SLAC-PUB-1104
(EXPI)
August 1972

# THE FIELD DISTRIBUTIONS IN THE VICINITY OF A SUPERCONDUCTING FLUX EXCLUSION TUBE*

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## Abstract

The distributions of the magnetic field inside and around a superconducting flux exclusion tube due to currents induced in the material by an external magnetic field are examined. A simple macroscopic model for the superconductor is assumed, based on the concept of a field-dependent critical current density, and London's equations are used to calculate the magnetic potentials in the three regions of interest. The magnetic field inside the tube was mapped using an electron beam, and the results compared with the calculations. It is concluded that the major contribution to the magnetic field present in the tube can be ascribed to the induced dipole and sextupole current distributions in the superconductor. By contrast, the effect of direct field penetration through the narrow slits in the structure is very small.

(Submitted to J. Appl. Phys.)

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*Work supported by the U. S. Atomic Energy Commission.

## INTRODUCTION

In previous reports,[1,2] we described the construction and the application of a magnetic flux exclusion tube 4 meters long and 1.3 cm in diameter to provide a field-free electron beam path through a 1-tesla transverse magnetic field. The general arrangement of the high-energy physics experiment is shown in Fig. 1.

The present paper examines the magnetic fields produced inside and around the superconducting tube by the currents induced in the superconductor by the external magnetic field. Although tests with sample tubes indicated that our structure should have been capable of shielding external fields in excess of 1.5 tesla, and although the apparatus was designed with a sufficient safety factor to exclude fields greater than these, the high-energy experiment had to be run at 1.0 tesla as we found that about 0.5% of the external field did in fact penetrate into the field-free region in spite of all our precautions. In order to determine the origin of this penetration, we made a series of measurements on the apparatus using the electron beam and we have calculated the relevant magnetic fields. We are able to account for the observed results and we offer some suggestions for reducing the internal field in possible future applications of this type of apparatus.

As the construction of the tube has been described elsewhere, we summarize only those features which are essential for this analysis. The tube consists of two half-cylinders formed from a number of layers of $Nb_3Sn$ tape, and separated from each other by two narrow slits as shown in cross section in Fig. 2. If we imagine the plane containing the two slits to be inclined at a small angle $\delta$ with respect to the external field $\vec{B}_0$, then we could speculate that the internal field is due to flux penetration through the slits. This, however, is too simple and inaccurate an answer. It turns out that the internal field is due to a superposition of a dipole and sextupole distribution in the current density, and, moreover, the field is approximately independent of the width of the slit. The current distributions are a direct consequence of the boundary conditions.

## THE THEORETICAL MODEL

In order to examine the relation between the current density and the local magnetic field in a superconductor, we make use of London's equations and apply them to Bean's picture[3] of a hard superconductor. This model assumes that the superconductor has filamentary structure capable of sustaining lossless macroscopic currents up to some critical current density, which is a function of the external magnetic field. Moreover, this critical current could be a direct consequence of Ampere's law as the flux is progressively driven into an inhomogeneous superconductor, or it could be, for example, an intrinsic property of the sponge walls in the Mendelssohn model. That is to say, it is not important to the present calculation to assume the detailed nature of the process generating the current density.

We solve London's equations

$$\nabla \times \Lambda \vec{J} = -\vec{B}$$

$$\nabla \times \nabla \times \vec{J} = -\vec{J}/\lambda^2 \qquad (1)$$

$$\lambda = M_e/n e^2 \mu_o = (\Lambda/\mu_o)^{1/2}$$

where $\lambda$ is the London penetration depth, n the number density of electrons and the other terms defined as usual.

(a) The problem can be treated as two-dimensional because the field external to the superconductor varies slowly along the z-axis of the cylinder.

(b) The permeability of the superconductor is equal to $\mu_0$.

(c) The current density $\vec{J}$ is quasi-constant in r between the limits $R-\Delta$ and R, the external radius of the tube. Here $\Delta$ is the thickness of a superficial layer just sufficient to reduce the local internal field to zero and it can be regarded as being a field-dependent penetration depth.

(d) The slit width between the two cylindrical shells is small compared to the actual thickness of the superconductor, so that direct flux penetration does not take place.

Before proceeding further, we should like to comment on the validity of assumptions which are the basis of our calculations. First, the assumption of a constant current density across the thickness of the superficial layers with infinitely sharply defined boundaries on the current density versus distance curve is only approximately correct. There must always be exponentially decreasing regions on the curve. This has been recognized by Bean[4] who has shown that the assumption will remain asymptotically valid provided the field-dependent penetration depth $\Delta$ greatly exceeds the London penetration depth, which is certainly the case in the present application. Second, the restrictive condition on the slit width is not as severe as would appear at first. As we will show in a subsequent section, the field can leak into the shielding tube when the width of the slit is much larger than the London penetration depth.

SOLUTION TO THE BOUNDARY VALUE PROBLEM

There are three regions in which to calculate the potentials, and because the current density is in the z direction, the vector potentials are also in the z direction. Internal to the tube, we expand the potential as:

$$\vec{A}_i = \hat{e}_z \sum_{n=0}^{\infty} r^n \left(C_n \cos n\theta + D_n \sin n\theta\right)$$

$$\frac{\partial \vec{A}_i}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} \left(C_n \cos n\theta + E_n \sin n\theta\right) \qquad (2)$$

This satisfies $\nabla \times \nabla \times \vec{A}_i = 0$ and $A_i$ is regular at $r = 0$.

The external potential is composed of two parts, one due to the external field ($B_o$) and one due to the current distribution:

$$\vec{A}_e = \hat{e}_z \left[ B_o\, r (\sin\theta \cos\delta - \cos\theta \sin\delta) + \sum_{n=0}^{\infty} \frac{1}{r^n} \left(A_n \cos n\theta + B_n \sin n\theta\right) \right]$$

$$\frac{\partial \vec{A}_e}{\partial r} = B_o (\sin\theta \cos\delta - \cos\theta \sin\delta) - \sum_{n=1}^{\infty} \frac{n}{r^{n+1}} \left(A_n \cos n\theta + B_n \sin n\theta\right) \qquad (3)$$

where $\delta$ is the angle between $B_o$ and the x-axis as shown in Fig. 2.
This is regular at $r = \infty$, contains the uniform dipole field, and satisfies $\nabla \times \nabla \times \vec{A} = 0$.

Inside the superconductor, the potential has the form:

$$\vec{A}_s = \hat{e}_z \sum \left(\alpha_n r^n + \beta_n / r^n\right) (\gamma_n \cos n\theta + \sin n\theta) + \hat{e}_z r^2 \sum_{n=0}^{\infty} \left(a_n' \cos n\theta + b_n' \sin n\theta\right) \qquad (4)$$

This must satisfy the field equation $\nabla \times \nabla \times \vec{A}_s = \mu_o \vec{J}$. The terms with $a_n'$ and $b_n'$ represent the solution to the inhomogeneous equation when the current density is defined by:

$$\vec{j} = \hat{e}_z j_o \sum (a_n \cos n\theta + b_n \sin n\theta) \ . \tag{5}$$

This satisfies London's equations and is in accord with the assumptions about the current density in the superconductor.

The boundary conditions on the current density are:

$$\int_{R-\Delta}^{R} \int_{\frac{-\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon} j\, r\, dr\, d\theta = 0 = \int_{R-\Delta}^{R} \int_{\frac{-\pi}{2}+\epsilon+\delta'}^{\frac{\pi}{2}-\epsilon} j\, r\, dr\, d\theta \tag{6}$$

Clearly the limit $\frac{-\pi}{2} + \epsilon + \delta'$ is the limit where the external field $B_o$ is parallel to the surface of the tube.

As we have assumed that the current density is independent of r, the integral over r contains no information. In addition, these boundary conditions are independent of whether the slits are narrow or wide. The $\cos n\theta$ terms are symmetric over the limits of integration and thus all terms of $a_n$ equal zero. This leaves us with the following conditions on the $b_n$'s:

$$\sum_{n=1}^{\infty} \frac{b_n}{n} \left[ \cos \frac{n\pi}{2} \Big( \cos n\epsilon - \cos n(\epsilon + \delta) \Big) + \sin \frac{n\pi}{2} \Big( \sin n\epsilon - \sin n(\epsilon + \delta) \Big) \right] = 0. \tag{7}$$

Consequently, Eq. (4), the potential inside the superconductor becomes:

$$\vec{A}_s = \hat{e}_z \sum_{n=0}^{\infty} \left( \alpha_n r^n + \beta_n / r^n \right) \left( \gamma_n \cos n\theta + \sin n\theta \right) + \hat{e}_z r^2 \sum_{n=0}^{\infty} \frac{b_n}{n^2 - 4} \sin n\theta \ . \tag{8}$$

The condition we seek is zero magnetic field inside the superconducting tube. This implies that the $D_1$ coefficient in $A_i$ is identically zero and the other coefficients will be zero unless there are some irregularities which make the shielding less than optimum.

The boundary conditions on the vector potentials are:

$$\begin{aligned} \vec{A}_i = \vec{A}_s \;;\; \frac{\partial A_i}{\partial r} = \frac{\partial A_s}{\partial r} \quad \text{at} \quad r = R - \Delta \\ \vec{A}_s = \vec{A}_e \;;\; \frac{\partial A_s}{\partial r} = \frac{\partial A_e}{\partial r} \quad \text{at} \quad r = R \end{aligned} \tag{9}$$

To satisfy these conditions, we find that all cosine terms are zero except for:

$$\alpha_1 \gamma_1 = -B_0 \sin\delta \quad , \quad C_1 = -B_o \sin \delta \tag{10}$$

The boundary conditions also dictate the first three coefficients of the sine terms: The n = 0 and n = 2 coefficients are zero in the potentials and the current density. The n = 1 coefficients are:

$$\begin{aligned} b_1 &= -2 B_0 \cos\delta/\Delta \\ \alpha_1 &= -B_o \cos\delta (R-\Delta)/\Delta \\ \beta_1 &= B_o \cos\delta (R-\Delta)^3/3\Delta \\ B_1 &= -B_o \cos\delta \left(R^3 - (R-\Delta)^3\right)/3\Delta \\ D_1 &= 0 \end{aligned} \tag{11}$$

What remains for the higher order terms are five coefficients, four equations, and the condition on the $b_n$'s specified by the boundary condition on the current density in Eq. (6). This leads to an arbitrary situation: since $b_1$, $b_0$, and $b_2$ are already determined, we can obtain a unique solution to this problem by setting $b_3$ according to the condition on the $b_n$'s and letting all $b_n = 0$ for $n > 3$. We find that

$$
\begin{aligned}
b_3 &= \frac{-6 B_o \cos}{5\Delta} \; \frac{\sin \epsilon - \sin(\epsilon + \delta)}{\sin 3\epsilon - \sin 3(\epsilon + \delta)} \\
\beta_3 &= -b_3 (R - \Delta)^5 / 30 \\
B_3 &= \frac{b_3}{30}\left(R^5 - (R - \Delta)^5\right) \qquad (12) \\
\alpha_3 &= \frac{-b_3}{6R} \\
D_3 &= \frac{b_3}{6}\left(\frac{1}{R - \Delta} - \frac{1}{R}\right)
\end{aligned}
$$

Collecting the coefficients in Eq. (10), (11), and (12), we obtain the following solutions for the potentials and an expression for $j_o$:

$$
\begin{aligned}
\vec{A}_i &= -r\hat{e}_z\left[B_o \sin\delta \cos\theta + b_3 \cos\delta \, r^2 \sin 3\theta / R(R-\Delta)\right] \\
\vec{A}_s &= \hat{e}_z\left[-B_o r \sin\delta \cos\theta + \frac{B_o 2\cos\delta \sin\theta}{\Delta}\left(\frac{r^2}{3} - \frac{r(R-\Delta)}{2} + \frac{(R-\Delta)^3}{6r}\right)\right. \\
&\qquad \left. - \frac{2 b_3 \cos\delta \sin 3\theta}{\Delta}\left(r^2 - \frac{5r^3}{6R} - \frac{(R-\Delta)^5}{6r^3}\right)\right] \qquad (13) \\
\vec{A}_e &= \hat{e}_z\left[-B_o r\cos\theta \sin\delta + B_o \sin\theta \, r\cos\delta\left(1 - \frac{R^3 - (R-\Delta)^3}{3r^2\Delta}\right) - \frac{\sin 3\theta \, b_3 \cos\delta}{3r^3\Delta}\left(R^5 - (R-\Delta)^5\right)\right] \\
\vec{j} &= -\hat{e}_z \frac{2B_o \cos\delta}{\Delta\mu_o}\left[\sin\theta + 3\,\frac{\sin\epsilon - \sin(\epsilon+\delta)}{\sin 3\epsilon - \sin 3(\epsilon+\delta)}\,\sin 3\theta\right]
\end{aligned}
$$

The simplest unique solution for the current density is a mixture of dipole and sextupole terms. In Fig. 3, we define the six current loops over the surface of the cylinder which reduce to two current loops when $\delta \leq \epsilon$.

## DOES THE EXTERNAL FIELD PENETRATE THE SLIT?

There exists a component $B_o \sin\delta$ of the external field in a direction along the axis of the slit. This component of the field penetrates the slit and superimposes itself upon the internal field generated by the sextupole current distribution. To determine the intensity of this field, let us consider what happens in a slit narrow compared to its length, with walls constructed from superconducting material. Assume that London's equations (1) are applicable.

In each wall, the local field generates a current distribution:

$$j = -H_o \sqrt{\Lambda}\, e^{-x/\sqrt{\Lambda}} \quad , \tag{14}$$

where $\Lambda$ is a characteristic of the material and x is measured from the edge of the material inwards. Integrating this equation over x, we can estimate the average field intensity in the slit induced by this current distribution:

$$H' = -H_o \Lambda/a \text{ per unit length} . \tag{15}$$

If the width "a" of the slit is considerably larger than the penetration depth of the superconductor, the field generated by the induced currents is negligible. Consequently the external field penetrates the slit. Only when the slit is narrower than the penetration depth of the superconductor is the external field impeded from entering the interior of the tube. These conclusions appear to be more or less independent of the shape of the slit.

Using a conformal transformation which maps the interior of a cylinder of unit radius in the z plane into a trough of width $\pi$ in the w plane, as shown in Fig. 4, we can calculate the shape of the field in the interior of the cylinder. The calculation is not particularly sensitive to the thickness of the superconducting shell.

The transformation of interest is:

$$Z = \frac{i - \tan \epsilon/2 \sin w}{i + \tan \epsilon/2 \sin w} \tag{16}$$

On the positive x-axis, the slit is transformed onto the u-axis. On the negative x-axis, the slit limits the upper extent of v to $\cot^2 \epsilon/2$. In the w-plane, the lines of flux run parallel to the v-axis with a density of $B_o \sin\delta$ between the superconducting walls at $\pm\pi/2$.

The inversion of Eq. (16) gives us a map of the B field and its corresponding potential in terms of x and y. In particular, we have:

$$\frac{4y^2}{\cosh^2 v} + \frac{(1 - x^2 - y^2)^2}{\sinh^2 v} = \tan^2 \epsilon/2 \left((1+x)^2 + y^2\right)^2$$

$$\frac{4y^2}{\sin^2 u} - \frac{(1 - x^2 - y^2)^2}{\cos^2 u} = \tan^2 \epsilon/2 \left((1+x)^2 + y^2\right)^2 \tag{17}$$

Since $\vec{B} = \nabla \times \vec{A} = \nabla \times (B \cdot v)$, the field lines are the loci generated for constant u in the second of the two equations (17).

## COMPARISON WITH EXPERIMENT

If $B_o$ is 1 tesla and the thickness of the superconductor is taken as 0.76 mm, then the maximum current density becomes $2 \times 10^9$ A/m$^2$, which is consistent with our measurements.

The residual internal field can be calculated from the vector potential and we find that

$$\vec{B}_i = B_o \sin\delta \left[ -\hat{e}_x \frac{9\cos\delta}{\sin 3\delta} \left(\frac{x^2 - y^2}{R(R-\Delta)}\right) + \hat{e}_y \left(1 + \frac{18\cos\delta}{\sin 3\delta} \frac{xy}{R(R-\Delta)}\right)\right] \tag{18}$$

We compare this expression with the measured displacements of the beam after it has passed through the "flux-free" region. The results are summarized in Table I:

Table I

| $B_o$ (T) | Displacement (mm) south | Displacement (mm) down | Bend Angles (mrad) south | Bend Angles (mrad) down | $B_{iy}$ (tesla) | $B_{ix}$ (tesla) |
|---|---|---|---|---|---|---|
| 0.5 | 0.52 | 0.45 | 0.097 | 0.084 | 0.0043 | 0.0037 |
| 1.0 | 1.27 | 0.75 | 0.237 | 0.140 | 0.0104 | 0.0064 |
| 1.25 | 1.85 | 0.75 | 0.346 | 0.140 | 0.0152 | 0.0064 |

Ignoring the dipole terms and considering only the horizontal displacement, we find that $\sin\delta$ is approximately 0.01 independent of $B_o$. The overall diameter of the apparatus is 8.9 cm which implies that our accuracy for rotational alignment was approximately 0.09 cm. This is certainly consistent with the techniques used to align the tube.

The field from the slit can be evaluated using the complex functions of Eq. (17). Alternatively, we can estimate the field intensity at $x = 0$ which is everywhere parallel to the x-axis. The field intensity in the slit is $B_o \sin\delta$. Now the equations (17) tell us that the field lines inside the cylinder spread out as they leave the slit. Assuming a uniform density along the y-axis ($x = 0$), the intensity becomes:

$$B_{x=0} = B_o 2 r \epsilon \sin\delta/2r = B_o \epsilon \sin\delta = .001 \text{ T} . \qquad (19)$$

Thus, for the conditions of our experiment, where $B_o = 1$ tesla, the slit width is 0.1 cm and the tube is 2 cm in diameter, the penetration appears inconsequential. We observe, parenthetically, that this result does not depend on the thickness of the superconducting shell.

The sign of $\delta$ can also be determined. The initial coordinate system is given such that a negative particle moving in the positive z-direction bends down, for negative y. This implies that $B_o$ is along the negative x-direction. The beam inside the tube is deflected in the negative x-direction (south), implying that $B_y$ is in the positive y-direction. Thus the sign of $\delta$ is negative.

The beam also bends down, but we cannot conclude much about the value of the x component of $B_i$ because the tube does not quite clear the fringe fields. However, the deflection is consistent with the sense of the dipole components along the y-axis.

Of particular interest is a possible modification to the present design which would further inhibit penetration of the external field. One possibility would be to add an inner shell of superconductor whose slits were rotated by $90^o$ with respect to the outside slits. Clearly this inner shield will exclude the uniform $B_{iy}$ component, but it seems fairly sure that some small percentage of the dipole field will actually penetrate into the interior.

We can also calculate the torque $\vec{\Gamma}$ and pressure on the tube:

$$\vec{\Gamma} = \int \vec{dr} \times \vec{j} \times \vec{B} = \hat{e}_z \int_{R-\Delta}^{R} \int_{-\pi/2}^{\pi/2} r\, j_z\, B_r\, r\, dr\, d\theta \tag{20}$$

This gives the torque per unit length of the tube, and the effect of $\epsilon$ is ignored.

The product $j_z B_r$, where $B_r$ is the field in the superconductor, contains products of $\sin\theta$, $\sin 3\theta$, $\cos\theta$, $\cos 3\theta$. These functions are orthogonal over the limits of integration and the only contribution to the torque comes from the $\sin^2\theta$ component.

$$\vec{\Gamma} = -\hat{e}_z \frac{2 B_o^2 \cos\delta \sin\delta}{\mu_o \Delta} \left(R^3 - (R - \Delta)^3\right) \tag{21}$$

The torque is thus $1.6 \times 10^5$ Nm/m, given the previous values of $\Delta$ and $\sin\delta$.

Similarly we can obtain the radial compression on the superconducting material, and we find that

$$\frac{d\vec{F}_n}{d\theta} = \hat{e}_r \frac{4 B_o^2 \cos^2\delta \sin^2\theta}{\mu_o \Delta^2} \int_{R-\Delta}^{R} \left( \frac{2r}{3} - \frac{R-\Delta}{2} - \frac{(R-\Delta)^3}{6r^2} \right) r\,dr \tag{22}$$

where the sinδ terms have been neglected. When the integration over r is made, we have:

$$\frac{dF_n}{d\theta} = \frac{-4 B_o^2 \sin^2\theta \cos^2\delta \, R\Delta^2}{2\mu_o \Delta^2} = \frac{-2 R B_o^2 \sin^2\theta \cos^2\delta}{\mu_o} \tag{23}$$

and the pressure

$$P = \frac{-2 B_o^2 \sin^2\theta}{\mu_o} \cos^2\delta \tag{24}$$

Integrating over $\theta$, the total force per unit length is:

$$F = \frac{-B_o^2 R\pi}{\mu_o} \cos^2\delta = -2.5 \times 10^4 \text{ N/m} \tag{25}$$

for a 1.0-T field. The negative sign indicates a compressive force.

## CONCLUSION

It appears that this analysis accounts for the major features observed in the behavior of the beam and the superconducting tube. Thus it seems that we are justified in ignoring the detailed physics of the superconductor in a magnetic field, and in using classical electromagnetic theory to evaluate the external fields created by this apparatus. We note that this type of apparatus is subject to considerable forces and as we found in our original attempts to construct it, the equipment must be designed to withstand them.

The principal question, however, is: can we completely eliminate the internal field given the small unavoidable misalignments? The complex nature of the field inside the tube suggests that an inner tube made of two semi-cylinders with the slits displaced by $90^{\circ}$ relative to those in the outer tube would shield most but not all of the field. As the internal fields are of the order of .02 tesla, this suggests that a thin continuous layer of a high $H_{c1}$ superconductor, such as lead plated on the inner stainless steel support tube, could exclude this residual field by the Meissner effect. The two semi-cylinders of $Nb_3Sn$ are then mounted rigidly on this tube.

We wish to thank Dr. M. Rabinowitz for his useful comments and Prof. M. Perl for his continued support of this project.

## References


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2. A. C. Newton, F. Martin, S. J. St. Lorant and W. T. Toner, SLAC-PUB-1102 (1972), submitted to the Review of Scientific Instruments.

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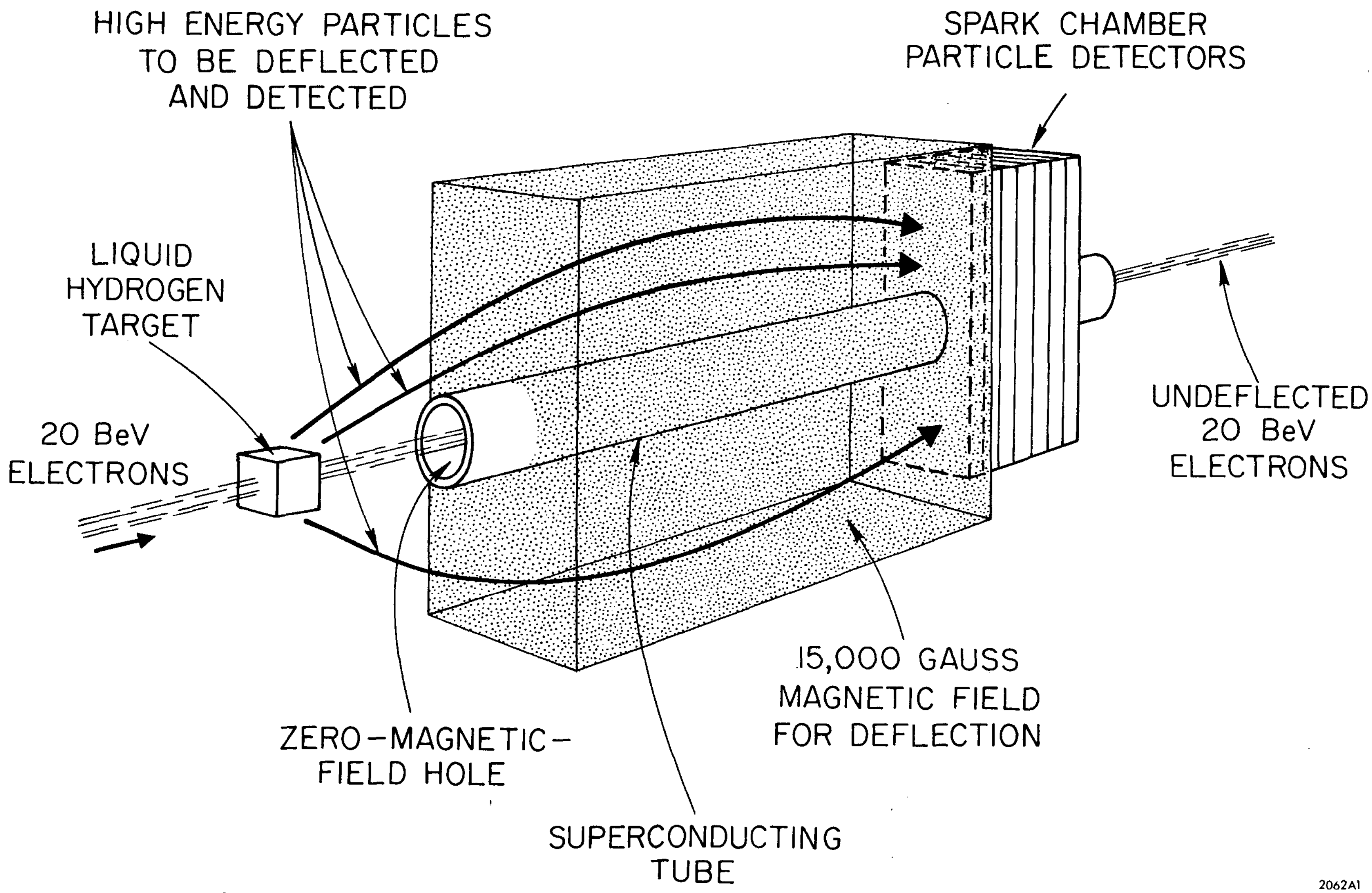
## Figure Captions

Figure 1. Schematic layout of the spectrometer with the superconducting tube.

Figure 2. Cross section of shielding tube showing the relative positions of the slit and external field.

Figure 3. Cross section of shielding tube showing the distribution of the sextupole current loops.

Figure 4. Transformation of the cylindrical geometry to a trough representing the width of the slit in the tube.


HIGH ENERGY PARTICLES
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AND DETECTED
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PARTICLE DETECTORS
LIQUID
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20 BeV
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15,000 GAUSS
MAGNETIC FIELD
FOR DEFLECTION
ZERO-MAGNETIC-
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SUPERCONDUCTING
TUBE
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Fig. 1

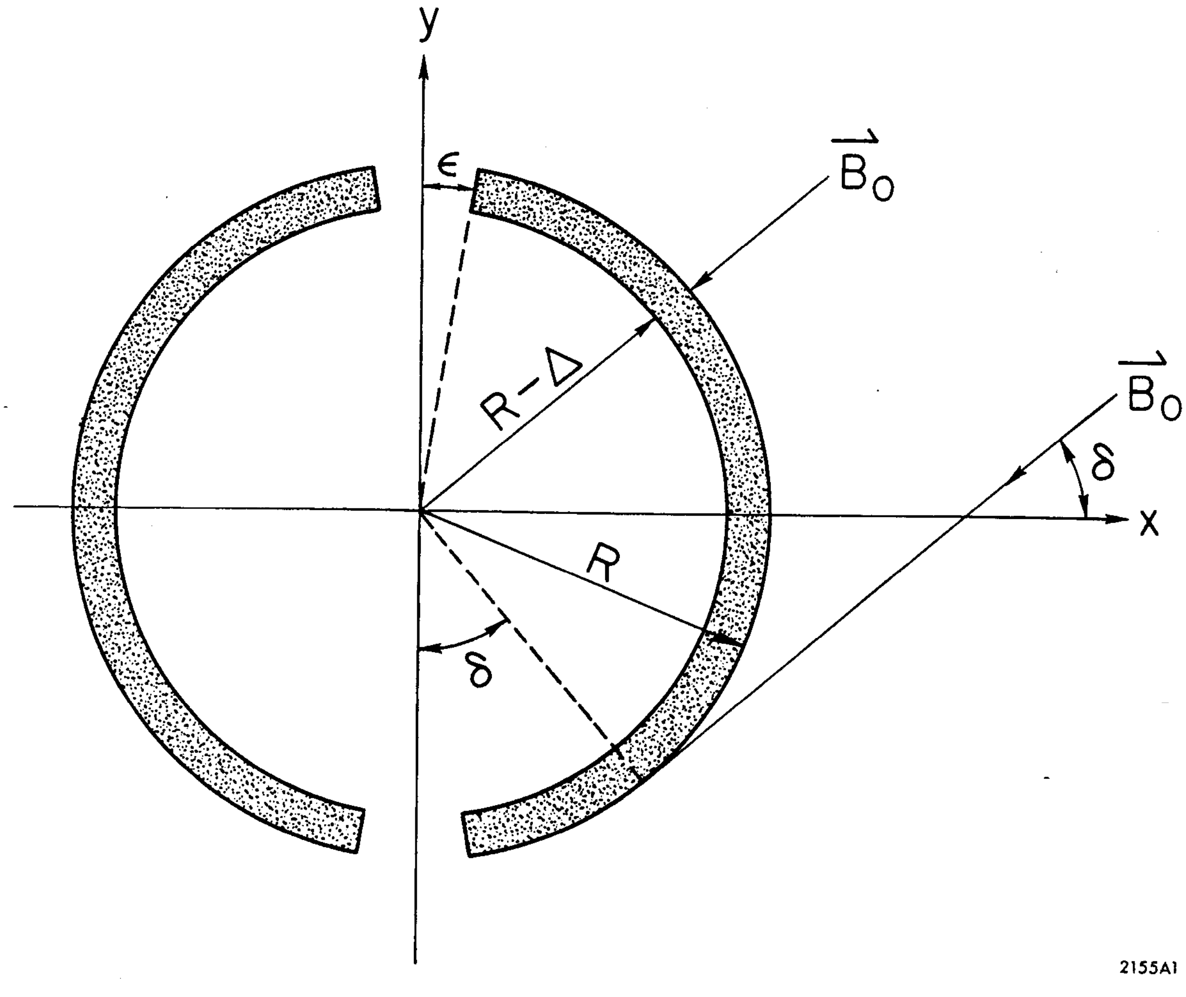


Fig. 2

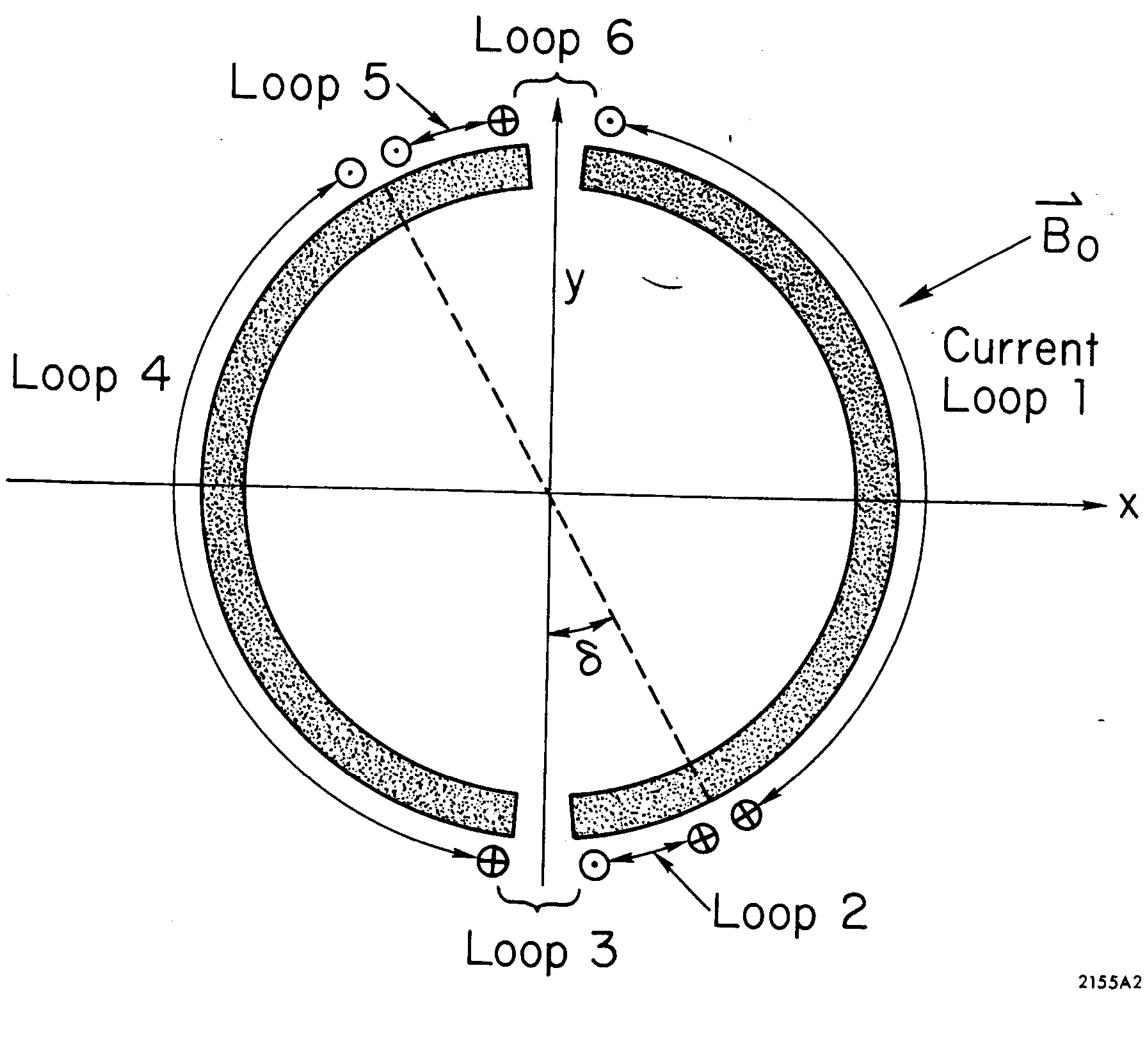
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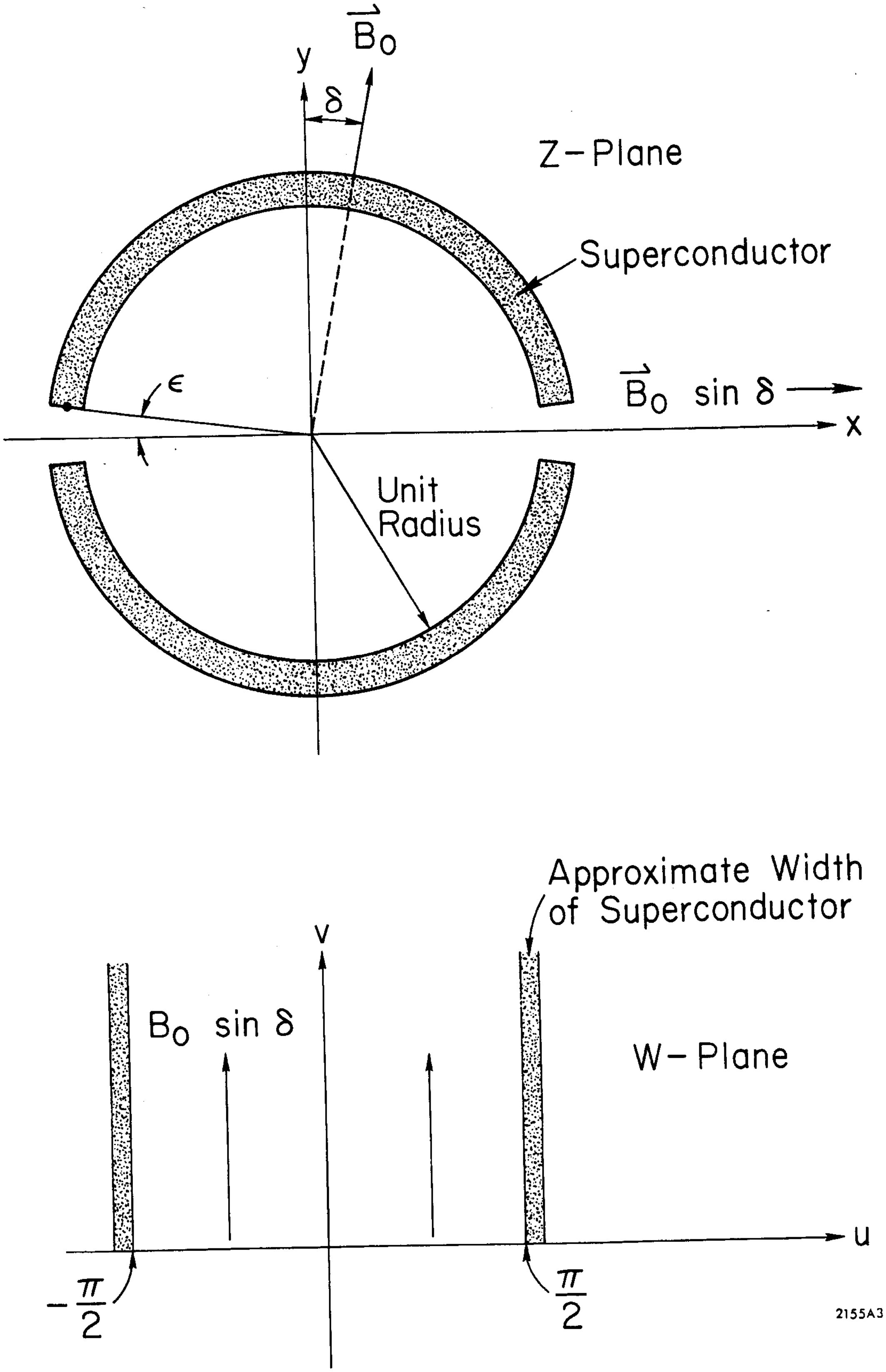
Fig. 3



B₀
y
δ
Z-Plane
Superconductor
ε
B₀ sin δ
x
Unit Radius
Approximate Width of Superconductor
v
B₀ sin δ
W-Plane
u
-π/2
π/2
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Fig. 4