Application of the phase integral method in some inflationary scenarios

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Abstract. We apply the phase-integral approximation devised by Fröman and Fröman in order to compute cosmological perturbations in the power-law inflationary model. The phase-integral formulas for the scalar and tensor power spectra are explicitly obtained up to ninth order of approximation. Our approximate expressions reproduce the shape of the power spectra as well as the spectral indices. We compare the accuracy of the power-integral approximation with the results for the power spectrum obtained with the slow-roll and uniform approximation methods.

1. Introduction

The results reported by WMAP favor inflation [1] over other cosmological scenarios. The data is consistent with a flat universe and with an almost scale invariant spectrum for the primordial perturbations. The spectrum of the perturbations generated during inflation depends on the model, therefore it is important to predict the energy spectrum of the cosmological perturbations for a variety of inflationary models. In general, most of the inflationary scenarios are not exactly solvable so approximate methods are mandatory. Traditionally, the method of choice for inflationary cosmology is the slow-roll approximation [2]. Recently, some authors have applied semiclassical methods, such as the WKB method with the Langer modification [3, 4, 5], and the method of uniform approximation [6, 7, 8]. In the present article we propose an alternative method of approximation for the study of cosmological perturbations during inflation, this method is based on the phase integral method [9, 10, 11] which has been succesfully applied in the study of quasinormal modes in black hole physics [12, 13].

The Friedmann-Robertson-Walker line element for a spatially flat universe can be written as

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

where a is the scale factor. In order to study cosmological perturbations we consider perturbations of the spatially-flat Friedmann-Robertson-Walker universe (1). The perturbative metric depends on new functions which depend on space and time.

The scalar density perturbations are described by the function $u_k = a\Phi/\phi'$, where Φ is a gauge-invariant variable corresponding to the Newtonian potential, and ϕ is the scalar field. The equations of motion for the perturbation u_k in a universe dominated by a scalar field ϕ can be obtained after linearizing the Einstein field equations [14], and are given by

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0, \tag{2}$$

where $z = a\phi'/\mathcal{H}$, $\mathcal{H} = a'/a$, where the prime indicates derivative with respect to the conformal time $\eta = \int dt/a$.

For the tensor perturbations (gravitational waves) we introduce a function $v_k = ah$, where h is the amplitude of the gravitational wave. The tensor perturbation obey a second order differential equation analogous to Eq. (2)

$$v_k'' + \left(k^2 - \frac{a''}{a} \right) v_k = 0. \tag{3}$$

Considering the limits $k^2 \gg |z''/z|$ (short wavelength) and $k^2 \ll |z''/z|$ (long wavelength), we have that the solutions to Eq. (2) and Eq. (3) have the following asymptotic behavior

$$u_k \;\; \rightarrow \;\; \frac{e^{-ik\eta}}{\sqrt{2k}} \quad \left(k^2 \gg |z''/z|, -k\eta \to \infty \right), \tag{4}$$

$$u_k \;\; \rightarrow \;\; A_k z \quad \left(k^2 \ll |z''/z|, -k\eta \to 0 \right), \tag{5}$$

the same asymptotic boundary conditions also hold for tensor perturbations.

Once the mode equations for scalar and tensor perturbations are solved for different momenta k, the power spectra for scalar and tensor modes are given by the expression

$$P_S(k) \;\; = \;\; \lim_{-k\eta \to 0} \frac{k^3}{2\pi^2} \left| \frac{u_k(\eta)}{z(\eta)} \right|^2, \tag{6}$$

$$P_T(k) \;\; = \;\; \lim_{-k\eta \to 0} \frac{k^3}{2\pi^2} \left| \frac{v_k(\eta)}{a(\eta)} \right|^2. \tag{7}$$

The spectral indices are defined as [15]

$$n_S(k) = 1 + \frac{d \ln P_S(k)}{d \ln k}, \; n_T(k) = \frac{d \ln P_T(k)}{d \ln k}. \tag{8}$$

The purpose of the next sections is compute approximate solutions for the scalar and tensor power spectra with the help of the phase-integral approximation method and show that the phase integral methods gives very good estimates for the power spectra when we compare them with those using other methods. The article is structured as follows: In section 2 we give an introductory review of the phase-integral method. In section 3 we apply the phase-integral approximation to the power-law inflationary model. In section 4, we numerically solve the equation governing the scalar and tensor perturbations. In section 5 we compare the results for the power-spectra obtained using the phase-integral approach with those computed with the slow-roll and uniform-approximation methods. Finally we summarize our results in section 6.

2. The phase-integral method
Let us consider the differential equation

$$\frac{d^2 u_k}{dz^2} + R(z) u_k = 0. \tag{9}$$

where $R(z)$ is an analytic function of z. In order to obtain an approximate solution to Eq. (9), we are going to use the phase integral method developed by Fröman [9, 16].

The phase integral approximation, generated using a non specified base solution $Q(z)$, is a linear combination of the phase integral functions [10, 17], which exhibit the following form

$$u_k = q^{-1/2}(z) \exp\left[\pm i\omega(z)dz\right], \tag{10}$$

where

$$\omega(z) = \int^z q(z)dz. \tag{11}$$

Substituting (10) into (9) we obtain that the exact phase integrand $q(z)$ must be a solution of the differential equation

$$q^{-3/2}(z)\frac{d^2}{dz^2}q^{-1/2}(z) + \frac{R(z)}{q^2(z)} - 1 = 0, \tag{12}$$

For any solution $q(z)$ of Eq. (12) the functions u_k in Eq. (10) are linearly independent and the linear combination of the functions u_k represents a local solution. In order to solve the global problem we choose a linear combination of phase integral solutions representing the same solution in different regions of the complex plane. This is known as the Stokes phenomenon [9].

If we have a function $Q(z)$ which is an approximate solution of Eq. (12), the quantity ϵ_0, obtained after substituting $Q(z)$ into Eq. (12)

$$\epsilon_0 = Q^{-3/2}(z)\frac{d^2}{dz^2}Q^{-1/2}(z) - \frac{R(z) - Q^2(z)}{Q^2(z)}, \tag{13}$$

is small compared to unity. We take into account the relative size of ϵ_0 by considering it proportional to λ^2, where λ is a small parameter. This is obtained when $Q(z)$ is proportional to $1/\lambda$ and $R(z) - Q^2(z)$ is independent of λ, i.e. if $R(z)$ is replaced by $Q^2(z)/\lambda^2 + [R(z) - Q^2(z)]$ in Eq. (9). Therefore, instead of using the original equation (9), we deal with the auxiliary differential equation

$$\frac{d^2 u_k}{dz^2} + \left\{\frac{Q^2(z)}{\lambda^2} + \left[R(z) - Q^2(z)\right]\right\}u_k = 0, \tag{14}$$

which reduces to Eq. (9) when $\lambda = 1$.

Inserting the solutions (10) into the auxiliary differential equation (14), we obtain the following equation for $q(z)$

$$q^{1/2}\frac{d^2}{dz^2}q^{-1/2} - q^2 + \frac{Q^2(z)}{\lambda^2} + R(z) - Q^2(z) = 0, \tag{15}$$

which is called the auxiliary q equation. After introducing the new variable $\xi = \int^z Q(z)dz$, we obtain the equation

$$1 - \left(\frac{q\lambda}{Q(z)}\right)^2 + \epsilon_0\lambda^2 + \left(\frac{q\lambda}{Q(z)}\right)^{1/2}\frac{d^2}{d\xi^2}\left(\frac{q\lambda}{Q(z)}\right)^{-1/2}\lambda^2 = 0, \tag{16}$$

where ϵ_0 is given by (13). A formal solution of Eq. (16) is obtained after the identification

$$\frac{q\lambda}{Q} = \sum_{n=0}^{\infty} Y_{2n}\lambda^{2n} \tag{17}$$

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By substituting (17) into (16), we obtain

$$1 - \left(\sum_n Y_{2n} \lambda^{2n} \right)^2 + \epsilon_0 \lambda^2 + \left(\sum_n Y_{2n} \lambda^{2n} \right)^{1/2} \frac{d^2}{d\xi^2} \left(\sum_n Y_{2n} \lambda^{2n} \right)^{-1/2} = 0. \tag{18}$$

Using computer manipulation algebra it is straightforward to obtain the coefficients Y_{2n} [16, 18]. If we truncate the series (17) at $n = N$ with $\lambda = 1$ we obtain

$$q(z) = Q(z) \sum_{n=0}^{N} Y_{2n}, \tag{19}$$

New substituting Eq. (19) into Eq. (11) we have that

$$\omega(z) = \sum_{n=0}^{N} \omega_{2n}(z), \tag{20}$$

where

$$\omega_{2n}(z) = \int^z Y_{2n} Q(z) dz. \tag{21}$$

From (19), (20) and (10) we obtain a phase integral approximation of order $2N + 1$ generated with the help of the base function $Q(z)$. The election of the base function $Q(z)$ depends on the problem in question. In many cases, it is enough to choose $Q^2(z) = R(z)$, and the first-order phase integral approximation reduces to the WKB result. In the first-order approximation it is convenient to choose a root of $Q^2(z)$ as the lower integration limit in expression (21). However, for higher orders, i.e. for $2N + 1 > 1$, this is not possible because the function $q(z)$ is singular at the zeros of $Q^2(z)$. In this case, $N > 0$, it is convenient to express $\omega_{2n}(z)$ as a contour integral over a two-sheet Riemann surface where $q(z)$ be single valued [16]. In order to compute $\omega_{2n}(z)$ we define

$$\omega_{2n}(z) = \frac{1}{2} \int_{\Gamma_t} Y_{2n}(z) Q(z) dz, \tag{22}$$

where t is a zero of $Q^2(z)$ and Γ_t is a contour of integration starting at the point corresponding to z over a Riemann sheet adjacent to the complex plane z, and that encloses the point t in the positive or negative sense and ends at the point z.

We assume that the function $Q^2(z)$ is real over the real axis z. Taking into account this restriction, we shall call turning point the zero of $Q^2(z)$. We want to know the connection formulas at both sides of an isolated turning point z_{ret}, i.e., a turning point which is located far from other turning points. We will adopt the terms "classically permitted region" and "classically forbidden region" in order to denote those regions over the real axis where $Q^2(z) > 0$ and $Q^2(z) < 0$, respectively.

3. Application to power-law inflation
The power-law inflationary model is a very simple model that allows one to solve the horizon and flatness problem. Since power-law model does not have a natural way of terminanting the inflationary epoch, this model is not physically acceptable, nevertheless the advantage lies in the possibility of analytically computing the solutions to the perturbation equations and the corresponding power spectra [19, 20]. The power-law model also allows testing approximations that are necessary in other models that do not exhibit analytic solutions. In the power-law model, the scale factor is given by

$$a(\eta) = l_0 \eta^{\frac{1}{2} - \nu}, \tag{23}$$

where $\nu = \frac{3}{2} + \frac{1}{p-1}$. We have to impose the condition $p > 1$ so that equation (23) satisfies the inflationary condition $\ddot{a} > 0$.

Using the power-law scale factor (23) we find that $z = l_0 M_{Pl} \sqrt{\frac{2}{p}} \eta^{\frac{1}{2} - \nu}$. Since, for this model, the differential equations governing the scalar and tensor perturbations are identical, we make the identification $u_k = v_k = w_k$.

$$\frac{d^2 u_k}{d\eta^2} + \left[k^2 - \frac{\left(\nu^2 - \frac{1}{4} \right)}{\eta^2} \right] u_k = 0, \tag{24}$$

where the function u_k in equation (24) satisfies the boundary conditions (4) and (5).

Equation (24) can be exactly solved. The exact solution, satisfying the boundary conditions (4) and (5) can be expressed in terms of a fractionary Hankel function [15].

$$u_k^{ex}(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[i \left(\nu + \frac{1}{2} \right) \frac{\pi}{2} \right] \sqrt{-\eta} H_\nu^{(1)}(-k\eta). \tag{25}$$

The exact power spectra are

$$P_S^{ex}(k) = \frac{1}{l_0^2 M_{Pl}^2} \mathrm{g}^{\mathrm{ex}}(\nu) k^{3-2\nu}, \quad P_T^{ex}(k) = \frac{1}{l_0^2} h^{ex}(\nu) k^{3-2\nu}, \tag{26}$$

where

$$g^{ex}(\nu) = \left(\frac{1 - 2\nu}{3 - 2\nu} \right) \left[\frac{2^{\nu-2} \Gamma(\nu)}{2\pi \Gamma(\frac{3}{2})} \right]^2, \tag{27}$$

$$h^{ex}(\nu) = \left[\frac{2^{\nu-3/2} \Gamma(\nu)}{2\pi \Gamma(\frac{3}{2})} \right]^2. \tag{28}$$

the corresponding spectral indices are

$$n_S^{ex}(k) = 2 - \frac{2p}{p-1}, \quad n_T^{ex}(k) = 3 - \frac{2p}{p-1}. \tag{29}$$

In order to apply the phase-integral method to Eq.(24), we introduce the variable $z = k\eta$. The function $R(z)$ has the form

$$R(z) = a_0 + \frac{a_{-2}}{z^2}, \tag{30}$$

where $a_0 = 1$ and $a_{-2} = \frac{1}{4} - \nu^2$ are constants. In order to solve Eq. (24) with the help of the phase integral approximation we need to choose the base function $Q(z)$. If we choose the square of the base function as $Q^2(z) = R(z)$ one obtains that the phase integral approximation fails at the origin, which is the place where the boundary condition (5) has to be imposed. We can circumvent this problem if we make the following choice for the square of $Q(z)$ [11]

$$Q^2(z) = R(z) - \frac{1}{4z^2}. \tag{31}$$

The equation governing the modes k for the scalar and tensor is

$$\frac{d^2 u_k}{dz^2} + Q^2(z) u_k = 0, \tag{32}$$

where

$$Q^2(z) = 1 - \frac{\nu^2}{z^2}, \tag{33}$$

therefore, the phase-integral approximation is valid as $-z \to 0$, where the boundary condition (5) should be imposed.

The square of the base function $Q^2(z)$ exhibits two turning points $z_{ret} = \pm\nu$. Since we are interested in the limit $-z \to 0$, we choose to work with the negative turning point. This turning point corresponds to the horizon $k = aH$ $(z = -\nu)$. The solution is defined in two ranges: On the left of the turning point, corresponding to scales lower than the horizon, we have the classically permitted region $Q^2(z) > 0$ and the solution oscillates. On the right of the turning point $-\nu < -z < 0$, corresponding to scales larger than the horizon, we have the classically forbidden region $Q^2(z) < 0$, and the solution grows or decreases exponentially.

After computing the coefficients Y_{2n} up to $N = 4$ we obtain a ninth order approximation for $q(z)$. In this case, the expression for $\omega(z)$ takes the form

$$\omega(z) = w_0(z) + \sum_{n=1}^{4} w_{2n}(z), \tag{34}$$

$$= \int_{-\nu}^{-z} Q(z) dz + \frac{1}{2} \sum_{n=1}^{4} \int_{\Gamma_{-\nu}} Y_{2n} Q(z) dz. \tag{35}$$

Using the connection formulas and the expression for $\omega(z)$, we obtain a ninth order phase integral approximation to the solution of the equation for scalar and tensor perturbations (24). In order to compute the power spectrum we need to evaluate the limit $-k\eta \to 0$ for the growing part of the phase integral solution. In this limit we have

$$u_k^{phi}(\eta) \to \exp\left[i\left(\nu - \frac{1}{2}\right)\frac{\pi}{2}\right] f_\nu^{pi} \frac{1}{\sqrt{k}} (-k\eta)^{\frac{1}{2}-\nu}, \tag{36}$$

where

$$f_\nu^{phi} = (2\nu)^{\nu - \frac{1}{2}} \exp\left(-\nu + \frac{1}{12\nu} - \frac{1}{360\nu^3} + \frac{1}{1260\nu^5} - \frac{1}{1680\nu^7}\right). \tag{37}$$

Using Eq. (36), we have that the scalar and tensor power spectra, given by equations (6) and (7) are

$$P_S^{phi}(k) = \frac{1}{l_0^2 M_{Pl}^2} g_\nu^{pi} k^{3-2\nu}, \quad P_T^{phi}(k) = \frac{1}{l_0^2} h_\nu^{pi} k^{3-2\nu}, \tag{38}$$

where

$$g_\nu^{phi} = \left(\frac{1-2\nu}{3-2\nu}\right)\left[\frac{f_\nu^{pi}}{2\pi}\right]^2, \quad h_\nu^{phi} = \left[\frac{2^{1/2} f_\nu^{pi}}{2\pi}\right]^2. \tag{39}$$

The index ν in f_ν^{pi} indicates the order of the approximation. If we only keep the first term, $-\nu$, in the exponential f_ν^{phi} (37), we obtain the first-order phase integral approximation which coincides with the WKB method after using the Langer modification [3, 4, 5]. If we keep the two first terms in the exponential (37), we obtain the third-order phase integral approximation.

It is worth mentioning that, for the power law model, the tensor and scalar spectral indices do not depend on the order of approximation. The spectral indices calculated with the help of Eq. (38) and Eq. (39) coincide with the exact ones.

4. Numerical implementation

In this section we carry out the numerical integration of Eq. (24) for the scalar and tensor perturbations. In order to integrate the equation (24) we apply the Adams predictor-corrector method of order 12. [21, 22]. Since Eq. (24) is a second order ordinary differential equation with real coefficients, we can take the real and imaginary parts of u_k as two linearly independent solutions.

We derive the asymptotic initial conditions for $u_k(\eta_i)$ and $u'_k(\eta_i)$ using two different approaches. The first method consists in obtaining from Eq. (4) the initial condition for starting the numerical integration: $\cos(-k\eta)/\sqrt{2k}$ for the real part of u_k and $\sin(-k\eta)/\sqrt{2k}$ for the imaginary part of u_k. We assume that, as $-k\eta \to \infty$, the solution oscillates. We carry out the integration in two steps, first, for a value of η_i calculated at 350 oscillations before the turning point, we integrate Eq. (24) assuming that $k^2 >> z''/z$. We solve the equation

$$\frac{d^2 u_k}{d\eta^2} + k^2 u_k = 0, \tag{40}$$

and let evolve the solution up to a point η_{eval} calculated at 250 oscillations before reaching the turning point. The value of the solution at this point is taken as initial condition of the exact differential equation for the perturbations (24). The number of oscillations is obtained from $n/2$, where n is the number of zeros which can be approximately calculated from $n = k/\pi(\eta_{ret} - \eta)$, with $\eta_{ret} = -\sqrt{\nu^2 - \frac{1}{4}}/k$. This procedure is called IC's exp. The second procedure consists in obtaining the initial conditions for the real and imaginary parts of u_k y u'_k from the ninth-order phase integral approximation solution w_k. We start the integration at a value η_i calculated 25 oscillations before reaching the turning point. We call this procedure IC's phi9.

Fig. 1 and Fig. 2 compare the real part of the analytic solution of u_k with the ninth order phase integral approximation and the numerical methods IC' exp and IC's phi, respectively. The graphics were made using the number of e-folds $N = \log(a(\eta))/a(\eta)$ as the independent variable. As expected, the phase integral solutions diverge at the root of $q(\eta)$. Using IC's phi9, the relative error is smaller than a part in 10^5, an error which is smaller than that obtained using the numerical method IC's exp. Therefore, we are going to use the phase integral initial condition for the numerical computation of $P_S(k)$ and $P_T(k)$. We stop the computation of $P_S(k)$ and $P_T(k)$ when the quotient u_k/z (scalar perturbations) or u_k/a (tensor perturbations) becomes constant, i.e., when the function u_k leaves the horizon.
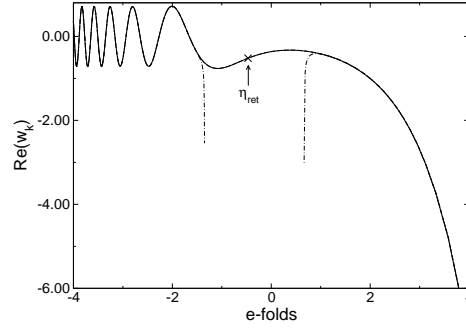
5. Results

In this section we proceed to compare the power spectra calculated using the phase integral approximation with the results obtained with the slow-roll and uniform approximation methods. From Ref. [8] (equations (63) and (64)) we obtain that the scalar and tensor power spectra in the slow-roll approximation are
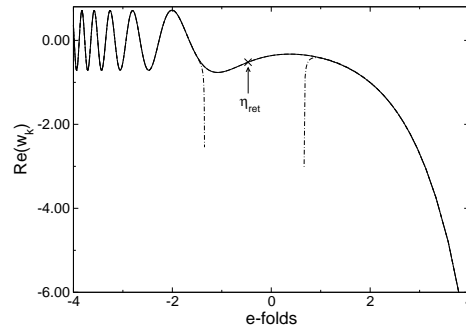
$$P_S^{sr}(k) = \frac{1}{l_0^2 M_{Pl}^2} g_\nu^{sr} k^{3-2\nu}, \ P_T^{sr}(k) = \frac{1}{l_0^2} h_\nu^{sr} k^{3-2\nu}, \tag{41}$$

with

$$g_\nu^{sr} = [1 + 2(2 - \ln 2 + b)(2\epsilon + \delta) - 2\epsilon] \left(\frac{1 - 2\nu}{3 - 2\nu}\right) \left[\frac{2^{-\nu} |1 - 2\nu|^{\nu - 1/2}}{2\pi}\right]^2, \tag{42}$$

Figure 1. $\Re(u_k)$ for the power-law inflationary model with $p = 10$ and $k = 1.389h\mathrm{Mpc}^{-1}$. The solid line indicates the analytic solution, the dashed line indicates the numerical result (IC's exp); the dot-dashed line indicates the ninth-order phase integral approximation.

Figure 2. $\Re(u_k)$ for the power-law inflationary model with $p = 10$ with $k = 1.389h\mathrm{Mpc}^{-1}$. The solid line indicates the analytic solution. The dashed line indicates the numerical result (IC's phi9). The dot-dashed line indicates the ninth-order phase integral approximation

$$h_\nu^{sr} = [1 - 2(\ln 2 + b - 1)\epsilon] \left[\frac{2^{1/2-\nu} |1 - 2\nu|^{\nu-1/2}}{2\pi} \right]^2 , \qquad (43)$$

where b is the Euler constant, $2 - \ln 2 - b \simeq 0.7296$, $\ln 2 + b - 1 \simeq 0.2704$, and $\epsilon = -\delta = \frac{1}{p}$. In the slow-roll approximation we have $\epsilon \ll 1$, therefore, for the power-law model, the slow-roll approximation is better suited for large values of the parameter p.

Using the result obtained in Ref. [7] (Eq. (109)), we obtain an expression for the second order uniform approximation for the power spectrum associated with the scalar and tensor perturbations. They are:

$$P_S^{ua}(k) = \frac{1}{l_0^2 M_{Pl}^2} g_\nu^{ua} k^{3-2\nu}, \ \ P_T^{ua}(k) = \frac{1}{l_0^2} h_\nu^{ua} k^{3-2\nu}, \qquad (44)$$
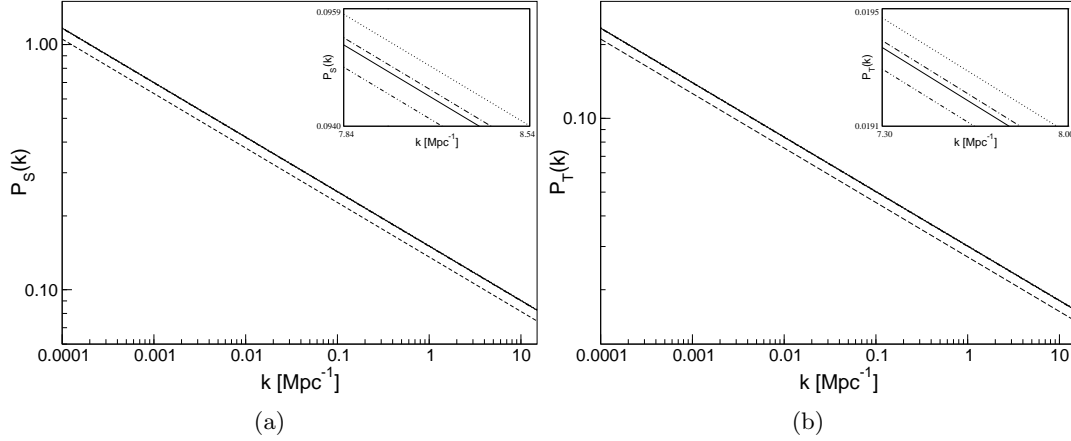
with

$$g_\nu^{ua} = \left(1 + \frac{1}{6\nu}\right) \left(\frac{1 - 2\nu}{3 - 2\nu}\right) \left[\frac{(2\nu)^{\nu-1/2} e^{-\nu}}{2\pi} \right]^2 , \ h_\nu^{ua} = \left(1 + \frac{1}{6\nu}\right) \left[\frac{2^{1/2}(2\nu)^{\nu-1/2} e^{-\nu}}{2\pi} \right]^2 \qquad (45)$$

where the index ν in g_ν^{ua} and h_ν^{ua} indicates the order of approximation of the method. Omitting the factor $1/6\nu$ in (45), we obtain the first-order uniform approximation, result that coincides

with the first-order phase-integral approximation and the WKB method with the Langer modification [3]. Keeping the second term of the expressions in (45) one gets the second-order uniform approximation.

Figure 3. (a) Scalar power spectrum $P_S(k)$ and (b) tensor power spectrum $P_T(k)$ for the power-law inflationary model with $p = 10$. Solid line: analytic solution; dotted line: slow-roll approximation; dashed line: first-order phase integral, WKB and first order uniform approximation; dot-dashed line: third-order phase integral approximation; two-dots dashed line: second-order uniform approximation.
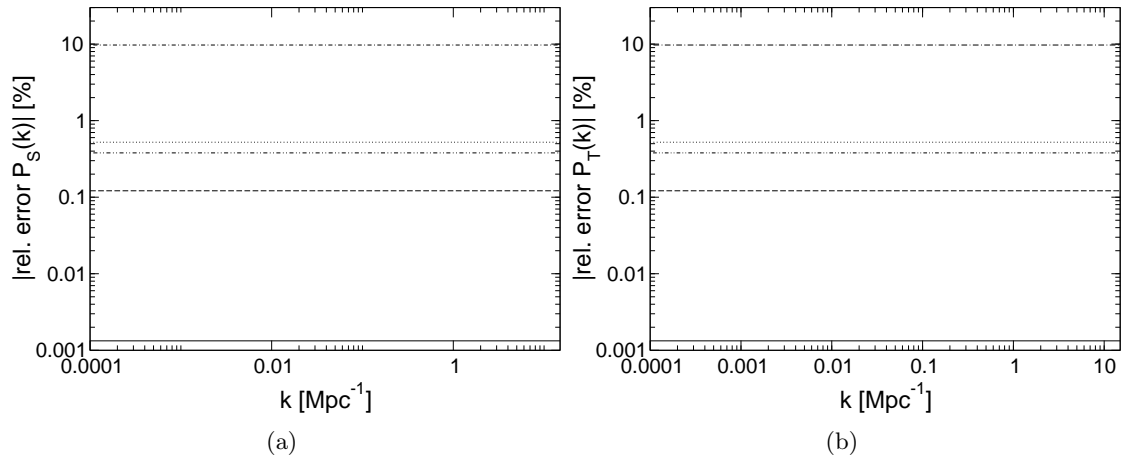
Figure 4. (a) Relative error for the scalar $P_S(k)$ and (b) tensor power spectra $P_T(k)$ calculated using different methods. The solid line indicates the ninth-order phase integral approximation. The dashed line indicates the third-order phase integral approximation. The two-dots dashed line indicates the second order uniform approximation. The dotted line indicates the slow-roll approximation. The dot-dashed line indicates the WKB and first-order uniform approximations.

We want to compare the analytic expression for the scalar and tensor power spectra for different values of k with the numerical result (IC's phi9), the ninth-order phase integral approximation, the slow-roll approximation and the first and second order uniform approximation. Fig. 3 shows the power spectra P_S y P_T calculated analytically and the

approximate spectra calculated using the the slow-roll, uniform approximation and phase integral methods. Fig. 4 shows the relative error between the analytic solution and the different approximation methods. It can be observed that the ninth-order phase integral approximation gives the best approximation among the different methods.

Acknowledgments

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