Simplicial Matter in Discrete and Quantum Spacetimes

by

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This dissertation was prepared under the direction of the candidate's dissertation advisor, Dr. Warner Allen Miller, Department of Physics, and has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Vita

Jonathan Ryan McDonald was born to James and Barbara McDonald in Honululu, Hawaii on 15 February 1983. He graduated from the Center for Air and Space Studies program at Booker T. Washington Magnet High School in May 2001. Jonathan attended The George Washington University as a Presidential Science Scholar between 2001 and 2005. During this time he was awarded the George Gamow Research Fellowship for his work with Dr. Mark Reeves. In May 2005, he graduated *cum laude* with a Bachelor of Science in Physics and a Bachelor of Arts in Mathematics. Jonathan has been a graduate student in Florida Atlantic University's Department of Physics since July of 2005. While a member of the Department of Physics, Jonathan has received the Nathan W. Dean Award and a National Science Foundation Fellowship through the Department of Chemistry and Biochemistry.

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Abstract

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A discrete formalism for General Relativity was introduced in 1961 by Tulio Regge in the form of a piecewise-linear manifold as an approximation to (pseudo-)Riemannian manifolds. This formalism, known as Regge Calculus, has primarily been used to study vacuum spacetimes as both an approximation for classical General Relativity and as a framework for quantum gravity. However, there has been no consistent effort to include arbitrary non-gravitational sources into Regge Calculus or examine the structural details of how this is done. This manuscript explores the underlying framework of Regge Calculus in an effort elucidate the structural properties of the lattice geometry most useful for incorporating particles and fields. Correspondingly, we first derive the contracted Bianchi identity as a guide towards understanding how particles and fields can be coupled to the lattice so as to automatically ensure conservation of source. In doing so, we derive a Kirchhoff-like conservation principle that identifies the flow of energy and momentum as a flux through the circumcentric dual boundaries. This circuit construction arises naturally from the topological structure suggested by the contracted Bianchi identity. Using the results of the contracted Bianchi identity we explore the generic properties of the local topology in Regge Calculus for arbitrary triangulations and suggest a first-principles definition that is consistent with the inclusion of source. This prescription for extending vacuum Regge Calculus is sufficiently general to be applicable to other approaches to discrete quantum gravity. We discuss how these findings bear on a quantized theory of gravity in which the coupling to source provides a physical interpretation for the approximate invariance principles of the discrete theory.

To my sister, Kelley.

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Chapter 1
Introduction

When mathematical propositions refer to reality they are not certain; when they are certain they do not refer to reality.
–Albert Einstein

One of the core problems in theoretical physics today is on how one unites the two shifts in physical thought of the 20th century, General Relativity (GR) and quantum mechanics (QM), into one consistent theory of quantum gravity. Early attempts to do this consistently found numerous obstacles in defining such a theory beyond a formal context. While formal definitions are possible practical and physical considerations of finiteness and uniqueness often doomed early attempts. It appeared from these early attempts that the fundamental axioms of GR were inconsistent with the interpretations of QM. For this reason two vastly different approaches were followed, background independent and perturbative quantum gravity. The first approach attempts to retain the core principles of GR as one pushes gravitation through a given quantization procedure. These approaches attempt to maintain a dynamical gravitational field which defines the background for matter fields. However, even the approaches that can be categorized as background independent can vary greatly in how they implement quantization and background independence. Despite the variations, the main attraction to these models is their reliance on 4-dimensional spacetime and their retention of spacetime as a dynamical field. The other category primarily refers to formulations of string theory, superstring theory and M-theory in which quantum gravity is predominately viewed as a theory of the graviton.

Perturbative approaches, such as superstring theory, use quantum field theory as the starting point for quantum gravity with point particles replaced by strings. However, these models are most often formulated with respect to some fixed background manifold. They implement the algebraic symmetries of the fields from the Standard Model and GR and recover theories with a increased parameter space. While background independence is broken, there is an attractiveness to these models since they form a mathematically consistent grand unified model of the forces in nature. Despite the beauty of these models, it has been difficult to extract out phenomenological evidence since the typical energy scale for Planck scale physics is currently beyond the technological advances of currently running particle accelerators.[1]

In the background independent models, the vast majority of the lines of research are aimed at understanding the geometric content of the vacuum quantum theory. The focus of the background independent models on geometry preserves the dynamical role of spacetime but there is less emphasis on the coupling of non-gravitational sources to spacetime. While it is vital to first understand the dynamics of quantum spacetimes, it is useful to also useful to understand how stress-energy couples to discrete quantum spacetimes. If geometry is indeed intrinsically linked to matter, then they must be understood together at a fundamental level. In the words of Wheeler, "Matter tells geometry how to curve while geometry tells matter how to move." This classical understanding should be preserved at the very least; however, one should expect that there is a more intricate and fundamental relationship between matter and geometry at the quantum level. Later we will outline some of the arguments why spacetime geometry cannot be altogether separated from the dynamics of the source particles but rather defined by them. It was argued by Synge that to extract observable content from GR, one naturally utilizes the paths of material sources to measure the spacetime geometry.[2] Arguments similar to those presented below form the basis for why a search for quantum gravity should inherently contain the inclusion of non-gravitational sources in addition to the geometric content of the gravitational field.

In particular, this thesis will be concerned primarily with background independent theories such as loop quantum gravity, spin foams, Regge Calculus, and causal dynamical triangulations. Although there is no completely successful quantization scheme, we find that there are numerous complementary approaches. Many of these come to similar conclusions regarding the underlying structure of spacetime as a discrete dynamical system. This underlying discrete simplicial lattice will thus be the focus of the majority of this thesis. Prior to introducing the framework for discrete spacetime, we will briefly discuss some of these various programs for quantizating background independent quantum gravity in §2.2.

Prior to quantizing gravity we must understand the gravitational field that we intend to quantize. Moreover, it is important to ask whether this field exists unto itself or relies on material fields to define its existence. Let us briefly examine this by exploring the meaning of a physical observable in spacetime geometry. In broaching the subject of relativity, one might introduce the concepts of relative and proper lengths or time by describing measurements using rods and clocks.[3] These are used to define the components of the metric which gives full definition to the geometry of spacetime. However, these mathematical constructs, e.g. the world function introduced by Synge,[2] must be used carefully examined. Only then can one understand what it means to measure lengths

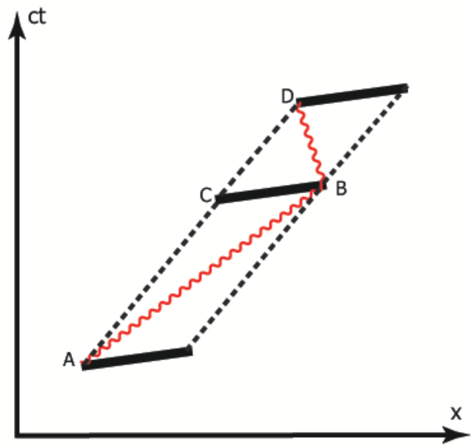

Figure 1.1: *Chronometric Spatial Measurements:* The measurement of lengths is shown to be an inferred observable in the view of the chronometric interpretation of General Relativity.[2] An observer following one end of a "rigid" rod is represented by the world-line $\overline{AD}$. The rod's length, $\overline{CB}$, can then be determined using the triangle ΔABD. This triangle is formed by sending a light pulse from A to B and back to the initial observer at D. The length of $\overline{CB}$ can then be determined using a clock carried by the observer on $\overline{AD}$ using the relation $|\overline{CB}|^2 = -\frac{1}{4}|\overline{AD}|^2$. This "chronometric measurement" also carries over to the measurement of the spatial separation of distinct events at C and B. Again, the spatial separation is obtained by a single temporal measurement, $\overline{AB}$ that can be made with the clock carried by the observer with world-line $\overline{AD}$.

with rods. In order to measure a length with a rod in a fundamental theory, we necessarily take into consideration the atoms that make up the the rod and define relations between such atoms. Each of these atoms is a self-defined time-like world-line and measuring distances with rods becomes the measurement of relative distances between time-like curves which are "rigidly" connected. But how do we determine the relative distances between the time-like curves in a sufficiently flat region of spacetime? One natural measurement is to use pulses of light from locations on the rod. One then measures these pulses of light as they are received by an independent observer. This situation is represented in Figure 1.1 where the measuring rod $\overline{CB}$ starts at A and one end of the rod follows the world line $\overline{AD}$. The independent observer sends off a photon at A and receives it at D. The length of the rod $\overline{CB}$ is then determined using the proper time of a clock carried along the world-line $\overline{AD}$. More simply put, the length of $\overline{CB}$ is determined by the the proper time as measured by the clock between the release and reception of the test photon. Indeed, this is analogous to the methodology of the global positioning system and it gives us initial indications that spatial measurements are derived measurements from the properties of clocks and photons.

This can be taken one step further if we look closely at the use of clocks and photons as measuring devises. Of the two pieces of this measurement scheme, one plays a particularly important role–the standard clock. The role of the light ray in these measurements is to help distinguish events and provide a means to determining spatial distances. Light is particularly useful in this regard since it is postulated in special and general relativity that the speed of light in a vacuum is a constant with regard to any inertial frame of reference. Clocks are of importance because they provide the formal measurement that give meaning to lengths. The ticks of the clock define the length of the time-like curve separating the transmission and reception of the light pulse. Therefore, it is crucial to understand the nature of the clock and it's relation to length of its time-like curve. There are many ways one may construct a standard clock; a rudimentary clock can be defined by simply using a free particle. Given a free particle one may prepare it in some initial state $|\psi(0)\rangle$ such that the particle's evolution in its rest frame with proper time parameterization, τ, is given by

$$|\psi(\tau)\rangle = e^{-\frac{i}{\hbar}\hat{H}\tau}\,|\psi(0)\rangle = e^{-\frac{i}{\hbar}mc^2\tau}\,|\psi(0)\rangle\,. \tag{1.1}$$

where H is the Hamiltonian for the particle. It's clear from this that the time dependence appears only as a global phase in the free-particle's evolution. The time evolution is therefore encoded in the overall accumulated phase of the quantum particle. This suggests that the local measurement of time amounts to a measurement of the local phase change of the quantum state of the particle. However we have defined the time parameterization as a global phase of the particle and global phases are not physically observable and therefore not wholly relevant to the measurement of time. This might appear to be a problem, yet this can be overcome by incorporating additional clocks such that we are able to measure relative phases. This allows a consistent and measurable picture of the parameterization of *each* clock's word-line. It is in this way that it appears that even time is not a directly observable physical quantity, but rather a quantity inferred from the dynamics of some underlying quantum system.

If we are to assume that the physical observables in quantum gravity are directly linked to the dynamics of particles and fields, then what does this imply about the fundamental structure of spacetime? For one, it tells us that geometry alone cannot complete the story of gravitation. Rather, there is no interpretation of local geometry without something to define and measure the geometry. This is the point of view taken in [4;5] where quantum dynamics serves as the dynamical foundation of spacetime geometry. However, this view of spacetime and measurement should not be focused only at alternative approaches to quantum gravity but should be applied to all established models. One cannot have spacetime geometry without matter just as there can be no physical description of the dynamics of matter without the inclusion of spacetime geometry. This creates a pseudo-Quantum Steering Principle for quantum gravity: *local spacetime geometry owes its existence to the flow of stress-energy, in turn, the derived local geometry determines the relative flux of stress-energy.* While this statement is not in particular true for all current models of quantum gravity, it is the author's belief that such a principle should be a partial goal of quantum gravity.

A clear separation between this Machian view of gravity and other approaches to quantum gravity we feel is self-evident. Much of the current effort in background independent quantum gravity is to study the geometric content of each model. While this is a necessary effort, it is undeniably insufficient in a complete fundamental theory. The picture outlined so far seems to indicate that a fundamental theory of gravitation should utilize and expand upon the intimate relationship between stress-energy and spacetime geometry. There is currently no consistent or coherent description for including arbitrary fields into discrete spacetimes. This incomplete picture persists despite discrete spacetimes providing numerous descriptions of dynamical, kinematical, and semiclassical models of quantum gravity. This manuscript aims to shine light on the mathematical and structural questions involved in coupling non-gravitational sources to lattice spacetimes. While continuous GR presents few problems to the coupling of a dynamical background to matter fields, defining fields on a lattice introduces unique obstacles. In the classical continuum one is able to clearly define the a continuous topological structure which serves as well-defined background to continuous fields. However, in the lattice one has to find a unique representation for stress-energy on either the primary simplicial lattice or its dual such that the geometric quantities coincide with the topology for non-gravitational quantities. Moreover, while there are many well-understood theorems that help us understand how to incorporate spin structure, spin-connections, and gauge fields into a continuous spacetime, it is not currently understood how a spin structure can be incorporated directly into a simplicial geometry.

We will briefly review the conceptual foundations of classical and quantum gravity, along with some of the current proposals for quantized gravity, in Chapter 2. This will include a discussion on the foundations of continuum and discrete formulations of spacetime as a precursor to understanding the many models of quantized spacetime. Following this overview, we will examine the issues related to the inclusion of sources in discrete formulations of classical and quantum gravity. We first examine the contracted Bianchi identity as the path towards automatic conservation of source. This identifies a necessary symmetry of the geometry to ensure the automatic conservation of source. This will be examined thoroughly in Chapter 3 with an eye towards identifying how source is conserved in the lattice. We then examine natural spacetime topologies as implied by the contracted Bianchi identity. Once a topology is defined, we search for a natural spinor structure on the lattice. This will be examined in Chapter 4 as we provide an outline and examples of how to couple arbitrary fields to the Regge lattice.

Chapter 2

Classical and Quantum Spacetimes

I weigh all that is.
Nowhere in the universe
Is there anything over which
I do not have dominion.
As a spacetime,
As curved all-pervading spacetime,
I reach everywhere.
My name is gravity.[6]

–John Archibald Wheeler

We have argued that spacetime is defined as a set of physically observable relationships whose existence are derived from the dynamics of the particles and fields in the universe. The poem leading into this chapter demonstrates a view of the gravitational field or the spacetime geometry as an independent field with its own dynamics. Despite the difference in philosophical or quantum points of view, the classical theory of spacetime should be recovered in the macroscopic (large-scale or coarse-granined) limit. Since any theory of quantum gravity should retain Einstein's general theory as the classical limit, we can start by understanding the vital aspects of the classical theory of the gravitational field. In this chapter we will examine the basic properties of the gravitational field and how these properties have been incorporated into discrete and continuum formulations of GR and quantum gravity. This will provide a necessary foundation for understanding how the gravitational interaction is quantized and why there are many distinct, yet complementary, approaches to quantum gravity. We first turn to the principles and foundations of gravity described as the geometry of spacetime.

2.1 Foundations of Classical and Quantum Gravity

The modern treatment of gravitation is embodied by the theory of relativity–both the special and general theories. While GR governs the global dynamics of matter in spacetime, special relativity describes physics locally at an observer. This separation is the heart of our current understanding of spacetime and is manifest in the guiding principles of GR. We start by defining the foundations of relativity theory in four principles before defining how each applies to either the global or local dynamics of nature.

P1. (Special) Principle of Relativity.

All (inertial) observers are equal under the laws of physics.

P2. Constancy of the Speed of Light.

All inertial observers measure the speed of light to be a constant value c.

These two principles define the structure of special relativity. Together these principles determine how inertial observers in special relativity observe physics in a unified space and time. Moreover, they imply that the geometry of this spacetime is that of Minkowski geometry. When we remove the *special* part of the Special Principle of Relativity, we must make another assumption in order to obtain the General Theory. We now refine our application of the Special Theory from general frames for inertial observers to local frames for general observers. We state this more clearly in the Principle of Equivalence;

P3. Principle of Equivalence.

The local form of all the laws of physics in a sufficiently small neighborhood of spacetime obey the laws of special relativity.

This is the most striking of the four principles as one makes the transition from continuum to discrete spacetimes. It tells us that there exist *local* neighborhoods of spacetime such that *all* the laws of physics obey the laws of special relativity. Assuming the Principle of Equivalence to be true, nature prescribes everywhere a sufficiently small neighborhood to which we can apply the geometry and principles of special relativity. We cannot emphasize this enough since it is one of the primary propositions when we make the transition to discrete spacetimes. In the continuum theory one interprets this principle in the sense of Leibniz's calculus: a sufficiently, maybe infinitesimal, neighborhood of a point in spacetime can always be chosen such that the geometry of the neighborhood is that of Minkowski spacetime. In the discrete theory, one presupposes that there are finite-sized neighborhoods in spacetime whose geometry is that of Minkowski space. In the large-scale, these small discrete, finite-sized neighborhoods are not typically noticeable or mathematically significant since the typical length scale is that of quantum gravity–the Planck length $l_P \approx 10^{-33}$cm.

To prescribe how we must model the physics that result from the previous principles, we turn to the Principle of General Covariance. The essential idea is that the physics must be independent of the system of coordinates that we use to describe the dynamics. Mathematically this implies that observable quantities in GR must be represented by tensorial fields which enforce the invariance of physical laws under local Lorentz transformations.

P4. Principle of General Covariance.
The laws of physics should be independent of which coordinate system one chooses.

These principles will serve as the guiding inspiration for how we search for a quantum theory of gravity. While the Principle of Equivalence tells us that spacetime is locally Minkowski, we do not make any assumptions regarding the large-scale geometry of the spacetime. As these locally Minkowski neighborhoods are strung together they build a curved or warped geometry. The motion of the particles on this warped geometry defines their motion through spacetime. Particles which are locally in inertial motion will then follow geodesics (or paths of extremal distance) on the large-scale curved geometry. In the discrete spacetime, we say that the particles follow straight lines within a single local Minkowski neighborhood and curved motion appears in trajectories that traverse at least one conic singularity of the lattice. We will see more of this in the next chapter as we discuss the geometric foundations of Regge Calculus.

For the remainder of this section we will outline the conceptual foundations that lie at the core of classical and quantum gravity. Most pivotal in the development of Einstein's relativity was the issue of background independence. However, we will first examine Mach's Principle (and Wheeler's Steering Principle) as a goal and source of inspiration for GR. This will lead us directly into a more detailed discussion on the meaning of background independence and its realization in GR. This will take us into a discussion on causality. We will close the section with a discussion on the roles of symmetries in classical and quantum gravity.

2.1.1 Mach's Principle

One of the great questions in physics since Newton formulated his laws of motion has been whether physical theories should explicitly rely on absolute space and time or on relations between material particles. This question was brought to the forefront with the work of Ernst Mach (1838-1916) and his views on absolute space have lead to what is now known as Mach's Principle. There is still much debate over the exact statement and interpretation of Mach's arguments; however, there are some common themes inherent in each point of view. The plethora of debate often forces one to preface any discussion of Mach's Principe with a description of how one chooses to interpret the argument. This prompted John Arhchibald Wheeler to coin the phrase *the Steering Principle*[7] as a physically motivated formulation of Mach's Principle as it is realized in GR. We will try to separate out the various interpretations involved in understanding Mach's Principle and outline what we believe it to mean for fundamental physical theories.[8;9]

Mach's Principles has grown out of the argument made by Mach in response to Newton's "Bucket Experiment".[10] Newton's setup consisted of a bucket hanging by a rope and filled with water. The argument stated that if the bucket is set in motion by twisting the rope and letting the system rotate, then the water will remain still until such a time when the walls of the bucket are able to transfer their motion to the water. At that time the water would begin climbing the walls of the bucket and the surface of the water would begin to curve. The question raised by this thought experiment is that when both the bucket and the water are spinning, when the bucket has completely transferred motion to the water, then why does the water's surface curve? Moreover, when bucket begins to slow, why does the surface of the water remain concave? At this time, it cannot be said that the water's surface is curved due to centrifugal forces on the water from the bucket. Instead, Newton argued that the water is in rotation with respect to the absolute space thus giving rise to the concavity of the water's surface. However, Mach railed against Newton's conclusion. He argued that all one can truly contend is that there are no centrifugal forces acting on the water *from the bucket.* Rather the water's surface becomes curved due to the influence of the Earth, the stars and galaxies with respect to which the water is rotating.

The argument forms the heart of relationalism and boils down to *stress-energy there determines inertia here*. This version of Mach's Principle has become known as the *Geometrodynamic Steering Principle* coined by Wheeler to avoid confusion with more philosophical interpretations of Mach's Principle which aim to say something about the ontological existence of spacetime itself. However this underlying principle of geometrodynamics relies explicitly on a (3+1)-dimensional split of spacetime as is clear in Wheeler's definition

> *The specifications of the relevant features of a 3-geometry and its time rate of change on a closed space-like hypersurface together with the energy density and the density of energy flow on that hypersurface together with the entire spacetime geometry and hence the inertial properties of every test particle and every field everywhere and for all time.*[7]

As Wheeler saw it the Geometrodynamic Steering Principle is a clear statement of how flow of energy-momentum affects spacetime geometry while spacetime geometry steers the flow of energy-momentum. Instead of making a claim that the spacetime is only derived from the existence of material fields, the Steering Principle embodies Mach's Principle as is realized in GR: spacetime geometry is determined from the stress energy of the material universe in addition to the stress-energy of the gravitational field.

In other interpretations, Mach's Principle challenges the concept of an ontologically real absolute space more explicitly. One example of such an interpretation of Mach's Principle can be stated as: "*The local inertial frame is completely determined by the matter content of the Universe.*" [8] In this form, Mach's Principle seems to state that there is no such thing as a gravitational field that exists unto itself. Rather, it emphasizes that the gravitational field is defined only by the relational properties of particles and fields. This is, perhaps, the ultimate form of Descartes' relationalism in which any measure of distance is wholly determined by the relations between particles and interactions of particles. Although the bucket debate focused on the question of absolute space, we saw in Chapter 1 that a measure of space is also a measure of time. Therefore, this form of Mach's Principle is also a statement on the ontological existence of spacetime and not just space.

The question remains whether these interpretations differ only in semantics or whether there is a true distinction. One point of view is that spacetime (or the gravitational field) is indeed a unique and ontologically real dynamical field, much like the matter fields of the Standard Model. In this view, one can make the argument that GR is indeed a fully relational theory since all physical observables are relational in terms of dynamical fields. However, as we have tried to argue in Chapter 1, the ontological existence of the gravitational field is an added assumption to the theory. Indeed, it is not clear whether one can attribute independent existence to spacetime or if spacetime is a derived physical observable at the microscopic level. Although there is no strict resolution between the forms of Mach's Principle, the insight of Mach's Principle tells us that the laws of physics must be formulated without respect to some absolute, untestable background space and time. Moreover physics must be defined with due regard to relations between distinguishable events as will become clear in a moment.

2.1.2 Background Independence

Mach's Principle brings to light a core component of Einstein's relativity: background independence. But what is background independence and how does that fit in GR and quantum gravity? To help make this clear, it is sometimes helpful to view natural phenomena and dynamics as a play taking place in spacetime. To say that GR is a background independent theory is to say that we cannot *a priori* define some fixed, non-dynamical geometry (a stage) on which the dynamical

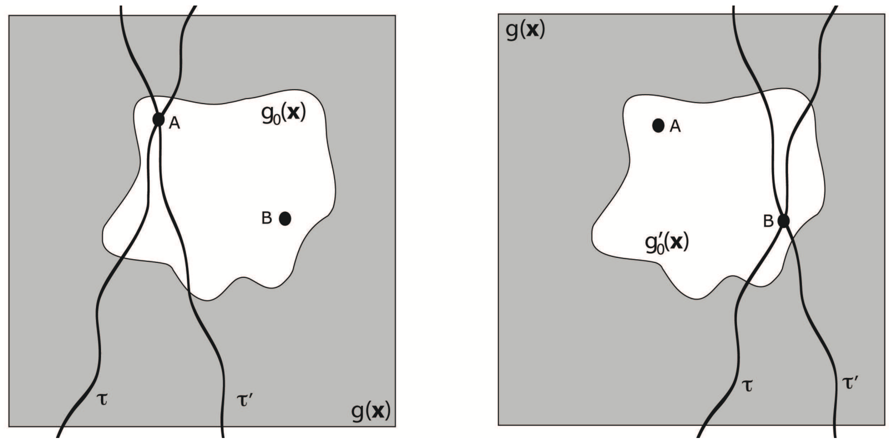

Figure 2.1: *The Hole Argument:* Diffeomorphism invariance in spacetime brings into question the physical meaning of points on the spacetime manifold. Here we illustrate a spacetime with a 'hole' of flat geometry–the unshaded region. Diffeomorphisms on trivially outside the hole and nontrivially inside the hole 'mix' up the spacetime points making them physically indistinguishable. These points only obtain physically relevant meaning when there is additional information used to distinguish them–such as the interaction of non-gravitational fields.[8]

fields (the actors) interact and evolve. Instead, spacetime is elevated from the non-participatory stage on which the play takes place to a core character who interacts and helps determine the path of the plot. More formally, it is common in the literature to put this in terms of gauge invariance: the gravitational field is a gauge invariant field under the group of diffeomorphisms on spacetime. Physically this tells us that the relevant information for the gravitational field is not the set of spacetime manifold $\mathcal{M}$–the geometry of spacetime– with its metric structure g–the information of distances and angles for the manifold–and all other fields F on the spacetime given by $(\mathcal{M}, g, F)$. Instead, the physically meaningful solutions to Einstein's equations are the equivalence classes of the manifold $\mathcal{M}$, the metric g and the field F, denoted $\{\mathcal{M}, g, F\}$, under smooth, invertible maps θ from $\mathcal{M}$ to itself;[11]

$$\theta : (\mathcal{M}, g, F) \rightarrow (\mathcal{M}', g', F'). \tag{2.1}$$

If we take all diffeomorphisms θ that leave a given manifold $\mathcal{M}$ invariant, then we build the diffeomorphism group $Diff(\mathcal{M})$. Moreover, if we take all the images of $\mathcal{M}$ under the elements of $Diff(\mathcal{M})$ we build the equivalence class $\{\mathcal{M}, g, F\}$. It is in this sense that the gravitational field is often regarded as a gauge theory with the gauge group $Diff(\mathcal{M})$.

The invariance of physical spacetimes under diffeomorphisms results in strong implications for observables in GR. Primary among these is that any observables cannot explicitly depend on spacetime points. This is the idea behind general covariance in GR. Einstein struggled with the implications of general covariance and even tried to reformulate the gravitational field equations in a non-generally covariant form.[8] The reason for Einstein's initial resistance to general covariance came from the physical meaning of spacetime points. This is often stated in the form of the *Hole Argument*:[8] given a spacetime with a closed, bounded region devoid of any matter and a metric $g(x^\mu)$ outside and on the boundary of the hole there does not exist a unique homogeneous solution to the gravitational field equations inside the hole. More precisely, one can apply a diffeomorphism to any solution inside the hole–as long as the diffeomorphism is trivial outside the hole–that transforms the homogeneous solution to a new solution. However, this produces a problem with the physical meaning of spacetime points. If we are able to apply limitless number of diffeomorphisms to the "hole" then how do we distinguish between points inside the region of spacetime? So one must make a decision at this point: either (1) spacetime points cannot be viewed as physically distinguishable by themselves or (2) we must abandon general covariance. However, if we introduce particles into this picture we can arrive at a satisfactory result. Suppose that there is one event attributed to the interaction of two test particles inside the hole. Is that event and point in spacetime still indistinguishable from the other points in the hole? By introducing this additional information, when we apply a diffeomorphism to the spacetime and to matter, we are able to still physically distinguish the point at which the two test particles interact from all other points in the spacetime. As such, we arrive at significant result in GR–any two spacetime points are physically indistinguishable unless there is additional information about the motion of test particles which distinguishes them. So is GR a theory of spacetime points and relations between spacetime point? No. It is not directly a theory of spacetime points at all but a theory of relations between events involving dynamical fields. Spacetime events rather than points on a spacetime manifold make up the physical observables of the theory.

Indistinguishability of spacetime points on a curved geometry void of stress-energy forms the core of the conceptual separation between quantum theory and GR. The problem stems from how each theory incorporates time. The background independence in GR tells us that spacetime is invariant under the group of diffeomorphisms that act on the entire spacetime. However, quantum theory in the Hamiltonian formulation manifestly requires that there be some notion of an evolution parameter for the variables being quantized which would lead to the ability to distinguish between different instances of the evolution. Meanwhile, the equivalent framework of Feynman's path integrals[12] only requires a distinction of 'paths' between two fixed boundaries. However, the problem of distinguishing a global time evolution parameter is not eliminated. It is transformed into a problem of identifying the unique paths over which one integrates. In either case, the problem of 4-diffeomorphism invariance represents a challenge to the overall program of producing a quantum theory of gravity. Below we will briefly state some of the methods commonly used to circumvent the problem of time in both the Hamiltonian and Feynman quantization schemes.[13]

Some of the first approaches to quantum gravity, instituted some notion of a (3+1)-dimensional split in GR in order to follow standard Hamiltonian quantization procedures familiar from particle physics. Yet doing this breaks the group of 4-diffeomorphisms into disjoint groups of diffeomorphisms on the spatial slices and diffeomorphisms in time. However complications arise from applying the Hamiltonian constraint for time evolution and the diffeomorphism constraint which constrains how slices are able to shift from one slice to the next. One approach is to enforce the constraints prior to quantization. While this method attempts to quantizes the dynamical degrees of freedom, it is not clear that the full dynamical Hilbert space of quantum gravity can be produced from effectively classical solutions. Moreover, this method relies strongly on the embedding of the spatial slices, but it appears that there is no unique way of performing this embedding or that the solutions are independent of the embedding variables.

Another option is to quantize the constraint equations and use the constraint operators to reduce a dynamical Hilbert space $\mathcal{H}_{\mathrm{dyn}}$ to the physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$. This produces problems in defining $\mathcal{H}_{\mathrm{dyn}}$ and an appropriate inner-product in it. The interested reader is referred to the more thorough account of the problem of time by Kiefer[13] or Isham.[14]

How one chooses to implement evolution and hence time into a quantum theory of gravity determines how the constraints are implemented on the states of the Hilbert space. However, one option is to avoid selecting out a canonical notion of evolution and instead focus on the possible "paths" of the quantum system between an initial and a final state. This other class of quantization procedures can sometimes be viewed as a 'timeless' option for quantum gravity. In the Feynman path integral approach the goal is to define an appropriate partition function and transition amplitude for the quantum system. The amplitude associated with the transition from a state, $|s\rangle$, at proper time, t, to a another state, $|s'\rangle$, at proper time, t', is defined by a superposition of the *quantum* paths connecting the two boundary states. We define the amplitude of each quantum path by $\frac{1}{N}\exp\left[-\frac{i}{\hbar}\int \mathcal{L}dt\right]$.[12] Here the amplitude is given by the action associated with the configuration of each unique path with $\frac{1}{N}$ used as a normalization factor. This approach manifestly retains the invariance under 4-diffeomorphisms of classical GR since there is no *a priori* split of the 4-geometry into a (3+1)-dimensional geometry. However, with this inclusion of the full symmetry class of the classical theory one has to determine how to integrate over various geometries.

The main problem with the path integral approach is thus contained in the difficulty in defining an appropriate measure for integration. This is most directly circumvented by imposing discretization from the beginning in which case a measure can be defined[15] or the measure is reduced to a problem of combinatorics.[16] Another concern in producing a continuum path integral formulation in quantum gravity is the explicit dependence on Wick rotations to Euclideanize the action. This again corresponds to the problem of time in that if we allow the background 4-geometry to be dynamical then such a Wick rotation is ill-defined in the continuum. This too has shown great progress in discrete quantum gravity[17] where causality (thus Lorentzian signature) is enforced in a dynamical building of spacetime topology. We will save a full discussion on the methods of discrete quantum gravity for §2.2. The path integral formally produces a well-defined construction of 4-dimensional GR and provides a conceptually simple framework for discrete quantum gravity. However there are still mathematical challenges in forming practical definitions a theory of quantum gravity based on path integrals.

2.1.3 Causality

Another important conceptual foundation to classical and quantum relativity is the inclusion of a causal structure into the theory. It is not clear how strongly causality should be encoded into quantum spacetimes; however, one is free to choose whether any given model should implement local causal principles weakly or strongly. It is known that strong causality naturally leads to fixed-topology spacetimes,[18;19] which, in turn, leads to the possibility of a (3+1)-dimensional split. Moreover it is clear from classical physics and observation that some form of global causality must emerge in an appropriate macroscopic limit. Moreover, it can be shown that a continuous causal structure on spacetime can give most of the geometric properties–up to a factor setting a global length scale–in addition to the topology of the spacetime.[18;20] What this seems to imply is that if a causal ordering can be defined in a theory of quantum gravity then almost all metric information is already encoded in a such a theory. It is useful to describe–albeit in a hand-waving fashion–how this comes about. One can think of the causal structure of the spacetime as defining a light cone at every point on the manifold and then setting ordering relations between points. This creates a partial ordering of points such that all the points satisfy properties of a poset (partially ordered set);

- *Reflexitivity:* If $x \prec x$, then $x = x$.
- *Transitivity:* If $x \prec y$ and $y \prec z$, then $x \prec z$ for all x, y, z.
- *Non-Circularity:* If $x \prec y$ and $y \prec x$, then $x = y$ for all x, y.

Once this poset is embedded into a manifold, one can attempt to recover the topology and conformal structure of the spacetime.[21;22] This can be seen by constructing tangent null cones at every point from the causal structure of the poset. The homeomorphisms for the poset then take null geodesics to null geodesics which makes them C^{∞}-diffeomorphisms. Moreover, these C^{∞}-diffeomorphisms preserve the local light-cone structure which makes conformal diffeomorphisms as well. Finally, since the local light-cone determines the local metric up to a conformal factor, we are able to use the poset structure of a spacetime to infer the local differential structure of spacetime, up to a conformal factor. How is this done? One approach is to build in spatial relations that are consistent with the causal structure. Once a global length scale is assumed, the relative measures of distances and angles can be deduced from the local light cone structure. This is an amazing result since we are not required to put any geometric information into the causal structure in order to recover almost all of the differential information about the manifold. A more thorough examination of how one can explicitly build the geometric content of a spacetime from the causal structure can be found in work by Penrose[23] or Sorkin.[24]

However, the continuous approach with poset or causal structure provides no intrinsic method for determining the complete spacetime structure. As we will examine later, there is at least one approach that claims to be able to use the causal nature of spacetime to reconstruct nearly the entire differential structure of the manifold–this approach is often referred to as Causal Sets or Causal Networks.[20] However, Causal Sets require one to infer a natural length scale from a framework beyond the scope of this theory. One proposal is to determine global scale factors from the dynamics of quantum particles embedded and defining the causal structure.[4;20] Since the phase of quantum particles provides a natural measure of the proper time, these phases give an external measure of the conformal scale of spacetime. This is the point of view illustrated in Chapter 1 and one we will see in §2.2.6. We must ask whether poset structure should be put in by hand or is it better to resort to histories of particles and fields to tell us the partial ordering of events in spacetime. We suggest here that using the properties of particles, e.g. their causal nature and the internal evolution of phase, one can construct the entirety of the geometric content of spacetime, and this ability is granted to us by the profound connection between causality and geometric structure.

2.1.4 Symmetry and Geometry

We have highlighted what we claim are some of the core characteristics of GR that are applicable to approaches to quantum gravity. However, it is also important to inject a short discussion of the role of invariance properties in fundamental theories of nature since we expect quantum gravity to exhibit some, if not all, of the symmetries of the classical gravitational field. Symmetry properties describe expected conservation laws of the fundamental theory, such as conservation of energy, via Noether's theorem.[25] In classical mechanics this is often seen in the relation between conservation of energy and time translation invariance or conservation of angular momentum and rotational invariance. As we understand these invariance principles, their conservation properties also lend themselves to a more unified and geometric interpretation through simple topological identities.

The basic premise behind invariance principles in dynamical theories hinges on the action principle for the dynamical field being described. We define the action functional for a dynamical field to be

$$I = \int \prod_{\mu} dx^{\mu} \;\; \mathcal{L}\left(q^i(x^{\mu}), q^i_{,\mu}(x^{\mu})\right) \tag{2.2}$$

where $\mathcal{L}$ is assumed to be a scalar, real function of the canonical dynamical variables q^i which may depend on the spacetime coordinates x^{μ}. Symmetries of the field are expressed as transformations applied to the action

$$I \to I^{\text{‘}} := I[\delta q^i] = I[q^i + \delta q^i] \tag{2.3}$$

which leave the action unchanged

$$\delta I := I'[\delta q^i] - I[q^i] = 0. \tag{2.4}$$

In modern field theories, we examine the invariance of the action under transformations generated by the action of elements of a Lie group on the field. The Lie algebra corresponding to these transformations forms the invariance algebra of the field. In the case of the Standard Model, it is typically understood that the invariance group is that of $U(1) \times SU(2) \times SU(3)$. Once we know the groups that generate symmetries of the action, we can also define conserved quantities for the field, as identified in Noether's theorem. Supposing that g is a generating element of the symmetry group of a field, then the conserved current associated with the field's action can be given by

$$J^\mu = \frac{\delta I}{\delta q^i_\mu} \frac{\delta q^i}{\delta g} \tag{2.5}$$

This provides an important result for classical fields as it provides a unique relationship between the algebra associated with a field and the invariant quantities of the field. We will see below that we can also translate this into a geometric picture for cases of physical importance, such as GR, electrodynamics, and Yang-Mills fields.

In the example of GR, we have already discussed how the field is invariant under elements of the diffeomorphism group *Diff*$(\mathcal{M})$ associated with the manifold. It can be shown that the generators of diffeomorphisms on spacetime define the symmetry transformations that lead to the conservation of energy-momentum, when Einstein's equations are assumed to hold. This conservation can also be formulated in terms of the "boundary of a boundary is zero" principle[26] (BBP) which takes the algebraic nature of the symmetries and gives them a geometric interpretation. In the BBP, a vanishingly small region of spacetime is decomposed into its 3-dimensional boundaries. These are in turn decomposed into their 2-dimensional boundaries. The integral over this boundary of the boundary is then topologically guaranteed to be trivial. This provides a glimpse at how a more fundamental theory can give new insight into symmetries of dynamics in nature. In particular, it shows us how the invariance group of GR, *Diff*$(\mathcal{M})$, which gives rise to the conservation properties of fields in spacetime and of spacetime itself can be viewed concretely in terms of the geometric interpretation of the field.

Moreover, the Standard Model picture of particles is based on the representation of fields which are invariant under transformations of the underlying symmetry group of spacetime. We will often refer to these as the external symmetries of particles as they are the symmetries due to the particles' motion in spacetime. The invariance of the field's action with respect to these symmetries gives rise to the already understood conservation properties found in GR. As such, we see that requiring this symmetry is a natural assumption that makes the particle fields and gravitational field compatible with one another. However, many fields also possess gauge symmetries, or internal symmetries, which are symmetries of the field degrees of freedom. The internal symmetries of a theory often possess a geometric interpretation similar to that in GR. In Yang-Mills theory, the invariance with respect to $SU(2)$ leads to a manifold interpretation of the internal space for the field. In this view, invariance can again be written in terms of the "boundary of a boundary is zero." Here the fundamental variables of the field can be viewed as connections on the principle fibre bundle (PFB) and the field itself is viewed as the curvature associated with the PFB. The symmetry of the field is then expressed in terms of changing the coordinates or applying diffeomorphisms on the PFB associated with the field. This geometric view provides an alternative understanding of the symmetries of the field that goes hand-in-hand with the algebraic structure of the field itself.

This brief account shows the dual nature of symmetries in physical theories. We can already see how the invariance of dynamical fields is inextricably linked to both the geometric and algebraic analysis of the fields. The algebraic picture gives insight through group actions on the action principle for a field, while the geometric picture relates invariances to geometric moves or properties of the manifold defining the field. The goal in developing a theory of quantum gravity is to incorporate these symmetry properties as much as possible while also providing a possible path towards unification of the internal symmetries of distinct fields. Symmetries are a vital component in physical theories as they point to the conserved quantities of the theory, whether it be energy-momentum or charge/current density. As such, they also provide crucial clues on how to connect existing theories or how to develop new models.

2.2 Approaches to Quantum Gravity

Background independence, local causal structure, and diffeomorphism invariance provide some of the fundamental properties of classical GR. Any model of quantum gravity should be able to recover these core features and the classical notion of spacetime no matter how the underlying notion of gravitational quantum dynamics is defined. However, one is often forced to implement one of these features more strongly or from the very beginning in order to define exactly what it means to quantize a theory of spacetime. The approaches to quantum gravity can be coarsely categorized into background independent and background dependent theories. If GR is to be the classical limit of a quantum theory of gravity, then it seems likely that a quantum theory should retain the background independence established in the continuum theory. We, therefore, will focus on background independent approaches to quantum gravity in this thesis. However, for an introduction to non-background independent or perturbative approaches to quantum gravity reader is pointed towards one of the many excellent overviews of string theory.[27;28] Among background independent theories we also make a distinction between canonical quantization techniques which often start with the continuum but must circumvent the ‘problem of time’ and path integral quantization which is most readily applied to discrete theories. We will first discuss one of the more prominent models, Loop Quantum Gravity (LQG), which will be the only direct quantization of continuum

GR in this thesis. Following LQG we will explore fundamentally discrete path integral–or 'sum over geometries'–approaches to quantum gravity. These discrete approaches provide a wide variety of methodology in how one quantizes the geometry and so we will attempt to distill the salient properties of each of these in this chapter.

2.2.1 Loop Quantum Gravity

Canonical quantization of GR has the unique problem of identifying how one implements the diffeomorphism invariance of spacetime on the Hilbert space of quantum spacetimes. This can either be done by initially imposing the constraint equations and quantizing the result or by quantizing the constraints and identifying states which are annihilated by the quantum constraints. Here we will overview the LQG canonical quantization of GR which uses an approach analogous to quantization of $SU(2)$ Yang-Mills fields. The LQG approach attempts to circumvent the problems with identifying an *a priori* direction of time by defining the gravitational fields in terms of a gauge-invariant connection on the 3-manifold spatial slice in such a way that many of previously discussed problems appear to vanish. We will try to provide a short introduction to the primary concepts associated with LQG here, but more thorough accounts of LQG are found in Rovelli's *Quantum Gravity*[8], Gambini and Pullin's reference for loops in gravity and Yang-Mills theories[29], or Thiemann's review of the mathematical foundations of LQG[30]. Our treatment will follow a conglomeration of the work presented in these expositions.

The LQG approach to quantizing gravity takes the underlying principles of the Arnowitt-Deser-Misner (ADM) formulation of gravitation[31] by using an embedding of a 3-dimensional spatial slice, $\mathcal{M}$, and defining the dynamics for the geometry of the slice. However, the ADM formalism treats the components of the 3-metric and their conjugate momenta as the canonical variables, whereas LQG defines connections of holonomies–or paths of parallel propagation–and their conjugate momenta as the canonical variables. This is done by introducing a triad bundle, $\mathbf{e}_a^i(\mathbf{x})$, on the 3-manifold instead of a metric at every point. With the use of the triad in the (3+1)-dimensional split we require the triad variables obey 3 sets of constraints: (1) the SU(2) Guass law constraints which require that the triad be invariant under arbitrary rotations within the $\mathcal{M}$, (2) the diffeomorphism constraints which require that the frame bundle be invariant under arbitrary active diffeomorphism on $\mathcal{M}$, and (3) the Hamiltonian constraint which acts as the generator of evolutions for the triad. One then defines a connection compatible with the triad, ω_a^i, which is used to construct covariant differentiation on $\mathcal{M}$;

$$\partial_{[a}e^i_{b]} + \epsilon^i{}_{jk}\omega^j_{[a}e^k_{b]} = 0. \tag{2.6}$$

However, in building the canonical pair of connections and their canonical momenta, we supplement the triad with the extrinsic curvature, K^i_a, to define a canonical pair $\{\mathbf{e}^i_a, K^i_a\}$. Using the old canonical pair, we define the Barbero connection[32;33] as

$$A^i_a = \omega a^i + \beta K^i_a \tag{2.7}$$

where β is, in general, a complex parameter known as the Immirzi parameter[34]. When we set $\beta = i$ we obtain the self-dual connection. We will assume that A^i_a is the self-dual connection from here on. We also find that the canonical conjugate momenta to the self-dual connection are the densitized triad variables $E^a_i = \frac{1}{2}\epsilon^{abc}_{ijk}e^j_b e^k_c$. So far everything is still classical and the next step is to cast this formalism into a quantum mechanical representation. When we do this we elevate the canonical pair to the level of operators on the space of functionals of the connection, $\Psi[A]$. We choose a representation such that the operators for the canonical pair are given by

$$\hat{A}^i_a\Psi[A] = A^i_a\Psi[A] \tag{2.8}$$

$$\frac{1}{8\pi G}\hat{E}^i_a\Psi[A] = -i\hbar\frac{\delta}{\delta A^i_a}\Psi[A]. \tag{2.9}$$

However, we have yet to define a Hilbert space for the functionals $\Psi[A]$. It was found that a useful representation of Hilbert space can be obtained from using holonomies of the connection

$$U(A,\gamma) = \mathcal{P}\exp\int_\gamma A \tag{2.10}$$

where γ is an oriented path in $\mathcal{M}$.[8] These holonomies define elements of the $su(2)$ Lie algebra in addition to being functionals on the space of smooth 3-dimensional connections. Moreover, if we generalize to smooth functions of holonomies $f\left(U(A,\gamma_1),\ldots,U(A,\gamma_n\right)$, where $\{\gamma_i\}$ is an ordered collection of paths, then we can define the linear vector space of all $\Psi[A]$ such that

$$\Psi[A]_{\gamma_i,f} = f\left(U(A,\gamma_1),\ldots,U(A,\gamma_n)\right). \tag{2.11}$$

Moreover, an inner-product can be consistently defined on this space of functionals. Additionally, it has become common to define closed holonomy or loop and knot states. Loop states are particularly useful in LQG since the equivalence class of knot states help define a discrete orthonormal basis of the kinematic Hilbert space of diffeomorphism invariance states, $\mathcal{H}_{diff}$. This basis can be constructed from knot states along with colorings of their nodes and links. These states are often referred to as spin-knot, or *s-knot* states. Moreover, multi-loop states have been found to help diagonalize geometric observables.[8] However, the linear vector space formed by functions of holonomies is not closed under the norm induced by any known inner-product, particularly the inner-product which gives observables as self-adjoint operators. As such, one defines an extended Hilbert space, $\mathcal{H}_{full}$ that closes the vector space under the norm. The Hilbert space $\mathcal{H}_{full}$ produced does not define the physical Hilbert space, $\mathcal{H}_{physical}$, rather the physical subspace is defined by the set of all functionals $\Psi[A]$ that are annihilated by the constraint equations. The remainder of the completed Hilbert space, $\mathcal{H}_{full}\backslash\mathcal{H}_{physical}$, contains diffeomorphic- and SU(2) gauge-invariant extensions of $\mathcal{H}_{physical}$. While the loops approach to quantum gravity is often likened to a quantized SU(2) Yang- Mills field theory, we must note one key difference. This quantization of GR also incorporates the 3-dimensional diffeomorphism group of the manifold $\mathcal{M}$ which must be taken into account when defining the physical subspace of the Hilbert space. In continuous SU(2) Yang-Mills theories, the reduced physical subspace of SU(2)-invariant functionals is still a non-separable Hilbert space which produces problems in the quantization of Yang-Mills theory on the continuum.[8] In the case of LQG, diffeomorphism invariance helps reduce the Hilbert space further into a separable physical subspace. In fact, LQG is more akin to a lattice Yang-Mills theory where the lattice loops are not definite loops but loops invariant under 3-dimensional diffeomorphisms.[29]

With a Hilbert space defined and the physical Hilbert space identified (conceptually) we need only define an orthonormal basis for the full Hilbert space. One way of doing this is to construct a basis from spin network states. The basis can be constructed by building a directed graph Γ with links, j, and vertices, v. To the links we assign irreducible representations ρ_j of SU(2), meanwhile we assign an intertwiner, ι_v, to the vertices. The intertwiner, ι_v, acts as a junction between the irreducible representations on the links, ρ_j, entering and exiting a given vertex, v. To be more precise, a Hilbert space is built at each vertex which consists of a tensor product of the Hilbert spaces associated with the representations of SU(2) assigned to the links. The intertwiner then becomes one element of an orthonormal basis in the vertex Hilbert space. We define an entire spin network by adopting a particular graph, Γ, with L links, j_l, and N vertices, v_n and write this as a spin network triple (Γ, j_l, v_n). The state of such a spin network is then given by

$$\Psi_{(\Gamma,j_l,v_n)}[A] = \langle A|\Gamma, j_l, v_n\rangle = \bigotimes_l \rho_{j_l}\left(U(A,\gamma)\right) \cdot \bigotimes_n \iota_n \tag{2.12}$$

where the direct product of intertwiners are suitably contracted with the direct product of representations of the links of the graph since the intertwiners occupy a space that is exactly dual to the space of the links. These states define for us a collection of kinematic states for LQG but the dynamics of the theory is still lacking.

In the kinematic theory there is no direct connection to the dynamical operators except to identify states which are annihilated by the constraint operators. This is only done to identify physically significant states out of the full Hilbert space of LQG. Dynamics is defined by how the Hamiltonian transforms one physical s-knot state into another. However, the full dynamics of LQG require techniques outside of the canonical theory as we will see in the coming subsections. We start with evolution of the states. In the (3+1)-dimensional formalism of GR, the Hamiltonian can be represented as the sum of the diffeomorphism and the Hamiltonian constraints. The proper time evolution is thus defined as the integral of the evolution operator from a slice of $\tau = 0$ to a slice of $\tau = 1$ where the integral is over the lapse and shift functions:

$$\hat{U}(T) = \int_T dN dN^a \exp\left[-i\int \hat{H}(N, N^a, \tau)\right] \tag{2.13}$$

We will not go into detail on the properties of this operator acting on spin network states,[35] but we will state that this time evolution operator acts, in an expansion of the operator with zero lapse, to either map links and edges to diffeomorphically equivalent links and edges or to add an edge and two vertices to the spin network when the Hamiltonian constraint fully acts on a node of the spin network. This produces the remarkable result that the transition amplitude between 3-geometries, ${}^{(3)}\mathcal{G}_i$ and ${}^{(3)}\mathcal{G}_f$, is given by a sum over topologically inequivalent mappings from the spin network on ${}^{(3)}\mathcal{G}_i$ to the spin network on ${}^{(3)}\mathcal{G}_f$.[35]

One of the important features of LQG is that geometric operators for volume, area and length are all represented by quantum operators with discrete eigenspectra. An example of this underlying discreteness comes from an attempt to define a gauge-invariant operator from the conjugate momenta. To do this, we follow Rovelli's construction[8] and break a given 2-dimensional surface into N smaller surfaces such that the area of these surfaces goes to zero as N increases and the union of all of these surfaces still recovers the entire surface. We can then define an operator

$\mathbf{A}(\mathcal{S})$ as

$$\hat{\mathbf{A}}(\mathcal{S}) = \lim_{N\to\infty} \sqrt{\hat{E}^2(\mathcal{S}_N)}. \tag{2.14}$$

This operator is analogous to the definition of the area of a surface in Riemannian geometry. The operator $\mathbf{E_i}(\mathcal{S})$ acting on a holonomy breaks the holonomy into two at the point of intersection between the holonomy and the $\mathcal{S}$. When $\hat{E}_i(\mathcal{S})$ acts on a spin network state with spin j twice, it returns $\hbar^2 j(j+1)$ for each place the spin network state puncture the surface. As such, the action of $\hat{\mathbf{A}}$ on a spin network state can be seen as a measurement of flux of a spin-network state through the surface $\mathcal{S}$. Therefore we can assign the physical meaning of this operator as the area of a surface punctured by a spin network. The calculation of the action of this gauge-invariant operator on a spin network state relies on the fact that for any sufficiently large number of cells on $\mathcal{S}$ the spin network will only intersect each cell once. This gives an eigenspectrum for the area operator

$$\hat{\mathbf{A}}(\mathcal{S})|S\rangle = \frac{8\pi G}{c^3}\hbar\beta \sum_{l\in S\cup\Gamma} \sqrt{j_l(j_l+1)}|S\rangle. \tag{2.15}$$

This immediately tells us that area is a quantized observable with a discrete spectrum and a minimum eigenvalue. One can further construct a volume operator[36] and a length operator[37] which are characterized by discrete eigenspectra as well.

LQG is a fully background independent theory which incorporates, at a minimum, 3-dimensional diffeomorphism invariance, a standard feature of Dirac quantization procedures. However, it is often argued that the fiducial choice in a foliation of the 4-dimensional manifold is arbitrary and affect the physical properties of the dynamics. It is not clear to the author whether or not one is still able to resurrect any sense of 4-diffeomorphism invariance once the arbitrary $(3+1)$-dimensional split has been made. It is also not clear how one obtains a correct classical limit from this formalism. Indeed, one would hope to be able to construct a low-energy limit from this approach , but the lack of a preferred or emergent concept of classical time makes it difficult to define what it even means to take a low-energy limit. Furthermore, a full understanding of the dynamics of the 3-geometry still requires the quantum relativist to resort to other methods to begin to grasp what it means to define a transition amplitude from one 3-geometry to another. Below we will examine one method for doing just this.

2.2.2 Spin Foams

An intimately related approach to LQG is that of spin foams–the name will become an obvious choice shortly–which has helped shed light on the issue of dynamics for states in LQG. As stated before, one of the core problems with LQG was that the canonical framework did not include an explicit method for calculating transition amplitudes from one 3-geometry, $|s\rangle$, to another, $|s'\rangle$. It is now known that one can gain some understanding of the evolution of the spin networks by deviating from the canonical quantum theory and resorting to an approach analogous to Feynman's path integrals.[12] Originally, spin networks were developed as an attempt to describe spacetime combinatorially,[38] using angular momentum. Later, Ponzano and Regge[39] used a similar approach on triangulated 3-manifolds to produce a 3-dimensional theory of quantum gravity.[40] We now know that these two approaches are fundamentally related.[8] For the purposes of this manuscript we will emphasize only the fundamental concepts along with a description of the 2-dimensional spin network relation to physical geometry.

The modern formalism for spin foams and spin networks often takes the results of LQG and attempts to build a propagator from one 3-geometry to another. Since it isn't known how to do this explicitly within the context of the canonical quantization, it is convenient to use Feynman's path integral treatment of the propagator. The basic premise behind path integral is that the one can calculate the propagator by summing (integrating) over intermediate steps in a increasingly large number of time steps between the two states. But let us put this into a little formalism in the context of geometry. We define the propagator between a two 3-geometries, $|s\rangle$ and $|s'\rangle$ at proper times of t and t' respectively, as

$$K(s_i, s_f) = \left\langle s_f \left| e^{-\frac{i}{\hbar}\int H(t-t')} \right| s_i \right\rangle \tag{2.16}$$

where H is the Hamiltonian constraint operator for LQG. We then take the proper time interval $t - t'$ and break it up into N non-overlapping intervals. One typically lets $N \to \infty$ but for this example we will only require that N be suitably large. In addition, we know from LQG that the spectrum of states satisfying the Hamiltonian constraint is produced by a discrete spectrum of spin networks.[8] Therefore, we can define the propagator as a sum over matrix elements of the Hamiltonian between intermediary states;

$$K(s_i, s_f) = \sum_{s_1, s_2, \ldots s_{N-1}} \left\langle s_f \left| e^{-\frac{i}{\hbar}\int_{t_{N-1}}^{t'} H(t-t')} \right| s_{N-1} \right\rangle \left\langle s_{N-1} \left| e^{-\frac{i}{\hbar}\int_{t_{N-2}}^{t_{N-1}} H(t-t')} \right| s_{N-2} \right\rangle$$

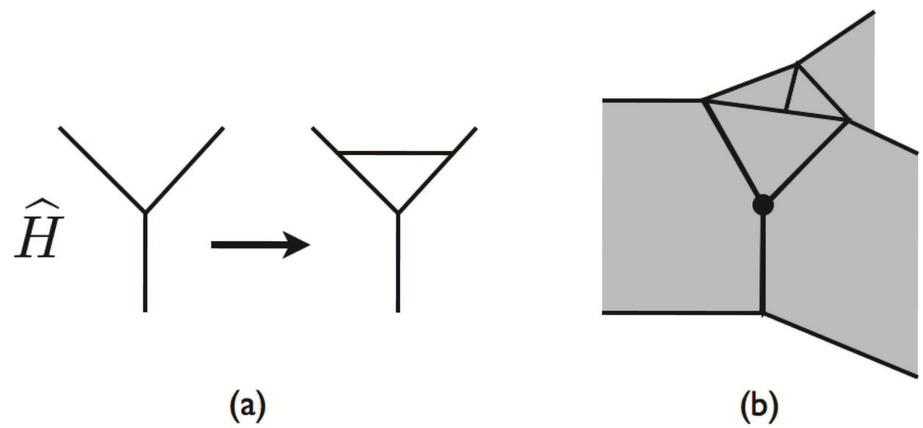

Figure 2.2: *The Generalized Action of the Hamiltonian on a Trivalent Node of a Spin Network:* The Hamiltonian operator acts on a trivalent node of a spin network by introducing two additional nodes with links connecting the original node with the new nodes. (a) We see the action of the Hamiltonian operator on a single trivalent node in (a). The history of this action on the trivalent node is represented in (b) where we see the introduction of a vertex in the spin foam history as the edge is split and expanded into three new edges.

$$= \qquad \cdots \left\langle s_1 \left| e^{-\frac{i}{\hbar}\int_{t_1}^{t} H(t-t')} \right| s_i \right\rangle . \tag{2.17}$$

We see from this expression the nature of spin foams as a sum over geometries that lead us from one spin network to another. This represents the formal core of spin foams, but the final piece of the puzzle is to evaluate the action of the Hamiltonian on spin network states. The action of the Hamiltonian operator has been found to act on trivalent spin network nodes in a remarkably straightforward manner.[41] We can represent it pictorially as expanding a single trivalent node into three trivalent nodes as in Figure 2.2. This is given explicitly by the relation

$$\hat{H}\left|s\right\rangle = \frac{-i}{\hbar}\sum_{n\epsilon s} N_n \sum_{l,l',l''} \epsilon_{l,l',l''} \mathrm{tr}\Big(h_{\gamma^{-1}_{x_n},l} h_{\alpha_{x_n},l',l''} \left[\mathbf{V}(n^*), h_{\gamma_{x_n},l}\right]\Big)\left|s\right\rangle . \tag{2.18}$$

where $\mathbf{V}(n^*)$ is the volume operator associated with the region dual to n, N_n is the lapse function at n, and the $h_{\gamma,l}$'s are operators that add holonomies to the graph. These latter operators act on the nodes in the following way: the $h_{\gamma,l}$ acts to add a link over l and therefore leaves l essentially unchanged while the $h_{\alpha,l',l''}$ acts to superpose links on l' and l'' and then connecting these two links at a finite distance down l' and l''. We see immediately that this transforms the node n into three nodes n, n', and n'' as in Figure 2.2.[8] It is important to note here that a general spin network may contain nodes of higher valence than 3 (more than three links meeting at the node) but these nodes are known to have a decomposition (via recoupling theory) in terms of trivalent nodes.[8] As such, we will only consider the action on trivalent nodes in general. With these basic ingredients in place we can construct a spin foam, or at the very least a history of a spin foam. The first step is to reinterpret the spin network elements, nodes and links, from their spatial geometric meaning to objects in spacetime. This is done by translating the spin network forward in time. Thus the nodes become edges and the links become 2-dimensional faces in the spin foam history. We define the fundamental objects in spin foams to be "two-complexes," Γ, which are individual histories of spin networks, i.e. spin networks together with an ordering, with faces given irreducible representations of $SU(2)$ and edges given intertwiners between the faces meeting at the edge. Using these histories we can construct a Feynman-like sum-over-histories

$$Z = \sum_{\Gamma} w(\Gamma) \sum_{\rho_f, \iota e} A_v(\rho_f, \iota_e) \tag{2.19}$$

where we sum over the two-complexes, Γ and $A_v(\rho_f, \iota_e)$ is the amplitude associated with the vertex that depends on the representations of the faces and edges adjacent to the vertex. While we now have formal expressions for the transition amplitude and the partition function, it is still not completely unambiguous in how one uses these. Both rely on the specific form of the Hamiltonian used–we have given one example. However, one could easily choose another form for Hamiltonian and possibly obtain different transition amplitudes associated with individual vertexes. Disregarding these problems for now, we will carry on to the connections with identifying relations to simplicial spacetimes.

One typically makes the connections to triangulations as a result of linking the theory more concretely with traditional physics of spacetime. In the spin foam formalism, we define only abstract spin networks without any notion of a spacetime manifold and expect these to define the geometry of spacetime. It is convenient to help recover the notion of spacetime by embedding spin networks into a geometric construct that relates more clearly with the spacetime geometry of classical GR. One can do this by embedding a spin network into a triangulated geometry as will be described in the next chapter and was first introduced by Regge.[42] This is done in 2-dimensional geometry by defining spin networks as the circumcentric dual 1-skeleton, Δ^*, to a triangulated 2-dimensional manifold, Δ. These circumcentric dual skeletons, Δ^*, are natural objects in triangulated manifolds as we will see in Chapter 3. One could alternatively take any spin network in its trivalent decomposition and construct a triangulation dual to the spin network via the methods of Voronoi and Delaunay dual tessellations.[43] For a triangulated 2-manifold, each link of the spin network is dual to a link of the triangulation while each node is dual to a triangle in the triangulation. The duality of the edges of the spin network and the triangulation allow us to define representations of $SU(2)$ to each edge of the triangulation. The spins associated with links of the spin network define the lengths of the edges in the triangulation[44] (in higher dimensions the spins associated with spin network links define the volume of the $(n-1)$–simplex dual to the links). Here again we see how the formalism for the quantum system imposes a discretized spacetime with a minimal length scale defined by the theory.

This can also be extended 3-dimensional gravity by taking a 3-dimensional simplicial lattice. Here, the edges of the dual lattice represent trivalent nodes of spin networks and vertexes of the dual lattice become 4-valent vertexes of the spin foam. Calculating the vertex amplitude amounts to just contracting the trivalent intertwiners associated with the edges adjacent to a vertex, v. This gives us a vertex amplitude that is precisely the Wigner-Racah 6j-symbol. Thus we can define a path integral for the spin foam as

$$Z = \sum_{\{j\}} \prod_{\text{face, f}\subset\Delta^*} \dim(\rho_f) \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{2.20}$$

where $\rho(f)$ is again the representation associated with a face of the dual lattice and the last product is over the 6j-symbols associated with v's. This is exactly the result arrived at by Ponzano and Regge[39] for the amplitude associated with 3d simplicial quantum gravity. We begin to see the converging relation between LQG, spin foams, and simplicial gravity. This is necessary step if we are to make any connection with the classical limit in the future as the simplicial representation makes the direct connection to spacetime geometry. In the coming subsections we see additional interpretations of quantized gravity in terms of discrete structures.

2.2.3 Group Field Theory

Background independence and local diffeomorphism invariance seem to indicate one need not only consider local geometry, as in Spin Foams, but also the topology of a spacetime in quantum gravity. In discrete 3-dimensional quantum gravity, it has been found that the dynamical variables are not local variables at all, but global variables defining the topological structure of spacetime.[8] Therefore it has been suggested that spin foams be generalized to include dynamic topology in addition to dynamic geometry.[35] The spin foam model just outlined is suited primarily as a sum-over-geometries quantization of spacetime. This has been extended to include the sum-over-topologies in group field theories (GFT).[45] A n-dimensional GFT is a quantum field theory of a field with n arguments g_i valued in some group G. In $n = 4$, the complex-valued field with $G = SO(3,1)$ can be viewed as the second quantization of a tetrahedron with the n triangles bounding the tetrahedron acting as the arguments of the field.[45] The action of this field, with the goal of making connections to spin foams, is given by

$$I_{GFT} = \frac{1}{2}\int dg_i d\bar{g}_i \phi(g_i)\mathcal{K}(g_i\bar{g}_i^{-1})\phi(\bar{g}_i) + \frac{\lambda}{5!}\int dg_{Ij}\phi(g_{1j})\phi(g_{2j})\phi(g_{3j})\phi(g_{4j})\phi(g_{5j})\mathcal{V}(g_{Ij}\bar{g}_{Ij}^{-1}) \tag{2.21}$$

where capital latin indices label the tetrahedra and $\mathcal{K}$ and $\mathcal{V}$ are kinetic and interaction (respectively) functions that are chosen to fit a given model. The interaction term is often viewed as the interaction of $(n{+}1)$-simplexes that meet to form a n-simplex, ie. 5 tetrahedra that meet to form a 4-simplex in this example. This action creates an ordinary quantum field theory and is often treated perturbatively like most other quantum field theories. This is done by writing the partition function as a sum over Feynman graphs, Γ, for the field;

$$Z = \int D\phi \exp\left[-I_{GFT}[\phi]\right] = \sum_{\Gamma}\frac{\lambda^N}{C[\Gamma]}\underbrace{\sum_{j_f,i_e}\prod_f \dim(j_f)\prod_v \{15j\}_v}_{Z[\Gamma]} \tag{2.22}$$

with $C[\Gamma]$ a factor from the symmetry of a given graph and the underbraced term is the partition function for the given Feyman graph. We see immediately that the GFT partition function gives both a sum over geometries–from $Z[\Gamma]$–and a sum over topologies–from the sum over graphs. Moreover, the $Z[\Gamma]$ is known to coincide with a 4-dimensional topological field theory known as BF theory.[46] BF theory is the direct generalization of the 3-dimensional action for gravity which discretizes to give the Ponzano-Regge model and 3-dimensional spin foams. This is a crucial understanding since it has been found that the 3-dimensional BF theory is actually a topological theory–the only dynamical variables turn out to be global variables.[8] We should point out that even the GFT approach seems to indicate a direct interpretation of quantum gravity in terms of discrete simplicial structures. It is also seen that there is a strong connection between GFT's and spin foams with GFT's acting as the more general framework for spin network graphs. From here we will take a step in an alternative direction and redefine the underlying fundamental structure of spacetime not as tetrad variables or connections but rely to the underlying causal structure of spacetime to define quantum gravity.

2.2.4 Causal Networks

While background independence, causal structure and symmetries are expected to be preserved in quantum gravity, we are free to choose which of these serves as the underlying fundamental structure. We saw in LQG and Spin Foams that the fundamental structure of spacetime was given by the geometric variables–the triad and connection. These directly implemented the local symmetries of GR on 3-dimensional space and sought to evolve each 3-geometry. An alternative is to assume that local causal structure is fundamental to dynamics of spacetime, not the symmetries of the spacetime. In this view, the symmetry relations and background independence are additional features that must be recovered in the final formulation of the theory but are not directly implemented in the axiomatic foundations. This is the conceptual road to causal networks.

The formalized framework implements causality through locally-finite posets, which are often referred to as causal sets or causets.[20] In addition to the properties for a poset, a locally-finite poset assumes that there are a finite number of elements between any two points of an ordered pair;

$$\textit{Locally Finite}: \quad \mathrm{card}\{z|x \prec z \prec y\} < \infty, \quad \forall x, y \tag{2.23}$$

With this last property, causets become strictly discrete collections of points with an ordering relation. We view the elements of a causet as a collection of events in spacetime and the ordering relation as a causality relationship between two events. However this creates an abstract structure without any direct geometric meaning. As with spin foams, geometric interpretation requires the abstract graph or causet to be embedded into a manifold. With causets one must first ask whether a given causal set can be *faithfully* embedded into a chosen manifold. A faithful embedding of a causet, C, into a manifold, M, is one which preserves the causal ordering, i.e. an embedding $i : C \to M$ is said to be faithful if

$$\forall x, y \in C; \quad x \prec y \Longrightarrow i(y) \in I^+(i(x)) \tag{2.24}$$

where $I^+(i(x))$ the set of points in the causal future of x embedded in M. In addition, the elements of C must be *uniformly* distributed in M, and the typical length scale of the embedding should be the minimum length scale of M. Of course there are some clear concerns that are open in causal sets: (1) does an M exist such that a given causet C can be faithfully embedded into M and (2) given a causet which can be faithfully embedded into two distinct manifolds M and M', are M and M' similar up to variations at the small scale? These two questions are still formally open but will not be a primary concern of this brief discussion. It will suffice to state these and move to the the practical constructions in causets.

As was stated in the discussion on causality, a strong causal structure in a manifold can be used to define the entire geometric structure of a Lorentzian geometry up to a global conformal factor. The faithful embedding of a causal set defines a causal structure on the manifold since the requirements of the embedding ensure a dense embedding of causal elements and ensures that the partial ordering is maintained. However, we can only determine the conformal geometry from this embedding. The original proposal for causets as spacetime[20] suggested that a volume element for the causet can be used to give the conformal factor. Such a construction should necessarily rely on information from non-gravitational sources, i.e. atomic radii, proper time, etc., to accurately give meaning to a density of causal elements such that one could translate between a number of events to spacetime volume. With this information given, one would be able to reconstruct the entire spacetime metric,[19] either by triangulating the embedded causet or some other suitable construction of metric components.

So far, there is no *quantum* in this construction of quantum gravity. Thus far, we have only outlined the general procedure for relating a causal network to a manifold's geometry. We have yet to define quantum dynamics to this theory of causality. To do so we would need some notion of probability amplitudes bounded by some compact boundary, or alternatively we would need to define a notion of quantum evolution of the causal set, i.e. define a causet Hamiltonian. One method of doing this requires us to define some new terminology. First, we define the causal future (past) of $x \in C$ as the set of all elements $y \in C$ such that there is a time-like curve with past (future) endpoint of $i(x)$ and future (past) endpoint of $i(y)$. We will denote this causal future as $I^+(x)$ and the causal past as $I^-(x)$. An acausal set $A \subset C$ is a collection of elements of C that are not causally related to one another. An acausal set A is said to be the complete future (past) of some event p when every event in $I^+(p)$ ($I^-(p)$) is related to some event in A. Moreover, two acausal sets A and B are said to form a complete pair when A is the complete past of B and B is the complete future of A.[47]

Using this new taxonomy of causal and acausal sets, we can form the poset of acausal sets. This poset is formed by all acausal sets of C with an ordering relations $A \to B$ which means that A and B form a complete pair. The quantum poset is constructed by identifying to each event x of the causal set a finite dimensional Hilbert space $H(x)$. The Hilbert space of an acausal set is defined to be the tensor product of the Hilbert spaces of all events that make up the acausal set;

$$H(A) = \bigotimes_{x_i \in A} H(x_i). \tag{2.25}$$

This is directly given in the quantum theory since the Hilbert spaces cannot be related through evolution as the events are not causally related. Evolution can be formally defined on the poset of acausal sets by defining an evolution operator E_{ab} between acausal sets that form a complete pair

$$E_{ab} : H(A) \to H(B). \tag{2.26}$$

Using this scheme the properties of a poset are translated into the language of evolution operators:

$$\text{Reflexitivity}: \; E_{aa} = \mathbb{I}_a \tag{2.27}$$

$$\text{Transitivity}: \; E_{ab}E_{bc} = E_{ac} \tag{2.28}$$

$$\text{Non}-\text{circularity}: \; E_{ab}E_{ba} = \mathbb{I}_a \Leftrightarrow E_{ab} = E_{ba} = \mathbb{I}_a. \tag{2.29}$$

Using these evolution operators a transition amplitude, or sum over causets, is assigned a quantum amplitude to each causet that connects two acausal sets:

$$\mathcal{A}_{C_i \to C_j} = \sum_m A_{C_m}(E) = \sum_m \prod E(C_m). \tag{2.30}$$

A perhaps more intuitive quantum evolution scheme is to associate unitary evolution to the nodes of a causet and Hilbert spaces to edges. We say this is more intuitive since we typically think of an event in spacetime as an interaction between fields which generate a change of state, and thus a mapping from one Hilbert space to another. We will see an example of this implementation in the form of the Computational Unvierse in §2.2.6. We have already seen a concrete version of quantum causal sets in the form of spin foams. In spin foams, the Hilbert spaces are given by the space intertwiners connection representations of SU(2), and the evolution is that of spin networks.

This provides an intuitive framework which locally encodes the causal structure throughout the spacetime. We have shown that there are also schemes for implementing quantization for the causal structure. However, to obtain complete, physical information about the geometry of spacetime the causal networks require additional information about length scales or volume that presumably comes from dynamics of matter in the causal set and the addition of new structure from which one can reconstruct the metric at an event. We already see how discretization of spacetime events and the inclusion of matter into quantum gravity assist in the interpretation and reconstruction of quantum histories of spacetime. However, there are still some unsolved questions and a lack of concrete constructions in the quantum dynamics using causal sets. In the rest of this chapter we will explore similar approaches that we hope can shed light on how causal structure, simplicial geometry, non-gravitational sources fit together to form a more complete picture of quantum gravity

2.2.5 Simplicial Quantum Gravity

Simplicial quantum gravity actually encompasses many distinct models of quantum gravity that can are based on a simplicial representation of the quantum geometry. In the previous subsection we emphasized that simplicial representations of spacetime can be naturally introduced as a way to make abstract graph representations more physically concrete. In this subsection, we highlight the other end of the spectrum with the simplicial geometry taking over the role as the determiner of geometric dynamics. There are two primary candidates for quantum gravity that follow this route: (1) quantum Regge Calculus (qRC) and (2) (Causal) Dynamical Triangulations (CDT).

The first of these, qRC, quantizes gravity by using the Regge action;

$$I_{\text{Regge}} = \sum_{\text{hinges,h}} \begin{pmatrix} \text{Area of} \\ \text{hinge, h} \end{pmatrix} \begin{pmatrix} \text{Deficit Angle of} \\ \text{hinge, h} \end{pmatrix} \tag{2.31}$$

to build a discrete Feynman path integral for the geometry. In the continuum, building such a path integral was formally straightforward but virtually impractical since defining a consistent choice of a measure in the path integral is currently beyond our reach. However, the situation is greatly simplified when one makes the transition from continuum to discrete dynamics–this may be yet another indication that a continuum formulation of quantum gravity is misleading when it comes to developing a physical theory. In the discrete case, the measure for the path integral is defined as a measure over squared edge-lengths such that they satisfy the triangle inequalities and appropriate higher dimensional analogs.[15] One then defines the simplical path integral as

$$\int_0^\infty \prod_l dl^2 \Theta[l^2] e^{I_{\text{Regge}}[l^2]} \tag{2.32}$$

where $\Theta[l^2]$ is a step function that is non-zero when the edges satisfy the above criteria. The transition amplitude between geometries is obtained by defining the boundaries of the path integral and integrating over all edge lengths of a given simplicial 4-geometry bounded by the two states. There have been several calculations along these lines carried out mainly by Hamber and Williams;[48] however, these calculations and formalism have only been implemented with respect to a Euclidean action. It is still not clear whether the Euclidean action for qRC formally equivalent to the path integral for Lorentizian spacetimes. Indirect evidence for the inequivalence of Lorentzian and Euclidean path integrals has been found in the (causal) dynamical triangulations approach discussed below. We will see in the coming discussion that the inclusion of causal structure–natural to Lorentzian signature spacetimes–produces significantly distinct path integrals from the Euclidean path integral without causality constraints.[49] We must, therefore, entertain the question as to whether or not a Euclidean path integral for gravity is provides qualitatively similar results to quantum Lorentzian spacetimes. It is also worth noting here that the issue of Euclidean versus Lorentzian path integral approaches can also be investigated from a dynamical perspective where one might attempt to develop a dynamical theory of spacetime signature.[50] However, we will not develop this line of thought further in this thesis. Instead, we will examine the results from dynamical triangulations and emphasize the distinction between results with Euclidean and Lorentzian signature path integrals.

This brings us to the question of how one builds spacetime quantum histories without resorting to Euclidean measures. One way to do this is to implement causality as the spacetime history is developed–that is to say that we insist that there is no ambiguity in the causal structure at any point in the geometry.[49] CDTs are implementations of the causal nature of Lorentzian spacetime to the previously developed Euclidean dynamical triangulations (DT). The implementation of causality as a selection criteria for simplicial geometries is motivated by the large-scale and classical limit sought. One should hope that in the large-scale limit–as edge lengths are suitably taken to zero–the simplicial geometry should give a classical limit in which a causal structure emerges with a dimension equal to the observed dimension of our universe. However, geometries that include topology change, an indication of acausal spacetimes, may not generally lead to a recovery of some large-scale sense of causality.[49] This was generically true in Euclidean DT and appears to be a generic feature of DT and CDT. Therefore, the CDT path integral directly excludes any topology changing geometries from the set of physical spacetimes. The DT quantum histories exhibited additional unphysical characteristics that preclude the viability of obtaining a coherent classical limit–crumbled or polymeric collections of simplexes. In particular, these curmbled or polymeric phases exhibited large-scale dimensionality, the Hausdorff dimension,[49] of either two or infinity, not four as one should expect in order to obtain a classical limit that matches our universe. However with the addition of causality criteria this dimensionality problem seems to have vanished and causality fixes the large-scale dimensionality to the dimensionality of the underlying simplexes.[17;51]

How does one build such a CDT quantum history and what is the path integral associated with a set of boundary conditions? The framework here is analogous to the qRC path integral. The basic premise remains the same–sum the Regge amplitudes $e^{I_{\rm Regge}}$ for each physical triangulation that connects two triangulated 3-geometries. The main difference is in how one defines the set of unique triangulations bounded by the two 3-geometries. The qRC approach took the point of view that we should integrate over all collections of squared edge lengths for a fixed simplicial complex as long as all physical criteria (the triangle inequalities and higher dimensional analogs) are satisfied. The CDT dynamics are based on fixed edge-lengths while changing how simplexes are 'glued' together–changing the simplicial complex for fixed topology. We can thus state the two underlying constraints on CDT: (1) any triangulation must be free of topology change–causal structure must be preserved–and (2) fix the edge lengths of a given simplicial complex such that all simplexes are regular (equilateral) simplexes. The first we have already discussed. The second assumption is a matter of simplification of the dynamical theory to constrain the state space of the theory to one with a combinatorial measure. With both these constraints of the theory we use the Regge action (along with the constraint on the edge lengths) to define the path integral;

$$Z = \sum_{\text{triangulations, } T} \frac{1}{C(T)} \exp\left[-\kappa_4 N_4(T) + \kappa_2 N_2(T)\right] \tag{2.33}$$

where $N_2,\ N_4$ are the number of triangles and 4-simplexes, respectively, κ_i are analogous to the gravitational and cosmological constants, and $C(T)$ is the order of the automorphism group of simplicial complex. This factor plays the important role of ensuring that one doesn't over count triangulations that are topologically and diffeomorphically equivalent. To see the meaning of the factor $C(T)$ we emphasize that triangulations related by elementary moves, called Pachner moves, are diffeomorphically equivalent triangulations with distinct but related incidence matrices.[52;53]

It seems likely that there will be some composite theory of simplicial spacetime that fully accounts for the sum over different simplicial complexes and a sum over possible edge lengths within a given simplicial complex. Currently, these two approaches are quite separate and each are characterized by their own strengths. The CDT approach gives a path integral which is conceptually simple but incorporates only regular simplexes. The qRC approach accounts for non-uniform edge lengths but is less conceptually simple in developing a measure for the path integral. However, if CDT's are to fully incorporate the expected dynamics of our universe, one should expect that the uniformity of edge lengths be relaxed and replaced by the determination of edge lengths by non-gravitational sources. In the next subsection we will explore a proposed model in which dynamics are fully derived from an underlying material quantum system.

2.2.6 The Computational Universe

While many paths toward quantum gravity seem to imply that geometric content of quantized geometry exists unto itself without any reliance on external sources, it appears possible that this is not the only choice one can make. In LQG, Spin Foams, causal sets and simplicial quantum gravity the geometry of spacetime or its underlying abstract structure is dynamically independent (contains independent degrees of freedom) from non-gravitational sources. This is fundamentally averse to the view of gravitation outlined in Chapter 1–that the dynamics of quantum system cannot be inextricably separated from the concept of measurement using physical, material devices. One of the first suggestions known to the author to completely base the construction of the spacetime geometry on the dynamics of a quantum non-gravitational system was proposed by Lloyd.[4] The original proposal by Lloyd can be viewed as a model of a quantum causal history with the evolution operators inheriting form from an underlying quantum computational system.

The construction starts with a directed quantum network built of links and nodes as in Figure 2.3. To each link we assign a Hilbert space for the qubit traveling along the link. To each node we assign a quantum gate that operates on the qubits entering the node. We assume that each node is 4-valent (2 qubits in and 2 qubits out) and that there is no evolution of the qubits along the links. So far, the picture is somewhat similar to spin networks except here the assignment of a representation of SU(2) to a link comes from the representation of the qubit associated with that link. Each quantum network defines multiple histories for the qubits by taking all possible evolutions through the gates of each qubit. One then embeds each quantum history into the simplicial complex of a manifold using the following critera: (1) each node gets embedded into a vertex , (2) each edge is embedded as a null edge, and (3) and each node whose action in the history is the identity is removed as in Figure 2.3. The dynamics on the embedded graph are captured by the stress-energy tensor associated to the graph. This is defined by associated kinetic terms, γ_l, to the edges and the interaction potentials to the vertices. Any edges and vertices which are not associated with the quantum graph are treated as fictitious and thus receive $\gamma = 0$ and no

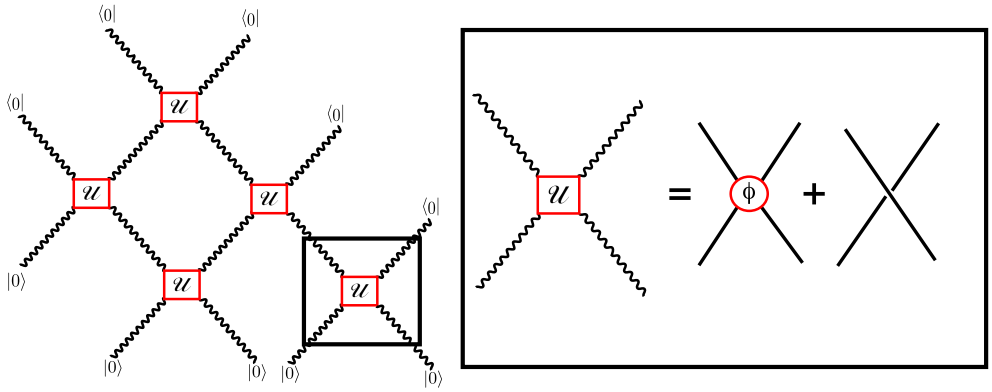

Figure 2.3: *A Gate in the Computational Universe:* A simple binary gate–a swap gate that rotates the qubits by a phase $e^{i\phi}$ or acts as the identity–in the computational universe can be broken up into the possible histories corresponding to the actions of the gate on the binary qubits. When no direct transformation takes place on the qubits the gate is removed and the qubits are said to not interact at all. As such, there is vertex for such a gate in the corresponding history and the qubits are unaware of the presence of each other.

interaction potential. We thus obtain quantum dynamics embedded into the topological structure of a manifold and all that is left is to solve for the local geometry. For quantum dynamics, one simply solve the inhomogeneous Einstein-Regge equations using the local stress-energies[1] defined by the quantum network.

$$\sum_{h \supset L} \frac{1}{2} L \cot{(\theta_{h,L})} \epsilon_h = \kappa T_{LL} \tag{2.34}$$

This methodology provides for a clear determination of quantum probability amplitudes and transition amplitudes since the probability for any given history is simply the product of the probability for each outcome at each node in that history. Furthermore, the transition amplitude is defined as the sum of all the probability amplitudes for the histories with the given boundary conditions.

In principle this proposal is extendable to any system of discrete quantum dynamics. However, the exact implementation will necessarily vary. To see why this is so, we can replace the quantum network by a system of N interacting quantum fermions. With these interacting fermions, we have a well-defined stress-energy tensor, and we can define an appropriate simplicial lattice defined by the piece-wise linear world-lines of the particles. However, Einstein's equations are solutions to the classical equations of motion for spacetime and along with local conservation of energy-momentum imply that the particles obey classical paths as well.[18] As such, we cannot arbitrarily

[1]The local stress-energy defined in the equations of motion are characterized a by a tensor with components directed along an edge L. We will not examine why the stress-energy takes this form at this time as this is the principle result derived in Chapter 3.

insist that the system automatically satisfy Einstein's equations for all paths.

A counter proposal is to allow the quantum particles to partially define the geometry. We take the world lines of the particles to define the proper length of the time-like line embedded in the simplicial complex. These lengths may or may not be classical solutions to the Einstein-Regge equations by themselves. However, a generic assortment of quantum particles will not fully complete the simplicial lattice, and we are left with edges which do not correspond to any dynamics of the quantum particles. However, the quantum dynamics do build a causet on the spacetime. Therefore, we are allowed, in principle, to reconstruct the entire geometry of the spacetime directly from the time-like edges of the quantum system. Lloyd has suggested that this be done by simply requiring that additional edges of the simplicial complex vary such that the Einstein-Regge equations (with the particle world-line edges of fixed length) be solved.[5] Moreover, the causet defined by the quantum particles is supplemented with information about the conformal scaling of the geometry by the proper times of the particles. We see that one path is to reconstruct a quantum spacetime by de-emphasizing the dynamical degrees of freedom of the gravitational field and relying on the quantum system to define the quantum histories. It is the hope that in the large scale, this freezing out of local dynamical degrees of freedom is not seen and the geometric degrees of freedom are largely reestablished. However, there is still much more work to be done in this regard and the model's construction is still in progress. In particular, we seek to understand exactly how one incorporates generic fields into the spacetime lattice. This question is the core theme of this manuscript.

2.2.7 Topological Quantum Field Theories

GFT's and CDT's have provided evidence that the dynamics of quantum spacetime should include reference to paths of different topologies. We will now explore a more encompassing theory of topological dynamics, topological quantum field theory (TQFT), which explores this more fully. TQFT's were first axiomatized by Atiyah[54] and have been found to include 3-dimensional gravity and spin foams. A TQFT is characterized by a functor Z which assigns to each (n-1)-dimensional compact oriented manifold, Σ, a Hilbert space $Z(\Sigma)$. The vectors in this Hilbert space are all n-dimensional compact oriented manifolds that contain Σ as their boundary. More formally, this functor Z is a mapping between catgeories–from d-cobordims to Hilbert spaces. The functor must satisfy several axioms as well:

1. $Z(\Sigma^{-}) = Z(\Sigma)^{*}$

2. $Z(\Sigma \cup \Sigma') = Z(\Sigma) \bigotimes Z(\Sigma')$

3. Given $M = M_1 \cup_{\Sigma_2} M_2$ with M_1 bounded by Σ_1 and Σ_2 and M_2 bounded by Σ_2 and Σ_3, $Z(M) = Z(M_2)Z(M_1) \in Homomorphisms(Z(\Sigma_1), Z(\Sigma_3))$.

4. $Z(\emptyset) = C$

5. $Z(\Sigma \times I)$ is the identity endomorphism of $Z(\Sigma)$

The first of axiom states that an oppositely oriented manifold gets mapped by Z to the dual space of $Z(\Sigma)$. The second defines how we map a disjoint union of two different (n-1)-dimensional manifolds, while the third prescribes how to map a composite cobordism to a composite Hilbert space. The last two enforce non-triviality for the functor Z. From a physical standpoint we must make a connection from these axioms to the spacetimes they are meant to describe. First let us motivate the functor Z. From the path integral formulation of the quantum gravity we know that the main goal is to define transition amplitudes between two (n-1)-dimensional boundaries of spacetime. The functor Z helps us by defining the category of spacetimes that contain one such boundary. We can then use the axiom for the mapping of a disjoint union of two boundaries to define the Hilbert space of spacetimes containing both boundaries. This is the Hilbert space over which the path integral should be evaluated. Moreover, the third axiom above describes how one should define a multi-layered path integral–a path integral with two end-point boundaries and one (or multiple) pre-defined intermediate slice(s). However, the last axiom greatly limits the dynamics allowed in TQFT's. To see this, we first point out that $Z(\Sigma \times I)$ is interpreted as the time evolution of the boundary Σ. The last axiom tells us that the time evolution must be the identity endomorphism and thus there can be no local dynamics on Σ. If this were not the case, the Hilbert space given by $Z(\Sigma)$ would not be trivially mapped to itself. Thus, TQFT's encompass the category of theories of quantum gravity that model quantum spacetimes with global dynamics. We have seen that this includes 3-dimensional spin foams and BF theory. For the purposes of this thesis, we wish to only introduce the reader to the main concept behind this approach as it helps shed light on another view of quantum dynamics of spacetime. We believe that this provides direction in how one constructs simplicial quantum spacetimes since the lesson is that the physical meaning in a TQFT comes from the collection of all manifolds that contain the end points of the path integral. This gives further evidence that a simplicial path integral should be one that accounts for different simplicial complexes for the 4-dimensional spacetime manifold as well as different edge lengths. However, it should be noted that TQFT's are theories of only the spacetime manifold; thus, they are theories of vacuum GR. The topological aspect of TQFT is therefore non-inclusive of the local dynamics which seem to come from being able to physically identify different points on the spacetime manifold. This ability is directly linked to the interaction between non-gravitational sources and the gravitational field.

2.3 Quantum Gravity is Discrete Gravity

We have identified some of influential ideas in developing approaches to quantum gravity: Mach's Principle, background independence, causality, and incorporation of symmetries. Since there is no guiding experimental evidence to direct development of quantum gravity, one must make a decision about what a quantum gravity model must incorporate from the beginning. The choice in which founding principle guides the development of a theory impacts the conceptual form the theory takes. For example, if one chooses to incorporate causality as the fundamental structure of the theory, then it naturally appears that a directed graph becomes the underlying foundation of the theory. However, given such a choice one can to some extent still recover or insist on the alternative conceptual features.

We focused here on many of the background approaches and distilled out the conceptual path to each. But how do these models relate to one another and are there some converging ideas behind them? LQG appeared as the only canonical approach to quantization discussed in this overview. It's primary objective was to follow Dirac quantization on the (3+1)-dimensional split of spacetime using a set of geometric variables which explicitly incorporate the gauge symmetry on the spatial slices. Here, symmetries play the vital role in the development of the theory and causal structure becomes inherent by way of the (3+1)-dimensional formulation. Spin foams and GFT's followed a similar path by directly and explicitly incorporating a gauge group on a spatial slice; however, the quantization procedure used follows the Feynman path integral view of quantum mechanics. We link these together at this stage since we have seen that the GFT approach encompasses the spin foam framework. If one wishes to incorporate causal structure in a more austere manner while letting symmetries play less of a forefront role, then there are a number of approaches to choose from. Causal Networks assume the underlying foundation of spacetime is embodied by the direct graph providing the temporal ordering of events in spacetime. There the causal structure is clearly the root of the formalism; however, the recovery of the local $SO(3,1)$ symmetry is obtained in the details of how these points of "sprinkled" in a given manifold.[55] The Computational Universe follows a similar path though the implementation is dramatically different. Here the embedding of the causal ordering follows a procedure closely linked to RC. The final approach that directly incorporates causal structure is that of CDT's. CDT's encode the local light cone directly into the fundamental building blocks of a simplicial lattice and restrain the theory to well-defined mappings of causal structure from one simplex to another. In both CDT's and the Computational Universe there is inherent reliance on the methods of RC and it is still debated how well this preserves the local Lorentz invariance and diffeomorphism invariance. Then there were the class of TQFT's which take background independence to its logical conclusion: define the Hilbert space of geometries between two states to be the category of spacetimes with those boundaries. In TQFT's the primary focus is directly on background independence while causal structure and incorporation of symmetries appears to be related to specific implementations of a TQFT.

One of the resultant features of many of these models is that individual states of the theories relies on a discretization of the spacetime. In some, such as CDT's and Causal Networks, the discretization is explicit. In others, such as LQG and GFT, the discretization comes through in the interpretation of states of the formalism. Either way, a simplicial representation of spacetime appears to be a theme throughout the background independent quantum gravity approaches. It, therefore, seems plausible that in order to grasp the incorporation of matter into quantum gravity, one should begin by understanding how matter naturally finds a role in a given discretization of spacetime, i.e. a given simplicial lattice history of the quantum theory. The rest of this thesis will take this route and explore how matter can most naturally be included in an arbitrary, but single, simplicial spacetime.

Chapter 3

The Geometry of Simplicial Spacetimes

My work always tried to unite the true with the beautiful, but when I had to choose one or the other, I usually chose the beautiful.[56]

–Hermann Weyl

One has a plethora of options in how to approach the dynamics of quantum gravity based on which properties of spacetime one identifies as most fundamental. Many of these approaches either included a simplicial geometry directly into the dynamical structure or used a simplicial lattice to give geometric meaning to the underlying dynamics. Henceforth in this manuscript we will not assume any particular model of quantum gravity. We will only assume that the fundamental structure of spacetime geometry is the piecewise-linear geometry originally described by Tullio Regge.[42] In this description–called Regge Calculus (RC)–an n-dimensional spacetime is inherently discretized into flat, Minkowski n-simplexes, e.g. triangles in two dimensions and tetrahedra in 3 dimensions. This is most clearly viewed as a finite-sized implementation of the Principle of Equivlance wherein we alter the assumption that there be some small neighborhood which acts like Minkowski spacetime to some small but finite neighborhoods which act like Minkowski spacetime. Here we stated the Principle of Equivalence in terms of geometry instead of in terms of special relativity so as to explore the interplay between the motion of material bodies with the underlying geometry. The continuum Principle of Equivalence allows one to choose a small enough neighborhood such that the underlying motion of test particles follow geodesics of Minkowski spacetime. The discrete Principle of Equivalence defines such neighborhoods *a priori* and allows the length scale of these neighborhoods to be determined by dynamics. For test particles in an otherwise vacuum spacetime, we should thus expect particles to follow straight-lines on the interior of the flat building blocks. Curved geodesics arise from discontinuous shifts of parallel world-lines to converging world-lines as the two paths pass around a conic singularity in the lattice spacetime. Thus for the purposes of this chapter, we will allow the Principle of Equivalence to be understood in either terms of geometry or the motion of test particles, as the two are interchangeable in this context.

Given this conceptual framework, we must now translate this into a discrete formulation of Einstein's equations. In the next section we will outline the geometric content of this discrete formulation of GR. This will include an overview of the inclusion of curvature into RC and a definition of the Regge action. We will then use this formulation to construct the Regge analog of the Einstein tensor before evaluating the contracted Bianchi identity on the lattice spacetime. This will provide a foundation for understanding how the structure of vacuum RC can be used to incorporate non-graviational sources naturally into lattice spacetimes.

3.1 The Geometric Structure of RC

The goal of RC is to define the local degrees of freedom of the gravitational field into the geometric content of a piecewise linear (PL) spacetime manifold. The metric tensor field, equivalently the frame bundle, define the geometry of spacetime in the continuum, and one must find a way to encode that content into the edges, faces, or volumes of the PL-manifold. Regge's core insight was that one need only use the edge lengths of a lattice to define the gravitational degrees of freedom. However, this places limitations on the type of PL-manifold used in RC. For example, if one attempts to use a polytope tiling of a curved n-dimensional manifold–such as a hypercubic lattice on a 4-dimensional spacetime–then edge lengths alone are insufficient to define the geometry. Instead, the n-polytope lattice generally requires one to define additional variables, such as angles, to fully specify the local geometry. However, as Regge emphasized, one can construct a piecewise linear manifold from simplicial building blocks such that only the edge lengths of each n-simplex are needed.[42] Once the edge lengths are identified, one can make a 1-1 correspondence between the $\frac{n(n+1)}{2}$ independent components of the metric and the $\frac{n(n+1)}{2}$ edge lengths of a n-simplex.

$$\underbrace{\mathbf{e}^{\mathbf{a}}_{\mu}(\mathbf{x})}_{\text{frame bundle}} \iff \underbrace{g_{\mu\nu}(\mathbf{x})}_{\text{metric field}} \iff \underbrace{\{l_i\}}_{\text{edge lengths}} \tag{3.1}$$

This completely determines the geometry of each simplex and how simplexes are connected together. If we require that each n-simplex be a representation of Minkowski spacetime, then we recover the finite-sized Principle of Equivalence in a PL manifold representation of spacetime.

However, the simplicial lattice of RC is not the only meaningful geometric structure in a discrete theory of gravity. It is often suggested that the circumcentric dual to the simplicial lattice is also natural,[57–67] and possibly fundamental, to RC, for an overview see the work by the author and Miller.[68] We will explore the roles of the circumcentric dual in the coming sections;

however, we can now state some geometric facts about the interplay between the simplicial lattice and its circumcentric dual. The circumcentric dual lattice is the unique lattice such that there is a (n-d)-dimensional polytope dual to each d-simplex in an n-dimensional PL-manifold. Moreover, the cells in the circumcentric dual are mutually orthogonal with the corresponding elements of the simplicial lattice. This provides for significant simplification of geometric results in RC since it allows for factoring 4-dimensional variables into a set of 2-dimensional quantities in each the simplicial and circumcentric dual lattices. Moreover this provides a unique factoring of the PL-manifold into flat and curved neighborhoods. Since the simplexes of the PL-manifold serve as the manifestation of the Principle of Equivalence, the circumcentric dual n-volumes cannot also take on this fundamental role. In general, these dual cells will embody finite-sized neighborhoods capturing the local geometry of the hinges containing a given vertex. We will see in the next section exactly how this comes about as we construct the curvature and geometric action for RC.

3.2 Curvature and Gravitation in RC

No discussion of GR would be complete without an explanation of how curvature is incorporated into the representation of spacetime. In RC, the curvature is concentrated at the (d-2) - dimensional hinges of the simplicial lattice, but how does this curvature come about? One measure of the curvature can be obtained by parallel transport of a vector–a physical gyroscope approximates this fairly well–around a closed loop. The rotation of a vector upon completion of transport around a loop defines the Guassian curvature, K, of the surface of transport. This provides a good measurement of the curvature of a 2-dimensional surface, but a different measure of curvature is needed in higher dimensions. To see this we examine how many independent 2-dimensional planes occupy a given surface. In 2-dimensions, the answer is clearly one. As such there is only one independent component of the curvature of the surface. In 3-dimensions we find 3 distinct planes for transport of vectors which, in principle, transform a vector differently. This is encoded in the Riemann curvature tensor, $R^{\mu\nu}{}_{\alpha\beta}$. However, in an Einsteinian space, such as RC, the twice contracted Riemann curvature tensor, $R^{\mu\nu}{}_{\mu\nu} = R$, is proportional to the Guassian curvature. Therefore, the parallel transport of a unit vector, $\hat{\vec{r}}$, around a closed loop with enclosed area, A_{loop} in an Einsteinian space is given by

$$^{(n)}R = n(n-1) \cdot {}^{(n)}K = \frac{n(n-1)}{A_{loop}} \cdot \delta\hat{r} \tag{3.2}$$

where ${}^{(n)}R$ is also called the Riemann scalar curvature. The factor of $n(n-1)$ comes about from the number of ways one can embed a plane into a n-dimensional spacetime. For undirected 2-dimensional planes one can choose $\binom{n}{2} = \frac{n(n-1)}{2}$ unique orientations in a n-dimensional manifold. When we assign orientation to the loop we obtain the total $n(n-1)$ ways to embed the oriented 2-dimensional plane into a n-dimensional spacetime.

Moreover, the curvature of RC, when present, appears in a unique way. First, parallel transport of a vector around a hinge, h, of the simplicial lattice comes back untransformed. Only when a vector is transported in a closed loop which lies in the plane orthogonal to h does the vector ordinarily come back rotated. How much does it rotate when transported around the loop? This is where RC is especially simple as a geometric theory. Only a vector with components in the plane of h and parallel transported around a closed loop orthogonal to h will ordinarily come back rotated and by an amount equal to the deficit of the rotated vector from 2π. To see this we take the hinge, h, and all $d-1$ facets hinging on h. In general, these will form a curved collection of $(d-1)$-simplexes. If we flatten them out by breaking apart the collection at a facet, then the collection will split by an amount equal to the deficit angle associated with h. This deficit angle is defined as the deficit from 2π of the sum of the angles between the $(d-1)$-dimensional faces hinging on h;

$$\epsilon_h = 2\pi - \sum_i \theta_i \tag{3.3}$$

When the vector is transported around the loop encircling h, the vector will accumulate an observed net rotation of $2\pi - \sum \theta_i$. This definition of curvature implies that the curvature associated with h is a conic singularity. This occurs since any loop orthogonal to h–no matter how vanishingly small–will result in the same rotation of the vector. Clearly this results in a curvature which can be as large as one chooses despite the well-definined notion of rotation of the vector. One possibility is to define an area that is more naturally associated with h than any other hinge, the circumcentric dual polygon, h^*. Using the dual area, A^*_h, as the loop of parallel transport we obtain the Regge expression for curvature associated with h;

$${}^{(d)}R_h = d(d-1) \cdot {}^{(d)}K_h = \frac{d(d-1)}{A^*_h} \cdot \epsilon_h. \tag{3.4}$$

We see immediately that the Riemann scalar curvature depends on quantities that originate from both the simplicial and dual lattices. As an example, the hinges in 4-dimensions are the triangle faces of the simplicial lattice. The deficit angle associated with the hinge h is defined with respect

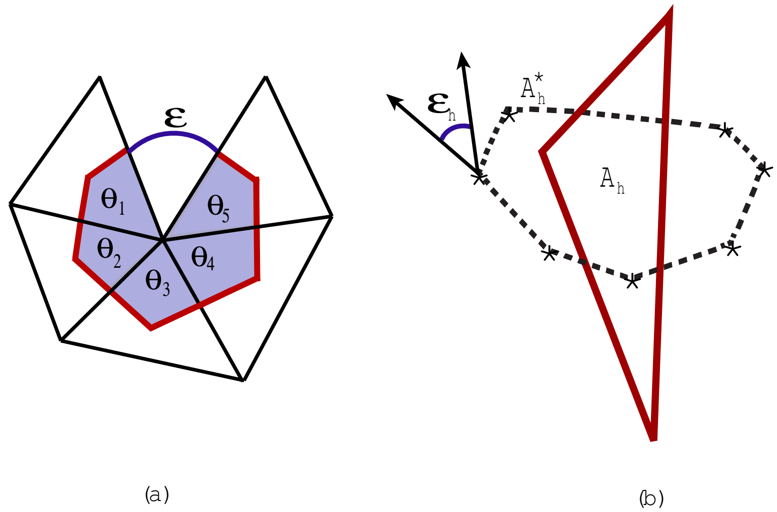

Figure 3.1: *Deficit Angles in Simplicial Geometry:* A vector parallel transported in the plane orthogonal to the (d-2)-dimensional hinge will ordinarily come back rotated by an amount equal to the deficit angle associated with the hinge. The deficit angle is defined to be the deficit of the sum of the angles on the hinge from 2π. In (a) we see the 2-dimensional case where the hinge is vertex and the θ_i are the angles between any adjacent edges containing h. In (b) we see the triangle hinge for a 4-dimensional simplicial lattice. The vector parallel transported around h^* will still come back rotated by an amount equal to ϵ_h.

to the hyperdihedral angles between tetrahedra containing h. The area dual to h, A^*_h, is defined as the area of the polygon connecting the circumcenters of the tetrahedra containing h. Parallel transport of the vector around the polygonal loop dual to h brings the vector rotated by ϵ_h as in Figure 3.1.

To turn this collection of simplexes into a theory of gravitation, we also define the Hilbert action of RC.[67] In Eq. (3.4) we have the Riemann scalar curvature associated with a given hinge of the simplicial lattice in terms of geometric quantities. This invariant measure of the geometry of a manifold plays a vital role in the definition of dynamics in GR. The continuum Hilbert action for GR is given by the integral of the scalar curvature over the manifold. Therefore, the Regge-Einstein-Hilbert action is thus given by the sum over the scalar curvature associated with each hinge in the simplicial lattice times an appropriate 4-volume for the hinge. How does one define such a 4-volume? One way is to define a new tessellation of the manifold using a hybrid between the circumcentric dual and simplicial lattices. The hyrbid cell is defined by connecting the vertices of the hinge to each vertex of the dual area and the volume is given by the n-dimensional version

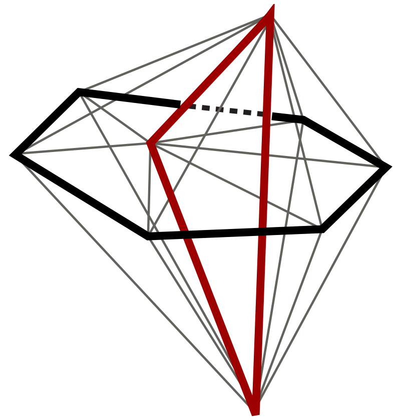

Figure 3.2: *Hybrid Building Blocks in RC:* The RC hybrid block that is essential to the construction of the Regge-Einstein-Hilbert action of Eq. (3.6) is constructed from elements of the simplicial and circumcentric dual lattices. The main construction of these cells comes from connecting the vertexes of the triangle hinge, h–the bold triangle above–with the vertexes of the dual to the hinge, h^*–the bold polygonal region above. These cells completely tile the spacetime and retain the rigidity inherited from the simplicial lattice.

of $\frac{1}{2}$base $\times$ height;

$$V_{hybrid,\,h} = \frac{2}{n(n-1)} A_h A_h^* \tag{3.5}$$

as is illustrated in Figure 3.2. Using the hyrbid volume and the scalar curvature for each hinge, we obtain the action for RC

$$I_{Regge} = \frac{1}{16\pi G} \sum_h R_h V_{hybrid,\,h} = \frac{1}{16\pi G} \sum_h \frac{n(n-1)}{A_h^*} \epsilon_h \cdot \frac{2}{n(n-1)} A_h A_h^* = \frac{1}{8\pi G} \sum_h \epsilon_h A_h. \tag{3.6}$$

One apparent result from this construction is that all reference to the circumcentric dual lattice is cancelled precisely because of the orthogonal decomposition of the 4-volume associated with a hinge. Thus we find that the circumcentric dual is an important tool in understanding the structure of RC, but it does not make its appearance explicit when studying the dynamics of the vacuum theory.

Field equations for gravitation can be obtained by varying the action with respect to the fundamental variables of the theory. In the continuum, the action can be varied with respect to the metric components of the tetrad variables to get Einstein's field equations

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}. \tag{3.7}$$

Since edge lengths of the Regge lattice replace the metric components as fundamental variables, we take the variation of the Regge action with respect to the edge lengths.

$$\delta I_R = \sum_{hinges,\, h} \delta(A_h)\epsilon_h + \sum_{hinges,\, h} A_h \cancelto{0}{\delta\epsilon_h} = 0 \tag{3.8}$$

However Regge showed that the variation of the deficit angle is exactly zero.[42] Therefore, only the first term remains and field equations result from only varying the areas of the triangle hinges. Varying the area with respect to an edge, L, returns an expression that depends on L and the interior angle opposite L;

$$\frac{\delta I_R}{\delta L} = \sum_{hinges,\ h} \frac{\delta A_h}{\delta L}\epsilon_h = \sum_{h \supset L} \frac{1}{2} L \cot(\theta_{hL})\epsilon_h = 0 \tag{3.9}$$

where the sum is taken over each triangular hinge h sharing edge L and θ_{Lh} is the interior angle of h opposite L. This results in the vacuum form of the Regge-Einstein equations. Comparison with the continuum, we preliminarily identify the expression above as an analog of the Einstein tensor for RC. In the next section we will provide geometric derivation of the Einstein tensor in the simplicial lattice as further evidence for this identification. However, we notice that so far there is no connection with matter. The stated goal of this thesis is to make such a link and identify an appropriate form of the stress-energy tensor such that

$$\sum_{h \supset L} \frac{1}{2} L \cot(\theta_{hL})\epsilon_h = \kappa \begin{pmatrix} \text{Flow of} \\ \text{Stress} - \text{Energy} \end{pmatrix} \tag{3.10}$$

As such, we will follow the geometric construction of the Einstein-Regge tensor with an analysis of its symmetry properties. This will provide the necessary link to understanding how stress-energy flows in the simplicial lattice such that the coupling of source to field protects conservation of source.

3.3 The Regge-Einstein tensor and the Cartan moment of rotation

To begin our geometric derivation we follow E. Cartan and examine the moment of rotation trivector. It is well known[69] that the dual of the moment of rotation trivector generates the Einstein tensor. Since vacuum GR is defined by setting the Einstein tensor to zero, the dual moment of rotation also generates the vacuum Regge equations. The Cartan moment-of-rotation trivector is defined through a moment arm, dP, reaching from a fulcrum to a rotation bivector. Each triangle hinge, h, in the simplicial spacetime has an associated rotation bivector, $\mathcal{R}_h$, located at the circumcenter, C, of the hinge h. The orientation of $\mathcal{R}_h$ is in a 2-plane, h^*, orthogonal to the hinge, h. The bivector is formed by two unit vectors separated by the usual RC deficit angle, ϵ_h.

It is convenient to locate the fulcrum at one of the two endpoints of edge L. We denote this fulcrum vertex as V, and by construction it is one of the vertexes of hinge h. This freedom of choice is guaranteed by the ordinary Bianchi identity, as we show below. This is in contrast to previous derivations of the Regge equations using the Cartan approach, where the fulcrum was taken halfway along edge L.[66;68;70]

With the fulcrum at V we can decompose the moment arm associated with hinge h into two vectors (Fig. 3.3),

$$(\text{Moment Arm})_{Lh} = P_L + dP_{Lh} \tag{3.11}$$

where $P_L = \frac{1}{2}\mathbf{L}$ is the vector from the fulcrum V to point O, located at the center of edge L. This is also the center of three-dimensional circumcentric polyhedron L^*, defined to be dual to edge L. The other component of the moment arm, dP_{Lh}, is the vector from O to the circumcenter C of the hinge. This gives us two vectorial contributions to the moment arm: one (P_L) is common to all 2-dimensional faces h^* of the dual polyhedron L^*, and another (dP_{Lh}) is distinct for each of these 2-dimensional faces. The contribution common to all faces of L^* can be factored out of the sum of moments of rotations, so that

$$\sum_{h \supset L} (\mathcal{P}_L + d\mathcal{P}_{Lh}) \wedge \mathcal{R}_h = \underbrace{\mathcal{P}_L}_{\text{common}} \wedge \sum_{h \supset L} \mathcal{R}_h \; + \; \sum_{h \supset L} d\mathcal{P}_{Lh} \wedge \mathcal{R}_h. \tag{3.12}$$

The resulting sum over all rotations around L^* is simply the ordinary Bianchi identity for RC,[42;66]

$$\sum_{h \supset L} \mathcal{R}_h = \mathcal{O}(L^2). \tag{3.13}$$

In Eqs. (3.12-3.13) the sum over the hinges, h_L, sharing edge L could have equally been taken over the bounding polygons, h^*, of the dual polygon V_L^*. There is a one-to-one correspondence between the h and h^*.

Using the approximate ordinary Bianchi identity[42] we are justified in removing the common contribution to the moment arm in our sum over the moments-of-rotation. We see that the ordinary Bianchi identity allows us to freely choose the position of the fulcrum. A natural choice for the fulcrum is the vertex V, and we use V^* to denote the dual 4-polytope to this vertex. Then each edge L emanating from vertex V has the moment arm $P_L = \frac{1}{2}\mathbf{L}$, which is directed along edge L. Since each edge L is dual to a corresponding 3-polytope L^*, the effective moment arm is

$$\begin{pmatrix} \text{Effective} \\ \text{Moment Arm} \end{pmatrix} = dP_{Lh} = \frac{1}{2}\, L\, \cot\theta_{Lh}\, \hat{\mathbf{n}}, \tag{3.14}$$

which is the segment from O to the circumcenter, C, of the hinge as in Figure 3.3.

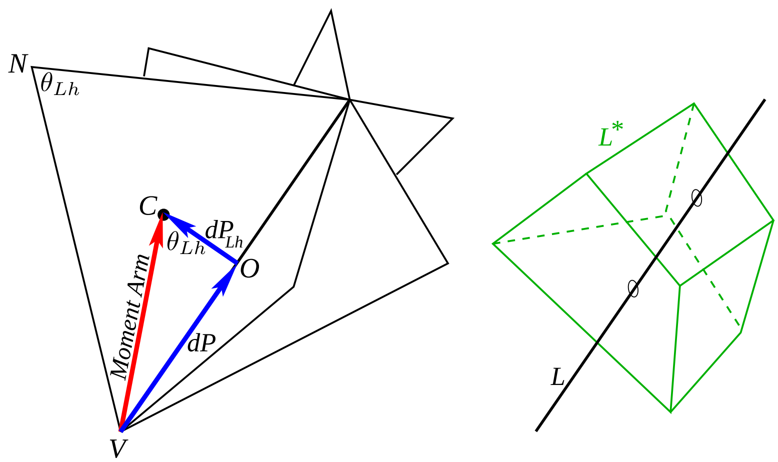

Figure 3.3: *Hinges and the Moment Arm:* In the simplicial lattice each edge is common to multiple hinges h (left). The circumcentric 3-volume L^* dual to edge L has 2-dimensional boundaries dual to each of the hinges h (right). The parallel transport of a vector around the perimeter of these dual areas will result in a net rotation by an angle equal to the deficit angle, ϵ_h, associated with the hinge, h. The moment of rotation is given by a moment arm $P_L + dP_{Lh}$ wedge the rotation associated with the parallel transport around the dual area. However, the first term does not contribute as it is equal to zero by the ordinary Bianchi identity. On a given hinge, the effective moment arm is the vector from the edge to the center of rotation, i.e. the circumcenter of the hinge C, which has length $(1/2)L\cot\theta_{Lh}$.

We are now in a position to explicitly reconstruct the Regge equation associated to an edge L, and to construct the corresponding Regge-Einstein tensor. To define the moment of rotation trivector associated to hinge h and edge L we need both the moment arm and the rotation bivector. Parallel transport of a unit vector around t)he 2-dimensional face h^* dual to the hinge h returns a unit vector rotated by an amount equal to the deficit angle, ϵ_h, associated with the hinge. Furthermore, the rotation bivector lies in the plane h^*, perpendicular to the hinge.

The dual of the Einstein tensor is expressible in terms of the moment of rotation trivector,[69]

$$\int_{V^*} {}^*\mathbf{G} = \int_{\partial \mathbf{V}^*} \star \left(\mathbf{dP} \wedge \mathcal{R}\right) = \mathbf{0}, \tag{3.15}$$

where the Hodge dual only acts in the space of values, i.e. on the moment of rotation trivector. In RC the moment of rotation trivector consists entirely of the parallelepiped formed by the moment arm dP_{Lh} and the two vectors defining the rotation bivector,

$$\begin{pmatrix} \text{Bivector} \\ \text{dual to} \\ \text{hinge } h \end{pmatrix} = \mathcal{R}_h = \frac{{}^*\left(\mathbf{L} \wedge \overrightarrow{VN}\right)}{2A_h}\,\epsilon_h, \tag{3.16}$$

which lies in the plane orthogonal to the triangular hinge h. This hinge is defined by the vectors $\mathbf{L}$ and $\overrightarrow{VN}$, and has area A_h. The star dual of the parallelepiped returns a vector of length ϵ_h and parallel to edge $\mathbf{L}$.

We can now construct the moment-of-rotation trivector. The dual moment of rotation associated with a hinge h containing the edge L is

$$\begin{pmatrix} \text{Dual Moment} \\ \text{of Rotation} \\ \text{for hinge } h \end{pmatrix}_L = \star\left(\mathrm{d}P_{Lh} \wedge R_h\right) \quad \longrightarrow \quad \underbrace{\frac{1}{2}\,\mathbf{L}\cot\theta_{Lh}}_{\text{Moment Arm}}\underbrace{\epsilon_h}_{\text{Rot'n}}. \tag{3.17}$$

The *total* dual moment of rotation over the Voronoi 3-volume L^* is then found by adding contributions from all hinges which share the edge L,

$$\int_{\partial \mathcal{V}^*} \star\left(\mathrm{d}P_{Lh} \wedge R_h\right) \quad \longrightarrow \quad \frac{1}{2}\sum_{h \supset L} \mathbf{L}\,\cot\theta_{Lh}\,\epsilon_h. \tag{3.18}$$

In the Cartan description of Einstein's theory[26;69;71] the Einstein tensor associated with a three dimensional region is the dual of the total moment of rotation tri-vector per unit three-volume. The two components of the Einstein tensor describe the orientation of the three volume, and the orientation of the moment of the rotation tri-vector. In RC there is one equation per edge L, as can be seen when the Cartan moment of rotation is calculated over the Voronoi three-volume L^*.[66] The orthogonality between the simplicial (Delaunay) lattice and its circumcentric dual (Voronoi lattice) yields an Einstein tensor which is doubly projected along edge L. That is,

$$\begin{pmatrix} \text{Integrated Einstein} \\ \text{Tensor associated} \\ \text{with edge } L \end{pmatrix} = \int_{V^*} {}^\star G \quad \longrightarrow \quad \frac{G_{LL}\; L^*}{L}\,\mathbf{L}, \tag{3.19}$$

which is directed along edge L and has magnitue $G_{LL}L^*$.

Combining Eqs. (3.15), (3.18) and (3.19) establishes the relationship between the Regge equations and the *integrated* simplicial Einstein equations,[66]

$$G_{LL}\, L^* = \frac{1}{2} \sum_{h \supset L} L\, \cot\theta_{Lh}\, \epsilon_h. \tag{3.20}$$

This effectively defines the simplicial Einstein tensor G_{LL} at edge L.

Finally, we note that the simplicial Einstein tensor along the edge L, constructed using the sum of moments of rotations for the dual 3-volume L^*, is simply the geometric portion of the familiar Regge equation.

3.4 Contracted Bianchi Identity in Discrete Geometry

Symmetries play a crucial role in understanding the physics behind our models for dynamical fields as they identify invariants and constraints of the theory. In GR, we know of no higher symmetry in spacetime than that of the contracted Bianchi identity (CBI). The CBI is the source of diffeomorphism invariance inherent in the gravitational field and the spacetime manifestation of conservation of stress-energy. This guarantees that the source of the gravitational field–material and bosonic fields–are automatically conserved. Recall that the driving aim of this thesis is that spacetime and matter are not wholly separable in fundamental theory of gravity. Rather we must view them as intimately linked. Thus, we will develop here the contracted Bianchi identity for lattice spacetime.

We know no shorter route to derive the contracted Bianchi identity than by using the topological tautology that the boundary of a boundary is zero, i.e. the boundary-of-a-boundary principle (BBP). The BBP appears twice over in each of nature's four fundamental interactions, once in its 1-2-3-dimensional form, and once in its 2-3-4-dimensional form.[26;69]

The BBP has been used in RC to obtain a discrete version of the contracted Bianchi identity.[66;72–74] However,the interpretation of this conservation law has been a source of some debate, particularly over the exactness of the identity. While the topological principle itself is exact and thoroughly studied in RC,[74] the transition from a continuum to a discrete spacetime forces one to apply these topological identities to non-infinitesimal rotations. Unlike the infinitesimal rotation operators in the continuum, finite rotations do not ordinarily commute. The transition from the continuum to the discrete case must be handled with care. We emphasize that the derivation presented here will not ordinarily produce an exact identity, due to the non-commuting nature of finite rotations. Nevertheless what one loses in exactness one gains in simplicity. In particular, the integrated Einstein tensor is doubly projected along its edge and this allows one to write down the CBI as a Kirchhoff-like conservation principle. These identities are second-order convergent[66] and valid locally at any event in a spacetime.

The contracted Bianchi identity for RC has clear implications for the coupling of energy-momentum to the lattice as well as to our understanding of diffeomorphism invariance in RC. Furthermore, if we expect the quantization of spacetime to produce an inherently discrete spacetime, then grasping the meaning of the BBP in a discrete theory becomes essential to understanding the quantum theory of gravity. RC serves naturally as an underlying framework since simplicial spacetimes provide one of the most elegant and universal descriptions of discrete spacetime.[75]

In §3.4.1 we review the BBP and its role in the fundamental forces of nature. The importance of this identity stems from its purely topological foundation. The Cartan construction of the moment of rotation trivector in RC is reviewed before applying the BBP directly to the simplicial lattice in §3.4.2. We conclude in §3.5 with our future plans to couple a generic stress-energy tensor to the geometric content of the Regge lattice.

3.4.1 Boundary of a Boundary Principle: The Guiding Topological Principle

In any fundamental field theory (electrodynamics, Yang − Mills, general relativity) source conservation is introduced in such a way that it is satisfied for any field. This is equivalent to saying that it does not impose any restrictions on the field itself, but rather puts constraints on the source of the field (charge in electrodynamics, energy–momentum in general relativity). This feature is conditioned only by the requirement that the field is described as the curvature of a connection on the appropriate vector bundle that is responsible for the correct implementation of the field symmetries.[76]

The universality of this feature follows from the fact that it is induced by (but not totally reduced to) the simple topological identity that the boundary-of-a-boundary is equal to zero.[71]

Application of this principle to spacetime is achieved by associating to it a chain complex (say by simplicial or any other triangulation) with the standard boundary operator based on the rules of orientations of the boundaries. As an example, we can examine a discrete representation spacetime wherein the spacetime geometry is tiled by 4-dimensional polytopes. The geometry interior to each of these infinitesimal polytopes is irrelevant and, for pictorial representations, can be thought of as flat Minkowski geometry. Let us examine one of these polytopes, V^*, which is the local neighborhood of an event, V. This polytope is bounded by 3-dimensional polyhedra (Figure 3.4). Any two adjacent polyhedra on the boundary of V^* share a common 2-dimensional face. In other words, in this 4-dimensional region of spacetime, no 2-dimensional polygonal faces are exposed. In general relativity, any flow of stress-energy (or equivalently the dual of the Cartan moment of rotation) into one of the 3-dimensional bounding polyhedra is exactly compensated by an equal flow of stress-energy (Cartan moment of rotation) out of an adjacent polyhedron. This guarantees conservation of source in V^*.

For each V^*, one would like to sum over each of its unexposed 2-dimensional boundaries – the meeting place of two of the polyhedral boundaries of V^* (Figure 3.5). These polyhedral boundaries induce opposing orientations on each of the 2-dimensional faces. Therefore when one sums over all of the 2-boundaries of all the 3-boundaries, two contributions are found for each polygon each of equal magnitude but opposite orientations. These identically cancel one another leaving the boundary-of-a-boundary identically equal to zero.

In applications to continuum field theories the boundary of a boundary relation of the chain complex is translated into the relation co-boundary of a co-boundary of the dual de Rham co-chain complex of differential forms. The exterior derivative acts as the co-boundary operator.[71] This duality is established by adding rotations caused by parallel translations of vectors around the 2-faces (or Cartan moments of these rotations) of an infinitesimal 3-simplex (or 4-simplex for moments of rotations). These rotations are expressed as products of Riemannian curvature tensors on each face and the oriented element of area associated with the face. This operation, when applied to finite structures, is ambiguous and cannot be performed in a consistent way. The ambiguity arises for two reasons: finite rotations do not commute, and tensor quantities are being computed at different points (on different faces) and then added. These difficulties disappear in the infinitesimal limit.

In particular, in general relativity, application of 1-2-3 dimensional BBP reduces to computing the sum of rotations caused by parallel transport around all 2-faces of a 3-simplex. When expanded in Taylor series with respect to displacements along the edges[69] it produces terms of

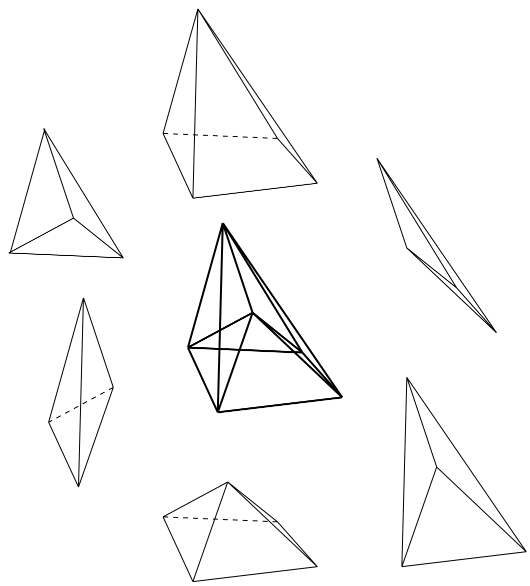

Figure 3.4: *The Polyhedral Boundary of a 4-polytope:* This illustration shows the 2-dimensional projection of a typical 4-dimensional polytope, V^*, of the circumcentric dual (Voronoi) spacetime. It is dual to a vertex, V, of the simplicial (Delaunay) spacetime. This 4-polytope is bounded by six polygons (shown exploded off into the perimeter of the polytope). These 4-polytopes are ordinarily not a 4-simplex nor are their bounding polyhedra. The orientation of V^* induces an orientation on each of its polyhedral faces, L^*. The orientation of each polyhedron consequently induces an orientation on each its polygonal faces. However, each 2-face is shared by two polyhedra thereby inducing equal and opposite orientations on it. In this sense, none of these polygonal faces are exposed and their orientations cancel. This is the origin of the BBP principle in its 2-3-4 dimensional form.

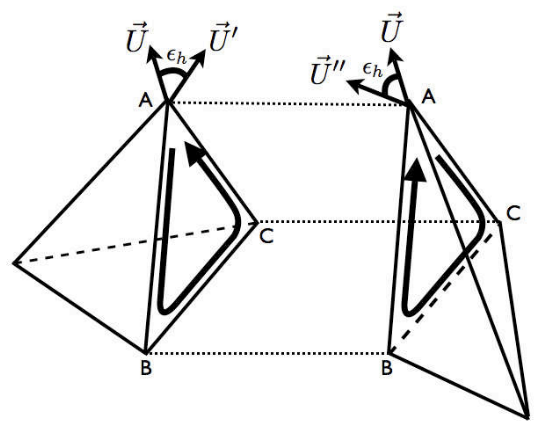

Figure 3.5: *BBP as a Geometric Identity:* Here two adjoining faces of the 3-dimensional boundary of a 4-dimensional volume are depicted with their induced orientation. The orientation of the 2-dimensional area is seen to be opposite for the adjoining 3-volumes such that in the sum over the boundary of the boundary these areas cancel one another. Furthermore, if a vector $\vec{U}$ is parallel transported around the area adjoining the two 3-volumes, then the vector will ordinarily come back rotated. When the area is associated with the left 3-volume, the vector $\vec{U}$ comes back rotated as $\vec{U}'$, but when the area is associated with the right 3-volume it will come back as $\vec{U}''$. The rotation in both cases is in the same plane and rotated by the same amount, but in opposite directions of rotation.

second and the third order (the higher orders are of no interest in this computation). The ambiguity caused by parallel transporting tensor quantities to a common point (necessary for addition) introduces errors of the fourth and higher order. These errors can be neglected. The requirement that the second order term vanishes leads to the conclusion that the Riemannian curvatures on the faces of the 3-simplex are not linearly independent, while the requirement that the third order term vanishes implies the ordinary Bianchi identities.

Likewise, application of 2-3-4 dimensional BBP amounts to adding Cartan moments of rotation over all 3-faces of a 4-simplex (or polytope). The Taylor series expansion proceeds as before, with terms up to the third order disappearing because of relations imposed by the 1-2-3 dimensional form of the BBP. The errors generated by the ambiguity of parallel translation are now of the fifth and higher order. The CBI arises from the fourth order term of the expansion.

3.4.2 Discrete Bianchi Identities

RC is based on a lattice of flat 4-dimensional simplexes that form a curved PL manifold. The curvature is concentrated as conical singularities on each of the co-dimension 2 triangular hinges.

To define the curvature we need to associate an "area-of-circumnavigation" to each triangular hinge. This provides a finite area over which we can distribute the curvature. Fortunately, the circumcentric dual lattice has been shown to arise naturally in RC and provides an appropriate area.[58–64;67;68;70;77] Correspondingly, it has been shown that the circumcentric 3-volumes naturally define the moment-of-rotation operators and discrete RC equations.[66] We postulate here that the circumcentric 4-volume (V^*) dual to a vertex (V) defines a natural domain to apply the Cartan BBP in its 2-3-4-dimensional form. Consequently the BBP in RC becomes the "co-boundary of the co-boundary" principle, although the geometric underpinnings are exactly the same.

In this section we derive the discrete form of the contracted Bianchi identity. We begin by emphasizing the central role that the Cartan moment-of-rotation trivector and the circumcentric dual lattice play in this derivation. In particular, we begin by re-expressing the familiar Regge equations in the Cartan prescription. This leads naturally to a Kirchhoff-like identity at each vertex inherently linked with the topological boundary-of-a-boundary identity. We will conclude with an analysis of the convergence properties of the identity with the typical lattice edge length L.

3.4.3 The Contracted Bianchi Identity

As shown above, the Regge equation and corresponding simplicial Einstein (or Regge-Einstein) tensor is naturally defined relative to the dual polyhedron L^*. It is therefore natural to define the 4-dimensional polytope V^* dual to the vertex V as the domain upon which we apply the Cartan BBP in its 2-3-4 dimensional form. To demand no net creation of source ($\nabla \cdot G = \nabla \cdot T = 0$) in this spacetime region is to embody the essence of the contracted Bianchi identity.

As in the continuum (Sec. 3.4.1), we can provide a finite sum ("integral") representation of the contracted Bianchi identity associated to the dual polytope V^* by summing over its polyhedral 3-boundaries L^*. By construction, each polyhedral 3-boundary of the dual polytope L^* is dual to one of the edges L of the simplicial lattice emanating from vertex V. This defines the domain of integration for the BBP. This completes two steps towards the BBP in RC, defining both the domain and the integrand (Sec. 3.3) for the BBP in RC.

The final step in deriving an expression for the conversation of moment of rotation in RC is achieved by summing over the dual 3-volumes L^* that bound the 4-volume V^*, which is dual to vertex V. However, care must be taken in evaluating this sum. Despite our choice of a common fulcrum, we have still decomposed the total moment arm into two vectors (one strictly in the tangent space of the vertex V, and one at the center of edge L). Yet, RC provides a simple solution to what could be problematic. Both of these vectors lie in the tangent space of their associated hinge h. As such, the decomposition of the total moment arm becomes the standard decomposition of a vector in flat Minkowski spacetime. Summation over terms at a common point can be achieved in two equivalent, but separate approaches: (1) by parallel transporting the effective moment arm prior to inclusion in the moment of rotation trivector, or (2) parallel transporting the net moment of rotation trivector. We will consider the second approach.

In RC the integrated Einstein tensor (3.20) is not only evaluated along the edge L, it is also directed along $\mathbf{L}$. Since each simplicial edge L is by definition a geodesic in the lattice, any vector parallel transported along $\mathbf{L}$ will maintain a constant angle with respect to $\mathbf{L}$. We take advantage of this property and individually transport each of the RC moment-of-rotation trivectors from the center of their respective edges to the vertex V which is common to all of these edges. We are then free to sum these moment-of-rotation vectors at V. Repeating this procedure across the lattice yields a 4-vector identity at each and every vertex V,

$$\begin{pmatrix} \text{Net Moment} \\ \text{of Rotation} \\ \text{at vertex } V \end{pmatrix} = \sum_{L \supset V} \sum_{h \supset L} \frac{1}{2} \mathbf{L} \cot\left(\theta_{Lh}\right) \epsilon_h. \tag{3.21}$$

This is the simplicial form of the net moment of rotation at vertex V, and must vanish by the 2-3-4 dimensional form of the BBP. However, as we have mentioned, the finite rotation operators do not ordinarily commute. This is important because we must apply our rotations in a given order. Nevertheless, the non-commutativity of the rotation operators can be made as small as one wishes by suitably refining the lengths of the simplicial lattice. Here, suitable refinement of the lattice is taken in the sense described in[78] where constant curvature barycentric subdivision is employed to refine the edge lengths by introducing new simplicial blocks and distributing curvature over the new subdivision of the simplexes. Under such refinements, the commutators for rotations scale as the deficit angles squared. Moreover, the deficit angles scale as the edge length squared as can be seen via their relation to the curvature (Curvature) $= K = \epsilon_h / A_h^*$. Consequently, the deficit angles scale as $\mathcal{O}(L^2)$ and the commutators for rotations scale as $\mathcal{O}(L^4)$. This second order convergence is the origin of the approximation being implemented. Therefore,

$$\underbrace{\sum_{L\supset V}\ \sum_{h\supset L}}_{\delta\circ\delta\equiv 0}\ \frac{1}{2}\mathbf{L}\,\cot\left(\theta_{Lh}\right)\epsilon_h + \mathcal{O}(L^5) = 0. \tag{3.22}$$

This is the RC formulation of the CBI. The first term in this expression scales with $\mathcal{O}(L^4)$, since the deficit angles ϵ_h scale as $\mathcal{O}(L^2)$. The contracted Bianchi identity is not identically zero because small, finite rotations do not necessarily commute. Consequently, the final term scales with both the edge length L and the rotation commutator $[\epsilon_h, \epsilon_{h'}]$, yielding an overall $\mathcal{O}(L^5)$ behaviour in the error term.

Two features are apparent in the discrete CBI. First, it has the form of a Kirchhoff-like conservation law. The analysis presented in this manuscript completes the derivation of this Kirchhoff-like property of the CBI. We have reduced the results of previous calculations from a non-local, boundary-valued sum to a vertex-based conservation equation. This was accomplished by utilizing the freedom we have in choosing the fulcrum for each of the moment of rotation trivectors, here we have chosen the vertex V common to all of the faces of the dual polytope V^*. Equivalently, this can be understood by our ability to parallel transport the moment of rotation trivector from the midpoint $\mathcal{O}_i$ the edge L_i to the vertex V. No higher order corrections than corrections already discussed in this manuscript are introduced. It is this understanding of the relation between the CBI give rise to conservation of source which is vital to understanding how source can be coupled to the lattice. Second, the appearance of a 4-vector identity at each vertex signals that there are exactly four "approximate" diffeomorphic degrees of freedom per vertex in the simplicial lattice. This last point has been important for understanding the dynamical degrees of freedom in RC,[66;72;73] and the resulting approximate diffeomorphism freedom has been utilized to solve the initial value problem.[79]

3.5 Regge Calculus Beyond the Vacuum

In applications to both classical and quantum gravity RC has, for the most part, been dominated by studies of the vacuum solutions. There have been some clear departures from this to study the coupling of source to arbitrary lattices,[80–82] although these methods have not provided a generic approach or understanding to incorporating non-gravitational fields into the lattice. Nor has there previously been developed a generic framework for the formulation of conservation of

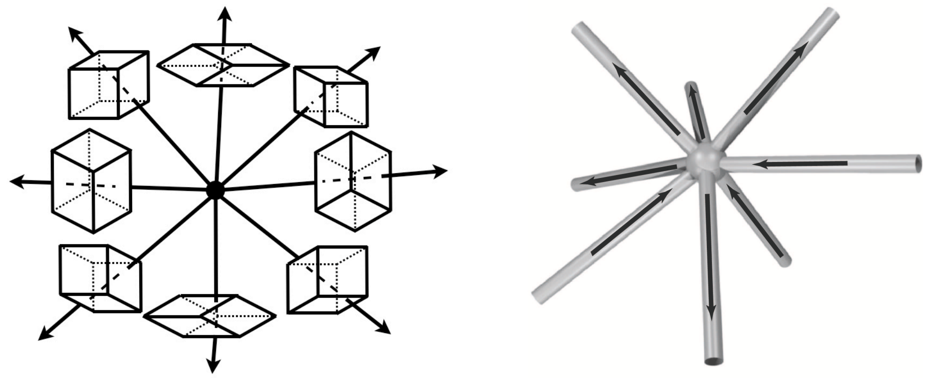

Figure 3.6: *The Kirchhoff-like Form of the Contracted Bianchi Identity in RC:* On the left is an exploded view of the edges meeting a vertex V and their dual 3-volumes, L^*. The first step towards the Kirchhoff-like conservation principle is constructing the total moment of rotation for each of the dual 3-volumes, L^*. On the right is a depiction of the flow of moment-of-rotation, or equivalently flow of stress-energy along each of the edges, L meeting at the vertex V. The flow of moment-of-rotation entering or leaving vertex, V, is conserved to second order in the lattice spacing, L. Since the Einstein equations, and their RC equivalent, equates the moment-of-rotation with the stress-energy, this contracted Bianchi identity can be viewed as a circuit-like conservation law. Here the "wires" of the circuit are the edges of the simplicial lattice, and the "current" in each of the wires is the doubly-projected stress-energy tensor T_{LL} along the given edge $\mathbf{L}$ emanating from vertex V.

source in the lattice. We chose to first study the geometric properties of the Einstein tensor so as to first fulfill conservation of source wired to field in RC. We know of no better way to study these properties than through the topological identity embodied in the BBP. Applying the moment-of-rotation trivector to the framework of the BBP has allowed us to better understand how the symmetries of the Einstein tensor are manifest in the lattice of RC.

We have found that the conservation of moment-of-rotation takes a form which is ideally suited for applications to matter: the CBI in RC become a Kirchhoff-like conservation principle for the edges emanating from a specific vertex. With this and Einstein's equations, we obtain an approximate conservation equation for energy-momentum on the lattice,

$$\sum_{L\supset V} G_{LL} L^* = \sum_{L\supset V} \kappa T_{LL} L^* \cong 0. \tag{3.23}$$

In particular, the doubly projected stress-energy along the edges emanating from a vertex must sum to zero, to at least second order in the length scale of the lattice. While not exact, this gives the interpretation of a Kirchhoff-like conservation principle for the geometry, and (with Einstein's equations) the flow of energy and momentum (Figure 3.6). As a result, we obtain a set of vertex-based constraints for edge-based expressions that constrain energy-momentum. This exercise indicates that energy-momentum is naturally wired to the simplicial lattice at each vertex, and is naturally wired to each edge in its coupling with the simplicial field equations.

For applications of RC to pre-geometric quantum spacetime, one must necessarily formulate an appropriate stress-energy tensor arising from the quantum dynamics. For applications to classical spacetimes a simplicial form of the stress-energy tensor must be constructed from the non-gravitational sources. This work indicates that the stress-energy will most naturally be expressed as a vertex-based tensor, and that its coupling to the RC equations will be through its double projection on the edges of the lattice. We will see in the coming chapter how this insight presents itself in the structure of RC and how we can construct a general framework for the coupling of course to field in RC.

Chapter 4

Matter in Simplicial Spacetimes

Although we must never confound a mathematical model of nature with nature itself, it certainly appears that the modern physicist finds a more realistic representation of matter in an assembly of particles than in a continuum.[2]

–J.L. Synge

The Geometrodynamic Steering Principle introduced in §2.1.1 elucidates the role of Mach's Principle in GR and is the core of modern understanding of the dynamical degrees of freedom in GR. Any discretization of spacetime must describe the coupling of non-gravitational sources to the gravitational degrees of freedom in addition to providing a description of the discrete spacetime. Consequently, we have illustrated how naturally the contracted Bianchi identity appears in RC. Only through this topological identity applied to the Einstein-Regge tensor can the *automatic* conservation of source be ensured in lattice spacetimes. There were two key lessons one should glean from this development of the conservation of source through the contracted Bianchi identity: (1) the natural domain for Einstein's equations in RC is defined through the locally defined circumcentric dual lattice and (2) conservation of source is embedded in RC as a set of four Kirchhoff-like conservation equations, accurate to second order in the edge lengths, at each vertex of the simplicial lattice.

The formulation of non-gravitational source matching the discrete nature of RC requires the circumcentric dual tessellation. This dual tessellation provides a concrete definition for the topology over which stress-energy and its conservation are defined. Interestingly, the necessity for the dual lattice only becomes important when non-gravitaitonal sources are included in RC. In vacuum RC we found in Eq. (3.6) that all dependence on the dual lattice exactly cancelled when contributions to the scalar curvature and the 4-volume at a hinge explicitly referenced the circumcentric dual. However, we cannot in general expect this beneficial cancellation to occur when one extends RC beyond the vacuum regime or when one analyzes other geometric structures in RC, e.g. scalar curvature at a vertex,[70] contracted Bianchi identity, or matter Lagrangians. When examining inclusion of non-gravitational sources, one must explicitly reference the flow of source through domains of the spacetime, i.e. the boundary to a dual cell. Therefore, one should expect the action for non-gravitational fields to be of the form;

$$I[\cdot] = \sum_L \text{K.E.}[\cdot, L] \times V_L^* + \sum_v \mathcal{L}_{int}[\cdot] \times V_v^* \tag{4.1}$$

where K.E.$[\cdot, L]$ is the finite-differenced "kinetic energy" contribution to the action and $\mathcal{L}_{int}[\cdot]$ is the interaction term from the Lagrangian of interacting fields–to include mass terms for free-fields. Clearly, one cannot expect there to be cancellations of the dual volumes (to edges or vertexes) in an action for arbitrary fields since the dynamics of the field defined in K.E.$[\cdot, L]$ and $\mathcal{L}_{int}[\cdot]$ will not explicitly reference the circumcentric dual as was the case with the scalar curvature in Eq. (3.6). This requires that we fully understand the circumcentric dual topology prior to the incorporation of source into the Regge lattice. This will be the primary focus of §4.1 which will enable us to incorporate matter into lattice spacetimes in §4.2.

4.1 The Topology of Matter in Simplicial Spacetimes

We have found that the inclusion of non-gravitational sources requires the complex of cells dual to the simplicial lattice. Although not explicit in the vacuum Regge action the dual tessellation appears naturally in the definition of geometric observables in RC.[57–67] However, there has not been a thorough study of the topological properties of the dual lattice. The systematic avoidance of triangles and simplexes with large *fatness* parameters[78] or *waste* functions[83] is often carried out in applications of RC for ease of computation and to avoid complications in determining convergence of the triangulation. This systematically avoids many of the subtleties of using arbitrary triangulations for the inclusion of source into RC. In the general case, one should carefully examine whether the use of arbitrary simplicial lattices will provide physically appropriate topologies, such as providing disjoint neighborhoods for distinct events in spacetime. If not, it will be necessary to define a subset of all possible triangulations such that the dual tessellations will always inherit physically meaningful characteristics.

Prior to studying the explicit properties of the circumcentric dual lattice, it is useful to provide a rigorous definition of the dual tessellation. The circumcentric dual to a n-dimensional simplicial lattice is defined with respect to the circumcenters of the d-dimensional simplexes, $\sigma^{(d)}$ ($d \leq n$): the circumcenter, $c(\sigma^{(d)})$, of $\sigma^{(d)}$ is defined as the center of the d-dimensional circumsphere (circumhyperboloid) in a Riemannian (resp. pseudo-Riemannian). That is the circumcenter to each vertex is the vertex itself, to each edge the circumcenter is the midpoint of the edge, and so on. The dual to $\sigma^{(d)}$ is then defined by the duality operator[84]

$$\star(\sigma^{(d)}) = \sum_{\sigma^{(d)} \supset \sigma^{(d+1)} \supset \ldots \supset \sigma^{(n)}} \epsilon_{\sigma^{(d)} \ldots \sigma^{(n)}} [c(\sigma^{(d)}), c(\sigma^{(d+1)}), \ldots, c(\sigma^{(n)})] \tag{4.2}$$

where $[a_1, a_2, \ldots, a_{n+1}]$ is a n-dimensional simplex with vertexes given by the a_i and the $\epsilon_{\sigma^{(d)} \ldots \sigma^{(n)}}$ is a pseudo-tensor whose purpose is to maintain proper orientation of the dual cells with respect to the orientation assigned to the simplicial lattice. This definition provides a direct correspondence between the d-dimensional simplexes and their (n-d)-dimensional dual cells, e.g. dual to a n-simplex is a vertex and dual to a (n-1)-simplex is an edge.

Given this definition of the dual complex to $\sigma^{(d)}$ we are able to examine the properties of the dual topology, given by the collection $\{\star\sigma^n\}$. We seek those properties that are important for us to understand the dual topology in RC. One physically meaningful property of a topology is that the topology define disjoint open neighborhoods for distinct events in spacetime. If this were not the case then two physically distinct events in spacetime would be described by subsets of the manifold which are not altogether distinct. This provides difficulties in accurately defining physical quantities such as densities of physical observables. A topology which provides such disjoint open neighborhoods and covers the manifold is said to be a Hausdorff topology. Since RC is a theory of events in spacetime which determine the spacetime, the topology induced by the lattice should require no additional structure to form the topology of the discrete manifold. The dual topology of the simplicial lattice *defines* the mathematical topology for the points and events, i.e. vertexes of the simplicial lattice, in the spacetime. It is with respect to this topology that one assesses the properties of functions on the spacetime. RC does not rely on a background smooth manifold with respect to which one can refer to open neighborhoods of points; therefore, one must define a dual topology having the properties expected of a physically relevant spacetime topology solely from the triangulation and/or events of the PL-spacetime manifold.

If RC is to be a theory of events in spacetime with conservation of source preserved at the events (vertexes of the simplicial lattice), then the topology assigned to the discrete collection of events cannot exactly coincide with the simplicial lattice. It is only with respect to the circumcentric dual that conservation of source and the contracted Bianchi identity are well-defined. Therefore, we should expect to define circumcentric dual lattices as a Hausdorff topology. However it can easily be shown in Euclidean geometry that a simple choice in triangulations where a rectangular region

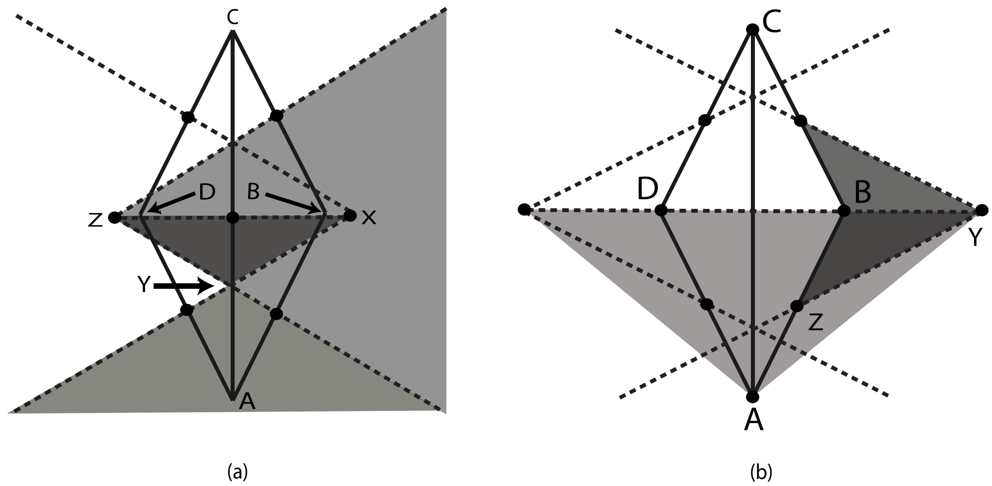

Figure 4.1: *Dual Topologies in 2-dimensions:* The dual topology to a simplicial spacetime is defined by constructing the dual cells to each vertex of the simplicial lattice according to Eq. (4.2). These dual cells do not ordinarily define a Hausdorff topology for the spacetime as can be seen above. In (a) we present a non-Hausdorff dual topology in Euclidean geometry. In (b) we present a similar example from the Minkowski plane. If we examine the dual to vertexes A and B closely, then we notice that there is an overlapping region defined by the simplex [XYZ] in (a) and [BYZ] in (b).

is triangulated along the longest diagonal will give circumcentric dual cells to the vertexes which overlap as in Figure 4.1. The situation is not averted in Minkowski spacetime as is also shown in Figure 4.1. For the purposes of a fundamental theory of nature, this cannot be explicitly acceptable as it points to regions of spacetime which, topologically, are not disjoint/distinct. Indeed, the dual topology constructed via the circumcentric dual tessellation does not explicitly form a topology either, as the dual tessellation is not closed under intersections of dual neighborhoods. Our aim for defining a dual topology that is consistently defined for an arbitrary collection of events and clearly the circumcentric dual cells do not meet this definition for an arbitrary triangulation of the events. Since one should aim for a mathematically consistent description of discrete dynamics, we cannot explicitly accept arbitrary triangulations that produce non-Hausdorff dual topologies in RC when sources are incorporated.

Since arbitrary lattices will not ordinarily induce meaningful dual topologies, we must identify a subset of all triangulations which will *always* produce Hausdorff coverings of the PL-manifold. Since triangulations and their duals are most readily and concretely understood in positive-definite metrics, we will describe such a subset in Euclidean geometry. This will be topic of §4.1.1 wherein we explain the general properties of the space of Delaunay triangulations and their dual Voronoi tessellations on $\mathbb{R}^m$. We are then able to conjecture as to the possible formulation of

Delaunay triangulations and their duals in Minkowski geometry. This will be done explicitly for the Minkowski plane, $\mathbb{M}^2$, in §4.1.2. These will provide the basis for the point-set topology for the vertex neighborhoods as we define the action and stress-energy on the simplicial lattice.

4.1.1 Delaunay Triangulations and Voronoi Tessellations

If an arbitrary lattice will not ordinarily produce a suitable dual topology, what triangulations are sufficient for the incorporation of matter in RC. It is known that there exists a collection of triangulations in $\mathbb{R}^n$ such that the circumcentric dual to the triangulation defines a unique, open neighborhood to each vertex of the triangulation such that no two neighborhoods overlap.[43] These triangulations and their duals appear in the literature as Delaunay triangulations and Voronoi tessellations, respectively. These tessellations have found various applications in physics, astronomy, sociology and other fields as evidenced by the various contributions to the *Tessellations in the Sciences.*[85] In this subsection we will outline the relevant definitions for Voronoi tessellations and Delaunay triangulations and distill out the key characteristics of Voronoi tessellations that are portable to arbitrary metrics. This will provide the necessary framework for us to generalize the Euclidean and Riemannian Voronoi tessellations to Minkowski spacetime.

The oft-mentioned meaning of the Voronoi tessellations in positive-definite metrics is that a Voronoi cell for a point $p_i \in \mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ is the set of all points in the manifold *closest* to p_i than any other member of $\mathcal{P}$;

$$V(p_i) = \{x \,|\, \|\mathbf{x} - \mathbf{x}_i\| < \|\mathbf{x} - \mathbf{x}_j\| \forall p_j \in \mathcal{P}\}. \tag{4.3}$$

This defines an open neighborhood around p_i; however, this can also be extended to closed neighborhoods, $\overline{V}(p_i)$

$$\overline{V}(p_i) = \{x \,|\, \|\mathbf{x} - \mathbf{x}_i\| \leq \|\mathbf{x} - \mathbf{x}_j\| \forall \mathcal{P}\} = V(p_0) \bigcup \partial V(p_0). \tag{4.4}$$

In each of these cases, one defines the distance between $\mathbf{x}$ and $\mathbf{x}_i$ using the standard metric on the manifold, i.e. $\|\mathbf{x} - \mathbf{x}_i\| = \sqrt{g_{\mu\nu}(x - x_i)^\mu (x - x_i)^\nu}$. The Voronoi cell can alternatively and equivalently be defined by introduce the notion of a half-space between two elements of $\mathcal{P}$. The half space between $p_i, p_j \in \mathcal{P}$ is defined as

$$H(p_i, p_j) = \{\mathbf{x} \,|\, \|\mathbf{x} - \mathbf{x}_i\| < \|\mathbf{x} - \mathbf{x}_j\|\}. \tag{4.5}$$

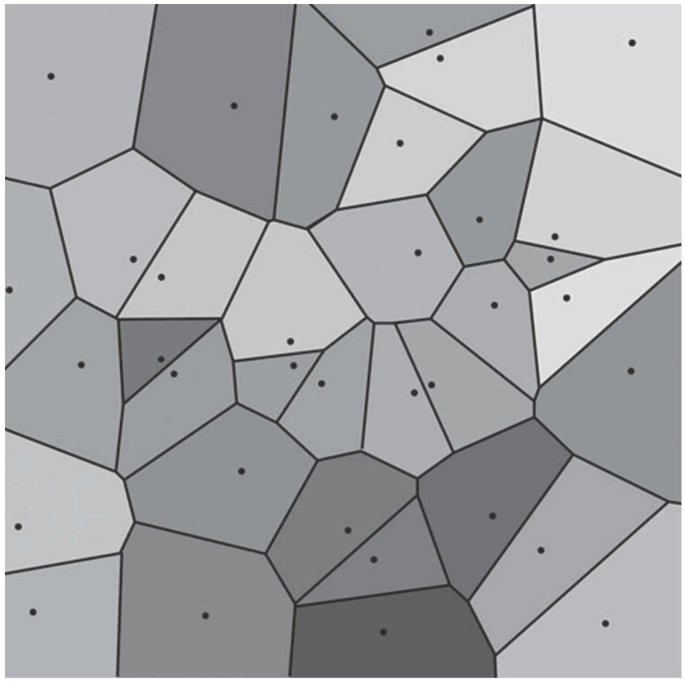

Figure 4.2: *Voronoi Tessellation in* $\mathbb{R}^2$*:* The Voronoi tessellation for a set $\mathcal{P} = \{p_1, \ldots, p_n\}$ is the collection of neighborhoods associated to each p_i. This figure illustrates the Voronoi tessellation for a randomly generated set of points $\mathcal{P}$ on the Euclidean plane. Each neighborhood, $V(p_i)$, consists of the points x which are closes to p_i than any other member of $\mathcal{P}$. As is clear from the figure, these neighborhoods form a disjoint covering of the plane. Moreover each $V(p_i)$ is guaranteed to be a convex set around the generator p_i of the neighborhood.

This transforms the Voronoi cell definition in a set of relations using half-planes, i.e.

$$V(p_i) = \bigcap_{p_j \in \mathcal{P}} H(p_i, p_j) \tag{4.6}$$

By definition, these cells define a territory to each p_i such that all points belonging to $V(p_i)$ are closer to p_i than any other $p_j \in \mathcal{P}$, as in Figure 4.2.

In addition to the remarkable feature that each cell defines a *closest* territory to each element of $\mathcal{P}$, the tessellation also induces a dual that is a triangulation of $\mathcal{P}$. This dual triangulation to the Voronoi tessellation is such that to every (n-1)-dimensional face of the Voronoi cell there is an edge connecting the p_i, $p_j \in \mathcal{P}$ that generated the boundary corresponding to the face. That is to say, given a sequence of points on a manifold, e.g. mailboxes, badger homes, spacetime events, etc., one can automatically associate a natural triangulation of the those points defined via the Voronoi tessellation. The duality relationship between the Delaunay triangulation and the Voronoi tessellation guarantees that the triangulation avoids, when possible, long, thin simplexes that do not minimize the waste function[83] or maximize the fatness parameter[78] for a given $\mathcal{P}$ in fixed position on some manifold. This can be seen more clearly if one examines the less instructive but more intuitive definition of a Delaunay triangulation: the Delaunay triangulation in $\mathcal{R}^m$ for a set $\mathcal{P}$ is the triangulation such that for any d-simplex, $\sigma^{(d)}$, the d-circumsphere for $\sigma^{(d)}$ only contains the vertexes of $\sigma^{(d)}$ on its boundary or interior.[43] This implies that any given choice of simplex will attempt to fill as much of its circumsphere as possible given the fixed locations of the elements of $\mathcal{P}$, in direct analogy with the waste function used in Null Strut Calculus.[83] However, since the elements of $\mathcal{P}$ are in fixed position, this choice is only capable of being made with respect to the incidence matrix which characterizes how members of $\mathcal{P}$ are linked together. The Delaunay triangulation does not minimize the waste function by varying edge lengths as it is assumed that the member of $\mathcal{P}$ are in fixed position.

Given these definitions, we are automatically able to select a subspace of all triangulations which will guarantee that each circumcentric dual tessellation defines a Hausdorff topology for the PL-manifold. This is guaranteed since it there is always a certain answer to the question of to which $p_i \in \mathcal{P}$ a given point on the manifold is closest for every point x on the manifold. The answer to this question is always certain as the positive-definite metric provides no ambiguity in the ordering of the distances between x and each p_i. We therefore restrict our application of RC to Delaunay triangulations as these are the only triangulations sufficiently suited to the incorporation of matter at a fundamental level. This matches with the current choices in the literature of avoiding elongated, thin simplexes, i.e. those which are not sufficiently spherical.

However, the definitions given thus far cannot carry full meaning directly in non-positive definite metrics. Only in positive-definite metrics can one put a lower-bound on the distance between two points or define a meaningful circum*sphere*. We must therefore find replacements for these notions and in particular limit our steadfast reliance on the notion of *closeness*. What, then, should be the guiding principles for Voronoi tessellations in arbitrary manifolds?

There are two essential properties of Voronoi tessellations that come directly from the aforementioned definition and preserve the character of the tessellation: (1) the closed Voronoi cells associated with $\mathcal{P}$ completely cover the manifold, $\mathcal{M}$, that is every point on $\mathcal{M}$ must be included in at least one $\overline{V}(p_i)$, and (2) the collection of Voronoi cells, $\mathcal{V} = \{V(p_i)\}$, forms a collection of disjoint open neighborhoods, i.e. no two cells overlap;

$$\begin{aligned} &(1) \bigcup_i \overline{V}_i = \mathcal{M} \\ &(2) \bigcap_i V_i = \emptyset \end{aligned} \tag{4.7}$$

These properties ensure that the set $\mathcal{V}$ forms a discrete Hausdorff topology for the manifold. Furthermore, we will also require that each p_i belong to its own Voronoi cell, $V(p_i)$, to ensure that an arbitrary disjoint covering is not created without reference to the point-set used to generate $\mathcal{V}$. We insist on one additional principle behind Voronoi tessellations: the Voronoi cell associated with p_i shall be defined with respect to geometric relations, e.g. metric distance, between p_i and all other $p_j \in \mathcal{P}$. Again, this requirement is to ensure maximal coincidence with the traditional meaning of Voronoi tessellations despite the lack of direct correspondence to the Riemannian definition. These requisite properties follow the traditional view of Voronoi diagrams for general metrics–at least from the well-studied examples with positive-definite inner-products.[43] Before moving on let us briefly state how each of properties affects the Voronoi definition for a given metric. The requirement that geometric relationships or ordering be used explicitly to construct the Voronoi cells amounts to the requirement that one be able to define half-surfaces that separate a manifold between any two given points. In the Euclidean case this is required to define the half-planes that are intersected to form the Voronoi cell. The requirement that a generator p_i belong to its own Voronoi cell can be seen as the statement on how one chooses which half-surface is associated to the generator–the geometric relations determine the boundary but do not automatically select which half-surface to assign to the generator. The Hausdorff covering conditions then provide guidance on how one pieces together the many half-planes associated with a given generator. In Euclidean geometry, these conditions indicate that one need only perform a series of intersections. In more general cases, this may not always be completely sufficient.

4.1.2 Voronoi Tessellations in Spacetime

If the Delaunay-Voronoi dual tessellations are indeed a sufficient structure for matter in RC, then our hope should be that such triangulations and their duals exist in arbitrary metrics, e.g. Lorentzian metrics. However, it has not been clear how one defines a Voronoi tessellation or Delaunay triangulation in spacetime from first principles. We now explore how we can apply the above principles of Voronoi tessellations to a random sprinkling on the Minkowski plane, $\mathbb{M}^2$. Since the Minkowski plane is only a tool to provide a simplified model of Voronoi cells in spacetime, it is necessary to only use features of $\mathbb{M}^2$ that extend to higher dimension. This way the construction will be sufficiently generic so as to allow for extension to higher dimension and possibly curved geometries.

We first note that while it is common to define a Euclidean topology on $\mathbb{M}^2$, this does not take into account the metric properties of the manifold that make $\mathbb{M}^2$ physically meaningful.[86] Instead, the Minkowski plane has everywhere defined a light cone structure that demarcates space-like, time-like and null distances. That is, we have two distinct classes of non-zero distances. It was proposed by Zeeman[86] that one could instead treat Minkowski spacetime as being described by a finer topology generated by two distinct sets of topologies–assign Euclidean topologies to each space-like surface and separately assign Euclidean topologies to each time-like direction. The motivation for Zeeman appeared to be to create a topology that accounts for the intrinsic local inhomogeneity in Minkowski spacetime, i.e. the demarcation of space- and time-like distances via the local light cone at each point. This is the primary distinction between Minkowski spacetime and Euclidean geometry that will be the focus of our construction of the spacetime Voronoi tessellations.

In addition, special relativity tells us that frames in Minkowski spacetime are physically indistinguishable from one another. Therefore, any construction of a Voronoi tessellation in space-time should also incorporate this observer independence, i.e. Lorentz invariance. This tells us that for any two time-like separated events A and B that there exists at least one observer who will observe A and B to be simultaneous. Likewise for two space-like separated events C and D there will be an observer who observes C and D to occur with equal spatial separation from the observer, i.e. occur at the same point but at different times. This property of flat spacetime should be directly incorporated into a spacetime Voronoi tessellation so as to preserve Lorentz invariance.

We now define an algorithm to construct Voronoi cells for a spinkling of points on the Minkowski plane. To simplify the exposition of this algorithm, we will only construct the Voronoi cell for a single element $p_i \in \mathcal{P}$. One can then follow the algorithm for each element of $\mathcal{P}$ independently. We first incorporate Zeeman's conception of the fine topology and ensure that our definition respects the distinction between space-like and time-like separated points. To do so, we decompose $\mathcal{P}\backslash p_i$ into two disjoint sets[1]

$$\begin{aligned} S_i[\mathcal{P}] &= \{p_j \in \mathcal{P} \mid \|\mathbf{x}_i - \mathbf{x}_j\| > 0\} \\ T_i[\mathcal{P}] &= \{p_j \in \mathcal{P} \mid \|\mathbf{x}_i - \mathbf{x}_j\| < 0\} \end{aligned} \tag{4.8}$$

where S_i contains the elements of $\mathcal{P}$ that are space-like separated from p_i and T_i contains those that are time-like separated from p_i. However, this is not sufficient to define an observer-independent Voronoi cell. Additional subsets are required to indicate events that are indistinguishable in either a space-like or time-like sense, i.e.

$$\begin{aligned} \tilde{S}_{ij}[\mathcal{P}] &= \left(S_i \bigcap T_j\right) \bigcup \{p_j\} \\ \tilde{T}_{ij}[\mathcal{P}] &= \left(T_i \bigcap S_j\right) \bigcup \{p_j\}. \end{aligned} \tag{4.9}$$

The content of these sets is immediately clear e.g. $\tilde{S}_{ij}$ ($\tilde{T}_{ij}$) contains all $p_k \in \mathcal{P}$ which are space-(time-)like separated from p_i while also time-(space-)like separated from p_j. We also add the point p_j for simplicity in the final definition of the Voronoi cell. These groupings of points allow us to ensure that the Voronoi cell for p_i will account for observers who see two time-(space-)like separated events as simultaneous (spatially equidistant) since in the temporal (spatial) topology these two events would be of equal time (equal distance and direction) from the observer passing through p_i.

Recall that a requisite property of a Voronoi cell is that the cell be dependent upon the geometric relationships between the elements of $\mathcal{P}$. This was required so that one could readily define half-surfaces that belong to one generator or another. However, due to the demarcation of distances defined in Minkowski spacetime, two notions of half-planes, $H_s(p_i, p_j)$ and $H_t(p_i, p_j)$ naturally arise;

$$\begin{aligned} H_s(p_i, p_j) &= \{x \mid \|\mathbf{x} - \mathbf{x}_i\| < \|\mathbf{x} - \mathbf{x}_j\|\} \\ H_t(p_i, p_j) &= \{x \mid \|\mathbf{x} - \mathbf{x}_i\| > \|\mathbf{x} - \mathbf{x}_j\|\}. \end{aligned} \tag{4.10}$$

These half-planes distinguish the space-(time-) like boundary between p_i and another time-(space-)like separated element of $\mathcal{P}$. It is possible to use a single half-plane definition by taking the absolute-value of the invariant metric distance; however, distinguishing between space-like and time-like half-planes provides more clarity in the final definition.

[1] We will explicitly disregard null separated elements of $\mathcal{P}$ as these produce degeneracies in the Delaunay Triangulations. In such a case where two elements of $\mathcal{P}$ are null-separated, then a choice of off-set of one of the points will break the degeneracy and the definition given here follows naturally.

The Hausdorff conditions require that we piece together these half-planes in such a way that the collection $\{V(p_i)\}$ forms a disjoint covering of $\mathbb{M}^2$. The subsets of $\mathcal{P}$ formed above provide a framework for deciding how the half-planes are incorporated into the Voronoi cell. To see how this is done, we look at the sets $\tilde{S}_{ij}$ and $\tilde{T}_{ij}$. These sets define the class of $p_k \in \mathcal{P}$ for which there is a class of observers passing through p_i containing observers who measure some pair of points in $\tilde{S}_{ij}$ ($\tilde{T}_{ij}$) to be spatially (temporally) equidistant from p_i. In order to keep $V(p_i)$ observer-independent the elements of $\tilde{S}_{ij}$ ($\tilde{T}_{ij}$) must be treated equally. This forces us to not intersect their half-planes with respect to p_i but to create a union of half-planes to define a PL boundary. This provides a path towards defining $V(p_i)$: combine half-planes for classes of points which are space-(time-)like separated from p_i but spatially (temporally) indistinguishable, but intersect distinguishable *classes* of points;

$$V_{\text{Mink.}}(p_i) \equiv \left[\bigcap_{p_j \in S_i} \left(\bigcup_{p_k \in \tilde{S}_{ij}} H_s(p_i, p_k)\right)\right] \bigcap \left[\bigcap_{p_j \in T_i} \left(\bigcup_{p_k \in \tilde{T}_{ij}} H_t(p_i, p_k)\right)\right]. \tag{4.11}$$

An example of the result of this definition is shown in Figure 4.3. Two features are directly apparent from the above definition: (1) the Euclidean definition of the Voronoi cell is obtained through the standard Wick rotation of all temporal axes to spatial axes and (2) the Minkowski Voronoi cells can in general no longer be defined as convex domains but will take on the more general geometric form of star domains. To see (1) we simply examine the effect of Wick rotation on $V(p_i)$. When the temporal axes are rotated to real spatial axes, there is no longer the clear distinction between time-like and space-like distances. This removes all elements of $\mathcal{P}$ from $T_i[\mathcal{P}]$ and creates the equivalence $S_i[\mathcal{P}] = \mathcal{P} \backslash p_i$. Moreover, since no two elements in the Wick rotated set are time-like separated we obtain the condition $\tilde{S}_{ij} = \{p_j\}$ leaving the union over a single half-space. Hence, under Wick rotation of Eq. (4.11) one recovers the standard planar Euclidean Voronoi definition:

$$V_{\text{Euclid.}}(p_i) = \bigcap_{p_j \in \mathcal{P}} H_s(p_i, p_j) \tag{4.12}$$

To understand the origin of (2) it is useful to remark on some properties of convex sets–a more complete accounting of the properties of the open sets discussed here can be found in Appendix A. First, we shall state the obvious: a half-plane in the Minkowski or Euclidean plane defines an infinite convex set. Second, the intersection of convex sets is again a convex set. In the planar Euclidean Voronoi polygon, since only intersections of half-planes are taken, the only result allowed is a convex set surrounding the generator of the polygon. However, the planar Minkowski Voronoi cells incorporate unions of half-planes which in general allow for infinite star domains with a convex kernel defined by the intersection of the half-planes. How are we guaranteed that the intersection of star domains are again star-shaped? In general, this is not true; however, when the kernels of the intersecting star domains overlap, then there exists a linear relation in the intersected set between a point in the intersection of the kernels to any point in the intersected star domain, i.e. there exists a kernel to the intersection of star domains defined by the intersection of the kernels. Since the kernel of each individual star domain is given by the intersection of the half-planes with respect to p_i, we ensure that the kernels overlap by at least one point, p_i. This implies that our Voronoi cells in the Minkowski plane will give star domains in the most general cases. This is a vital property since conservation principles, e.g. the contracted Bianchi identity and conservation of stress-energy, rely on the existence of exact, closed forms. When star-domains are the most general type of domain in the topology we are guaranteed that all closed forms are exact via the Poincaré lemma.[87] Thus we preserve the existence of our invariance principles.

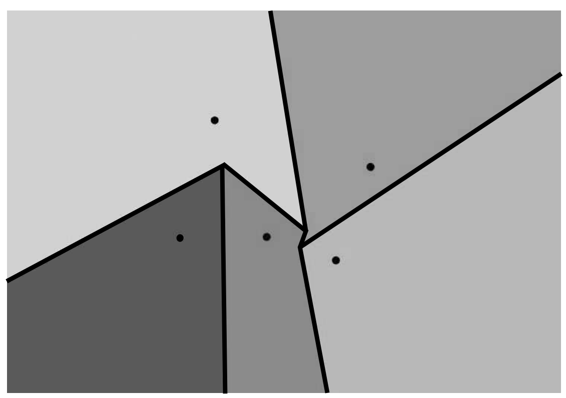

Figure 4.3: *Planar Minkowski Voronoi Diagrams:* An example of the planar Minkowski Voronoi diagrams is presented above. We conjecture that this new definitions exhibit exactly the correct properties expected of a Voronoi diagram. While an exact correspondence with the usually cited meaning of a Voronoi diagram–the region *closest* to any $p_i \in \mathcal{P}$ than any other point in $\mathcal{P}$– is lost, there remains a strong correlation between distance and the resultant Voronoi cell for p_i. We note that convexity of Voronoi cells is lost and replaced by the generalization to star domains. This retains the vital geometric properties necessary for the definition of physical observables and conservation properties.

We leave this definition at the stage of a conjecture for the time being. It should be proven that for a distribution of points lying in general position that this definition of the Voronoi cell to a vertex will produce a Hausdorff covering of the manifold; however, there are unique problems to providing such a proof in manifolds with non-positive definite metrics. One can construct a series of computational examples to convince one's self of these results for arbitrarily many and randomly distributed elements of $\mathcal{P}$. Instead, we will comment on extending this work to higher dimension and curved geometries. The above construction has been explicitly carried out in the Minkowski plane, $\mathbb{M}^2$; however, it was constructed in such a way that it could readily be extended to higher dimensions. In the above construction we only rely on two properties of flat spacetime: (1) there exists a well-defined light-cone structure separating space-like and time-like events at every point and (2) distances are defined by the invariant measure using the Minkowski metric, $\eta_{\mu\nu} = \text{diag}(-, +, \ldots, +)$. In principle one could apply Eqs. (4.8 - 4.11) directly to $\mathbb{M}^n$ and still preserve all of the properties described above. This, however, has not been directly tested and requires a full proof for the Minkowski plane prior to further application to higher dimensions. The extension of this algorithm to arbitrary Lorentzian geometries and causal structure is a topic for future work which we only mention as a possibility in this thesis.

4.2 Flow of Fields and Matter in the Discrete Lattice

The contracted Bianchi identity not only identified the circumcentric (Voronoi) dual lattice as the topological structure of RC, but it also illuminated how stress-energy should be embedded on the lattice. We now take the insight gained from the conservation of source and apply it directly to the discrete stress-energy tensor. Just as in the continuum, one should hope to identify an underlying prescription for embedding a continuous field into spacetime and obtaining a stress-energy tensor for it. In the classical theory, the components of the stress-energy tensor identify how a given component of the 4-momentum flows through a surface of constant x^μ, i.e. $T^{\mu\nu}$ provides the flux of p^μ through a surface of constant x^ν.[69] In RC, there is a well-defined set of surfaces bounding the dual 4-volume to a vertex through which one can identify the flux of 4-momentum. As such there are four components of stress-energy per edge of the Regge lattice which identify the flux of the 4-momentum through the dual L^* to the edge, L. The stress-energy is thus projected along the edge to which the stress-energy tensor is associated.

If a fundamental theory of nature is our goal, then an action principle is the most readily identified path from particle dynamics to stress-energy. Indeed, we already have a well-defined notion of an action principle for RC and it is fitting that we start with an action principle for the matter in RC as well. The mapping from the action principle for the matter to the stress-energy is obtained in classical GR by varying the Lagrangian density with respect to the metric

$$T^{\mu\nu} = (-g)^{-\frac{1}{2}} \frac{\delta \mathcal{L}_{field}}{\delta g_{\mu\nu}} \tag{4.13}$$

or via variation with respect to the tetrad. This definition provides a clear and meaningful correspondence between the dynamics of particles and fields and the the flow of stress-energy, particularly when one adapts the view of the action defining the *virtual paths* of particles and fields in spacetime. If we assign such meaning to the action, the Lagrangian density defines the local flow of the field in a infinitesimal region of spacetime. Varying this with respect to the metric then tells us how each component of the of momentum *flows* through the boundary of the local domain. In RC, the variation is taken with respect to the edge lengths instead of the metric and so we obtain a direct association between the action for a field and the stress-energy on the lattice;

$$T_{LL} V_L^* = \frac{\delta I_{\text{field}}}{\delta L}. \tag{4.14}$$

This recovers the notion of an integrated measure of the flux of the components of the 4-momentum through L^*. There is a clear direct correspondence between the stress-energy in the continuum and the discrete stress-energy. We do not restrict the components of the stress-energy to surfaces of constant x^ν since the 3-volumes dual to L provide a natural 3-volume over which one defines flux. We thus have a notion of the stress-energy in RC.

Does this also provide guidance for the construction of an action principle for arbitrary fields? Yes, it automatically instructs one how to go from definition of field to action principle to stress-energy associated with an edge. However, we will follow the opposite direction since we so far only know the structure of stress-energy in RC. It is already clear how energy-momentum is embedded on the lattice and it is only required that a scalar action be constructed so as to mimic the flow of the field and particles on the lattice.

Since I_{field} is defined as a Lorentz scalar without any *a priori* reference to the discrete structure, any well-defined scalar action would produce a stress-energy tensor that is doubly-projected along the edge, L. However, we have outlined some additional considerations that will aide in understanding the action in RC. We identify that the discrete structure, either the simplicial or the dual, should define the pathways for fields and particles. However, the previous results have identified that the energy-momentum associated with a field flows along the simplicial 1-skeleton. Given this view of stress-energy, one should hope that the action, as the definer of virtual paths of fields, should mimic the natural flow of the stress-energy of the field. How can one define the action so as to best mimic the flux of energy momentum through the dual lattice? Embed the field variables such that they are in 1-1 correspondence with the geometric structure of the simplicial lattice, e.g. d-forms get mapped to d-simplexes. This ensures that the fields are defined only on the simplicial skeleton such that the corresponding flux is directed on boundaries of simplexes alone. Once the identification of the mathematical correspondence between the field structure and the spacetime lattice is made, an assignment of a tangent space to the fields must be decided upon. It is a natural option to associate fields directly to the simplexes as these automatically define unambiguous tangent spaces. However, the interior of simplexes are *a priori* defined to have the geometry of flat Minkowski spacetime–a vacuum solution to Einstein's equation. And as Wheeler once wrote

> ...that geometry cannot be a God-given perfection standing on high above the battles of matter and energy. It is instead a participant, an actor, an object on equal terms with the other fields of physics...Geometry can't be part of physics in some regions and not a part of physics in others.[7]

Here we have defined such a region; however, the flat subspaces of the Regge lattice are still in full compliance with the dynamics of curvature coupled to matter and field when non-gravitational sources are relegated to the boundaries of the simplexes. Rather than allowing geometry to not be a part of physics on the interior of simplexes, we shift the dynamics from the interior of simplexes to their boundaries. This automatically allows us to define a natural flow of field strength on the simplicial lattice. Moreover, this is in direct agreement with the identification of the dual lattice as the topology of matter in RC. By assigning the topology of fields to be the Voronoi cells, they become naturally defined at the vertexes of the simplicial lattie. This automatically allows for the identification of flux through the boundary of the circumcentric dual cell by describing the flow of field strength along the simplicial 1-skeleton.

Given this discussion we define a procedure for describing discrete stress-energy in RC:

1. *Map field variables to the geometric variables of the simplicial lattice:* Taking the mathematical structure of the field in the continuum, one should make the correspondence to the exterior calculus structure of the Regge lattice, e.g. d-forms get mapped to d-simplexes. For fields not defined via tensors, e.g. spinor fields, one need only map the simplicial geometric variables into the representation of the field variables, see §4.2.1.

2. *Assign field strengths to vertexes:* The natural domain of conservation of source is the circumcentric (Voronoi) dual to a vertex. We therefore assign field strengths and coupling of fields over the dual volume which naturally assigns the field strengths to vertexes, i.e. the "center" of the volume.

3. *Flow of stress-energy is directed on edges:* The Kirchhoff-like conservation principle provides an explicit representation of the flow of the field energy-momentum through the boundary of the Voronoi cell for a vertex. The kinetic contribution to the action is thus defined as a finite-differencing of field strengths at adjoined vertexes.

These steps ensure that stress-energy are coupled to source so as to automatically ensure conservation while also providing direct correspondence between flow of energy-momentum and the virtual paths of fields defined in the action principle. The outline above is sufficiently general so as to apply to any field (e.g. scalar, vector, or spinor) and directly matches the scalar action defined by Hamber and Williams.[82] We show how this prescription for the stress-energy can be applied to a field with non-trivial embedding into the discrete spacetime. It serves as an example of the methodology and a first step towards describing fundamental fields in RC.

4.2.1 The Dirac Lagrangian

With regard to our current understanding of observable physical interactions, it is hard to overstate the importance of fermionic fields (defined via Dirac spinors) in the universe. Of all the known fundamental particles, fermions describe the majority of constituents of the ordinary matter content of the universe. It is therefore necessary to prescribe exactly how fermions are embedded into RC. This provides a unique challenge for describing stress-energy in the lattice since fermion states are described by spinor representations of the double covering of the Lorentz group. Hence we are required to map the edges, i.e. representations of the Lorentz group, to spinor states. By providing a description of fermions in the lattice we are able to provide a non-trivial example of the prescription given in the previous section.

A consistent mapping from the simplicial representation of the Lorentz group to its double covering, $SL(2,\mathbb{C})$, the spacetime must admit a spin structure, i.e. an orthonormal *vielbein*, or tetrad in 4-dimensions, at every point. Moreover, a *consistent* mapping requires that there be a continuous mapping from an *oriented vielbein* defined on one tangent space to the *oriented vielbein* on another tangent space. This requires that the spacetime be orientable[88] such that *vielbeins* can be mapped into one another and still preserve the orientation of the volume form for each *vielbein*.

For the purposes of this thesis we will map fermions only onto completely oriented simplicial lattices, i.e. mappings from one simplex to another preserve orientation.

Since each simplex is *a priori* a subspace of flat Minkowski spacetime, it is guaranteed that each simplex automatically permits a well-defined orthonormal frame on its interior. Moreover, we assume that the lattice is oriented such that there is a consistent mapping of a tetrad in one simplex to the tetrad in another simplex. RC therefore inherently admits a spin structure inside of each simplex and a consistent spin structure over the lattice as a whole. However, our prescription above requires the field be defined at the vertex where there is not an unambiguous choice in defining a tangent space and hence an orthonormal frame. In this subsection we will examine the use of vertex-based tetrads in the definition of spinor structure and how these can be used to construct a vertex-based action for fermionic matter. This will provide an action which is internally consistent and a more direct match with the underlying assumptions in RC.

Understanding the fermion action first requires an understanding of how spin-$\frac{1}{2}$ representations of the double covering of $SO(3,1)$ can be represented via the spacetime lattice. To create a concrete geometric picture we will frame the issue in the context of the Penrose spin-flag formalism[89] then transition to arbitrary orthonormal and null frames. The Penrose spin-flag representation of a 2-component spinor is based first and foremost on the identification of a null vector, l^a, with a basis spinor, ξ^A, where lower-case latin letters represent abstract spacetime indices, upper-case latin letter represent spin indices in the representation of $SL(2,\mathbb{C})$ and dotted upper-case latin letters indicate indices in the adjoint represenation of $SL(2,\mathbb{C})$. If one first maps the null vector into representation of $SL(2,\mathbb{C})$ through the mapping:

$$l^{A\dot{B}} := l^a \cdot \sigma_a^{A\dot{B}} = \begin{bmatrix} ct+z & x-iy \\ x+iy & ct-z \end{bmatrix} \tag{4.15}$$

where $\sigma_0 = \mathbb{I}$ and $\vec{\sigma}$ are the Pauli matrices, then we make a correspondence between ξ^A and l^a;

$$l^{AB} = \xi^A \bar{\xi}^{\dot{B}}. \tag{4.16}$$

Under the standard inner-product of spinors using the anti-symmetric tensor as the metric, $\epsilon_{AB}\xi^A\xi^B = 0$ one sees immediately that this definition is consistent with the interpretation of l^a as a null vector. However, we cannot identify ξ^A uniquely via this relation as the relative phase factor, $\tilde{\xi}^A = e^{i\phi}\xi^A$, leaves l^a unchanged. This implies that we need to define an additional set of conditions to pin down the exact phase for ξ^A. Penrose's solution to this is to introduce a unit spacelike-vector orthogonal to l^a which can be defined in terms to ξ^A and an additional basis spinor η^A:

$$y^{A\dot{B}} := y^a \sigma_a^{A\dot{B}} = \xi^A \bar{\eta}^{\dot{B}} + \eta^A \bar{\xi}^{\dot{B}}. \tag{4.17}$$

Viewing the set $\{\xi^A, \eta^A\}$ as a basis in spin space we assign the standard normalization conditions:

$$\epsilon_{AB}\eta^A\eta^B = 0 = \epsilon_{AB}\xi^A\xi^B \tag{4.18}$$

$$\epsilon_{AB}\eta^A\xi^B = 1 = -\epsilon_{AB}\xi^A\eta^B. \tag{4.19}$$

With the addition of this new space-like vector we are able to define a spin basis complete with appropriate phase. However, η^A is not a unique basis vector mate to our originally defined ξ^A since an addition of a linear factor in ξ^A will still satisfy the normalization conditions above. However, we need only pick one such space-like vector which is orthonormal to l^a to define a spinor mate to ξ^A to complete the spin basis. We therefore find, a class of spin bases defined by a single null vector and a collection of orthonormal space-like vectors. This collection of space-like vectors define a "flag" orthogonal to the null vector–the flag-pole–to create the spin-flag representation.[69;89] This provides us with a geometric picture of a spin basis in terms of a bivector, $F^{ab} = l^a y^b - y^a l^b$, defined by the flag-pole and spin-flag. This is a natural geometric picture to use in RC, as has been illustrated in the example of Null-Strut Calculus,[90] since we can make a direct analogy between the spin-flag bivectors and the hinges of 4-dimensional RC. However, it is not always the case that we will have a null edge in every simplex in the spacetime lattice and so we require a more general framework for the inclusion of spin structure in RC.

To generalize the spin-flag representation we will rely on the more familiar representation of an orthonormal frame in terms of a spin basis. The procedure for obtaining a spin basis follows the above geometric picture, except one now defines an orthonormal frame and inverts the frame to obtain basis spinors. We use null tetrads consisting of two real null vectors, l^a and k^a, and two complex null vectors which are conjugate to one another, m^a and $\overline{m}^a$ to provide the most clear connection between the tetrad and the spin basis. These vectors are designed such that the only non-vanishing inner-products are

$$-l^a k_a = 1 = m^a \overline{m}_a. \tag{4.20}$$

Moreover, since these vectors are null, we can define simple relations between the tetrad and a spin basis:

$$\begin{aligned} l^{A\dot{B}} &= \xi^A \overline{\xi}^{\dot{B}} \\ k^{A\dot{B}} &= \eta^A \overline{\eta}^{\dot{B}} \\ m^{A\dot{B}} &= \xi^A \overline{\eta}^{\dot{B}} \end{aligned} \tag{4.21}$$

Yet we are only concerned with this null frame if it can be defined in the lattice. For the more general case, we can build linear combinations of the null frame to link the spin basis to the simplex-defined orthonormal tetrad. In the case of a simplex with a well-defined time-like vector and three space-like vectors the linear mapping from null tetrad to standard orthonormal tetrad is given by;

$$\begin{aligned} t^a &= \frac{1}{\sqrt{2}}\left(l^a - k^a\right) \\ x^a &= \frac{1}{\sqrt{2}}\left(l^a + k^a\right) \\ y^a &= \frac{1}{\sqrt{2}}\left(m^a + \overline{m}^a\right) \\ z^a &= \frac{i}{\sqrt{2}}\left(m^a - \overline{m}^a\right) \end{aligned} \tag{4.22}$$

otherwise one must develop additional linear combinations to provide a direct link between the edges and a spin basis. In principle, these are well-defined and only require some basic spinor algebra and solving a linear system of equations mapping the tetrad to the spin basis.

The question remains how do we define a tetrad in the simplicial lattice at a vertex? The argument made previously[81] was that the vertex based definition of spinors provides undue ambiguity in the definition of the tangent space and so the spin structure was argued to be naturally located at the circumcenter of the simplexes. The underlying issue is whether one can consistently define a tetrad for each term of the Lorentz invariant scalar action. How does one choose between such tangent spaces *and* what is the mapping between two tangent spaces at a given vertex?

To choose a tetrad we examine the general form of the action and each of the terms included therein. We have seen from Eq. (4.1) that the general form of the action contains two distinct parts: (1) edge-based kinetic terms and (2) vertex-based interaction/mass terms. Therefore, our main focus should be on the writing of spinors such that finite-differencing can be well-defined for the field at adjacent vertexes. All vertex-based terms in the action are scalar invariants and require only one to have a clear connection between distinct spin frames at a vertex. However, the edge-based kinetic terms require a common tangent space for the two spinors being differenced. Again, we have a number of options for choice in tetrad which are common to the edge containing the two vertexes as endpoints. One need only choose one such tetrad to perform the finite-differencing and construct the scalar invariant for the field at the two adjacent vertexes. Moreover, the terms in the action corresponding to vertex based quantities, such as mass or interaction terms, require only that the tangent space chosen matches a simplex containing the vertex.

While the choices in tetrad are not unique, we expect there to be well-defined transformations between one choice and another if the action is truly to be a Lorentz scalar. This topic has already been broached in the framework of geodesic deviation[91;92] which is mainly concerned with mapping vectors from one tangent space to another. In 2-dimensions the result obtained is that two initially parallal vectors propagating on either side of a hinge will be seen to converge with an angle of convergence equal to the deficit angle of that hinge. ϵ_h. However, for a single vector, or spinor, being transformed from one tangent space to another there is no net rotation as rotations only occur when the vector, tensor, or spinor traverses the conic singularity. To rotate a spinor from one simplex at the vertex to another simplex hinging on the same vertex, one need only transform from one tetrad to the other via a coordinate transformation. These transformations are by definition transformations of basis on the vector space and hence leave the scalar action invariant. These transformations encode the information on how the vector space in one simplex is transformed into the vector space at the neighboring simplex, thus providing essentially the same content as the spin connection in continuum curved geometries.

With the construction of basis spinors defined, we construct the Dirac spinors and the resulting action. The Dirac spinors are defined in the Hilbert space of the direct sum of two irreducible representations of $SL(2,\mathbb{C})$ with bases given by $\{\xi^A, \eta^A\}$ and $\{\overline{\xi}^A, \overline{\eta}^A\}$. The Dirac spinors are the well-known four component spinors given;

$$\psi = \left(\alpha\xi^A + \beta\eta^B \ \ \right) \oplus \left(\delta\overline{\xi}^{\dot{A}} + \gamma\overline{\eta}^{\dot{B}}\right). \tag{4.23}$$

We now assign the Dirac action via the method obtained in §4.2;

$$I\left[\psi, \overline{\psi}\right] = \sum_L i\left(\overline{\psi}_{v|_{L^-}} \gamma^a_L \frac{\psi_{v|_{L^+}} - \psi_{v|_{L^-}}}{L^a} + \text{c.c.}\right) - \sum_v m\overline{\psi}_v \psi_v. \tag{4.24}$$

where L is the oriented edge and the vertexes, $v|_{L^+}$ and $v|_{L^-}$, are the vertexes at the endpoints of L. This action, while a straight-forward embedding into the Regge lattice, provides a vertex-based action which is consistent with the conservation of source and the resulting topology. Moreover, it is also consistent with the general construction of RC in defining building blocks which are *untouched* subspaces of Minkowski spacetime. Moreover, each term in this action provides a scalar invariant which requires that we need only ensure that there be a consistent tangent space for each term. This provides the internally consistent Dirac action for RC which requires no references to masses or field on the interior or any simplex. Fields are relegated to the boundary and we obtain the requisite flow of field strength along the egdes of the simplicial lattice.

4.3 The Lattice Defines the Paths of Particles and Fields

Automatic conservation of source provides a natural guide for understanding how stress-energy is embedded into the simplicial lattice of RC while making use of the geometric content of the lattice. This suggested that the natural topology of RC is not the simplicial complex but the circumcentric dual. However, we found that the circumcentric dual does not ordinarily produce a Hausdorff topology for the PL-manifold that is consistent with conservation of source. There are, however, a category of triangulations which *always* induce a Hausdorff dual topology–the Delaunay triangulations.

In response to this observation, we define the Voronoi tessellation in Minkowski spacetime from first principles which focuses directly on the light-cone structure of spacetime. The spacetime Voronoi tessellations preserve the portable characteristics of the Voronoi tessellations as defined in Euclidean geometry. However, the resultant Voronoi cells are found to produce regions which are no longer convex but retain the necessary features required for conservation principles, i.e. the domains are still star domains.

Given the Voronoi topology for a spinkling of points, we are able to embed the stress-energy in a consistent fashion so as to preserve the Kirchhoff-like conservation law that results from the contracted Bianchi identity. This has provided a direct description of the paths of particles and fields in terms of the edges of the lattice. This prescription allows us to take an arbitrary field defined in the continuum and embed it, using the geometric variables of the lattice, into the simplicial lattice. The picture obtained is the flux of field strength, i.e. particles, along edges of the spacetime lattice as defined directly into the action for the field. Since there is no explicit reference to anything other than the simplicial structure, this prescription is sufficiently general for application to both RC and simplicial representations of quantum gravity, e.g. spin foams and GFT.

This methodology has been applied to Dirac particles such that the fields are consistent with the vertex-based topology and conservation of source. Moreover, we have addressed some of the underlying issues of assigning tetrads to vertexes in the action principle. It is found that there is no internal inconsistency by defining tetrads at the vertex. We thus have a description of Dirac fermions in terms of the simplicial lattice. However, it is still necessary to carry this prescription forward to gauge fields in order to create a consistent picture for the Dirac field coupled to electromagnetism.

In the coming chapter, we will illuminate a picture of RC in light of the results of this chapter as a realization of the goal of a matter-dependent description of RC. We believe this will shed more light on the *approximate* diffeomorphism invariance described by the Kirchhoff-like conservation law and the issue of Lorentz invariance of discrete spacetimes. We believe that the lessons learned from the incorporation of matter illuminate the role of matter in understanding these *approximate* symmetries in discrete dynamical spacetimes.

Chapter 5

Regge Calculus as the Paths of Fields and Particles

But since the positions in space of the material parts [of the universe] can be recognized only by their states, we can also say that all the states of the material universe depend upon one another.[93]

–Ernst Mach

We have argued that discrete spacetimes provide a compelling description of classical and quantum theories of gravity. In many models of quantum gravity the simplicial lattice serves not as a direct, ontological geometric object but bridge between quantum gravity and the geometry of spacetime. However, it has not been clear how the Geometrodynamic Steering Principle and the determiners of inertia are manifest in suchdiscrete theories of spacetime. The work in this thesis has attempted to bring the dynamics of the material and measurable universe to these models by investigating the structures amenable to the inclusion of source in discrete spacetimes. While the results of Chapters 3 & 4 are sufficiently general to be applicable to many different models of discrete spacetimes, one could argue that it is in the connection with source in which RC can become elevated to a complete dynamical picture of spacetime. We have seen how the edges are natural descriptions for the flow of energy-momentum in the lattice and how vertexes become the foci for interactions between fields. We now wish to examine the insight gained from this understanding of matter in the lattice for the symmetry properties of discrete spacetimes.

It is still not well understood to what extent classical symmetries can be incorporated into the discrete spacetime or how this affects the constraint algebra for the evolution of (3+1)-dimensional formulations of discrete gravity. Can the association of particle dynamics with the simplicial lattice shed light on these open issues? If we are to accept the approximate conservation laws or open constraint algebras inherent to RC, then how are we to reconcile this with a classical theory that exhibits exact symmetries and a well-defined constraint algebra?

5.1 Towards an Understanding of Diffeomorphism and Local Lorentz Invariance in Discrete Spacetimes

Through the CBI, the manifestation of diffeomorphism invariance in GR, we obtain an approximate symmetry in RC due to non-commutativity of finite rotations. Since any discrete spacetime is necessarily a theory of finite rotations, the lack of an exact identity should be expected to be a generic feature of any such theory. However, the exactness of the diffeomorphism invariance has been the subject of much debate. There have been efforts to identify variations of the Regge action with respect to vertexes which leave the action invariant;[94] however, these too have produced inexact symmetries in the Regge action. The question remains whether or not exact diffeomorphism invariance need be a property of the discrete spacetime or whether a discrete theory can incorporate such an exact symmetry? We do not believe this to be so nor do we expect that this is ordinarily possible for a given discretization.

Similarly, it has been a topic of debate about whether or not an imposed or derived minimum length scale inherent in discrete spacetimes breaks local Lorentz invariance(LLI).[55;95;96] In favor of preserving LLI, it is argued that discrete structures in a Lorentzian spacetime require an inherent randomness in the underlying structure of the spacetime so as to not choose any preferred reference frame. This is the argument placed in favor Causal Sets since the embedding of the causet into a spacetime manifold is done via a Poisson sprinkling which preserves the Lorentz symmetry.[55] RC, on the otherhand, cannot, in general, be said to be a “sprinkling” of vertexes via some random process which can preserve LLI. Yet, the Regge lattice provides only one discretization of a spacetime to which there may be many triangulations with the same coarse-grained limit. So again we have a question on the meaning of a fundamental symmetry of spacetime and its manifestation in a discrete theory.

These two issues represent a core question about the underlying structure of spacetime: if a discrete structure is more fundamental than our classical notion of spacetime, should we expect exact infinitesimal invariances? If not, how are these recovered in the classical limit? The one statement that comes to mind on the question of diffeomorphism invariance is Wheeler’s adage, “A Ford fender is still a Ford fender no matter how one paints coordinate lines on it.” In terms of RC, this says that a simplicial representation is just one of many discretizations that result in the same continuum limit. Although Regge’s original manuscript on RC was titled “General Relativity Without Coordinates,” a chosen discretization with all edge lengths fixed represents a choice of the “painting of coordinate lines” on the spacetime. By choosing a given simplicial lattice a partial gauge-fixing on the spacetime is already defined. But how does one make physical sense of choosing one lattice over another to make this partial gauge-fixing? How can we make sense of this choice of partially breaking the diffeomorphism invariance–evident by the *approximate* symmetry that remains?

One possibility worth exploring is to acknowledge that when we select a particular triangulation of the spacetime there are many "essentially equivalent" triangulations. But what can be used to select one triangulation over another? This is where the main thrust of this thesis comes into play. The idea proposed before[4] and being pushed forward again here is that the motion of particles allows us to associate a given triangulation to which some of the edges of the simplicial lattice follow the world-lines of the interacting particles. We know from the "Hole Argument" that the inclusion of additional information about the interaction of non-gravitational fields or gravitational waves, on an otherwise diffeomorphically invariant spacetime, will identify spacetime points which are completely distinguishable from one another. This reduces the diffeomorphism symmetry and restricts the allowable transformation under which the full action is invariant. Moreover, the motions of the interacting particles or fields select preferred frames of reference, given by the simplicial building blocks with their minimum length scales, out from the sea of possible reference frames (triangulations). This manifests itself in RC through the *approximate* Kirchhoff-like conservation principle in which the finite rotations (LLI breaking) produce non-vanishing commutation relations between rotations and a partial breaking of diffeomorphism invariance for the given triangulation.

Therefore, it appears that in order to fully investigate the issue of exact symmetry properties in RC one must be able to explore the space of triangulations that give the same approximate smooth manifold in an appropriate coarse-grained limit. A single triangulation of the spacetime cannot provide the necessary framework to study exact symmetries since a partial gauge-fixing and preferred path of particles is already selected out from the space of all possible paths of particles. This is not altogether inconsistent with our continuum notions of symmetries and it would be worthwhile to investigate the meaning of these partial symmetries more fully. However, to do so would require a far better understanding of what it means to construct a complete class of "equivalent" triangulations, something that is not well understood at this time and is a subject of future research.

5.2 Constraint Algebras in Discrete Theories

A related question to the invariance principles described above is whether or not one can define a full constraint algebra in a canonical formulation of discrete theories.[97] However, the constraint algebra is directly related to the existence of exact symmetries in the dynamical system. The principle here is that in GR the diffeomorphism invariance imposes four constraints on the dynamical evolution, the Hamiltonian constraint and the Diffeomorphism constraints on the spatial slices. When the exact symmetries hold, the constraints form a constraint algebra[98] that constrains the dynamical evolution such that the invariance properties hold on all future slices.

In the discrete theory, the constraints are not ordinarily exact, but we do have a sense of approximate constraints which on the limit of small edge lengths provide the corresponding continuum limit. Gambini and Pullin have provided an alternative way to look at these discretizations in an effort to develop a "consistent discretizations" technique for classical and quantum gravity.[99] In this procedure, one treats the choice in the lapse and shift for (3+1)-dimensional spacetimes as a variables in the discrete theory and not as freely specifiable. This gives the theory new degrees of freedom not present in the continuum theory. The variables are then solved with the full set of equations of the theory. This amounts to giving the discretization more freedom to consistently discretize the theory. The collection of solutions to these equations for a given set of boundary conditions and sources then form the set of allowable discretizations of the theory. In RC, we see this manifest in the lack of exact conservation equations tantamount to a "smearing" of the constraints.

The constraint algebra is an issue directly related to the canonical (3+1)-dimensional formulation of GR, but the underlying issues are the same. Any given discretization of the spacetime will be a partial gauge-fixing of the spacetime which produces an incomplete constraint algebra. This is directly analogous to the partial-gauge fixing in the full 4-dimensional discrete theory in which the invariance properties become "fuzzy" symmetries on the individual spacetime lattice. The issue of why these approximate identities appear readily in the canonical formulation is explicit in how the freedom of lapse and shift is removed and one must now solve for both lapse and shift. In "consistent discretizations" these "coordinization choices" must now be satisfied in solving for the dynamics of the theory.

5.3 A View of Regge Calculus For Quantum Gravity

We have argued that a discrete spacetime must necessarily select choices of the gauge freedom to fit with a consistent discrete evolution that best preserves the constraints. We propose that this makes sense from a physical level if the theory selects out paths for the source of the spacetime curvature. The spacetime is partially gauge-fixed by the fields that determine the geometry of the spacetime. The edges then become defined by the inertial motion of particles as they propagate freely between interactions with other non-gravitational fields and particles. Indeed, one might say that the degrees of freedom of the gravitational field are expanded only to then be forced to satisfy the motion of particles of fields in the lattice. With this dynamics, the particles and fields select a discretization that corresponds to their paths in the spacetime. This elevates the discretization to an ontological existence out of the Hilbert space of triangulations.

However, not every edge in the triangulation need correspond exactly to the motion of particles in the material universe. If we are to only allow the paths and interactions of particles to define the simplicial structure, then we will, in general, not obtain a triangulation. Indeed we must allow for additional edges to exist in the triangulation. These additional edges serve to rigidify and completely determine the conformal geometry, much in the way that the causet structure requires the addition of *beams* and *struts* to the predefined ordering of events to connect back to geometry.[19] It seems likely that the natural role of RC in quantum gravity is through defining the relational ordering of events between particles. We allow the time-ordering of events in the histories of the particles to construct the partial ordered structure; however, a more rigid local structure–the light cone structure– must be constructed in order to define the local conformal geometry. Moreover, the local dynamics of the non-gravitational fields also allows for the definition of the conformal scale–the missing information given only a partial ordering– thus, the entire local geometry can be reconstructed from the assignment of particle paths to the simplicial lattice.

With this we can begin to make sense of the approximate nature of the symmetries of discrete formulations of spacetime. We have a correspondence between the selected discrete geometry and the dynamics of source to the geometry. This provides a selection rule for connectivities in the simplicial lattice and solutions to the Regge equations determine the entire geometry given the already defined incidence matrix. For this reason, one can say that RC may provide a natural and consistent prescription for quantum gravity coupled to source, and that RC acts as the definer of virtual paths of particles and fields in an emergent spacetime.

Appendix A

Convex and Star Domains and their Properties

There are two categories of domains or sets that are important for our study of Voronoi tessellations in Euclidean and Minkowksi geometries. We will explore their generic properties and outline the basic facts related to the use of Voronoi tessellations as topologies for spacetime.

A.1 Convex Domains

A convex set or a convex domain, C is a collection of elements of a vector space, $\mathbb{V}$, such that for any two elements $a, b \in C$ then for $t \in [0, 1]$,

$$(1-t)a + tb \in C. \tag{A.1}$$

In Euclidean Voronoi tessellations, the Voronoi cells are always guaranteed to be convex polygons, i.e. convex domains with PL boundaries.[43;100] From a geometric point of view, this says that a convex set in a flat geometry–Euclidean or Minkowski–is one such that the line segment connecting any two points in the set is also entirely contained in the set. It is clear from this definition that the intersection of convex sets is again convex. To prove this, one need only prove it for two intersecting convex sets and the generalization follows directly. Suppose we have two convex sets A and B in the vector space $\mathbb{V}$. Moreover, let the two points x and y exist in both A and B. By definition, the line segment connecting x and y exists entirely in both A and B; therefore, the line segment is also in the intersection $A \bigcap B$.

Half-spaces are also convex sets. This is clear since the half-space has only one linear boundary, the boundary of the hyperplane, any points in the half-space are connected by a line segment which at most lies on the hyperplane (which occurs only for points also on the hyperplane). Therefore, the intersection of half-spaces is also a convex set by the above theorem.

The framework of convex domains can be easily extended to arbitrary geometries by considering geodesics between two points rather than straight line-segments. Such convex sets are said to be geodesically convex. Here the boundaries of the geodesically convex domain are also geodesics of the non-Euclidean geometry.

A.2 Star Domains

An extension of convex domains are star domains or star-convex sets. A star domain,[43] S, in a vector space $\mathbb{V}$ is one containing at least one $x_0 \in S$ such that

$$(1-t)x_0 + tx \in \mathbb{V}, \quad \forall x \in S \quad \& \quad \forall t \in [0,1]. \tag{A.2}$$

In the context of the Voronoi diagrams, the Minkowski Voronoi cells will be star polytopes, i.e. star domains with PL boundaries.[100] The collection of all $x_0 \in S$ for which Eq. (A.2) remains true is the kernel of the star domain. The kernel of a star domain can be constructed by identifying the interior of each linear edge (containing the region of S local to the edge) of the boundary of the domain a segment of a half-space. By considering the full half-space defined by a edge of the boundary, the kernel of the star domain can be seen to be the intersection of all such half-spaces. Via this construction, the kernel of a star domain creates a convex subset of the star domain. It is therefore clear that a convex domain is also a star domain whose kernel is the entire convex domain.

Unlike convex sets, the intersection or union or star domains does not ordinarily produce another star domain. This is clear since the intersection of two star domains which overlap in two disjoint neighborhoods not in the kernel of either domain will produce a domain which is not connected. Hence, there can be no element of this intersection that can be linearly connected to each point in the domain. This would produce a disjoint domain which clearly cannot be a star domain. Similarly for unions if we consider two disjoint star domains. However, for any two star domains, A and B, whose kernels overlap by at least one point $x_0 \in A, B$, then $A \bigcap B$ is also a star domain. This proof of this is evident: since x_0 is in the kernel of both A and B then for all $y \in A \bigcap B$ the segment $(1-t)x_0 + ty \in A \bigcap B$ for all t in the unit interval since this line segment entirely exists in both A and B.

As with convex sets, star domains are easily generalized to curved geometries by substituting the line segment $(1-t)x_0 + tx$ by the geodesic connecting x_0 and any y in the domain. In addition, all properties of the star domains regarding unions and intersections automatically carry over to the generalizations to curved geometries. Moreover, the kernel to a star domain in a curved geometry remains a geodesically convex set.

The physical importance of star domains comes about in the proof of the Poincaré lemma: for a star-shaped region $\mathcal{U}$, a p-form on $\mathcal{U}$ is closed if and only if it is exact.[88] This lemma is often stated for contractible domains as well, though this generalization of the star domain that is not necessary in this context. It is sufficient, then, to say that if we have a star domain then the Poincaré lemma holds exactly. This is essential for our models of spacetime since our conservation principles, such as conservation of stress-energy, rely on the exactness of closed forms, i.e. $d\beta = dd\alpha = 0$.

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