# Propagators in Quantum Mechanics on Multiply Connected Spaces

Norman E. Hurt

## Introduction.

In several recent papers [ 4 ], [ 5 ], [ 6 ], [11], [15] quantum mechanics and quantum mechanical propagators on multiply connected spaces have been discussed. The quantum mechanics of a free particle moving on the Riemannian manifold M = G, a simple Lie group, was examined by J. S. Dowker in [ 5 ]. In this paper he showed that the quasi-classical approximation is "exact" and that the propagator may be calculated either by "the sum over classical paths' or by the stationary state method. In a later paper [ 6 ] Dowker suggested a natural framework (following [11]) for quantum mechanical propagators on multiply connected, homogeneous spaces.

In this paper we indicate the interconnections of Dowker's formulation and the yoga of the Selberg trace formula. The Selberg trace formula plays a very fundamental role in a broad cross section of mathematics. Below we demonstrate its vitality in modern physics.

## §1. Quantum Mechanical Propagators.

Various compact and noncompact homogeneous manifolds occur in the world of mathematical physics: e.g., the spherical top, the particle with spin, the classical energy level of the Kepler problem, etc. This leads naturally to the consideration of homogeneous manifolds of the form M = G/H, where initially G is a separable, locally compact unimodular group with compact isotropy subgroup $H = \{g \in G \mid gm_o = m_o\}$. Let $C_c(G)$ denote the associative algebra over $\mathbb{C}$ of continuous functions with compact support, under the multiplication $(f_1*f_2)(g) = \int_G f_1(gx^{-1})f_2(x)dx$. The associated space of <u>spherical</u> functions, $C_c(H\backslash G/H)$, is a subalgebra of $C_c(G)$. If the convolution algebra of spherical functions is commutative, then (G,H) is a <u>Gelfand pair</u>.

The quantum mechanical propagator for a homogeneous Riemannian manifold $M = G/H$, with Laplace-Beltrami operator $\Delta$, is denoted by $K_{G/H}(t;x,y)$. It is characterized by the equations

$$(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x)K_{G/H}(t;x,y) = 0 \text{ for } t > 0$$

and

$$\psi(x) = \lim_{t\to 0^+} \int_{G/H} K(t;x,y)\psi(y)dy$$

for $\psi$ in a suitable class of functions on M. What is essential here is that the propagator is a two-point invariant function:

$$K_{G/H}(t;gxH,gyH) = K_{G/H}(t;xH,yH),$$

which follows from the invariance of $\Delta_{G/H}$ by the action of G on $M = G/H$: $G \times M \to M$, $(g,xH) \to gxH$. Any two-point invariant function, e.g., $K_{G/H}$, defines a rotationally symmetric function on M, i.e., $f(hm) = f(m)$ for all h in H. Viz,. $f(m) = K(t;m,m_o)$ (or $m = zH \to K_{G/H}(t;zH) = K_{G/H}(t;y^{-1}xH,H)$). And conversely. Furthermore, any rotationally symmetric function f(m) may also be considered as a function on G, viz., $\tilde{f}(g) = f(m)$ if $gm_o = m$. It is easily checked that $\tilde{f}$ is H-bi-invariant on G; and conversely, every spherical function on G defines a r.s. function f on M.

Under the Gelfand pair hypothesis, there is a special class of spherical functions on G which are "orthogonal and span" the space of all spherical functions. These spherical functions are the <u>elementary</u> or <u>zonal spherical</u> (z.s.) <u>functions</u>. The z.s. functions can be characterized in several ways: e.g., $\phi$ in $C(H\backslash G/H)$ is a z.s. function iff $\phi(e) = 1$ and $f*\phi = \lambda_f\phi$ for all f in $C_c(H\backslash G/H)$.

The relation to representation theory is well-known also. There is a one-one correspondence between the set of positive definite (p.d.) z.s. functions on G and the set $\hat{G}(1)$ of equivalence classes of class one representations of G The notion of spanning is in the sense of a Fourier expansion of a suitable spherical function f:

$$f(g) = \int_{\hat{G}(1)} \hat{f}(\lambda)\phi_\lambda d\hat{\mu}(\lambda)$$

where $\hat{f}(\lambda) = \int_G f(g)\phi_\lambda^* \, dg$.

In the case G is compact and so $\hat{G}$ is discrete, then for every f in $L^2(H\backslash G/H)$ there is a Fourier series expansion

$$f = \sum_{\lambda\in\hat{G}(1)} f(\lambda)\phi_\lambda$$

where $d(\lambda)\hat{f}(\lambda) = \int_G f(g)\phi_\lambda^*(g)\, dg = \int_{G/H} f\phi_\lambda^* = \int_{H\backslash G/H} f\phi_\lambda^*$; and if $U_\phi$ is the (class one) representation associated to the p.d.z.s. function $\phi$, then $\phi(gH) = \int_H \chi_{U_\phi}(g\cdot h)\, dh$ where $\chi$ is the character of the representation $U_\phi$ of G on $\mathcal{H}(U_\phi)$.

Since $K_{G/H}(t;m,m_o)$ gives a spherical function on G, we might consider an expansion in terms of the z.s. functions on G, viz.,

$$K_{G/H}(t;m,n) = \int_{\hat{G}(1)} \phi_\lambda(g)d\hat{\mu}_t(\lambda), \quad gn = m,$$

$$\text{or} = \sum_{\lambda\in\hat{G}(1)} \phi_\lambda(g)F_\lambda(t)$$

if G is compact. The function $F_\lambda(t)$ is the spectral density.

When M = G is a compact connected Lie group, it is easy to calculate the spectral density. Knowing $\Delta_G\chi_\lambda = h_\lambda\chi_\lambda$, then $F_\lambda(t) = d(\lambda)\exp(-ih_\lambda t)$ (where $F_\lambda(0) = \lim_{t\to 0^+} \int_G K_G(t;g)\chi_\lambda(g)^* \, dg = \chi_\lambda(e) = d(\lambda)$).

Thus $K_G(t;g) = \sum_{\lambda\in\hat{G}(1)} d(\lambda)\exp(-ih_\lambda t)\chi_\lambda(g)$, (i.e., eq. (19) in [ 5 ]).

As sketched by Dowker [ 6 ] if $K_G(t;g)$ is the propagator on G, then $K_{G/H}(t;zH) = \int_H K_G(t;zh)\,dh$. Viz., if $\pi: G \to M = G/H$ is the natural projection then $\Delta_M(f)\circ\pi = \Delta_G(f\circ\pi)$. Thus $(\Delta_M K_M)(gH) = [\Delta_G \int_H K_G(t;\cdot,h^{-1})\,dh](g) = [\int_H \Delta_G K_G(t;\cdot,h^{-1})\,dh](g) = -i\,\partial/\partial t\,[\int_H K_G(\cdot)\,dh] = -i\,[\partial/\partial t\,K_M](gH)$; and $\lim_{t\to 0^+} \int f\circ\pi(g)\,K_G\,(t;g)\,dg = f\circ\pi(e) = f(H)$. Thus $K_M(t;gH) =$

$$\sum_{\lambda\in\hat{G}(1)} d(\lambda)\exp(-ih_\lambda t)\phi_\lambda(gH), \text{ or } K_M(t;xH,xH) = \sum_{\lambda\in\hat{G}(1)} d(\lambda)\dim Z\,(H(\lambda))\exp(-ih_\lambda t) \quad (1.1)$$

where $Z(H(\lambda)) = \{v\in H_\lambda \mid U_\lambda(h)v = v \text{ for all } h \text{ in } H\}$, where $H(\lambda)$ is the representation space of $\lambda$, $d(\lambda) = \dim H(\lambda)$.

We now turn our attention to the relation between the propagator $K_M$ on $M = G/H$ and the propagator $K_{\mathcal{M}}$ on $\mathcal{M} = \Gamma\backslash G/H$, where $\Gamma$ is a discrete subgroup of G. The best way to look at this question is as follows. Suppose the propagator is known only over the subspace $\gamma m_o$ where $\gamma$ varies over $\Gamma$. Then what is the relation of the spectral density of the propagator we observe to that of the unobserved propagator on the whole space M? The connection is given by the Selberg trace formula which we review next.

## §2. The Selberg Trace Fromula.

Let $\Gamma$ be a closed subgroup of finite index of a separable locally compact group G. Let L be a finite dimensional unitary representation of $\Gamma$. The induced representation $U^L$ acting on $H(U^L)$, [12]. Let $\chi_L$, resp $\chi_{U^L}$, denote the associated characters $g\to \mathrm{Tr}L(g)$, resp. $g\to \mathrm{Tr}U^L(g)$. For f in $L^1(G)$, define $U^L_f$ by $(U^L_f\,\alpha_1,\alpha_2) = \int_G f(g)(U^L(g)\,\alpha_1,\alpha_2)dg$ for $\alpha_i$ in $H(U^L)$. Then $\mathrm{Tr}(U^L_f) =$

$\int_G \chi_{U^L}(g)f(g)\ dg$ can be computed:

$$\mathrm{Tr}(U^L_f) = \frac{1}{\sigma(\Gamma\backslash G)} \sum_{\tilde{y}\in\Gamma\backslash G} \int_\Gamma f(y^{-1}\gamma y)\chi_L(\gamma)\ d\gamma$$

$$\text{or } \int_{\Gamma\backslash G} [\int_\Gamma f(y^{-1}\gamma y)\chi_L(\gamma)\ d\gamma] \times d\nu(\tilde{y}) \quad (*)$$

If $U^L$ decomposes discretely as $U^L = \sum_{\lambda\ G} \oplus\ n(\lambda)\ U^\lambda$ with multiplicity $n(\lambda)$, then

$$\mathrm{Tr}(U^L_f) = \sum_{\lambda\in\hat{G}} n(\lambda)\ \mathrm{Tr}\ U^\lambda_f = \int_{\hat{G}} \mathrm{Tr}\ U_f\ d\hat{\mu}(M) \quad (**)$$

The Selberg trace formula (STF) in its first version is that (*) = (**). These hypotheses are met when $\Gamma$ is a discrete subgroup of G such that $\Gamma\backslash G$ is compact. The induced representation $U^L$ is a discrete direct sum of IUR of G, each occurring with finite multiplicity $n_\Gamma(\lambda,L)$; so $[U^L] = \sum_{\lambda\in\hat{G}} n_\Gamma(\lambda,L)\lambda$ where $\hat{G}$ is the set of equivalence classes of IUR's. If f is an <u>admissable</u> function--i.e., (a) the series $\sum_{\gamma\in\Gamma} f(y^{-1}\gamma x)L(\gamma)$ converges absolutely, uniformly on compacts of $G \times G$, to a continuous function $F(x,y,L)$ and (b) $U_f$ is of trace class, then the STF holds

$$\sum_{\lambda\in\hat{G}} n_\Gamma(\lambda,L)\ \mathrm{Tr}\ U^\lambda_f = \int_{\Gamma\backslash G} \mathrm{Tr}\ F(x,x,L)\ d\dot{x} = \sum_{\gamma\in C_\Gamma} \chi_L(\gamma)\ \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\ J_\gamma(f)$$

where $C_\Gamma$ is a complete set of representatives in $\Gamma$ of G-conjugacy classes of elements of $\Gamma$, $G_\gamma$ is the centralizer of $\gamma$ in G, $\Gamma_\gamma = \Gamma\cap G_\gamma$, $J_\gamma(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\ dx_\gamma$

The connection of the STF with class one representations is as follows. A $C_1$-<u>spherical</u> function f has the property that $\text{Ⓗ}_\lambda(f) = \mathrm{Tr}\ U^\lambda_f = 0$ unless $\lambda$ is of type one in which case $\text{Ⓗ}_\lambda(f) = \hat{f}(\lambda)$. Thus if f is also admissable and if $\sum_{\lambda\in\hat{G}(1)} n_\Gamma(\lambda,L)\hat{f}(\lambda)$ converges absolutely then the Selberg-Tamagawa trace formula (STTF) states that

$$\sum_{\lambda\in\hat{G}(1)} n_\Gamma(\lambda,L)\hat{f}(\lambda) = \sum_{\gamma\in C_\Gamma} \chi_L(\gamma)\ \mathrm{Vol}(\Gamma_\gamma\setminus G_\gamma)\ J_\gamma(f).$$

§3. Back to Propagators.

In the situation above where $\Gamma$ is a discrete subgroup of G with $\Gamma\setminus G$ compact, $\Gamma$ acts on G/H by left translation and the quotient space $\Gamma\setminus G/H$ is compact. As above with L = I, $L_2(\Gamma\setminus G/H) = \sum\oplus\mathcal{H}(\lambda)$ where the Laplacian $\Delta\in\mathcal{D}(G/H)$ acts on $\mathcal{H}(\lambda)$ by the scalar $h_\lambda(\Delta)$. Formally the sum $\sum_{\lambda\in\hat{G}(1)} d(\lambda)\exp(-ith_\lambda(\Delta))$ is the trace of the operator $\exp(-it\Delta)$ on $L_2(\Gamma\setminus G/H)$. Of course $\exp(-it\Delta)$ for $-\infty < t < \infty$ is a group with distributional kernel $\sum_\lambda \sum_{j=1}^{d(\lambda)} \exp(-ith_\lambda(\Delta))\phi_{\lambda j}(x)\overline{\phi_{\lambda j}(y)}$, where $\{\phi_{\lambda j}\}$, $j=1,\ldots,d(\lambda)$ is an ONB for $\mathcal{H}(\lambda)$.

If the Euclidean case is mimicked one expects that on $L_2(H\setminus G/H)$ the operator $\exp(-it\Delta)$ has an integral kernel $k(t;y^{-1}x) = g(it;y^{-1}x)$ where $g_t(m)$, m G/H, is the fundamental solution of the heat equation $\Delta u = \partial u/\partial t$ on G/H. The analogous integral operator on $L_2(\Gamma\setminus G/H)$ should then be obtained by "periodizing" k by "wrapping it around" $\Gamma$ G/H along the orbits of $\Gamma$ on G/H, i.e., (for L nontrivial) $K_{\Gamma\setminus G/H}(t;y,x) = \sum_{\gamma\in\Gamma} L(\gamma)k_t(y^{-1}\gamma x)$ with trace being $\sum_{\gamma\in\hat{G}(1)} n_\Gamma(\lambda,L)\exp(-ith_\lambda(\Delta)) = \int_{\Gamma\setminus G/H} K(t;x,x)dx$. (Requiring the wave functions to transform according to $\psi(\gamma m) = L(\gamma)\psi(m)$, one is led to $\int_M k(t;x,y)\psi(y) = \int_{\Gamma\setminus M} K(t;x,y)\psi(y)$ where $K(t;x,y) = \sum_{\gamma\in\Gamma} L(\gamma)k(t;x,\gamma y)$ as observed by Selberg [16]. Later this was the motivation for eq. (5') of [ 6 ] and [11].)

This leads naturally to the conjecture that $K_t$ has a trace $\int_{\Gamma\setminus G/H} K_t(x,x)dx =$

$$\sum_{\lambda\in\hat{G}(1)} n_\Gamma(\lambda,L)\exp(-ith_\lambda(\Delta)) = \int_{\Gamma\setminus G/H} \sum_{\gamma\ \Gamma} \chi_L(\gamma)k_t(x^{-1}\gamma x)dx = \sum_{\gamma\in\Gamma} \chi_L(\gamma) \int_F k_{G/H}(t;x^{-1}\gamma x)dx \tag{3.1}$$

where F is a measurable fundamental domain for $\Gamma$ in G/H. In this generalized STF we emphasize that everything is understood only formally.

The conjecture crystallizes the status of the remarks in [ 4 ], [ 5 ], [ 6 ], [11], and elsewhere. The evidence which leads us to expect a GSTF to be true is as follows.

Of course (3.1) reduces to (1.1) in the case $\Gamma$ = id and G compact. Furthermore, it was observed by Schulman [15] for SU(2) and Dowker [ 5] for a compact Lie group G that the propagator is formally

$$K_G(t;x) = \sum_{\substack{\text{classical}\\ \text{paths}}} k(t;\gamma x)$$

where k(t;exp H) satisfies the "radial equation"

$$i \frac{\partial}{\partial t} j - \left( \sum_{k=1}^{n} \frac{\partial^2}{\partial h_k^2} + 4\pi^2 \, |\rho|^2 \right) j = 0$$

for H regular, where: T is the maximal torus in G; the Lie algebra of T is $\mathcal{T}$ with basis $H_1,\ldots,H_n$; $\rho = \frac{1}{2} \sum_{r=1}^{\frac{N-n}{2}} \theta_r$ where $\theta_r$ are the positive roots of G (N = dim G); $|\cdot|$ is the Cartan-Killing norm; $j(H) = \Pi_r \, [\exp(\pi i \theta_r(H) - \exp(-\pi i \theta_r(H))]$ for H in $\mathcal{T}$; T' is the set of regular (i.e., $j(H) \neq 0$) points of $\mathcal{T}$; $H = \sum_k h_k H_k$. This fits precisely in our framework as hinted by Dowker. Viz., the propagator K(t;x,y) on G satisfies $K(t;x,y) = K(t;y^{-1}x,e) = K(t;y^{-1}x)$; and the map $z \to K(t,z)$ is invariant by inner automorphisms of G. Thus it is determined by the restriction to the maximal torus $T \subset G$ and K(t;exp ) satisfies the radial equation for $h = \exp H \in T'$. The positive roots $\theta_r$ of G are linear forms on $\mathcal{T}$ which take integral values on a discrete subgroup $\Gamma$ of $\mathcal{T}$. The function

$$T' \ni h \to k_{\mathcal{T}}(t;h) = \exp \frac{(4\pi^2 i \, |\rho|^2 \, t)}{(it)^{n/2}} \sum_{\gamma \in \Gamma} \frac{[L(h) \, \exp(-i|h|^2/4t)](h)}{j(h)}$$

is a W-invariant solution of the radial equation (where W is the Weyl group with order $|W|$). Then $\int_G K(t;g)\,dg = \frac{1}{|W|}\int_T |j(u)|^2 \times [\int_G K(t;gug^{-1})\,dg]\,du =$

$\frac{1}{\mathrm{Vol}(T/\Gamma)|W|} \int_{T/\Gamma} \sum_{\gamma\in\Gamma} |j(h+\gamma)|^2\, k_T(h+\gamma)\,dh$. This is given by (16-(18) in [ 5 ].

(Note that $|\rho|^2 = R/6$ where R is the scalar curvature of G.)

For the heat equation on these spaces, the analogous formulae have been proven in [ 1 ], [ 7 ], and elsewhere.

In the simple case that G = R, H = {e}, and Γ = Z, the GSTF reduces to the following analogue of the Poisson-Jacobi (theta) formula

$$\sum_{n\in Z} e^{-itn^2} = \sqrt{\pi/t} \sum_{m\in Z} \exp[\frac{i(2\pi m)^2}{4t}] \exp(\frac{-i\pi}{4}) \tag{3.2}$$

where $n^2$ is the spectrum of $-\Delta$ on $S_1^1 = R/Z$ and $2\pi m$ is the length of the iterated closed geodesic (length $2\pi$ on $S_1^1$).

Colin de Verdiere has generalized the classical Poisson-Jacobi formula

$$\sum_{n\in Z} \exp(-n^2/z) = \sqrt{\pi z} \sum_{m\in Z} \exp(-\pi^2 m^2 z) \qquad \mathrm{Re}\ z > 0$$

for certain Riemannian manifolds with negative sectional curvature to

$$\sum_{k\geq 0} \exp(-\lambda_k/z) = \sum_{\ell\ L^+\cup\{0\}} f_\ell(z) \exp(-z\ell^2/4) \quad (F)$$

where F denotes the techniques of nonlinear Fourier transform, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of the Laplacian $-\Delta$ on the compact connected Riemannian manifold M, $L$ = set of lengths (and their opposites) of periodic geodesics on M. Note also Chazarain's formula [ 2 ].

The author is unable at present to prove the GSTF in the most general case. (Perhaps via techniques of Nelson and Ray it has been suggested.) However in the case in [11], G = R, H = {e} and Γ = Z, the GSTF can be proven. The dynamics of the situation here is the fixed axis rigid rotator with Lagrangian $L = \frac{I}{2}\dot{\phi}^2$, $0 \leq \phi < 2\pi$, on $M = SO(2)$; a CONB for the wave functions is given by $\{\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}\}$, $m \in Z$, with $E_m = m^2/2I$. Then $K(t,\phi) = \frac{1}{2\pi}\sum \exp(i\phi n)\exp(-in^2/2\gamma)$ $= \frac{1}{2\pi}\Theta_3(\frac{\phi}{2}, -\frac{1}{2\gamma\pi})$ where $\gamma = I\hbar t$ and $\Theta_3(z,t) = \sum_{n\in Z} e^{i\pi tn^2} e^{2inz}$. For Im$t > 0$, the Poisson summation formula gives

$$K(t,\phi) = (\frac{\gamma}{2\pi i})^{1/2} \sum_{n\in Z} e^{i\gamma(\phi-2n\pi)^2/2} = \sum_{\gamma\in\Gamma} k(t;\phi+\gamma)$$

where k is the free particle propagator. The GSTF is just the case $\phi = 0$.

Theorem. The GSTF is true for the case $G = R$, $H = \{e\}$ and $\Gamma = Z$, i.e., (3.2) holds in the sense of Wiener's fourier transform.

The proof is straight forward.

## §4. Geodesics and Propagators.

As noted above the philosophy in Schulman, Dowker and elsewhere is to express the propagator as a "sum over all classical paths." If $M$ is a compact Riemannian manifold, then each closed path g (distinct from the identity) of $\Pi_1(M)$ corresponds to a closed geodesic $\gamma_g$ of class g whose length is minimal among the closed curves of the same class as g. If M is of negative sectional curvature then there is only one closed geodesic of each homotopy type and every closed geodesic is so obtained. So there is a biunique correspondence between closed geodesics and nontrivial elements of $\Pi_1(M)$, or between the free homotopy classes of closed paths and the set $C_\Gamma$ of conjugacy classes of elements of $\Gamma$.

In the situation above we have $\Gamma$ a discrete torsion-free subgroup of Lie group G with $\Gamma$  G compact. Then $M = \Gamma$  $G/H$ is a compact Riemannian with simply connected covering space $G/H$ and $\Gamma \simeq \Pi_1(M)$. From these remarks, if $M$ has negative sectional curvature the GSTF(3.1) is modified by writing

$$K_{\Gamma\backslash G/H}(t;y,x) = \sum_{\substack{\text{"all closed} \\ \text{geodesics } \gamma\text{"}}} k(t;y^{-1}\gamma x)$$

A large class of manifolds of negative sectional curvature in this form are $\Gamma\backslash G/H$ where G is a noncompact connected simple Lie group of R-rank one and finite center, H is a maximal compact subgroup of G, $\Gamma$ is a discrete subgroup of G acting freely on G/H; and G/H is a rank one symmetric space of noncompact type. This is an extremely interesting case for then the length spectrum (lengths of the periodic geodesics $\gamma_g$ and their multiplicities) is determined by the (harmonic) spectrum of $\Delta$ on $M$. (Huber [10], Atiyah and Duistermaat (to appear), Gangolli (to appear)) and a "generalized" length spectrum plus $\mathrm{Vol}(\Gamma\backslash G)$ determines the (harmonic) spectrum.

§5. STF and Geometric Quantization.

As we know, an important object in geometric quantization is the quantized Hilbert space associated with a Kählerian polarization F, $H^\circ(M,\mathcal{O}(E))$ where $E \to M$ is a holomorphic line bundle over $(M,\Omega)$ with the curvature of the connection on E being $\Omega$, etc. This situation arises when $M = G/H$ is a bounded symmetric domain with cocompact $\Gamma$ acting freely; then $M = \Gamma\backslash G/H$ is an algebraic manifold. In this case the first version of the STF applies. E.G., if $E_\lambda \to M$ is the bundle corresponding to the holomorphic discrete series of Harish Chandra then the multiplicity of the "energy levels" for the "energy manifold" $M$ is

$$n_\Gamma(\lambda) = \dim H^\circ(M,\mathcal{O}(E_\lambda)) = \sum_\gamma \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)J_\lambda(\gamma) = \text{dimension}$$

of the space of automorphic forms for such a representation. (Cf. Hotta-Parthasarathy, et al.)

In general in the situation at the end of §5 if $\Gamma$ has no elliptic elements $n_\Gamma(\lambda) = \mathrm{Vol}(\Gamma\backslash G)d(\lambda)$ when $\lambda$ is integrable. (Cf. Langlands, Schmid, et al.)

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