v \in V: \qquad \alpha \circ \alpha = \mathrm{id}, \qquad \text{and} \qquad \alpha(\iota(v)) = -\iota(v). \tag{3.51}$$

[10] Actually, with four building blocks and a couple of additional relations, every Clifford algebra $Cl(p,q)$ can be constructed. We restrict ourselves to the blocks we need and refer, again, to [47] for a description of the others.

A proof of this can, again, be found in [47]. In layman's terms, if $\{e'_i\}_{i\in I}$ is a basis of $\mathbb{R}^{p+q}$, then the Clifford algebra $Cl(p,q)$ consists of "polynomials" in e_i, where $e_i = \iota(e'_i)$. α then "counts" the number of basis vectors in each monomial and assigns a sign to each:

$$\alpha(e_{i_1}\cdots e_{i_k}) = (-1)^k e_{i_1}\cdots e_{i_k}. \tag{3.52}$$

Note that since $e_i^2 = \phi(e'_i)\cdot\mathbf{1}$, there is only a finite number of different monomials for every (p,q) and therefore the Clifford algebra is always of finite dimension as a vector space.[11] We see that $Cl(p,q) = Cl^0(p,q)\oplus Cl^1(p,q)$ where

$$\forall i\in\{0,1\}: \qquad Cl^i(p,q) = \{x\in Cl(p,q)|\ \alpha(x) = (-1)^i x\}. \tag{3.53}$$

Hence, $Cl(p,q)$ is graded and decomposes into an even and an odd part. The even part $Cl^0(p,q)$ is a subalgebra, while the odd part $Cl^1(p,q)$ is not (the product of two monomials with an odd number of factors yields a monomial with an even number of factors). From the fact that simple rotations are generated by products of two vectors, it immediately follows that all rotations can be created by products of an even number of vectors. Therefore, the group which creates all rotations on $\mathbb{R}^{p+q}$ should lie within $Cl^0(p,q)$ and indeed, it does.

Definition 3.9 *We define the norm of $Cl(p,q)$ by its action on the monomials:*

$$N: \qquad e_{i_1}\cdots e_{i_k}\mapsto \phi(e'_{i_1})\cdots\phi(e'_{i_k})\cdot\mathbf{1}. \tag{3.54}$$

Note that this is not a norm in the strict sense as it does not map to the reals but back into the algebra. Now we have everything assembled to define the spin group which we will use to represent rotations.

Definition 3.10 *The spin group of a Clifford algebra $Cl(p,q)$ is*

$$\mathrm{Spin}(p,q) = \left\{x\in Cl^0(p,q)\middle|\ \forall v\in\mathbb{R}^{p+q}:\ xvx^{-1}\in\mathbb{R}^{p+q}\ \text{ and }\ N(x) = \mathbf{1}\right\}. \tag{3.55}$$

In our case, we want to represent rotations in six dimensions. Therefore, we find the vectors which define the hyperplane-reflections of the rotation, inject them into $Cl(0,6)$ and multiply them there. Hence the only thing left is to project the result down into Spin(0,6). Here, another beautiful isomorphism helps us. It can be shown that $Cl(p,q)\cong Cl^0(p,q+1)$. This can be seen as any even monomial on the left hand side can be identified with the same monomial on the right hand side, whereas

[11]In fact, its dimension is 2^{p+q}.

an odd monomial gets identified with the same monomial multiplied by e_{p+q+1}. Our construction is now complete. It is well-known and can be proved directly from this construction that $\mathrm{SU}(4) \cong \mathrm{Spin}(0,6) \subsetneq Cl(0,5) \cong Cl^0(0,6)$ ([49]). This is also called an "accidental" isomorphism, since it relates a spinny structure to a Lie group, which is surprising at first sight and does not generalise well. In fact, there are a couple such isomorphisms in low dimensions, but a general correlation between Lie groups and spin groups does not exist.

A Mathematica-package for this task has been developed and is available online at http://users.ph.tum.de/fischmax/clifford.m. It is also printed in Appendix C.

3.8 Character tables

In this section we present an alternative approach to the same problem, i. e. to decide whether a given point group P is a subgroup of SU(3), which does not require the direct spinor representation of any point group elements. This method has been adopted in [8].

Firstly, one has to check whether P is in SO(6). In our representation of choice, the point group is a unimodular matrix group in $\mathrm{GL}(6,\mathbb{Z})$, which represents the rotations from O(6) in a lattice basis. But since the determinant is independent of choice of basis, it can be directly verified to be +1 without any further ado.

Following that, we have to break P as a discrete subgroup of the **6** of $\mathrm{SO}(6) \cong \mathrm{SU}(4)$ into representations of SU(3). Since **6** is a real representation, the branching goes [50]

$$\mathbf{6} \to \mathbf{3} \oplus \overline{\mathbf{3}}\,, \tag{3.56}$$

However, P can, in general, form a reducible representation. Thus it generally decomposes into

$$\mathbf{6} \to \boldsymbol{a} \oplus \boldsymbol{b} \oplus \cdots \tag{3.57}$$

with irreducible representations $\boldsymbol{a}$, $\boldsymbol{b}$ and so on. To find this decomposition, the character table of P can be utilised: for each representation $\boldsymbol{\rho}$ of P, the character $\chi_{\boldsymbol{\rho}}(g)$ of an element $g \in P$ is the trace of its image $\boldsymbol{\rho}(g)$ in that representation,

$$\chi_{\boldsymbol{\rho}}(g) \;=\; \mathrm{Tr}(\boldsymbol{\rho}(g))\,. \tag{3.58}$$

This is the same for all elements of a conjugacy class of P, since the trace is invariant under cyclic permutations and therefore conjugations of the element in question. Hence, the character table of any finite group possesses one row for each irreducible representation and one column for each of its conjugacy classes. Also, the character table is always square, which is related to the group theoretic fact that for any finite group, the number of conjugacy classes matches the number of irreducible representations. Now we can compute the character $\chi_{\mathbf{6}}(g)$ of the six-dimensional representation **6** for each conjugacy class $[g]$ of P and subsequently determine their multiplicities in the decomposition:

$$\mathbf{6} \rightarrow \bigoplus_{i=1}^{c} n_i \, \boldsymbol{\rho}_i \quad \text{with} \quad n_i \;=\; \frac{1}{|P|} \sum_{g \in P} \chi_{\boldsymbol{\rho}_i}(g) \overline{\chi_{\mathbf{6}}(g)} \, . \tag{3.59}$$

In order for P to be a subgroup of SU(3), Equation (3.59) needs to yield something of the form

$$\mathbf{6} \rightarrow \boldsymbol{a} \oplus \overline{\boldsymbol{a}} \, , \tag{3.60}$$

plus, possibly, some singlet representations. If this is indeed the case, we have at least $P \subsetneq \text{U}(3)$. Finally, we will have to verify that the determinants of all elements of P in the three-dimensional matrix representation of $\boldsymbol{a}$ do actually equal $+1$. If this is in fact the case, then we have $P \subsetneq \text{SU}(3)$ and at least $\mathcal{N} = 1$ supersymmetry generator survives the compactification procedure.

In [8], the computer algebra system GAP [51] was used to create the character tables, along with the package Repsn [52] for the construction of the three-dimensional representations. GAP identifies finite groups by their so-called GAPID $[N, M]$, where N is the order of the group and M a sequential index. In the tables in Appendix B, this GAPID is always given together with a flat index into the CARAT data sets.

3.9 Fundamental groups

In compactifications of heterotic string theories, when questioned with GUT-breaking mechanisms, one often wants to look for non-trivial freely acting space group elements which can not be written as a composition of non-freely acting ones. This is, because the gauge embedding of said elements can fertilise so-called non-local GUT breaking [53, 54, 55], which is phenomenologically appealing.

Formalising this approach, the fundamental group π_1 of a topological space X measures its "connectedness", i. e. how many classes of non-trivial non-contractible loops exist on that space, together with their nature:

Definition 3.11 *The set of homotopy classes of loops at $x_0 \in X$ is denoted by $\pi_1(X, x_0)$. It is called the fundamental group of X at x_0.*

If X is arcwise connected—and all our orbifolds are—then, for any x_0, x_1 in X, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic and the notation is shortened to $\pi_1(X)$. If $\pi_1(X)$ is trivial, then X is called simply connected. In that case, X has no non-contractible loops.

In order to compute the fundamental groups of toroidal orbifolds, we make use of a technique introduced in [23, 22] and formalised in [56]. Let S be a space group, defining an orbifold $\mathcal{O} = \mathbb{R}^n/S$. Then the fundamental group of $\mathcal{O}$ is given by

$$\pi_1(\mathcal{O}) \;=\; S/\langle F \rangle \;, \tag{3.61}$$

where $F \subseteq S$ is the set of all space group elements which leave at least one point fixed, i. e. which are not freely acting. This is not necessarily a group itself, but the group $\langle F \rangle$ generated by it, is always a normal subgroup of S and hence Equation (3.61) is well defined.

We find, that among space groups whose orbifolds preserve at least $\mathcal{N} = 1$ supersymmetry, most fundamental groups are trivial, that is, in those cases $\langle F \rangle = S$ holds. Only about 10 to 15 percent of supersymmetry admitting space groups possess at least a $\mathbb{Z}_2$ fundamental group. A detailed tabulation of all non-trivial fundamental groups and the space groups which give rise to them can be found in Appendix B.2.

3.10 Cohomology

Besides the fundamental groups of orbifolds, we will be interested in other topological invariants of them as well. In particular, we would like to know their non-trivial Hodge numbers, which describe their Kähler and complex structure moduli and give, by means of the Euler characteristic $\chi = 2\left(h^{(1,1)} - h^{(2,1)}\right)$ the number of chiral generations of matter obtained from a standard embedding of the point group in the gauge degrees of freedom [43, 42, 57, 30]. Note however, that non-standard embeddings can reduce this number further, by means of discrete Wilson lines. In this section, we will, for convenience, review some of the definitions leading to Hodge numbers, state how they can be applied to orbifolds and present our algorithm for calculating them from [9]. This recapitulation is only a brief one and readers interested in the details are encouraged to consult for instance [58] or the less formal, but maybe more accessible treatment given in [59].

3.10.1 Definitions

Let $(X_n)_{n\in I}$ be a sequence of topological spaces. Now any sequence of operators $O_n: X_n \to X_{n+1}$ which satisfy $O_n \circ O_{n-1} = 0$, that is $\Im O_{n-1} = \ker O_n$, defines an exact sequence and therefore, when an additional homotopy criterion is met, a (co)homology structure [58]. We will construct these operators O as Dolbeaut-operators acting on spaces of differential forms of complex manifolds.

Definition 3.12 *Let M be a m-dimensional complex manifold and $T_p(M)$ its tangent space*[12] *at some point $p \in M$. Then, the space of r-forms is*

$$\Omega^r(M) \;=\; \bigwedge^r T_p(M) \;=\; \bigotimes^r T_p(M)/\mathrm{span}\left(\bigotimes^r v\right)_{v\in T_p(M)} , \tag{3.62}$$

that is the totally antisymmetric r-fold tensor product of the tangent space with itself. Designate a basis of differentials $\mathrm{d}z^\mu = \mathrm{d}x^\mu + i\mathrm{d}y^\mu$ and $\mathrm{d}\bar{z} = \mathrm{d}x^\mu - i\mathrm{d}y^\mu$ for $T_p(M)$. Then, a (r,s)-form is a multilinear object

$$\omega \;=\; \frac{1}{r!s!}\omega_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\mathrm{d}z^{\mu_1}\wedge\ldots\wedge\mathrm{d}z^{\mu_r}\wedge\mathrm{d}\bar{z}^{\nu_1}\wedge\ldots\wedge\mathrm{d}\bar{z}^{\nu_s} . \tag{3.63}$$

The space of (r,s)-forms is called $\Omega^{r,s}(M)^{\mathbb{C}}$ or short $\Omega^{r,s}$.

The Dolbeaut-$\bar{\partial}$ operator now maps any (r,s)-form to a $(r,s+1)$-form. Let ω be as in Equation (3.63). Then,

$$\bar{\partial}_{r,s} : \omega \mapsto \frac{1}{r!s!}\frac{\partial}{\partial\bar{z}^\lambda}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}\mathrm{d}\bar{z}^\lambda\wedge\mathrm{d}z^{\mu_1}\wedge\ldots\wedge\mathrm{d}z^{\mu_r}\wedge\mathrm{d}\bar{z}^{\nu_1}\wedge\ldots\wedge\mathrm{d}\bar{z}^{\nu_s} . \tag{3.64}$$

It can be shown that for any $\omega \in \Omega^{r,s}$, $\bar{\partial}\bar{\partial}\omega = 0$ holds, making the sequence

$$\cdots \longrightarrow \Omega^{r,s-1} \xrightarrow{\bar{\partial}_{r,s-1}} \Omega^{r,s} \xrightarrow{\bar{\partial}_{r,s}} \Omega^{r,s+1} \longrightarrow \cdots \tag{3.65}$$

exact. We now have everything in place to properly define the cohomology groups.

Definition 3.13 *The set*

$$Z^{(r,s)}_{\bar{\partial}}(M) \;=\; \left\{\omega \in \Omega^{r,s}(M)|\bar{\partial}\omega = 0\right\} = \ker\bar{\partial}_{r,s} \tag{3.66}$$

[12] A tangent space is a m-dimensional vector space “tangential” to the manifold at the given point. It can be defined in many ways, but the details will not be relevant to us and are therefore omitted.

is called the (r,s)-cocycle and its elements are called $\bar{\partial}$-closed forms. Further, the set

$$B_{\bar{\partial}}^{(r,s)}(M) \;=\; \left\{\omega \in \Omega^{r,s}(M) | \exists \eta \in \Omega^{r,s-1}(M) : \omega = \bar{\partial}\eta\right\} = \text{im}\ \bar{\partial}_{r,s-1} \tag{3.67}$$

is called the (r,s)-coboundary and its elements are called $\bar{\partial}$-exact forms.

Finally, the (r,s)-th $\bar{\partial}$-cohomology group is

$$H_{\bar{\partial}}^{(r,s)}(M) \;:=\; Z_{\bar{\partial}}^{(r,s)}(M)/B_{\bar{\partial}}^{(r,s)}(M)\ . \tag{3.68}$$

The exact sequence property in Equation (3.65) renders the definition in Equation (3.68) well defined. An element $[\omega] \in H_{\bar{\partial}}^{(r,s)}(M)$ is an equivalence class of $\bar{\partial}$-closed (r,s)-forms which differ from ω by a $\bar{\partial}$-exact form at the most. In addition to its inherent group structure, $H_{\bar{\partial}}^{(r,s)}(M)$ inherits the vector space structure from the tangential space of M. Its complex dimension is called the (r,s)-th Hodge number $h^{(r,s)}$. In total, all cohomology groups of a manifold M combined give rise to the famous Hodge diamond [59],

$$\begin{pmatrix} & & & h^{m,m} & & & \\ & & h^{m,m-1} & & h^{m-1,m} & & \\ & & & \vdots & & & \\ h^{m,0} & h^{m-1,1} & \cdots & & \cdots & h^{1,m-1} & h^{0,m} \\ & & & \vdots & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{pmatrix}, \tag{3.69}$$

which conveniently sums up the cohomology structure of M. However, this diamond is already a redundant description, since not all Hodge numbers are independent quantities: one can prove that $h^{(r,s)} = h^{(s,r)}$ and $h^{(r,s)} = h^{(m-r,m-s)}$. In addition, the outer layers $h^{(\ell,k)}$, where either ℓ or k equals zero (and of course their dependent values with ℓ or k equalling m) are trivial for toroidal orbifolds and will not concern us. Hence, together with the fact that the spaces we are interested in only have three complex dimensions, we will only have to compute the values of $h^{(1,1)}$ and $h^{(2,1)}$ to recognise the full cohomology structure of these orbifolds.

3.10.2 Orbifold cohomology

The definitions in the previous section are only declared on manifolds and it is not a priori clear how to apply them to orbifolds with non-trivial fixed loci. However, since all of our orbifolds originate from a torus $\mathbb{R}^6/\Lambda$, they are “good” [21] in the sense that they possess covering spaces which are manifolds. Hence one can define the orbifold cohomology as the cohomology of the covering space [57]. Although this is not strictly necessary, as the notion of orbifold cohomology can be defined as well [60], there is no reason not to take this shortcut. We are interested in the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$ which, from a physical point of view, count the Kähler and complex structure moduli of the orbifold and therefore quantise the possible deformations of the underlying geometry. The methods we develop in this section have been published in [9].

First, we split the Hodge numbers into contributions from the bulk and the twisted sectors of the string,

$$(h^{(1,1)},\ h^{(2,1)}) \ = \ (h_{\mathrm{U}}^{(1,1)},\ h_{\mathrm{U}}^{(2,1)}) + (h_{\mathrm{T}}^{(1,1)},\ h_{\mathrm{T}}^{(2,1)}) \ . \tag{3.70}$$

Untwisted sectors

The contributions to the Hodge numbers from the untwisted moduli can be directly computed from the point group P, using only representation theory of finite groups. Since they are only dependent on P, they do not change within one $\mathbb{Q}$-class and can therefore be used for all affine classes in that $\mathbb{Q}$-class. $h_{\mathrm{U}}^{(1,1)}$ and $h_{\mathrm{U}}^{(2,1)}$ arise from the internal degrees of freedom of the ten-dimensional supergravity multiplet corresponding to the string excitations

$$|q\rangle_{\mathrm{R}} \otimes \tilde{\alpha}^{\bar{\jmath}}_{-1}|0\rangle_{\mathrm{L}} \qquad \text{for Kähler moduli, and} \tag{3.71a}$$

$$|q\rangle_{\mathrm{R}} \otimes \tilde{\alpha}^{j}_{-1}|0\rangle_{\mathrm{L}} \qquad \text{for complex structure moduli.} \tag{3.71b}$$

Here, $j \in \{1,2,3\}$ denotes the three complexified coordinates on the orbifold and $|q\rangle_{\mathrm{R}}$ is the right-moving ground state of the bosonized string, whereas $\tilde{\alpha}^{\bar{\jmath}}_{-1}$ and $\tilde{\alpha}^{j}_{-1}$ are excitations of the left-moving ground state $|0\rangle_{\mathrm{L}}$.

Only the invariant combinations of the untwisted states survive the orbifolding procedure. Furthermore, they are uncharged under the gauge group and thus are only acted upon by the point group P. Fortunately, we already obtained the explicit three-dimensional, in general reducible, representation $\boldsymbol{\rho}$ of P in Section 3.8 and therefore we can immediately write down the transformation properties of the ground state and the excitation modes as

$$|q\rangle_{\mathrm{R}} \quad \text{lives in} \quad \boldsymbol{\rho}\ , \tag{3.72a}$$

$$\tilde{\alpha}^{\bar{\jmath}}_{-1} \quad \text{lives in} \quad \overline{\boldsymbol{\rho}}\ , \tag{3.72b}$$

$$\tilde{\alpha}^{j}_{-1} \quad \text{lives in} \quad \boldsymbol{\rho}\ . \tag{3.72c}$$

Hence, using Equations (3.71) and (3.72), the untwisted Hodge numbers $h_{\mathrm{U}}^{(1,1)}$ and $h_{\mathrm{U}}^{(2,1)}$ can be read off of the tensor product decompositions

$$\boldsymbol{\rho} \otimes \overline{\boldsymbol{\rho}} \;\rightarrow\; h_{\mathrm{U}}^{(1,1)}\mathbf{1} \oplus \cdots \quad \text{and} \quad \boldsymbol{\rho} \otimes \boldsymbol{\rho} \;\rightarrow\; h_{\mathrm{U}}^{(2,1)}\mathbf{1} \oplus \cdots \;. \tag{3.73}$$

There, **1** is the trivial one-dimensional representation of P and the Hodge numbers are their respective multiplicities. They can be read off of Equation (3.59) from Section 3.8 using the orthogonality of the character table of P. Again, GAP and Mathematica were used to perform the necessary computations. For non-Abelian point groups, often only one (untwisted) Kähler modulus survives the orbifolding, leaving only the overall size of $\mathcal{O}$ open and leaving no further geometrical freedom. This might be interesting for no-scale supergravity [61, 62, 63, 64].

Twisted sectors

Now that the untwisted sector—or bulk—of the orbifold is accounted for, we can focus on the contributions to the Hodge diamond from the twisted sectors, that is from non-trivial constructing elements of the space group, which are related to their respective fixed points or tori (cf. also [30]). We will do this by "thinking of" the standard embedding of the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group, without actually performing it. Hence all computations are closely tied to the geometry and only the geometry of $\mathcal{O}$. This standard embedding would be performed by diagonalising the constructing element (see below) in question, which could then be expressed as a diagonal matrix

$$\vartheta \;=\; \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} v_1} & & \\ & \mathrm{e}^{2\pi \mathrm{i} v_2} & \\ & & \mathrm{e}^{2\pi \mathrm{i} v_3} \end{pmatrix} , \tag{3.74}$$

acting on three complex coordinates. One would then embed the twist vector $v = (v_1, v_2, v_3)$ into the sixteen gauge degrees of freedom as

$$V \;=\; (v_1, v_2, v_3, 0^5)(0^8)\;. \tag{3.75}$$

Note that for non-Abelian point groups, no one basis exists for which all constructing elements are diagonal simultaneously. However, since for computations in one twisted sector, only its constructing element and the corresponding centraliser are of interest, most standard techniques like level-matching go through unhampered, cf. also [65, 18].

This approach would then break the first E_8 gauge group factor to an E_6 times some (S)U(ℓ) factors, which are irrelevant for us. Now, world-sheet supersymmetry correlates the number of **27**- and $\overline{\mathbf{27}}$-plets of E_6 to $h_{\mathrm{T}}^{(1,1)}$ and $h_{\mathrm{T}}^{(2,1)}$ respectively [66, 67, 68].

The twisted sectors are enumerated by the conjugacy classes $[\vartheta]$ of the point group P and are named $T_{[\vartheta]}$ accordingly. Now, a space group element $g = (\vartheta, \lambda) \in S$ with $\mathrm{id} \neq \vartheta$ is a constructing element iff it possesses a fixed locus. That is, if the equation

$$g f \;=\; f \qquad \Leftrightarrow \qquad \vartheta f + \lambda \;=\; f \;, \tag{3.76}$$

has a solution for some $f \in \mathbb{R}^6$. For space groups which preserve at least $\mathcal{N} = 1$ supersymmetry in four dimensions, this solution is always either zero- or two-dimensional, defining a discrete set of fixed points or a fixed torus, respectively.

Following that, the conjugacy classes of P give rise to the constructing elements of the orbifold. However, since the fundamental domain of the orbifold $\mathcal{O} = \mathbb{T}/G$ is only a subset of the fundamental torus, one has to be careful not to over-count constructing elements. Two elements $g_1,\ g_2 \in S$ of the space group are called conjugate to each other iff $\exists h \in S \;:\; h g_1 h^{-1} = g_2$, i. e. when they lie in the same conjugacy class of the space group. Fixed loci of elements from the same conjugacy class are identified on the orbifold. This can easily be seen by taking elements g_1 and g_2 as above which leave points f_1 and f_2 fixed:

$$g_1 f_1 \;=\; f_1 \qquad \text{and} \qquad g_2 f_2 \;=\; f_2 \;. \tag{3.77}$$

Now, since $g_2 = h g_1 h^{-1}$, we have $(h^{-1} f_2) = (h^{-1} g_2)\, f_2 = (g_1 h^{-1})\, f_2 = g_1\, (h^{-1} f_2)$. Therefore, $f_1 = h^{-1} f_2$ and thus, f_1 and f_2 are identified after orbifolding.

Once all inequivalent constructing elements are obtained, we could start creating their associated matter spectra. Each constructing element $g = (\vartheta, \lambda)$ defines a boundary condition for a string closing on $\mathcal{O}$,

$$X(t, \sigma + 2\pi) = g X(t, \sigma) = \vartheta X(t, \sigma) + 2\pi\lambda \;, \tag{3.78}$$

where we normalised the string length ℓ to the dimensionless value 2π, compare the boundary condition in Equation (2.1) for a string living in the bulk, i. e. with trivial constructing element.

Twisted matter invariance

Now, with each constructing element g, the full spectrum of massless twisted strings on the orbifold can be constructed. However, not all states obtained this way survive the orbifold projection conditions. In layman's terms, the Hilbert space of massless strings needs to be projected on an invariant subspace regarding all with g commuting space group elements h, i. e. $gh = hg$. The set of all such h is the centraliser of g. Since g and all elements of its centraliser commute, they can pairwise be diagonalised simultaneously.

Let $g \in [g]$ be a representant of a conjugacy class of S. It will be our constructing element. We now distinguish two cases by the dimension of the nullspace—and thus the fixed point set—of g.

If the nullspace of g is zero-dimensional, it possesses a fixed point, which relates to a singularity of curvature on the orbifold. There, the $\mathcal{N} = 1$ supersymmetry in ten dimensions is broken down to $\mathcal{N} = 1$ in four dimensions. Such a point contributes one twisted **27**-plet of E_6 to the spectrum, which is related to one twisted Kähler modulus and therefore a contribution of 1 to $h_{\mathrm{T}}^{(1,1)}$ and 0 to $h_{\mathrm{T}}^{(2,1)}$. Finally, the Hilbert space of $[g^{-1}]$ contributes the CPT partner of $[g]$ and therefore needs not to be considered independently in the computation.

The only other possibility for the nullspace of g is to be two-dimensional[13]: in that case, g has a fixed torus and therefore, the spectrum on this torus has $\mathcal{N} = 1$ supersymmetry in six dimensions, corresponding to $\mathcal{N} = 2$ supersymmetry in four dimensions. There, in standard embedding an E_7 gauge group factor survives and yields a twisted **56**-dimensional multiplet, giving either a $\mathcal{N} = 2$ hypermultiplet or a “half-hypermultiplet”, depending on the reality of the representation. If the conjugacy classes of g and g^{-1} are identical, $[g] = [g^{-1}]$, then the **56**-plet is real (meaning a vanishing U(1) charge) and transforms as a half-hypermultiplet. Otherwise, $[g^{-1}]$ contributes a second left-chiral superfield, transforming in the complex conjugated representation, which combines with the first to a full ($\mathcal{N} = 2$) hypermultiplet. Once the fixed torus gets compactified, the **56**-plet splits into $\mathbf{27} \oplus \overline{\mathbf{27}} \oplus \mathbf{1} \oplus \mathbf{1'}$. Therefore, g yields one twisted **27**-plet and one twisted $\overline{\mathbf{27}}$-plet, which would result in one Kähler modulus and one complex structure modulus.

However, the projection on orbifold-invariant states might remove some of these states from the spectrum. If a centraliser element of g exists, which breaks $\mathcal{N} = 1$ supersymmetry in six dimensions to $\mathcal{N} = 1$ supersymmetry in four dimensions, i. e. if it leaves a different torus or no torus at all fixed, then the $\overline{\mathbf{27}}$-plet is projected out of the spectrum and thus the corresponding complex structure modulus ceases to exist. If this is indeed the case, then the conjugacy class $[g]$ contributes $(1, 0)$ to the twisted Hodge numbers $\left(h_{\mathrm{T}}^{(1,1)}, h_{\mathrm{T}}^{(2,1)}\right)$. Otherwise, if all elements of the centraliser of g maintain six-dimensional supersymmetry, both the **27**- and the $\overline{\mathbf{27}}$-plet survive the orbifold projection and so the contribution to the twisted Hodge numbers is $(1, 1)$.

Notice that the above derivation only uses geometrical data from the space group and does not carry out the thought of standard embedding directly. This has the benefit of being able to compute the Hodge numbers without the need for a complete

[13]The other thinkable cases are excluded by the supersymmetry condition, see Section 3.6.

analysis of the resulting particle spectrum in four dimensions; the latter being a feat which has only recently being achieved on general grounds [18]. The results for all 520 orbifolds which yield $\mathcal{N} \geq 1$ supersymmetry in four dimensions, first published in [8] for Abelian point groups and [9] for non-Abelian ones are tabulated in appendix B. There, we find heuristically that most space groups satisfy the rule

$$h^{(1,1)} - h^{(2,1)} = 0 \bmod 6 \ , \tag{3.79}$$

for which we have no explanation or intuition yet, cf. [39]. The Euler characteristic $\chi = 2\left(h^{(1,1)} - h^{(2,1)}\right)$, which is related to the number of generations of chiral matter in four dimensions thus hints at realistic models, once discrete Wilson lines are added to the gauge embedding [22, 69].

Chapter 4

Running of gauge couplings

In this chapter, we analyse the predictions for gauge coupling unification obtained from heterotic string theory. After a short recapitulation of β-function coefficients in supersymmetric and non-supersymmetric scenarios, we will first demonstrate the principle of our calculation at a toy model with one extra dimension. There we will find that a Kaluza Klein tower of states, produced by the compact extra dimension, leads to a zig-zag pattern in the differential running of gauge couplings, yielding precision gauge coupling unification at high energies.

Afterwards we will try to apply the same principle to a real string model, known as the Blaszczyk model, which we compactify anisotropically with two large extra dimensions. Although we will find a similar pattern in the differential running, the beautiful picture of the toy model will be thwarted by the two-dimensional nature of the KK-tower.

4.1 β-function coefficients

In field theory, gauge couplings are not constant, but depend on the energy scale at which they are measured. Their behaviour, or running, is described by the so–called β-function [3, 70]

$$\beta(g) \;:=\; Q' \left.\frac{\partial g^2}{\partial Q'}\right|_Q \,, \tag{4.1}$$

where g is the coupling in question and Q the energy at which it is measured. The left–hand side of (4.1) can be computed by calculating loop–corrections to gauge boson propagators and evaluates to

$$\beta(g) \;=\; \frac{g^4}{8\pi^2} b \,, \tag{4.2}$$

where b is the relevant β-function coefficient. For a non–supersymmetric theory, b equals

$$-\frac{11}{3}C_2(G) + \frac{2}{3}\sum_i \ell_i(\mathcal{R}) + \frac{1}{3}\sum_\alpha \ell_\alpha(\mathcal{R}) \,, \tag{4.3}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group and $\ell(\mathcal{R})$ is the Dynkin index of a representation $\mathcal{R}$. The index i runs over all left–chiral fermions and the α over all complex scalar fields in the theory. In a $\mathcal{N} = 1$ supersymmetric non-Abelian gauge theory, fermions and scalars come in chiral multiplets with equal degrees of freedom each. This, together with some additional gauge-loops modifies (4.3) to

$$b \;=\; -3C_2(G) + \sum_i \ell_i(\mathcal{R}) \,, \tag{4.4}$$

where the index i runs over the chiral superfields in question. Finally, in a $\mathcal{N} = 2$ supersymmetric theory, each vector superfield is accompanied by a chiral superfield and chiral hyperfields come in sets of two $\mathcal{N} = 1$ chiral superfields. Together, this leads to

$$b \;=\; -2C_2(G) + 2\sum_i \ell_i(\mathcal{R}) \,. \tag{4.5}$$

Here, the index i runs over all chiral hyperfields.

4.2 $S^1/\mathbb{Z}_2$ orbifold

4.2.1 Construction

We consider a supersymmetric SU(5) gauge theory with minimal matter content in five–dimensional spacetime with coordinates x^μ, $\mu \in \{0, 1, 2, 3\}$ and $y = x^5$, where the x^μ describe the classical Minkowski space $\mathbb{M}^{1,3}$ and the fifth dimension is taken to be compactified on a circle S^1 with radius R. Such a setting has has been explored in the literature, e. g. [71, 72]. Therefore, a generic field $\Phi(x^\mu, y)$ can be Fourier expanded as

$$\Phi(x^\mu, y) \;=\; \sum_{n\in\mathbb{Z}} \phi^{(n)} \mathrm{e}^{\mathrm{i}ny/R} \,. \tag{4.6}$$

The Fourier–modes in (4.6) have KK–masses $|n|/R$ as can be easily verified by evaluating the Klein-Gordon equation for them. Thus they are degenerate, with two complete states with $\pm n$ at each mass level $|n|/R$. We now divide out a shift $\tau :\; y \mapsto y + \pi R$. This shift is accompanied by a parity transformation $\mathcal{P}$ which acts on the fermionic and gauge degrees of freedom. Notice that $\mathcal{P}$ itself needs not to be an element of the gauge group. We choose $\mathcal{P}$ to act as the identity transformation on the matter fields and as diag$(-,-,-,+,+)$ on the gauge supermultiplet. Clearly, $\mathcal{P}$ has eigenvalues ± 1. The projection condition on the fields reads as

$$\Phi(\tau(y)) \;=\; \mathcal{P}\Phi(y)\;. \tag{4.7}$$

Here, we suppressed the dependence on the Minkowski space coordinates. Expanding (4.7) yields

$$\sum_{n\in\mathbb{Z}} \phi^{(n)} \mathrm{e}^{\mathrm{i}ny/R}(-1)^n \;=\; \mathcal{P}\left(\sum_{n\in\mathbb{Z}} \phi^{(n)} \mathrm{e}^{\mathrm{i}ny/R}\right)\;. \tag{4.8}$$

Hence, for fields even/odd under $\mathcal{P}$, we find $\phi^{(n)}(-1)^n \;=\; \pm\phi^{(n)}$ and therefore $\phi_+^{(2n+1)} \;=\; \phi_-^{(2n)} \;=\; 0$.

4.2.2 Theory content and breaking

We consider an SU(5) theory with minimal content: a **24**-plet A containing gauge bosons, a **5**-plet H and a $\bar{\mathbf{5}}$-plet H^C containing the Higgs, as well as N_f $\overline{\mathbf{10}}$- and **5**-plets ψ containing N_f generations of chiral matter. $\mathcal{P}$ acts on the 24 SU(5)-generators T^a as following:

$$\mathcal{P}T^a\mathcal{P}^{-1} \;=\; T^a\;, \qquad \mathcal{P}T^{\hat{a}}\mathcal{P}^{-1} \;=\; -T^{\hat{a}}\;; \tag{4.9}$$

here, indices a run over the 12 subgroup-generators of $G_{\mathrm{SM}} \;=\; \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, while indices $\hat{a}$ run over the remaining SU(5)-generators. This breaks SU(5) to the standard model and our fields as depicted in table 4.1.

Note that neither the Higgs triplets nor the SU(5)/G_{SM} gauge multiplets acquire zero-modes. Therefore, the doublet-triplet problem is solved in an elegant manner. Also, all chiral matter ψ has always positive parity under $\mathcal{P}$ and thus stays in complete SU(5) multiplets at all Kaluza-Klein levels. Therefore, they will not contribute to a differential running and can be ignored in our analysis in the next subsection.

4.2.3 Differential running

Equating (4.1) and (4.2) and integrating over Q' from a reference scale Q_0 at which the couplings are measured to Q, we find

$$g^2(Q) \;=\; g^2(Q_0)\left[1-\frac{b\,g^2(Q_0)}{8\pi^2}\ln\frac{Q}{Q_0}\right]^{-1}\;. \tag{4.10}$$

4D field	SU(5)	$G_{\rm SM}$	$\mathcal{P}$-parity
A^a_μ		$(\mathbf{8},\mathbf{1})_0+(\mathbf{1},\mathbf{3})_0+(\mathbf{1},\mathbf{1})_0$	+
$A^{\hat{a}}_5$	**24**	$+(\mathbf{3},\mathbf{2})_{-5/6}+(\overline{\mathbf{3}},\mathbf{2})_{5/6}$	-
H		$(\mathbf{1},\mathbf{2})_{1/2}$	+
	5	$+(\mathbf{3},\mathbf{1})_{1/3}$	-
H^C		$(\mathbf{1},\mathbf{2})_{-1/2}$	+
	$\overline{\mathbf{5}}$	$+(\overline{\mathbf{3}},\mathbf{1})_{1/3}$	-
$\psi_{\overline{10}}$	$\overline{\mathbf{10}}$	$(\mathbf{1},\mathbf{1})_1+(\overline{\mathbf{3}},\mathbf{1})_{-2/3}+(\mathbf{3},\mathbf{2})_{1/6}$	+
ψ_5	**5**	$(\mathbf{1},\mathbf{2})_{1/2}+(\mathbf{3},\mathbf{1})_{-1/3}$	+

Table 4.1: Field content of the $S^1/\mathbb{Z}_2$ model.

We will now concentrate on the differential running

$$\alpha_{ij}(Q) \;:=\; g_i^{-2}(Q)-g_j^{-2}(Q) \;=\; \frac{1}{g_i^2(Q_0)}-\frac{1}{g_j^2(Q_0)}+\frac{b_j-b_i}{8\pi^2}\ln\frac{Q}{Q_0}\,. \tag{4.11}$$

We will in the following only take on-shell contributions to the β-functions into account, i. e. we only allow KK-modes with masses below the available energy Q to contribute. In our model, (4.11) holds true for energies above $M_Z = Q_0$ and below the compactification scale M_c. This is the same running as in the MSSM. At M_c, KK-modes with $n = \pm 1$ get "turned on" and, since full multiplets of the GUT-group cannot contribute to the differential running, precisely invert this running until $Q = 2M_c$, where α_{ij} reads

$$\alpha_{ij}(2M_c) \;=\; \alpha_{ij}(M_Z)+\frac{b_j-b_i}{8\pi^2}\ln\frac{M_c}{2M_Z}\,. \tag{4.12}$$

The complete picture (as can be seen in figure 4.1) is one where the α_{ij} run up and down, switching directions at every KK–level:

$$\alpha_{ij}(n\cdot M_c) \;=\; \alpha_{ij}(M_Z)+\frac{b_j-b_i}{8\pi^2}\ln\left[\prod_{0<m\le n-1}\left(\frac{2m+(-1)^m+1}{2m+(-1)^{m+1}+1}\right)\frac{M_c}{M_Z}\right]\,, \tag{4.13}$$

which, for even $n = 2k$ simplifies to

$$\alpha_{ij}(2k\cdot M_c) \;=\; \alpha_{ij}(M_Z)+\frac{b_j-b_i}{8\pi^2}\ln\left[\begin{pmatrix}2k\\k\end{pmatrix}^2\frac{2k+1}{2^{4k}}\frac{M_c}{M_Z}\right] \tag{4.14}$$

$$\stackrel{k\to\infty}{\longrightarrow}\alpha_{ij}(M_Z)+\frac{b_j-b_i}{8\pi^2}\ln\left(\frac{2M_c}{\pi M_Z}\right)\,. \tag{4.15}$$

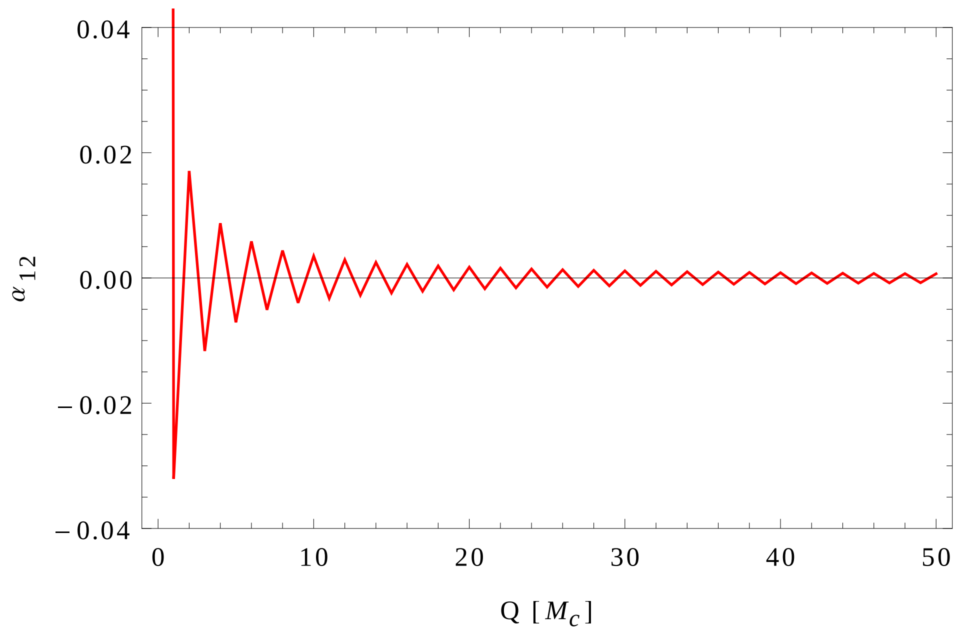


Figure 4.1: Differential running of couplings in the $S^1/\mathbb{Z}_2$ model. Since the β-function coefficients precisely invert at each KK-level, α_{12} converges.

We now demand unification at high energies, i. e. $\alpha_{12}(Q) \xrightarrow{Q\to\infty} 0$, which, in turn, yields us a prediction of the compactification scale, which can be easily read off (4.15) as $M_c = {}^{\pi}/_{2} \cdot M_{MSSM}$, where M_{MSSM} is the GUT-scale of the MSSM. If we take α_{12} as the benchmark for our fit and $g_1^2(M_Z) = {}^5/_3 \cdot g_Y^2(M_Z) = {}^5/_3 \cdot 0.1277$ and $g_2^2(M_Z) = 0.424$ [70], we find

$$M_c \approx 3.35 \times 10^{16}\,\text{GeV}\;. \tag{4.16}$$

4.3 The Blaszczyk model

4.3.1 Construction

We now focus on an anisotropic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ orbifold model with a freely acting shift, known in the literature as the Blaszczyk model [73]. The full gauge group $E_8 \times E_8$ of the heterotic string gets broken by ϑ and Wilson lines perpendicular to the large extra dimensions to SU(6), which is then further reduced by ω to SU(5) and finally broken down to the Standard Model gauge group $G_{\text{SM}} = \text{SU}(3)\times\text{SU}(2)\times\text{U}(1)$ times some hidden U(1) factors by τ. We obtained the massless spectrum in the field theoretic limit of this orbifold model using the C++ orbifolder [19]; the untwisted

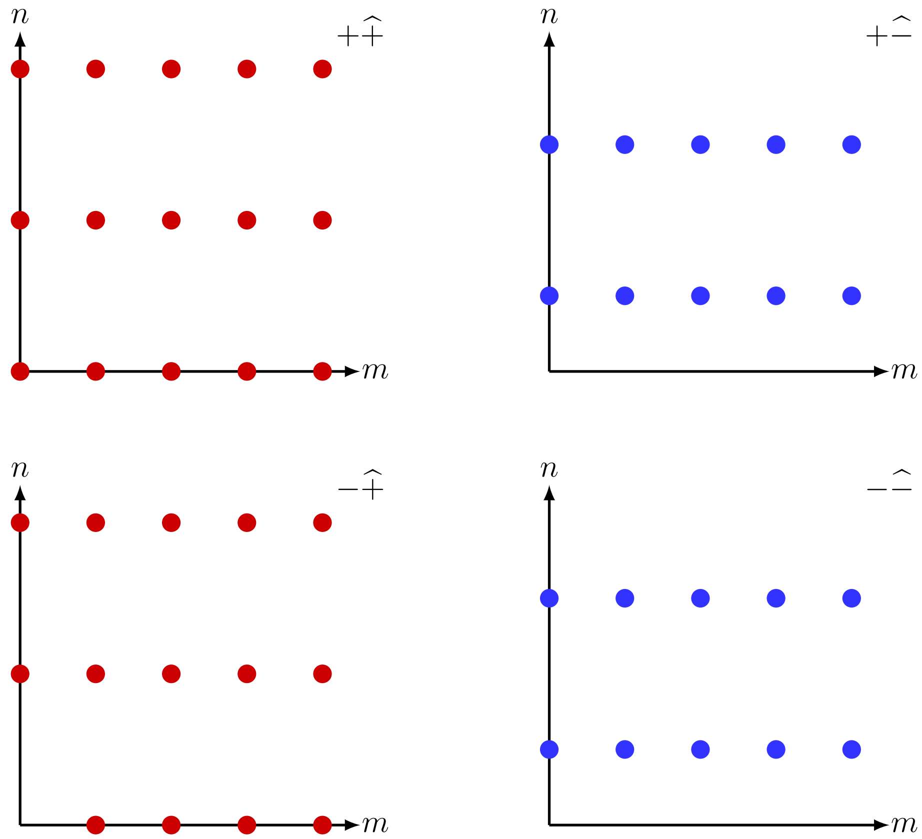


Figure 4.2: KK modes of fields on the Blaszczyk geometry. Only fields which are even under both parities possess zero-modes.

part is summed up in table 4.2. In the first torus $(x_5, x_6) \equiv (x_5+2\pi m R_5, x_6+2\pi n R_6)$, the orbifold actions ϑ, ω and τ act as

$$\vartheta : x_5 \mapsto x_5, \quad x_6 \mapsto x_6 \,, \tag{4.17}$$

$$\omega : x_5 \mapsto -x_5, \quad x_6 \mapsto -x_6 \,, \tag{4.18}$$

$$\tau : x_5 \mapsto x_5, \quad x_6 \mapsto x_6 + \pi R_6 \,. \tag{4.19}$$

A general Fourier expansion of a field on the two torus coordinates x_5 and x_6 looks like

$$\varphi(x_5, x_6) = \frac{1}{\sqrt{2\pi R_5 R_6}} \sum_{m,n\in\mathbb{Z}} \varphi^{(m,n)} \mathrm{e}^{\mathrm{i}\left(\frac{m x_5}{R_5} + \frac{n x_6}{R_6}\right)} \,. \tag{4.20}$$

Enforcing the orbifold boundary conditions

$$\omega : \varphi_{\pm\hat{\pm}}(-x_5, -x_6) \;=\; \pm\varphi_{\pm\hat{\pm}}(x_5, x_6), \quad \text{and} \tag{4.21}$$

$$\tau : \varphi_{\pm\hat{\pm}}(x_5, x_6 + \pi R_6) \;=\; \hat{\pm}\varphi_{\pm\hat{\pm}}(x_5, x_6) \,, \tag{4.22}$$

we obtain

$$\varphi_{\pm\hat{\pm}}(x_5, x_6) = \frac{1}{4\sqrt{2\pi R_5 R_6}}. \tag{4.23}$$

$$\sum_{m,n\in\mathbb{Z}} \left(\varphi^{(m,n)} \pm \varphi^{(-m,-n)}\right) \left(1\hat{\pm}(-1)^n\right) \mathrm{e}^{\mathrm{i}\left(\frac{mx_5}{R_5}+\frac{nx_6}{R_6}\right)} . \tag{4.24}$$

These modes have squared masses

$$(\mu)^2 = \left(\frac{m}{R_5}\right)^2 + \left(\frac{n}{R_6}\right)^2 . \tag{4.25}$$

We additionally introduce the quotient of the compactification radii as a new parameter ζ,

$$\zeta = \frac{R_6}{R_5} , \tag{4.26}$$

and introduce the dimensionless Kaluza Klein mass parameter M

$$M^2 := \mu^2 R_6^2 = \underbrace{\left(\frac{R_6}{R_5}\right)^2}_{\zeta^2} m^2 + n^2 , \tag{4.27}$$

which measures the KK–mass in units of the inverse radius R_6.

4.3.2 Theory content and breaking

Since all fields originating from twisted sectors of the full string theory come in complete representations of the GUT group, we can drop them from our analysis of the differential running of couplings. The spectrum of the untwisted sector is summarised in Table 4.2.

In the following, we will assume a mechanism, which gives all matter which does not arise from a **24**–plet of SU(5), i. e. all fields which were not present in the toy model (cf. Table 4.1), a mass v/R_6, which we also denote in multiples of R_6^{-1} for later convenience. This mechanism could for instance be achieved by giving a VEV to some extra-dimensional component of the gauge field. This removes the zero-modes of these fields from the massless spectrum and, depending on v some of the higher KK-modes too. But at very high energies, these fields might start contributing to the differential running.

SU(6) field		$SU(3) \times SU(2) \times U(1)$ fields and parities
$\mathcal{N}=2$	$\mathcal{N}=1$	under ω and τ
$\mathbf{35_V}$	$\mathbf{35_V}$	$(\mathbf{8},\mathbf{1})_0^{++} + (\mathbf{1},\mathbf{3})_0^{++} + (\mathbf{1},\mathbf{1})_0^{++} + (\mathbf{3},\mathbf{2})_{-5/6}^{+-} + (\overline{\mathbf{3}},\mathbf{2})_{5/6}^{+-} + \left[(\mathbf{1},\mathbf{2})_{1/2}^{-+} + (\mathbf{3},\mathbf{1})_{-1/3}^{--} + (\mathbf{1},\mathbf{2})_{-1/2}^{-+} + (\overline{\mathbf{3}},\mathbf{1})_{1/3}^{--} + (\mathbf{1},\mathbf{1})_0^{++}\right]$
	$\mathbf{35_C}$	$(\mathbf{1},\mathbf{2})_{1/2}^{++} + (\mathbf{3},\mathbf{1})_{-1/3}^{+-} + (\mathbf{1},\mathbf{2})_{-1/2}^{++} + (\overline{\mathbf{3}},\mathbf{1})_{1/3}^{+-} + \left[(\mathbf{1},\mathbf{3})_0^{-+} + (\mathbf{8},\mathbf{1})_0^{-+} + (\mathbf{1},\mathbf{1})_0^{-+} + (\mathbf{3},\mathbf{2})_{-5/6}^{--} + (\overline{\mathbf{3}},\mathbf{2})_{5/6}^{--} + (\mathbf{1},\mathbf{1})_0^{-+}\right]$
$\mathbf{6_H}$	$\mathbf{6_C}$	$\left[(\mathbf{1},\mathbf{2})_{1/2}^{+-} + (\mathbf{3},\mathbf{1})_{-1/3}^{++} + (\mathbf{1},\mathbf{1})_0^{+-}\right]$
$(\times 2)$	$\overline{\mathbf{6}}_\mathbf{C}$	$\left[(\mathbf{1},\mathbf{2})_{-1/2}^{+-} + (\overline{\mathbf{3}},\mathbf{1})_{1/3}^{++} + (\mathbf{1},\mathbf{1})_0^{+-}\right]$

Table 4.2: Untwisted field content in SU(6) and SM language. Fields in square brackets are additional fields not present in the SU(5) toy model.

4.3.3 Differential running

To compute the contributions to the β-function coefficients of fields with various parities, we remember that vector superfields contribute with -3 times the Dynkin index of their representation (which coincides with the value of the Casimir operator for fields in the adjoint), whereas chiral superfields contribute with one times their Dynkin index. For the Abelian U(1) subgroup, we simply have to sum the square of the hypercharge. Two remarks are in order: first, note that we do not need to take the twisted sector into account, since all twisted fields come in complete GUT-multiplets and therefore do not contribute to the differential running. Second, remember the normalisation factor of 3/5 between b_Y and b_1 to correctly compute the β-function coefficients.

The contributions from the gauge **35**-plet are:

35	b_1	b_2	b_3	(b_1-b_2)	(b_2-b_3)	(b_3-b_1)
$++$	$3/5$	-5	-9	$28/5$	4	$-48/5$
$-+$	$-9/5$	-1	3	$-4/5$	-4	$24/5$
$+-$	$-73/5$	-9	-5	$-28/5$	-4	$48/5$
$--$	$38/10$	3	-1	$4/5$	4	$-24/5$
Σ	-12	-12	-12	0	0	0

Since the zeroth Kaluza Klein level $(m,n) = (0,0)$ only has a $++$-mode (cf. Figure 4.2), the first row in this table ought to reproduce the MSSM running, which is easily verified.

The contributions from the hyper **6**-plets read:

$\mathbf{6}$, $\overline{\mathbf{6}}$	b_1	b_2	b_3	$(b_1 - b_2)$	$(b_2 - b_3)$	$(b_3 - b_1)$
$++$	$2/5$	0	1	$2/5$	-1	$3/5$
$-+$	0	0	0	0	0	0
$+-$	$3/5$	1	0	$-2/5$	1	$-3/5$
$--$	0	0	0	0	0	0
Σ	-1	-1	-1	0	0	0

These contributions have to be multiplied by two, since we get two hyper **6**–plets.

To sum the contributions at a specific energy scale M/R_6, we follow this procedure:

1. Generate all tuples $(m, n) \in \mathbb{Z}^2$ with $|m|, |n| \leq M$. These are sufficient for $\zeta \geq 1$.

2. Select all tuples for which $\sqrt{(\zeta|m|)^2 + |n|^2} \leq M$ (SU(5) matter, not in brackets in Table 4.2) or
$\sqrt{v^2 + (\zeta|m|)^2 + |n|^2} \leq M$ (bracketed in Table 4.2) holds.

3. For each such pair, extract the contributing parities from figure 4.2.

4. Use the small tables above to look up the contribution from this tuple.

From this we see that, as in the toy model, a reversal of the differential running at consecutive Kaluza Klein levels is achieved. However, since the contributions from successive levels do grow in absolute size, an additional "drift" in the differential running can be observed and thus precision gauge unification is lost again at higher energies, cf. Figure 4.3.

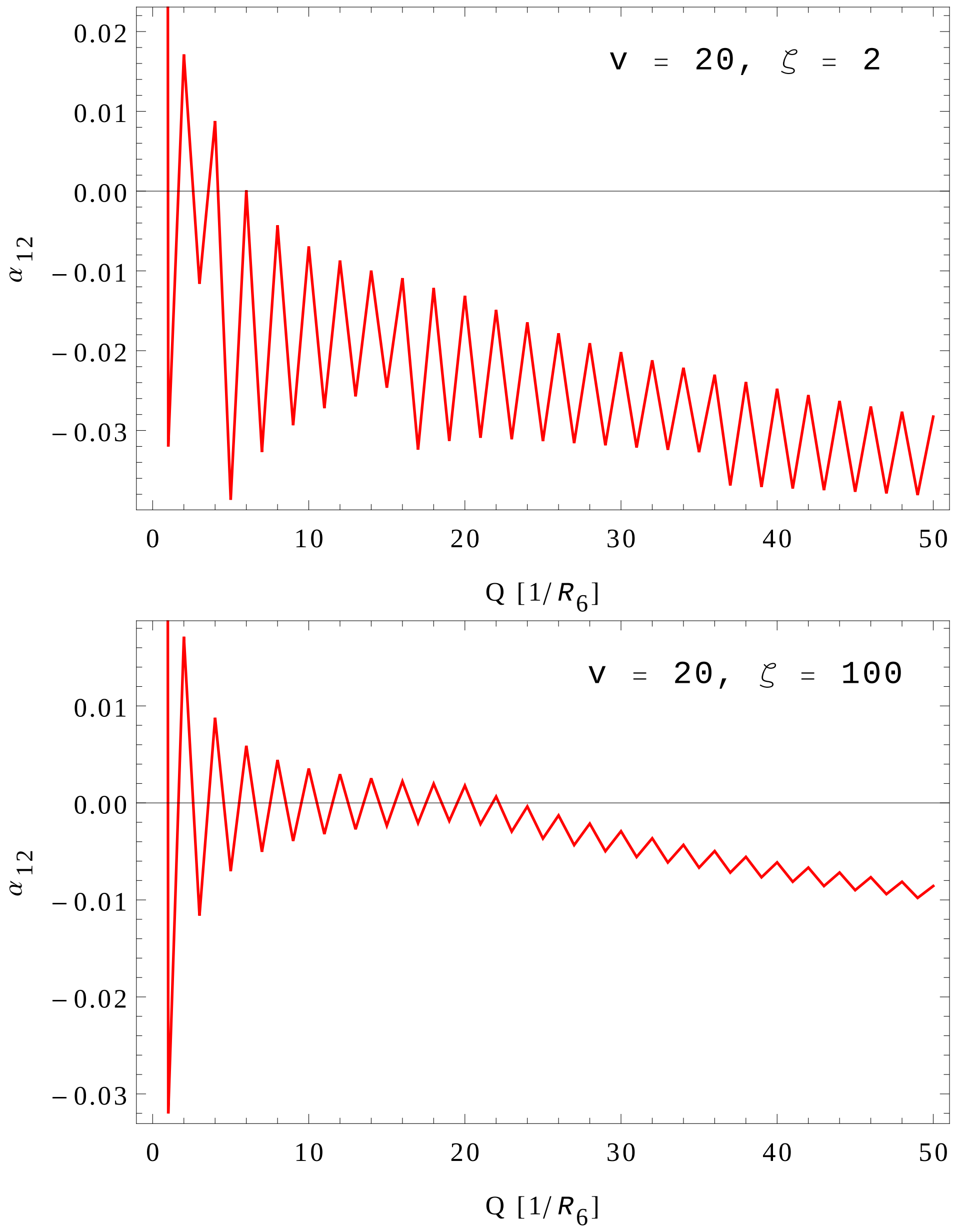


Figure 4.3: The differential running of the Blaszczyk Model. For higher values of $\zeta = R_6/R_5$, the running resembles the toy model more closely. Once the non-SU(5)-states are available, an additional shift is introduced into the running, since the contribution of the ++-mode at the lowest level has no counterpart.

Chapter 5

Conclusion

In this thesis we have studied orbifold compactifications of the heterotic string. After a short introduction to the latter, where we mentioned the possibility of compactifying on Calabi-Yau manifolds, we focused on orbifold compactifications and, for the time being, put physics aside and ventured into the purely mathematical aspects of orbifolds. We soon found the general definition too tedious to work with and constrained ourselves to orbifolds which stem from genus one tori by modding out an additional discrete symmetry group. We closely examined their building blocks, namely the space group and its constituents, the lattice and the point group as well as the orbifolding group for cases where roto-translations are involved. After looking into some basic examples, were we made use of the augmented matrix notation, we took on the task of classifying all possible six-dimensional toroidal orbifolds with genus one. In order to do so, we reviewed crystallographic textbook knowledge about affine, $\mathbb{Z}$- and $\mathbb{Q}$-classes of unimodular groups, how they are related to space groups, how they are defined and which geometric properties of the resulting orbifold spaces are invariants within one such class. At this point we took the opportunity to make a short remark about the common practice of naming torus lattices after the Lie algebras whose simple roots span said lattice and why we find that practice problematic. We also illustrated the difference between these types of classes with some example space groups that were equivalent in one class but not another.

We then realised that in order to classify orbifolds in some dimension, we would have to classify all affine classes of that degree, the first step of which would be the classification of subgroups of $\mathrm{GL}(n, \mathbb{Z})$. Luckily, an old theorem by Zassenhaus stated the finiteness of the problem and therefore the feasibility of a counting approach. We then utilised the powerful computer program Carat which comes pre-packed with a list of all $\mathbb{Q}$-classes in six-dimensions, which were obtained from splitting irreducible maximally finite, or short i. m. f. subgroups of $\mathrm{GL}(6, \mathbb{Z})$. Carat also provided us with algorithms to split these $\mathbb{Q}$-classes into $\mathbb{Z}$- and affine classes. The great benefit of using Carat was not only that it did all the heavy lifting for us, but also that it assured us we would not miss out on some of the classes. We mentioned previous work in the field and how, in the past, due to insufficient understanding of crystallography, exactly that had happened.

We then returned to physics and proceeded to develop the means to compute the number of conserved supersymmetry generators on the classes of orbifolds CARAT gave us. Here the problem boiled down to deciding whether the point group is a subgroup of SU(3). We developed two strategies of doing this; the first one made use of the accidental isomorphism $\mathrm{SO}(6) \to \mathrm{SU}(4)$ which could be implemented explicitly using some Clifford algebra machinery. The second method skipped that machinery and directly used the character tables of the point group to achieve results. This enabled us to compile a complete list of all toroidal genus one orbifolds, Abelian as well as non-Abelian, which yield $\mathcal{N} \geq 1$ SUSY in four dimensions in symmetric compactifications. Such a list has not existed before. Its significance becomes clear once one realises that if nature can indeed be described by a heterotic string theory compactified on a symmetric orbifold, then that orbifold will necessarily be found in our list. We also computed the fundamental groups and cohomology Hodge numbers for these orbifolds – results which where mostly non-existent for non-Abelian orbifolds before.

Finally, we had a deeper look into the so-called Blaszczyk-model, which served as a demonstrator for the zig-zag behaviour the differential running of the gauge couplings develop above the compactification scale. We first stated the principle in a toy model and then showed the behaviour in the full string construction of the Blaszczyk-model and briefly explored its consequences for gauge-coupling unification.

As important as these results are, much work lies still ahead: the C++ orbifolder for instance, can, at the moment, only handle orbifolds with Abelian point groups. Although the principal mechanisms of how to compute the spectra of orbifolds with non-Abelian point groups have been devised in the meantime, they have not yet been implemented into the orbifolder and hence they are still not broadly accessible.

In the past years, asymmetric orbifold constructions gained heightened interest. There, the left- and the right-moving part of the string are compactified on almost completely independent geometries, which opens up a whole new plethora of possibilities, e. g. with gauge groups $\subset \mathrm{SO}(44)$, while, at the same time simplifying the description of the setting, by describing the geometry as well as the gauge-embedding with just one big matrix. Classifying these is, however, unfortunately an enormous task, since the size of the problem grows to up to 28 dimensions. Although the i. m. f. groups are known in that dimension, the currently available computational power renders the naive attempt of extending the previous analysis to that dimension unrealistic. Also, a complete knowledge of all 28-dimensional space groups will not be needed, since the asymmetric construction installs harsh constraints on many parts of the matrix. A more sophisticated approach would thus try to incorporate these constraints, notably the Narrain condition for the gauge lattice, directly into the sublattice algorithm of Carat to vastly reduce the amount of $\mathbb{Z}$-classes one has to consider. Also, Carat would need to be rewritten to incorporate long integer arithmetic as well as the possibility to parallelise. However, if this approach will reduce the number of candidate $\mathbb{Z}$-classes to something, that can be handled in a meaningful way remains to be seen.

Lastly, the possibility of supersymmetry already being broken at the string scale, i. e. $\mathcal{N} = 0$ SUSY in four dimensions from the model builder's perspective, has not been explored in great detail in the past. These models would be constructed from any six-dimensional orbifold not in our list of SUSY-preserving geometries, cf. [74].

Appendix A

Two-dimensional orbifolds

To illustrate the concepts from Chapter 3 we present here the list of two-dimensional space groups and the resulting quotient spaces, or orbifolds. This is taken from appendix B of [8].

The possible orders m of (irreducible) point group elements in n dimensions are given by the equation

$$\phi(m) \ \leq \ n \ , \tag{A.1}$$

where ϕ is the Euler ϕ–function. For dimension two, this leaves only elements with order in $\{1, 2, 3, 4, 6\}$ as possible point group elements.

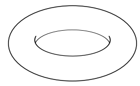
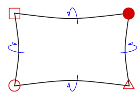
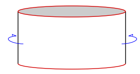
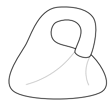
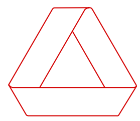
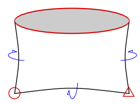
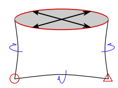
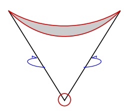
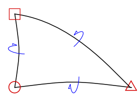
label of $\mathbb{Q}$–class	# of $\mathbb{Z}$–classes	# of affine classes
id	1	1
$\mathbb{Z}_2$–I	1	1
$\mathbb{Z}_2$–II	2	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \ \cong D_2$	2	4
$\mathbb{Z}_4$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_4 \ \cong D_4$	2	2
$\mathbb{Z}_3$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_3 \ \cong S_3 \cong D_3$	2	2
$\mathbb{Z}_6$	1	1
$\mathbb{Z}_2 \ltimes \mathbb{Z}_6 \ \cong D_6$	1	1

Table A.1: $\mathbb{Q}$–classes in two dimensions.

One can classify the 17 two-dimensional space groups by their $\mathbb{Q}$-classes. Those can be found in Table A.1. There, D_n is the dihedral group of order $2n$ and S_n is the symmetric group of order $n!$. In Table A.2 the specific information of every affine class is shown: the $\mathbb{Q}$-, $\mathbb{Z}$- and affine class to which they belong, its Bravais type of lattice, its orbifolding group generators in augmented matrix notation and a name, description and image of its topology.

Sometimes it is of interest to know the fundamental groups of the resulting orbifolds. Among the two-dimensional space groups, most of the fundamental groups are trivial with the following exceptions: the torus has a fundamental group of $(\mathbb{Z})^2$, the pipe and the Möbius strip $\mathbb{Z}$, the cross-cap pillow (a projective plane) $\mathbb{Z}_2$ and the Klein bottle's one is its own space group, with group structure

$$S \;=\; \left\{a^n b^m \mid m, n \in \mathbb{Z}\,,\; b\,a \;=\; a^{-1}\,b\right\}\;. \tag{A.2}$$

$\mathbb{Q}$-$\mathbb{Z}$-aff. class lattice	generators	name & description	image
id-1-1 Oblique		Torus Manifold	
$\mathbb{Z}_2$-I-1-1 Oblique	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Pillow Orbifold, 4 singularities with cone-angle π	
$\mathbb{Z}_2$-II-1-1 p-Rectangular	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Pipe Manifold, 2 boundaries	
$\mathbb{Z}_2$-II-1-2 p-Rectangular	$\left(\begin{array}{cc\|c} 1 & 0 & 1/2 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Klein bottle Manifold, non-orientable	
$\mathbb{Z}_2$-II-2-1 c-Rectangular	$\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Möbius strip Manifold, non-orientable, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$-1-1 p-Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Rectangle Manifold, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$-1-2 p-Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right)$	Cut pillow Orbifold, 2 singularities with cone-angle π, 1 boundary	
$\mathbb{Z}_2 \times \mathbb{Z}_2$-1-3 p–Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right)$	Cross-cap pillow Orbifold, 2 singularities with cone-angle π	
$\mathbb{Z}_2 \times \mathbb{Z}_2$–2–1 c-Rectangular	$\left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Jester's hat Orbifold, 1 singularity with cone-angle π, 1 boundary	
$\mathbb{Z}_4$-1-1 Square	$\left(\begin{array}{cc\|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$	Triangle pillow Orbifold, 2 singularities with cone–angle $\pi/2$, 1 singularity with cone-angle π	

continued . . .

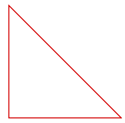
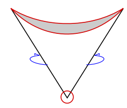
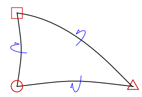
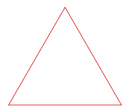
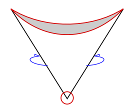
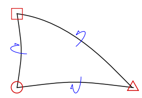
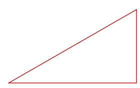
<table>
<tr><th>$\mathbb{Q}$-$\mathbb{Z}$-aff. class
lattice</th><th>generators</th><th>name & description</th><th>image</th></tr>
<tr><td>$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$-1-1
Square</td><td>$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Triangle
Manifold, one boundary, 1 angle of $\pi/2$ and 2 of $\pi/4$</td><td></td></tr>
<tr><td>$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$–1–2
Square</td><td>$\left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Jester's hat
Orbifold, 1 singularity with cone-angle $\pi/2$, 1 boundary</td><td></td></tr>
<tr><td>$\mathbb{Z}_3$-1-1
Hexagonal</td><td>$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Triangle pillow
Orbifold, 3 singularities with cone-angle $2\pi/3$</td><td></td></tr>
<tr><td>$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$-1-1
Hexagonal</td><td>$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Triangle
Manifold, 3 boundary, all angles $\pi/3$</td><td></td></tr>
<tr><td>$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$-2-1
Hexagonal</td><td>$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Jester's hat
Orbifold, 1 singularity with cone-angle $2\pi/3$, 1 boundary</td><td></td></tr>
<tr><td>$\mathbb{Z}_6$-1-1
Hexagonal</td><td>$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Triangle pillow
Orbifold, 3 singularities with cone-angles $2\pi/3$, $\pi/3$ and π</td><td></td></tr>
<tr><td>$\mathbb{Z}_2 \ltimes \mathbb{Z}_6$-1-1
Hexagonal</td><td>$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$</td><td>Triangle
Manifold, 1 boundary, with angles $\pi/2$, $\pi/3$ and $\pi/6$</td><td></td></tr>
</table>

Table A.2: List of all possible two-dimensional orbifolds. $\mathbb{Q}$–classes are separated by double lines.

Appendix B

Tabulation of results

In this appendix, we present the results of the classification procedure described in Chapter 3 together with the Hodge numbers arisinf from the different sectors and all non-trivial fundamental groups. We also provide some overview statistics. These results have been published in [8] (appendix C therein) and [9] (appendix A therein).

B.1 Summary

	$\mathcal{N}=1$			$\mathcal{N}=2$			$\mathcal{N}=4$			$\sum$		
	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.
Abelian	17	60	138	4	10	23	1	1	1	22	71	162
Non-Abelian	35	108	331	3	7	27	0	0	0	38	115	358
$\sum$	52	168	469	7	17	50	1	1	1	**60**	**186**	**520**

Table B.1: Distribution of $\mathbb{Q}$-, $\mathbb{Z}$- and affine classes of space groups which are consistent with at least $\mathcal{N}=1$ supersymmetry in four dimensions.

B.2 Fundamental groups

We only list affine classes of space groups with non-trivial fundamental groups. All affine classes not listed here have $\pi_1 = \{0\}$, i. e. are simply connected and thus do not admit non-local breaking of the GUT group.

$\mathbb{Q}$–class	$\mathbb{Z}$– and affine class	Hodge numbers $(h^{(1,1)}, h^{(2,1)})$	$\pi_1 = S/\langle F\rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1–3	$(11, 11)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	1–4	$(3, 3)$	S
	2–3	$(11, 11)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	2–4	$(11, 11)$	$\mathbb{Z}_2$
	2–5	$(7, 7)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	2–6	$(3, 3)$	S
	3–3	$(7, 7)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	3–4	$(3, 3)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	4–2	$(7, 7)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	5–1	$(27, 3)$	$\mathbb{Z}_2$
	5–2	$(11, 11)$	$\mathbb{Z}_2$
	5–4	$(7, 7)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	5–5	$(3, 3)$	S
	6–2	$(9, 9)$	$\mathbb{Z}_2$
	6–3	$(5, 5)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	7–2	$(7, 7)$	$\mathbb{Z}_2$
	9–1	$(17, 5)$	$\mathbb{Z}_2$
	9–2	$(7, 7)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	10–1	$(15, 3)$	$\mathbb{Z}_2$
	12–1	$(15, 3)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	12–2	$(9, 9)$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1–6	$(17, 5)$	$\mathbb{Z}_2$
	2–4	$(17, 5)$	$\mathbb{Z}_2$
	3–6	$(14, 2)$	$\mathbb{Z}_2$
	4–4	$(15, 3)$	$\mathbb{Z}_2$
	6–5	$(14, 2)$	$\mathbb{Z}_2$
	8–3	$(13, 1)$	$\mathbb{Z}_2$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1–4	$(12, 0)$	$\mathbb{Z}_3$
	2–4	$(12, 0)$	$\mathbb{Z}_3$
	3–3	$(12, 0)$	$\mathbb{Z}_3$
	4–3	$(12, 0)$	$\mathbb{Z}_3$
S_3	1–2	$(6, 6)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
	2–2	$(6, 6)$	$\mathbb{Z}_3$
	3–2	$(6, 6)$	$\mathbb{Z}_3$
D_4	1–3	$(11, 11)$	$\mathbb{Z}_2 \ltimes \mathbb{Z}^2$
	1–5	$(6, 6)$	$\mathbb{Z}_4$
	1–6	$(2, 2)$	S
	1–8	$(17, 5)$	$\mathbb{Z}_2$
	1–9	$(7, 7)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}^2$
			continued ...

$\mathbb{Q}$–class	$\mathbb{Z}$– and affine class	Hodge numbers $(h^{(1,1)}, h^{(2,1)})$	$\pi_1 = S/\langle F\rangle$
	2–4	$(9,3)$	$\mathbb{Z}_2$
	2–6	$(4,4)$	$\mathbb{Z}_4$
	2–8	$(10,4)$	$\mathbb{Z}_2$
	5–4	$(4,4)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
	5–6	$(12,6)$	$\mathbb{Z}_2$
	6–3	$(12,6)$	$\mathbb{Z}_2$
	6–4	$(6,6)$	$\mathbb{Z}_2$
	6–6	$(4,4)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
	6–8	$(10,4)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	8–2	$(6,6)$	$\mathbb{Z}_2$
	9–1	$(17,5)$	$\mathbb{Z}_2$
	9–2	$(6,6)$	$\mathbb{Z}_2$
	9–3	$(15,3)$	$\mathbb{Z}_2$
A_4	2–1	$(11,3)$	$\mathbb{Z}_2$
	2–2	$(3,3)$	$\mathbb{Z}_4$
	4–1	$(7,3)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4–2	$(5,5)$	$\mathbb{Z}_2$
	5–1	$(7,3)$	$\mathbb{Z}_2$
	6–2	$(3,3)$	$\mathbb{Z}_2$
QD_{16}	3–4	$(17,5)$	$\mathbb{Z}_2$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	1–11	$(27,3)$	$\mathbb{Z}_2$
	1–12	$(15,3)$	$\mathbb{Z}_2$
	1–18	$(17,5)$	$\mathbb{Z}_2$
	1–19	$(15,3)$	$\mathbb{Z}_2$
	1–21	$(12,6)$	$\mathbb{Z}_2$
	1–22	$(10,4)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\Delta(27)$	1–3	$(12,0)$	$\mathbb{Z}_3$
	1–4	$(4,0)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
	3–3	$(12,0)$	$\mathbb{Z}_3$
	3–4	$(4,0)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$

Table B.2: List of all non-trivial fundamental groups. The first column specifies the $\mathbb{Q}$-class and the second column enumerates the respective $\mathbb{Z}$- and affine classes. In the third column we list the Hodge numbers in order to identify those cases which allow for chiral spectra, c.f. [68]. Finally, the last column lists π_1.

B.3 Point groups

label of $\mathbb{Q}$-class	twists	GAPID	CARAT index	# $\mathbb{Z}$	# aff.	page
Abelian $\mathcal{N}=1$			$\sum$	60	138	
$\mathbb{Z}_3$	$(1,1,-2)/3$	$[3,1]$	1965	1	1	80
$\mathbb{Z}_4$	$(1,1,-2)/4$	$[4,1]$	4667	3	3	80
$\mathbb{Z}_6$–I	$(1,1,-2)/6$	$[6,2]$	1997	2	2	80
$\mathbb{Z}_6$–II	$(1,2,-3)/6$	$[6,2]$	944	4	4	80
$\mathbb{Z}_7$	$(1,2,-3)/7$	$[7,1]$	2950	1	1	80
$\mathbb{Z}_8$–I	$(1,2,-3)/8$	$[8,1]$	5600	3	3	81
$\mathbb{Z}_8$–II	$(1,3,-4)/8$	$[8,1]$	5567	2	2	81
$\mathbb{Z}_{12}$–I	$(1,4,-5)/12$	$[12,2]$	3346	2	2	81
$\mathbb{Z}_{12}$–II	$(1,5,-6)/12$	$[12,2]$	3307	1	1	81
$\mathbb{Z}_2\times\mathbb{Z}_2$	$(0,1,-1)/2\ ,\ (1,0,-1)/2$	$[4,2]$	4625	12	35	81
$\mathbb{Z}_2\times\mathbb{Z}_4$	$(0,1,-1)/2\ ,\ (1,0,-1)/4$	$[8,2]$	2377	10	41	84
$\mathbb{Z}_2\times\mathbb{Z}_6$–I	$(0,1,-1)/2\ ,\ (1,0,-1)/6$	$[12,5]$	871	2	4	88
$\mathbb{Z}_2\times\mathbb{Z}_6$–II	$(0,1,-1)/2\ ,\ (1,1,-2)/6$	$[12,5]$	1745	4	4	88
$\mathbb{Z}_3\times\mathbb{Z}_3$	$(0,1,-1)/3\ ,\ (1,0,-1)/3$	$[9,2]$	1964	5	15	89
$\mathbb{Z}_3\times\mathbb{Z}_6$	$(0,1,-1)/3\ ,\ (1,0,-1)/6$	$[18,5]$	1759	2	4	90
$\mathbb{Z}_4\times\mathbb{Z}_4$	$(0,1,-1)/4\ ,\ (1,0,-1)/4$	$[16,2]$	2629	5	15	91
$\mathbb{Z}_6\times\mathbb{Z}_6$	$(0,1,-1)/6\ ,\ (1,0,-1)/6$	$[36,14]$	1859	1	1	93
Abelian $\mathcal{N}=2$			$\sum$	10	23	
$\mathbb{Z}_2$	$(1,-1,0)/2$	$[2,1]$	5	3	5	
$\mathbb{Z}_3$	$(1,-1,0)/3$	$[3,1]$	1968	3	5	
$\mathbb{Z}_4$	$(1,-1,0)/4$	$[4,1]$	4668	3	9	
$\mathbb{Z}_6$	$(1,-1,0)/6$	$[6,2]$	1970	1	4	
non–Abelian $\mathcal{N}=1$			$\sum$	108	331	
S_3	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{-2\pi\,i\,\frac{1}{3}}&0\\0&0&e^{2\pi\,i\,\frac{1}{3}}\end{pmatrix}$	$[6,1]$	2262	6	11	93
D_4	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$[8,3]$	4682	9	48	94
A_4	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$[12,3]$	4893	9	15	97
D_6	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{2\pi\,i\,\frac{1}{6}}&0\\0&0&e^{-2\pi\,i\,\frac{1}{6}}\end{pmatrix}$	$[12,4]$	2258	2	8	98
$\mathbb{Z}_8\rtimes\mathbb{Z}_2$	$\begin{pmatrix}-i&0&0\\0&0&1\\0&-i&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$[16,6]$	6222	6	18	99
QD_{16}	$\begin{pmatrix}1&0&0\\0&0&e^{-2\pi\,i\,\frac{1}{8}}\\0&e^{-2\pi\,i\,\frac{3}{8}}&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$[16,8]$	5650	4	14	101
						continued ...

label of $\mathbb{Q}$-class	twists	GAPID	CARAT index	# $\mathbb{Z}$	# aff.	page
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}$	[16, 13]	5645	5	55	102
$\mathbb{Z}_3 \times S_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}$	[18, 3]	4235	6	16	108
Frobenius T_7	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{4}{7}} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{7}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{7}} \end{pmatrix}$	[21, 1]	2935	3	3	109
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$\begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	[24, 1]	6266	1	1	110
SL(2, 3)–I	$\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} & 0 & 0 \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \end{pmatrix},$	[24, 3]	6743	4	7	110
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$					
$\mathbb{Z}_4 \times S_3$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{5}{12}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{12}} \end{pmatrix}$	[24, 5]	3414	1	2	111
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	[24, 8]	3408	2	6	111
$\mathbb{Z}_3 \times D_4$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}$	[24, 10]	4326	2	2	111
$\mathbb{Z}_3 \times Q_8$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \end{pmatrix}$	[24, 11]	6735	2	2	112
S_4	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	[24, 12]	4895	6	19	112
$\Delta(27)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	[27, 3]	2864	3	10	113
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	$\begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	[32, 11]	6337	5	30	114
$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \end{pmatrix}$	[36, 6]	4353	1	1	118
$\mathbb{Z}_3 \times A_4$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	[36, 11]	2875	3	3	118
$\mathbb{Z}_6 \times S_3$	$\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{6}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{6}} \end{pmatrix}$	[36, 12]	4356	2	4	119
$\Delta(48)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}$	[48, 3]	2774	4	8	119
GL(2, 3)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \\ 0 & -\frac{1}{2}(1+\mathrm{i}) & -\frac{1}{2}(1+\mathrm{i}) \end{pmatrix},$	[48, 29]	5713	1	4	120
	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{1}{8}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{3}{8}}) \end{pmatrix}$					

continued . . .

label of $\mathbb{Q}$-class	twists	GAPID	CARAT index	# $\mathbb{Z}$	# aff.	page
$SL(2,3) \rtimes \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{pmatrix}$	[48, 33]	5712	1	3	121
$\Delta(54)$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{2\pi i \frac{1}{3}} & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 1 & 0 & 0 \end{pmatrix}$	[54, 8]	2897	3	10	121
$\mathbb{Z}_3 \times SL(2,3)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \\ 0 & -\frac{1}{2}(1-i) & -\frac{1}{2}(1-i) \end{pmatrix}, \begin{pmatrix} e^{-2\pi i \frac{1}{3}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{2\pi i \frac{1}{6}} & 0 \end{pmatrix}$	[72, 25]	6988	1	2	122
$\mathbb{Z}_3 \times \text{GAPID}\,[24, 8]$	$\begin{pmatrix} e^{2\pi i \frac{1}{6}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{-2\pi i \frac{1}{3}} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{-2\pi i \frac{1}{3}} & 0 \\ 0 & 0 & e^{2\pi i \frac{1}{3}} \end{pmatrix}$	[72, 30]	4533	1	1	123
$\mathbb{Z}_3 \times S_4$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-2\pi i \frac{1}{3}} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 0 & e^{2\pi i \frac{1}{6}} & 0 \end{pmatrix}$	[72, 42]	2924	3	3	123
$\Delta(96)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$	[96, 64]	2802	4	12	123
$SL(2,3) \rtimes \mathbb{Z}_4$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ 0 & -\frac{1}{2}(1+i) & -\frac{1}{2}(1+i) \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & -\frac{1}{2}(1+i) & \frac{1}{2}(1-i) \\ 0 & \frac{1}{2}(1-i) & -\frac{1}{2}(1+i) \end{pmatrix}$	[96, 67]	6512	1	2	125
$\Sigma(36\phi)$	$\begin{pmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} & \frac{1}{2} + \frac{i}{2\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ -\frac{1}{2} + \frac{i}{2\sqrt{3}} & \frac{1}{3} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ -\frac{1}{2} + \frac{i}{2\sqrt{3}} & \frac{1}{3} & \frac{1}{2} + \frac{i}{2\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	[108, 15]	2806	2	4	125
$\Delta(108)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-\frac{2\pi i}{6}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	[108, 22]	2810	1	1	126
$PSL(3,2)$	$\begin{pmatrix} \frac{1}{18}\left(-5+4i\sqrt{7}\right) & \frac{1}{36}\left(11+5i\sqrt{7}\right) & \frac{1}{18}\left(-1-4i\sqrt{7}\right) \\ -\frac{1}{36}i\left(-25i+\sqrt{7}\right) & -\frac{1}{9}i\left(-i+\sqrt{7}\right) & \frac{1}{36}i\left(23i+\sqrt{7}\right) \\ \frac{1}{18}\left(-1+2i\sqrt{7}\right) & \frac{1}{36}\left(-5-11i\sqrt{7}\right) & \frac{1}{18}\left(7-2i\sqrt{7}\right) \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	[168, 42]	2934	1	3	126
$\Sigma(72\phi)$	$\begin{pmatrix} \frac{1}{6}\left(3+i\sqrt{3}\right) & \frac{e^{2\pi i \frac{5}{12}}}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,i\right) & -\frac{i}{\sqrt{3}} & \frac{e^{2\pi i \frac{5}{12}}}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,i\right) & \frac{1}{6}\left(3+\sqrt{3}\,i\right) & \frac{1}{6}\left(3+\sqrt{3}\,i\right) \end{pmatrix}, \begin{pmatrix} -\frac{i}{\sqrt{3}} & \frac{1}{6}\left(3+\sqrt{3}\,i\right) & -\frac{i}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,i\right) & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{6}\left(3+\sqrt{3}\,i\right) & \frac{1}{6}\left(3+\sqrt{3}\,i\right) & \frac{e^{2\pi i \frac{5}{12}}}{\sqrt{3}} \end{pmatrix}$	[216, 88]	2846	2	2	126
$\Delta(216)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{2\pi i \frac{1}{3}} & 0 \\ 0 & 0 & e^{-2\pi i \frac{1}{3}} \\ 1 & 0 & 0 \end{pmatrix}$	[216, 95]	2851	1	1	127
continued …						

label of $\mathbb{Q}$-class	twists	GAPID	CARAT index	# $\mathbb{Z}$	# aff.	page
non–Abelian $\mathcal{N} = 2$			$\sum$	7	27	
Q_8	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}$	$[8, 4]$	5750	5	20	
Dic_3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{1}{3}} \end{pmatrix}$	$[12, 1]$	3374	1	3	
$\mathrm{SL}(2,3)$–II	$\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} & 0 & 0 \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & \frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} + \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \\ 0 & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) & -\frac{1}{2}(\mathrm{e}^{2\pi\,\mathrm{i}\,\frac{2}{3}} - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{11}{12}}) \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$[24, 3]$	5669	1	4	

Table B.3: Summary of all space groups with $\mathcal{N} \geq 1$ supersymmetry in four dimensions.

For Abelian point groups, twist vectors (cf. Equation (3.18)) are given, whereas for non-Abelian point groups, the SU(3) representations are given. Columns # 3 and 4 identify the $\mathbb{Q}$-classes: "GAPID" denotes the internal GAP name for the finite group in question and "CARAT index" is a flat index (starting at 1) into the list of all 7103 six-dimensional $\mathbb{Q}$-classes obtained from CARAT. Lastly, for $\mathbb{Q}$-classes with $\mathcal{N} = 1$ SUSY, "page" gives the first page of the $\mathbb{Q}$-class in Table B.4.

B.4 All space groups with $\mathcal{N} = 1$ supersymmetry in 4D

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
$\mathbb{Z}_3$	1	1	$(\theta, 0)$	
		local	$(9,0)U + (27,0)T_1$	$(36,0)$
$\mathbb{Z}_4$	1	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (10,6)T_2$	$(31,7)$
	2	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (6,2)T_2$	$(27,3)$
	3	1	$(\theta, 0)$	
		local	$(5,1)U + (16,0)T_1 + (4,0)T_2$	$(25,1)$
$\mathbb{Z}_6$–I	1	1	$(\theta, 0)$	
		local	$(5,0)U + (3,0)T_1 + (15,0)T_2 + (6,5)T_3$	$(29,5)$
	2	1	$(\theta, 0)$	
		local	$(5,0)U + (3,0)T_1 + (15,0)T_2 + (2,1)T_3$	$(25,1)$
$\mathbb{Z}_6$–II	1	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (6,3)T_2 + (8,4)T_3 + (6,3)T_4$	$(35,11)$
	2	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (6,3)T_2 + (4,0)T_3 + (6,3)T_4$	$(31,7)$
	3	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (3,0)T_2 + (8,4)T_3 + (3,0)T_4$	$(29,5)$
	4	1	$(\theta, 0)$	
		local	$(3,1)U + (12,0)T_1 + (3,0)T_2 + (4,0)T_3 + (3,0)T_4$	$(25,1)$
$\mathbb{Z}_7$	1	1	$(\theta, 0)$	
		local	$(3,0)U + (7,0)T_1 + (7,0)T_2 + (7,0)T_4$	$(24,0)$
				continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
$\mathbb{Z}_8$–I	1	1	$(\theta,0)$	
		local	$(3,0)U+(4,0)T_1+(10,0)T_2+(6,3)T_4+(4,0)T_5$	$(27,3)$
	2	1	$(\theta,0)$	
		local	$(3,0)U+(4,0)T_1+(10,0)T_2+(4,1)T_4+(4,0)T_5$	$(25,1)$
	3	1	$(\theta,0)$	
		local	$(3,0)U+(4,0)T_1+(10,0)T_2+(3,0)T_4+(4,0)T_5$	$(24,0)$
$\mathbb{Z}_8$–II	1	1	$(\theta,0)$	
		local	$(3,1)U+(8,0)T_1+(3,1)T_2+(8,0)T_3+(6,4)T_4+(3,1)T_6$	$(31,7)$
	2	1	$(\theta,0)$	
		local	$(3,1)U+(8,0)T_1+(2,0)T_2+(8,0)T_3+(4,2)T_4+(2,0)T_6$	$(27,3)$
$\mathbb{Z}_{12}$–I	1	1	$(\theta,0)$	
		local	$(3,0)U+(3,0)T_1+(3,0)T_2+(2,1)T_3+(9,0)T_4+(4,3)T_6+(3,0)T_7+(2,1)T_9$	$(29,5)$
	2	1	$(\theta,0)$	
		local	$(3,0)U+(3,0)T_1+(3,0)T_2+(1,0)T_3+(9,0)T_4+(2,1)T_6+(3,0)T_7+(1,0)T_9$	$(25,1)$
$\mathbb{Z}_{12}$–II	1	1	$(\theta,0)$	
		local	$(3,1)U+(4,0)T_1+(1,0)T_2+(8,0)T_3+(3,2)T_4+(4,0)T_5+(4,2)T_6+(3,2)T_8+(1,0)T_{10}$	$(31,7)$
$\mathbb{Z}_2\times\mathbb{Z}_2$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(16,0)T_{0,1}+(16,0)T_{1,0}+(16,0)T_{1,1}$	$(51,3)$
		2	$\left(\theta,\frac{1}{2}e_2\right)$,$(\omega,0)$	
		local	$(3,3)U+(8,8)T_{0,1}+(8,8)T_{1,1}$	$(19,19)$
		3	$\left(\theta,\frac{1}{2}(e_2+e_6)\right)$,$(\omega,0)$	
		non-local	$(3,3)U+(8,8)T_{0,1}$	$(11,11)$
				continued …

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4	$\left(\theta, \frac{1}{2}(e_2+e_6)\right), \left(\omega, \frac{1}{2}e_4\right)$	
		non-local	$(3,3)U$	$(3,3)$
	2	1	$(\theta,0), (\omega,0)$	
		local	$(3,3)U + (12,4)T_{0,1} + (8,0)T_{1,0} + (8,0)T_{1,1}$	$(31,7)$
		2	$\left(\theta, \frac{1}{2}e_3\right), (\omega,0)$	
		local	$(3,3)U + (8,8)T_{0,1} + (4,4)T_{1,1}$	$(15,15)$
		3	$\left(\theta, \frac{1}{2}(e_3+e_6)\right), (\omega,0)$	
		non-local	$(3,3)U + (8,8)T_{0,1}$	$(11,11)$
		4	$(\theta,0), \left(\omega, \frac{1}{2}e_5\right)$	
		non-local	$(3,3)U + (4,4)T_{1,0} + (4,4)T_{1,1}$	$(11,11)$
		5	$\left(\theta, \frac{1}{2}e_3\right), \left(\omega, \frac{1}{2}e_5\right)$	
		non-local	$(3,3)U + (4,4)T_{1,1}$	$(7,7)$
		6	$\left(\theta, \frac{1}{2}(e_3+e_6)\right), \left(\omega, \frac{1}{2}e_5\right)$	
		non-local	$(3,3)U$	$(3,3)$
	3	1	$(\theta,0), (\omega,0)$	
		local	$(3,3)U + (8,0)T_{0,1} + (8,0)T_{1,0} + (8,0)T_{1,1}$	$(27,3)$
		2	$\left(\theta, \frac{1}{2}e_6\right), (\omega,0)$	
		local	$(3,3)U + (4,4)T_{0,1} + (4,4)T_{1,0}$	$(11,11)$
		3	$\left(\theta, \frac{1}{2}e_6\right), \left(\omega, \frac{1}{2}e_5\right)$	
		non-local	$(3,3)U + (4,4)T_{1,0}$	$(7,7)$
		4	$\left(\theta, \frac{1}{2}(e_4+e_6)\right), \left(\omega, \frac{1}{2}e_5\right)$	
		non-local	$(3,3)U$	$(3,3)$
	4	1	$(\theta,0), (\omega,0)$	
		local	$(3,3)U + (10,6)T_{0,1} + (4,0)T_{1,0} + (4,0)T_{1,1}$	$(21,9)$
		2	$(\theta,0), \left(\omega, \frac{1}{2}e_4\right)$	
		non-local	$(3,3)U + (2,2)T_{1,0} + (2,2)T_{1,1}$	$(7,7)$
	5	1	$(\theta,0), (\omega,0)$	
		non-local	$(3,3)U + (8,0)T_{0,1} + (8,0)T_{1,0} + (8,0)T_{1,1}$	$(27,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		2	$(\theta,\frac{1}{2}e_4)$,$(\omega,0)$	
		non-local	$(3,3)U+(4,4)T_{0,1}+(4,4)T_{1,1}$	$(11,11)$
		3	$(\theta,\frac{1}{2}(e_2+e_3))$,$(\omega,0)$	
		local	$(3,3)U+(4,4)T_{0,1}+(4,4)T_{1,0}+(4,4)T_{1,1}$	$(15,15)$
		4	$(\theta,\frac{1}{2}e_4)$,$(\omega,\frac{1}{2}e_5)$	
		non-local	$(3,3)U+(4,4)T_{1,1}$	$(7,7)$
		5	$(\theta,\frac{1}{2}(e_4+e_6))$,$(\omega,\frac{1}{2}e_5)$	
		non-local	$(3,3)U$	$(3,3)$
	6	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(6,2)T_{1,1}$	$(19,7)$
		2	$(\theta,0)$,$(\omega,\frac{1}{2}e_5)$	
		non-local	$(3,3)U+(2,2)T_{1,0}+(4,4)T_{1,1}$	$(9,9)$
		3	$(\theta,\frac{1}{2}e_6)$,$(\omega,\frac{1}{2}e_5)$	
		non-local	$(3,3)U+(2,2)T_{1,0}$	$(5,5)$
	7	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(17,5)$
		2	$(\theta,0)$,$(\omega,\frac{1}{2}e_6)$	
		non-local	$(3,3)U+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(7,7)$
	8	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,3)U+(4,0)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(15,3)$
	9	1	$(\theta,0)$,$(\omega,0)$	
		non-local	$(3,3)U+(6,2)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(17,5)$
		2	$(\theta,0)$,$(\omega,\frac{1}{2}e_6)$	
		non-local	$(3,3)U+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(7,7)$
		3	$(\theta,\frac{1}{2}(e_2+e_3))$,$(\omega,0)$	
		local	$(3,3)U+(4,4)T_{0,1}+(2,2)T_{1,0}+(2,2)T_{1,1}$	$(11,11)$
	10	1	$(\theta,0)$,$(\omega,0)$	
		non-local	$(3,3)U+(4,0)T_{0,1}+(4,0)T_{1,0}+(4,0)T_{1,1}$	$(15,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$\left(\theta, \frac{1}{2}(e_1+e_2)\right)$,$(\omega, 0)$	
		local	$(3,3)U + (2,2)T_{0,1} + (2,2)T_{1,0} + (2,2)T_{1,1}$	$(9,9)$
	11	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,3)U + (3,1)T_{0,1} + (3,1)T_{1,0} + (3,1)T_{1,1}$	$(12,6)$
	12	1	$(\theta, 0)$,$(\omega, 0)$	
		non-local	$(3,3)U + (4,0)T_{0,1} + (4,0)T_{1,0} + (4,0)T_{1,1}$	$(15,3)$
		2	$\left(\theta, \frac{1}{2}(e_5+e_6)\right)$,$(\omega, 0)$	
		non-local	$(3,3)U + (2,2)T_{0,1} + (2,2)T_{1,0} + (2,2)T_{1,1}$	$(9,9)$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,1)U + (4,0)T_{0,1} + (10,0)T_{0,2} + (4,0)T_{0,3} + (12,0)T_{1,0} + (16,0)T_{1,1} + (12,0)T_{1,2}$	$(61,1)$
		2	$\left(\theta, \frac{1}{2}(e_1+e_2)\right)$,$\left(\omega, \frac{1}{2}(e_1+e_2)\right)$	
		local	$(3,1)U + (2,2)T_{0,1} + (6,4)T_{0,2} + (2,2)T_{0,3} + (8,0)T_{1,1} + (4,4)T_{1,2}$	$(25,13)$
		3	$\left(\theta, \frac{1}{2}(e_1+e_2+e_4+e_5)\right)$, $\left(\omega, \frac{1}{2}(e_1+e_2+e_4+e_5)\right)$	
		local	$(3,1)U + (2,2)T_{0,1} + (6,4)T_{0,2} + (2,2)T_{0,3} + (8,0)T_{1,1}$	$(21,9)$
		4	$\left(\theta, \frac{1}{2}e_4\right)$,$\left(\omega, \frac{1}{2}e_4\right)$	
		local	$(3,1)U + (10,0)T_{0,2} + (8,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2}$	$(37,1)$
		5	$\left(\theta, \frac{1}{2}(e_1+e_2+e_4)\right)$, $\left(\omega, \frac{1}{2}(e_1+e_2+e_4)\right)$	
		local	$(3,1)U + (6,4)T_{0,2} + (8,0)T_{1,0} + (4,4)T_{1,2}$	$(21,9)$
		6	$\left(\theta, \frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\right)$, $\left(\omega, \frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\right)$	
		non-local	$(3,1)U + (6,4)T_{0,2} + (8,0)T_{1,1}$	$(17,5)$
	2	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,1)U + (4,0)T_{0,1} + (8,2)T_{0,2} + (4,0)T_{0,3} + (8,0)T_{1,0} + (16,0)T_{1,1} + (8,0)T_{1,2}$	$(51,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\theta, \frac{1}{2}e_6), (\omega, \frac{1}{2}e_6)$	
		local	$(3,1)U + (8,2)T_{0,2} + (4,0)T_{1,0} + (8,0)T_{1,1} + (4,0)T_{1,2}$	$(27,3)$
		3	$(\theta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}(e_1+e_2))$	
		local	$(3,1)U + (2,2)T_{0,1} + (6,4)T_{0,2} + (2,2)T_{0,3} + (8,0)T_{1,1}$	$(21,9)$
		4	$(\theta, \frac{1}{2}(e_1+e_2+e_6)), (\omega, \frac{1}{2}(e_1+e_2+e_6))$	
		non-local	$(3,1)U + (6,4)T_{0,2} + (8,0)T_{1,1}$	$(17,5)$
		5	$(\theta, \frac{1}{2}(e_3+e_4)), (\omega, \frac{1}{2}(e_3+e_4))$	
		local	$(3,1)U + (2,2)T_{0,1} + (8,2)T_{0,2} + (2,2)T_{0,3} + (4,0)T_{1,0} + (8,0)T_{1,1} + (4,0)T_{1,2}$	$(31,7)$
		6	$(\theta, \frac{1}{2}(e_3+e_4+e_6)), (\omega, \frac{1}{2}(e_3+e_4+e_6))$	
		local	$(3,1)U + (8,2)T_{0,2} + (4,0)T_{1,0} + (8,0)T_{1,1} + (4,0)T_{1,2}$	$(27,3)$
	3	1	$(\theta, 0), (\omega, 0)$	
		local	$(3,1)U + (2,0)T_{0,1} + (6,0)T_{0,2} + (2,0)T_{0,3} + (6,0)T_{1,0} + (12,0)T_{1,1} + (8,2)T_{1,2}$	$(39,3)$
		2	$(\theta, \frac{1}{2}(e_5+e_6)), (\omega, \frac{1}{2}(e_5+e_6))$	
		local	$(3,1)U + (1,1)T_{0,1} + (4,2)T_{0,2} + (1,1)T_{0,3} + (2,2)T_{1,0} + (8,0)T_{1,1}$	$(19,7)$
		3	$(\theta, \frac{1}{2}e_4), (\omega, \frac{1}{2}e_4)$	
		local	$(3,1)U + (6,0)T_{0,2} + (4,0)T_{1,0} + (8,0)T_{1,1} + (6,2)T_{1,2}$	$(27,3)$
		4	$(\theta, \frac{1}{2}(e_4+e_5+e_6)), (\omega, \frac{1}{2}(e_4+e_5+e_6))$	
		local	$(3,1)U + (4,2)T_{0,2} + (2,2)T_{1,0} + (8,0)T_{1,1}$	$(17,5)$
		5	$(\theta, \frac{1}{2}(e_1+e_3)), (\omega, \frac{1}{2}e_1)$	
		local	$(3,1)U + (3,1)T_{0,2} + (8,0)T_{1,1} + (4,4)T_{1,2}$	$(18,6)$
		6	$(\theta, \frac{1}{2}(e_1+e_3+e_5+e_6)), (\omega, \frac{1}{2}(e_1+e_5+e_6))$	
		non-local	$(3,1)U + (3,1)T_{0,2} + (8,0)T_{1,1}$	$(14,2)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	4	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,1)U + (2,0)T_{0,1} + (6,0)T_{0,2} + (2,0)T_{0,3} + (6,0)T_{1,0} + (12,0)T_{1,1} + (6,0)T_{1,2}$	$(37,1)$
		2	$\big(\theta, \frac{1}{2}e_6\big)$,$\big(\omega, \frac{1}{2}e_6\big)$	
		local	$(3,1)U + (6,0)T_{0,2} + (4,0)T_{1,0} + (8,0)T_{1,1} + (4,0)T_{1,2}$	$(25,1)$
		3	$\big(\theta, \frac{1}{2}(e_1+e_2+e_3+e_4)\big)$, $\big(\omega, \frac{1}{2}(e_1+e_2+e_3+e_4)\big)$	
		local	$(3,1)U + (1,1)T_{0,1} + (4,2)T_{0,2} + (1,1)T_{0,3} + (8,0)T_{1,1}$	$(17,5)$
		4	$\big(\theta, \frac{1}{2}(e_1+e_2+e_3+e_4+e_6)\big)$, $\big(\omega, \frac{1}{2}(e_1+e_2+e_3+e_4+e_6)\big)$	
		non-local	$(3,1)U + (4,2)T_{0,2} + (8,0)T_{1,1}$	$(15,3)$
		5	$\big(\theta, \frac{1}{2}(e_3+e_4)\big)$,$\big(\omega, \frac{1}{2}(e_2+e_4+e_5)\big)$	
		local	$(3,1)U + (3,1)T_{0,2} + (2,2)T_{1,0} + (8,0)T_{1,1}$	$(16,4)$
	5	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,1)U + (3,1)T_{0,1} + (7,3)T_{0,2} + (3,1)T_{0,3} + (4,0)T_{1,0} + (12,0)T_{1,1} + (4,0)T_{1,2}$	$(36,6)$
		2	$\big(\theta, \frac{1}{2}e_6\big)$,$\big(\omega, \frac{1}{2}e_6\big)$	
		local	$(3,1)U + (7,3)T_{0,2} + (2,0)T_{1,0} + (8,0)T_{1,1} + (2,0)T_{1,2}$	$(22,4)$
	6	1	$(\theta, 0)$,$(\omega, 0)$	
		local	$(3,1)U + (2,0)T_{0,1} + (6,0)T_{0,2} + (2,0)T_{0,3} + (6,0)T_{1,0} + (12,0)T_{1,1} + (6,0)T_{1,2}$	$(37,1)$
		2	$\big(\theta, \frac{1}{2}(e_4+e_5)\big)$,$\big(\omega, \frac{1}{2}(e_4+e_5)\big)$	
		local	$(3,1)U + (1,1)T_{0,1} + (4,2)T_{0,2} + (1,1)T_{0,3} + (2,2)T_{1,0} + (8,0)T_{1,1} + (2,2)T_{1,2}$	$(21,9)$
		3	$\big(\theta, \frac{1}{2}e_6\big)$,$\big(\omega, \frac{1}{2}e_6\big)$	
		local	$(3,1)U + (6,0)T_{0,2} + (4,0)T_{1,0} + (8,0)T_{1,1} + (4,0)T_{1,2}$	$(25,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		4	$\left(\theta,\frac{1}{2}(e_4+e_5+e_6)\right)$, $\left(\omega,\frac{1}{2}(e_4+e_5+e_6)\right)$	
		local	$(3,1)U+(4,2)T_{0,2}+(2,2)T_{1,0}+(8,0)T_{1,1}+(2,2)T_{1,2}$	$(19,7)$
		5	$\left(\theta,\frac{1}{2}e_2\right)$,$\left(\omega,\frac{1}{2}(e_1+e_3)\right)$	
		non-local	$(3,1)U+(3,1)T_{0,2}+(8,0)T_{1,1}$	$(14,2)$
	7	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(2,0)T_{0,1}+(5,1)T_{0,2}+(2,0)T_{0,3}+(4,0)T_{1,0}+(12,0)T_{1,1}+(4,0)T_{1,2}$	$(32,2)$
		2	$\left(\theta,\frac{1}{2}e_6\right)$,$\left(\omega,\frac{1}{2}e_6\right)$	
		local	$(3,1)U+(5,1)T_{0,2}+(2,0)T_{1,0}+(8,0)T_{1,1}+(2,0)T_{1,2}$	$(20,2)$
		3	$\left(\theta,\frac{1}{2}(e_3+e_4+e_5)\right)$,$\left(\omega,\frac{1}{2}(e_3+e_5)\right)$	
		local	$(3,1)U+(4,0)T_{0,2}+(2,0)T_{1,0}+(8,0)T_{1,1}+(2,0)T_{1,2}$	$(19,1)$
	8	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(1,0)T_{0,1}+(4,0)T_{0,2}+(1,0)T_{0,3}+(4,1)T_{1,0}+(10,0)T_{1,1}+(4,1)T_{1,2}$	$(27,3)$
		2	$\left(\theta,\frac{1}{2}(e_1+e_3)\right)$,$\left(\omega,\frac{1}{2}e_2\right)$	
		local	$(3,1)U+(2,0)T_{0,2}+(8,0)T_{1,1}+(2,2)T_{1,2}$	$(15,3)$
		3	$\left(\theta,\frac{1}{2}(e_1+e_3)\right)$, $\left(\omega,\frac{1}{2}(e_2+e_5)\right)$	
		non-local	$(3,1)U+(2,0)T_{0,2}+(8,0)T_{1,1}$	$(13,1)$
	9	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U+(2,0)T_{0,1}+(5,1)T_{0,2}+(2,0)T_{0,3}+(4,0)T_{1,0}+(12,0)T_{1,1}+(4,0)T_{1,2}$	$(32,2)$
		2	$\left(\theta,\frac{1}{2}e_6\right)$,$\left(\omega,\frac{1}{2}e_6\right)$	
		local	$(3,1)U+(5,1)T_{0,2}+(2,0)T_{1,0}+(8,0)T_{1,1}+(2,0)T_{1,2}$	$(20,2)$
		3	$\left(\theta,\frac{1}{2}(e_1+e_2+e_4+e_5)\right)$, $\left(\omega,\frac{1}{2}(e_1+e_2+e_4+e_5)\right)$	
		local	$(3,1)U+(1,1)T_{0,1}+(5,1)T_{0,2}+(1,1)T_{0,3}+(2,0)T_{1,0}+(8,0)T_{1,1}+(2,0)T_{1,2}$	$(22,4)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	10	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U + (1,0)T_{0,1} + (4,0)T_{0,2} + (1,0)T_{0,3} + (3,0)T_{1,0} + (10,0)T_{1,1} + (3,0)T_{1,2}$	$(25,1)$
		2	$(\theta,0)$,$(\omega,\frac{1}{2}e_6)$	
		local	$(3,1)U + (2,0)T_{0,2} + (1,1)T_{1,0} + (8,0)T_{1,1} + (1,1)T_{1,2}$	$(15,3)$
$\mathbb{Z}_2 \times \mathbb{Z}_6$–I	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U + (1,0)T_{0,1} + (4,1)T_{0,2} + (6,0)T_{0,3} + (4,1)T_{0,4} + (1,0)T_{0,5} + (8,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (8,0)T_{1,3}$	$(51,3)$
		2	$(\theta,\frac{1}{2}e_4)$,$(\omega,\frac{1}{2}e_4)$	
		local	$(3,1)U + (4,1)T_{0,2} + (4,1)T_{0,4} + (4,2)T_{1,0} + (6,0)T_{1,1} + (6,0)T_{1,2} + (4,2)T_{1,3}$	$(31,7)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,1)U + (1,0)T_{0,1} + (4,1)T_{0,2} + (4,2)T_{0,3} + (4,1)T_{0,4} + (1,0)T_{0,5} + (4,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (4,0)T_{1,3}$	$(41,5)$
		2	$(\theta,\frac{1}{2}e_6)$,$(\omega,\frac{1}{2}e_6)$	
		local	$(3,1)U + (4,1)T_{0,2} + (4,1)T_{0,4} + (2,0)T_{1,0} + (6,0)T_{1,1} + (6,0)T_{1,2} + (2,0)T_{1,3}$	$(27,3)$
$\mathbb{Z}_2 \times \mathbb{Z}_6$–II	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (6,0)T_{0,3} + (6,0)T_{1,0} + (2,0)T_{1,1} + (6,0)T_{1,3} + (2,0)T_{1,4}$	$(36,0)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0} + (2,0)T_{1,1} + (4,2)T_{1,3} + (2,0)T_{1,4}$	$(26,2)$
	3	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (2,0)T_{0,1} + (9,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0} + (2,0)T_{1,1} + (2,0)T_{1,3} + (2,0)T_{1,4}$	$(24,0)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
	4	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(2,0)T_{0,1}+(9,0)T_{0,2}+(2,0)T_{0,3}+(2,0)T_{1,0}+(2,0)T_{1,1}+(2,0)T_{1,3}+(2,0)T_{1,4}$	$(24,0)$
$\mathbb{Z}_3\times\mathbb{Z}_3$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(9,0)T_{0,1}+(9,0)T_{0,2}+(9,0)T_{1,0}+(27,0)T_{1,1}+(9,0)T_{1,2}+(9,0)T_{2,0}+(9,0)T_{2,1}$	$(84,0)$
		2	$\big(\theta,\frac{1}{3}(2e_5+e_6)\big)$,$\big(\omega,\frac{1}{3}(e_5+2e_6)\big)$	
		local	$(3,0)U+(3,3)T_{0,1}+(3,3)T_{0,2}+(3,3)T_{1,0}+(9,0)T_{1,1}+(3,3)T_{2,0}$	$(24,12)$
		3	$\big(\theta,\frac{1}{3}(2e_1+e_2+2e_5+e_6)\big)$,$\big(\omega,\frac{1}{3}(e_1+2e_2+e_5+2e_6)\big)$	
		local	$(3,0)U+(3,3)T_{0,1}+(3,3)T_{0,2}+(9,0)T_{1,1}$	$(18,6)$
		4	$\big(\theta,\frac{1}{3}(2e_1+e_2+2e_3+e_4+2e_5+e_6)\big)$, $\big(\omega,\frac{1}{3}(e_1+2e_2+e_3+2e_4+e_5+2e_6)\big)$	
		non-local	$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(5,2)T_{0,1}+(5,2)T_{0,2}+(3,0)T_{1,0}+(15,0)T_{1,1}+(3,0)T_{1,2}+(3,0)T_{2,0}+(3,0)T_{2,1}$	$(40,4)$
		2	$\big(\theta,\frac{1}{3}(2e_5+e_6)\big)$,$\big(\omega,\frac{1}{3}(e_5+2e_6)\big)$	
		local	$(3,0)U+(1,1)T_{1,0}+(9,0)T_{1,1}+(1,1)T_{1,2}+(1,1)T_{2,0}+(1,1)T_{2,1}$	$(16,4)$
		3	$\big(\theta,\frac{1}{3}(2e_3+e_4)\big)$,$\big(\omega,\frac{2}{3}(e_1+e_2+e_4)\big)$	
		local	$(3,0)U+(3,3)T_{0,1}+(3,3)T_{0,2}+(9,0)T_{1,1}$	$(18,6)$
		4	$\big(\theta,\frac{1}{3}(e_1+2e_2+2e_3+e_6)\big)$, $\big(\omega,\frac{1}{3}(2e_1+e_2+e_4+e_5+e_6)\big)$	
		non-local	$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	3	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(3,0)T_{0,1}+(3,0)T_{0,2}+(3,0)T_{1,0}+(15,0)T_{1,1}+(3,0)T_{1,2}+(3,0)T_{2,0}+(3,0)T_{2,1}$	$(36,0)$
		2	$\big(\theta,\frac{1}{3}(e_3+2e_4)\big)$, $\big(\omega,\frac{1}{3}(2e_1+2e_2+e_3+e_4)\big)$	
		local	$(3,0)U+(9,0)T_{1,1}+(1,1)T_{1,2}+(1,1)T_{2,1}$	$(14,2)$

continued . . .

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		3	$\left(\theta, \frac{1}{3}(2e_1+e_2+2e_3+e_4+e_5+2e_6)\right)$, $\left(\omega, \frac{1}{3}(e_1+2e_2+e_3+2e_4+2e_5+e_6)\right)$	
		non-local	$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	4	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(3,0)T_{0,1}+(3,0)T_{0,2}+(3,0)T_{1,0}+(15,0)T_{1,1}+(3,0)T_{1,2}+(3,0)T_{2,0}+(3,0)T_{2,1}$	$(36,0)$
		2	$\left(\theta, \frac{1}{3}(e_2+2e_3)\right)$, $\left(\omega, \frac{1}{3}(2e_2+e_3)\right)$	
		local	$(3,0)U+(1,1)T_{0,1}+(1,1)T_{0,2}+(1,1)T_{1,0}+(9,0)T_{1,1}+(1,1)T_{1,2}+(1,1)T_{2,0}+(1,1)T_{2,1}$	$(18,6)$
		3	$\left(\theta, \frac{1}{3}(e_1+e_3+2e_4+2e_5)\right)$, $\left(\omega, \frac{1}{3}(2e_1+2e_2+e_4+2e_5+2e_6)\right)$	
		non-local	$(3,0)U+(9,0)T_{1,1}$	$(12,0)$
	5	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(1,0)T_{0,1}+(1,0)T_{0,2}+(1,0)T_{1,0}+(11,0)T_{1,1}+(1,0)T_{1,2}+(1,0)T_{2,0}+(1,0)T_{2,1}$	$(20,0)$
$\mathbb{Z}_3 \times \mathbb{Z}_6$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(1,0)T_{0,1}+(5,0)T_{0,2}+(4,1)T_{0,3}+(5,0)T_{0,4}+(1,0)T_{0,5}+(6,0)T_{1,0}+(6,0)T_{1,1}+(15,0)T_{1,2}+(6,0)T_{1,3}+(6,0)T_{1,4}+(6,0)T_{2,0}+(3,0)T_{2,1}+(6,0)T_{2,2}$	$(73,1)$
		2	$\left(\theta, \frac{1}{3}(e_3+2e_4)\right)$,$\left(\omega, \frac{1}{3}(2e_3+e_4)\right)$	
		local	$(3,0)U+(4,1)T_{0,3}+(2,1)T_{1,0}+(4,0)T_{1,1}+(5,0)T_{1,2}+(4,0)T_{1,3}+(2,1)T_{1,4}+(2,1)T_{2,0}+(1,0)T_{2,1}+(2,1)T_{2,2}$	$(29,5)$
	2	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U+(1,0)T_{0,1}+(3,1)T_{0,2}+(4,1)T_{0,3}+(3,1)T_{0,4}+(1,0)T_{0,5}+(3,0)T_{1,0}+(6,0)T_{1,1}+(9,0)T_{1,2}+(6,0)T_{1,3}+(3,0)T_{1,4}+(3,0)T_{2,0}+(3,0)T_{2,1}+(3,0)T_{2,2}$	$(51,3)$
			continued …	

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		2	$\big(\theta,\frac{1}{3}(e_5+2e_6)\big),\big(\omega,\frac{1}{3}(2e_5+e_6)\big)$	
		local	$(3,0)U+(4,1)T_{0,3}+(1,0)T_{1,0}+(4,0)T_{1,1}+(5,0)T_{1,2}+(4,0)T_{1,3}+(1,0)T_{1,4}+(1,0)T_{2,0}+(1,0)T_{2,1}+(1,0)T_{2,2}$	$(25,1)$
$\mathbb{Z}_4\times\mathbb{Z}_4$	1	1	$(\theta,0),(\omega,0)$	
		local	$(3,0)U+(4,0)T_{0,1}+(9,0)T_{0,2}+(4,0)T_{0,3}+(4,0)T_{1,0}+(12,0)T_{1,1}+(12,0)T_{1,2}+(4,0)T_{1,3}+(9,0)T_{2,0}+(12,0)T_{2,1}+(9,0)T_{2,2}+(4,0)T_{3,0}+(4,0)T_{3,1}$	$(90,0)$
		2	$\big(\theta,\frac{1}{2}(e_5+e_6)\big),(\omega,0)$	
		local	$(3,0)U+(2,0)T_{0,1}+(7,0)T_{0,2}+(2,0)T_{0,3}+(2,0)T_{1,0}+(4,0)T_{1,1}+(8,0)T_{1,2}+(7,0)T_{2,0}+(8,0)T_{2,1}+(9,0)T_{2,2}+(2,0)T_{3,0}$	$(54,0)$
		3	$\big(\theta,\frac{1}{2}(e_1+e_2+e_5+e_6)\big),(\omega,0)$	
		local	$(3,0)U+(2,0)T_{0,1}+(5,0)T_{0,2}+(2,0)T_{0,3}+(4,0)T_{1,1}+(4,0)T_{1,2}+(7,0)T_{2,0}+(8,0)T_{2,1}+(7,0)T_{2,2}$	$(42,0)$
		4	$\big(\theta,\frac{1}{2}(e_1+e_2+e_4+e_5+e_6)\big),\ \big(\omega,\frac{1}{2}(e_3+e_4)\big)$	
		local	$(3,0)U+(5,0)T_{0,2}+(4,0)T_{1,1}+(4,0)T_{1,2}+(5,0)T_{2,0}+(4,0)T_{2,1}+(5,0)T_{2,2}$	$(30,0)$
	2	1	$(\theta,0),(\omega,0)$	
		local	$(3,0)U+(3,0)T_{0,1}+(6,1)T_{0,2}+(3,0)T_{0,3}+(2,0)T_{1,0}+(8,0)T_{1,1}+(8,0)T_{1,2}+(2,0)T_{1,3}+(6,0)T_{2,0}+(10,0)T_{2,1}+(6,0)T_{2,2}+(2,0)T_{3,0}+(2,0)T_{3,1}$	$(61,1)$
		2	$\big(\theta,\frac{1}{2}(e_1+e_4)\big),\big(\omega,\frac{1}{2}(e_1+e_3)\big)$	
		local	$(3,0)U+(1,1)T_{0,1}+(4,1)T_{0,2}+(1,1)T_{0,3}+(4,0)T_{1,1}+(4,0)T_{1,2}+(3,0)T_{2,0}+(4,0)T_{2,1}+(3,0)T_{2,2}$	$(27,3)$
		3	$\big(\theta,\frac{1}{2}e_6\big),\big(\omega,\frac{1}{2}(e_5+e_6)\big)$	
		local	$(3,0)U+(6,1)T_{0,2}+(1,0)T_{1,0}+(6,0)T_{1,1}+(6,0)T_{1,2}+(1,0)T_{1,3}+(4,0)T_{2,0}+(4,0)T_{2,1}+(4,0)T_{2,2}+(1,0)T_{3,0}+(1,0)T_{3,1}$	$(37,1)$

continued . . .

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4	$\big(\theta, \frac{1}{2}(e_1+e_4+e_6)\big)$, $\big(\omega, \frac{1}{2}(e_1+e_3+e_5+e_6)\big)$	
		local	$(3,0)U + (4,1)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (3,0)T_{2,0} + (4,0)T_{2,1} + (3,0)T_{2,2}$	$(25,1)$
	3	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (2,0)T_{0,1} + (5,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (2,0)T_{1,3} + (5,0)T_{2,0} + (8,0)T_{2,1} + (5,0)T_{2,2} + (2,0)T_{3,0} + (2,0)T_{3,1}$	$(54,0)$
		2	$\big(\theta, \frac{1}{2}(e_1+e_2+e_3+e_5)\big)$, $\big(\omega, \frac{1}{2}(e_3+e_4+e_5+e_6)\big)$	
		local	$(3,0)U + (1,0)T_{0,1} + (3,0)T_{0,2} + (1,0)T_{0,3} + (4,0)T_{1,1} + (4,0)T_{1,2} + (4,0)T_{2,0} + (6,0)T_{2,1} + (4,0)T_{2,2}$	$(30,0)$
	4	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (2,0)T_{0,1} + (5,0)T_{0,2} + (2,0)T_{0,3} + (2,0)T_{1,0} + (8,0)T_{1,1} + (8,0)T_{1,2} + (2,0)T_{1,3} + (5,0)T_{2,0} + (8,0)T_{2,1} + (5,0)T_{2,2} + (2,0)T_{3,0} + (2,0)T_{3,1}$	$(54,0)$
		2	$\big(\theta, \frac{1}{2}e_2\big)$,$\big(\omega, \frac{1}{2}(e_1+e_2)\big)$	
		local	$(3,0)U + (1,0)T_{0,1} + (5,0)T_{0,2} + (1,0)T_{0,3} + (1,0)T_{1,0} + (6,0)T_{1,1} + (6,0)T_{1,2} + (1,0)T_{1,3} + (5,0)T_{2,0} + (6,0)T_{2,1} + (5,0)T_{2,2} + (1,0)T_{3,0} + (1,0)T_{3,1}$	$(42,0)$
		3	$\big(\theta, \frac{1}{2}(e_2+e_5+e_6)\big)$, $\big(\omega, \frac{1}{2}(e_2+e_4+e_5)\big)$	
		local	$(3,0)U + (2,1)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (2,1)T_{2,0} + (4,0)T_{2,1} + (2,1)T_{2,2}$	$(21,3)$
	5	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (1,0)T_{0,1} + (3,0)T_{0,2} + (1,0)T_{0,3} + (1,0)T_{1,0} + (6,0)T_{1,1} + (6,0)T_{1,2} + (1,0)T_{1,3} + (3,0)T_{2,0} + (6,0)T_{2,1} + (3,0)T_{2,2} + (1,0)T_{3,0} + (1,0)T_{3,1}$	$(36,0)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$\left(\theta, \frac{1}{2}(e_3+e_4)\right)$, $\left(\omega, \frac{1}{2}(e_1+e_2+e_3+e_6)\right)$	
		local	$(3,0)U + (1,0)T_{0,2} + (4,0)T_{1,1} + (4,0)T_{1,2} + (1,0)T_{2,0} + (4,0)T_{2,1} + (1,0)T_{2,2}$	$(18,0)$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	1	$(\theta,0)$,$(\omega,0)$	
		local	$(3,0)U + (1,0)T_{0,1} + (4,0)T_{0,2} + (4,0)T_{0,3} + (4,0)T_{0,4} + (1,0)T_{0,5} + (1,0)T_{1,0} + (2,0)T_{1,1} + (4,0)T_{1,2} + (4,0)T_{1,3} + (2,0)T_{1,4} + (1,0)T_{1,5} + (4,0)T_{2,0} + (4,0)T_{2,1} + (9,0)T_{2,2} + (4,0)T_{2,3} + (4,0)T_{2,4} + (4,0)T_{3,0} + (4,0)T_{3,1} + (4,0)T_{3,2} + (4,0)T_{3,3} + (4,0)T_{4,0} + (2,0)T_{4,1} + (4,0)T_{4,2} + (1,0)T_{5,0} + (1,0)T_{5,1}$	$(84,0)$
S_3	1	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (9,9)T_{[\omega]}$	$(15,15)$
		2	$(\vartheta,0), (\omega, \frac{1}{3}e_5)$	
		non-local	$(2,2)U + (4,4)T_{[\vartheta]}$	$(6,6)$
	2	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (9,9)T_{[\omega]}$	$(15,15)$
		2	$(\vartheta,0), (\omega, \frac{1}{3}e_5)$	
		non-local	$(2,2)U + (4,4)T_{[\vartheta]}$	$(6,6)$
	3	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (3,3)T_{[\omega]}$	$(9,9)$
		2	$(\vartheta,0), (\omega, \frac{1}{3}e_1)$	
		non-local	$(2,2)U + (4,4)T_{[\vartheta]}$	$(6,6)$
	4	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (9,9)T_{[\omega]}$	$(15,15)$
		2	$(\vartheta,0), (\omega, \frac{1}{3}e_5)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]}$	$(6,6)$
	5	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (3,3)T_{[\omega]}$	$(9,9)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, 0), (\omega, \frac{1}{3}e_1)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]}$	$(6,6)$
	6	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (1,1)T_{[\omega]}$	$(7,7)$
D_4	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (8,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (4,4)T_{[\vartheta\omega]} + (9,1)T_{[\vartheta\omega\vartheta\omega]}$	$(31,7)$
		2	$(\vartheta, \frac{1}{2}e_1), (\omega, 0)$	
		local	$(2,2)U + (8,0)T_{[\omega]} + (4,4)T_{[\vartheta\omega]} + (7,3)T_{[\vartheta\omega\vartheta\omega]}$	$(21,9)$
		3	$(\vartheta, \frac{1}{2}e_1), (\omega, \frac{1}{2}e_2)$	
		non-local	$(2,2)U + (4,4)T_{[\vartheta\omega]} + (5,5)T_{[\vartheta\omega\vartheta\omega]}$	$(11,11)$
		4	$(\vartheta, \frac{1}{4}e_5), (\omega, 0)$	
		local	$(2,2)U + (4,4)T_{[\vartheta]} + (4,4)T_{[\omega]}$	$(10,10)$
		5	$(\vartheta, \frac{1}{2}e_1 + \frac{1}{4}e_5), (\omega, 0)$	
		non-local	$(2,2)U + (4,4)T_{[\omega]}$	$(6,6)$
		6	$(\vartheta, \frac{1}{2}e_1 + \frac{1}{4}e_5), (\omega, \frac{1}{2}e_2)$	
		non-local	$(2,2)U$	$(2,2)$
		7	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,2)U + (8,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (9,1)T_{[\vartheta\omega\vartheta\omega]}$	$(27,3)$
		8	$(\vartheta, \frac{1}{2}(e_1 + e_5)), (\omega, 0)$	
		non-local	$(2,2)U + (8,0)T_{[\omega]} + (7,3)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
		9	$(\vartheta, \frac{1}{2}(e_1 + e_5)), (\omega, \frac{1}{2}e_2)$	
		non-local	$(2,2)U + (5,5)T_{[\vartheta\omega\vartheta\omega]}$	$(7,7)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (2,2)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
		2	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,4)T_{[\omega]} + (3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(13,7)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}e_1)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (4,2)T_{[\vartheta\omega\vartheta\omega]}$	$(12,6)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4	$(\vartheta, \frac{1}{2}e_5), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (4,0)T_{[\vartheta]} + (3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(9,3)$
		5	$(\vartheta, \frac{1}{4}e_3), (\omega, 0)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (4,4)T_{[\omega]}$	$(8,8)$
		6	$(\vartheta, \frac{1}{4}e_3), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (2,2)T_{[\vartheta]}$	$(4,4)$
		7	$(\vartheta, \frac{1}{2}e_3), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]}$	$(20,2)$
		8	$(\vartheta, \frac{1}{2}e_3), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,2)T_{[\vartheta\omega\vartheta\omega]}$	$(10,4)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (2,2)T_{[\vartheta\omega]} + (5,1)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
		2	$(\vartheta, \frac{1}{2}e_3), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(14,2)$
		3	$(\vartheta, \frac{1}{4}e_1), (\omega, 0)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (2,2)T_{[\omega]}$	$(6,6)$
		4	$(\vartheta, \frac{1}{2}e_1), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (5,1)T_{[\vartheta\omega\vartheta\omega]}$	$(15,3)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(15,3)$
		2	$(\vartheta, \frac{1}{2}e_3), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (2,2)T_{[\omega]} + (2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(10,4)$
		3	$(\vartheta, \frac{1}{2}e_3), (\omega, \frac{1}{2}e_4)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (2,2)T_{[\omega]} + (1,1)T_{[\vartheta\omega\vartheta\omega]}$	$(7,7)$
	5	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (16,0)T_{[\omega]} + (4,4)T_{[\vartheta\omega]} + (10,0)T_{[\vartheta\omega\vartheta\omega]}$	$(36,6)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_1)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,4)T_{[\vartheta\omega]} + (6,4)T_{[\vartheta\omega\vartheta\omega]}$	$(16,10)$
		3	$(\vartheta, \frac{1}{4}e_5), (\omega, 0)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (8,8)T_{[\omega]}$	$(12,12)$
		4	$(\vartheta, \frac{1}{4}e_5), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (2,2)T_{[\vartheta]}$	$(4,4)$
		5	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (16,0)T_{[\omega]} + (10,0)T_{[\vartheta\omega\vartheta\omega]}$	$(32,2)$
		6	$(\vartheta, \frac{1}{2}e_5), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (4,0)T_{[\vartheta]} + (6,4)T_{[\vartheta\omega\vartheta\omega]}$	$(12,6)$
	6	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (2,2)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_4)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(16,4)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (4,0)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (4,2)T_{[\vartheta\omega\vartheta\omega]}$	$(12,6)$
		4	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1 + e_4))$	
		non-local	$(2,2)U + (2,2)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega\vartheta\omega]}$	$(6,6)$
		5	$(\vartheta, \frac{1}{4}e_3), (\omega, 0)$	
		local	$(2,2)U + (2,2)T_{[\vartheta]} + (4,4)T_{[\omega]}$	$(8,8)$
		6	$(\vartheta, \frac{1}{4}e_3), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (2,2)T_{[\vartheta]}$	$(4,4)$
		7	$(\vartheta, \frac{1}{2}e_3), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]}$	$(20,2)$
		8	$(\vartheta, \frac{1}{2}e_3), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,2)T_{[\vartheta\omega\vartheta\omega]}$	$(10,4)$
	7	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(15,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		2	$(\vartheta,0),(\omega,\frac{1}{2}e_4)$	
		local	$(2,2)U+(2,2)T_{[\vartheta]}+(4,0)T_{[\omega]}+(2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(10,4)$
	8	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(4,4)T_{[\vartheta\omega]}+(7,3)T_{[\vartheta\omega\vartheta\omega]}$	$(21,9)$
		2	$(\vartheta,\frac{1}{4}e_5),(\omega,0)$	
		non-local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\omega]}$	$(6,6)$
		3	$(\vartheta,\frac{1}{2}e_5),(\omega,0)$	
		local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(7,3)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
	9	1	$(\vartheta,0),(\omega,0)$	
		non-local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(2,2)T_{[\vartheta\omega]}+(5,1)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
		2	$(\vartheta,\frac{1}{4}e_1),(\omega,0)$	
		non-local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\omega]}$	$(6,6)$
		3	$(\vartheta,\frac{1}{2}e_1),(\omega,0)$	
		non-local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(5,1)T_{[\vartheta\omega\vartheta\omega]}$	$(15,3)$
		4	$(\vartheta,0),(\omega,\frac{1}{2}(e_2+\frac{1}{2}e_3+e_5))$	
		local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\omega]}+(2,2)T_{[\vartheta\omega\vartheta\omega]}$	$(8,8)$
		5	$(\vartheta,0),(\omega,\frac{1}{2}e_3)$	
		local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(2,2)T_{[\vartheta\omega]}+(5,1)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
A_4	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(1,1)U+(1,1)T_{[\vartheta]}+(3,1)T_{[\omega]}+(1,1)T_{[\vartheta^2]}$	$(6,4)$
	2	1	$(\vartheta,0),(\omega,0)$	
		non-local	$(1,1)U+(1,1)T_{[\vartheta]}+(8,0)T_{[\omega]}+(1,1)T_{[\vartheta^2]}$	$(11,3)$
		2	$(\vartheta,0),(\omega,\frac{1}{2}(e_1+e_2))$	
		non-local	$(1,1)U+(1,1)T_{[\vartheta]}+(1,1)T_{[\vartheta^2]}$	$(3,3)$
		3	$(\vartheta,0),(\omega,\frac{1}{2}(e_1+e_3))$	
		local	$(1,1)U+(1,1)T_{[\vartheta]}+(4,4)T_{[\omega]}+(1,1)T_{[\vartheta^2]}$	$(7,7)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	3	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(7,3)$
	4	1 non-local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(7,3)$
		2 non-local	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1 + e_3))$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(5,5)$
	5	1 non-local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(7,3)$
		2 local	$(\vartheta, \frac{1}{2}(e_1 + e_5)), (\omega, \frac{1}{2}(e_2 + e_5))$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(5,5)$
	6	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (16,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(19,3)$
		2 non-local	$(\vartheta, 0), (\omega, \frac{1}{2}(e_2 + e_4))$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\vartheta^2]}$	$(3,3)$
	7	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (8,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(11,3)$
		2 local	$(\vartheta, \frac{1}{2}(e_1 + e_3 + e_5)), (\omega, \frac{1}{2}(e_1 + e_2))$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\vartheta^2]}$	$(3,3)$
	8	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(7,3)$
	9	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(7,3)$
D_6	1	1 local	$(\vartheta, 0), (\omega, 0)$	
			$(2,2)U + (4,0)T_{[\vartheta]} + (1,1)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (5,5)T_{[\omega^2]} + (5,1)T_{[\omega^3]}$	$(21,9)$
		2 local	$(\vartheta, 0), (\omega, \frac{1}{6}e_5)$	
			$(2,2)U + (2,2)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]}$	$(6,6)$
		3 local	$(\vartheta, 0), (\omega, \frac{1}{3}e_5)$	
			$(2,2)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (5,1)T_{[\omega^3]}$	$(15,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		4	$(\vartheta,0),(\omega,\frac{1}{2}e_5)$	
		local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\vartheta\omega]}+(5,5)T_{[\omega^2]}$	$(11,11)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,2)U+(4,0)T_{[\vartheta]}+(1,1)T_{[\omega]}+(4,0)T_{[\vartheta\omega]}+(5,5)T_{[\omega^2]}+(5,1)T_{[\omega^3]}$	$(21,9)$
		2	$(\vartheta,0),(\omega,\frac{1}{6}e_5)$	
		local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\vartheta\omega]}$	$(6,6)$
		3	$(\vartheta,0),(\omega,\frac{1}{3}e_5)$	
		local	$(2,2)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\vartheta\omega]}+(5,1)T_{[\omega^3]}$	$(15,3)$
		4	$(\vartheta,0),(\omega,\frac{1}{2}e_5)$	
		local	$(2,2)U+(2,2)T_{[\vartheta]}+(2,2)T_{[\vartheta\omega]}+(5,5)T_{[\omega^2]}$	$(11,11)$
$\mathbb{Z}_8\rtimes\mathbb{Z}_2$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(8,0)T_{[\omega]}+(10,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(4,0)T_{[\vartheta^2\omega]}+(5,1)T_{[\vartheta^4]}$	$(37,1)$
		2	$(\vartheta,\frac{1}{2}(e_2+e_4)),(\omega,\frac{1}{2}(e_1+e_2+e_3+e_4))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(6,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(2,2)T_{[\vartheta^2\omega]}+(5,1)T_{[\vartheta^4]}$	$(27,3)$
		3	$(\vartheta,0),(\omega,\frac{1}{2}(e_5+e_6))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(6,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(5,1)T_{[\vartheta^4]}$	$(25,1)$
		4	$(\vartheta,\frac{1}{2}(e_2+e_4)),(\omega,\frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(6,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(5,1)T_{[\vartheta^4]}$	$(25,1)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(6,0)T_{[\omega]}+(8,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^2\omega]}+(4,0)T_{[\vartheta^4]}$	$(30,0)$
		2	$(\vartheta,\frac{1}{2}(e_2+e_4+e_5)),(\omega,\frac{1}{2}(e_2+e_3+e_4+e_6))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(2,2)T_{[\omega]}+(6,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta^4]}$	$(21,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1+e_3))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\omega]} + (8,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]}$	$(30,0)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (7,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\vartheta^4]}$	$(24,0)$
		2	$(\vartheta, \frac{1}{2}e_1), (\omega, \frac{1}{2}(e_1+e_3+e_4+e_6))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (1,1)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^4]}$	$(19,1)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (12,0)T_{[\omega]} + (10,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta^2\omega]} + (6,0)T_{[\vartheta^4]}$	$(42,0)$
		2	$(\vartheta, \frac{1}{2}(e_3+e_4)), (\omega, \frac{1}{2}(e_1+e_2+e_3+e_4))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta^2\omega]} + (4,2)T_{[\vartheta^4]}$	$(22,4)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}(e_5+e_6))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta^4]}$	$(30,0)$
		4	$(\vartheta, \frac{1}{2}(e_3+e_4)), (\omega, \frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (4,2)T_{[\vartheta^4]}$	$(20,2)$
	5	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\omega]} + (8,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]}$	$(30,0)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_2+e_3+e_4)), (\omega, \frac{1}{2}(e_1+e_6))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\omega]} + (8,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]}$	$(30,0)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}(e_1+e_2+e_3+e_4+e_5))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta^4]}$	$(19,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
	6	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,1)T_{[\omega]}+(7,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta^2\omega]}+(3,0)T_{[\vartheta^4]}$	$(25,1)$
		2	$(\vartheta,\frac{1}{2}(e_1+e_2+e_4)),(\omega,\frac{1}{2}(e_1+e_4+e_5+e_6))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(6,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^4]}$	$(18,0)$
QD_{16}	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(4,4)T_{[\vartheta]}+(4,0)T_{[\omega]}+(5,2)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(26,8)$
		2	$(\vartheta,\frac{1}{2}e_6),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\omega]}+(5,2)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
		3	$(\vartheta,\frac{1}{2}e_1),(\omega,0)$	
		local	$(2,1)U+(4,4)T_{[\vartheta]}+(4,0)T_{[\omega]}+(5,2)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(26,8)$
		4	$(\vartheta,\frac{1}{2}(e_1+e_6)),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\omega]}+(5,2)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(2,2)T_{[\vartheta]}+(4,0)T_{[\omega]}+(4,1)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
		2	$(\vartheta,\frac{1}{2}e_1),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\omega]}+(4,1)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(20,2)$
		3	$(\vartheta,\frac{1}{2}(e_2+e_3+e_6)),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\omega]}+(3,0)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(19,1)$
	3	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(4,4)T_{[\vartheta]}+(8,0)T_{[\omega]}+(6,1)T_{[\vartheta^2]}+(8,0)T_{[\vartheta\omega]}+(3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(31,7)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}e_6), (\omega, 0)$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (6,1)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(27,3)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}e_1)$	
		local	$(2,1)U + (4,4)T_{[\vartheta]} + (4,3)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(21,9)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_2+e_6)), (\omega, \frac{1}{2}e_1)$	
		non-local	$(2,1)U + (4,3)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta\omega\vartheta\omega]}$	$(17,5)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,1)U + (2,2)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,1)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
		2	$(\vartheta, \frac{1}{2}e_1), (\omega, 0)$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,1)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(20,2)$
		3	$(\vartheta, \frac{1}{2}(e_2+e_6)), (\omega, 0)$	
		local	$(2,1)U + (2,2)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,1)T_{[\vartheta^2]} + (8,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta\omega]}$	$(22,4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	1	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(2,1)U + (8,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (16,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta\rho]} + (4,0)T_{[\omega\rho]} + (8,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(61,1)$
		2	$(\vartheta, \frac{1}{2}e_5), (\omega, 0), (\rho, \frac{1}{2}e_5)$	
		local	$(2,1)U + (8,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (8,0)T_{[\rho]} + (4,0)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(37,1)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_3))$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (8,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta\rho]} + (4,0)T_{[\omega\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (6,1)T_{[\rho^2]}$	$(36,6)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_3+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_3+e_5))$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (6,1)T_{[\rho^2]}$	$(22,4)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		5	$(\vartheta, \frac{1}{2}e_1), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4))$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (8,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta\rho]} + (4,0)T_{[\omega\rho]} + (5,2)T_{[\rho^2]}$	$(31,7)$
		6	$(\vartheta, \frac{1}{2}(e_1+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_5))$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (5,2)T_{[\rho^2]}$	$(17,5)$
		7	$(\vartheta, \frac{1}{2}(e_3+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_3+e_4+e_5))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (6,1)T_{[\rho^2]}$	$(22,4)$
		8	$(\vartheta, \frac{1}{2}e_6), (\omega, 0), (\rho, \frac{1}{2}(e_5+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(25,1)$
		9	$(\vartheta, \frac{1}{2}(e_1+e_3+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_3))$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (8,0)T_{[\rho]} + (4,0)T_{[\omega\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (6,1)T_{[\rho^2]}$	$(32,2)$
		10	$(\vartheta, \frac{1}{2}(e_1+e_3+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_3+e_5+e_6))$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (6,1)T_{[\rho^2]}$	$(20,2)$
		11	$(\vartheta, \frac{1}{2}(e_1+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4))$	
		non-local	$(2,1)U + (8,0)T_{[\omega]} + (8,0)T_{[\rho]} + (4,0)T_{[\omega\rho]} + (5,2)T_{[\rho^2]}$	$(27,3)$
		12	$(\vartheta, \frac{1}{2}(e_1+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_5+e_6))$	
		non-local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (5,2)T_{[\rho^2]}$	$(15,3)$
		13	$(\vartheta, \frac{1}{2}(e_2+e_4)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_4))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta\rho]} + (2,2)T_{[\omega\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(31,7)$
		14	$(\vartheta, \frac{1}{2}(e_2+e_4+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_4+e_5))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(27,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		15	$(\vartheta, \frac{1}{2}(e_1+e_2+e_4)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2))$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta\rho]} + (2,2)T_{[\omega\rho]} + (5,2)T_{[\rho^2]}$	$(21, 9)$
		16	$(\vartheta, \frac{1}{2}(e_1+e_2+e_4+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2+e_5))$	
		local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (5,2)T_{[\rho^2]}$	$(17, 5)$
		17	$(\vartheta, \frac{1}{2}(e_2+e_4+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_4+e_5+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(25, 1)$
		18	$(\vartheta, \frac{1}{2}(e_1+e_2+e_4+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2))$	
		non-local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\omega\rho]} + (5,2)T_{[\rho^2]}$	$(17, 5)$
		19	$(\vartheta, \frac{1}{2}(e_1+e_2+e_4+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2+e_5+e_6))$	
		non-local	$(2,1)U + (4,0)T_{[\omega]} + (4,0)T_{[\rho]} + (5,2)T_{[\rho^2]}$	$(15, 3)$
		20	$(\vartheta, \frac{1}{2}e_1), (\omega, \frac{1}{2}(e_1+e_4)), (\rho, \frac{1}{2}e_4)$	
		local	$(2,1)U + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (2,2)T_{[\vartheta\rho]} + (2,2)T_{[\omega\rho]} + (4,3)T_{[\rho^2]}$	$(16, 10)$
		21	$(\vartheta, \frac{1}{2}(e_1+e_5)), (\omega, \frac{1}{2}(e_1+e_4)), (\rho, \frac{1}{2}(e_4+e_5))$	
		non-local	$(2,1)U + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\rho]} + (4,3)T_{[\rho^2]}$	$(12, 6)$
		22	$(\vartheta, \frac{1}{2}(e_1+e_6)), (\omega, \frac{1}{2}(e_1+e_4)), (\rho, \frac{1}{2}(e_4+e_5+e_6))$	
		non-local	$(2,1)U + (4,0)T_{[\rho]} + (4,3)T_{[\rho^2]}$	$(10, 4)$
	2	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (10,0)T_{[\rho]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\rho]} + (2,0)T_{[\omega\rho]} + (6,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(37, 1)$
		2	$(\vartheta, \frac{1}{2}e_6), (\omega, 0), (\rho, \frac{1}{2}(e_5+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (2,0)T_{[\vartheta\rho]} + (3,0)T_{[\rho^2]}$	$(19, 1)$
continued …				

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		3	$(\vartheta,0),(\omega,0),(\rho,\frac{1}{2}e_1)$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(6,0)T_{[\rho]}+(2,0)T_{[\vartheta\omega]}+(6,0)T_{[\vartheta\omega\rho^3]}+(5,0)T_{[\rho^2]}$	$(25,1)$
		4	$(\vartheta,\frac{1}{2}e_6),(\omega,0),(\rho,\frac{1}{2}(e_1+e_5+e_6))$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(4,0)T_{[\rho]}+(3,0)T_{[\rho^2]}$	$(13,1)$
		5	$(\vartheta,\frac{1}{2}e_6),(\omega,\frac{1}{2}(e_2+e_6)),(\rho,\frac{1}{2}e_5)$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(4,0)T_{[\rho]}+(1,1)T_{[\vartheta\omega]}+(2,2)T_{[\vartheta\omega\rho^3]}+(3,0)T_{[\rho^2]}$	$(16,4)$
		6	$(\vartheta,\frac{1}{2}e_1),(\omega,0),(\rho,\frac{1}{2}e_1)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(6,0)T_{[\rho]}+(2,0)T_{[\vartheta\rho]}+(4,0)T_{[\vartheta\omega\rho^3]}+(5,0)T_{[\rho^2]}$	$(25,1)$
		7	$(\vartheta,\frac{1}{2}(e_1+e_6)),(\omega,\frac{1}{2}(e_2+e_6)),(\rho,\frac{1}{2}(e_1+e_5))$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(4,0)T_{[\rho]}+(2,2)T_{[\vartheta\omega\rho^3]}+(3,0)T_{[\rho^2]}$	$(15,3)$
	3	1	$(\vartheta,0),(\omega,0),(\rho,0)$	
		local	$(2,1)U+(3,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(7,0)T_{[\rho]}+(1,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta\rho]}+(1,0)T_{[\omega\rho]}+(3,0)T_{[\vartheta\omega\rho^3]}+(4,0)T_{[\rho^2]}$	$(25,1)$
		2	$(\vartheta,\frac{1}{2}(e_2+e_6)),(\omega,0),(\rho,\frac{1}{2}(e_2+e_6))$	
		local	$(2,1)U+(3,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(7,0)T_{[\rho]}+(1,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta\rho]}+(1,0)T_{[\omega\rho]}+(3,0)T_{[\vartheta\omega\rho^3]}+(4,0)T_{[\rho^2]}$	$(25,1)$
		3	$(\vartheta,\frac{1}{4}(e_1+2e_2+2e_3+3e_4+3e_5+3e_6)),(\omega,0),(\rho,\frac{1}{4}(e_1+3e_2+e_3+2e_4+3e_6))$	
		local	$(2,1)U+(1,1)T_{[\vartheta]}+(1,1)T_{[\omega]}+(4,0)T_{[\rho]}+(1,1)T_{[\vartheta\omega\rho^3]}+(1,0)T_{[\rho^2]}$	$(10,4)$
		4	$(\vartheta,\frac{1}{2}(e_1+e_4+e_5+e_6)),(\omega,0),(\rho,\frac{1}{2}(e_1+e_2+e_3+e_6))$	
		local	$(2,1)U+(3,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(7,0)T_{[\rho]}+(1,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta\rho]}+(1,0)T_{[\omega\rho]}+(3,0)T_{[\vartheta\omega\rho^3]}+(4,0)T_{[\rho^2]}$	$(25,1)$

continued …

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	4	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (12,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta\rho]} + (3,1)T_{[\omega\rho]} + (12,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(51,3)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (8,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta\rho]} + (3,1)T_{[\omega\rho]} + (5,2)T_{[\rho^2]}$	$(31,7)$
		3	$(\vartheta, 0), (\omega, 0), (\rho, \frac{1}{2}e_5)$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (8,0)T_{[\rho]} + (4,0)T_{[\vartheta\omega]} + (12,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(37,1)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_5))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (2,2)T_{[\vartheta\omega]} + (5,2)T_{[\rho^2]}$	$(17,5)$
		5	$(\vartheta, \frac{1}{2}e_6), (\omega, 0), (\rho, \frac{1}{2}e_6)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (3,1)T_{[\vartheta\rho]} + (8,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(32,2)$
		6	$(\vartheta, \frac{1}{2}(e_1+e_2+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (3,1)T_{[\vartheta\rho]} + (5,2)T_{[\rho^2]}$	$(22,4)$
		7	$(\vartheta, \frac{1}{2}e_6), (\omega, 0), (\rho, \frac{1}{2}(e_5+e_6))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (8,0)T_{[\vartheta\omega\rho^3]} + (7,0)T_{[\rho^2]}$	$(25,1)$
		8	$(\vartheta, \frac{1}{2}(e_1+e_2+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_5+e_6))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (5,2)T_{[\rho^2]}$	$(15,3)$
	5	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (10,0)T_{[\rho]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\rho]} + (2,0)T_{[\omega\rho]} + (6,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(37,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}e_1), (\omega, 0), (\rho, \frac{1}{2}e_1)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (2,0)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(25,1)$
		3	$(\vartheta, \frac{1}{2}(e_2+e_3+e_4+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_3+e_4+e_5))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (2,0)T_{[\vartheta\omega]} + (1,1)T_{[\vartheta\rho]} + (1,1)T_{[\omega\rho]} + (6,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(27,3)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_2+e_3+e_4+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2+e_3+e_4+e_5))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (1,1)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(20,2)$
		5	$(\vartheta, \frac{1}{2}(e_3+e_4)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_3))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (1,1)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\rho]} + (1,1)T_{[\omega\rho]} + (4,1)T_{[\rho^2]}$	$(22,4)$
		6	$(\vartheta, \frac{1}{2}(e_1+e_3+e_4)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_2+e_3))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (6,0)T_{[\rho]} + (2,0)T_{[\vartheta\rho]} + (4,1)T_{[\rho^2]}$	$(20,2)$
		7	$(\vartheta, \frac{1}{2}(e_1+e_2+e_5)), (\omega, 0), (\rho, \frac{1}{2}(e_1+e_4+e_5))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (1,1)T_{[\vartheta\rho]} + (4,1)T_{[\rho^2]}$	$(15,3)$
		8	$(\vartheta, \frac{1}{2}(e_2+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (8,0)T_{[\rho]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\rho]} + (1,1)T_{[\omega\rho]} + (6,0)T_{[\vartheta\omega\rho^3]} + (5,0)T_{[\rho^2]}$	$(32,2)$
		9	$(\vartheta, \frac{1}{2}(e_2+e_3+e_4+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_3+e_6))$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (8,0)T_{[\rho]} + (1,1)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta\rho]} + (2,0)T_{[\omega\rho]} + (4,1)T_{[\rho^2]}$	$(27,3)$
		10	$(\vartheta, \frac{1}{2}(e_5+e_6)), (\omega, 0), (\rho, \frac{1}{2}(e_2+e_4+e_5+e_6))$	
		local	$(2,1)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\rho]} + (1,1)T_{[\vartheta\omega]} + (1,1)T_{[\vartheta\rho]} + (1,1)T_{[\omega\rho]} + (4,1)T_{[\rho^2]}$	$(17,5)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		11	$(\vartheta,0),(\omega,0),(\rho,\frac{1}{2}e_1)$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(6,0)T_{[\rho]}+(2,0)T_{[\vartheta\omega]}+(6,0)T_{[\vartheta\omega\rho^3]}+(5,0)T_{[\rho^2]}$	$(25,1)$
		12	$(\vartheta,\frac{1}{2}(e_3+e_4)),(\omega,0),(\rho,\frac{1}{2}(e_1+e_2+e_3))$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(4,0)T_{[\rho]}+(1,1)T_{[\vartheta\omega]}+(4,1)T_{[\rho^2]}$	$(15,3)$
		13	$(\vartheta,\frac{1}{2}(e_2+e_6)),(\omega,0),(\rho,\frac{1}{2}(e_1+e_2+e_6))$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(6,0)T_{[\rho]}+(2,0)T_{[\vartheta\omega]}+(6,0)T_{[\vartheta\omega\rho^3]}+(5,0)T_{[\rho^2]}$	$(25,1)$
		14	$(\vartheta,\frac{1}{2}(e_2+e_3+e_4+e_6)),(\omega,0),(\rho,\frac{1}{2}(e_1+e_3+e_6))$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(4,0)T_{[\rho]}+(1,1)T_{[\vartheta\omega]}+(4,1)T_{[\rho^2]}$	$(15,3)$
$\mathbb{Z}_3\times S_3$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(9,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(15,0)T_{[\omega^2]}+(9,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(9,0)T_{[\vartheta^2\omega^4]}$	$(49,1)$
		2	$(\vartheta,0),(\omega,\frac{1}{3}(e_3+e_4))$	
		local	$(2,0)U+(3,3)T_{[\vartheta]}+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(16,4)$
		3	$(\vartheta,\frac{1}{3}(e_5+e_6)),(\omega,0)$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(3,3)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(3,3)T_{[\vartheta^2\omega^4]}$	$(19,7)$
		4	$(\vartheta,\frac{1}{3}(e_5+e_6)),(\omega,\frac{1}{3}(e_3+e_4))$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(13,1)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(3,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(9,0)T_{[\omega^2]}+(3,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(3,0)T_{[\vartheta^2\omega^4]}$	$(25,1)$
		2	$(\vartheta,\frac{1}{3}(2e_1+e_5)),(\omega,\frac{1}{3}(2e_2+e_3))$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(13,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	3	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(5,2)T_{[\vartheta]}+(3,0)T_{[\omega]}+(9,0)T_{[\omega^2]}+(3,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(3,0)T_{[\vartheta^2\omega^4]}$	$(27,3)$
		2	$(\vartheta,0),(\omega,\frac{1}{3}(e_1+e_3))$	
		local	$(2,0)U+(3,3)T_{[\vartheta]}+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(16,4)$
		3	$(\vartheta,\frac{1}{3}(2e_5+e_6)),(\omega,0)$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(1,1)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(1,1)T_{[\vartheta^2\omega^4]}$	$(15,3)$
		4	$(\vartheta,\frac{1}{3}(2e_5+e_6)),(\omega,\frac{1}{3}(e_1+e_3))$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(13,1)$
	4	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(3,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(9,0)T_{[\omega^2]}+(3,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(3,0)T_{[\vartheta^2\omega^4]}$	$(25,1)$
		2	$(\vartheta,0),(\omega,\frac{1}{3}(e_2+e_4))$	
		local	$(2,0)U+(1,1)T_{[\vartheta]}+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(1,1)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(1,1)T_{[\vartheta^2\omega^4]}$	$(16,4)$
		3	$(\vartheta,\frac{1}{3}(e_5+e_6)),(\omega,\frac{1}{3}(e_1+e_2))$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(2,1)T_{[\omega^3]}$	$(13,1)$
	5	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(1,0)T_{[\vartheta]}+(3,0)T_{[\omega]}+(7,0)T_{[\omega^2]}+(1,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(1,0)T_{[\vartheta^2\omega^4]}$	$(17,1)$
	6	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(5,2)T_{[\vartheta]}+(3,0)T_{[\omega]}+(9,0)T_{[\omega^2]}+(3,0)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(3,0)T_{[\vartheta^2\omega^4]}$	$(27,3)$
		2	$(\vartheta,\frac{1}{3}(2e_5+e_6)),(\omega,0)$	
		local	$(2,0)U+(3,0)T_{[\omega]}+(6,0)T_{[\omega^2]}+(1,1)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\omega^3]}+(1,1)T_{[\vartheta^2\omega^4]}$	$(15,3)$
Frobenius T_7	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(1,0)U+(1,1)T_{[\vartheta]}+(7,0)T_{[\omega]}+(1,1)T_{[\vartheta^2]}$	$(10,2)$
				continued …

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (7,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(10, 2)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (7,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]}$	$(10, 2)$
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (3,2)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta^3]} + (4,0)T_{[\vartheta^2\omega]} + (3,1)T_{[\vartheta^4]} + (1,0)T_{[\vartheta^4\omega]}$	$(27, 3)$
SL$(2,3)-$I	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (4,4)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(29, 5)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(25, 1)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (4,4)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(29, 5)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(25, 1)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (4,4)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(29, 5)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (5,1)T_{[\omega^2]}$	$(25, 1)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (1,1)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (12,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\omega^2]}$	$(25, 1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
$\mathbb{Z}_4\times S_3$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(3,1)T_{[\vartheta\omega]}+(1,0)T_{[\omega^2]}+(4,0)T_{[\vartheta\omega^6]}+(8,0)T_{[\omega^3]}+(3,1)T_{[\vartheta\omega^{11}]}+(3,2)T_{[\omega^4]}+(4,1)T_{[\omega^6]}$	$(36,6)$
		2	$(\vartheta,0),(\omega,\frac{1}{2}e_5)$	
		local	$(2,1)U+(2,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(1,0)T_{[\omega^2]}+(2,0)T_{[\vartheta\omega^6]}+(4,0)T_{[\omega^3]}+(3,2)T_{[\omega^4]}+(4,1)T_{[\omega^6]}$	$(22,4)$
$(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(8,0)T_{[\omega]}+(4,4)T_{[\vartheta\omega]}+(4,1)T_{[\omega^2]}+(4,0)T_{[\omega^3]}+(4,1)T_{[\vartheta\omega\vartheta\omega]}+(1,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(31,7)$
		2	$(\vartheta,0),(\omega,\frac{1}{4}e_5)$	
		local	$(2,1)U+(2,2)T_{[\vartheta]}+(6,0)T_{[\omega]}+(4,1)T_{[\omega^2]}+(2,0)T_{[\omega^3]}$	$(16,4)$
		3	$(\vartheta,0),(\omega,\frac{1}{2}e_5)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(8,0)T_{[\omega]}+(4,1)T_{[\omega^2]}+(4,0)T_{[\omega^3]}+(4,1)T_{[\vartheta\omega\vartheta\omega]}+(1,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(27,3)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(8,0)T_{[\omega]}+(4,4)T_{[\vartheta\omega]}+(4,1)T_{[\omega^2]}+(8,0)T_{[\omega^3]}+(5,0)T_{[\vartheta\omega\vartheta\omega]}+(1,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(36,6)$
		2	$(\vartheta,0),(\omega,\frac{1}{4}e_5)$	
		local	$(2,1)U+(2,2)T_{[\vartheta]}+(6,0)T_{[\omega]}+(4,1)T_{[\omega^2]}+(4,2)T_{[\omega^3]}$	$(18,6)$
		3	$(\vartheta,0),(\omega,\frac{1}{2}e_5)$	
		local	$(2,1)U+(4,0)T_{[\vartheta]}+(8,0)T_{[\omega]}+(4,1)T_{[\omega^2]}+(8,0)T_{[\omega^3]}+(5,0)T_{[\vartheta\omega\vartheta\omega]}+(1,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(32,2)$
$\mathbb{Z}_3\times D_4$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(6,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(3,0)T_{[\vartheta\omega]}+(6,0)T_{[\omega^2]}+(2,0)T_{[\omega\vartheta\omega]}+(2,0)T_{[\omega^3]}+(2,1)T_{[\vartheta\omega^3]}+(2,0)T_{[\vartheta\omega\vartheta\omega]}+(4,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(31,1)$
continued …				

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (2,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (3,0)T_{[\vartheta\omega]} + (6,0)T_{[\omega^2]} + (2,0)T_{[\omega\vartheta\omega]} + (2,0)T_{[\omega^3]} + (1,0)T_{[\vartheta\omega^3]} + (2,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta\omega^5]}$	$(24, 0)$
$\mathbb{Z}_3 \times Q_8$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,2)T_{[\omega^6]} + (3,0)T_{[\vartheta\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\vartheta\omega^8]} + (6,0)T_{[\omega^8]} + (2,1)T_{[\omega^3]} + (2,1)T_{[\vartheta\omega^3]}$	$(29, 5)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,1)T_{[\omega^6]} + (3,0)T_{[\vartheta\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\vartheta\omega^8]} + (6,0)T_{[\omega^8]} + (1,0)T_{[\omega^3]} + (1,0)T_{[\vartheta\omega^3]}$	$(25, 1)$
S_4	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,4)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (10,0)T_{[\omega^2]}$	$(20, 6)$
		2	$(\vartheta, \frac{1}{4}(e_1 + e_3)), (\omega, \frac{1}{4}(e_1 + 3e_2))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]}$	$(4, 4)$
		3	$(\vartheta, \frac{1}{2}(e_1 + e_3)), (\omega, \frac{1}{2}(e_1 + e_2))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (10,0)T_{[\omega^2]}$	$(16, 2)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\omega^2]}$	$(14, 4)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_5)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\omega^2]}$	$(9, 3)$
		3	$(\vartheta, \frac{1}{4}(e_1 + e_2)), (\omega, \frac{1}{4}(e_2 + 3e_3))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]}$	$(4, 4)$
		4	$(\vartheta, \frac{1}{2}(e_1 + e_2)), (\omega, \frac{1}{2}(e_2 + e_3))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\omega^2]}$	$(12, 2)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\omega^2]}$	$(14, 4)$
			continued ...	

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}e_4), (\omega, \frac{1}{2}(e_5+e_6))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (4,0)T_{[\omega^2]}$	$(8,4)$
		3	$(\vartheta, \frac{1}{4}(e_1+e_2)), (\omega, \frac{1}{4}(e_2+3e_3))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]}$	$(4,4)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}(e_2+e_3))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\omega^2]}$	$(12,2)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (4,0)T_{[\omega^2]}$	$(11,3)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}e_4)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]}$	$(8,2)$
	5	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (4,0)T_{[\omega^2]}$	$(11,3)$
		2	$(\vartheta, \frac{1}{2}e_2), (\omega, \frac{1}{2}e_1)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]}$	$(6,4)$
		3	$(\vartheta, \frac{1}{2}(e_4+e_5)), (\omega, \frac{1}{2}e_4)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]}$	$(8,2)$
		4	$(\vartheta, \frac{1}{2}(e_2+e_4+e_5)), (\omega, \frac{1}{2}(e_1+e_4))$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (1,1)T_{[\omega^2]}$	$(5,5)$
	6	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (4,0)T_{[\omega^2]}$	$(11,3)$
		2	$(\vartheta, \frac{1}{2}e_4), (\omega, \frac{1}{2}e_3)$	
		local	$(1,1)U + (1,1)T_{[\vartheta]} + (2,2)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]}$	$(6,4)$
$\Delta(27)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (3,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\omega\vartheta\omega]} + (3,0)T_{[\omega\vartheta^2\omega]} + (11,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(36,0)$
			continued …	

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2 local	$(\vartheta, \frac{1}{3}(e_2+e_5)), (\omega, 0)$ $(1,0)U + (1,1)T_{[\omega]} + (1,1)T_{[\vartheta\omega]} + (1,1)T_{[\omega^2]} + (1,1)T_{[\omega\vartheta^2\omega]} + (3,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(8,4)$
		3 non-local	$(\vartheta, \frac{1}{3}(2e_1+2e_2+e_5)), (\omega, \frac{1}{3}e_1)$ $(1,0)U + (3,0)T_{[\vartheta\omega]} + (3,0)T_{[\omega\vartheta^2\omega]} + (5,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(12,0)$
		4 non-local	$(\vartheta, \frac{1}{3}(2e_2+e_3+2e_5)), (\omega, \frac{1}{3}e_1)$ $(1,0)U + (3,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(4,0)$
	2	1 local	$(\vartheta, 0), (\omega, 0)$ $(1,0)U + (1,0)T_{[\vartheta]} + (9,0)T_{[\omega]} + (1,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (9,0)T_{[\omega^2]} + (1,0)T_{[\omega\vartheta\omega]} + (1,0)T_{[\omega\vartheta^2\omega]} + (11,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(36,0)$
		2 local	$(\vartheta, \frac{1}{3}(2e_3+e_4)), (\omega, \frac{1}{3}(e_1+e_4))$ $(1,0)U + (1,0)T_{[\vartheta]} + (1,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (1,0)T_{[\omega\vartheta\omega]} + (1,0)T_{[\omega\vartheta^2\omega]} + (5,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(12,0)$
	3	1 local	$(\vartheta, 0), (\omega, 0)$ $(1,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (3,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\omega\vartheta\omega]} + (3,0)T_{[\omega\vartheta^2\omega]} + (11,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(36,0)$
		2 local	$(\vartheta, \frac{1}{3}(e_2+e_4+e_6)), (\omega, 0)$ $(1,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (3,0)T_{[\omega^2]} + (7,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(20,0)$
		3 non-local	$(\vartheta, \frac{1}{3}(2e_2+2e_4+e_5+e_6)), (\omega, 0)$ $(1,0)U+(3,0)T_{[\omega]}+(3,0)T_{[\omega^2]}+(5,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(12,0)$
		4 non-local	$(\vartheta, \frac{1}{3}(e_2+e_4+e_5)), (\omega, \frac{1}{3}(e_1+e_3))$ $(1,0)U + (3,0)T_{[\vartheta\omega\vartheta^2\omega^2]}$	$(4,0)$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	1	1 local	$(\vartheta, 0), (\omega, 0)$ $(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (9,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\vartheta^2\omega]} + (6,0)T_{[\vartheta^4]} + (12,0)T_{[\vartheta^3\omega]} + (4,0)T_{[\vartheta^7\omega]} + (9,0)T_{[\vartheta\omega\vartheta\omega]} + (4,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(61,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}e_6), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (7,0)T_{[\vartheta\omega\vartheta\omega]}$	$(36,0)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (7,0)T_{[\vartheta^2]} + (3,1)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (7,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(37,1)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_3+e_6)), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega]}$	$(24,0)$
		5	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (7,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta^4]} + (12,0)T_{[\vartheta^3\omega]} + (4,0)T_{[\vartheta^7\omega]} + (9,0)T_{[\vartheta\omega\vartheta\omega]} + (4,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(54,0)$
		6	$(\vartheta, \frac{1}{2}(e_1+e_3+e_5)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (5,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (7,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(30,0)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (7,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(42,0)$
		2	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (5,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (1,1)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(37,1)$
		3	$(\vartheta, \frac{1}{2}e_2), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (5,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(36,0)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\vartheta^2]} + (1,1)T_{[\vartheta^2\omega]} + (3,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(25,1)$
		5	$(\vartheta, \frac{1}{2}(e_1+e_3+e_5)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(30,0)$
		6	$(\vartheta, \frac{1}{2}(e_1+e_2+e_3)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\vartheta^2]} + (3,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(24,0)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (8,0)T_{[\omega]} + (10,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta^2\omega]} + (5,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(54,0)$
		2	$(\vartheta, \frac{1}{2}e_5), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta\omega]} + (5,0)T_{[\vartheta^4]} + (8,0)T_{[\vartheta^3\omega]} + (2,0)T_{[\vartheta^7\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(42,0)$
		3	$(\vartheta, \frac{1}{2}e_6), (\omega, \frac{1}{2}e_5)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (5,0)T_{[\vartheta^4]} + (6,0)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta^7\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(30,0)$
		4	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (2,2)T_{[\vartheta^2\omega]} + (4,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (1,1)T_{[\vartheta\omega\vartheta^3\omega]}$	$(27,3)$
		5	$(\vartheta, \frac{1}{2}(e_1+e_3+e_5)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (1,1)T_{[\vartheta\omega\vartheta^3\omega]}$	$(25,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)},h^{(2,1)})$ from U and T sectors	$(h^{(1,1)},h^{(2,1)})$
		6	$(\vartheta,\frac{1}{2}(e_1+e_3+e_6)),(\omega,\frac{1}{2}e_5)$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(3,0)T_{[\vartheta^2]}+(4,0)T_{[\vartheta^4]}+(4,0)T_{[\vartheta^3\omega]}+(3,0)T_{[\vartheta\omega\vartheta\omega]}$	$(24,0)$
		7	$(\vartheta,\frac{1}{2}e_4),(\omega,\frac{1}{2}(e_1+e_2))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(6,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,2)T_{[\vartheta^2\omega]}+(4,1)T_{[\vartheta^4]}+(8,0)T_{[\vartheta^3\omega]}+(2,0)T_{[\vartheta^7\omega]}+(6,0)T_{[\vartheta\omega\vartheta\omega]}+(3,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(39,3)$
		8	$(\vartheta,\frac{1}{2}(e_4+e_5)),(\omega,\frac{1}{2}(e_1+e_2))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(6,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(4,1)T_{[\vartheta^4]}+(8,0)T_{[\vartheta^3\omega]}+(2,0)T_{[\vartheta^7\omega]}+(6,0)T_{[\vartheta\omega\vartheta\omega]}+(3,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(37,1)$
		9	$(\vartheta,\frac{1}{2}(e_4+e_6)),(\omega,\frac{1}{2}(e_1+e_2+e_5))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(3,0)T_{[\vartheta^2]}+(1,0)T_{[\vartheta\omega]}+(4,1)T_{[\vartheta^4]}+(6,0)T_{[\vartheta^3\omega]}+(1,0)T_{[\vartheta^7\omega]}+(4,0)T_{[\vartheta\omega\vartheta\omega]}$	$(25,1)$
		10	$(\vartheta,\frac{1}{2}(e_1+e_3+e_4)),(\omega,\frac{1}{2}(e_1+e_2))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(3,0)T_{[\vartheta^2]}+(2,2)T_{[\vartheta^2\omega]}+(3,1)T_{[\vartheta^4]}+(4,0)T_{[\vartheta^3\omega]}+(3,0)T_{[\vartheta\omega\vartheta\omega]}+(1,1)T_{[\vartheta\omega\vartheta^3\omega]}$	$(22,4)$
		11	$(\vartheta,\frac{1}{2}(e_1+e_3+e_4+e_5)),(\omega,\frac{1}{2}(e_1+e_2))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(3,0)T_{[\vartheta^2]}+(3,1)T_{[\vartheta^4]}+(4,0)T_{[\vartheta^3\omega]}+(3,0)T_{[\vartheta\omega\vartheta\omega]}+(1,1)T_{[\vartheta\omega\vartheta^3\omega]}$	$(20,2)$
		12	$(\vartheta,\frac{1}{2}(e_1+e_3+e_4+e_6)),(\omega,\frac{1}{2}(e_1+e_2+e_5))$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(3,0)T_{[\vartheta^2]}+(3,1)T_{[\vartheta^4]}+(4,0)T_{[\vartheta^3\omega]}+(3,0)T_{[\vartheta\omega\vartheta\omega]}$	$(19,1)$
	4	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(4,0)T_{[\vartheta]}+(4,0)T_{[\omega]}+(7,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^2\omega]}+(4,0)T_{[\vartheta^4]}+(8,0)T_{[\vartheta^3\omega]}+(2,0)T_{[\vartheta^7\omega]}+(5,0)T_{[\vartheta\omega\vartheta\omega]}+(2,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(42,0)$
continued …				

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}(e_2+e_3+e_4+e_5+e_6)), (\omega, \frac{1}{2}(e_1+e_2+e_4))$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (2,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (2,1)T_{[\vartheta\omega\vartheta\omega]}$	$(19,1)$
		3	$(\vartheta, \frac{1}{2}e_3), (\omega, \frac{1}{2}e_2)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (4,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (4,0)T_{[\vartheta^4]} + (6,0)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta^7\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(30,0)$
	5	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (5,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\vartheta^4]} + (6,0)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta^7\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(30,0)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_2) + \frac{1}{4}(e_3+3e_4)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (1,1)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta^4]} + (4,0)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta\omega\vartheta\omega]}$	$(16,1)$
		3	$(\vartheta, \frac{1}{2}(e_3+e_4)), (\omega, 0)$	
		local	$(2,0)U + (4,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (5,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\vartheta^4]} + (6,0)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta^7\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (1,0)T_{[\vartheta\omega\vartheta^3\omega]}$	$(30,0)$
$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (5,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\omega^2]} + (6,0)T_{[\vartheta^2\omega^2]} + (2,1)T_{[\omega^3]} + (9,0)T_{[\omega^4]} + (6,0)T_{[\vartheta^2\omega^4]} + (3,0)T_{[\omega^5]} + (3,1)T_{[\omega^6]} + (1,0)T_{[\vartheta^2\omega^6]} + (6,0)T_{[\vartheta^2\omega^8]} + (2,1)T_{[\omega^9]}$	$(51,3)$
$\mathbb{Z}_3 \times A_4$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (2,0)T_{[\omega]} + (1,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (5,0)T_{[\omega^2]} + (1,0)T_{[\omega\vartheta\omega]} + (1,0)T_{[\omega\vartheta^2\omega]} + (6,0)T_{[\omega^3]}$	$(20,0)$

continued …

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(1,0)U+(1,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(1,0)T_{[\vartheta^2]}+(1,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta^2\omega]}+(5,0)T_{[\omega^2]}+(1,0)T_{[\omega\vartheta\omega]}+(1,0)T_{[\omega\vartheta^2\omega]}+(2,0)T_{[\omega^3]}$	$(16,0)$
	3	1	$(\vartheta,0),(\omega,0)$	
		local	$(1,0)U+(1,0)T_{[\vartheta]}+(2,0)T_{[\omega]}+(1,0)T_{[\vartheta^2]}+(1,0)T_{[\vartheta\omega]}+(1,0)T_{[\vartheta^2\omega]}+(5,0)T_{[\omega^2]}+(1,0)T_{[\omega\vartheta\omega]}+(1,0)T_{[\omega\vartheta^2\omega]}+(2,0)T_{[\omega^3]}$	$(16,0)$
$\mathbb{Z}_6\times S_3$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(2,0)T_{[\vartheta]}+(1,0)T_{[\omega]}+(9,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^3]}+(6,0)T_{[\vartheta^2\omega]}+(2,0)T_{[\vartheta^3\omega]}+(5,0)T_{[\omega^2]}+(6,0)T_{[\vartheta^2\omega^2]}+(6,0)T_{[\vartheta^4\omega^2]}+(3,0)T_{[\omega^3]}+(2,0)T_{[\vartheta^2\omega^3]}$	$(48,0)$
		2	$(\vartheta,0),(\omega,\frac{1}{3}(e_5+e_6))$	
		local	$(2,0)U+(2,0)T_{[\vartheta]}+(4,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^3]}+(4,0)T_{[\vartheta^2\omega]}+(2,0)T_{[\vartheta^3\omega]}+(2,1)T_{[\vartheta^2\omega^2]}+(2,1)T_{[\vartheta^4\omega^2]}+(3,0)T_{[\omega^3]}+(1,0)T_{[\vartheta^2\omega^3]}$	$(26,2)$
	2	1	$(\vartheta,0),(\omega,0)$	
		local	$(2,0)U+(2,0)T_{[\vartheta]}+(1,0)T_{[\omega]}+(6,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^3]}+(6,0)T_{[\vartheta^2\omega]}+(2,0)T_{[\vartheta^3\omega]}+(3,1)T_{[\omega^2]}+(3,0)T_{[\vartheta^2\omega^2]}+(3,0)T_{[\vartheta^4\omega^2]}+(3,0)T_{[\omega^3]}+(2,0)T_{[\vartheta^2\omega^3]}$	$(37,1)$
		2	$(\vartheta,0),(\omega,\frac{1}{3}(2e_5+e_6))$	
		local	$(2,0)U+(2,0)T_{[\vartheta]}+(4,0)T_{[\vartheta^2]}+(2,0)T_{[\vartheta\omega]}+(2,0)T_{[\vartheta^3]}+(4,0)T_{[\vartheta^2\omega]}+(2,0)T_{[\vartheta^3\omega]}+(1,0)T_{[\vartheta^2\omega^2]}+(1,0)T_{[\vartheta^4\omega^2]}+(3,0)T_{[\omega^3]}+(1,0)T_{[\vartheta^2\omega^3]}$	$(24,0)$
$\Delta(48)$	1	1	$(\vartheta,0),(\omega,0)$	
		local	$(1,0)U+(1,1)T_{[\vartheta]}+(4,0)T_{[\omega]}+(1,1)T_{[\vartheta^2]}+(9,0)T_{[\omega^2]}+(4,0)T_{[\omega^3]}+(12,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(32,2)$
			continued …	

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}(e_1+e_3))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\vartheta^2]} + (5,0)T_{[\omega^2]} + (4,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(12,2)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (2,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]} + (5,0)T_{[\omega^2]} + (2,0)T_{[\omega^3]} + (8,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(20,2)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1+e_3))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\vartheta^2]} + (2,1)T_{[\omega^2]} + (4,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(9,3)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]} + (5,0)T_{[\omega^2]} + (1,0)T_{[\omega^3]} + (6,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(16,2)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (2,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]} + (5,0)T_{[\omega^2]} + (2,0)T_{[\omega^3]} + (8,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(20,2)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,0)T_{[\omega]} + (1,1)T_{[\vartheta^2]} + (3,0)T_{[\omega^2]} + (1,0)T_{[\omega^3]} + (6,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(14,2)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, \frac{1}{2}(e_1+e_2))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\vartheta^2]} + (1,0)T_{[\omega^2]} + (4,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]}$	$(8,2)$
GL(2,3)	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,1)U + (5,5)T_{[\vartheta]} + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\omega^2]} + (1,1)T_{[\omega\vartheta^2\omega]} + (3,0)T_{[\omega^4]}$	$(26,8)$
		2	$(\vartheta, \frac{1}{3}e_6), (\omega, \frac{1}{3}e_6)$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\omega^2]} + (3,0)T_{[\omega^4]}$	$(20,2)$
		3	$(\vartheta, \frac{1}{2}e_2 + \frac{2}{3}e_5), (\omega, \frac{1}{2}e_2 + \frac{2}{3}e_5)$	
		local	$(2,1)U + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\omega^2]} + (3,0)T_{[\omega^4]}$	$(20,2)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		4	$(\vartheta, \frac{1}{2}e_2), (\omega, \frac{1}{2}e_2)$	
		local	$(2,1)U + (5,5)T_{[\vartheta]} + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (3,1)T_{[\omega^2]} + (1,1)T_{[\omega\vartheta^2\omega]} + (3,0)T_{[\omega^4]}$	$(26,8)$
$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,1)U + (4,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\omega^6]} + (4,0)T_{[\omega^7]} + (1,0)T_{[\omega^2]} + (3,2)T_{[\omega\vartheta\omega]} + (8,0)T_{[\omega^3]} + (8,0)T_{[\omega^2\vartheta\omega]} + (3,2)T_{[\omega^4]} + (1,0)T_{[\omega^{10}]}$	$(41,5)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_3)), (\omega, \frac{1}{2}(e_1+e_3))$	
		local	$(2,1)U + (2,2)T_{[\vartheta]} + (4,0)T_{[\omega]} + (2,1)T_{[\omega^6]} + (4,0)T_{[\omega^7]} + (1,0)T_{[\omega^2]} + (3,2)T_{[\omega\vartheta\omega]} + (4,0)T_{[\omega^3]} + (3,2)T_{[\omega^4]} + (1,0)T_{[\omega^{10}]}$	$(26,8)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_2+e_3+e_4)), (\omega, \frac{1}{2}(e_1+e_2+e_3+e_4))$	
		local	$(2,1)U + (2,2)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\omega^6]} + (4,0)T_{[\omega^7]} + (1,0)T_{[\omega^2]} + (3,2)T_{[\omega\vartheta\omega]} + (4,0)T_{[\omega^3]} + (4,0)T_{[\omega^2\vartheta\omega]} + (3,2)T_{[\omega^4]} + (1,0)T_{[\omega^{10}]}$	$(31,7)$
$\Delta(54)$	1	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (9,0)T_{[\omega]} + (1,0)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (1,0)T_{[\omega\rho]} + (1,0)T_{[\omega^2\rho]} + (7,0)T_{[\omega\rho^2\omega^2\rho]}$	$(25,1)$
		2	$(\vartheta, 0), (\omega, \frac{1}{3}(e_1+e_2+2e_3+2e_4+e_5+e_6)), (\rho, \frac{1}{3}(e_1+e_2+2e_3+2e_4+e_5+e_6))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (1,0)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (1,0)T_{[\omega\rho]} + (1,0)T_{[\omega^2\rho]} + (4,0)T_{[\omega\rho^2\omega^2\rho]}$	$(13,1)$
	2	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega\rho]} + (3,0)T_{[\omega^2\rho]} + (7,0)T_{[\omega\rho^2\omega^2\rho]}$	$(25,1)$
continued …				

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{3}(2e_1 + e_2 + e_3 + 2e_5)), (\omega, \frac{2}{3}(e_1 + e_3 + e_4)), (\rho, \frac{2}{3}(e_1 + e_3 + e_4))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega^2\rho]} + (5,0)T_{[\omega\rho^2\omega^2\rho]}$	$(17,1)$
		3	$(\vartheta, \frac{1}{3}(2e_1 + 2e_3 + e_4 + 2e_5 + e_6)), (\omega, \frac{1}{3}(2e_2 + 2e_3 + e_6)), (\rho, \frac{1}{3}(2e_2 + 2e_3 + e_6))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega^2\rho]} + (4,0)T_{[\omega\rho^2\omega^2\rho]}$	$(13,1)$
		4	$(\vartheta, 0), (\omega, \frac{1}{3}(e_2 + e_3 + e_4)), (\rho, \frac{1}{3}(e_2 + e_5 + 2e_6))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega\rho^2\omega^2\rho]}$	$(9,1)$
	3	1	$(\vartheta, 0), (\omega, 0), (\rho, 0)$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega\rho]} + (3,0)T_{[\omega^2\rho]} + (7,0)T_{[\omega\rho^2\omega^2\rho]}$	$(25,1)$
		2	$(\vartheta, \frac{1}{3}(2e_1 + e_3 + e_4 + 2e_6)), (\omega, \frac{1}{3}(2e_1 + 2e_3 + e_4)), (\rho, \frac{1}{3}(2e_1 + 2e_3 + e_4))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (1,1)T_{[\rho]} + (3,0)T_{[\vartheta\rho]} + (1,1)T_{[\omega^2\rho]} + (3,0)T_{[\omega\rho^2\omega^2\rho]}$	$(11,3)$
		3	$(\vartheta, \frac{1}{3}(e_2 + e_5)), (\omega, \frac{1}{3}(e_2 + 2e_3)), (\rho, \frac{1}{3}(e_2 + 2e_3))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega^2\rho]} + (4,0)T_{[\omega\rho^2\omega^2\rho]}$	$(13,1)$
		4	$(\vartheta, \frac{1}{3}(e_1 + e_2 + e_3 + e_4 + 2e_6)), (\omega, \frac{1}{3}(2e_1 + e_2 + 2e_3 + e_4)), (\rho, \frac{1}{3}(e_1 + 2e_3 + 2e_4 + 2e_5 + e_6))$	
		local	$(1,0)U + (2,1)T_{[\vartheta]} + (3,0)T_{[\vartheta\rho]} + (3,0)T_{[\omega\rho^2\omega^2\rho]}$	$(9,1)$
$\mathbb{Z}_3 \times \mathrm{SL}(2,3)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (3,0)T_{[\vartheta^2]} + (3,1)T_{[\vartheta\omega]} + (6,0)T_{[\vartheta^2\omega]} + (3,0)T_{[\omega^2]} + (1,0)T_{[\omega\vartheta^2\omega]} + (2,1)T_{[\omega^3]} + (6,0)T_{[\vartheta\omega^3]} + (6,0)T_{[\omega^4]} + (3,0)T_{[\omega\vartheta^2\omega^3]} + (3,0)T_{[\vartheta\omega^5]} + (3,0)T_{[\omega^6]} + (1,0)T_{[\vartheta\omega^7]} + (3,1)T_{[\omega\vartheta^2\omega^7]}$	$(51,3)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{3}(e_5+e_6)), (\omega, \frac{1}{3}(e_5+e_6))$	
		local	$(2,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (1,0)T_{[\vartheta^2]} + (4,0)T_{[\vartheta^2\omega]} + (1,0)T_{[\omega^2]} + (2,1)T_{[\omega^3]} + (4,0)T_{[\vartheta\omega^3]} + (2,0)T_{[\omega^4]} + (1,0)T_{[\omega\vartheta^2\omega^3]} + (1,0)T_{[\vartheta\omega^5]} + (3,0)T_{[\omega^6]}$	$(25,1)$
$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$	1	1	$(\vartheta,0), (\omega,0)$	
		local	$(2,0)U + (2,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (6,0)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^3]} + (4,0)T_{[\vartheta^2\omega]} + (2,1)T_{[\vartheta^3\omega]} + (1,0)T_{[\vartheta^4\omega]} + (4,0)T_{[\omega^2]} + (4,0)T_{[\vartheta^2\omega^2]} + (4,0)T_{[\vartheta^4\omega^2]} + (4,0)T_{[\omega^3]} + (2,0)T_{[\vartheta^2\omega^3]} + (1,0)T_{[\vartheta^2\omega^5]} + (2,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^5\omega]} + (4,0)T_{[\vartheta\omega\vartheta\omega^5]} + (1,0)T_{[\vartheta\omega\vartheta^5\omega^5]}$	$(55,1)$
$\mathbb{Z}_3 \times S_4$	1	1	$(\vartheta,0), (\omega,0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]} + (1,0)T_{[\vartheta^2\omega^8]} + (2,1)T_{[\omega^3]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (1,0)T_{[\omega^2\vartheta^2\omega^8]} + (4,0)T_{[\omega^6]}$	$(23,1)$
	2	1	$(\vartheta,0), (\omega,0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]} + (1,0)T_{[\vartheta^2\omega^8]} + (1,0)T_{[\omega^3]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (1,0)T_{[\omega^2\vartheta^2\omega^8]} + (2,0)T_{[\omega^6]}$	$(20,0)$
	3	1	$(\vartheta,0), (\omega,0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,0)T_{[\vartheta\omega]} + (2,0)T_{[\omega^2]} + (1,0)T_{[\vartheta^2\omega^8]} + (1,0)T_{[\omega^3]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (1,0)T_{[\omega^2\vartheta^2\omega^8]} + (2,0)T_{[\omega^6]}$	$(20,0)$
$\Delta(96)$	1	1	$(\vartheta,0), (\omega,0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,1)T_{[\vartheta\omega]} + (9,0)T_{[\omega^2]} + (4,0)T_{[\omega^2\vartheta^2\omega]} + (6,0)T_{[\omega^4]} + (4,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(32,2)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{2}(e_3+e_4)), (\omega, \frac{1}{2}(e_4+e_5+e_6))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]}$	$(15,1)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}(e_4+e_6))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (7,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (6,0)T_{[\omega^4]} + (4,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(25,1)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (2,0)T_{[\vartheta\omega]} + (7,0)T_{[\omega^2]} + (4,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (2,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(25,1)$
		2	$(\vartheta, \frac{1}{2}(e_1+e_2)), (\omega, \frac{1}{2}(e_3+e_4+e_5))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (1,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(17,1)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1+e_3+e_6))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (2,0)T_{[\omega^4]}$	$(13,1)$
	3	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (2,0)T_{[\vartheta\omega]} + (7,0)T_{[\omega^2]} + (4,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (2,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(25,1)$
		2	$(\vartheta, \frac{1}{2}(e_2+e_3+e_4)), (\omega, \frac{1}{2}(e_4+e_5+e_6))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,1)T_{[\vartheta\omega]} + (5,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (2,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(20,2)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}e_4)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (5,0)T_{[\omega^2]} + (2,0)T_{[\omega^2\vartheta^2\omega]} + (4,0)T_{[\omega^4]} + (2,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(19,1)$
	4	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,0)T_{[\vartheta\omega]} + (5,0)T_{[\omega^2]} + (3,0)T_{[\omega^2\vartheta^2\omega]} + (3,0)T_{[\omega^4]} + (1,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(19,1)$

continued . . .

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{4}(3e_1+3e_2+3e_3+e_5)), (\omega, \frac{1}{4}(3e_2+e_5))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (3,0)T_{[\omega^2]} + (1,1)T_{[\omega^2\vartheta^2\omega]} + (1,0)T_{[\omega^4]}$	$(11,2)$
		3	$(\vartheta, \frac{1}{2}(e_1+e_2+e_3+e_5)), (\omega, \frac{1}{2}(e_2+e_5))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,0)T_{[\vartheta\omega]} + (5,0)T_{[\omega^2]} + (3,0)T_{[\omega^2\vartheta^2\omega]} + (3,0)T_{[\omega^4]} + (1,0)T_{[\vartheta\omega^3\vartheta\omega]}$	$(19,1)$
$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_4$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(2,0)U + (3,2)T_{[\vartheta]} + (8,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (2,0)T_{[\vartheta^2\omega]} + (6,0)T_{[\omega^2]} + (4,0)T_{[\omega\vartheta\omega]} + (2,0)T_{[\omega^2\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega^3\vartheta\omega]} + (3,0)T_{[\vartheta^2\omega^3\vartheta^2\omega]} + (6,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^2\omega\vartheta\omega^2]}$	$(44,2)$
		2	$(\vartheta, \frac{1}{2}e_1), (\omega, \frac{1}{2}(e_1+e_2+e_3))$	
		local	$(2,0)U + (3,2)T_{[\vartheta]} + (4,0)T_{[\omega]} + (4,0)T_{[\vartheta\omega]} + (3,0)T_{[\omega^2]} + (4,0)T_{[\omega\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega^3\vartheta\omega]} + (1,1)T_{[\vartheta^2\omega^3\vartheta^2\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (2,0)T_{[\vartheta\omega\vartheta^2\omega\vartheta\omega^2]}$	$(27,3)$
$\Sigma(36\phi)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^3]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta^3\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega\vartheta\omega\vartheta\omega]}$	$(25,1)$
		2	$(\vartheta, \frac{2}{3}(e_1+e_4)), (\omega, \frac{1}{3}(e_1+2e_2+2e_3+e_4+e_5+e_6))$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^3]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta\omega\vartheta\omega]}$	$(17,1)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^3]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta^3\omega]} + (5,0)T_{[\vartheta\omega\vartheta\omega\vartheta\omega\vartheta\omega]}$	$(25,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
		2	$(\vartheta, \frac{1}{3}(2e_3 + e_4)), (\omega, \frac{1}{3}(e_1 + 2e_4 + 2e_5 + 2e_6))$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^3]} + (3,0)T_{[\vartheta\omega\vartheta\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta^3\omega]} + (3,0)T_{[\vartheta\omega\vartheta\omega\vartheta\omega\vartheta\omega]}$	$(17,1)$
$\Delta(108)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (4,0)T_{[\omega]} + (1,0)T_{[\vartheta^2]} + (1,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (4,0)T_{[\omega^2]} + (1,0)T_{[\omega\vartheta\omega]} + (1,0)T_{[\omega\vartheta^2\omega]} + (4,0)T_{[\omega^3]} + (4,0)T_{[\omega^4]} + (2,0)T_{[\vartheta\omega^4\vartheta^2\omega^5]} + (1,0)T_{[\vartheta\omega^3\vartheta^2\omega^4]} + (4,0)T_{[\vartheta\omega^2\vartheta^2\omega^3]} + (1,0)T_{[\vartheta\omega^4\vartheta^2\omega]} + (5,0)T_{[\vartheta\omega^2\vartheta^2\omega^4]}$	$(36,0)$
PSL$(3,2)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (4,0)T_{[\omega]} + (7,0)T_{[\vartheta\omega]} + (1,1)T_{[\vartheta^2\omega\vartheta\omega]}$	$(14,2)$
		2	$(\vartheta, 0), (\omega, \frac{1}{2}(e_3 + e_6))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (1,1)T_{[\omega]} + (7,0)T_{[\vartheta\omega]}$	$(10,2)$
		3	$(\vartheta, 0), (\omega, \frac{1}{2}(e_1 + e_2 + e_3 + e_5))$	
		local	$(1,0)U + (1,1)T_{[\vartheta]} + (2,0)T_{[\omega]} + (7,0)T_{[\vartheta\omega]}$	$(11,1)$
$\Sigma(72\phi)$	1	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (1,0)T_{[\vartheta^3\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\omega\vartheta^2\omega]} + (3,0)T_{[\omega\vartheta^3\omega]} + (4,0)T_{[\omega^4]}$	$(25,1)$
	2	1	$(\vartheta, 0), (\omega, 0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (3,0)T_{[\omega]} + (2,1)T_{[\vartheta^2]} + (3,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta^2\omega]} + (1,0)T_{[\vartheta^3\omega]} + (3,0)T_{[\omega^2]} + (3,0)T_{[\omega\vartheta^2\omega]} + (3,0)T_{[\omega\vartheta^3\omega]} + (4,0)T_{[\omega^4]}$	$(25,1)$

continued ...

$\mathbb{Q}$-class (P)	$\mathbb{Z}$-class (Λ)	affine class, breaking	generators of G / contributions to $(h^{(1,1)}, h^{(2,1)})$ from U and T sectors	$(h^{(1,1)}, h^{(2,1)})$
$\Delta(216)$	1	1	$(\vartheta,0),(\omega,0),(\rho,0)$	
		local	$(1,0)U + (1,0)T_{[\vartheta]} + (2,1)T_{[\omega]} + (1,0)T_{[\rho]} + (2,0)T_{[\vartheta\omega]} + (1,0)T_{[\vartheta\rho]} + (4,0)T_{[\vartheta^2\rho]} + (3,0)T_{[\omega^2]} + (2,0)T_{[\omega\rho]} + (3,0)T_{[\vartheta^2\omega^3\rho]} + (1,0)T_{[\omega\rho\vartheta\omega^3\rho^2]} + (4,0)T_{[\omega^2\rho\vartheta\omega^2\rho]} + (4,0)T_{[\omega\rho\omega\rho]} + (2,0)T_{[\omega^3\rho\omega^3\rho]}$	$(31,1)$

Table B.4: List of all space groups which preserve $\mathcal{N}=1$ supersymmetry in four dimensions. Listed are all constructing conjugacy classes and their contributions to the Hodge number $h^{(1,1)}$ and $h^{(2,1)}$ as well as the contribution from the untwisted sector and the summed up Hodge numbers.

Appendix C

Clifford. m Mathematica package

```
(* ::Package:: *)

(* Revelation 22:13
"I am the A and the #[#]&[#[#]&], the First[] and the
   Last[],
the Begin[] and the End[]."

File: clifford.m
Purpose: do computations with Clifford and Spinor algebras.
Author: Maximilian Fischer, May 2012

*)

BeginPackage["clifford`"]
Begin["`Private`"]

Cl[p_, q_] := Module[{f, g, r2, r11, h, c, rr},
    r2  = { {{0, 1}, {1, 0}}, {{1, 0}, {0, -1}} };
    r11 = { {{1, 0}, {0, -1}}, {{0,-1}, {1, 0}} };
    h = { {{0, I}, {I, 0}}, {{0, 1}, {-1, 0}} };
    c = { {{I}} };
    rr = { {{0, 1}, {1, 0}} };
    f = Function[{big, small}, Join[
        KroneckerProduct[#, small[[1]].small[[2]]] & /@ big,
        KroneckerProduct[IdentityMatrix[Length[big[[1]]]], #] &
           /@ small]];
    g = Function[{clpq, cl11},
       Table[Switch[i,
          _?(# ≤ p - 1 &), KroneckerProduct[clpq[[i]],
             cl11[[1]].cl11[[2]]],
```

```
        p, KroneckerProduct[IdentityMatrix[Length[clpq[[1]]]],
            cl11[[1]]],
        _?(# ≤ p + q − 1 &), KroneckerProduct[clpq[[i − 1]],
            cl11[[1]].cl11[[2]]],
        p + q,
            KroneckerProduct[IdentityMatrix[Length[clpq[[1]]]],
            cl11[[2]]]],
    {i, 1, p + q}]];
  If[¬IntegerQ[p] ∨ ¬IntegerQ[q] ∨ p < 0 ∨ q < 0,
   Print["ERROR: p and q have to be integers ≥ 0 !!!"];
      False,
     Switch[{p, q},
      {0, 0}, {},
      {0, 2}, h,
      {2, 0}, r2,
      {0, 1}, c,
      {1, 0}, rr,
      {1, 1}, r11,
      {0, _}, f[Cl[q − 2, 0], Cl[0, 2]],
      {_, 0}, f[Cl[0, p − 2], Cl[2, 0]],
      {_?(# > 0 &), _?(# > 0 &)}, g[Cl[p − 1, q − 1], Cl[1,
         1]]
      ]]];

(* Inject and project scalars *)
FromScalarFunc[p_,q_] := Function[{s},
   IdentityMatrix[Length[Cl[p,q][[1]]]]*s];
ToScalar              := Function[{c}, Re[Tr[c]/Length[c]]];
(* Inject and project vectors *)
DotProduct[cl1_, cl2_] :=
   −Re[ComplexExpand[Tr[cl1.cl2]/Length[cl1]]]
FromVectorFunc[p_, q_] :=  Function[{v}, Plus @@ Times[v,
   Cl[p, q]]];
ToVectorFunc[p_, q_]   :=  Function[{cl}, Plus @@
   (DotProduct[cl, Cl[p, q][[#]]]*UnitVector[p + q, #]) &
   /@ Range[p + q]];
(* Inject and project rotations *)
ToRotationFunc[p_, q_] := Function[{c},
   FullSimplify[Transpose[FullSimplify[
  ToVectorFunc[p,q][c.FromVectorFunc[p,q][#].Inverse[c]]&
     /@ IdentityMatrix[p+q]]]]]
```

```
FromRotationFunc[p_, q_] := Function[{mat},
  Module[{realize, Rotor, planes},
  realize = Function[{pair},
     Select[DeleteDuplicates[Flatten[{Re[#], Im[#]} & \
    /@ pair, 1], #1 == #2 ∨ #1 == -#2 &], # ≠
       ConstantArray[0, Length[#]] &]];
  Rotor = Function[{b1, b2}, Module[{y, t}, t = mat.b1;
    y = If[t == -b1, b2, Normalize[t + b1]];
    FullSimplify [Dot @@ (FromVectorFunc[p, q] /@ {-y,
       b1})]]];
  planes = Partition[Flatten[FullSimplify /@
     Orthogonalize[realize[#[[2]]]] & \
    /@ Transpose /@ Gather[ComplexExpand[Transpose[Chop @
       Eigensystem[mat]]] \
    , #1[[1]] == #2[[1]] ∨ #1[[1]] == Conjugate[#2[[1]]] &], 1], 2];
  ComplexExpand[Dot @@ (Rotor @@@ planes)]]]

(* Inject stuff *)
IntoFunc[p_,q_] := Function[{x},Switch[Depth[N[x]],
  1,FromScalarFunc[p,q][x],
  2,FromVectorFunc[p,q][x],
  3,FromRotationFunc[p,q][x],
  _,Print["ERROR: Depth of object is too high !!!"];
     False]];

(* Inject and project spinor rotations *)
ConversionTable[p_,q_] := If[p+q < 3 ∨ q < 1,
  Print["ERROR: Functionality for these values of (p,q) has
     not been implemented yet !!!"]; False,
  Function[{ist},{
    Fold[Dot, IntoFunc[p, q] [1],Cl[p, q] [[#]] & /@ ist],
    Fold[Dot, IntoFunc[p,q-1][1],Cl[p,q-1][[#]] & /@
       Select[ist, # ≠ p+q &]]}] \
      /@ Select[Subsets[Range[p + q]], EvenQ[Length[#]] &]]
ToSpinFunc[p_,q_]    := Function[{c},
  ComplexExpand[Plus @@ (DotProduct[c, #[[1]]]*#[[2]] & /@
     ConversionTable[p,q])]]
FromSpinFunc[p_,q_] := Function[{n},FullSimplify[
  ComplexExpand[Plus @@ (DotProduct[n, #[[2]]]*#[[1]] & /@
     ConversionTable[p,q])]]]
End[]
EndPackage[]
```

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Bibliography

[1] A. Einstein, “Relativitätsprinzip und die aus demselben gezogenen folgerungen,” *Jahrbuch der Radioaktivität*, vol. 4, p. 411–462, 1907. now in CPAE 2, Doc. 47.

[2] E. Noether, “Invarianten beliebiger Differentialausdrücke.,” *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, vol. 1918, pp. 37–44, 1918.

[3] M. E. Peskin and D. V. Schroeder, “An introduction to quantum field theory,” 1995. Reading, USA: Addison-Wesley (1995) 842 p.

[4] G. Aad *et al.*, “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” *Phys.Lett.*, vol. B716, pp. 1–29, 2012.

[5] A. Zee, *Quantum Field Theory in a Nutshell.* Princeton University Press, 2003.

[6] J. A. Shapiro, “Reminiscence on the birth of string theory,” 2007.

[7] W. Plesken, “The carat computer package,” 2008. `http://wwwb.math.rwth-aachen.de/carat/index.html`.

[8] M. Fischer, M. Ratz, J. Torrado, and P. K. Vaudrevange, “Classification of symmetric toroidal orbifolds,” *JHEP*, vol. 1301, p. 084, 2013.

[9] M. Fischer, S. Ramos-Sánchez, and P. K. S. Vaudrevange, “Heterotic non-Abelian orbifolds,” *JHEP*, vol. 1307, p. 080, 2013.

[10] M. Green, J. Schwarz, and E. Witten, *Superstring Theory: Volume 1, Introduction.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1988.

[11] K. Becker, M. Becker, and J. Schwarz, *String Theory and M-Theory.* Cambridge: Cambridge University Press, 2007.

[12] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String.*

Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1998.

[13] J. Polchinski, *String Theory: Volume 2, Superstring Theory and Beyond*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1998.

[14] L. e. Ibáñez and A. M. Uranga, *String Theory and Particle Physics*. Cambridge University Press, 2012.

[15] S. P. Martin, “A Supersymmetry primer,” *Adv.Ser.Direct.High Energy Phys.*, vol. 21, pp. 1–153, 2010.

[16] P. Goddard and C. B. Thorn, “Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model,” *Phys.Lett.*, vol. B40, pp. 235–238, 1972.

[17] P. K. S. Vaudrevange, “Grand Unification in the Heterotic Brane World,” 2008.

[18] T. Steininger, “Heterotic strings on non-abelian orbifolds,” master thesis, Technical University Munich, June 2014. Available online at http://einrichtungen.ph.tum.de/T30e/research/theses/MasterthesisSteininger.pdf;.

[19] H. P. Nilles, S. Ramos-Sánchez, P. K. Vaudrevange, and A. Wingerter, “The Orbifolder: A Tool to study the Low Energy Effective Theory of Heterotic Orbifolds,” *Comput.Phys.Commun.*, vol. 183, pp. 1363–1380, 2012. 29 pages, web page http://projects.hepforge.org/orbifolder/.

[20] M. Fischer, “On $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ orbifolds,” diploma thesis, Technical University Munich, February 2011. Available online at http://einrichtungen.physik.tu-muenchen.de/T30e/research/theses/FischerDiplomarbeit.pdf;.

[21] W. P. Thurston, “The geometry and topology of three-manifolds.” Website, 2002. Available online at http://www.msri.org/publications/books/gt3m/;.

[22] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, “Strings on orbifolds. 2,” *Nucl. Phys.*, vol. B274, pp. 285–314, 1986.

[23] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, “Strings on orbifolds,” *Nucl. Phys.*, vol. B261, pp. 678–686, 1985.

[24] Brown, Buelow, Neubueser, Wondratchek, and Zassenhaus, *Crystallographic Groups of Four-Dimensional Space*. Wiley, 1978.

[25] E. Fedorov, "Simmetrija na ploskosti," *Zapiski Imperatorskogo Sant-Petersburgskogo Mineralogicheskogo Obshchestva*, vol. 28, pp. 245–291, 1891.

[26] J. Burckhardt, "Zur geschichte der entdeckung der 230 raumgruppen," *Archive for History of Exact Sciences*, vol. 4, no. 3, pp. 235–246, 1967.

[27] W. Plesken and T. Schulz, "Counting crystallographic groups in low dimensions," *Experimental Mathematics*, vol. 9, no. 3, pp. 407–411, 2000.

[28] W. Plesken and M. Pohst, "On maximal finite irreducible subgroups of gl $(n, \mathbf{Z})$. i: The five and seven dimensional case. ii: The six dimensional case," *Bulletin of the American Mathematical Society*, vol. 82, pp. 757–758, 09 1976.

[29] W. Plesken and W. Hanrath, "The lattices of six-dimensional euclidean space," *Mathematics of computation*, pp. 573–s584, 1984.

[30] R. Donagi and K. Wendland, "On orbifolds and free fermion constructions," *J. Geom. Phys.*, vol. 59, pp. 942–968, 2009.

[31] J. Dillies, "Toroidal orbifolds a la Vafa-Witten," *Adv.Theor.Math.Phys.*, vol. 11, pp. 683–705, 2007.

[32] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*. New York: Interscience Publishers, 1962.

[33] J. Opgenorth, W. Plesken, and T. Schulz, "Crystallographic Algorithms and Tables," *Acta Crystallographica Section A*, vol. 54, pp. 517–531, Sep 1998.

[34] K. Narain, M. Sarmadi, and C. Vafa, "Asymmetric Orbifolds," *Nucl.Phys.*, vol. B288, p. 551, 1987.

[35] F. Beye, T. Kobayashi, and S. Kuwakino, "Three-generation Asymmetric Orbifold Models from Heterotic String Theory," *JHEP*, vol. 1401, p. 013, 2014.

[36] C. Condeescu, I. Florakis, C. Kounnas, and D. Lüst, "Gauged supergravities and non-geometric Q/R-fluxes from asymmetric orbifold CFT's," *JHEP*, vol. 1310, p. 057, 2013.

[37] C. Condeescu, I. Florakis, and D. Lüst, "Asymmetric Orbifolds, Non-Geometric Fluxes and Non-Commutativity in Closed String Theory," *JHEP*, vol. 1204, p. 121, 2012.

[38] G. Nebe and W. Plesken, “Finite rational matrix groups,” *AMS*, vol. 116, no. 556, 1995.

[39] D. Bailin and A. Love, “Orbifold compactifications of string theory,” *Phys.Rept.*, vol. 315, pp. 285–408, 1999.

[40] S. Förste, T. Kobayashi, H. Ohki, and K.-j. Takahashi, “Non-Factorisable Z(2) x Z(2) Heterotic Orbifold Models and Yukawa Couplings,” *JHEP*, vol. 0703, p. 011, 2007.

[41] T. Kimura, M. Ohta, and K.-J. Takahashi, “Type IIA orientifolds and orbifolds on non-factorizable tori,” *Nucl.Phys.*, vol. B798, pp. 89–123, 2008.

[42] R. Donagi and A. E. Faraggi, “On the number of chiral generations in Z(2) x Z(2) orbifolds,” *Nucl. Phys.*, vol. B694, pp. 187–205, 2004.

[43] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, “Vacuum Configurations for Superstrings,” *Nucl. Phys.*, vol. B258, pp. 46–74, 1985.

[44] G. Aragón-González, J. Aragón, M. Rodríguez-Andrade, and L. Verde-Star, “Reflections, rotations, and pythagorean numbers,” *Advances in Applied Clifford Algebras*, vol. 19, pp. 1–14, 2009.

[45] A. Richard, L. Fuchs, and S. Charneau, “An algorithm to decompose n-dimensional rotations into planar rotations,” *CompIMAGE*, vol. 6026, pp. 60–71, 2010.

[46] J. Gallier, *Geometric Methods and Applications For Computer Science and Engineering*. Springer, 2011.

[47] J. Gallier, “Clifford Algebras, Clifford Groups, and a Generalization of the Quaternions,” 2009.

[48] D. Hestenes, *New Foundations for Classical Mechanics*. Fundamental Theories of Physics, Springer, 1999.

[49] J. Figueroa-O’Farrill, “Pg course on spin geometry.” Website, 2010. Available online at http://empg.maths.ed.ac.uk/Activities/Spin/;.

[50] R. Slansky, “Group theory for unified model building,” *Phys. Rept.*, vol. 79, pp. 1–128, 1981.

[51] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.5.5*, 2012.

[52] V. Dabbaghian-Abdoly, "Repsn - A Package for Constructing Representations of Finite Groups," 2004.

[53] G. Ross, "Wilson line breaking and gauge coupling unification," 2004.

[54] A. Hebecker and M. Trapletti, "Gauge unification in highly anisotropic string compactifications," *Nucl.Phys.*, vol. B713, pp. 173–203, 2005.

[55] A. Anandakrishnan and S. Raby, "SU(6) GUT Breaking on a Projective Plane," *Nucl.Phys.*, vol. B868, pp. 627–651, 2013.

[56] R. Brown and P. J. Higgins, "The fundamental groupoid of the quotient of a hausdorff space by a discontinuous action of a discrete group is the orbit groupoid of the induced action," 2002. Available online at http://arxiv.org/abs/math/0212271v1;.

[57] C. Vafa and E. Witten, "On orbifolds with discrete torsion," *J. Geom. Phys.*, vol. 15, pp. 189–214, 1995.

[58] T. tom Dieck, *Algebraic Topology*. Zürich: EMS Textbooks in Mathematics, 2008.

[59] M. Nakahara, *Geometry, Topology and Physics*. Abingdon: Taylor & Francis Group, 2003.

[60] E. Zaslow, "Topological orbifold models and quantum cohomology rings," *Commun. Math. Phys.*, vol. 156, pp. 301–332, 1993.

[61] J. R. Ellis, A. Lahanas, D. V. Nanopoulos, and K. Tamvakis, "No-Scale Supersymmetric Standard Model," *Phys.Lett.*, vol. B134, p. 429, 1984.

[62] M. Cvetič, J. Louis, and B. A. Ovrut, "A String Calculation of the Kahler Potentials for Moduli of Z(N) Orbifolds," *Phys. Lett.*, vol. B206, p. 227, 1988.

[63] A. Brignole, L. E. Ibáñez, and C. Munoz, "Soft supersymmetry breaking terms from supergravity and superstring models," 1997.

[64] L. Covi, M. Gomez-Reino, C. Gross, J. Louis, G. A. Palma, *et al.*, "de Sitter vacua in no-scale supergravities and Calabi-Yau string models," *JHEP*, vol. 0806, p. 057, 2008.

[65] S. J. Konopka, "Non Abelian orbifold compactifications of the heterotic string," 2012.

[66] M. Dine, N. Seiberg, X. Wen, and E. Witten, "Nonperturbative Effects on the String World Sheet," *Nucl.Phys.*, vol. B278, p. 769, 1986.

[67] L. J. Dixon, V. Kaplunovsky, and J. Louis, "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String," *Nucl.Phys.*, vol. B329, pp. 27–82, 1990.

[68] S. Groot Nibbelink and P. K. S. Vaudrevange, "Schoen manifold with line bundles as resolved magnetized orbifolds," *JHEP*, vol. 1303, p. 142, 2013.

[69] L. E. Ibáñez, H. P. Nilles, and F. Quevedo, "Orbifolds and Wilson Lines," *Phys.Lett.*, vol. B187, pp. 25–32, 1987.

[70] M. Drees, R. Godbole, and P. Roy, "Theory and phenomenology of sparticles: An account of four-dimensional N=1 supersymmetry in high energy physics," 2004. Hackensack, USA: World Scientific (2004).

[71] Y. Kawamura, "Gauge symmetry reduction from the extra space S(1)/Z(2)," *Prog. Theor. Phys.*, vol. 103, pp. 613–619, 2000.

[72] L. J. Hall and Y. Nomura, "Gauge unification in higher dimensions," *Phys. Rev.*, vol. D64, p. 055003, 2001.

[73] M. Blaszczyk, S. Groot Nibbelink, M. Ratz, F. Ruehle, M. Trapletti, *et al.*, "A Z2xZ2 standard model," *Phys.Lett.*, vol. B683, pp. 340–348, 2010.

[74] M. Blaszczyk, S. G. Nibbelink, O. Loukas, and S. Ramos-Sánchez, "Non-supersymmetric heterotic model building," 2014.

[75] A. E. Faraggi and D. V. Nanopoulos, "Naturalness of three generations in free fermionic $\mathbb{Z}_2^n \otimes \mathbb{Z}_4$ string models," *Phys. Rev.*, vol. D48, pp. 3288–3296, 1993.

[76] N. Lambert, "String theory 101," 2010. Lectures at the International School of Stirngs and Fundamental Physics.

[77] M. G. Danner, "The spectrum of orbifolds," 2010. Diploma thesis at Technical University of Munich.

[78] H. McKean and V. Moll, *Elliptic Curves.* Cambridge University Press, 1999.

[79] R. Hartshorne, *Algebraic Geometry.* New York: Springer Science, 2006.

[80] Bröcker and Jänich, *Einführung in die Differentialtopologie.* Berlin Heidelberg: Springer, 1973.

[81] G. Fischer, *Lineare Algebra.* Wiesbaden: vieweg, 15 ed., 2005.

[82] D. Huybrechts, *Complex Geometry.* Heidelberg: Springer, 2005.

[83] C. Birkenhake and H. Lange, *Complex Abelian Varieties.* Berlin: Springer, 2004.

[84] C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras.* Berlin Heidelberg: Springer, 1997.

[85] F. Schwabl, *Quantenmechanik.* Berlin Heidelberg: Springer, 7th ed., 2007.

[86] T. Frankel, *The Geometry Of Physics.* Cambridge: Cambridge University Press, 2nd ed., 2007.

[87] R. Cahn, *Semi-Simple Lie Algebras and Their Representations.* New York: Dover Publications, 2006.

[88] R. Boas, *Lion Hunting and Other Mathematical Pursuits: A Collection of Mathematics, Verse, and Stories by Ralph P. Boas, Jr.* The Mathematical Association of America, 1995.

[89] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl. Phys.*, vol. B256, p. 253, 1985.

[90] J. Erler and P. Langacker, "Electroweak Physics," *Acta Phys. Polon.*, vol. B39, pp. 2595–2610, 2008.

[91] S. J. Konopka, "Non Abelian orbifold compactifications of the heterotic string," *JHEP*, vol. 1307, p. 023, 2013.

[92] M. Blaszczyk, S. Nibbelink Groot, F. Ruehle, M. Trapletti, and P. K. S. Vaudrevange, "Heterotic MSSM on a Resolved Orbifold," *JHEP*, vol. 1009, p. 065, 2010.

[93] L. E. Ibáñez and D. Lüst, "Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4-D strings," *Nucl.Phys.*, vol. B382, pp. 305–364, 1992.

[94] H. Zassenhaus, "Über einen algorithmus zur bestimmung der raumgruppen," *Commentarii Mathematici Helvetici*, vol. 21, no. 1, pp. 117–141, 1948.

[95] D. Holt and W. Plesken, *Perfect groups.* Oxford mathematical monographs, Clarendon Press, 1989.

[96] S. Groot Nibbelink and F. Ruehle, "Torus partition functions and spectra of gauged linear sigma models," 2014.