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An Estimate of the Sixth Order Contribution to the Anomalous Magnetic Moment of the Electron*

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Abstract

An improvement of the estimate by Drell and Pagels of the anomalous magnetic moment of the electron is given. Use is made of a sidewise dispersion relation in which the mass W^2 of one of the external electron lines is analytically continued off its mass shell. Only one-electron one-photon states are retained in the absorptive amplitude, but this is sufficient to obtain all terms in the absorptive part of the anomalous moment proportional to $(W^2 - m^2)$ and $(W^2 - m^2)^2$ as $W^2 \to m^2$. This calculation relates $\frac{1}{2}(g-2)$ to the Compton amplitude in its exact threshold region. An expansion is made in powers of the photon energy in the Compton amplitude rather than a perturbation calculation in powers of $\alpha \approx 1/137$. The Schwinger term $\alpha/2\pi$ is reproduced exactly. A cutoff is chosen so that the fourth order term $-0.328\ \alpha^2/\pi^2$ is approximately reproduced. This cutoff leads to an estimate of the sixth order term of $\approx +0.13\ \alpha^3/\pi^3$.

I. INTRODUCTION

Recently, Drell and Pagels[1] (hereafter referred to as DP) have given an estimate of the sixth order contribution to the anomalous magnetic moment of the electron. It is the purpose of this paper to improve this estimate and to make more plausible the approximations involved.

Drell and Pagels wrote a sidewise dispersion relation for the anomalous magnetic moment in which the mass W^2 of one of the external electron lines is analytically continued off its mass shell. They kept relativistic kinematics and used the full content of the <u>exact</u> low energy theorem on Compton scattering including the magnetic moment terms which are linear in the energy. Their calculation reproduced exactly the Schwinger[2] term $\alpha/2\pi$. When they included the moment to order α in the Compton amplitude, they obtained the correct sign and approximately the correct magnitude for the Sommerfield-Petermann-Terent'ev[3] term $-0.328\,\alpha^2/\pi^2$. When the moment to order α^2 was included in the Compton amplitude, a prediction of $\approx +\,0.15\,\alpha^3/\pi^3$ was obtained for the sixth order contribution to the moment. These predictions were obtained by using a cutoff of $\approx 6m^2$ on the dispersion integral for the anomalous moment. Thus it was hoped that the magnetic moment was dominated by the low energy region of the dispersion integral.

In this paper, we calculate additional terms which contribute to the dispersion relation to the same order in the energy as the terms linear in energy in the low energy theorem for Compton scattering. In fact, we obtain <u>all</u> terms in the absorptive part of the anomalous moment proportional to $(W^2 - m^2)$ <u>and</u> $(W^2 - m^2)^2$ as $W^2 \longrightarrow m^2$. These terms will improve the accuracy of the dispersion relation in the low energy region and lead to a better estimate for $\frac{1}{2}(g-2)$.

II. CALCULATIONS

Let us review the calculation of Drell and Pagels.[1] They used the fact that the Feynman amplitude for Fig. 1 satisfies a dispersion relation in the invariant mass $W^2 = (p + \ell)^2$ of one of the external lines, with the other two lines on their respective mass shells. This property is valid to all finite orders in perturbation theory. The assumption was made that the dispersion relation for the anomalous magnetic moment part of the interaction requires no subtractions. Otherwise, the anomalous magnetic moment $\frac{1}{2}(g-2)$, like the charge e, would be a parameter of the theory.

The scalar functions multiplying the spinor factors in the amplitude for Fig. 1 are analytic functions of W^2 in the cut W^2 plane with a branch cut from m^2 to $+\infty$.[4] The absorptive parts of these amplitudes are given by the discontinuities across the branch cut and are obtained by setting the internal photon and electron lines on their respective positive energy mass shells. This is done by replacing the propagators of these internal particles by[5,6]

$$(q^2 + i\epsilon)^{-1} \ (k^2 - m^2 + i\epsilon)^{-1} \longrightarrow 2\pi^2 \ \delta(k^2 - m^2) \ \theta(k_o) \ \delta(q^2) \ \theta(q_o) \ . \tag{1}$$

The absorptive amplitude is then given by

$$\text{Abs}\left\{ \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \ (2\pi)^4 \ \delta^4 \ (q + k - p - \ell) \ \bar{u}(p) \ N(q, \ k, \ p) \ \frac{1}{q^2 + i\epsilon} \ \frac{1}{k^2 - m^2 + i\epsilon} \right\}$$

$$= \frac{W^2 - m^2}{32\pi W^2} \int_{-1}^{1} dx \ \bar{u}(p) \ N(W^2, x) \ , \tag{2}$$

where $x = \underset{\sim}{q} \cdot \underset{\sim}{\ell} / |\underset{\sim}{q}||\underset{\sim}{\ell}|$ in the frame where $p + \ell = (W, \underset{\sim}{0})$ and $\underset{\sim}{q} = -\underset{\sim}{k}$. The function N is the numerator factor in the amplitude and is a polynomial which does not influence the analytic properties of the amplitude when the two ends of the internal photon are tied together by[7] $g_{\mu\nu}$. The "cut" graphs corresponding to the absorptive amplitude of Eq. (2) are shown in Fig. 2. These graphs illustrate how the absorptive part of the amplitude is obtained by multiplying the electromagnetic current by the Compton amplitude and then integrating over the scattering angle $\cos\theta = x$ as in Eq. (2).

In DP, the electromagnetic current was represented by $\bar{u}(k)\,\gamma^{\nu}\,u(p)$, while in this paper the most general possible expression[4] for a current of this type will be used. Thus while DP had the threshold behavior of the absorptive part of the amplitude correct, they obtained only some of the terms proportional to the next order in energy $[(W^2 - m^2)^2]$. It is the purpose of this paper to obtain all terms in the absorptive part of the amplitude proportional to $(W^2 - m^2)^2$ in the limit $W \to m$. This will increase the accuracy of the calculation in the low energy region and will make the cutoff procedure more plausible.

The most general expression for an electromagnetic vertex with the emerging photon and electron on their respective mass shells ($\ell^2 = 0$ and $p^2 = m^2$) is of the form[4]

$$e\,\bar{u}(p)\Gamma_{\mu}(p, p+\ell) =$$

$$e\,\bar{u}(p)\Big[\Big\{F_1^+(W^2)\,\gamma_\mu + F_2^+(W^2)\,(-i\sigma_{\mu\nu}\,\ell^\nu/2m) + F_3^+(W^2)\,\ell_\mu\Big\}\,(\not{p} + \not{\ell} + m)/2m$$

$$+ \Big\{F_1^-(W^2)\,\gamma_\mu + F_2^-(W^2)\,(-i\sigma_{\mu\nu}\,\ell^\nu/2m) + F_3^-(W^2)\,\ell_\mu\Big\}\,(-\not{p} - \not{\ell} + m)/2m\Big]. \quad (3)$$

The scalar functions $F_i^{\pm}(W^2)$ are functions of the invariant $W^2 = (p+\ell)^2$ and are analytic functions in the cut W^2 plane with a branch point at $W^2 = m^2$. This is illustrated in Fig. 3. For $W^2 > m^2$, $F_i^{\pm}(W^2) \equiv \lim_{\epsilon \to 0^+} F_i^{\pm}(W^2 + i\epsilon)$. In addition, the Ward-Takahashi[8] identity requires

$$\bar{u}(p)\Gamma_\mu(p, p+\ell)\,\ell^\mu = \bar{u}(p)\,\not{\ell} \;.$$

It follows from this that $F_1^+(W^2) = F_1^-(W^2) = 1$ for all W^2. Thus the charge as well as the normal Dirac moment $g = 2$ are subtraction constants in the sidewise dispersion approach. The anomalous magnetic moment of the electron $\frac{1}{2}(g-2)$ can be found by evaluating F_2^+ on the mass shell[9], $W^2 = m^2$;

$$\begin{aligned} & e\,\bar{u}(p)\Gamma_\mu(p, p+\ell)\,u(p+\ell) = \\ = \;& e\,\bar{u}(p)\left[\gamma_\mu + F_2^+(m^2)\,(-i\sigma_{\mu\nu}\ell^\nu/2m)\right]u(p+\ell) = \\ = \;& e\,\bar{u}(p)\left[\gamma_\mu + \frac{1}{2}(g-2)\,(-i\sigma_{\mu\nu}\ell^\nu/2m)\right]u(p+\ell)\,. \end{aligned} \tag{4}$$

In order to calculate $F_2^{\pm}(W^2)$, we define two projection operators $\nu_\mu^{(2\pm)}(p, \ell, s)$ such that

$$\begin{aligned} & Tr\sum_{\pm s} e\,\bar{u}(p, s)\,\Gamma^\mu(p, p+\ell)\;\;\nu_\mu^{(2\pm)}(p, \ell, s) \\ & = -(e/2m)\,F_2^{\pm}(W^2)\;\;. \end{aligned} \tag{5}$$

For $\ell^2 = 0$,

$$\nu_\mu^{(2+)}(p, \ell, s) = \frac{-m}{2(W^2-m^2)^2} \left[(\not p + \not \ell + m)(-i\sigma_{\mu\tau}\ell^\tau) \right] u(p, s) ,$$

$$\nu_\mu^{(2-)}(p, \ell, s) = \frac{m}{2(W^2-m^2)^2} \left[(-\not p - \not \ell + m)(-i\sigma_{\mu\tau}\ell^\tau) - 6m\,\ell_\mu \right] u(p, s) . \qquad (6)$$

The dispersion relations for $F_2^\pm(W^2)$ are assumed to be unsubtracted, i.e.,

$$F_2^\pm(W^2) = \frac{1}{\pi} \int_{m^2}^{\infty} \frac{dW'^2 \; \mathrm{Im}\; F_2^\pm(W'^2)}{W'^2 - W^2 - i\epsilon} . \qquad (7)$$

The anomalous moment is thus to be calculated from the radiative corrections.

In line with the assumption that the low energy region is dominant in the dispersion relation, we keep only the two-particle intermediate state of one photon and one electron as in Fig. 2. We ignore all three-body and higher intermediate states, such as in Fig. 4. This will be justified below. In this approximation, we write, using Eq. (2),

$$\mathrm{Im}\; F_2^\pm(W^2) = (m^2/8\pi)\left((W^2 - m^2)/W^2\right)$$
$$\times \sum_{\text{spins}} \int_{-1}^{1} dx\, \bar{u}(p,s)\, T^{\mu\nu} u(k, s') \Gamma_\nu(k, p+\ell)\, \nu_\mu^{(2\pm)}(p, \ell, s) \qquad (8)$$

where $\epsilon_\mu(\ell)\epsilon_\nu(q)\,\bar{u}(p,s)\,T^{\mu\nu}\,u(k, s')$ is the Compton amplitude for an initial photon of momentum q and polarization $\epsilon_\nu(q)$ and an electron of momentum k and spin s' to scatter to a final state of a photon of momentum ℓ and polarization $\epsilon_\mu(\ell)$ and an electron of momentum $p = k + q - \ell$ and spin s. Equation (8) is evaluated in the center-of-mass of the Compton scattering process; $\underset{\sim}{p} = -\underset{\sim}{\ell}$, $\underset{\sim}{q} = -\underset{\sim}{k}$, $p_0 + \ell_0 = W$, and $x = \underset{\sim}{q} \cdot \underset{\sim}{\ell} / |\underset{\sim}{q}| |\underset{\sim}{\ell}|$.

The exact Compton amplitude or any satisfactory approximation to it will satisfy the requirements of current conservation, i.e.,

$$\begin{aligned} \ell_\mu \bar{u}(p) \, T^{\mu\nu} \, u(k) &= 0 , \\ q_\nu \bar{u}(p) \, T^{\mu\nu} u(k) &= 0 . \end{aligned} \tag{9}$$

The second of these relations permits us to drop terms proportional to $F_3^{\pm}(W^2)$ in Γ_ν (k, p + ℓ) in Eq. (8).

It was shown by Thirring, Low, and by Gell-Mann and Goldberger[10] that the Compton amplitude through first order terms in the energy $\omega = W - m$ may be written exactly, i.e.

$$\begin{aligned} &\epsilon_\mu(\ell) \; \epsilon_\nu(q) \; \bar{u}(p, s) \; T^{\mu\nu} u(k, s') \xrightarrow[(\omega = W - m \to 0)]{} \\ &\chi_s^* \Big[-(e^2/m) \, \underset{\sim}{\epsilon}(\ell) \cdot \underset{\sim}{\epsilon}(q) + (i\, e^2/m)(\omega/2m) \\ &\times \Big\{ (g-1) \, \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}(\ell) \times \underset{\sim}{\epsilon}(q) + (\tfrac{1}{4} g^2) \, \underset{\sim}{\sigma} \cdot \Big[(\underset{\sim}{\epsilon}(q) \times \hat{q}) \times (\underset{\sim}{\epsilon}(\ell) \times \hat{\ell}) \Big] \\ &+ (\tfrac{1}{2} g) \Big[(\underset{\sim}{\epsilon}(q) \cdot \hat{\ell}) \, \underset{\sim}{\epsilon}(\ell) \cdot (\sigma \times \hat{\ell}) - (\underset{\sim}{\epsilon}(\ell) \cdot \hat{q}) \, \underset{\sim}{\epsilon}(q) \cdot (\underset{\sim}{\sigma} \times \hat{q}) \Big] \Big\} \Big] \chi_{s'} , \end{aligned} \tag{10}$$

where $\omega = |\underset{\sim}{\ell}| = |\underset{\sim}{q}|$, $\hat{\ell} = \underset{\sim}{\ell}/\omega$, and $\hat{q} = \underset{\sim}{q}/\omega$. For our purposes, it is simpler to write the Compton amplitude in a relativistic notation which reduces to Eq. (10) in the limit $W - m \to 0$. This can be done by writing the Compton amplitude as in Fig. 5.

$$\bar{u}(p)\; T_{\mu\nu}\; u(k) = -e^2\,\bar{u}(p)\left[\Gamma_\mu(-\ell)\,(\not{p}-\not{\ell}-m)^{-1}\;\Gamma_\nu(q)\right.$$

$$\left. +\,\Gamma_\nu(q)\;(\not{p}-\not{q}-m)^{-1}\,\Gamma_\mu(-\ell)\right]\,u(k)\;, \qquad (11)$$

where $\Gamma_\mu(\ell) = \gamma_\mu - \big((g-2)/8m\big)\,[\gamma_\mu, \not{\ell}\,]$. This includes the entire content of the pole terms.

The result of performing the operations indicated in Eq. (8) is two coupled equations relating $\mathrm{Im}\,F_2^+(W^2)$ and $\mathrm{Im}\,F_2^-(W^2)$ to both $F_2^+(W^2)$ and $F_2^-(W^2)$. When Eq. (7) is used, the two equations in Eq. (8) are transformed into two coupled integral equations of the Omnés type.[11] Within the spirit of our approximation, i.e., dropping terms in $\mathrm{Im}\,F_2^{\pm}(W^2)$ of order $(W^2 - m^2)^3$ and higher as $W \to m$, these two coupled equations can be solved by iteration. Also, since $\mathrm{Im}\,\bar{u}(p, s)\,T^{\mu\nu}\,u(k, s')$ is proportional to ω^2 as $\omega \to 0$, we may replace the two complex functions $F_2^{\pm}(W^2)$ which occur in $\Gamma_\nu(k, p+\ell)$ in Eq. (8) by $\mathrm{Re}\;F_2^{\pm}(W^2)$, again dropping terms of order $(W^2 - m^2)^3$ as $W \to m$.

The right hand side of Eq. (8) was evaluated using the algebraic computer program REDUCE.[12] This program performed the necessary traces, substituted the appropriate functions of W^2 and x for the four-vector invariants and carried out the integration over x. The result is

$$\text{Im}\ F_2^+(W^2) = \alpha \left\{ \left[\frac{W^2 - m^2}{2W^2} \quad \frac{m^2}{W^2} \right] \right.$$

$$+ \left(\frac{g-2}{2}\right) \left[\frac{m^2 \ell n\,(W^2/m^2)}{4(W^2 - m^2)} + \frac{-6\ W^4 + 9\,W^2 m^2 - 5\ m^4}{8\ W^4} \right]$$

$$\left. + \left(\frac{g-2}{2}\right)^2 \left[\frac{m^2\ \ell n\,(W^2/m^2)}{4(W^2 - m^2)} + \frac{-2W^4 + W^2\ m^2 - m^4}{8\ W^4} \right] \right\}$$

$$+ \alpha\ F_2^+(W^2) \left\{ \left[\frac{3\ m^2 \ell n\ (W^2/m^2)}{4\ (W^2 - m^2)} + \frac{-\ 4\ W^4 - W^2\ m^2 - m^4}{8\ W^4} \right] \right.$$

$$+ \left(\frac{g-2}{2}\right) \left[\frac{m^2\ \ell n\,(W^2/m^2)}{2(W^2 - m^2)} + \frac{-\ W^6 - W^4\ m^2 - W^2\ m^4 - m^6}{8W^4\ m^2} \right]$$

$$\left. + \left(\frac{g-2}{2}\right)^2 \left[\frac{(W^2 + 3m^2)\ \ell n\,(W^2/m^2)}{16\ (W^2 - m^2)} + \frac{-\ 4\ W^6 + W^4 m^2 - 4W^2\ m^4 - m^6}{32\ W^4\ m^2} \right] \right\}$$

$$+ \alpha\ F_2^-(W^2) \left\{ \left[\frac{-m^2\ \ell n\ (W^2/m^2)}{4\ (W^2 - m^2)} + \frac{3\ W^2\ m^2 - m^4}{8\ W^4} \right] \right.$$

$$+ \left(\frac{g-2}{2}\right) \left[\frac{(W^2 - m^2)^3}{8\ W^4\ m^2} \right]$$

$$\left. + \left(\frac{g-2}{2}\right)^2 \left[\frac{-\ \ell n\ (W^2/m^2)}{16} + \frac{2W^6 - 3W^4\ m^2 + 2W^2 m^4 - m^6}{32\ W^4\ m^2} \right] \right\}, \qquad (12a)$$

$$\begin{aligned}
\mathrm{Im}\ F_2^-(W^2) = \alpha \Bigg\{ & \left[\frac{-2\, m^4\, \ell n\, (W^2/m^2)}{(W^2 - m^2)^2} + \frac{5\, W^4\, m^2 - m^6}{2W^4 (W^2 - m^2)} \right] \\
& + \left(\frac{g-2}{2}\right) \left[\frac{-m^2 (W^2 + 7m^2) \ell n (W^2/m^2)}{4\, (W^2 - m^2)^2} + \frac{-6\, W^6 + 17\, W^4\, m^2 + 10\, W^2\, m^4 - 5\, m^6}{8\, W^4\, (W^2 - m^2)} \right] \\
& + \left(\frac{g-2}{2}\right)^2 \left[\frac{(W^2 - m^2)^2}{8\, W^4} \right] \Bigg\} \\
& + \alpha\, F_2^+(W^2) \Bigg\{ \left[\frac{m^2\, \ell n\, (W^2/m^2)}{4\, (W^2 - m^2)} + \frac{-3\, W^2\, m^2 + m^4}{8\, W^4} \right] \\
& + \left(\frac{g-2}{2}\right) \left[\frac{-(W^2 - m^2)^3}{8\, W^4\, m^2} \right] \\
& + \left(\frac{g-2}{2}\right)^2 \left[\frac{\ell n\, (W^2/m^2)}{16} + \frac{(W^2 - m^2)(-2W^4 + W^2\, m^2 - m^4)}{32\, W^4\, m^2} \right] \Bigg\} \\
& + \alpha\, F_2^-(W^2) \Bigg\{ \left[\frac{m^2\, (W^2 - 5\, m^2)\, \ell n\, (W^2/m^2)}{4\, (W^2 - m^2)^2} + \frac{-4\, W^6 + 9\, W^4 m^2 + 4\, W^2 m^4 - m^6}{8\, W^4\, (W^2 - m^2)} \right] \\
& + \left(\frac{g-2}{2}\right) \left[\frac{m^2\, \ell n\, (W^2/m^2)}{2\, (W^2 - m^2)} + \frac{W^6 - 3W^4\, m^2 - 3W^2\, m^4 + m^6}{8\, W^4\, m^2} \right] \\
& + \left(\frac{g-2}{2}\right)^2 \left[\frac{-\ell n\, (W^2/m^2)}{16} + \frac{(W^2 - m^2)(3W^2 - m^2)}{32\, W^4} \right] \Bigg\} \quad . \qquad \text{(12b)}
\end{aligned}$$

In both Eqs. (12a) and (12b), the coefficient of $F_2^+(W^2)$ is proportional to $(W^2 - m^2)^2$ as $W \longrightarrow m$. Thus, consistent with dropping terms of $\mathrm{Im}\, F_2^+(W^2)$ proportional to $(W^2 - m^2)^3$ as $W \longrightarrow m$, we may replace $F_2^+(W^2)$ by $F_2^+(m^2) \equiv \frac{1}{2}(g-2)$. Similarly, it would appear that it is sufficient to replace $F_2^-(W^2)$ in Eq. (12a) by $F_2^-(m^2)$. However, $F_2^-(m^2)$ is infinite. If we solve for $\mathrm{Re}\, F_2^-(W^2)$ using Eqs. (12b) and (7) and expand it as $W \longrightarrow m$, we obtain

$$\mathrm{Re}\ F_2^-(W^2) \xrightarrow[W \longrightarrow m]{} -\frac{2\alpha}{\pi} \ln \left((W^2-m^2)/m^2\right) + O\left(\text{constant},\ (W^2-m^2)\right). \tag{13}$$

This singularity comes entirely from the first term in square brackets in Eq. (12b). Since the singularity is logarithmic, it is integrable and will not cause further complications. The contribution to $\mathrm{Re}\, F_2^-(W^2)$ of all other terms of Eq. (12b) is finite as $W \longrightarrow m$. Thus we will separate $\mathrm{Re}\, F_2^-(W^2)$ into two parts: (a) the contribution of the first term in square brackets of Eq. (12b) and, (b) the contribution of all the other terms. These two parts will be denoted by $\mathrm{Re}\, F_2^-(W^2)_{(a)}$ and $\mathrm{Re}\, F_2^-(W^2)_{(b,i)}$ respectively, where the subscript i indicates that a particular term was derived from the i-th term in square brackets of Eq. (12b). For part (a) we must calculate $\mathrm{Re}\, F_2^-(W^2)$ while for part (b) it is sufficient to calculate $\mathrm{Re}\, F_2^-(m^2)$. Since our object is to calculate the α^3 term of $\frac{1}{2}(g-2)$, it is sufficient to insert for $\mathrm{Re}\, F_2^-(W^2)$ in Eq. (12b) only that part we call part (a); the other part contributes to $\frac{1}{2}(g-2)$ to order α^4 and higher.

The first term in square brackets in each of Eqs. (12a) and (12b) is exact in perturbation theory. For these two terms, we use Eq. (7) with its correct upper limit, infinity. All other terms are not exact, i.e., they are correct only in their low energy limit. These terms diverge when the integration of Eq. (7) is performed and the integral must be cut off with an upper limit $\lambda^2 m^2$. In every case, however, the divergence is only logarithmic.

Let us now solve for $\mathrm{Re}\, F_2^-(W^2)$ in order to insert it into Eq. (12a). The contribution of the first term in square brackets in Eq. (12b), i.e., part (a), is

$$\mathrm{Re}\, F_2^-(W^2)_{(a)} = \frac{2\alpha}{\pi} \Bigg\{ \frac{m^4 \left[f(1 - W^2/m^2) + (W^2/m^2 - 1) \right]}{(W^2 - m^2)^2}$$

$$+ \frac{m^2 \,\ell n \left((W^2 - m^2)/m^2 \right)}{(W^2 - m^2)} \left[\frac{m^2 \,\ell n\, (W^2/m^2)}{(W^2 - m^2)} + \frac{m^4}{4W^4} - \frac{5}{4} \right] - \frac{m^2}{4W^2} \Bigg\} \qquad (14)$$

where[13]

$$f(x) = \int_0^x \frac{-\ell n\, |1 - y|}{y}\, dy \,. \qquad (15)$$

The limit of Eq. (14) as $W \longrightarrow m$ is Eq. (13). The contribution of the remaining terms of Eq. (12b) is calculated in part analytically and in part numerically. The second and fourth terms of Eq. (12b) give

$$\mathrm{Re}\; F_2^-(m^2)_{(b,2,4)} =$$

$$= \frac{\alpha}{\pi} \left(\frac{g-2}{2}\right) \left[\frac{3}{4\lambda^2} - \frac{1}{\lambda^2 - 1} - \frac{1}{4} + \ell n\ \lambda^2 \left(\frac{1}{(\lambda^2 - 1)^2} - \frac{3}{4} \right) \right], \qquad (16)$$

where $F_2^+(m^2)$ has been replaced by $\frac{1}{2}(g-2)$ and the integral in Eq. (7) has been cut off at $\lambda^2 m^2$. The contribution of the seventh term of Eq. (12b) was computed numerically using Eq. (14) for $\mathrm{Re}\, F_2^-(W^2)$. The result is shown in Table I. All other terms of Eq. (12b) contribute to $\frac{1}{2}(g-2)$ to order α^4 or higher.

We now turn our attention to the calculation of $F_2^+(m^2) = \frac{1}{2}(g-2)$ from Eq. (12a). Again part of the calculation can be done analytically while some must be done numerically. The contribution of the first three terms of Eq. (12a) is identical with the result in DP (see their Eq. (31)). They got (the subscripts indicate which terms of Eq. (12a) give rise to a particular contribution to $\frac{1}{2}(g-2)$)

$$\left(\frac{g-2}{2}\right)_{(1,2,3)} = \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi}\,\frac{g(g-2)}{4}\left[(\ell n\lambda^2)\left\{\frac{3}{2}+\frac{1}{2(\lambda^2-1)}\right\} - \frac{7}{4}+\frac{5}{4\lambda^2}\right] + \frac{\alpha}{2\pi}\left(\frac{g-2}{2}\right)^2\left[\ell n\,\lambda^2 - 1 + \frac{1}{\lambda^2}\right]. \tag{17}$$

The fourth and fifth terms contribute

$$\left(\frac{g-2}{2}\right)_{(4,5)} = \frac{\alpha}{\pi}\left(\frac{g-2}{2}\right)\left[\frac{7\lambda^2-1}{8\lambda^2} - \frac{2\lambda^2+1}{4(\lambda^2-1)}\,\ell n\,\lambda^2\right] + \frac{\alpha}{\pi}\left(\frac{g-2}{2}\right)^2\left[\frac{1}{2}-\frac{1}{8}\left(\lambda^2-1-\frac{\lambda^2-1}{\lambda^2}\right)-\frac{1}{4}\,\frac{\lambda^2+1}{\lambda^2-1}\,\ell n\lambda^2\right]. \tag{18}$$

The sixth tern contributes only to order α^4. The contribution of the seventh term is calculated in two parts: (a) that associated with the logarithmic divergent part of Re $F_2^-(W^2)$, i.e., Eq. (14); and (b) that associated with the non-divergent part of Re $F_2^-(W^2)$, i.e., Eq. (16) and Table I. The former was calculated numerically from Eqs. (7) and (14) and the seventh term of Eq. (12a). The result is shown in Table II. For the latter, use is made of the fact that Re $F_2^-(W^2)$ may be replaced by the non-divergent part of Re $F_2^-(m^2)$. This is then a constant with respect to the integration over W'^2 in Eq. (7). Thus, for this part, we get

$$\left(\frac{g-2}{2}\right)_{(7b)} = \frac{\alpha}{\pi}\left[\mathrm{Re}\, F_2^-(m^2)_{(b,2,4)} + \mathrm{Re}\, F_2^-(m^2)_{(b,7)}\right]$$

$$\times \int_{m^2}^{\lambda^2 m^2} \frac{dW'^2}{W'^2 - m^2}\left[\frac{-m^2 \ln(W'^2/m^2)}{4(W'^2-m^2)} + \frac{3W'^2 m^2 - m^4}{8\,W'^4}\right]$$

$$= \frac{\alpha}{\pi}\left[\mathrm{Re}\, F_2^-(m^2)_{(b,2,4)} + \mathrm{Re}\, F_2^-(m^2)_{(b,7)}\right]$$

$$\times\left[\frac{\ln\lambda^2}{4(\lambda^2-1)} - \frac{1}{8\lambda^2} - \frac{1}{8}\right] \qquad (19)$$

The eighth term of Eq. (12a) is dropped because it is of order $(W^2-m^2)^3$ as $W \to m$. The ninth term of Eq. (12a) is not included in this calculation since it first contributes to $\frac{1}{2}(g-2)$ to order α^4.

The estimate for the α^3 term of $\frac{1}{2}(g-2)$ is then obtained by inserting the value of $\frac{1}{2}(g-2)$ "accurate" to order α^2, i.e., $\frac{1}{2}(g-2) = \alpha/2\pi - 0.328\,\alpha^2/\pi^2$, into the right hand side of Eqs. (17), (18), and (19). The coefficients of α/π, α^2/π^2, and α^3/π^3 are then calculated as functions of λ^2 from the sum of Eqs. (17), (18), and (19) and the function tabulated in Table II. We then get an estimate for $\frac{1}{2}(g-2)$ of the form

$$\left(\frac{g-2}{2}\right) = \frac{\alpha}{2\pi} - A(\lambda^2)\frac{\alpha^2}{\pi^2} + B(\lambda^2)\frac{\alpha^3}{\pi^3} + O(\alpha^4) .$$

The functions $A(\lambda^2)$ and $B(\lambda^2)$ are plotted in Fig. 6. We choose the cutoff λ^2 to approximately reproduce the perturbation calculation for the α^2 term, i.e., $\lambda^2 \approx 4$. Using the same cutoff for the α^3 term, we estimate the coefficient of α^3/π^3 to be $\approx +0.13$. This is very close to the DP result of $\approx +0.15$. They obtained a cutoff of $\lambda^2 \approx 6$ which reproduced $-0.328\,\alpha^2/\pi^2$ and predicted $+0.17\,\alpha^3/\pi^3$.

Since we retained <u>all</u> terms of order $(W^2-m^2)^2$ in Im $F_2^{\pm}(W^2)$, we would have expected a lower cutoff that DP if the assumption that the low energy region dominated the dispersion integral is correct. This is indeed the case.

The one-electron, two-photon intermediate states such as in Fig. 4 have a threshold behavior such that their contribution to Im $F_2^{\pm}(W^2)$ is of order $(W^2-m^2)^3$ as $W \to m$. This is due to the fact that all virtual photon radiative corrections to the Compton amplitude have a threshold behavior of[14] $\omega^2(-e^2/m)\,\underset{\sim}{\epsilon}(\ell)\cdot\underset{\sim}{\epsilon}(q)$ where $\omega = W - m$. Thus all one-electron, n-photon intermediate states can be ignored. Intermediate states of n electrons and any number of photons have a threshold of n^2m^2 (n is odd). Thus for $n \geq 3$, the threshold is greater than or equal to $9\,m^2$ which is higher than our cutoff on the dispersion integral, Eq. (7), and therefore these states can also be ignored.

III. CONCLUSION

An improved estimate for the sixth order contribution to the electron's anomalous magnetic moment has been given. This estimate is

$$\left(\frac{g-2}{2}\right) = \frac{\alpha}{2\pi} - 0.328 \frac{\alpha^2}{\pi^2} + 0.13 \frac{\alpha^3}{\pi^3} .$$

The fact that this estimate differs little from the result of DP ($\approx +0.15\,\alpha^3/\pi^3$) and the fact that by including all terms of order $(W^2-m^2)^2$, the cutoff necessary to reproduce the fourth order result ($-0.328\,\alpha^2/\pi^2$) was smaller than the cutoff of DP, gives one a certain degree of confidence in the estimate. In addition, the success of DP in calculating the magnetic moments of the nucleons by a similar method gives one confidence in the assumption that the low energy region is the dominant contributor to the magnetic moment.

ACKNOWLEDGMENT

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Table I

Table of values of $(\pi^2/\alpha^2)\ \mathrm{Re}\ F_2^-(m^2)_{(b,7)}$ as a function of the cutoff parameter λ^2.

λ^2	$(\pi^2/\alpha^2)\ \mathrm{Re}\ F_2^-(m^2)_{(b,7)}$
2.0	-0.4766
3.0	-0.4517
3.7	-0.4199
3.8	-0.4154
3.9	-0.4109
4.0	-0.4065
4.1	-0.4021
4.2	-0.3978
4.3	-0.3935
4.4	-0.3893
4.5	-0.3851
4.6	-0.3811

Table II

Table of values of $(\pi^2/\alpha^2)\frac{1}{2}(g-2)_{(7a)}$ as a function of the cutoff parameter λ^2.

λ^2	$(\pi^2/\alpha^2)\frac{1}{2}(g-2)_{(7a)}$
2.0	-0.0159
3.0	-0.0133
3.7	-0.0099
3.8	-0.0095
3.9	-0.0090
4.0	-0.0086
4.1	-0.0081
4.2	-0.0077
4.3	-0.0073
4.4	-0.0068
4.5	-0.0064
4.6	-0.0060

FOOTNOTES

1. S. D. Drell and H. R. Pagels, Phys. Rev. 140, B397 (1965).
2. J. Schwinger, Phys. Rev. 73, 416 (1948).
3. C. Sommerfield, Phys. Rev. 107, 328 (1957); Ann. Phys. (N. Y.) 5, 26 (1958); A. Petermann, Helv. Phys. Acta 30, 407 (1957); M. V. Terent'ev, Soviet Physics JETP, 16, 444 (1963).
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6. We use the notation $p^2 = p_o^2 - \underset{\sim}{p}^2 = p_\mu p^\mu$,
   $g^{\mu\nu} = (1, -1, -1, -1), \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$,
   $\sigma^{\mu\nu} = \frac{1}{2} i \, (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \not p = p_\mu \gamma^\mu$,
   $\hbar = c = 1$, and $\alpha = e^2/4\pi \approx 1/137$.
7. It is necessary to specify this particular gauge choice in computing the absorptive part because the amplitude for an off-mass-shell particle with arbitrary W^2 to come onto the mass shell upon radiation of a photon, $\ell^2 = 0$, is not in general gauge-invariant. By tying the internal photon line endings together with a $g_{\mu\nu}$ we ensure that the numerator $N(q, k, p)$ in Eq. (2) is purely a polynomial and does not influence the analytic properties of the Feynman amplitude which are determined by the denominator factors and for which there is a Nambu representation.
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FIGURE CAPTIONS

1. Feynman graphs contributing to the first-order radiative corrections of the electron current.
2. Cut Feynman graphs contributing to the absorptive amplitude.
3. Analytic properties of the invariant functions $F_i^{\pm}(W^2)$.
4. Three-body intermediate state contributing to the absorptive amplitude to order α^2.
5. Pole term contribution to the Compton amplitude.
6. Graph of coefficients of α^2/π^2 and α^3/π^3 as a function of the cutoff parameter λ^2.

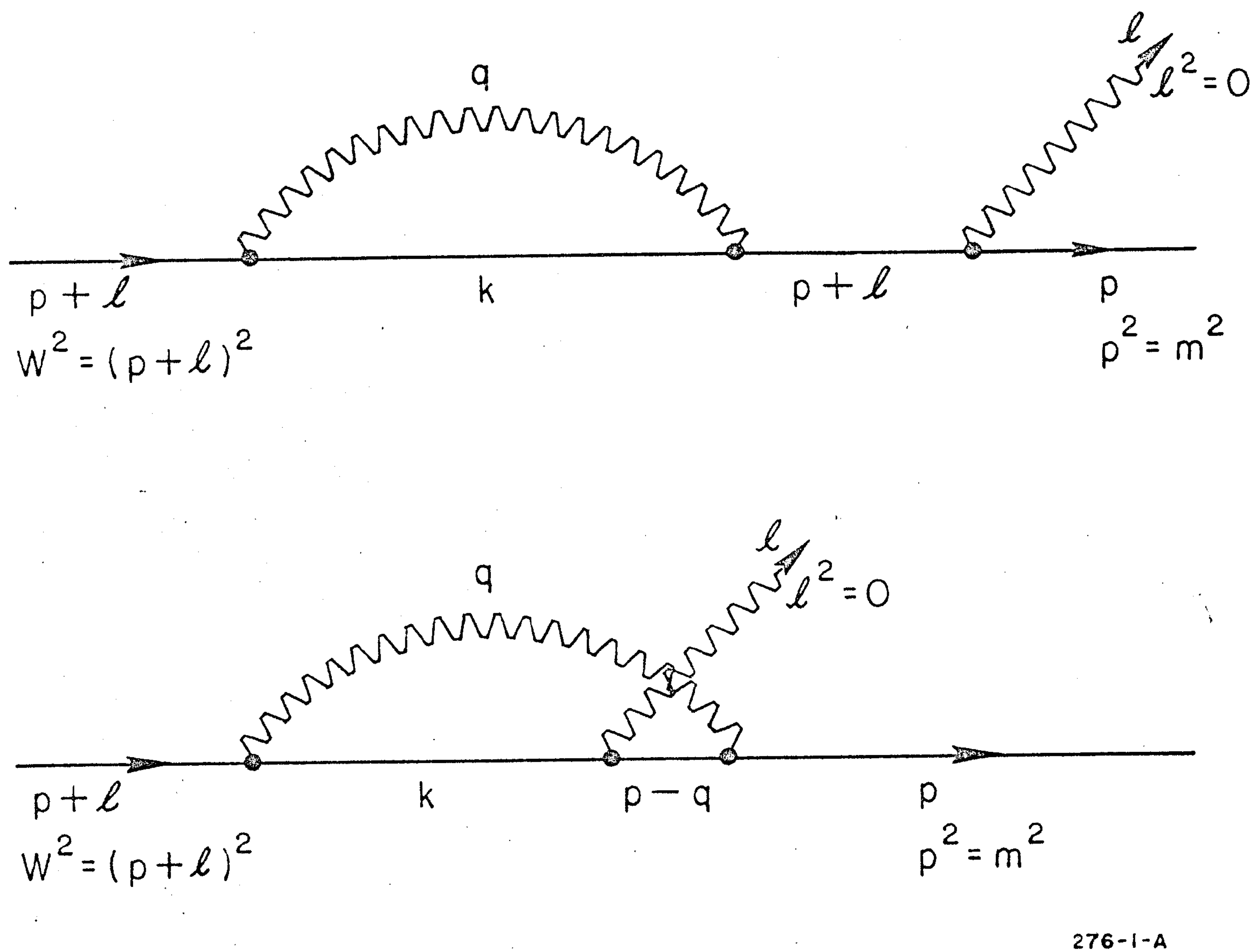


FIG. 1

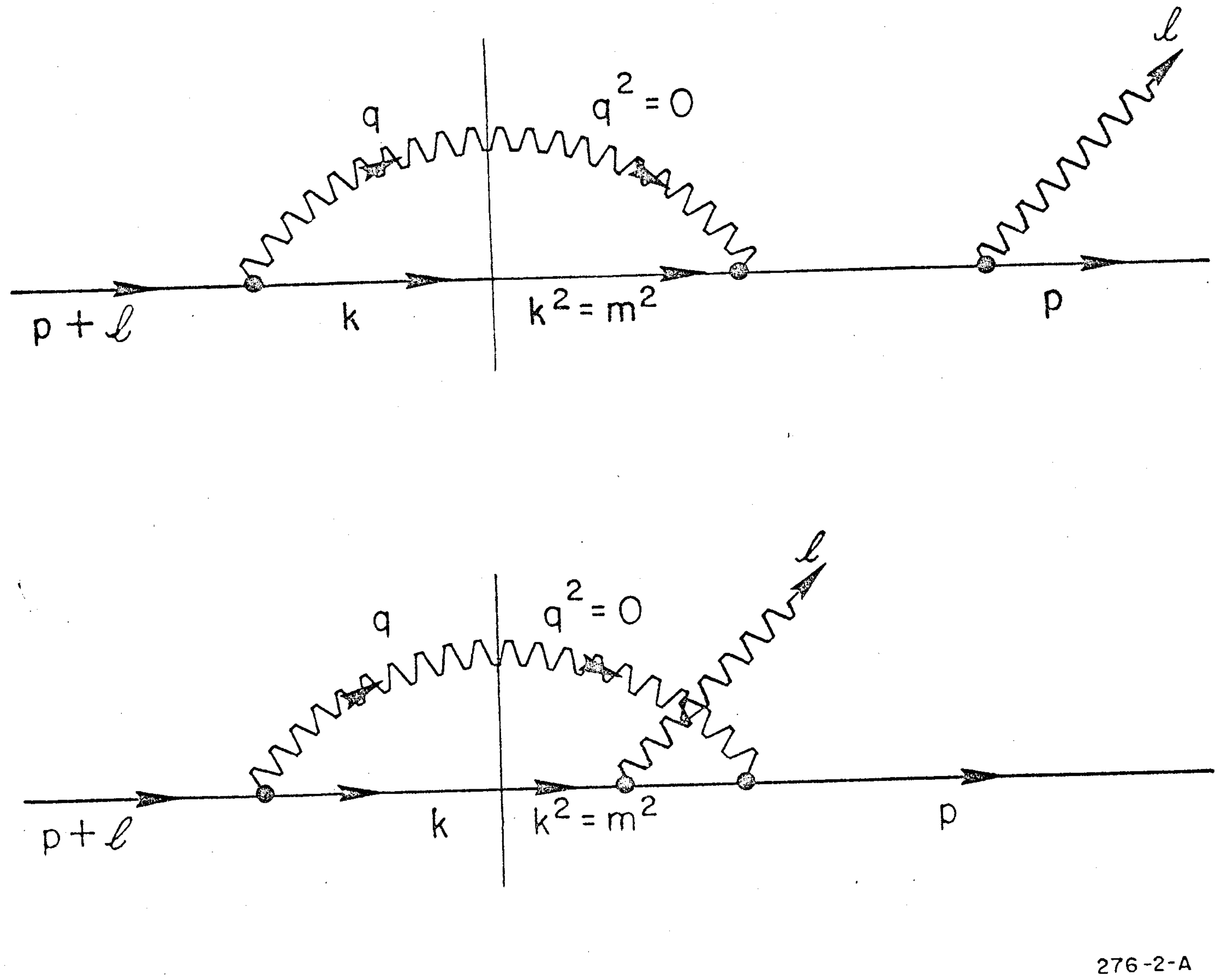


FIG. 2

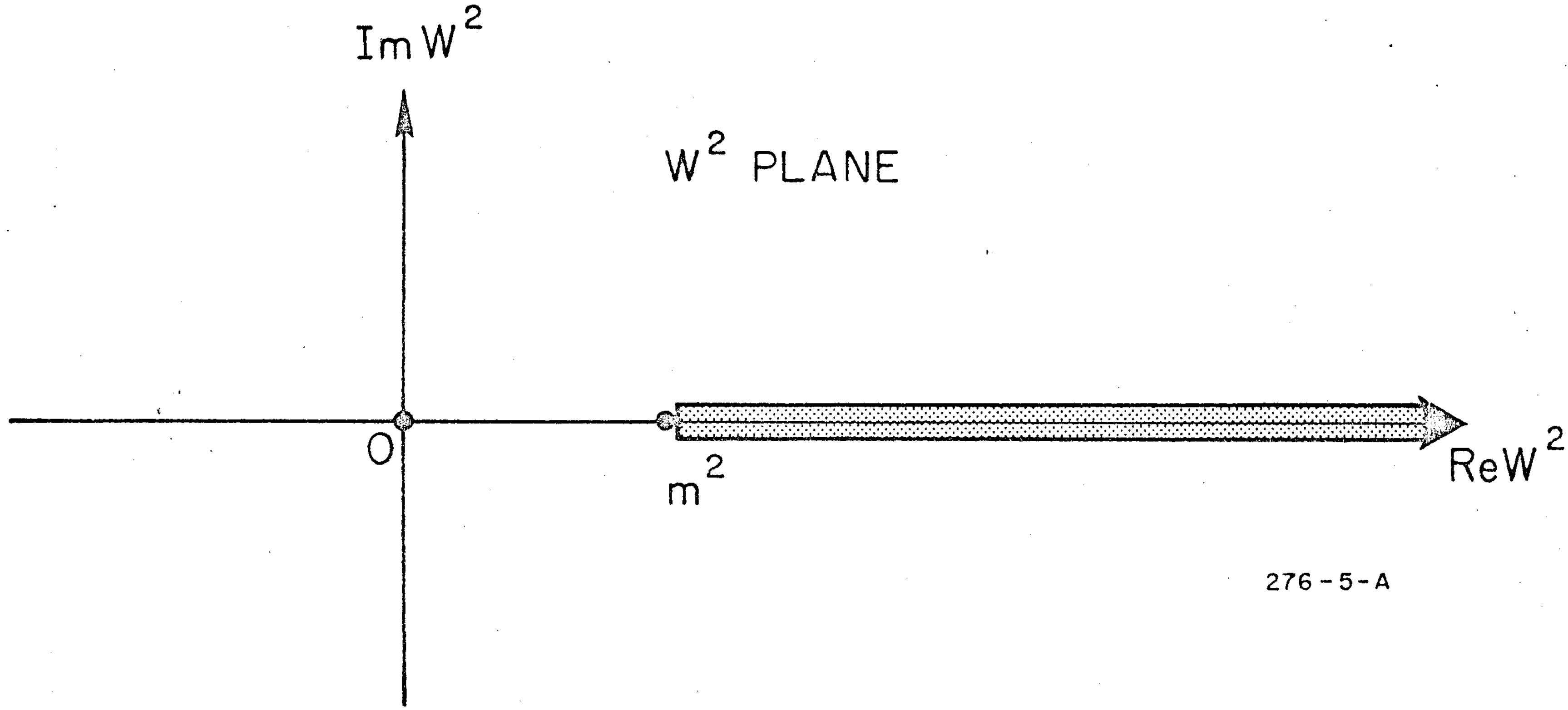


FIG. 3

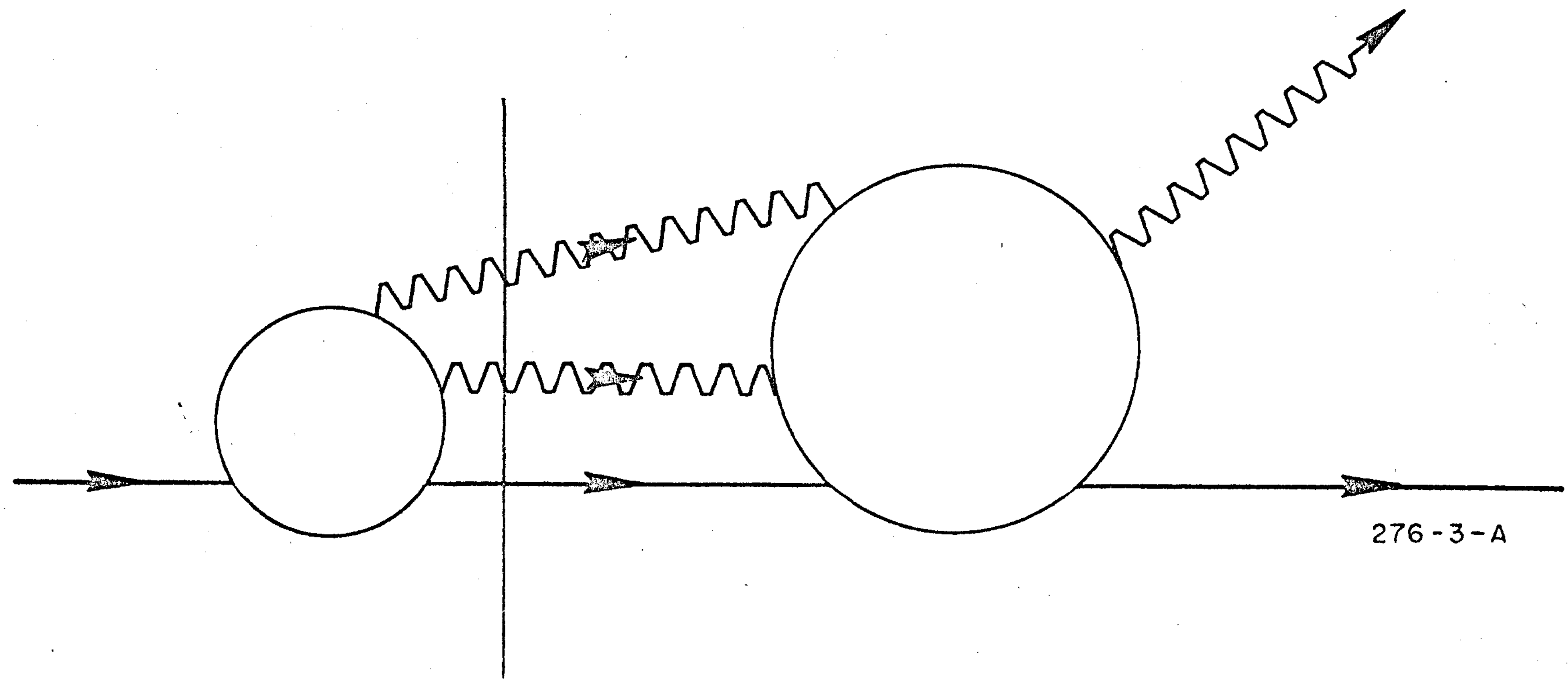
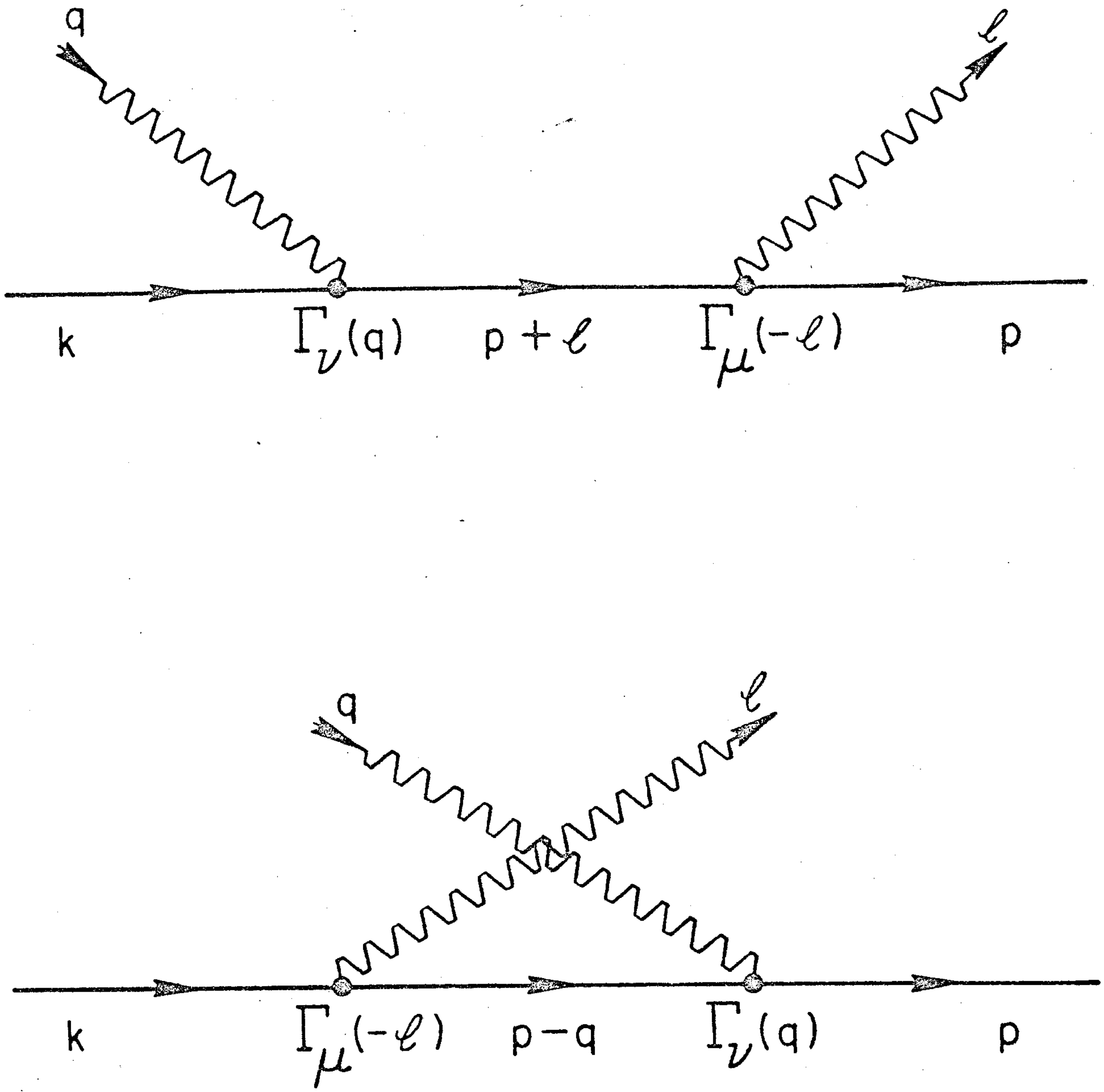


FIG. 4



$$\Gamma_\mu(\ell) \equiv \gamma_\mu - \frac{(g-2)}{8m}\left[\gamma_\mu , \mathbf{1}\right]$$

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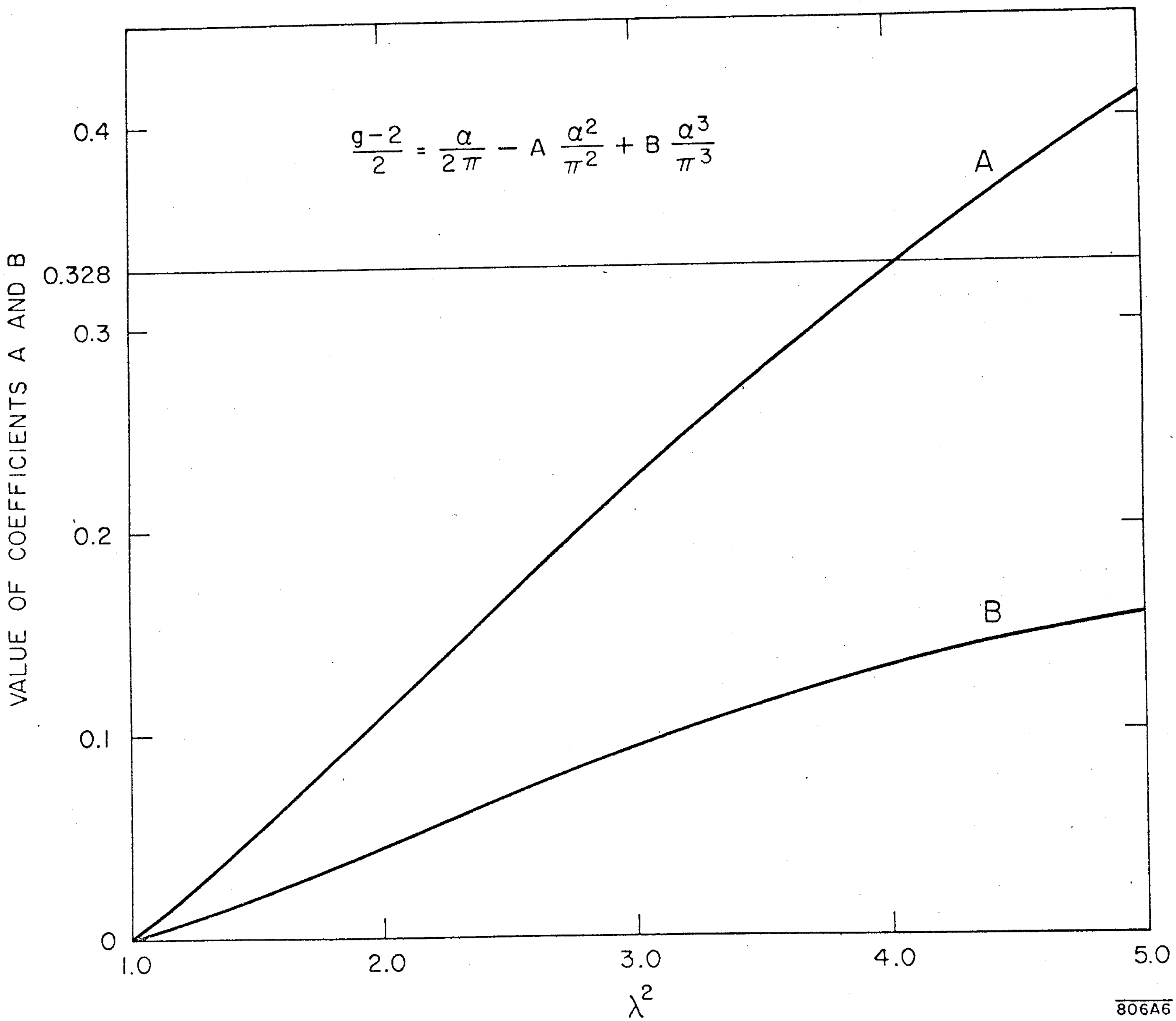
FIG. 5



$\frac{g-2}{2} = \frac{\alpha}{2\pi} - A\frac{\alpha^2}{\pi^2} + B\frac{\alpha^3}{\pi^3}$
A
B
0.4
0.328
0.3
0.2
0.1
0
VALUE OF COEFFICIENTS A AND B
1.0
2.0
3.0
4.0
5.0
λ^2
806A6

FIG. 6