THE MOTION OF CHARGED PARTICLES IN SLIGHTLY INHOMOGENEOUS HIGH-FREQUENCY FIELDS

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I. DESCRIPTION OF THE PROBLEM

We shall discuss a non-relativistic movement of the particle in an external electromagnetic field

$$\vec{E}(\vec{r},t) = \sum_{n=1}^{N} \vec{E}_n(\vec{r})e^{i\omega_n t} + \vec{E}_0(\vec{r})$$

$$\vec{H}(\vec{r},t) = \sum_{n=1}^{N} \vec{H}_n(\vec{r})e^{i\omega_n t} + \vec{H}_0(\vec{r}) . \quad (1.1)$$

Frequencies ω_n are considered large but limited by the following condition :

$$\omega_n \ll \frac{mc^3}{e^2} \quad (1.2)$$

where m is the mass, e is the charge of the particle, and c is the velocity of light in vacuum. In this case, as is known [1], it is possible to neglect the retarding force due to the self-radiation of the particle and to write the non-relativistic equation of motion

$$\ddot{\vec{r}} = \eta\vec{E}(\vec{r},t) + \frac{\eta}{c}[\dot{\vec{r}}\,\vec{H}(\vec{r},t)] \quad (1.3)$$

where $\eta = e/m$.

Our task is to find an approximate solution of Eq. (1.3) in the case of slightly inhomogeneous electromagnetic fields, i.e. under the assumption that $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are slowly changing functions of co-ordinates.

Let us try to represent a solution of Eq. (1.3) as a superposition of a smooth motion $\vec{r}^{(0)}(t)$ and a rapidly changing (by direction or velocity) motion $\vec{r}^{(1)}(t)$

$$\vec{r}(t) = \vec{r}^{(0)} + \vec{r}^{(1)}. \quad (1.4)$$

The method of obtaining an averaged equation of motion described hereinafter is analogous to that proposed by P. L. Kapiza [2,3] for the analysis of the oscillations of a pendulum with vibrating suspension. An application of this method to the motion of particles in electromagnetic fields is described in other papers [4-10].

Let us put the following limitations to the solution, Eq. (1.4) and the field of Eq. (1.1)

$$\frac{\dot{r}^{(0)}}{c} \sim \frac{\dot{r}^{(1)}}{c} \ll 1 \quad (1.5)$$

$$\frac{\dot{r}^{(0)}}{\omega L_E} \sim \frac{\dot{r}^{(1)}}{\omega L_E} \ll 1 \quad (1.6)$$

$$\frac{r^{(1)}}{L_E} \ll 1 \quad (1.7)$$

where L_E is the characteristic distance at which the amplitude of the external field undergoes noticeable changes

$$L_E \sim \left|\frac{E}{\nabla E}\right| \sim L_H \sim \left|\frac{H}{\nabla H}\right|. \quad (1.8)$$

Condition (1.6) means that the transit time of the particle in the zone where the field is noticeably inhomogeneous exceeds considerably the period of the high-frequency field $2\pi/\omega_n$. Due to the condition of Eq. (1.6) amplitudes of the rapidly changing part of the solution Eq. (1.4) are proposed to be small (in the scale L_E). Inequalities (1.6) and (1.7) are also at the same time criteria of the slow space variation of the fields. It is clear that in this sense a field may be slightly inhomogeneous, or essentially inhomogeneous, depending on the nature of the motion (1.4) which in its turn is determined by this field.

Assuming beforehand that conditions (1.5), (1.6), (1.7) are fulfilled we shall search for solutions of

Equation (1.3) satisfying them. For this purpose let us represent the field Eq. (1.1) as a series of a small parameter, Eq. (1.7), keeping in the expansion only terms of the first and second orders.

$$\vec{E}(\vec{r},t) \simeq \vec{E}(\vec{r}^{\,(0)},t)+(\vec{r}^{\,(1)}\nabla)\vec{E}+\tfrac{1}{2}\sum_{i,j} r_i^{(1)}r_j^{(1)}\frac{\partial^2\vec{E}}{\partial r_i\partial r_j}$$
$$\vec{H}(\vec{r},t) \simeq \vec{H}(\vec{r}^{\,(0)},t)+(\vec{r}^{\,(1)}\nabla)\vec{H}+\tfrac{1}{2}\sum_{i,j} r_i^{(1)}r_j^{(1)}\frac{\partial^2\vec{H}}{\partial r_i\partial r_j} \tag{1.9}$$

where r_i, r_j are components of the radius vector along the orthogonal co-ordinate directions. Substitution of Eq. (1.9) into Eq. (1.3) gives the following equation:

$$\ddot{\vec{r}}^{\,(0)}+\frac{\partial^2 r^{(1)}}{\partial t^2}+2(\dot{\vec{r}}^{\,(0)}\nabla)\frac{\partial\vec{r}^{\,(1)}}{\partial t}+(\ddot{\vec{r}}^{\,(0)}\nabla)\vec{r}^{\,(1)}+(\dot{\vec{r}}^{\,(0)}\nabla)^2\vec{r}^{\,(1)} =$$
$$=\eta\vec{E}+\eta(\vec{r}^{\,(1)}\nabla)\vec{E}+\eta/2\sum_{i,j} r_i^{(1)}r_j^{(1)}\frac{\partial^2\vec{E}}{\partial r_i\partial r_j}+\eta/c[\dot{\vec{r}}^{\,(0)}\vec{H}]+\eta/c\left[\frac{\partial\vec{r}^{\,(1)}}{\partial t}\vec{H}\right]+$$
$$+\eta/c[(\dot{\vec{r}}^{\,(0)}\nabla)\vec{r}^{\,(1)}\vec{H}]+\eta/c[\dot{\vec{r}}^{\,(0)}(\vec{r}^{\,(1)}\nabla)\vec{H}]+\eta/c\left[\frac{\partial\vec{r}^{\,(1)}}{\partial t}(\vec{r}^{\,(1)}\nabla)\vec{H}\right]. \tag{1.10}$$

It is taken into account here that

$$\frac{d\vec{r}}{dt}=\dot{\vec{r}}^{\,(0)}+\frac{\partial\vec{r}^{\,(1)}}{\partial t}+(\dot{\vec{r}}^{\,(0)}\nabla)\vec{r}^{\,(1)} \tag{1.11}$$

$$\frac{d^2\vec{r}}{dt^2}=\ddot{\vec{r}}^{\,(0)}+\frac{\partial^2\vec{r}^{\,(1)}}{\partial t^2}+2(\dot{\vec{r}}^{\,(0)}\nabla)\frac{\partial\vec{r}^{\,(1)}}{\partial t}+(\ddot{\vec{r}}^{\,(0)}\nabla)\vec{r}^{\,(1)}+(\dot{\vec{r}}^{\,(0)}\nabla)^2\vec{r}^{\,(1)} \tag{1.12}$$

Equation (1.10) is the starting equation in examining the motion of particles in slightly inhomogeneous fields. The possibility of its further simplification depends on the relation between the amplitudes of vectors of the electric and magnetic fields. The motion in purely static and quasi-static fields is discussed, for example, in the papers by Hellwig [11] and Bogolyubov and Mitropolskij [12]. Our attention will be focused on the systems with high-frequency fields. We shall assume, in particular, that the motion takes place in the zones where $E \sim H$ or $E \gg H$. (As it will become clear from the following, a particle cannot be held inside zones where $E \ll H$ and inevitably will get into zones where $E \sim H$ and where its movement goes on in conformity with the Equation (2.5). About the nature of the motion of particles in the zone $H \gg E$ see, for example, the paper by Vedenov and Rudakov [13].)

One may easily see that the conditions (1.5), (1.6) and (1.7) are not independent. Thus, at $L_E \sim \lambda\!\!\!^{-} = \lambda/2\pi$ (a typical case for high-frequency distributed systems) limitations (1.5) and (1.6) become equivalent and, if we take into account that $r^{(1)} \sim \eta E/\omega$, then the inequality (1.7) will also come to (1.6). Hence, in Equation (1.10) which is as valid as Eq. (1.3) is, only to the terms of the first order $\dot{r}/c$ inclusive, it is necessary to leave out of the sequence the terms of second order according to any of the parameters given in Eqs. (1.5), (1.6) and (1.7). This circumstance will simplify considerably further calculations.

2. AVERAGED EQUATIONS OF MOTION IN A MONOCHROMATIC ELECTROMAGNETIC FIELD

We shall begin with the simplest case. Let us assume that the field Eq. (1.1) is purely monochromatic. Then the following will be obtained directly from Equation (1.10) for a rapidly oscillating motion, taking into account the infinitesimal terms of the first order

$$\frac{\partial^2\vec{r}^{\,(1)}}{\partial t^2}+2(\dot{\vec{r}}^{\,(0)}\nabla)\frac{\partial\vec{r}^{\,(1)}}{\partial t}=\eta\vec{E}+\{\eta(\vec{r}^{\,(1)}\nabla)\vec{E}\}_{\omega,2\omega}+\left\{\eta/c\left[\frac{\partial\vec{r}^{\,(1)}}{\partial t}\vec{H}\right]\right\}_{\omega,2\omega}+\eta/c[\dot{\vec{r}}^{\,(0)}\vec{H}]. \tag{2.1}$$

Here the function $\vec{r}^{(0)}(t)$ is treated as a parameter and the sign of $\{\}_\alpha$ determines the operation of separating harmonics, frequencies of which are given in the index.

The solution of Eq. (2.1) may be found with the required accuracy by the perturbation method and has the following form :

$$\vec{r}^{(1)} = \left\{ -\frac{\eta \vec{E}}{\omega^2} + \frac{2\eta}{i\omega^3}(\dot{\vec{r}}^{(0)}\nabla)\vec{E} - \frac{\eta}{\omega^2 c}[\dot{\vec{r}}^{(0)}\vec{H}] \right\}_\omega + \\ + \frac{\eta^2}{8\omega^4}\{\vec{A}_a \cos 2\omega t + \vec{A}_b \sin 2\omega t\}_{2\omega} . \quad (2.2)$$

The first braces in Eq. (2.2) contain terms written in complex form changing with time at a frequency ω : the second braces contain terms which, in the actual form, change at a frequency 2ω; we have

$$\begin{aligned} \vec{E} &= \vec{E}_a \cos \omega t + \vec{E}_b \sin \omega t \\ \vec{H} &= \vec{H}_a \cos \omega t + \vec{H}_b \sin \omega t \end{aligned} \quad (2.3)$$

$$\begin{aligned} \vec{A}_a &= \{(\vec{E}_a\nabla)\vec{E}_a - (\vec{E}_b\nabla)\vec{E}_b + [\vec{E}_b \text{rot} \vec{E}_a] - [\vec{E}_a \text{rot} \vec{E}_b] \\ \vec{A}_b &= \{(\vec{E}_a\nabla)\vec{E}_b + (\vec{E}_b\nabla)\vec{E}_a - [\vec{E}_a \text{rot} \vec{E}_b] - [\vec{E}_b \text{rot} \vec{E}_a] \end{aligned} \quad (2.4)$$

Substitution of Eq. (2.2) into (1.10) and averaging over a period $2\pi/\omega$ brings one to the averaged equation for $\vec{r}^{(0)}(t)$; in this case it appears that the terms $\{\}_{2\omega}$, in the approximation $(\dot{r}/c)$, do not give any contribution to the averaged force : their contribution is essentially of the second order when the initial Equation (1.10) is already incorrect. Omitting the infinitesimal terms of the second order in all the parameters of Eqs. (1.5), (1.6) and (1.7) we obtain

$$\ddot{\vec{r}}^{(0)} = -\nabla\Phi + \vec{F}(\dot{\vec{r}}^{(0)}, \vec{E}) \quad (2.5)$$

where

$$\Phi = (\eta/2\omega)^2 |\vec{E}|^2 \quad (2.6)$$

$$\vec{F} = \frac{\eta^2}{2\omega^3} Im\{2[(\dot{\vec{r}}^{(0)}\nabla)\vec{E}\nabla)]\vec{E}^* + [(\dot{\vec{r}}^{(0)}\nabla)\vec{E} \,\text{rot}\, \vec{E}^*] + \\ + ([\dot{\vec{r}}^{(0)} \text{rot} \vec{E}]\nabla)\vec{E}^* - [\dot{\vec{r}}^{(0)}(\vec{E}\nabla)\text{rot}\vec{E}^*]\} . \quad (2.7)$$

We shall call function $\Phi(\vec{r})$ the high-frequency potential. As $F/|\nabla\Phi| \sim \dot{r}^{(0)}/\omega L_E$ the averaged motion appears to derive from a potential with an accuracy to the terms of the first order. (Precisely this case is discussed in our papers [4, 6, 7] and in papers by Boot et al. [8, 9].

In this approximation Equation (2.5) has a clear physical sense : instead of the exact equation for the particle Eq. (1.3) we discuss an approximated Equation averaged (over the period) of the oscillator (quasi-particle).

$$\vec{r}^{(1)} = -\frac{\eta}{\omega^2}\vec{E}, \quad \dot{\vec{r}}^{(1)} = -i\frac{\eta}{\omega}\vec{E} . \quad (2.8)$$

The averaged force acting upon this oscillator is composed of the ponderomotive force of the inhomogeneous electric field acting upon the electric dipole $\vec{p}^{\,e} = e\vec{r}^{(1)}$

$$\widetilde{\vec{f}_{el}}^{2\pi/\omega} = \widetilde{(\vec{p}^{\,e}\nabla)\vec{E}}^{2\pi/\omega} = -\widetilde{\frac{e\eta}{\omega^2}(\vec{E}\nabla)\vec{E}}^{2\pi/\omega} \quad (2.9)$$

and of the Lorentz force acting upon the current element $\vec{j}^{\,e} = e\dot{\vec{r}}^{(1)}$

$$\widetilde{\vec{f}}_{mag} = \frac{e}{c}[\dot{\vec{r}}^{(1)}\vec{H}] = -\frac{e\eta}{\omega^2}[\vec{E}\,\text{rot}\,\vec{E}] \quad (2.10)$$

Combining Eqs. (2.9) and (2.10) we obtain the following equation

$$\ddot{\vec{r}}^{(0)} = -\nabla\Phi \quad (2.11)$$

which is a particular case of Eq. (2.5). Naturally the above derivation of this equation, does not contain any new statement but permits the adopted approximations, to be followed in sequence.

3. AVERAGED EQUATIONS OF MOTION IN MULTI-FREQUENCY FIELDS

It is not difficult to make a generalization of Equation (2.5) to a system with multi-frequency fields, static fields included. Further, for simplicity, we shall confine ourselves to accounting only for the main terms in the averaged Equation (2.5), i.e. to deal, practically, with the generalization of Equation (2.11).

In case the outside field is presented as a superposition of several monochromatic fields, the frequencies of which differ from zero and satisfy the requirements of Eqs. (1.6), (1.7) the oscillatory motion $\vec{r}^{(1)}(t)$ may be written in a form analogous to that of Eq. (2.8), namely

$$\vec{r}^{(1)} = -\eta \sum \frac{\vec{E}_n(\vec{r}^{(0)})}{\omega_n^2} e^{i\omega_n t} . \quad (3.1)$$

Now, when substituting into Eq. (1.10) and averaging relative to time it becomes necessary to take into account the cross terms containing products of fields with various frequencies. The latter disappear only upon averaging over an interval of time exceeding greatly the periods of all the " partial " oscillations $2\pi/\omega$, as well as the periods of all the combination frequencies $2\pi/\omega_n \pm \omega_m$ which should also satisfy the condition of Eq. (1.6). The averaged equation will then coincide with (2.11) and the high-frequency potential Φ, which it contains, is composed of potentials corresponding to the " partial " fields.

$$\Phi = \sum_{n=1}^{N} \Phi_n = (\eta/2)^2 \sum_{n=1}^{N} \left|\frac{E_n}{\omega_n}\right|^2 . \tag{3.2}$$

This principle of the superposition of potentials avoids the different frequencies used in the papers by Knox[14,15] and broadens considerably the possibility of making potential simplifications of arbitrary kind.

The case is a little different when static fields are present. We shall explain this by means of two examples.

Let a monochromatic high-frequency field be added to a slightly inhomogeneous electrostatic field

$$\vec{E}_0 = \vec{E}_0(\vec{r}^{(0)}) + (\vec{r}^{(1)}\nabla)\vec{E}_0(\vec{r}^{(0)}) . \tag{3.3}$$

If the characteristic distance L_0, at which this field changes considerably, satisfies the condition

$$\frac{r^{(1)}}{L_E} \sim \frac{r^{(1)}}{L_0}\frac{E_0}{E_1} \ll 1 \tag{3.4}$$

then the equation for the fast oscillating motion written as

$$\ddot{\vec{r}}^{(1)} - \eta(\vec{r}^{(1)}\nabla)\vec{E}_0 = \eta\vec{E}_1(\vec{r}^{(0)})e^{i\omega t} \tag{3.5}$$

will have the following solution with an accuracy of the order of $\sim \frac{r^{(1)}}{L_0}\frac{E_0}{E_1}$:

$$\vec{r}^{(1)} = -(\eta/\omega^2)\vec{E}_1(\vec{r}^{(0)})e^{i\omega t} + (\eta/\omega^2)^2(\vec{E}_1\nabla)\vec{E}_0 e^{i\omega t}.$$

By substituting Eq. (3.6) into Eq. (1.10) and taking into account Eq. (3.4), after averaging over the period $2\pi/\omega$, we obtain

$$\ddot{\vec{r}}^{(0)} = -\nabla\Phi + \tfrac{1}{2}(\eta^3/\omega^4)Re\{[(\vec{E}_1\nabla)\vec{E}_1\nabla]\vec{E}_1^* + [(\vec{E}_1\nabla)\vec{E}_0 \mathrm{rot}\,\vec{E}_1^*]\} \tag{3.7}$$

where

$$\Phi = (\eta/2\omega)^2|\vec{E}_1|^2 + \eta\phi_{st}$$
$$\vec{E}_0 = -\nabla\phi_{st} . \tag{3.8}$$

(We are supposing that the frequency ω and its subharmonics are far from the free frequencies of the homogeneous Equation (3.5).)

The expression on the right hand side of Eq. (3.7) cannot be readily represented in the general case as a potential vector.

Let us discuss the second example. Let the system contain the monochromatic high-frequency field and a slightly inhomogeneous magnetic-static field

$$\vec{H}_0(\vec{r}) = \vec{H}_0(\vec{r}^{(0)}) + (\vec{r}^{(1)}\nabla)\vec{H}_0 \tag{3.9}$$

Assuming $r^{(1)} \ll L_0$ we shall conserve, however, in the averaged equation terms of order $\dot{r}^{(1)}/c \cdot r^{(1)}/L_0 \cdot H_0/H_1$, i.e. at $L_0 \sim L_E$ we shall consider that

$$\frac{r^{(1)}}{L_0}\frac{H_0}{H_1} \lesssim 1 . \tag{3.10}$$

This means that the static field intensity may exceed considerably the amplitude of the variable field.

Let us introduce the orthogonal trihidron of unit vectors $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3$, directing $\vec{\tau}_3$ along the lines of $\vec{H}$ and marking the $\vec{\tau}_3$-components with the index " $\|$ ", and the orthogonal vectors—with the index $\perp$. We shall represent the latter as a sum of vectors with a right and left circular polarization

$$\vec{E}_\perp = \vec{E}_\perp^{(+)}(\vec{r}) + \vec{E}_\perp^{(-)}(\vec{r})$$
$$\vec{H}_\perp = \vec{H}_\perp^{(+)}(\vec{r}) + \vec{H}_\perp^{(-)}(\vec{r}) \tag{3.11}$$

whereupon

$$\vec{E}_\perp^{(+)} = \frac{E^{(+)}}{\sqrt{2}}(\vec{\tau}_1 - i\vec{\tau}_2)$$
$$\vec{E}_\perp^{(-)} = \frac{E^{(-)}}{\sqrt{2}}(\vec{\tau}_1 - i\vec{\tau}_2) . \tag{3.12}$$

Then the equation for the rapidly oscillating motion may be written from Eq. (1.10) as

$$\ddot{\vec{r}}^{(1)}-\omega_H[\dot{\vec{r}}^{(1)}\vec{\tau}_3]=\eta\vec{E}_{||}(\vec{r}^{(0)})e^{i\omega t}+\eta\vec{E}_{\perp}(\vec{r}^{(0)})e^{i\omega t}. \tag{3.13}$$

It is evident, that generally speaking, the fast oscillating motion will contain oscillations with the field frequency ω and also with the cyclotron frequency $\omega_H=\eta H_0/c$

$$\vec{r}(t)=\vec{r}^{(0)}(t)+\vec{r}_{\omega}^{(1)}(t)+\vec{r}_{\omega H}^{(1)}(t). \tag{3.14}$$

Therefore, depending on the relation between the frequencies ω and ω_H it is necessary if $\omega\gg\omega_H$, to first average over $2\pi/\omega$ and the motion $r_{\omega H}^{(1)}(t)$ may be then added to the smooth motion $\vec{r}^{(0)}(t)$, or, if $\omega\ll\omega_H$, to first average over $2\pi/\omega_H$ and combine $r_{\omega}^{(1)}(t)$ to the smooth motion $\vec{r}^{(0)}(t)$; then, in the obtained drift equation, one has to average anew over the period $2\pi/\omega$ as well as over the period of all the difference frequencies. However, as the equation for $\vec{r}^{(0)}(t)$ before its averaging over $2\pi/\omega_H$ has a free solution itself which corresponds to the rotation of the particle at a cyclotron frequency, we may in the second case formally include the solution of $\vec{r}_{\omega H}^{(1)}(t)$ into the non-averaged part of the solution $\vec{r}^{(0)}(t)$:

$$\vec{r}_{\omega H}^{(0)}(t)=\vec{r}^{(0)}(t)+\vec{r}_{\omega H}^{(1)}(t) \tag{3.15}$$

and discuss only the forced solution of Eq. (3.13), where ω_H is a slowly changing function of the co-ordinates.

Let us write this solution in the component form along the co-ordinate directions $\vec{\tau}_1$, $\vec{\tau}_2$, $\vec{\tau}_3$ as

$$\vec{r}_{\omega}^{(1)}=a_1\vec{\tau}_1+a_2\vec{\tau}_2+a_3\vec{\tau}_3$$

$$a_1=\frac{i\eta}{\omega\sqrt{2}}\left(\frac{E_{\perp}^{(+)}}{\omega+\omega_H}-\frac{E_{\perp}^{(-)}}{\omega-\omega_H}\right)$$

$$a_2=-\frac{\eta}{\omega\sqrt{2}}\left(\frac{E_{\perp}^{(+)}}{\omega+\omega_H}+\frac{E_{\perp}^{(-)}}{\omega-\omega_H}\right).$$

$$a_3=-(\eta/\omega^2)E_{||} \tag{3.16}$$

Substitution of Eq. (3.16) into Eq. (1.10) with further averaging over the period $2\pi/\omega$ leads, after a number of calculations treated in detail in another paper [7] to the averaged equation for $\vec{r}_{\omega H}^{(0)}$

$$\overbrace{\ddot{\vec{r}}_{\omega H}^{(0)}-\omega_H[\dot{\vec{r}}_{\omega H}^{(0)}\vec{\tau}_3]}^{2\pi/\omega}=-\nabla\Phi+1/m(\tilde{\vec{p}}_{\perp}^{m}\nabla)\vec{H}_0 \tag{3.17}$$

where

$$\Phi=(\eta/2\omega)^2\left\{|\vec{E}_{||}|^2+\frac{\omega}{\omega+\omega_H}|\vec{E}_{\perp}^{(+)}|^2+\frac{\omega}{\omega-\omega_H}|\vec{E}_{\perp}^{(-)}|^2\right\} \tag{3.18}$$

$$\tilde{\vec{p}}^{m}=\tilde{\vec{p}}_{||}^{m}+\tilde{\vec{p}}_{\perp}^{m}=\frac{e}{2c}[\vec{r}_{\omega}^{(1)}\dot{\vec{r}}_{\omega}^{(1)}]=\frac{e\omega}{4c}Im[\vec{r}_{\omega}^{(1)}\vec{r}_{\omega}^{(1)*}] \tag{3.19}$$

i.e.

$$\tilde{\vec{p}}_{||}^{m}=\frac{e\omega}{4c}\vec{\tau}_3(|\vec{r}_{\perp}^{(+)(1)}|^2-|\vec{r}_{\perp}^{(-)(1)}|^2) \tag{3.20}$$

$$\tilde{\vec{p}}_{\perp}^{m}=\frac{e\omega}{2c}Im\,a_3(a_1^*\vec{\tau}_2-a_2^*\vec{\tau}_1) \tag{3.21}$$

Thus, in the general case the right hand side of Eq. (3.17) appears to be non-potential due to the appearance of the additional force due to the action of the inhomogeneous magnetostatic field upon the time independent magnetic dipole, Eq. (3.19), determined by the rotation of the particle at a frequency ω. This force is absent in the case of a purely homogeneous magnetic field. (The presence of $\vec{p}_{\perp}^{m}$ causes a precession of the dipole $\vec{p}^{m}$ in the homogeneous magnetic field but it does not lead to a displacement under the approximation considered of the centre of oscillations and therefore, the precession movement is averaged and falls out from the equation for $\vec{r}^{(0)}(t)$. It is absolutely necessary to take into account the precession when examining the movement in the quasi-stationary magnetic field ($H_1\gg E_1$) and in the homogeneous constant magnetic fields. For example see the paper by Vedenov and Rudakov [13].

The right hand side of Eq. (3.17) derives also from a potential for $a_3=0$, i.e. when $\vec{E}_{\perp}\vec{H}_0$. In such a field there is only a rotation of the particles in a plane perpendicular to the lines of H_0.

Let us clarify, at last, the meaning of the left hand side of Eq. (3.17). As already mentioned, for $\omega\gg|\omega_H|$ the function $\vec{r}_{\omega H}^{(0)}(t)$ is a slow time function (in the scale $2\pi/\omega$), and averaging over $2\pi/\omega_H$ is of no special interest; the Equation (3.17) may then be used as a usual averaged equation without any reservations. When ω tends to ω_H, as in the case of $\omega\ll\omega_H$, averaging over the period $2\pi/\omega_H$ (and relative to the periods of the combination frequencies) is absolutely necessary. As is known, this averaging brings us to the equation of the drift approximation.

In this case the Equation (3.17) practically reduces to the following:

$$\ddot{\vec{r}}^{(0)} - \omega_H[\dot{\vec{r}}^{(0)}\vec{\tau}_3] = -\nabla\Phi + \frac{1}{m}(\vec{p}_\perp^{\,m}\nabla)\vec{H}_0 + \frac{1}{m}(\vec{p}_{||0}^{\,m}\nabla)\vec{H}_0 \tag{3.22}$$

where

$$\vec{p}_{||0}^{\,m} = \frac{e}{4c}\mathrm{Re}[\vec{r}_{\omega H}^{\,(1)}\dot{\vec{r}}_{\omega H}^{\,(1)*}] = -\frac{\omega_H}{4}|\vec{r}_{\omega H}^{\,(1)}|^2\vec{\tau}_3$$

$$\vec{r}_{\omega H}^{\,(1)}(t) = \frac{r_{\omega H}^{(1)}}{\sqrt{2}}(\vec{\tau}_1 + i\vec{\tau}_2)e^{i\omega_H t}$$

and $\vec{r}^{\,(0)}(t)$ is a slowly changing function of time.

In the same way it is possible to obtain the averaged equations of more complex systems and also to draw their generalization on account of infinitesimal terms of the next order according to parameters of Eqs. (1.5)-(1.7), as was made in Section 2 in the example of a single-frequency field.

4. THE INTEGRAL OF AVERAGED ENERGY

The averaged equations of motion are essentially simpler than the initial equations (1.10); their right hand side is not time dependent and in some cases may even be a vector potential. The latter circumstance permits to obtain the first integral of motion in the general form. We shall explain this, first, by the example of motion in the single-frequency field. Multiplying Eq. (2.11) by $\dot{\vec{r}}^{(0)}$ and integrating we find

$$\frac{\dot{\vec{r}}^{(0)2}}{2} + \Phi(\vec{r}^{\,(0)}) = \text{const.} \tag{4.1}$$

This expression should be interpreted as an integral of the averaged energy. As a matter of fact, substituting the value $\vec{E}$ from Eq. (2.8) into Eq. (2.6) we can give the relation (4.1) the following appearance

$$\frac{(\dot{\vec{r}}^{(0)})^2}{2} + \frac{(\dot{\vec{r}}^{(1)})^2}{2} = \frac{(\dot{\vec{r}}^{(0)})^2}{2} + \frac{|\dot{\vec{r}}^{(1)}|^2}{4} = \text{const.} \tag{4.2}$$

Thus, the sum of the kinetic energy of the smooth motion $\vec{r}^{\,(0)}(t)$ and the kinetic energy of the rapidly oscillating motion (averaged over the time) is conserved. The most intensive oscillations at the frequency of the external field take place at a full stop of the particle (in the scale of the smooth motion), i.e. for $\dot{\vec{r}}^{(0)} = 0$. This ability to transform the energy of the progressive motion of the particle *into that of oscillatory motion without energy contribu*tion from the external field is the basis of all the proposals for using slightly inhomogeneous high-frequency fields in various devices. In this case an important role is played by the independence of the potential $\Phi(r)$, and hence of the averaged movement of particles, from the sign of their charge.

It is interesting to note that the integral Eq. (4.1) is also valid for Equation (3.5), as it is not difficult to prove the validity of the equality

$$\dot{\vec{r}}^{(0)}\vec{F}(\vec{r}^{\,(0)}, \vec{E}) = 0\,.$$

However, taking into account terms of the next order of Eqs. (1.5)-(1.7) does not permit the interpretation of Eq. (4.1) as the integral of the averaged energy, as the relation between r and E, given by the expressions Eq. (2.2), is now essentially different.

The same condition is observed in the systems with the inhomogeneous high-frequency and the homogeneous magnetostatic fields where the integral Eq. (4.1) may be given the following form:

$$\frac{\dot{\vec{r}}^{(0)2}}{2} + \frac{|\dot{\vec{r}}_{||}^{(1)}|^2}{4} + \frac{|\dot{\vec{r}}_{\perp}^{(1)}|^2}{4} + \frac{W_{interact}}{m} = \text{constant.} \tag{4.3}$$

where

$$\frac{W_{interact}}{m} = \frac{\vec{p}_{||}^{\,m}\vec{H}_0}{m} = \frac{\omega_H\omega}{4}(|\vec{r}_\perp^{\,(+)(1)}|^2 - |\vec{r}_\perp^{\,(-)(1)}|^2)$$

is the interaction energy between the magnetic dipole, Eq. (3.19), and the field H_0. Hence, it is necessary in this case to also include in the energy integral additional terms connected with the interaction between the moving oscillator and the external field. This makes it impossible to transform fully the kinetic energy of the smooth motion into the kinetic energy of the oscillatory motion.

5. POSSIBLE APPLICATIONS

An investigation of the solutions of the averaged equations permit one to obtain quite a visible outline of the particle motion in the general form, and, hence, to analyse from a sufficiently general point of view the problem of applicability of the inhomogeneous high-frequency fields in controlling the movement of charged particles.

Thus, a high-frequency optics for charged particles can be created and in particular the high-frequency electronic optics.

If the conditions Eqs. (1.5)-(1.7) are observed and the averaged Equation (2.11) is valid, then the following expression can be taken as the electron-optical index of refraction.

$$n(\vec{r}) = \sqrt{v_0^2 - \Phi(\vec{r})}\,. \qquad (5.1)$$

(When we speak of the electronic optics, we have in mind that in principle all the statements refer to the movement of any charged particle, as well as, within certain approximations, to the movement of clusters of the quasineutral plasma.)

This index of refraction is independent of the charge sign because of Eq. (2.6), but depends on the initial velocity v_0, which is defined by Eq. (5.1) as the velocity of particles at points where $\Phi(\vec{r}) = 0$. In the high-frequency fields as well as in the multi frequency fields, even arbitrary different distributions of the index of refraction, Eq. (5.1), occur. Consequently, an analogous high-frequency distribution may be created corresponding to any non-relativistic static electron-optical device. Naturally, almost all the methods of calculations for such devices must be transferred from static optics to the high-frequency optics.

There are however important differences. From the point of view of distribution of $n(\vec{r})$ the high-frequency systems are more numerous due to the less strict limitations on $\Phi(\vec{r})$. In particular, with the help of the high-frequency potential we may have absolute wells and absolute peaks—a situation which is impossible in electronic optics with purely electrostatic controlling fields without space charge. On the other hand, the high-frequency systems are more limited than the static ones mainly due to the approximate nature of the averaged description of the movement of particles. Firstly, this makes the oscillating " undetermination " of the averaged trajectory inevitable and, hence, limits the accuracy of the electron-optical image ($\sim\eta E/\omega^2$) and secondly, it does not allow the use of particles with relativistic velocities. Lastly, the high-frequency methods of controlling the movement of particles are worse than the static methods from energy considerations. In order to form the required high-frequency field, high, and sometimes even gigantic, power sources ($>10^8$W) should be necessary. Therefore, these methods may be applied only when the static ones are unsuitable for certain reasons, or when the presence of the high-frequency field is at any rate required for other purposes. A good example is the control system of the quasineutral plasma clusters or electronic instruments designed for the amplification and generation of powerful high-frequency oscillations.

With the help of the high-frequency fields it is possible to make reflecting mirrors for charged particles or plasma clusters [16, 17] as well as lenses. By forming two-dimensional wells of the high-frequency potential Φ, it is possible to build a system focusing the plasma beams [5, 18, 19], and inside three-dimensional wells to localise plasma clusters [4, 6, 8, 9, 10, 14, 15]. By displacing potential wells with the plasma located inside, it is possible to obtain an acceleration of the latter [20, 21]. (This problem is discussed in detail in another of our reports " Plasma acceleration in slightly inhomogeneous high-frequency fields ", see p. 167). At last, to make a coherent selection of particles passing towards a gradually increasing potential barrier at the moment when their full speed becomes zero, it is possible to transform the whole kinetic energy of the beam into the high-frequency energy, thus building a high-frequency oscillator by using this principle [22, 23].

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