Invariances of generalized uncertainty principles

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Abstract

In Ref.[1] we use two different generalized uncertainty principles to compute mass thresholds and lifetimes for micro black holes close to their Planck phase. Motivated by that paper, we study here in detail the conditions for the translation and rotation invariance of these two different kinds of deformed commutation relations.

1 Introduction

When we consider a high energy collision, we know that Heisenberg principle $\Delta p \Delta x \geq \hbar/2$ can be casted in the form $\Delta E \Delta x \geq \hbar c/2$ (since $\Delta E \simeq c\Delta p$). Actually, the main reason since larger and larger energies are required to explore smaller and smaller details is that the size of the smallest detail theoretically detectable with a beam of energy E is $\delta x = \hbar c/(2E)$. An equivalent argument comes from considering the resolving power of a "microscope": the smallest resolvable detail goes roughly as the wavelength of the employed photons, and therefore $\delta x \simeq \lambda = \frac{c}{\nu} = \frac{\hbar c}{\epsilon}$.

The research on viable generalizations of the Heisenberg uncertainty principle traces back to many decades (see for early approaches [2], etc. See for a review [3] and for more recent approaches [4]). In the last 20 years, there have been seminal studies in string theory [5] suggesting that for very high energy scattering the uncertainty relation (ST GUP) should be written (in $4+n$ dimensions) as

$$\delta x \geq \frac{\hbar c}{2E} + \beta \ell_{4n} \frac{E}{\mathcal{E}_{4n}}\,, \tag{1}$$

where ℓ_{4n} is the $4+n$ dimensional Planck length and E is the energy of the colliding beams (we use the relation $\mathcal{E}_{4n}\ell_{4n} = \hbar c/2$). If however we take into account the possibility of a formation of micro black holes in the scattering, with a gravitational radius of $R_S \sim (E)^{1/(n+1)}$, then we easily see that in $4+n$ dimensions (and $n \geq 1$) the stringy principle seems to forbid the very observation of the micro hole itself. In fact, at high energy the error predicted by the stringy GUP goes like $\delta x \sim E$, while the size of the hole goes like $R_S \sim (E)^{1/(n+1)}$. For E large enough and $n \geq 1$, we always have $E > (E)^{1/(n+1)}$, thereby loosing the possibility of observing micro black holes, just when they become massive (that is, when they should approach the classicality). Also to avoid this state of affairs, and on the ground of gedanken experiments involving the formation of micro black holes, it has been proposed [6] a modification of the uncertainty principle, that in $4+n$ dimensions reads

$$\delta x \geq \frac{\hbar c}{2E} + \beta R_{4n}(E)\,, \tag{2}$$

where R_{4n} is the $4+n$ dimensional Schwarzschild radius associated with the energy E (see [7])

$$R_{4n} = \left[\frac{16\pi G_{4n} E}{(N-1)\Omega_{N-1} c^4}\right]^{\frac{1}{N-2}} = \ell_{4n}\left(\omega_n \frac{E}{\mathcal{E}_{4n}}\right)^{\frac{1}{n+1}} \tag{3}$$

and $N = 3+n$ is the number of space-like dimensions, $\omega_n = 8\pi/((N-1)\Omega_{N-1})$, $\Omega_{N-1} = 2\pi^{N/2}/\Gamma(N/2) =$ area of the unit $\mathrm{S}^{\mathrm{N}-1}$ sphere. Thus, the GUP originating from micro black hole gedanken experiments (MBH GUP) can be written as

$$\delta x \geq \frac{\hbar c}{2E} + \beta \ell_{4n}\left(\omega_n \frac{E}{\mathcal{E}_{4n}}\right)^{\frac{1}{n+1}}\,, \tag{4}$$

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where β is the deformation parameter, generally believed of $O(1)$. Remarkably, in 4 dimensions ($N = 3$, $n = 0$) the two principles coincide. The deformation parameter β, supposed independent from the dimensions N, can be therefore chosen as the same for both principles.

2 Translation and rotation invariance of the GUPs

In this section we shall prove that the GUPs previously introduced do respect the constraints posed by requiring the conventional translation and rotation invariance of the commutation relations. First, we show what these kinematic constraints imply about the structure, in $4 + n$ dimensions, of the $[x, p]$ commutations relations. In this, we follow closely Ref. [8]. As a general ansatz for the x, p commutation relation in $4 + n$ dimensions we take

$$[x_i, p_j] = i\,\hbar\ \Theta_{ij}(p) \tag{5}$$

and we require that $\Theta_{ij}(p)$ differs significantly from δ_{ij} only for large momenta. We assume also $[p_i, p_j] = 0$ and we compute the remaining commutation relation through the Jacobi identities, obtaining

$$[x_i, x_j] = i\hbar\{x_a,\ \Theta^{-1}_{ar}\Theta_{s[i}\Theta_{j]r,s}\} \tag{6}$$

where $\{\}$ are the anti-commutators and $Q_{,s} := \partial Q/\partial s$. The commutation relations (5) are translation invariant (they are preserved under the transformations $x_i \to x_i + d_i, \quad p_i \to p_i$). However, the commutation relations (6) are not invariant under translation, unless we require $\Theta_{ij}(p)$ to be such that it yields $[x_i, x_j] = 0$. Thus, in order to implement translation invariance, Θ_{ij} must satisfy the necessary and sufficient condition (read off from the (6))

$$\Theta_{ia}\partial_{p_i}\Theta_{bc} = \Theta_{ib}\partial_{p_i}\Theta_{ac} \tag{7}$$

where sum over i is understood. The rotation invariance can be implemented by requiring Θ_{ij} to have the form

$$\Theta_{ij}(p) = f(p^2)\delta_{ij} + g(p^2)p_i p_j\,. \tag{8}$$

Together, conditions (7) and (8) imply that f and g must satisfy the differential equation

$$2f'f + (2p^2 f' - f)g = 0 \tag{9}$$

where $f'(p^2) = df/d(p^2)$. Under these conditions, commutation relations do obey translation and rotation invariance. Considering, for sake of simplicity, the mono-dimensional case $i = j$, we write for the main commutator

$$[x, p] = i\hbar(f(p^2) + g(p^2)p^2)\,. \tag{10}$$

The usual Heisenberg commutator is recovered by choosing, for example, $f(p^2) = 1$. Then Eq.(9) implies $g(p^2) = 0$ and $[x, p] = i\hbar$. The stringy inspired commutator is obtained, to the first order in β, by choosing $g(p^2) = \beta$ (see [8]). Then, in fact, solving (9) (a Manfredi equation, in such case), we find

$$f(p^2) = \frac{\beta p^2}{\sqrt{1 + 2\beta p^2} - 1} \simeq 1 + \frac{\beta}{2}p^2 + O((\beta p^2)^2) \tag{11}$$

and, to the first order in β (or, equivalently, for small p) we have

$$[x, p] = i\hbar\left(1 + \frac{3}{2}\beta p^2 + O(\beta^2)\right)\,. \tag{12}$$

The MBH GUP (4) can be written in terms of momentum transferred as $p\,\delta x \gtrsim \frac{\hbar}{2}\left(1 + \gamma p^{\frac{n+2}{n+1}}\right)$ where $\gamma = \beta(\omega_n)^{\frac{1}{n+1}}\left(\frac{2\ell_{4n}}{\hbar}\right)^{\frac{n+2}{n+1}}$ and this in terms of commutators becomes

$$[x, p] = i\hbar\left(1 + \gamma p^{\frac{n+2}{n+1}}\right)\,. \tag{13}$$

To show that MBH GUP is translation and rotation invariant we must show that the commutator (13) is of the same form of commutator (10) (when $p \to 0$), with f and g satisfying (9) (in particular we would like to have $f(p^2) \to 1$ for $p \to 0$). However, the previous strategy, namely to fix a priori a given form for $g(p^2)$ and then to compute $f(p^2)$ by solving (9) (as we did for HUP, $g(p^2) = 0$, and for stringy GUP, $g(p^2) = \beta$), in this case does not work. Even if one puts $p^2 g(p^2) = \gamma p^{(n+2)/(n+1)}$, Eq.(9) becomes however rather complicated (it is an Abel equation of 2^{nd} kind), and hardly we can hope it gives $f(p^2) \to 1$ for $p \to 0$. Moreover, an explicit solution could not be so useful, since we are mainly interested in an asymptotic behaviour. Therefore we ask the following general properties to be satisfied by the functions f and g

$$\begin{cases} [f(p^2) + g(p^2)p^2] \to [1 + \gamma p^{\frac{n+2}{n+1}}] \quad \text{for} \quad p \to 0 \\ \\ 2f'f + (2f'p^2 - f)g = 0\,, \end{cases} \tag{14}$$

We shall look if there actually exist f and g such that the above two properties can be simultaneously satisfied. In this way the rotational and translational invariance of GUP (4) will result proved. In what follows such solutions are proved to exist, provided we allow g to develop poles (of course, the function f and the whole function $f + gp^2$ remain perfectly finite).

In the differential equation (9) everything is function of p^2 and $f'(p^2) = df/d(p^2)$. So, let's set $y := p^2$ ($y > 0$, $p = y^{1/2}$) and, to avoid fractionary powers, set also $y^{\frac{1}{2(n+1)}} =: \lambda$, $y = \lambda^{2(n+1)}$. Then $f'(y) = \frac{1}{2(n+1)}\lambda^{-(2n+1)}F'(\lambda)$, and the system (14) becomes

$$\begin{cases} [F(\lambda) + G(\lambda)\lambda^{2(n+1)}] \to [1 + \gamma\lambda^{n+2}] \quad \text{for} \quad \lambda \to 0 \\ \\ F'(\lambda)F(\lambda) + [\lambda F'(\lambda) - (n+1)F(\lambda)]G(\lambda)\lambda^{2n+1} = 0\,. \end{cases} \tag{15}$$

We have to see if the two conditions are compatible, and what this implies for f and g. To check this compatibility we can use power series representations of the functions $F(\lambda)$, $G(\lambda)$. We allow $G(\lambda)$ to develop poles. Since the factor $\lambda^{2(n+1)}$ multiplies $G(\lambda)$ in the boundary condition, we could allow poles until $\lambda^{-2(n+1)}$ and still the combination $[F + G\lambda^{2(n+1)}]$ would remain analytical. However, we'll show that the result can be obtained by allowing poles just until λ^{-n} only. So we write

$$F(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k \qquad \text{and} \qquad G(\lambda) = \sum_{k=-n}^{\infty} b_k \lambda^k\,. \tag{16}$$

and we look for what the two conditions imply on the coefficients a_k, b_k. We have

$$F(\lambda) + G(\lambda)\lambda^{2(n+1)} = \sum_{k=0}^{\infty}(a_k + b_{k-2(n+1)})\lambda^k \tag{17}$$

where $b_{k-2(n+1)} = 0$ for $k = 0, 1, 2, ..., n+1$ and $b_{-n} \neq 0$, $b_{-n+1} \neq 0$, etc.
At small λ we should have the matching, for $\lambda \to 0$,

$$\sum_{k=0}^{\infty}(a_k + b_{k-2(n+1)})\lambda^k \longrightarrow 1 + \gamma\lambda^{n+2}\,. \tag{18}$$

This means

$$\begin{array}{lcllll} k & = & 0; & [a_0 + b_{-2(n+1)}] = 1 & \Rightarrow & a_0 = 1 \\ k & = & 1; & [a_1 + b_{1-2(n+1)}] = 0 & \Rightarrow & a_1 = 0 \\ k & = & 2; & [a_2 + b_{2-2(n+1)}] = 0 & \Rightarrow & a_2 = 0 \\ & & ... & ... & ... & \\ k & = & n+2; & [a_{n+2} + b_{-n}] = \gamma & \Rightarrow & (*) \\ k & = & n+3; & [a_{n+3} + b_{-n+1}] = \text{any quantity} & & \\ & & ... & ... & ... & \end{array} \tag{19}$$

(*) here at least b_{-n} is $\neq 0$, therefore at least b_{-n} can be chosen equal to γ.

Now let's see if the conditions on a_k, b_k just found above, required by the first of Eqs.(15), are compatible with those required by the differential equation (15). Since $F'(\lambda) = \sum_{k=0}^{\infty}(k+1)a_{k+1}\lambda^k$, $G(\lambda) = \sum_{k=-n}^{\infty} b_k\lambda^k = \sum_{k=0}^{\infty} b_{k-n}\lambda^{k-n}$ (with $b_{-n} \neq 0$, $b_{-n+1} \neq 0$,..., $b_0 \neq 0$), we have $F(\lambda)F'(\lambda) = \sum_{k=0}^{\infty} C_k\lambda^k$, $G(\lambda)F'(\lambda) = \sum_{k=0}^{\infty} D_k\lambda^{k-n}$, $F(\lambda)G(\lambda) = \sum_{k=0}^{\infty} E_k\lambda^{k-n}$, where $C_k = \sum_{q=0}^{k}(q+1)a_{k-q}a_{q+1}$, $D_k = \sum_{q=0}^{k}(q+1)b_{k-n-q}a_{q+1}$, $E_k = \sum_{q=0}^{k} a_{k-q}b_{q-n}$ and the differential equation (15) becomes

$$\sum_{k=0}^{\infty}\left[C_k\lambda^k + D_k\lambda^{k+n+2} - (n+1)E_k\lambda^{k+n+1}\right] = 0 \tag{20}$$

Reshuffling indexes a bit in Eq.(20) we get

$$\sum_{k=0}^{\infty}\left[C_k + D_{k-n-2} - (n+1)E_{k-n-1}\right]\lambda^k = 0 \tag{21}$$

where $D_{-n-2} = 0$, $D_{1-n-2} = 0$, ..., $D_{-1} = 0$, and $E_{-n-1} = 0$, $E_{1-n-1} = 0$, ..., $E_{-1} = 0$, and it is easy to see that these relations are direct consequences of the definitions for b_k in Eq.(17) and of relations (19). Equation (21) can be satisfied only if all the coefficients of λ^k are identically zero. We can now check explicitly that this requirement is in full agreement with conditions (19). In fact: $k = 0$; $\quad C_0 + D_{-n-2} - (n+1)E_{-n-1} = a_0a_1 = 0 \;\Rightarrow\; a_1 = 0$ (since $a_0 = 1$) and this agrees with (19). And then $k = 1$; $\quad C_1 + D_{1-n-2} - (n+1)E_{1-n-1} = a_1a_1 + 2a_0a_2 = 0 \;\Rightarrow\; a_2 = 0$ (since $a_0 = 1$) and this agrees with (19). Again, for $k = 2$ we have $C_2 + D_{2-n-2} - (n+1)E_{2-n-1} = 0 \;\Rightarrow\; a_3 = 0$ and so on for $k = 3, 4, \ldots$. For $k = n$ we find $a_{n+1} = 0$ in agreement with (19). For $k = n+1$ we have

$$C_{n+1} + D_{-1} - (n+1)E_0 = \sum_{q=0}^{n+1}(q+1)a_{n+1-q}a_{q+1} - (n+1)\sum_{q=0}^{0} a_{0-q}b_{q-n} = (n+2)a_0a_{n+2} - (n+1)a_0b_{-n} = 0$$

Since $a_0 = 1$, then $(n+2)a_{n+2} - (n+1)b_{-n} = 0$ and this equation *is* compatible with the "$k = n+2$" condition of (19). In fact, we have two equations in two unknowns

$$(n+2)a_{n+2} - (n+1)b_{-n} = 0 \qquad \text{and} \qquad a_{n+2} + b_{-n} = \gamma\,, \tag{22}$$

which allow us to compute a_{n+2} (the first non zero coefficient for $F(\lambda)$, after $a_0 = 1$) and b_{-n} (pole of order n of $G(\lambda)$). For the next case, $k = n+2$, we don't have evidently any problem, since Eq.(19) simply gives $(a_{n+3} + b_{-n+1}) =$*any quantity*. Therefore any relation between a_{n+3}, b_{-n+1} required by the differential equation in (15) is acceptable. Note moreover that if we allowed poles for $G(\lambda)$ with a degree less than n, we would find contradiction between the conditions (18)-(19), *and* the differential equation in (15). Thus, we conclude that the two conditions (15) are compatible (if we allow $G(\lambda)$ to develop poles). So the MBH GUP, as well as the ST GUP, are translational and rotational invariant. Q.E.D.

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