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Maxwell's Equations, Fields or Potentials?

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Abstract. Fields and potentials of Maxwell's Equations transform differently under the Lorentz transformation. Electric and magnetic fields are tridimensional objects which transform like massless fields as a subgroup of the Lorentz group. On the other hand, potentials are four-vectors. Therefore interactions of the electromagnetic fields with four-currents of massive particles cannot be achieved with fields, but only with potentials. Thus potentials seemed to be preferable and a correction to Maxwell's equations seemed unavoidable. We have generalized the Maxwell's equations, with 4-vector fields to allow interaction with currents. We suggest a new way to define the potentials and relate them to the new fields. Key words: Maxwell's Equations, Generalized Maxwell's Equations, massless particles, any spin.

1. Introduction

Maxwell's Equations are one of the world's wonders. Created in 1862 they satisfy today the rules of Spacial Relativity[5], and of Relativistic Quantum Mechanics[1],[11],[9],[13]. Four exceptional scientists contributed to them, Gauss -the king of mathematicians, Ampere, who at 12 years of age mastered the calculus, and later laid the foundations of electrodynamics, Michael Faraday, the paradoxical scientist who was not acquainted with elementary mathematics, who could not write a simple equation, whose books contain no mathematical formulas, and Maxwell, who corrected Ampere's law demanding charge conservation and who unified the electric and magnetic laws into Maxwell's Equations and predicted the existence of electromagnetic waves travelling with the speed of light. Maxwell, a Scottish gentleman, gave most of the credit to Michael Faraday. Michael Faraday not only discovered the Faraday law (without mathematics) but also imagined the existence of fields of energy which led Maxwell to formulate his theory as field theory. Potentials were introduced, which simplified Maxwell's Equations.

Everything seemed to work perfectly (but for the Planck radiation law and second quantization, which we do not treat here), until in 1958, a Ph.D. student of David Bohm, by the name of Yakir Aharonov found an effect which questioned the foundations of Maxwell's Equations. The Aharonov-Bohm^[2] effect demonstrated, that different results are obtained using the electromagnetic fields compared to electromagnetic potentials. Namely, outside an electromagnetically isolated object the fields were equal to zero while the potentials were different from zero. With potentials, measurable results were reproduced within the framework of quantum mechanics. The conclusion was that potentials were preferred.

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We will show that Maxwell's Equations without sources can be considered as quantum type equations of the helicity operator of a massless particle, the photon. The fields are 3-dimensional objects, while potentials are 4-vectors. It is difficult to formulate an interaction of 3-dimensional fields with 4-currents in a covariant way, while it is possible with potentials. Electrodynamics with charges and electric currents is presently formulated with potentials and not with fields.

Wigner[3] has shown that wavefunctions of massless particles of any spin may have only two non-zero helicity components corresponding to the forward or backward directions of motion. Dirac [6], [8] in 1936 derived wavefunctions for massless particles of any spin. The angular momentum basis of the wavefunction of the helicity operator has 2s + 1 components of spin s. But for massless particles only two components, for forward and backward directions, are nonzero. The other 2s-1 components become equal to zero by imposing 2s-1 subsidiary conditions. We have developed a formalism equivalent to the Dirac equations using the angular momentum basis of the $[s - \frac{1}{2}, \frac{1}{2}]$ representation of the Lorentz Group[25],[26] . The angular momentum basis consists of two spins s and s - 1, and has (2s + 1) + 2(s - 1) + 1 = 4s components. By equating to zero the spin s - 1 components of the wave function, the 2s-1 subsidiary conditions (needed to eliminate the non-forward and non-backward helicities.), are fulfilled. The zeros will remain under Lorentz transformations and the wave function transforms under a subgroup of the Lorentz group with only two non-zero helicity projections.

We will show that (in Gaussian units system), that Maxwell's Equations without sources can be considered as quantum equations of a massless particle of spin 1. The wavefunction is proportional to

$$\begin{pmatrix} 0\\ E_x + iB_x\\ E_y + iB_y\\ E_z + iB_z \end{pmatrix},$$
(1)

where E and B are electric and magnetic fields. The complex 3-vector E+iB transforms under the Lorentz transformation like a massless particle. Thus the electric and magnetic fields are 3 dimensional. This is an obstacle for constructing interactions with 4-currents. On the other hand, the introduction of 4-potentials allows the formulation of interactions with 4-currents. All Lagrangian formulations of electrodynamics are done with 4-potentials. Thus we come to the conclusion that Maxwell's equations should be generalized, so that fields should be 4-vectors.

There have been attempts by Majorana in the 1930's[21], to reformulate Maxwell's Equations with 4-fields in order to replace the treatment with potentials. It did not replace, it only gave a new presentation of Maxwell's equations without sources. Nevertheless, the ideas of Majorana are very interesting and may lead to new developments. It allowed Ivezic[20] to use them for developing electromagnetic field equations for moving media.

Our generalization of Maxwell's Equations is straightforward with a wave function proportional to

$$\begin{pmatrix} E_0 + iB_0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}.$$
(2)

where the component $E_0 + iB_0$ is the spin 0 contribution to the wavefunction. Here we suggest a new method of defining potentials for generalized Maxwell's Equations.

In next sections we will show that Maxwell's Equations can be considered as quantum equations, we will discuss our presentation of massless fields with two helicity projections and we will present our generalization of Maxwell's Equations.

2. Maxwell's Equations as wave equations of the helicity operator

Minkowski [[5]] found the covariant form of Maxwell's Equations [10], which in the metric [1, -1, -1, -1] and Gaussian system units, are

$$\partial_{\mu}F^{\mu\nu} = \frac{4\pi}{c}j^{\nu}, \quad \partial_{\mu}\tilde{F}^{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3,$$
(3)

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where $\partial_0 = \frac{1}{c} \frac{\partial}{\partial t}$ and the antisymmetric tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$ are defined via the electric and magnetic fields **E** and **B** respectively as,

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$
(4)

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix},$$
(5)

where $\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor ($\epsilon^{0123} = 1$, $\epsilon^{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$). The sum of the $F^{\mu\nu}$ and $i\tilde{F}^{\mu\nu}$ is the self-dual antisymmetric tensor, it depends only on the combination,

$$\Psi = (\mathbf{E} + i\mathbf{B}), \tag{6}$$

$$F^{\mu\nu} + i\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x - iB_x & -E_y - iB_y & -E_z - iB_z \\ E_x + iB_x & 0 & iE_z - B_z & B_y - iE_y \\ E_y + iB_y & B_z - iE_z & 0 & iE_x - B_x \\ E_z + iB_z & iE_y - B_y & B_x - iE_x & 0 \end{pmatrix},$$
(7)

$$= \begin{pmatrix} 0 & -\Psi_x & -\Psi_y & -\Psi_z \\ \Psi_x & 0 & i\Psi_z & -i\Psi_y \\ \Psi_y & -i\Psi_z & 0 & i\Psi_x \\ \Psi_z & i\Psi_y & -i\Psi_x & 0 \end{pmatrix}$$
(8)

The self-dual Maxwell Equations take the form,

$$\partial_{\mu} \left(F^{\mu\nu} + i\tilde{F}^{\mu\nu} \right) = -\left(\Gamma^{(4)}_{\mu} \right)^{\nu k} \partial_{\mu} \Psi_{k} = \frac{4\pi}{c} j^{\nu}, \qquad (9)$$

$$\mu = 0, x, y, z, = 0, 1, 2, 3 \qquad k = x, y, z, = 1, 2, 3,$$

where,

,

$$\Gamma_0^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_1^{(4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\Gamma_2^{(4)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Gamma_3^{(4)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
(10)

and the Ψ_k is one of the three components of the wave function,

$$\Phi = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} 0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_{zz} \end{pmatrix}.$$
(11)

Multiplying Eq. (9) by $i\hbar c$, where \hbar is the reduced Planck constant, we obtain,

$$\left(\Gamma_{\mu}^{(4)}\right)^{\nu k} \left(i\hbar c\partial_{\mu}\right)\Psi_{k} = -i\hbar c\frac{4\pi}{c}j^{\nu}.$$
(12)

Replacing $i\hbar\partial_{\mu}$ by their quantum energy-momentum operators, Eq. (12) takes the form,

$$\left(\Gamma_0^{(4)}E - c\mathbf{\Gamma}^{(4)} \cdot \boldsymbol{p}\right)\Psi = -i\hbar 4\pi J,\tag{13}$$

where J is the column of the four vector j^{ν} . Maxwell's (quantum) Equations without sources (the photon equation) for the helicity 1 are,

$$\left(\Gamma_0^{(4)}E - c\mathbf{\Gamma}^{(4)} \cdot \boldsymbol{p}\right)\Psi = 0.$$
(14)

The equation for helicity -1 can be obtained using exactly the same procedure, but starting from the anti-self-dual tensor $F^{\mu\nu} - i\tilde{F}^{\mu\nu}$.

We obtain,

$$\partial_{\mu} \left(F^{\mu\nu} - i\tilde{F}^{\mu\nu} \right) = -\left(\Gamma^{(4)}_{\mu} \right)^{\nu k} \partial_{\mu} \Psi^{*}_{k} = \frac{4\pi}{c} j^{\nu}, \qquad (15)$$
$$\mu = 0, x, y, z, = 0, 1, 2, 3 \qquad k = x, y, z, = 1, 2, 3,$$

where Ψ_k^* is the complex conjugate of Ψ_k . The helicity -1 equation is,

$$\left(\Gamma_0^{(4)}E + c\mathbf{\Gamma}^{(4)} \cdot \boldsymbol{p}\right)\Psi^* = -i\hbar 4\pi J.$$
(16)

As we have stated in the introduction, it is difficult to couple 3-vctors with 4 currents. Therefore we can write a Lagrangian formalism only for Maxwell's Equations without sources,

$$\left[E\Gamma_0^{(4)} - c\mathbf{p}\cdot\mathbf{\Gamma}^{(4)}\right] \begin{pmatrix} 0\\ E_x + iB_x\\ E_y + iB_y\\ E_z + iB_{zz} \end{pmatrix} = 0, \ Helicity/s = 1; \tag{17}$$

$$\left[E\Gamma_0^{(4)} + c\mathbf{p}\cdot\mathbf{\Gamma}^{(4)}\right] \begin{pmatrix} 0\\ E_x - iB_x\\ E_y - iB_y\\ E_z - iB_{zz} \end{pmatrix} = 0, \ Helicity/s = -1.$$
(18)

2.1. The structure of the $\Gamma^{(4)}$ matrices

The $\Gamma^{(4)}$ matrices have the following structure,

$$\Gamma_0^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (I_0) \\ 0 & & & \end{pmatrix}, \text{ is the unit matrix}$$
(19)

and

$$\Gamma_x^{(4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & & & \\ 0 & & (S_x) \\ 0 & & & \end{pmatrix}, \ \Gamma_y^{(4)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & & & \\ -1 & & (S_y) \\ 0 & & & \end{pmatrix},$$
(20)

$$\Gamma_z^{(4)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & & \\ 0 & (S_z) \\ -1 & & \end{pmatrix},$$
(21)

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
(22)

$$I_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
(23)

where S_i are the spin 1 matrices with the properties,

$$[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad \mathbf{S}^2 = 2I_0.$$
(24)

Substituting Eqs. (19-20) in Eq. (14) we obtain for the spin 1 projection Φ of the wave function

$$-c\boldsymbol{p}\cdot\boldsymbol{\Phi}=\boldsymbol{0},\tag{25}$$

$$\left(I_0 E - c \frac{\boldsymbol{S} \cdot \boldsymbol{p}}{s}\right) \boldsymbol{\Phi} = \boldsymbol{0}, \quad Helicity/s = 1,$$
(26)

where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix} = \begin{pmatrix} E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}, \tag{27}$$

and

$$c\mathbf{p} \cdot \mathbf{\Phi}^* = \mathbf{0}, \ \left(I_0 E + c \frac{\mathbf{S} \cdot \mathbf{p}}{s}\right) \mathbf{\Phi}^* = \mathbf{0}, \ Helicity/s = -1,$$
 (28)

Eq. (26) is the unconstrained helicity equation and Eq. (25) is the subsidiary condition (the perpendicularity condition) which eliminates the zero helicity component.

The $\Gamma^{(4)}$ matrices satisfy the same relations as the Pauli matrices, namely

$$\left(\Gamma_x^{(4)}\right)^2 = \left(\Gamma_y^{(4)}\right)^2 = \left(\Gamma_z^{(4)}\right)^2 = \left(\Gamma_0^{(4)}\right)^2 = \Gamma_0^{(4)},\tag{29}$$

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$$\Gamma_x^{(4)} \Gamma_y^{(4)} = i \Gamma_z^{(4)} \quad \text{(and cyclic permutation)}, \tag{30}$$

$$\Gamma_k^{(4)} \Gamma_l^{(4)} = -\Gamma_l^{(4)} \Gamma_k^{(4)}, \quad \text{for } k \neq l, \quad k, l = x, y, z,$$
(31)

therefore the $\Gamma^{(4)}$ matrices form a reducible representation of the Pauli matrices. Furthermore

$$\left(\Gamma_0^{(4)}E - c\mathbf{\Gamma}^{(4)} \cdot \boldsymbol{p}\right) \left(\Gamma_0^{(4)}E + c\mathbf{\Gamma}^{(4)} \cdot \boldsymbol{p}\right) = \left(E^2 - c^2\mathbf{p}^2\right)\Gamma_0^{(4)},\tag{32}$$

As a result we obtain for the photon the two helicity equations:

$$\left[E\Gamma_0^{(4)} - c\mathbf{p}\cdot\mathbf{\Gamma}^{(4)}\right]\psi_R = 0, \quad Helicity/s = 1.$$
(33)

$$\left[E\Gamma_0^{(4)} + c\mathbf{p}\cdot\mathbf{\Gamma}^{(4)}\right]\psi_L = 0, \quad Helicity/s = -1.$$
(34)

One should note that both equations are equivalent to the free Maxwell's Equations.

2.2. Conserved current of probability

In our previous paper [23] we have evaluated the conserved current of probability of the free Maxwell's Equations, we obtained,

$$\frac{\hat{E}}{c} \left(\mathbf{\Phi}^H \mathbf{\Phi} \right) - \hat{\mathbf{p}} \cdot \left(\mathbf{\Phi}^H \mathbf{S} \mathbf{\Phi} \right) = 0, \qquad (35)$$

or

$$\partial_t \left(\mathbf{\Phi}^H \mathbf{\Phi} \right) + c \nabla \cdot \left(\mathbf{\Phi}^H \mathbf{S} \mathbf{\Phi} \right) = 0.$$
(36)

Eq.(36) is the conserved current of probability and $\Phi^H \Phi$ is interpreted as the density of probability which should be normalized to unity,

$$\int \int \int dx dy dz \left(\mathbf{\Phi}^H \mathbf{\Phi} \right) = 1$$

Using Eqs.(24,27)The conserved probability current is

$$\partial_t \left(\boldsymbol{\Phi}^H \boldsymbol{\Phi} \right) + c \boldsymbol{\nabla} \cdot \left(\boldsymbol{\Phi}^H \boldsymbol{S} \boldsymbol{\Phi} \right) = \mathcal{N}^2 \left[\partial_t \left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right) + 2c \boldsymbol{\nabla} \cdot \left(\boldsymbol{E} \times \boldsymbol{B} \right) \right] = 0, \quad (37)$$

which coincides with the Poynting theorem. The normalization coefficient \mathcal{N} should correspond to the normalization factor.

2.3. Lagrangian Formalism

Here we develop a formalism for the photon wavefunction

$$\Phi = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}, \tag{38}$$

The Hermitian conjugate of Φ is

$$\Phi^H = \begin{pmatrix} 0 & \Psi_x^* & \Psi_y^* & \Psi_z^* \end{pmatrix}$$
(39)

With these new definitions one can define the Lagrangian density L

$$L = \frac{\Phi^H}{2} \left(E\Gamma_0^{(4)} - c\mathbf{p} \cdot \mathbf{\Gamma}^{(4)} \right) \Phi - \frac{\Phi^T}{2} \left(E\Gamma_0^{(4)} + c\mathbf{p} \cdot \mathbf{\Gamma}^{(4)} \right) \Phi^*, \tag{40}$$

where Φ^T is the transpose of Φ and Φ^* its complex conjugate.

$$\Phi^T = \begin{pmatrix} 0 & \Psi_x & \Psi_y & \Psi_z \end{pmatrix}.$$
(41)

The probability current is

$$\partial_t \left(\boldsymbol{\Psi}^H \boldsymbol{\Psi} \right) + c \boldsymbol{\nabla} \cdot \left(\boldsymbol{\Psi}^H \boldsymbol{S} \boldsymbol{\Psi} \right) = 0.$$
(42)

Thus $\Psi^{H}\Psi$ is interpreted as the density of probability which should be normalized to unity (and is a constant of motion),

$$\int \int \int dx dy dz \left(\boldsymbol{\Psi}^{H} \boldsymbol{\Psi} \right) = 1 \tag{43}$$

Having a Lagrangian, one can compute the corresponding energy-momentum tensor,

$$T_{\mu\nu} = \sum_{j} \frac{\partial L}{\partial \left(\frac{\partial u_j}{\partial x\mu}\right)} \frac{\partial u_j}{\partial x_\nu} - L\delta_{\mu\nu},\tag{44}$$

where u_j stand for the components of Ψ and Ψ^* . For T^{00} we obtain,

$$\frac{2T^{00}}{c} = 2\frac{T_{00}}{c} = \boldsymbol{\Psi}^{H}\left(\boldsymbol{\hat{p}}\cdot\boldsymbol{S}\right)\boldsymbol{\Psi} + \boldsymbol{\Psi}^{T}\left(\boldsymbol{\hat{p}}\cdot\boldsymbol{S}\right)\boldsymbol{\Psi}^{*} = \boldsymbol{\Psi}^{H}\frac{\hat{E}}{c}\boldsymbol{\Psi} - \boldsymbol{\Psi}^{T}\frac{\hat{E}}{c}\boldsymbol{\Psi}^{*}$$
(45)

and

$$2\int \int \int dx dy dz T_{00} = \int \int \int \int dx dy dz \left[\mathbf{\Psi}^{H} \hat{E} \mathbf{\Psi} - \mathbf{\Psi}^{T} \hat{E} \mathbf{\Psi}^{*} \right]$$
(46)

Applying the energy operator on Eq.(43) we obtain

$$\hat{E} \int \int \int dx dy dz \left(\mathbf{\Psi}^{H} \mathbf{\Psi} \right) = \int \int \int dx dy dz \left[\mathbf{\Psi}^{H} \hat{E} \mathbf{\Psi} + \mathbf{\Psi}^{T} \hat{E} \mathbf{\Psi}^{*} \right] = 0$$
(47)

Hence

$$\int \int \int \Psi^T \hat{E} \Psi^* dx dy dz = -\int \int \int \Psi^H \hat{E} \Psi dx dy dz.$$
(48)

Substituting Eq.(48) into Eq.(46) we finally obtain

$$\int \int \int dx dy dz T^{00} = \int \int \int dx dy dz \Psi^H \hat{E} \Psi = \langle \hat{E} \rangle$$
(49)

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which is the expectation value of the energy operator. For the T^{0k} components we have

$$\frac{T^{0k}}{c} = -\frac{T_{0k}}{c} = \Psi^H \frac{\hat{p}_k}{2} \Psi - \Psi^T \frac{\hat{p}_k}{2} \Psi^* = \Psi^H \frac{\hat{p}_k}{2} \Psi + \left(\Psi^H \frac{\hat{p}_k}{2} \Psi\right)^* = \operatorname{Re}\left(\Psi^H \hat{p}_k \Psi\right), \quad (50)$$

and

$$\int \int \int dx dy dz T^{0k} = \int \int \int \int dx dy dz \Psi^H(c\hat{p}_k) \Psi = \langle c\hat{p}_k \rangle, \quad k = 1, 2, 3, \tag{51}$$

which are the expectation values of the momenta.

Let us prove that \hat{E} is a self-adjoint operator. Indeed, from Eq.(48) $\hat{E}^* = -\hat{E}$, we obtain,

$$\int \int \int dx dy dz \left(\hat{E}\Psi\right)^{H} \Psi = -\int \int \int \int dx dy dz \left(\hat{E}\Psi^{H}\right) \Psi$$
(52)

$$= -\int \int \int \Psi^T \hat{E} \Psi^* dx dy dz = \int \int \int dx dy dz \Psi^H \hat{E} \Psi.$$
(53)

In a similar way one can prove that the momenta \hat{p}_k are self adjoint. Therefore Eq.(14), with the normalization condition Eq.(43), is the first quantized equation of the photon.

3. Generalization to any spin

In previous papers we have generalized the helicity equations to any spin [23],[24],[25]]. We have used the angular momentum basis of the $D^{(s-1/2,1/2)}$ representation of the Lorentz group. This basis is the sum of the bases of spins s and spin s - 1 with

$$(2s+1) + (2s-1) = 4s$$
 components.

In this basis the wavefunction is,

$$\Phi^{(4s)} = \begin{pmatrix} \psi_{s-1}^{(2s-1)} \\ \vdots \\ \psi_{-s+1}^{(2s-1)} \\ \psi_{s}^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s+1)} \end{pmatrix} \implies \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_{s}^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s+1)} \end{pmatrix}.$$
(54)

The advantage of this formalism, as we have shown in Refs. [24],[25] is that by equating to zero the spin (s-1) components of the wavefunction $\psi_{\mu}^{(2s-1)}$, the 2s-1 subsidiary conditions (needed to eliminate the non-forward and non-backward helicities.) are automatically satisfied. The equations for any spin have the form

$$\left[\hat{E}\Gamma_{0}^{(4s)} - c\mathbf{\Gamma}^{(4s)} \cdot \hat{\boldsymbol{p}}\right] \Phi_{R}^{(4s)} = 0,$$
(55)

$$\left[\hat{E}\Gamma_0^{(4s)} + c\mathbf{\Gamma}^{(4s)} \cdot \hat{\boldsymbol{p}}\right] \Phi_L^{(4s)} = 0,$$
(56)

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where the $(4s \times 4s)$ gamma matrices have the same multiplication table as the Pauli matrices, therefore they are reducible representation of the Pauli matrices. The equations have built in subsidiary conditions, which are needed to eliminate the non-forward and non-backward helicities. By equating to zero the spin s-1 components of the wave function, the 2s-1 subsidiary conditions (needed to eliminate the non-forward and non-backward helicities.), are automatically satisfied. For spins $\frac{1}{2}$ and spin 1 they are the σ and the $\Gamma^{(4)}$ matrices respectively. The spin 3/2 and spin 2 gamma matrices are given in Refs.[14],[23].

3.1. Lagrangian formalism

The continuity equation for any spin s is

$$\partial_t \left(\left(\Phi^{(4s)} \right)^H \Phi^{(4s)} \right) + \frac{c}{s} \nabla \cdot \left(\left(\Phi^{(4s)} \right)^H \tilde{\mathbf{S}} \Phi^{(4s)} \right) = 0.$$
(57)

Thus the probability density (which should be normalized) is,

$$\rho = \left(\Phi^{(4s)}\right)^H \Phi^{(4s)}, \quad \int \int \int dx dy dz \rho = 1.$$
(58)

In the above formula the density is non-negative and the scalar product is positive definite. Having this result we can find a Lagrangian density,

$$\mathcal{L} = i\hbar \left(\Phi^{(4s)}\right)^{H} \left(\Gamma_{0}^{(4s)} \frac{\partial}{\partial t} + c\Gamma^{(4s)} \cdot \boldsymbol{\nabla}\right) \Phi^{(4s)},\tag{59}$$

and using the definition of the energy momentum tensor $T^{\mu\nu}$, we find,

$$\int \int \int dx dy dz T^{00} = \int \int \int dx dy dz \left(\Phi^{(4s)}\right)^H \mathcal{H}\Phi^{(4s)} = \langle \mathcal{H} \rangle, \tag{60}$$

$$\int \int \int dx dy dz T^{0k} = \int \int \int \int dx dy dz \left(\Phi^{(4s)}\right)^H (c\hat{p}_k) \Phi^{(4s)} = \langle c\hat{p}_k \rangle, \qquad (61)$$

$$k = 1, 2, 3,$$

i.e. consistent with the expectation values of energy and momentum.

4. Generalized Maxwell's Equations

As we have mentioned earlier, Maxwell's Equations have to be generalized in order to have an interaction term proportional to $\Psi_{\mu}j^{\mu}$.

The generalization of Maxwell's Equations is straightforward with replacing the zero of the spin zero component with $\Psi_0 = E_0 + iB_0$,

$$\left(\Gamma_0^{(4)}\partial_0 + \mathbf{\Gamma}^{(4)} \cdot \mathbf{\nabla}\right) \Phi^{(4)} = -\alpha J, \quad \Phi^{(4)} = \begin{pmatrix} \Psi_0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}, \quad J = \begin{pmatrix} j_0 \\ j_x \\ j_y \\ j_z \end{pmatrix}.$$
(62)

from which one can extract generalized Maxwell's Equations in the form

$$\partial_0 E_0 + \boldsymbol{\nabla} \cdot \boldsymbol{E} = \alpha j_0, \tag{63}$$

$$\partial_0 \mathbf{E} + \boldsymbol{\nabla} E_0 - \boldsymbol{\nabla} \times \mathbf{B} = -\alpha \mathbf{j},\tag{64}$$

$$\partial_0 B_0 + \nabla \cdot \boldsymbol{B} = 0, \tag{65}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \boldsymbol{\partial}_0 \boldsymbol{B} + \boldsymbol{\nabla} B_0 = 0. \tag{66}$$

Similar equations were derived by Dvoeglazov [16] as we show below.

4.1. Gersten's method[15],[18],[19] and Dvoeglazov's[16] generalization of Maxwell's Equations Gersten[15] has derived the photon equation from the following decomposition

$$\left(\frac{E^2}{c^2} - \tilde{\mathbf{p}}^2\right) I^{(3)} =$$

$$= \left(\frac{E}{c}I^{(3)} - \tilde{\mathbf{p}} \cdot \tilde{\mathbf{S}}\right) \left(\frac{E}{c}I^{(3)} + \tilde{\mathbf{p}} \cdot \tilde{\mathbf{S}}\right) - \left(\begin{array}{cc}p_x^2 & p_x p_y & p_x p_z\\p_y p_x & p_y^2 & p_y p_z\\p_z p_x & p_z p_y & p_z^2\end{array}\right) = 0, \tag{67}$$

where $I^{(3)}$ is a 3 × 3 unit matrix, and $\tilde{\mathbf{S}}$ is a spin one vector matrix with components

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
(68)

and with the properties

$$[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad \tilde{\mathbf{S}}^2 = 2I^{(3)}.$$
(69)

The decomposition Eq.(67) can be verified directly by substitution.

The matrix on the right hand side of Eq.(67) can be rewritten as:

$$\begin{pmatrix} p_x^2 & p_x p_y & p_x p_z \\ p_y p_x & p_y^2 & p_y p_z \\ p_z p_x & p_z p_y & p_z^2 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}.$$
(70)

From Eqs.(67 - 68) and Eq.(70), the photon equation can be obtained form

$$\left(\frac{E^2}{c^2} - \tilde{\mathbf{p}}^2\right)\tilde{\mathbf{\Psi}} = \left(\frac{E}{c}I^{(3)} - \tilde{\mathbf{p}}\cdot\tilde{\mathbf{S}}\right)\left(\frac{E}{c}I^{(3)} + \tilde{\mathbf{p}}\cdot\tilde{\mathbf{S}}\right)\tilde{\mathbf{\Psi}} - \begin{pmatrix}p_x\\p_y\\p_z\end{pmatrix}\left(\tilde{\mathbf{p}}\cdot\tilde{\mathbf{\Psi}}\right) = 0, \quad (71)$$

where $\tilde{\Psi}$ is a 3 component (column) wave function. Eq. (71) will be satisfied if the two equations

$$\left(\frac{E}{c}I^{(3)} + \tilde{\mathbf{p}} \cdot \tilde{\mathbf{S}}\right)\tilde{\mathbf{\Psi}} = 0, \qquad (72)$$

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{\Psi}} = 0, \tag{73}$$

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will be simultaneously satisfied.

Dvoeglazov[16] has found other solution to Eqs.(71),

$$\left(\frac{E}{c}I^{(3)} + \tilde{\mathbf{p}} \cdot \tilde{\mathbf{S}}\right)\tilde{\boldsymbol{\Psi}} = \tilde{\mathbf{p}}\chi,\tag{74}$$

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{\Psi}} = \frac{E}{c} \chi,\tag{75}$$

where χ is an arbitrary scalar function. It can be shown[16] that $\chi = \Psi_0$ in Eq.(62), i.e. the solution of Dvoeglazov is equivalent to the generalized Maxwell's Equations as in Eq.(62).

5. New way of defining potentials

Let us introduce the 4-potentials,

$$A^{(4)} = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix}, \tag{76}$$

where the scalar potential is $V = cA_0$. Their relation to the wavefunction $\Phi^{(4)}$, generalizing previous definitions, can be obtained by requiring that they satisfy the wave equation,

$$\partial_{\mu}\partial^{\mu}A^{(4)} = \alpha J. \tag{77}$$

The Γ matrices, which are a representation of the Pauli matrices, will factorize the d'Alembertian the same way as the Pauli matrices factorize it, i.e.,

$$\left(\Gamma_0^{(4)}\partial_0 - \mathbf{\Gamma}^{(4)} \cdot \boldsymbol{\nabla}\right) \left(\Gamma_0^{(4)}\partial_0 + \mathbf{\Gamma}^{(4)} \cdot \boldsymbol{\nabla}\right) = \Gamma_0^{(4)}\partial_\mu\partial^\mu \tag{78}$$

In order to get Eq. (77), using Eqs. (78) and (62) the relation between the $\Phi^{(4)}$ and $A^{(4)}$ has to be

$$\Phi^{(4)} = \begin{pmatrix} E_0 + iB_0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix} = -\left(\Gamma_0^{(4)}\partial_0 - \Gamma^{(4)} \cdot \nabla\right) \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix}$$
(79)

$$= - \begin{pmatrix} \partial_0 A_0 + \partial_1 A_x + \partial_2 A_y + \partial_3 A_z \\ \partial_0 A_x + \partial_1 A_0 + i (\partial_2 A_z - \partial_3 A_y) \\ \partial_0 A_y + \partial_2 A_0 - i (\partial_1 A_z - \partial_3 A_x) \\ \partial_0 A_z + \partial_3 A_0 + i (\partial_1 A_y - \partial_2 A_x) \end{pmatrix},$$
(80)

from which, assuming that the potentials are real, the relation between the fields and the potentials is,

$$E_0 = -\partial_0 A_0 - \nabla \cdot \mathbf{A},\tag{81}$$

$$B_0 = 0, \tag{82}$$

$$\mathbf{E} = -\boldsymbol{\nabla}A_0 - \partial_0 \mathbf{A},\tag{83}$$

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$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.\tag{84}$$

These are new relations which can now link two four vectors! With this new formalism we have generalized the fields from 3-vectors to 4-vectors as should be expected from a reliable relativistic theory. The case $E_0 + iB_0 = 0$ brings us back to the usual Maxwell's equations in the Lorenz gauge[17],

$$\partial_0 A_0 + \nabla \cdot \mathbf{A} = 0, \tag{85}$$

In the generalized equations the Lorentz condition does not have to be satisfied. If it is satisfied it enforces the spin 0 component E_0 to be zero. Thus only when the Lorenz condition is not satisfied (while Eq. (81) is valid) the E_0 can be different from zero.

When the Lorenz condition is satisfied the Generalized Maxwell's Equations reduces to the regular spin 1 Maxwell's Equations.

5.1. Wave equations for the Generalized Maxwell's Equations Let us multiply Eq. (62) by $\left(\Gamma_0^{(4)}\partial_0 - \mathbf{\Gamma}^{(4)} \cdot \boldsymbol{\nabla}\right)$,

$$\left(\Gamma_0^{(4)}\partial_0 - \mathbf{\Gamma}^{(4)} \cdot \mathbf{\nabla}\right) \left(\Gamma_0^{(4)}\partial_0 + \mathbf{\Gamma}^{(4)} \cdot \mathbf{\nabla}\right) \Phi^{(4)} = \Gamma_0^{(4)}\partial_\mu \partial^\mu \Phi^{(4)} \tag{86}$$

$$= \Box \Phi^{(4)} = -\alpha \left(\Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \mathbf{\nabla} \right) J, \tag{87}$$

where $\Box = \partial_{\mu} \partial^{\mu}$, from which we obtain,

$$\Box E_0 = \alpha \left(\partial_0 j_0 + \boldsymbol{\nabla} \cdot \mathbf{j} \right), \tag{88}$$

$$\Box B_0 = 0, \tag{89}$$

$$\Box \mathbf{E} = \alpha \left(\partial_0 \mathbf{j} + \nabla j_0 \right), \tag{90}$$

$$\Box \mathbf{B} = -\alpha \nabla \times \mathbf{j},\tag{91}$$

Current conservation requires,

$$\partial_0 j_0 + \boldsymbol{\nabla} \cdot \mathbf{j} = \mathbf{0},\tag{92}$$

therefore when the current is conserved

$$\Box E_0 = 0, \tag{93}$$

and only if the current is not conserved the possibility of $\Box E_0 \neq 0$ will exist. As indicated before, a spin changing interaction can induce the E_0 component.

We will now look for new features of these equations.

5.2. Energy density

According to Eq.60) the energy density is proportional to,

$$\begin{pmatrix} \Psi_0^* & \Psi_x^* & \Psi_y^* & \Psi_z^* \end{pmatrix} (i\hbar c\partial_0) \begin{pmatrix} \Psi_0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = i\hbar c \left(\Psi_0^* \partial_0 \Psi_0 + \Psi^* \cdot (\partial_0 \Psi) \right)$$
(94)

$$= i\hbar c \left[(E_0 - iB_0) \partial_0 \left(E_0 + iB_0 \right) + (\mathbf{E} - i\mathbf{B}) \cdot \left(\partial_0 \left(\mathbf{E} + i\mathbf{B} \right) \right) \right], \tag{95}$$

which is different from the conventional form proportional to

$$\Psi^* \cdot \Psi = (\mathbf{E} - i\mathbf{B}) \cdot (\mathbf{E} + i\mathbf{B}) = \mathbf{E}^2 + \mathbf{B}^2$$
(96)

6. Summary

We have shown that Maxwell's Equations without sources, can be considered as wave equations of the helicity operator of a massless particle of spin 1, the photon. The fields are 3-vectors, while potentials are 4-vectors. It is almost impossible to formulate an interaction of 3-dimensional fields with 4-currents, while it was possible with potentials. Thus electrodynamics with charges and electric currents can be formulated with potentials, but not with fields. Therefore there are contradictions in Maxwell's Equations that should be corrected.

Wigner[3] has shown that wavefunctions of massless particles of any spin may have only two non-zero helicity components in the forward or backward directions of motion. Dirac[6], [8] in 1936 has derived wavefunctions for massless particles of any spin. The angular momentum basis of the wavefunction of the helicity operator had 2s + 1 components of spin s. But for massless particles only two components, for forward and backward directions, are non-zero. The other 2s - 1 components become equal to zero by imposing 2s - 1 subsidiary conditions.

We have developed a formalism equivalent to Dirac equations using the angular momentum basis of the $[s - \frac{1}{2}, \frac{1}{2}]$ representation of the Lorentz Group[25],[26]. The angular momentum basis consists of two spins s and s - 1, and has (2s + 1) + 2(s - 1) + 1 = 4s base vectors. we found that all these equations, are equations of the helicity operator and are similar in form to the spin one-half equation. In these equations the helicity operators are reducible representations of the helicity operator of spin one-half. Moreover, for any spin, there are only two helicity equations similar to the spin one-half equations.

By equating to zero the spin s - 1 components of the wave function, the 2s-1subsidiary conditions (needed to eliminate the non-forward and non-backward helicities.), are created. The zeros will remain under Lorentz transformations and the wave function will transform under a subgroup of the Lorentz group with only two non-zero helicity projections.

We have generalized the Maxwell's Equations with fields being 4-vectors, allowing thus to have an interaction term of the fields with the electromagnetic currents. We found a new way to define the potentials, applicable to the generalized Maxwell's Equations, and to massless field equations of any spin.

We did not resolve all difficulties in the covariant form of Maxwell's Equations with our generalization. For example we found (see Eq.(90))

$$\Box E_0 = \alpha \left(\partial_0 j_0 + \boldsymbol{\nabla} \cdot \mathbf{j} \right) = 0,$$

when the electromagnetic current is conserved. Although the fields are 4-vectors, but $\Psi_0 = 0$ brings us back to previous problems of Maxwell's Equations.

More work is needed in order to have equivalent results with fields and potentials.

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