Group-Theoretical Aspects of Dimensional Reduction

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ABSTRACT

The problem of lifting space-time symmetries to automorphisms of principal bundles is discussed. Bundles admitting such lifts are classified for a case more general than that considered by Harnad,Shnider and Vinet. Next, the classification of invariant connections on bundles admitting lifts is performed. Finally, a group-theoretical interpretation of the constraint equation for scalar fields appearing in the dimensional reduction scheme is given and a method for solving this equation is shortly discussed.

0. INTRODUCTION

One of the interesting examples of field theories on higher dimensional space-time (multidimensional universe) - usually referred to as Kaluza-Klein theories - is that of a pure Yang-Mills theory. After its dimensional reduction one obtains a gauge model which includes scalar fields, minimally coupled to the reduced gauge field and having a self interaction term up to the fourth power in the fields /1-4/, see also /5/ and references therein.

There were numerous attempts to construct by this method realistic Higgs-models, for example, the bosonic sector of the

Weinberg-Salam model /6,7/ or of grand unification models, see /8/ and references therein.

It is well-known that the appropriate mathematical language for studying problems related to this subject is that of differential geometry: Gauge potentials are connection forms on a principal bundle P(M,G) over space-time M with structure (gauge) group G. The additional space-time symmetry group K acts to the left on M:

$$\delta : K \times M \longrightarrow M \quad . \qquad (0.1)$$

Thus, the first question arising is: What is a K-invariant gauge potential? To answer this question one has to define a lift of δ to the group of automorphisms Aut(P) of P :

$$\sigma : K \times P \longrightarrow P \quad , \qquad (0.2)$$

$\sigma_k \in \mathrm{Aut}(P)$, for all $k \in K$.

Unfortunately, such a lift does not always exist /9/, see also /10,11/. No problems arise if the bundle is trivial or if it is the frame bundle over M - therefore, for gravity this problem does not occur.

A simple example demonstrating the existence of an obstruction is the following: Consider the real line as a principal bundle $R^1(U(1),\mathbb{Z})$ over U(1) with structure group $\mathbb{Z}$. For K take the discrete reflection group given by complex conjugation: $f(z) = \bar{z}$. Obviously, the unique bijective homomorphism $\varkappa$ of R^1, which projects onto f is reflection with respect to the origin: $\varkappa(t) = -t$. Now, observe that $\varkappa$ does not commute with the right (principal) action of $\mathbb{Z}$ on R^1 , showing that $\varkappa$ is not an automorphism of $R^1(U(1),\mathbb{Z})$, /9/.

Thus, one may formulate the following problem: Classify all principal bundles P(M,G) with K-action (as automorphisms) projecting on a given K-action on M. If K acts transitively on M, the answer is well-known /9,12/. A general solution to the problem is not known to us. In the first section we shall give a generalization of the classical result (transitive action), including but generalizing the case considered in /13/. Our treatment will be based on /14/. In section 2 we shall briefly discuss the classification of K-invariant connections on bundles admitting lifts and comment on dimensional reduction of the gauge field action. As already mentioned at the beginning, after dimensional reduction one obtains in addition to the reduced gauge field a set of scalar fields. These fields have to fulfill a certain (algebraic) constraint equation, which can be interpreted in terms of group theory: The set of scalar fields form an operator intertwining certain representations. Solving the constraint equation and finding the explicit form of the self interaction potential for scalar fields amounts to constructing this operator. In section 3 we shall make some remarks on this problem. For a detailed discussion we refer to /15/.

1. THE PROBLEM OF LIFTING SPACE-TIME SYMMETRIES

From the very beginning we restrict ourselves to the case when (0.2) treated as a mapping $\delta : K \longrightarrow Aut(P)$, is a homomorphism. If one drops this assumption, then the problem becomes very complicated - as a simple example discussed in /16/ shows. Now, let K be a connected, compact Lie group. Suppose that K acts on M to the left with one orbit type [H] /17/, ([H] - conjugacy class of stabilizers of the K-action), and that the bundle M $\longrightarrow$ M/K admits a global section

$$s: \ M/K \longrightarrow M \ . \tag{1.1}$$

This section can be chosen such that:

$$\delta(h,s(b)) = s(b) \quad , \tag{1.2}$$

for all $h \in H$, $b \in M/K$.

This shall be called the case of simple K-action. In /13/ the above classification problem was solved for this case under the <u>additional</u> assumption that M/K is contractible.

Let us denote $\breve{M} := s(M/K)$ and $\breve{P} := \pi^{-1}(\breve{M})$. Obviously, $\breve{P}(\breve{M},G)$ is a G-principal bundle over $\breve{M}$. We denote the restriction of the right group action Ψ and the canonical projection π to $\breve{P}$ by $\breve{\Psi}$ and $\breve{\pi}$ and the vertical automorphisms by $Aut_o(\breve{P})$.

<u>Proposition 1:</u>

Let there be chosen a section (1.1) satisfying (1.2).

1. Let P(M,G) be a G-principal bundle and $\sigma : K \longrightarrow Aut(P)$ an action of K on P projecting onto a simple action on M. Then there exists a K-equivariant diffeomorphism

$$\chi : K \times_H \breve{P} \longrightarrow P \quad , \tag{1.3}$$

where K is treated as an H-principal bundle K(K/H,H).

2. Conversely, let $\delta : K \longrightarrow Diff(M)$ be a simple action of K on M and let $\breve{P}$ be a G-principal bundle over $\breve{M}$. Moreover, let there be given a homomorphism:

$$\hat{\sigma} : H \longrightarrow \mathrm{Aut}_o(\tilde{P}) \quad . \tag{1.4}$$

Then $P := K \times_H \tilde{P}$ is naturally a G-principal bundle over M and the natural action of K on P is a homomorphism $\bar{\sigma} : K \longrightarrow \mathrm{Aut}(P)$, projecting onto δ .

<u>Proof:</u> See /14/ .

This Proposition reduces the lift problem for simple K-action to the problem of analyzing the structure of principal bundles $\tilde{P}(\tilde{M},G)$ admitting homomorphisms (1.4). We are able to solve this problem only after making an additional regularity assumption. First, observe that $\hat{\sigma}$ defines (and is completely characterized by) a mapping $\tau : H \times \tilde{P} \longrightarrow G$, given by:

$$\hat{\sigma}_h(\tilde{p}) = \tilde{\Psi}_{\tau(h,\tilde{p})}(\tilde{p}) \ , \quad h \in H, \ \tilde{p} \in \tilde{P} \ . \tag{1.5}$$

For every $\tilde{p}$, $\tau_{\tilde{p}} : H \longrightarrow G$, is a group homomorphism. Now, denote $\tau_o := \tau_{\tilde{p}_o}$ for a fixed $\tilde{p}_o \in \tilde{P}$ and assume that for every $\tilde{p}$ there exists a $g(\tilde{p})$ with

$$\tau_{\tilde{p}} = g(\tilde{p}) \cdot \tau_o \cdot g(\tilde{p})^{-1} \quad . \tag{1.6}$$

Clearly, this assumption does not imply any restriction on the action of K on M - but, nevertheless, it would be interesting to drop it and try to investigate the general case. Finally, let us denote the centralizer of $\tau_o(H)$ in G by C.

<u>Proposition 2:</u>

1. Let $\tilde{P}(\tilde{M},G)$ be a principal bundle and $\hat{\sigma} : H \longrightarrow \mathrm{Aut}_o(\tilde{P})$ a homomorphism satisfying (1.6). Then $\tilde{P}$ is reducible to a principal subbundle $\hat{P}$ over $\tilde{M}$ with structure group C.

Moreover, there exists a G-equivariant diffeomorphism

$$\tilde{\varkappa}: G \times_C \hat{P} \longrightarrow \tilde{P} \ , \qquad (1.7)$$

where G is treated as a C-principal bundle G(G/C,C).

2. Conversely, let $\hat{P}(\tilde{M},C)$ be a C-principal bundle and $\tau_o \in \mathrm{Hom}(H,G)$. Then $\tilde{P} := G \times_C \hat{P}$ is naturally a G-principal bundle over $\tilde{M}$, $\hat{P}$ a subbundle, and there exists a natural homomorphism $\hat{\sigma}: H \longrightarrow \mathrm{Aut}_o(\tilde{P})$, satisfying (1.6).

<u>Proof:</u> See /14/ .

It follows from this Proposition that a homomorphism (1.4) satisfying (1.6) is implementable iff $\tilde{P}$ is reducible to $\hat{P}$. This has been already shown in /18/. As a result of our discussion we obtain that bundles admitting lifts of simple group actions are (for a fixed immersion of $\tilde{M}$) classified by pairs $(\tau_o,\hat{P})$, with $\hat{P}$ being in general - of course - non-trivial. In the case considered in /13/ bundles admitting lifts were (for a fixed immersion) - as in the transitive case - classified just by homomorphism τ_o .
In the next section we shall perform the classification of K-invariant connections on bundles of the above type.

2. CLASSIFICATION OF K-INVARIANT GAUGE POTENTIALS

A K-invariant gauge potential is a connection form ω on P satisfying

$$\sigma_k^{*}\omega = \omega \quad , \text{ for all } k \in K \ . \qquad (2.1)$$

First, observe that a connection satisfying (2.1) is completely given by its values on $\tilde{P}$. Now, let us fix an AdKinvariant scalar product on $\mathfrak{k}$ (Lie algebra of K) and take the corresponding orthogonal (reductive) decomposition:

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} , \qquad (2.2)$$

with $\mathfrak{h}$ being the Lie algebra of the stabilizer H. Obviously, (2.2) induces naturally a connection in K(K/H,H), which in turn induces a connection in the associated bundle $K \times_H \tilde{P}$. Taking its image under χ gives a splitting of the tangent bundle TP:

$$T_pP = \sigma'_k \{ T_{\tilde{p}}\tilde{P} \oplus \sigma'_{\tilde{p}}(\mathfrak{h}^{\perp}) \} \quad , \qquad (2.3a)$$

with $p = \sigma(k,\tilde{p})$ and $\sigma'_{\tilde{p}}(\mathfrak{h}^{\perp})$ being the subspace spanned by Killing vectors of the lifted group action. The decomposition of 1-forms corresponding to (2.3a) will be denoted by

$$\alpha = \alpha^v + \alpha^h \quad . \qquad (2.3b)$$

<u>Proposition 3:</u>

1. A connection form ω on P is completely characterized by a pair $(\tilde{\omega}, \tilde{\Phi})$, with
   a) $\tilde{\omega} := \omega^v \restriction \tilde{P}$ being a connection form on $\tilde{P}$,
   b) $\tilde{\Phi}(\tilde{p}) := \sigma^*_{\tilde{p}} \cdot \omega^h \restriction \tilde{P}$ being an equivariant mapping

$$\tilde{\Phi}: \tilde{P} \longrightarrow (\mathfrak{h}^{\perp})^* \otimes \mathfrak{g} , \qquad (2.4a)$$

$$\tilde{\Phi} \circ \tilde{\Psi}_g = \mathrm{Ad}g^{-1} \circ \tilde{\Phi} \quad , g \in G. \qquad (2.4b)$$

2. A K-invariant connection form ω is characterized by a pair $(\tilde{\omega}, \tilde{\Phi})$ satisfying:
   a) $\hat{\sigma}^*_h \tilde{\omega} = \tilde{\omega}$, (2.5a)
   b) $\tilde{\Phi}(\tilde{p}) \circ \mathrm{Ad}h = \mathrm{Ad}\tau_{\tilde{p}}(h) \circ \tilde{\Phi}(\tilde{p})$, for all $h \in H$. (2.5b)

<u>Proof:</u> See /3,4/.

Now, (2.5a) means exactly that $\tilde{\omega}_{\restriction \hat{P}}$ has to take values in C and, therefore, we have:

Proposition 4:

A K-invariant connection form is in 1-1-correspondence with a pair $(\hat{\omega}, \hat{\Phi})$, with

$$\hat{\omega} := \tilde{\omega}_{\restriction \hat{P}} \quad , \tag{2.6a}$$

$$\hat{\Phi} := \tilde{\Phi}_{\restriction \hat{P}} \quad , \text{ satisfying} \tag{2.6b}$$

$$\hat{\Phi}(\hat{p}) \circ \mathrm{Ad}h = \mathrm{Ad}\, \tau_0(h) \circ \hat{\Phi}(\hat{p}) \ . \tag{2.6c}$$

Proof: See /3,4/ .

We see that K-invariant gauge potentials are in a natural way characterized by objects living on the bundle $\hat{P}$ obtained in section 1. In order to consider field dynamics one needs an additional structure, namely a (pseudo)-Riemannian metric γ on M. Dimensional reduction of the gauge field action is possible if one assumes γ to be also K-invariant:

$$\delta^*_k \gamma = \gamma \quad , \text{ for all } k \in K. \tag{2.7a}$$

Let us restrict ourselves here to the simplest case when, additionally, the splitting

$$T_{\tilde{x}}M = T_{\tilde{x}}\tilde{M} \oplus \delta'_{\tilde{x}}(\mathfrak{h}^{\perp}) \quad , \qquad \tilde{x} \in \tilde{M} \ , \tag{2.7b}$$

is orthogonal with respect to γ . ($\delta'_{\tilde{x}}(\mathfrak{h}^{\perp})$ is the space tangent to the K-orbit through $\tilde{x}$.) One can show easily that a sufficient condition for (2.7b) to be orthogonal with respect to γ is that N/H (N - normalizer of H in K) is a discrete group. Under this additional assumption one obtains:

Proposition 5:

The canonical action of the pure Yang-Mills theory on P reduces - due to K-invariance (2.1) and (2.7) - to the following action on $\hat{P}$:

$$S = 1/\lambda^2 \int_{\hat{P}} \{ \langle\langle \hat{\Omega}, \hat{\Omega} \rangle\rangle_{(1)} + 1/2 \langle\langle D\hat{\Phi}, D\hat{\Phi} \rangle\rangle_{(2)} - V(\hat{\Phi}) \} \, dv_{\hat{P}} \, , \quad (2.8)$$

where $\hat{\Omega}$ is the curvature form of $\hat{\omega}$, $D\hat{\Phi}$ the covariant derivative of $\hat{\Phi}$ with respect to $\hat{\omega}$ and

$$V(\hat{\Phi}) = - \langle\langle \kappa(\hat{\Phi}), \kappa(\hat{\Phi}) \rangle\rangle_{(3)} \quad , \qquad (2.9a)$$

$$\kappa(\hat{\Phi}) = 1/2 \{ [\hat{\Phi}, \hat{\Phi}] - \hat{\Phi} \circ [\cdot, \cdot] \restriction_{\mathfrak{h}^\perp} - \tau' \circ [\cdot, \cdot] \restriction_{\mathfrak{h}^\perp} \} , \qquad (2.9b)$$

$\tau' : \mathfrak{h} \longrightarrow \mathfrak{g}$ being the Lie algebra homomorphism induced by τ_0. Moreover, $\langle\langle \cdot, \cdot \rangle\rangle_{(i)}$ denote the scalar products in the spaces of horizontal forms on $\hat{P}$ with values in $\mathfrak{g}$, $(\mathfrak{h}^\perp)^* \otimes \mathfrak{g}$ and $\overset{2}{\Lambda}(\mathfrak{h}^\perp)^* \otimes \mathfrak{g}$ respectively.

Proof: See /4/, (but, restricted to a subbundle of $\hat{P}$ over a contractible piece of $\tilde{M}$ (2.8) is identical with the result obtained in /1,2/.)

In /4/ we performed the reduction of the gauge field action without the above mentioned orthogonality assumption. In that case additional terms in the reduced action appear, describing non-minimal interaction of gauge and Higgs fields. For similar results see also /5/. For an application of these fibre bundle reduction techniques to gravitational theories see /19/.

It would be interesting to drop the assumption that $M \longrightarrow M/K$ admits a section and to do the classification of bundles admitting lifts of K-actions and of K-invariant connections in this more general case. In /20/ the classification of K-invariant connections for this case has been done - but with the a priori assumption that a lift exists.

3. REMARKS ON THE CONSTRAINT EQUATION FOR SCALAR FIELDS AND MODEL BUILDING

In this section we assume the groups G and K to be simple. Then there are unique, up to a constant, Ad-invariant scalar products on $\mathfrak{g}$ and $\mathfrak{k}$, which we denote by $\langle\, ,\, \rangle$ and $(\cdot,\cdot)$. Due to (2.9) - the potential $V(\hat{\Phi})$ is formally of fourth power in $\hat{\Phi}$. Its explicit form, however, can be found only after solving the constraint equation (2.6c). This equation has the following group theoretical interpretation: $\hat{\Phi}$ is an operator intertwining the representations $\mathrm{Ad}G|_{\tau_0(H)}$ and $\mathrm{Ad}K|_H(\mathfrak{h}^{\perp})$. Thus, solving (2.6c) means constructing this intertwining operator explicitely. Technically, it is more convenient to use the infinitesimal version of (2.6c) :

$$\mathrm{ad}\,\tau'(\,\cdot\,)\circ\hat{\Phi}(\hat{p}) = \hat{\Phi}(\hat{p})\circ\mathrm{ad}(\,\cdot\,) \quad . \tag{3.1}$$

It is also useful to complexify the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, and to continue $\hat{\Phi}$ linearly to the complexified algebras:

$$\hat{\Phi}^{\mathbb{C}}(u_1 + iu_2) := \hat{\Phi}(u_1) + i\,\hat{\Phi}(u_2) \ , \quad u_1, u_2 \in \mathfrak{h}^{\perp} . \tag{3.2a}$$

Then

$$\hat{\Phi}^{\mathbb{C}}(u) = \hat{\Phi}^{\mathbb{C}}(\bar{u}) \ , \quad u \in (\mathfrak{h}^{\perp})^{\mathbb{C}} \ , \tag{3.2b}$$

with "bar" denoting complex conjugation. Obviously, $\hat{\Phi}^{\mathbb{C}}$ fulfills (3.1). Continuing this equation linearly to $\mathfrak{h}^{\mathbb{C}}$ we get

$$\mathrm{ad}\,\tau'(\mathfrak{h}^{\mathbb{C}})\circ\hat{\Phi}^{\mathbb{C}} = \hat{\Phi}^{\mathbb{C}}\circ\mathrm{ad}\,\mathfrak{h}^{\mathbb{C}} \quad . \tag{3.2c}$$

If one has constructed an operator $\hat{\Phi}^{\mathbb{C}}$, satisfying (3.2c), then one obtains $\hat{\Phi}$ satisfying (3.1) by restricting $\hat{\Phi}^{\mathbb{C}}$ to $\mathfrak{h}^{\perp} \subset (\mathfrak{h}^{\perp})^{\mathbb{C}}$. In /7/ there has been proposed a nice graphical method to solve (3.2c), based on a graphical representation of the lattice of positive roots of Lie algebras. Moreover, the authors of /7/ investigated the case K/H being symmetric quite in detail. They found - provided τ is injective and the subalgebras $\mathfrak{h} \subset \mathfrak{k}$ and $\tau(\mathfrak{h}) \subset \mathfrak{g}$ are regular - that in this case one has one irreducible multiplet of scalar fields. They discussed also several options for constructing the bosonic sector of the Weinberg-Salam model and found that - under the above assumptions - the best one is choosing $K/H = \mathbb{C}P^m$ and $G = Sp(m+1)$. In that case one gets for the Weinberg angle

$$\sin^2 \Theta_W = 1/(m+1) \ , \qquad (3.3)$$

which gives reasonable values for m=3 or m=4. If one assumes additionally /15/ that the symmetric space is of rank 1, then one gets a nice geometrical formula for the potential:

<u>Proposition 6:</u>

Let K/H be symmetric, rank(K/H) = 1, $\mathfrak{h}$ - simple, τ - injective and the subalgebras $\mathfrak{h} \subset \mathfrak{k}$ and $\tau(\mathfrak{h}) \subset \mathfrak{g}$ regular. Then

$$V(\hat{\Phi}) = (1 - 1/\varepsilon \cdot |\hat{\Phi}|^2)^2 \cdot \varepsilon \cdot m^2 \cdot R \ , \qquad (3.4)$$

where R is the scalar curvature of K/H corresponding to γ. Moreover, ε denotes the ratio of indices of $\tau(\mathfrak{h})$ in $\mathfrak{g}$ and of $\mathfrak{h}$ in $\mathfrak{k}$, and m is defined by

$$(\delta^*_{(\cdot)} \gamma) \restriction_{\mathfrak{h}^{\perp}} = -1/m^2 \ (\cdot \ , \cdot) \restriction_{\mathfrak{h}^{\perp}} \ . \qquad (3.5)$$

Proof: See /15/ .

An interesting example of this type is:

$$K/H = SO(l+1)/SO(l) = S^l \quad , \qquad (3.6a)$$

$$G = SO(l+p) \quad . \qquad (3.6b)$$

For p=3 , one gets the bosonic sector of the Georgi-Glashow model and one can estimate the Higgs and the `t Hooft-Polyakov monopole mass in terms of the above parameters:

$$m_H = m\sqrt{(l-1)\cdot 2} \qquad (3.7a)$$

$$m_{mon.} = (4\pi/g^2)\sqrt{l}\cdot m\cdot c(2(l-1)/l) \ , \qquad (3.7b)$$

with c being a slowly varying function given in /22/.

For a discussion of the case K/H - not symmetric and (or) $\tau^{\backslash}$ - not injective we refer to /15/. It turns out that in this case one gets interesting new possibilities of model building.

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