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ANGULAR MOMENTUM PARADOXES WITH SOLENOIDS AND MONOPOLES

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Abstract

The Poynting vector produced by crossing the Coulomb field from a charged particle with a distant external magnetic field gives rise to a physical angular momentum which must be included in applications of angular momentum conservation and quantization. Simple examples show how the neglect of the return flux in an infinite solenoid or in two-dimensional models can lead to unphysical effects, how the Dirac charge quantization is obtained and can be modified by the presence of additional long range forces, and why the origin must be excluded in describing the motion of a point charge in the field of a fixed point monopole.

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The renewed interest in magnetic monopoles [1,2] has called attention to peculiar problems arising in the application of the conservation and quantization of angular momentum to the motion of a charged particle in an external magnetic field [3,4] like that of a solenoid [5,6,7] or monopole [8,9,10]. Errors arise when the angular momentum in the electromagnetic field approaches a finite constant value either at very large distances or at very small distances and is not taken properly into account. In particular, one must not overlook contributions from the return magnetic flux which completes a flux loop outside a solenoid, often at large distances outside the orbit of the charged particle under consideration. This paper considers several paradoxes and presents simple and general results based only on the conservation and quantization of angular momentum and the description of the momentum density in the electromagnetic field by the Poynting vector. Those results therefore hold in any theory or model which incorporates these principles, and are independent of the formalism used; e.g. the Schroedinger equation [10], the Dirac equation, non-Abelian gauge field theory [1], or descriptions using strings [11] or sections [12].

Consider a point charge +e located at the origin in the presence of an arbitrary external magnetic field $\vec{B}(\vec{r})$. The angular momentum in the crossed electric and magnetic fields is [8]

$$\vec{F} = \frac{1}{4\pi c} \int \vec{r} \times [\vec{E} \times \vec{B}(\vec{r})]\ d^3r = \frac{e}{4\pi c} \int \vec{r} \times [\vec{r} \times \vec{B}(\vec{r})]\ d^3r/r^3. \qquad (1)$$

In cases where the magnetic field $\vec{B}$ is only in the z-direction, as in models with one or more infinite solenoids or in two dimensional models, Eq. (1) can easily be evaluated to give

$$F_z = \frac{-e}{4\pi c} \iiint \frac{(x^2 + y^2)\, B_z(x,y)}{(x^2 + y^2 + z^2)^{3/2}}\, dx\, dy\, dz = \frac{-e\phi}{2\pi c} \tag{2a}$$

where

$$\phi = \iint B_z(x,y)\, dx\, dy \tag{2b}$$

is the total magnetic flux.

A striking feature of this result is that the value of the angular momentum depends only upon the charge e and the total magnetic flux ϕ, and is independent of the spatial distribution of the flux. This is the physical basis underlying the paradoxes arising at large and small distances. The angular momentum (2a) remains constant and finite even when the flux is all pushed out to infinity or when the flux is concentrated in a tiny region around the origin. This angular momentum must be carefully considered in the limiting cases where flux is pushed out to infinity or where a magnetic source is located exactly at the origin and must have zero angular momentum.

The constancy of the angular momentum at small and large distances is a general feature of electromagnetism and is independent of the simple geometry used to derive Eq. (2a). This can be seen by noting that the expression (1) is invariant under the scale transformation

$$B(\vec{r}) \rightarrow \kappa^2 B(\kappa \vec{r})$$

where κ is an arbitrary scaling factor. Thus for example if $\vec{B}(\vec{r})$ is produced by an assembly of magnetic monopoles, the angular momentum in the crossed fields is unchanged by scale transformations which move all the monopoles out to infinity or move them all arbitrarily close to the origin.

Our first paradox applies Eq. (2a) to the system of a deuteron at the origin and an infinitely long solenoid located beyond the moon. The total angular momentum of the system is

$$\vec{J} = \vec{L} + \vec{S} + \vec{F} \qquad (3)$$

where $\vec{S}$ denotes the sum of the spins of the neutron and proton, and $\vec{L}$ denotes the mechanical orbital angular momentum $(\vec{r} \times m\,\vec{v})_{rel}$ in the deuteron.*

Since the value of the flux is not restricted to particular values, the electromagnetic angular momentum F_z is not restricted to integral or half-integral multiples of $\hbar$. This leads to the following paradox:

1. If the total angular momentum of the system is required to be an integral or half-integral multiple of $\hbar$, the allowed values of the relative orbital angular momentum, the centrifugal barrier, and the binding energy of the deuteron all depend upon the field of the solenoid beyond the moon. If the centrifugal barrier in the deuteron for the value $L_z = \hbar/2$ is so high that no bound state exists, it becomes possible to break up a deuteron on earth by turning on a small magnetic field beyond the moon.

2. If the dynamics of the deuteron are required to be independent of what is happening beyond the moon by requiring L_z to be an integral multiple of $\hbar$, then the total angular momentum of the system takes on peculiar values which are not integral or half-integral multiples of $\hbar$.

3. The conventional treatment using the Schroedinger equation and a vector potential which remains finite at infinity correctly shows no effect on

*We treat the deuteron as if its center of mass is held at the origin. A more accurate treatment would confine the deuteron to a sphere. The result is the same.

the deuteron from the field of the solenoid at large distances. However, it has no consistent definition of the total angular momentum to include the angular momentum (2a) in the crossed fields, which is a physical effect and cannot simply be ignored.

The paradox remains when the Hamiltonian is rotationally invariant about the z-axis and the total angular momentum in the z direction is a constant of the motion, since the result (2a) also applies for a cylindrically symmetric configuration of several concentric solenoids of very large radius, tailored to give any desired cylindrically symmetric field at large distances.

The paradox is resolved by correcting the improper treatment of the finite flux at infinity. Consider the infinite solenoid as the limiting case of a long finite solenoid. A return flux exists outside the solenoid and moves out to infinity as the length of the solenoid is increased. Eqs. (2) show that the total angular momentum in the fields including the return flux is zero. Since the contribution of the return flux remains finite as the length of the solenoid approaches infinity, contradictions arise when the return flux is ignored in the infinite limit. The apparent unphysical effects of fields at large distances disappear when the contributions of return flux are properly included.

For the long finite solenoid the treatment with the Schroedinger equation and the vector potential is known to include all effects of the angular momenta in the crossed fields [3,4]. This is most easily seen in the convenient gauge where div $\vec{A} = 0$ and $\vec{A}(\infty) = 0$. In this gauge the total angular momentum (3) is the generator of rotations and the additional angular momentum introduced by the vector potential is exactly equal to the angular momentum in the crossed fields,

$$\vec{r}\times (e/c)\vec{A} = \vec{F} \qquad \text{if div } \vec{A} = \vec{A}(\infty) = 0. \qquad (4)$$

If the fields are rotationally invariant the Hamiltonian is manifestly rotationally invariant and the total angular momentum (3) is conserved.

Equation (4) holds for any field configuration which is a solution of Maxwell's equations, produces this value of the vector potential at the position of the particle and vanishes sufficiently rapidly at infinity. This formalism therefore implicity includes the angular momenta in the return fluxes at large distances which are required by Maxwell's equations to close all flux loops and make the fields go to zero at infinite distance, even when these return fluxes are not specified in the statement of the problem. The paradox arises only when the angular momentum in the crossed fields is calculated directly from the Poynting vector as in Eq. (2) for systems with finite flux at infinity and no return flux.

This analysis leads to the following conclusions:

1. The angular momentum in the field is a physical angular momentum which must be included in the total angular momentum. In a rotationally invariant system it is this total angular momentum which is conserved, and which generates rotations and is therefore quantized.

2. The contributions from the angular momentum in the return flux from a long solenoid remains finite and cannot be ignored in the limit of an infinitely long solenoid, or in extrapolating the results of a two-dimensional model to three dimensions.

3. The conventional formalism with the vector potential for describing the motion of charged particles in magnetic fields includes the contributions from crossed fields to the angular momentum properly if the fields decrease rapidly enough at infinite distance. If the Hamiltonian is

invariant under totations, the total angular momentum operator (3) generates rotations and is conserved. It includes not only the mechanical angular momentum $\vec{L}$, which enters the dynamics via the centrifugal barrier, but also the angular momentum in the fields given by Eq. (4). However, inconsistencies in the treatment of rotations and of angular momentum quantization and conservation arise if fields at infinity are not properly treated. All these physical results are independent of the choice of gauge, but they are most transparent in the gauge (4) where the Hamiltonian is manifestly rotationally invariant.

Another example where these conclusions are particularly significant is in the motion of an electron in the field of an infinitely long solenoid. All the correct dynamics are again in the solution of the Schroedinger equation using the appropriate vector potential [4]. However, peculiar results are obtained when the mechanical angular momentum of the electron is interpreted as being the total angular momentum of the system, without taking into account the angular momentum in the return flux which is present when the infinite solenoid is considered as the limit of a finite solenoid. There is no paradox and no peculiar value of the total angular momentum if the angular momentum of the field is properly computed and includes the contribution from the return flux. The angular momentum in the crossed fields is calculated in a manner similar to Eqs. (1) and (2), with corrections when the origin is taken to be the center of the orbit of the particle rather than the position of the particle. In this case, there are three regions of physical interest:

I. The electron is completely outside the return flux of the solenoid (or where the field has decreased to a negligible value as some power of the distance). This occurs in all practical cases of remote solenoids. The angular momentum and total flux (2) are zero, since the integral of the magnetic field includes all of the return flux. The solenoid has no effect on the motion or angular momentum of the electron.

II. The electron is in a region where there is a finite magnetic field due to the solenoid, either in the center of the solenoid or in the return flux. The electron feels a Lorentz force and exchanges momentum and angular momentum with the field. Angular momentum is conserved between the electron and the crossed fields, and the total angular momentum always is an integral or half-integral multiple of $\hbar$.

III. The electron is in a field-free region between the solenoid and the return flux. The electron feels no Lorentz force. When the origin is chosen to be the center of the solenoid, the angular momentum in the crossed fields within the solenoid obviously vanishes in the limit of an infinitely thin solenoid. When the return flux is neglected this leads to the erroneous conclusion that the total angular momentum in the crossed fields is zero and not related to the vector potential by Eq. (4). Detailed calculations [4] show that the angular momentum in the fields is all in the return flux, agrees with Eqs. (2a) and (4), and can take on any value depending upon the value of the flux. However, the allowed values of the mechanical angular momentum of the electron also depend upon the flux and are required to be just the right peculiar values to make the total angular momentum equal to an integral or half-integral multiple of $\hbar$. This dependence of the electron dynamics on the flux elsewhere is commonly known as the Aharonov-Bohm [13] effect. If the strength of the magnetic field is changed; i.e. by changing the current through the solenoid, the electron experiences a torque by the well-known betatron effect and its angular momentum and the angular momentum in the field are both changed by exactly the same amount in opposite directions to keep the total angular momentum constant.

When the electron is moved from the external region (I) through the return flux region (II) into the field-free region (III), angular momentum is exchanged between the electron and the field and is completely conserved between them. There is no torque on the solenoid, nor any angular momentum transfer between the electron and the solenoid [4-6], in contrast to the erroneous conclusions obtained by improper extrapolation from a two-dimensional model [7].

Further paradoxes arise in treating the angular momentum and statistics for n identical composite systems each consisting of a particle of charge e and a flux-tube or solenoid with flux ϕ and no return flux [7]. The total electromagnetic angular momentum is $-n(n-1)e\phi/4\pi c$ and arises from the Poynting vectors from the coulomb fields of the charges and the magnetic fields of solenoids in different composites. This angular momentum is independent of the choice of origin or the distances between systems. The n-dependence implies that allowed values of angular momentum are changed by introducing composites beyond the moon. Such contradictions arise when return fluxes are not introduced in a consistent fashion, with a return flux attached to each composite system and included in the permutations which define the statistics.

We next consider the case of the motion of an electron in the presence of a magnetic monopole beyond the moon. Evaluation of Eq. (1) for this system gives the well known result [8-10]

$$F_z = eg/c \qquad (5a)$$

where g is the monopole charge and the z axis is along the line between the two particles. This angular momentum is independent of distance as expected from the scale invariance of Eq. (1). Both the long-range and short-range behavior have interesting implications. Here the necessity to restrict the values of the total angular momentum to either an integral or a half-integral multiple of $\hbar$ leads to the Dirac quantization condition [11] for electric and magnetic charges,

$$eg/\hbar c = \text{integer or half integer.} \qquad (5b)$$

The close analogy between the monopole and the solenoid problems can be seen by examining the motion of an electron in the presence of a monopole-antimonopole pair separated by a large distance. In the Dirac description of the pair, with a string singularity in the vector potential along a line joining the pair [11], the magnetic field outside the string is equivalent to the field of an infinitely thin solenoid joining the pair. We can again define the three regions discussed above for the electron-solenoid problem. Here, all the physics is in the monopole fields outside the string, which is analogous to the return flux in the solenoid problem. The Dirac quantization condition (5b) is exactly the same as the Aharonov-Bohm condition to make the flux in the string unobservable in any field-free region. The magnetic field in the solenoid, i.e. along the string, is unobservable when condition (5b) is obeyed.

For the physical solenoid beyond the moon, the flux inside the solenoid and the return flux outside contribute with opposite signs to the angular momentum in the crossed fields of the solenoid and a distant electron, and there is no constraint on the allowed values of the solenoid flux. For the monopole beyond the moon, there is only the magnetic flux emanating from the monopole source and no physical return flux. The flux carried by the

Dirac string is fictitious. The condition that angular momentum must be quantized in units of $\hbar/2$ thus leads to the quantization condition (5b) on the charges, which is equivalent to the condition for the unobservability of the fictitious string flux by an Aharonov-Bohm experiment.

It is noteworthy that the argument leading to the quantization condition (5b) assumes that the electromagnetic field is the only long range field that can give a finite angular momentum to a system of two particles separated by large distances. If another long range field, such as color, can give a finite angular momentum to the system of a charged particle on earth and a monopole beyond the moon, then the quantization condition (5a) must be modified. The value of the monople strength reported by Cabrera is consistent with the Dirac quantization condition (5b) where e is the electronic charge. The apparent contradiction between Cabrera's result [2] and the fractional charge e/3 reported by La Rue et. al. [14] can be resolved if there is an additional unscreened long range field which is coupled to both particles.

Another paradox arises when the charged particle and the monopole are at exactly at the same point. In this case the electric and magnetic fields are exactly in the same direction and the angular momentum in the field is exactly zero. However, the finite angular momentum (5a) must hold for any finite separation of the two particles. The angular momentum in the field thus has a singularity at the origin which must be reflected in any dynamical description. In the quantum-mechanical case this can be seen by examining the behavior of the wave function at the origin for a system of a point charge moving in the field of an infinitely heavy point monopole.

The total angular momentum of the system $\vec{J}$ is again given by Eq. (3) with $\vec{L}$ and $\vec{S}$ now denoting the orbital angular momentum and the total spin of the particle - monopole system. Let $\psi(\vec{x})$ denote the wave function for the charged particle, obtained by solving some wave equation in the monopole field. It could be a Dirac spinor, a Pauli spinor, or some boson tensor. We need not assume any particular dynamics at this point. Our result follows only from kinematics alone. Then

$$(\vec{J} - \vec{S})\ \psi(\vec{x}) = (\vec{L} + \vec{F})\ \psi(\vec{x})\ . \tag{6}$$

Since the angular momentum in the fields $\vec{F}$ is parallel to the vector $\vec{x}$ if $x \neq 0$

$$\vec{x}\cdot(\vec{J} - \vec{S})\ \psi(\vec{x}) = \vec{x}\cdot\vec{F}\ \psi(\vec{x}) = (eg/\hbar c)\ x\ \psi\ (\vec{x}) \tag{7a}$$

At $x = 0$, both $\vec{L}$ and $\vec{F}$ vanish. Thus, from Eq. (6),

$$(\vec{J} - \vec{S})\ \psi(0) = (\vec{L} + \vec{F})\ \psi(0) = 0. \tag{7b}$$

If $\psi\ (\vec{x})$ is an eigenfunction of the total angular momentum and the total spin with the eigenvalues j and s,

$$(\vec{J} + \vec{S})\cdot(\vec{J} - \vec{S})\ \psi(0) = \left[j(j + 1) - s(s + 1)\right]\ \psi(0) = 0. \tag{7c}$$

Thus

$$\psi(0) = 0 \text{ unless } j = s \neq 0. \tag{8}$$

(The partial wave $j = s = 0$ is excluded by the condition (7a)).

The wave function for any state of a charged particle moving in the field of a fixed point multipole must therefore vanish at the origin as a result of the kinematics of angular momentum conservation, except for partial waves with $j = s \neq 0$, which occur only for integral values of F and of the quantization condition (5b). Even in this case the wave function is highly singular at the origin if it does not vanish there. The operator S is a matrix in spin space and is independent of $\vec{x}$. For any wave function which is an eigenfunction of J_z, Eq. (7b) shows that ψ (0) is a spinor which is an eigenfunction of S_z with the same eigenvalue. But Eq. (7a) shows that ψ ($\vec{x}$) must jump discontinuously to a spinor which is not an eigenfunction of S_z with the same eigenvalue. There is a discontinuous spin flip when the particle passes through the origin. This result is independent of the detailed dynamics; e.g. whether the wave function is obtained by solving the Schroedinger equation, the Dirac equation or has an additional spherically symmetric field which conserves angular momentum (e.g. if the monopole is a dyon and also has an electric charge). In simple models, this exclusion of the particle from the origin is brought about by the presence of a centrifugal barrier in the wave equation, which exists [10] even for the case of $j = 0$. But the essential physics is in the discontinuity in the angular momentum of the crossed fields $\vec{F}$ at the origin.

Additional insight into this problem at the origin is obtained by examining the classical orbit of a head-on collision of a point charge with a monopole of finite size described by a magnetic charge density confined within a finite radius. Since the charged particle moves along the radius and is parallel to the magnetic field, it experiences no force and goes through the monopole with constant velocity. The angular momentum in the crossed fields is directed toward the monopole and is zero when the particle is at the origin. But the transition from finite angular momentum to zero is now continuous, rather than singular, since the angular momentum begins to decrease when the particle approaches the monopole radius. At the same time there is a torque on the monopole charge density due to the magnetic field around the moving electric charge. The angular momentum in the field thus decreases to zero and then reverses sign as the particle goes through the monopole, and the monopole's internal angular momentum (spin) changes accordingly as a result of the torque. Since the two-body problem is completely symmetric with respect to electric and magnetic charge, the same situation would occur for a finite sized electric charge and a point monopole. This spin excitation might explain the possibility of spin flip in certain partial waves for the quantum-mechanical case with point particles [15], as indicated by Eqs. (7).

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