Symmetries of Super Wilson Loops and Fishnet Feynman Graphs

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To my parents and my sister

Executive Summary

Quantum integrability has turned out to be an important concept in overcoming the limitations of perturbation theory and reaching a more profound understanding of particular four-dimensional quantum field theories. The algebraic structure that underlies integrability in field and string theory is the Yangian, which can be understood as an infinite-dimensional extension of a Lie algebra. Here, we investigate the Yangian symmetry of super Maldacena–Wilson loops and fishnet Feynman graphs.

In the first part of this thesis, we discuss Maldacena–Wilson loops in $\mathcal{N} = 4$ super Yang–Mills theory. Utilizing the non-chiral superspace formulation of the $\mathcal{N} = 4$ SYM model, we construct the full supersymmetric completion of this operator, which is the natural object dual to a minimal surface described by the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. We show that the super loop operator enjoys global superconformal as well as local kappa symmetry, the latter being related to the 1/2 BPS property of the bosonic Maldacena–Wilson loop. Using a convenient type of transversal gauge, we establish the operators one-loop expectation value and prove it to be finite.

We then perform a detailed study of the Yangian symmetries of smooth super Maldacena–Wilson loops. Focusing on a generic gauge theory setup, we analyze in detail the different options for representing the Yangian generators and argue for a representation in terms of gauge-covariant operator insertions. Subsequently, we utilize this approach to prove the Yangian invariance of the full one-loop expectation value. Importantly, we find that the Yangian symmetry of Wilson loop operators heavily relies on two features of the underlying gauge theory: first, the vanishing of the dual Coxeter number of the underlying Lie algebra; and second, a novel identity that we call $\mathcal{G}$-identity, basically stating that the field strength two-form vanishes when contracted with two Lie algebra-generating vector fields.

The second part of this thesis is devoted to the study of four-dimensional fishnet Feynman graphs, which are built from four-valent vertices that are joined by scalar propagators. We show that these diagrams feature a conformal all-loop Yangian symmetry, which we phase in terms of generators annihilating these graphs as well as in terms of inhomogeneous monodromy eigenvalue relations. The Yangian symmetry results in novel differential equations for this family of largely unsolved Feynman integrals and we shall study their form by considering the box integral as an example.

Zusammenfassung

Integrabilität hat sich als ein wichtiges Konzept erwiesen, um die Grenzen einer störungstheoretischen Beschreibung zu überwinden und ein tiefer gehendes Verständnis von speziellen vierdimensionalen Quantenfeldtheorien zu erlangen. Die der Integrabilität zugrunde liegende algebraische Struktur ist der Yangian, welchen man als eine unendlichdimensionale Erweiterung einer Lie Algebra auffassen kann. In der vorliegenden Arbeit untersuchen wir die Yang'sche Symmetrie von super Wilson Schleifen und Fischnetz Feynman Graphen.

Im ersten Teil dieser Arbeit diskutieren wir Maldacena–Wilson Schleifen in $\mathcal{N} = 4$ supersymmetrischer Yang–Mills Theorie. Unter Ausnutzung der nicht-chiralen Superraumbeschreibung des $\mathcal{N} = 4$ supersymmetrischen Yang–Mills Modells konstruieren wir den supersymmetrisch vervollständigten Schleifenoperator, welcher dual ist zu einer durch den vollen $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring beschriebenen Minimalfläche. Wir zeigen, dass dieser Schleifenoperator sowohl globale superkonforme als auch lokale kappa Symmetrie besitzt, wobei wir letztere zur 1/2 BPS Eigenschaft der bosonischen Maldacena–Wilson Schleife in Beziehung setzen. Weiterhin berechnen wir den Einschleifenerwartungswert des Operators und beweisen dessen Endlichkeit.

Anschließend beschäftigen wir uns detailliert mit der Yang'schen Symmetrie von glatten super Maldacena–Wilson Schleifen. Wir untersuchen anhand einer generischen Eichtheorie die verschiedenen Möglichkeiten, die Yang'schen Generatoren zu realisieren und begründen unsere Wahl einer Darstellung in Form von eichkovarianten Operatoreinsetzungen. Unter Verwendung dieser Darstellung beweisen wir nachfolgend die Yang'sche Invarianz des vollen Einschleifenerwartungswertes der super Maldacena–Wilson Schleife. Ein wichtiges Resultat unserer Analyse ist von übergeordneter Natur und besteht aus zwei Konsistenzbedingungen, die Aufschluss darüber geben, in welchen Eichtheorien Wilson Schleifen integrabel sein könnten: Erstens muss die duale Coxeter Zahl der zugrunde liegenden Lie Algebra verschwinden, und zweitens muss das innere Produkt des Feldstärketensors mit zwei Lie Algebra Vektorfeldern null ergeben. Letztere Relation scheint in der Literatur nicht bekannt zu sein und wird von uns als $\mathcal{G}$-Identität bezeichnet.

Im zweiten Teil dieser Arbeit beschäftigen wir uns mit Fischnetz Feynman Graphen, welche aus viervalenten Vertizes bestehen, die durch skalare Propagatoren miteinander verbunden sind. Wir zeigen, dass diese Diagramme zu allen Schleifenordnungen eine konforme Yang'sche Symmetrie aufweisen und konstruieren explizit die Yang'schen Generatoren, die diese Diagramme vernichten. Für Vielschleifendiagramme gelingt uns Letzteres durch eine Umformulierung der Symmetrie in Form von Eigenwertgleichungen inhomogener Monodromiematrizen, aus deren Entwicklung sich die Generatoren ablesen lassen. Die Yang'sche Symmetrie impliziert, dass Fischnetz Integrale partielle Differenzialgleichungen erfüllen, deren Form wir anhand des Boxintegrals illustrieren.

Publications

This first part of this thesis describes the continuation of a line of research that was initiated in 2013 and published in [1],

[1] D. Müller, H. Münkler, J. Plefka, J. Pollok and K. Zarembo, *"Yangian Symmetry of smooth Wilson Loops in $\mathcal{N} = 4$ super Yang-Mills Theory"*, JHEP 1311, 081 (2013), `arxiv:1309.1676` ,

which was included in the authors' master's thesis. It is based on the peer-reviewed publications [2, 3],

[2] N. Beisert, D. Müller, J. Plefka and C. Vergu, *"Smooth Wilson loops in $\mathcal{N} = 4$ non-chiral superspace"*,JHEP 1512, 140 (2015), `arxiv:1506.07047` ,

[3] N. Beisert, D. Müller, J. Plefka and C. Vergu, *"Integrability of smooth Wilson loops in $\mathcal{N} = 4$ superspace"*,JHEP 1512, 141 (2015), `arxiv:1509.05403` .

The second part of this thesis is based on the peer-reviewed publication [4],

[4] D. Chicherin, V. Kazakov, F. Loebbert, D. Müller and D.-l. Zhong, *"Yangian Symmetry for Fishnet Feynman Graphs"*, Phys. Rev. D96, 121901 (2017), `arxiv:1708.00007` ,

as well as on the arXiv preprint [5],

[5] D. Chicherin, V. Kazakov, F. Loebbert, D. Müller and D.-l. Zhong, *"Yangian Symmetry for Bi-Scalar Loop Amplitudes"*, `arxiv:1704.01967` ,

which is currently under peer review.

Some rather technical parts of the manuscripts [2–5] have been included in this thesis with only minor modifications.

Contents

1. Introduction and Motivation

In the past half-century, Yang–Mills [6] theories have clearly revolutionized our understanding of the world of elementary particles and their interactions. In fact, three out of the four fundamental forces of nature — the weak nuclear force, the strong nuclear force and the electromagnetic force — are accurately described by a unified Yang–Mills theory with gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, which is known as the Standard Model of particle physics. The accuracy to which its predictions agree with high precision measurements carried out at colliders, such as the Large Hadron Collider (LHC), is spectacular and leaves no doubt that the Standard Model is one of the most successful theories ever devised. Lately, also the long-sought Higgs boson has been discovered at the LHC [7, 8], so that by now all particles that the Standard Model predicts have been observed. Nonetheless, despite its great success in predicting the outcome of scattering experiments, the Standard model is not free of flaws or inadequacies as it fails to explain certain mysteries of nature, such as neutrino oscillations, the asymmetry of matter and antimatter as well as what dark matter is made of. Furthermore, the model only unifies three of the four fundamental forces and therefore lacks a description of (quantum) gravity.

But even apart from the aforementioned conceptual problems, Yang–Mills theories in general still pose major challenges to theoretical physicists and mathematicians. Many of these challenges are related to the theoretical and practical limitations of our most reliable tool of investigation in any quantum field theory — perturbation theory. For instance, as perturbation theory requires the coupling constants to be small, large areas of the parameter space are simply inaccessible to this method. Inherently strongly-coupled quantum phenomena, such as quark confinement in quantum chromodynamics (QCD), can therefore not be studied by using perturbation theory. But even within the range of applicability, exact results for observables most often remain inaccessible by all practical means. On the one hand, this is related to the fact that the number of Feynman diagrams grows rapidly with both the number of loops and the number of legs. On the other hand, also the Feynman integrals that need to be evaluated become progressively more complex as the number of loops increases and constantly probe (often exceed) the limits of our knowledge and methods.

One of the most promising approaches to reach a more profound understanding of Yang–Mills theories and four-dimensional quantum field theories is to study field theories which allow for exact results. The prime example of such a theory is $\mathcal{N} = 4$ super Yang–Mills theory ($\mathcal{N} = 4$ SYM) [9] with gauge group $\mathrm{SU}(N)$, which is the unique maximally supersymmetric gauge theory in four dimensions and a close cousin of QCD. Devised already in the late '70s, the theory was soon abandoned due to its scale invariance, which makes the model incompatible with particle phenomenology. New interest in $\mathcal{N} = 4$ SYM theory was triggered by one of the breakthrough discoveries of the late '90s: the Anti-de Sitter/Conformal Field theory (AdS/CFT) correspondence [10–12], see, e.g. [13] for a textbook treatment. In its strongest form and applied to the case at hand, the conjectured correspondence states that the $\mathcal{N} = 4$ SYM model has a dual description in terms of type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The (conformal) gauge theory lives on the four-dimensional boundary of the AdS_5 space, which is conformally equivalent to four-dimensional Minkowski space. Since the information of the string theory is conjectured to be completely encoded in the lower-dimensional field theory, one also speaks of the holographic duality. A striking feature of the duality is that it relates the strongly-coupled sector of the gauge theory to the weakly-coupled sector of the string theory and vice versa. As the latter is computationally under control using string perturbation theory, the dual description can be used to gain novel insights into the strong-coupling regime of $\mathcal{N} = 4$ SYM theory. Another interesting aspect of the AdS/CFT correspondence is that it relates a theory that naturally contains gravity (string theory) to a pure gauge theory. Thus, we can also hope to learn something about the long-standing problem of quantum gravity by studying this intriguing duality.

The renewed interest in the $\mathcal{N} = 4$ SYM model that was sparked by the advent of the AdS/CFT correspondence soon led to another groundbreaking discovery: the integrability (exact solvability) of the planar gauge theory. The word planar refers to a particular limit, also called 't Hooft limit [14], which is characterized by sending the rank of the gauge group $N \to \infty$ while keeping the product of the (squared) Yang–Mills coupling and number of colors $\lambda = g^2 N$ fixed. In $\mathcal{N} = 4$ SYM theory, integrable structures were first observed in the context of the spectral problem. The spectral problem consists of finding the anomalous scaling dimensions γ of local gauge-invariant composite operators, which is the only dynamical information in the two-point function of two such operators since the functional form is completely fixed by (super)conformal symmetry

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{C}{(x-y)^{2\Delta}}\,, \qquad \Delta = \Delta_0 + \gamma\,. \tag{1.1}$$

In the breakthrough work [15], it was shown that the one-loop spectral problem of a certain subset of operators can be reformulated as an eigenvalue problem of an integrable spin chain Hamiltonian. This result was soon generalized [16, 17] and led to spectacular progress in the study of two-point functions, see also reference [18] for an extensive review. The state-of-the-art method for computing the anomalous scaling dimensions of local operators is provided by the so-called Quantum Spectral Curve (QSC) [19, 20]. The QSC allows for the extraction of scaling dimensions at any value of the 't Hooft coupling λ with almost unlimited precision, so that the spectral problem can be considered as being solved.

Although the progress in applying methods of integrability to the $\mathcal{N} = 4$ SYM model has been tremendous, the field-theoretic origin of integrability still remains somewhat obscure. A more intuitive understanding of why $\mathcal{N}=4$ SYM theory is integrable comes from the AdS/CFT correspondence. The model which is dual to $\mathcal{N}=4$ SYM theory is type IIB superstring theory on $AdS_5 \times S^5$, which is a two-dimensional non-linear sigma model on a (semi)symmetric coset space [21]. For such theories integrability is a common phenomenon. In fact, it was shown in [22] that the equations of motion of the non-linear sigma model can be phrased as a zero-curvature condition for a one-parameter family of Lax connections. From this construction, an infinite number of conserved charges follows immediately, which is a hallmark of integrability.

The appearance of an infinite number of conserved charges at strong coupling raises the question whether similar structures are present on the field theory side of the duality. Such structures have indeed been observed in $\mathcal{N}=4$ SYM theory, e.g. for the one-loop dilatation operator [23] as well as for scattering amplitudes [24, 25] and lately also for the field equations of motion and the action [26]. At weak coupling, the integrability manifests itself as an infinite-dimensional extension of the underlying Lie algebra symmetry of the model, which is called a Yangian. In fact, the existence of an infinite number of hidden symmetries is what constrains the model so tightly that exact analytic results come within reach. The Yangian therefore sits at the heart of integrability in $\mathcal{N}=4$ SYM theory and to fully understand its role and implications is a major desideratum. In this work, we establish and study the Yangian for Wilson loops and fishnet Feynman graphs and we will now give a brief introduction to both parts of this thesis.

In the first part of this work, we will be concerned with smooth Maldacena–Wilson loops. The Maldacena–Wilson loop operator [27, 28] represents a specific generalization of an ordinary Wilson loop operator as typically considered in generic Yang–Mills theories. It is defined as the path-ordered exponential of the gauge field plus a scalar term both integrated along a path γ,

$$W_M(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \Big(\int \mathrm{d}\tau \big(\dot{x}^\mu A_\mu + n^i \sqrt{\dot{x}^2} \phi_i \big) \Big) \,. \tag{1.2}$$

Here, $x^\mu(\tau)$ parametrizes a path in Minkowski space and n^i is a unit six-vector that characterizes a point on S^5. The loop operator can be defined for any closed path in space, thus providing a huge class of gauge-invariant observables. Importantly, the Maldacena–Wilson loop possesses a dual description in terms of strings in AdS_5: At strong coupling, the expectation value is determined by the area of the minimal surface that ends on the contour γ at the conformal boundary of AdS_5. Maldacena–Wilson loops are thus highly natural observables to study within the context of the AdS/CFT correspondence. Curiously, the Maldacena–Wilson loop was also the first observable for which an exact result at finite coupling was available. In fact, two years before the discovery of integrability it was conjectured that the expectation value of a circular Maldacena–Wilson loop is given by a simple Bessel function $\langle W_M(\bigcirc)\rangle \propto I_1(\sqrt{\lambda})/\sqrt{\lambda}$ [29]. This result was later proven using a technique called localization [30]. The fact that the expectation value is finite is not special to circular Wilson loops but rather a generic feature of Maldacena–Wilson loops depending on a smooth non-intersecting contour.

The appeal of (Maldacena–)Wilson loops[1] in $\mathcal{N} = 4$ SYM theory is further raised by an intriguing duality between Wilson loops and scattering amplitudes. Concretely, the duality states that

$$\ln M_n = \ln\langle W_n\rangle + \text{const.}\,, \tag{1.3}$$

where M_n is the ratio of the planar MHV amplitude and its tree-level value and W_n is a polygonal Wilson loop with light-like edges, which are defined through the relation $p_i = x_i - x_{i+1}$, cf. references [31–34]. Soon after its discovery, the duality was extended to amplitudes of arbitrary helicity configurations by introducing chiral light-like Wilson loops which also couple to the fermions [35, 36].[2] Motivated by an attempt to understand the $\bar{\mathrm{Q}}$-anomaly of these chiral Wilson loops, non-chiral null Wilson loops were constructed and studied to first order in $\bar{\theta}$ in [38] and later on more thoroughly in [39, 40]. A common feature of all null-polygonal Wilson loops is that their vacuum expectation value (VEV) is divergent due to light-like edges and cusps. These divergences parallel the IR divergences of massless scattering amplitudes with which they get in fact identified by means of the duality map [41].

The Yangian symmetry of scattering amplitudes [25] in combination with the aforementioned duality makes it natural to ask whether Wilson loops could be Yangian symmetric as well. Since divergences typically obscure certain symmetries, it is natural to begin by focusing on smooth Maldacena–Wilson loops as their expectation value is finite, which prevents the symmetries from becoming anomalous. Further support for this approach comes from strong coupling as the string sigma model is known to be integrable. On the downside, it is clear that the Maldacena–Wilson loop as stated above does not even respect all superconformal symmetries of $\mathcal{N} = 4$ SYM theory because the operator couples only to the bosonic fields of the model. It is therefore most likely that the Maldacena–Wilson loop operator must first be lifted to a fully supersymmetric operator before the challenge of exposing its non-local symmetries can be attempted.

Indeed, in reference [1] as well as in the authors' master's thesis [42] it was shown that bosonic Maldacena–Wilson loops do not possess Yangian symmetry at weak coupling. Instead, one needs to consider the supersymmetric completion of this loop operator, which we constructed through quadratic order in an expansion in the anticommuting Graßmann variables. Subsequently, the one-loop expectation value of the supersymmetrized loop operator was demonstrated to be Yangian invariant to leading order in the Graßmann expansion. Here, we aim at completing this analysis. We begin by reviewing the $\mathcal{N} = 4$ non-chiral superspace formulation of the $\mathcal{N} = 4$ SYM model, thereby laying the foundations for the definition of the full super Maldacena–Wilson loop. Importantly, the $\mathcal{N} = 4$ non-chiral superspace is an on-shell superspace in the

[1] For light-like contours, the Maldacena–Wilson loop reduces to the ordinary Wilson loop.

[2] Note that in reference [37] it was pointed out that the generalized duality might actually be spoiled due to subtle anomalies on the Wilson loop side.

sense that the superspace constraints imply the equations of motion for the fields. While at first sight this feature seems to be in tension with quantization, we show how to establish one-loop perturbation theory within this framework and derive various Graßmann exact free two-point functions. Subsequently, we shall define the full super Maldacena–Wilson loop as a loop operator in non-chiral superspace. Interestingly, the classical operator has a local fermionic symmetry, which in spirit is very close to kappa symmetry of string theory. We expose this symmetry and clarify the relation to the 1/2 BPS property of the bosonic Maldacena–Wilson loop. The remaining part of the chapter is devoted to establishing the one-loop expectation value of the operator and discussing its superconformal symmetries. With the super Maldacena–Wilson loop established, we then turn to an investigation of its non-local symmetries. In contrast to the path-derivative approach pursued in reference [1], we argue for a formulation of the Yangian generators in terms of gauge-covariant field insertions. This definition will then be shown to pass various algebraic consistency checks before we use it to prove the Yangian invariance of the super Maldacena–Wilson loop at the one-loop level and to all orders in the Graßmann expansion. The Yangian symmetry of super Maldacena–Wilson loops is backed up by an investigation at strong coupling in which the Yangian symmetries of minimal supersurfaces were derived from the integrability of the non-linear sigma model that describes the full-fledged $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. This result was obtained in a parallel line of research and published in [43–45].

Although $\mathcal{N} = 4$ SYM theory is arguably the simplest interacting gauge theory in four dimensions, the model is not simple on an absolute scale. To better understand the origins and implications of Yangian symmetry in four-dimensional field theories, it would be advantageous to have easier models which still display integrability. Recently, such a class of theories was constructed [46] by studying a particular class of double-scaling limits of the γ_i-deformed $\mathcal{N} = 4$ SYM model [47, 48]. The simplest of these models is a theory with just two scalars interacting via a specific four-valent vertex. Besides being conformal and integrable in the large-N limit [4, 5, 46, 49–51], this theory has the remarkable feature that all its non-vanishing planar single-trace correlators are in one-to-one correspondence with a single Feynman diagram of fishnet type. These fishnet graphs correspond to scalar high-loop Feynman integrals, which play an important role also beyond the above-mentioned context. The pairing of integrability and single-graphness potentially provides us with unique opportunity to use integrability to compute this family of largely unsolved scalar Feynman integrals and we shall here take the first step and expose their infinite-dimensional symmetry algebra.

We begin by reviewing the relation between the γ_i-deformed $\mathcal{N} = 4$ SYM model and the bi-scalar model that generates these graphs. Subsequently, we shall investigate the local and non-local symmetries of four-dimensional fishnet diagrams using the language of Lie algebra and Yangian generators. In this context, we will explicitly demonstrate that the cross integral and the double-cross integral feature a conformal Yangian symmetry with non-trivial evaluation parameters. Further progress concerning exposing the symmetry is achieved by phrasing the Yangian symmetry in terms of monodromy eigenvalue relations for these graphs. This will allow us to demonstrate the Yangian symmetry of generic high-loop fishnet Feynman integrals by using simple graphical arguments. The Yangian symmetry results in novel differential equations for these integrals as will be illustrated on the example of the cross integral. Finally, we study on-shell limits of fishnet Feynman graphs and clarify the relation between the Yangian symmetry and the conformal/dual conformal symmetry of theses graphs.

We conclude by giving a brief summary of the results that have been obtained and comment on their impact.

2. Symmetry, Field Theory and Wilson Loops

The aim of this chapter is to provide an introduction to the fundamental concepts and objects which are relevant for this thesis. The first part of this chapter is devoted to algebraic preliminaries. We shall start by introducing the conformal algebra as well as its superconformal extension. Subsequently, we define an important algebraic structure: the Yangian algebra. In the second part of this chapter, we discuss the component formulation of the $\mathcal{N} = 4$ SYM model and introduce one of the most interesting observables in a gauge theory, the Wilson loop operator. The focus in this latter part lies on the Wilson loop with scalar extension, which is called the Maldacena–Wilson loop. It represents one of the central objects that we shall investigate in this thesis. The presentation here is in parts based on that in the authors' master's thesis [42] and the one in [45]. Further references will be cited in the sections below.

2.1. Symmetry

Symmetry is one of the most important concepts in theoretical physics. It can not only serve as a guiding principle for the construction of new theories and objects but is often also invaluable with respect to finding the solution to a given problem. Symmetry and in particular Yangian symmetry is in fact also the main theme of this thesis and it therefore seems fitting to begin our discussion by introducing the algebraic foundations of conformal symmetry and Yangian symmetry.

2.1.1. Conformal Symmetry

In this section, we discuss conformal transformations of flat four-dimensional Minkowski space $\mathbb{R}^{1,3}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Conformal transformations are by definition transformations $x^\mu \to x'^\mu$ which leave the metric tensor invariant up to a local scaling factor

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \, \eta_{\rho\sigma} = \Omega^2(x) \, \eta_{\mu\nu} \, , \tag{2.1}$$

where $\Omega(x)$ denotes an arbitrary smooth function. Conformal transformations thus preserve angles, while lengths are generically altered. In what follows, we shall discuss conformal transformations from the infinitesimal point of view as well as from the viewpoint of finite transformations.

Infinitesimal transformations. We begin by focusing on infinitesimal conformal transformations. In order to determine the form of the most general conformal transformation, we make the ansatz $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$. Plugging this ansatz into the defining equation (2.1) and expanding

$$\Omega(x) \approx 1 + \sigma(x)\,, \tag{2.2}$$

yields the conformal Killing equation, which explicitly reads

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \tfrac{1}{2}\,(\partial \cdot \xi)\,\eta_{\mu\nu}\,. \tag{2.3}$$

Note that in the above equation we have already taken the trace in order to express the function $\sigma(x)$ in terms of the vector ξ^μ, i.e. $\sigma(x) = 1/4\,(\partial_\mu \xi^\mu)$. It is easy to show that the most general solution to this equation is given by

$$\xi_\mu(x) = -a_\mu + \omega_{\mu\nu}\,x^\nu - \lambda\,x_\mu + 2\,(b\cdot x)\,x_\mu - b_\mu\,x^2\,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{2.4}$$

The first two terms correspond to infinitesimal translations and Lorentz transformations, while the latter represent infinitesimal scale transformations and special conformal transformations. On general grounds, it is clear that the vector fields $\xi = \xi^\mu \partial_\mu$ form a Lie algebra with the Lie bracket given by the vector field commutator. In what follows, we shall refer to it as the conformal algebra. To study this Lie algebra, we introduce the following convenient basis

$$\xi = \xi^\mu \partial_\mu = a^\mu \mathrm{P}_\mu + \tfrac{1}{2}\omega^{\mu\nu}\mathrm{L}_{\mu\nu} + \lambda \mathrm{D} + b^\mu \mathrm{K}_\mu\,, \tag{2.5}$$

where

$$\begin{aligned}
&\mathrm{P}_\mu = -\partial_\mu && \text{(translations)}\,,\\
&\mathrm{D} = -x^\mu \partial_\mu && \text{(dilatations)}\,,\\
&\mathrm{L}_{\mu\nu} = -\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) && \text{(Lorentz transformations)}\,,\\
&\mathrm{K}_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu && \text{(special conformal transformations)}\,.
\end{aligned} \tag{2.6}$$

These generators obey the following algebra relations:

$$\begin{aligned}
[\mathrm{L}_{\mu\nu}, \mathrm{L}_{\rho\sigma}] &= \eta_{\mu\rho}\mathrm{L}_{\nu\sigma} + \eta_{\nu\sigma}\mathrm{L}_{\mu\rho} - \eta_{\mu\sigma}\mathrm{L}_{\nu\rho} - \eta_{\nu\rho}\mathrm{L}_{\mu\sigma}\,,\\
[\mathrm{L}_{\mu\nu}, \mathrm{P}_\lambda] &= \eta_{\mu\lambda}\mathrm{P}_\nu - \eta_{\nu\lambda}\mathrm{P}_\mu\,, \qquad [\mathrm{D}, \mathrm{K}_\mu] = -\mathrm{K}_\mu\,,\\
[\mathrm{P}_\mu, \mathrm{K}_\nu] &= 2\eta_{\mu\nu}\mathrm{D} - 2\mathrm{L}_{\mu\nu}\,, \qquad [\mathrm{D}, \mathrm{P}_\mu] = \mathrm{P}_\mu\,,\\
[\mathrm{L}_{\mu\nu}, \mathrm{K}_\rho] &= \eta_{\mu\rho}\mathrm{K}_\nu - \eta_{\nu\rho}\mathrm{K}_\mu\,.
\end{aligned} \tag{2.7}$$

All the remaining commutators vanish. One can show that the Lie algebra defined by the above commutation relations is isomorphic to the Lie algebra $\mathfrak{so}(2,4)$. For this reason, we will refer to the conformal algebra as $\mathfrak{so}(2,4)$.

Finite transformations. In the above, we have seen that infinitesimal conformal transformations form a Lie algebra. For this reason, there exists a clear conceptual pathway for obtaining finite conformal transformations, which goes under the name of exponentiation [52]. However, since we will be mainly concerned with infinitesimal transformations in this thesis, we refrain from discussing this method here in detail and merely state the final result. Exponentiating the infinitesimal transformations in equation (2.6) yields

$$\begin{aligned} x'^{\mu} &= x^{\mu} - a^{\mu} && \text{(translations)}\,, \\ x'^{\mu} &= \Lambda^{\mu}{}_{\nu}\, x^{\nu} && \text{(Lorentz transformations)}\,, \\ x'^{\mu} &= \alpha\, x^{\mu} && \text{(dilatations)}\,, \\ x'^{\mu} &= \frac{x^{\mu} - c^{\mu}\, x^2}{1 - 2\,(c\cdot x) + c^2\, x^2} && \text{(special conformal transformations)}\,. \end{aligned} \tag{2.8}$$

An important point to note is that only those group elements belonging to the connected component of the identity can be reached by exponentiation. Quite often it is completely sufficient to focus on the connected component of the (symmetry) group, but when dealing with the conformal group it is convenient to additionally take into account one further element, which is called the conformal inversion. This discrete transformation is defined by

$$\mathrm{I}_b[x^{\mu}] = \frac{x^{\mu}}{x^2}\,, \tag{2.9}$$

and does, due to the absence of an infinitesimal analogue, not belong to the identity-connected component of the conformal group. An important feature of the conformal inversion is that it relates translations and special conformal transformations. More precisely, a special conformal transformation can be represented as an inversion, followed by a finite translation with shift vector $-c^{\mu}$, followed by another inversion

$$(\mathrm{I}_b \circ \mathrm{T}_{-c} \circ \mathrm{I}_b)\,[x^{\mu}] = \frac{\frac{x^{\mu}}{x^2} - c^{\mu}}{\left(\frac{x^{\nu}}{x^2} - c^{\nu}\right)\left(\frac{x_{\nu}}{x^2} - c_{\nu}\right)} = \frac{x^{\mu} - c^{\mu}\, x^2}{1 - 2\,(c\cdot x) + c^2\, x^2}\,. \tag{2.10}$$

At this point, let us note an important consequence of the relation given above: The conformal invariance of a Poincaré-invariant quantity follows trivially if the considered quantity is invariant under inversion. In practice, one therefore considers quite often the conformal inversion instead of the more complicated conformal boosts.

A common feature of all conformal transformations is that their Jacobian is up to a scaling factor a local Lorentz transformation/rotation

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x) O^{\mu}{}_{\nu}(x)\,, \qquad \text{with} \qquad \det\left(\frac{\partial x'}{\partial x}\right) = \Omega(x)^4\,. \tag{2.11}$$

Note that this is in complete agreement with the defining equation (2.1). For translations, Lorentz transformations and dilatations the above-stated property is obvious.

For conformal boosts, it follows from considering the conformal inversion for which one finds

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{1}{x^2}\left(\delta^{\mu}_{\nu} - \frac{2x^{\mu}x_{\nu}}{x^2}\right), \qquad \Omega(x) = \frac{1}{x^2}, \qquad O^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \frac{2x^{\mu}x_{\nu}}{x^2}. \tag{2.12}$$

Finally, let us point out that the inversion can also assist in constructing representations of the conformal algebra. From equation (2.10), which relates elements of the conformal group, we infer the following algebra relation

$$\mathrm{K}_{\mu} = \mathrm{I}_b \circ \mathrm{P}_{\mu} \circ \mathrm{I}_b \,. \tag{2.13}$$

This notation is slightly imprecise but is frequently used. What it means is that the conformal boost generators can be replaced by the sequence $\mathrm{I}_b \circ \mathrm{P}_{\mu} \circ \mathrm{I}_b$ when applied to an element of the vector space on which they act. Given the action of the inversion, it is thus a straightforward exercise to find the generators K_{μ}. The remaining conformal generators then follow by computing commutators, see the algebra relations (2.7). This concludes the discussion of finite conformal transformations.

Transformation of the fields. So far our considerations were entirely of geometrical concern as we discussed the action of conformal transformations on points in Minkowski space. To make contact to field theory, we first need to discuss the action of conformal transformations on the fields. We will do this following to a certain extend the presentation in references [53, 54] and the one in [55].

Contrary to the approach pursued above, we shall begin our discussion by considering finite conformal transformations from which we shall then derive the infinitesimal transformation laws. Under a finite conformal transformation, a field ϕ^M of arbitrary spin-tensor structure transforms as

$$\phi^M(x) \to \phi'^M(x') = \Omega(x)^{-\Delta} R[O^{\mu}{}_{\nu}(x)]^M{}_N \phi^N(x) \,. \tag{2.14}$$

Here, Δ is the scaling dimension of the field ϕ^M and $R[O^{\mu}{}_{\nu}(x)]$ denotes the representation of the Lorentz transformation $O^{\mu}{}_{\nu}(x)$ (2.11) that acts on the vector space of the fields ϕ^M. For a scalar field with scaling dimension $\Delta = 1$, the above equation reduces to

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1/4} \phi(x) \,, \tag{2.15}$$

where we have used equation (2.11) to express the function $\Omega(x)$ in terms of the Jacobian of the transformation. Fields or operators transforming according to equation (2.14) are called primary fields/operators. Every operator in a conformal field theory (CFT) which is not primary is either a descendant operator or can be expressed as a linear combination of primaries and descendants. Descendants are primary operators to which momentum generators have been applied, i.e. derivatives of primary operators. Since their transformations follow from the relations presented above, one usually focuses on primary operators and so do we in this work. Given the finite transformation laws (2.14) and (2.15), let us now derive the action of the generators on the fields. For this, we consider the difference of the transformed field and the original one both evaluated at position x,

$$\delta_\xi \phi^M = \phi'^M(x) - \phi^M(x)\,. \tag{2.16}$$

By expanding equation (2.14) for an infinitesimal transformation of the form $x'^\mu = x^\mu + \xi^\mu(x)$, we obtain

$$\delta_\xi \phi^M = -\xi^\mu \partial_\mu \phi^M(x) - \frac{\Delta}{4}(\partial \cdot \xi)\phi^M(x) - \frac{1}{2}(\partial_{[\mu}\xi_{\nu]})(\Sigma^{\mu\nu})^M{}_N \phi^N(x)\,, \tag{2.17}$$

where the square brackets denote antisymmetrization including a factor of 1/2. Here, we have already used the Killing equation (2.3) to express the function $\sigma(x)$ that appears in the expansion of $\Omega(x)$ as a function of the Killing vectors ξ_μ. The $\Sigma^{\mu\nu}$-matrices form a representation of the Lorentz algebra: For scalars we have $\Sigma^{\mu\nu} = 0$, while vector fields transform under the representation $(\Sigma^{\mu\nu})^\rho{}_\sigma = 2\eta^{\rho[\mu}\delta^{\nu]}_\sigma$. In order to obtain the representation of the generators on the fields, we plug in the explicit form of the Killing vectors (2.4) and compare equation (2.17) to the defining relation (2.5). This yields

$$\begin{aligned}
&\mathbb{P}_\mu \, \phi^M = \partial_\mu \, \phi^M\,, \\
&\mathbb{D} \, \phi^M = (x^\mu \partial_\mu + \Delta)\, \phi^M\,, \\
&\mathbb{L}_{\mu\nu} \, \phi^M = (x_\mu \partial_\nu - x_\nu \partial_\mu)\, \phi^M + (\Sigma_{\mu\nu})^M{}_N \, \phi^N\,, \\
&\mathbb{K}_\mu \, \phi^M = (x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu)\, \phi^M - 2x^\nu (\Sigma_{\mu\nu})^M{}_N \, \phi^N\,.
\end{aligned} \tag{2.18}$$

Note the extra minus signs appearing in front of the differential operator terms compared to equation (2.6). These signs are necessary in order to ensure that the generators of field transformations satisfy the same algebra relations as the generators in the coordinate representation (2.6). For a more detailed discussion on this point see section 5.1.1. Having set the stage, we are now ready to discuss the implications of conformal symmetry.

Implications of conformal symmetry. Correlations functions in a CFT obey relations due to conformal symmetry. These are especially tight for two- and three-point functions of scalar primary operators as their functional form is completely fixed by conformal symmetry. In what follows, we shall briefly review the implications of conformal symmetry on correlation functions. For convenience, we shall employ the path integral formulation of the underlying field theory. Note, however, that the derived relations are valid in any CFT independent of whether or not the theory admits a description in terms of an action. For an exhaustive discussion on the implications of conformal symmetry the reader is referred to the textbook [55].

As before, we begin by focusing on finite conformal transformations. Let ϕ be a collection of scalar primary fields with conformal dimensions Δ and $S[\phi]$ be short for the action of the underlying theory which we assume to be conformally invariant. Using the path integral formulation, an n-point Greens function can be expressed as

$$G_n(x_1,\ldots,x_n) = \langle\phi_1(x_1')\ldots\phi_n(x_n')\rangle = \frac{1}{Z}\int[\mathcal{D}\phi]\,\phi_1(x_1')\ldots\phi_n(x_n')\,e^{iS[\phi]}\,, \tag{2.19}$$

where Z is the vacuum functional. For simplicity, we shall assume that the coordinates x_i' are all different and that the Green's functions are finite. Relabeling the integration variables and using equation (2.14) yields

$$\begin{aligned}\langle\phi_1(x_1')\ldots\phi_n(x_n')\rangle &= \frac{1}{Z}\int[\mathcal{D}\phi']\,\phi_1'(x_1')\ldots\phi_n'(x_n')\,e^{iS[\phi']}\\ &= \frac{1}{Z}\left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{-\Delta_1/4}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{-\Delta_n/4}\int[\mathcal{D}\phi]\,\phi_1(x_1)\ldots\phi_n(x_n)\,e^{iS[\phi]}\,. \end{aligned}\tag{2.20}$$

Here, we have used that the transformation leaves both the action and the measure invariant, i.e.

$$S[\phi'] = S[\phi]\,, \qquad\qquad [\mathcal{D}\phi'] = [\mathcal{D}\phi]\,. \tag{2.21}$$

The correlators thus satisfy the following identity:

$$\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/4}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/4}\langle\phi(x_1')\ldots\phi(x_n')\rangle\,. \tag{2.22}$$

Note that the above relations hold for any correlation function of scalar primary operators. Whether the operators are fundamental fields in the sense of the path integral or composite objects does not play a role.

The infinitesimal analogs of the equations (2.22) are called conformal Ward identities. These can be derived as follows. Consider, for example, a dilatation, which acts on the coordinates as $x' = e^{\lambda}x$. Under such a transformation, a Green's function transforms according to

$$G_n(x_1,\ldots,x_n) = e^{\lambda(\Delta_1+\ldots+\Delta_n)}G_n(e^{\lambda}x_1,\ldots,e^{\lambda}x_n)\,. \tag{2.23}$$

Taking the derivative with respect to λ and setting $\lambda = 0$ yields

$$\sum_{i=1}^{n}\left(x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}} + \Delta_i\right)G_n(x_1,\ldots,x_n) = 0\,. \tag{2.24}$$

This is the Ward identity for dilatations for correlators of primary fields. In a completely similar manner one can derive Ward identities for the other conformal generators. Denoting the conformal generators (2.18) by $\mathbb{J}^{\delta}$, we can summarize the conformal Ward identities as

$$\sum_{i=1}^{n}\Big\langle\phi_1(x_1)\ldots(\mathbb{J}_i^{\delta}\phi_i)(x_i)\ldots\phi_n(x_n)\Big\rangle = 0\,. \tag{2.25}$$

The Ward identities have severe consequences for the functional form of correlation functions. In particular, two- and three-point functions are heavily constrained by conformal symmetry. As we will employ a similar logic later on in order to obtain two-point functions of superfields, let us examine in detail how conformal symmetry can be used to fix these correlation functions. For this, we consider the correlator of two scalar primary fields

$$G_2(x_1, x_2) = \langle \phi_1(x_1)\phi_2(x_2)\rangle \,. \tag{2.26}$$

From invariance under Poincaré transformations we deduce that the two-point function must be a function of the variable $x_{12}^2 := (x_1 - x_2)^2$. Taking into account the scaling identity (2.23), we find that the two-point correlator has to be of the following form

$$G_2(x_1, x_2) = \frac{c_{12}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}} \,, \tag{2.27}$$

where c_{12} is an undetermined constant, which reflects the normalization of the two-point function. What remains is to exploit the consequences of conformal boost symmetry. Instead of studying special conformal transformations directly, we again focus on the inversion

$$G_2(x_1, x_2) = (x_1^2)^{-\Delta_1}(x_2^2)^{-\Delta_2}\, G_2\left(\frac{x_1}{x_1^2}, \frac{x_2}{x_2^2}\right) \,. \tag{2.28}$$

By inserting the result (2.27) into the former equation we learn that the conformal dimensions of the two operators have to coincide, i.e. $\Delta_1 = \Delta_2 = \Delta$. If the conformal dimensions are different, the two-point correlator vanishes. Thus, we deduce that

$$G_2(x_1, x_2) = \frac{c_{12}}{(x_{12}^2)^{\Delta}} \,, \qquad \text{for} \quad \Delta_1 = \Delta_2 = \Delta \,. \tag{2.29}$$

The functional form of the two-point function is obviously completely fixed. The only dynamically determined quantities are the scaling dimensions Δ. For three-point functions of scalar operators the situation is similar. The same reasoning as employed above leads us to the conclusion that

$$G_3(x_1, x_2, x_3) = \frac{c_{123}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \,, \tag{2.30}$$

where c_{123} is a so-called structure constant. At four points, the impressive performance of conformal symmetry stops. Indeed, with four points or more one can construct conformal invariants in the form of cross-ratios, which in general read

$$\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \,. \tag{2.31}$$

From the point of view of conformal symmetry, the n-point function may have an arbitrary dependence on these ratios and is thus not fixed by conformal covariance.

2.1.2. Superconformal Symmetry

In this section, we discuss the $\mathcal{N} = 4$ supersymmetric extension of the conformal algebra. For a general introduction to supersymmetry the reader is referred to the textbooks [56, 57]. A more formal introduction to Lie superalgebras can be found in [58]. For more information on the particular extension to be discussed below see also [59].

In general, superconformal algebras are Lie superalgebras which contain the conformal algebra $\mathfrak{so}(2,4) \simeq \mathfrak{su}(2,2)$ as a subalgebra but which have additional odd generators that map bosonic fields to fermionic ones and vice versa. These algebras are denoted as $\mathfrak{su}(2,2|\mathcal{N})$, where $\mathcal{N}$ is related to the number Poincaré supercharges. Since the supercharges carry spinor indices, it is convenient to use spinor indices for the conformal generators as well. Spinor indices can be introduced by contracting the conformal generators with the four-dimensional sigma matrices σ^μ and $\bar{\sigma}^\mu$, see appendix A.1 for our conventions. More precisely, we define

$$\begin{aligned} \mathrm{P}_{\alpha\dot{\alpha}} &= \mathrm{P}_\mu \sigma^\mu_{\alpha\dot{\alpha}}\,, & \mathrm{L}^\alpha{}_\beta &= -\tfrac{1}{2}\mathrm{L}^{\mu\nu}(\sigma_{\mu\nu})_\beta{}^\alpha\,, \\ \mathrm{K}^{\dot{\alpha}a} &= \mathrm{K}^\mu \bar{\sigma}^{\dot{\alpha}\alpha}_\mu\,, & \bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}} &= \tfrac{1}{2}\mathrm{L}^{\mu\nu}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\,. \end{aligned} \tag{2.32}$$

Translating the commutation relations (2.7) into spinor language is a straightforward exercise although a bit tedious. For completeness, let us restate them here. The commutation relations involving rotations can be conveniently phrased by noting that their form does only depend on the spinor indices and their position. Indeed, the indices of any generator J transform canonically according to

$$\begin{aligned} \left[\mathrm{L}^\alpha{}_\beta, \mathrm{J}^\gamma\right] &= -2\delta^\gamma_\beta \mathrm{J}^\alpha + \delta^\alpha_\beta \mathrm{J}^\gamma\,, & \left[\mathrm{L}^\alpha{}_\beta, \mathrm{J}_\gamma\right] &= 2\delta^\alpha_\gamma \mathrm{J}_\beta - \delta^\alpha_\beta \mathrm{J}_\gamma\,, \\ \left[\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathrm{J}^{\dot{\gamma}}\right] &= -2\delta^{\dot{\gamma}}_{\dot{\beta}} \mathrm{J}^{\dot{\alpha}} + \delta^{\dot{\alpha}}_{\dot{\beta}} \mathrm{J}^{\dot{\gamma}}\,, & \left[\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathrm{J}_{\dot{\gamma}}\right] &= 2\delta^{\dot{\alpha}}_{\dot{\gamma}} \mathrm{J}_{\dot{\beta}} - \delta^{\dot{\alpha}}_{\dot{\beta}} \mathrm{J}_{\dot{\gamma}}\,. \end{aligned} \tag{2.33}$$

For the commutator of translations and conformal boosts, we find

$$\left[\mathrm{P}_{\beta\dot{\beta}}, \mathrm{K}^{\dot{\alpha}\alpha}\right] = 2\delta^{\dot{\alpha}}_{\dot{\beta}} \mathrm{L}^\alpha{}_\beta + 2\delta^\alpha_\beta \bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}} + 4\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^\alpha_\beta \mathrm{D}\,. \tag{2.34}$$

As before, the dilatation generator D acts diagonally on all basis elements of the conformal algebra, i.e.

$$\left[\mathrm{D}, \mathrm{J}\right] = \dim(\mathrm{J})\,\mathrm{J}\,, \tag{2.35}$$

with

$$\dim(\mathrm{P}) = -\dim(\mathrm{K}) = 1\,, \qquad \text{and} \qquad \dim(\mathrm{L}) = -\dim(\bar{\mathrm{L}}) = 0\,. \tag{2.36}$$

As mentioned above, we are interested in the $\mathcal{N} = 4$ supersymmetric extension of the conformal algebra. To obtain this algebra, we supplement the conformal algebra by four pairs of Poincaré supercharges $\mathrm{Q}_{\alpha a}$ and $\bar{\mathrm{Q}}^b{}_{\dot{\alpha}}$ with $a, b = 1, \ldots, 4$ being an index which enumerates these pairs. These generators commute with the generator of translations and their anticommutator is given by

$$\left\{\mathrm{Q}_{\alpha a}, \bar{\mathrm{Q}}^{b}{}_{\dot{\alpha}}\right\} = 2\delta^{b}_{a}\mathrm{P}_{\alpha\dot{\alpha}}\,. \tag{2.37}$$

Under Lorentz transformations, the supercharges transform according to (2.33) and it follows immediately from (2.37) that they have half the dimension of the momentum generator, i.e.

$$\dim(\mathrm{Q}) = \dim(\bar{\mathrm{Q}}) = \tfrac{1}{2}\,. \tag{2.38}$$

The additional Latin index that is carried by the supercharges gives rise to a further set of bosonic generators, which are called R-symmetry generators. Under an R-symmetry transformation, the supercharges get rotated into each other, while the bosonic generators stay unaffected. In analogy to the case of spacetime rotations (2.33), the R-symmetry generators act on an arbitrary basis element of the algebra by transforming the index according to

$$\left[\mathrm{R}^{a}{}_{b}, \mathrm{J}^{c}\right] = -2\delta^{c}_{b}\mathrm{J}^{a} + \tfrac{1}{2}\delta^{a}_{b}\mathrm{J}^{c}\,, \qquad \left[\mathrm{R}^{a}{}_{b}, \mathrm{J}_{c}\right] = 2\delta^{a}_{c}\mathrm{J}_{b} - \tfrac{1}{2}\delta^{a}_{b}\mathrm{J}_{c}\,. \tag{2.39}$$

The above commutation relations show that the trace $\sum \mathrm{R}^{a}{}_{a}$ vanishes, so that we have in total fifteen linearly independent generators which form an $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ algebra. An additional set of odd charges arises from commuting the generators of supertranslations with the conformal boost generator $\mathrm{K}^{\dot{\alpha}\alpha}$,

$$\left[\mathrm{Q}_{\beta b}, \mathrm{K}^{\dot{\alpha}\alpha}\right] = -2\delta^{\alpha}_{\beta}\bar{\mathrm{S}}^{\dot{\alpha}}{}_{b}\,, \qquad \left[\bar{\mathrm{Q}}^{b}{}_{\dot{\beta}}, \mathrm{K}^{\dot{\alpha}\alpha}\right] = -2\delta^{\dot{\alpha}}_{\dot{\beta}}\mathrm{S}^{b\alpha}\,. \tag{2.40}$$

We shall refer to the elements S and $\bar{\mathrm{S}}$ as the generators of superboosts. The naming serves the purpose of indicating that these generators are related to K in the same way as the Poincaré supercharges are related to P,

$$\left\{\mathrm{S}^{a\alpha}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\right\} = -2\delta^{a}_{b}\mathrm{K}^{\dot{\beta}\alpha}\,. \tag{2.41}$$

The superboost generators thus have half the dimension of $\mathrm{K}^{\dot{\alpha}\alpha}$, i.e.

$$\dim(\mathrm{S}) = \dim(\bar{\mathrm{S}}) = -\tfrac{1}{2}\,. \tag{2.42}$$

Finally, let us note the remaining non-vanishing commutators. In analogy to (2.37), we have

$$\left[\mathrm{P}_{\alpha\dot{\alpha}}, \mathrm{S}^{b\beta}\right] = +2\delta^{\beta}_{\alpha}\bar{\mathrm{Q}}^{b}{}_{\dot{\alpha}}\,, \qquad \left[\mathrm{P}_{\alpha\dot{\alpha}}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\right] = +2\delta^{\dot{\beta}}_{\dot{\alpha}}\mathrm{Q}_{\alpha b}\,. \tag{2.43}$$

For the anticommutators of supertranslations and superboosts, one finds

$$\begin{aligned} \left\{\mathrm{Q}_{\alpha a}, \mathrm{S}^{b\beta}\right\} &= -2\delta^{b}_{a}\mathrm{L}^{\beta}{}_{\alpha} + 2\delta^{\beta}_{\alpha}\mathrm{R}^{b}{}_{a} - 2\delta^{b}_{a}\delta^{\beta}_{\alpha}(\mathrm{D}-\mathrm{C})\,, \\ \left\{\bar{\mathrm{Q}}^{a}{}_{\dot{\alpha}}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\right\} &= -2\delta^{a}_{b}\bar{\mathrm{L}}^{\dot{\beta}}{}_{\dot{\alpha}} - 2\delta^{\dot{\beta}}_{\dot{\alpha}}\mathrm{R}^{a}{}_{b} - 2\delta^{a}_{b}\delta^{\dot{\beta}}_{\dot{\alpha}}(\mathrm{D}+\mathrm{C})\,. \end{aligned} \tag{2.44}$$

Here, C is the central charge of the superconformal algebra $\mathfrak{su}(2,2|4)$, which commutes with all the other generators. If the central charge C is absent, the algebra is denoted by $\mathfrak{psu}(2,2|4)$. Conversely, we can also promote the algebra $\mathfrak{su}(2,2|4)$ to $\mathfrak{u}(2,2|4)$. This is achieved by adding the hypercharge generator B to $\mathfrak{su}(2,2|4)$. The hypercharge generator is diagonal in the chosen basis, i.e.

$$\Big[\mathrm{B},\mathrm{J}\Big] = \mathrm{hyp}(\mathrm{J})\,\mathrm{J}\,, \tag{2.45}$$

with the non-vanishing hypercharges given by

$$\mathrm{hyp}(\bar{\mathrm{Q}}) = -\,\mathrm{hyp}(\mathrm{Q}) = \mathrm{hyp}(\mathrm{S}) = -\,\mathrm{hyp}(\bar{\mathrm{S}}) = \tfrac{1}{2}\,. \tag{2.46}$$

This concludes our formal discussion of the $\mathcal{N} = 4$ supersymmetric extension of the conformal algebra.

A natural question that arises is how the superconformal algebra can be represented in terms of vector fields, cf. equation (2.6). Constructing such a representation is in general a quite difficult task. In fact, the odd generators require the introduction of anticommuting coordinates, which together with the coordinates of Minkowski space form a superspace. We shall introduce this space in chapter 3, thereby laying the foundations for a detailed discussion of the superspace representation of the superconformal algebra in section 4.2.2.

2.1.3. Yangian Symmetry

A common feature of many integrable models is the appearance of an infinite-dimensional symmetry algebra of Yangian type. In general, the Yangian can be viewed as an extension of an underlying Lie algebra symmetry and as such it was introduced by Drinfel'd in references [60, 61]. It represents one of the central algebraic structures underlying the quantum inverse scattering method (QISM) [62, 63], which forms the foundation for a set of tools that have proven to be incredibly useful when analyzing quantum integrable systems.

The most common occurrence of Yangian symmetry is within models which are at most two-dimensional. These can be discrete, e.g. spin chain models, or continuous, such as two-dimensional sigma models. One reason why Yangian symmetry is mainly restricted to one- and one-plus-one-dimensional models is that the definition of the Yangian requires an ordering prescription, which is typically absent in higher-dimensional theories. It is thus remarkable that Yangian symmetry has been encountered within the class of four-dimensional gauge theories, namely for $\mathcal{N} = 4$ SYM theory. Most prominently, Yangian symmetry has been detected for the spectral problem of local gauge-invariant operators [15, 16, 23, 64, 65] as well as for the tree-level S-matrix [25, 66–71]. The appearance of the Yangian in $\mathcal{N} = 4$ SYM theory is intimately related to the large-N limit or planar limit [14]. This becomes most transparent within the context of tree-level scattering amplitudes. Only in the large-N limit single-trace structures become dominant and the order of fields is unambiguously[1] determined in terms of

[1]More precisely, the position is unambiguous modulo cyclic shifts.

their position within the matrix product that spans the color trace. We shall come back to this point in section 5.1.2.

Below, we give a brief and pedagogical introduction to the so-called first realization of the Yangian [60, 61], thereby laying the foundations for our discussion of Yangian symmetry of super Wilson loops and fishnet Feynman graphs. Later on, we shall also encounter the so-called RTT realization, which is related to the QISM. However, as this formulation of the Yangian becomes only relevant in chapter 6, we shall introduce it there. Our presentation is based on references [42, 45, 72–74].

Preliminaries. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra[2] with structure constants ${f^{\delta\rho}}_{\kappa}$ and generators J^{δ}, i.e.

$$\left[\mathrm{J}^{\delta}, \mathrm{J}^{\rho}\right] = {f^{\delta\rho}}_{\kappa}\mathrm{J}^{\kappa}\,, \qquad \left[\mathrm{J}^{\delta}, \left[\mathrm{J}^{\rho}, \mathrm{J}^{\kappa}\right]\right] + \text{cyclic} = 0\,. \tag{2.47}$$

Importantly, there is a priori no associative product operation in a Lie algebra. The only product is the abstract Lie bracket. Products of two arbitrary elements, such as $x \star y$, are thus undefined unless we are working in a representation, which specifies the action of Lie algebra elements on some vector space V. Regarding the construction of the Yangian this puts us in a slightly inconvenient position as the definition of the Yangian naturally requires such a product. In order to define the Yangian in complete generality, one therefore needs to rely on the concept of the universal enveloping algebra, which is, simply speaking, an embedding of a Lie algebra into a much bigger associative algebra such that the abstract bracket operation in $\mathfrak{g}$ corresponds to the commutator $x \star y - y \star x$. However, we refrain from introducing the universal enveloping algebra and instead content ourselves with working in a representation so that the product of Lie algebra elements is well-defined. For the moment, we do not specify the representation of the underlying Lie algebra, but we shall argue in section 5.1.2 that the field representation is in fact the most natural one for defining the Yangian.

Before we can define the Yangian, we need to introduce a symmetric bilinear form on the Lie algebra, i.e. we need to define a metric that we can use to raise and lower Lie algebra indices. A convenient choice for such a symmetric bilinear form is the Killing form, which is defined as

$$K(x, y) = \mathrm{tr}\Big(\mathrm{ad}(x)\ \mathrm{ad}(y)\Big)\,, \qquad x, y \in \mathfrak{g}\,, \tag{2.48}$$

where $\mathrm{ad}(x)$ denotes a Lie algebra element in the adjoint representation. Evaluated on basis elements, the Killing form reads

$$K^{\delta\rho} = K(J^{\delta}, J^{\rho}) = \mathrm{tr}\Big(\mathrm{ad}(J^{\delta})\ \mathrm{ad}(J^{\rho})\Big) = {f^{\delta\kappa}}_{\sigma}\,{f^{\rho\sigma}}_{\kappa}\,. \tag{2.49}$$

Unfortunately, the Killing form vanishes for certain superalgebras including $\mathfrak{psu}(2,2|4)$. We thus cannot use it to raise and lower indices of the superconformal structure constants. Fortunately, any non-degenerate symmetric bilinear form will do the job and

[2] For reasons of clarity, we shall assume all the generators to be bosonic. The generalization to the case of a superalgebra is straightforward, see, for example, reference [25].

we can thus simply pick the trace in the fundamental representation

$$g^{\delta\rho} = g(\mathrm{J}^\delta, \mathrm{J}^\rho) = \mathrm{tr}\Big(M[\mathrm{J}^\delta]\, M[\mathrm{J}^\rho]\Big)\,. \tag{2.50}$$

For a simple Lie algebra, there is after all not much difference because in this case all symmetric bilinear forms are proportional to the Killing form, i.e. $g^{\delta\rho} = cK^{\delta\rho}$. As the metric (2.50) is non-degenerate by assumption, its inverse exists and we will denote it by $g_{\delta\rho}$. Using this metric, we can now raise and lower algebra indices, for example

$$f^{\delta}{}_{\rho\kappa} = g_{\rho\sigma}\, f^{\delta\sigma}{}_{\kappa}\,. \tag{2.51}$$

Having set out the framework, we are now ready to define the Yangian algebra.

Definition of the Yangian algebra. The Yangian algebra $Y[\mathfrak{g}]$, as it was introduced by Drinfel'd [60, 61], is defined as the algebra which is generated by the Lie algebra generators J^δ and a second set of generators $\widehat{\mathrm{J}}^\delta$ obeying the following commutation relations:

$$\big[\mathrm{J}^\delta, \mathrm{J}^\rho\big] = f^{\delta\rho}{}_{\kappa}\mathrm{J}^\kappa\,, \qquad\qquad \big[\mathrm{J}^\delta, \widehat{\mathrm{J}}^\rho\big] = f^{\delta\rho}{}_{\kappa}\,\widehat{\mathrm{J}}^\kappa\,. \tag{2.52}$$

The generators $\widehat{\mathrm{J}}^\delta$ are typically called level-one generators for reasons that will become clear momentarily. From the above commutation relations two sets of Jacobi identities follow. However, a third Jacobi-like identity is quantum deformed to the following Serre relation:[3]

$$\Big[\widehat{\mathrm{J}}^\rho, \big[\mathrm{J}^\kappa, \widehat{\mathrm{J}}^\delta\big]\Big] + \Big[\widehat{\mathrm{J}}^\delta, \big[\mathrm{J}^\rho, \widehat{\mathrm{J}}^\kappa\big]\Big] + \Big[\widehat{\mathrm{J}}^\kappa, \big[\mathrm{J}^\delta, \widehat{\mathrm{J}}^\rho\big]\Big] = \tfrac{1}{4}\, f^{\rho\sigma}{}_{\omega} f^{\kappa\alpha}{}_{\beta} f^{\delta\gamma}{}_{\varepsilon} f_{\sigma\alpha\gamma}\, \mathrm{J}^{(\omega}\mathrm{J}^{\beta}\mathrm{J}^{\varepsilon)}\,. \tag{2.53}$$

For $\mathfrak{g} = \mathfrak{su}(2)$, this relation has to be replaced by another relation which is otherwise implied, see reference [73]. Using the commutation relations (2.52), it is straightforward to rewrite the Serre relation in the following way:

$$f^{[\rho\kappa}{}_{\sigma}\big[\widehat{\mathrm{J}}^{\delta]}, \widehat{\mathrm{J}}^\sigma\big] = \tfrac{1}{12}\, f^{\rho\sigma}{}_{\omega} f^{\kappa\alpha}{}_{\beta} f^{\delta\gamma}{}_{\varepsilon} f_{\sigma\alpha\gamma}\, \mathrm{J}^{(\omega}\mathrm{J}^{\beta}\mathrm{J}^{\varepsilon)}\,. \tag{2.54}$$

As opposed to the more familiar case of Lie algebras, the above relations do not completely specify the commutation relations of all generators. To see this, let us note that the commutator of two level-one generators does not lie in the span of $\mathfrak{g} \oplus \{\widehat{\mathrm{J}}^\delta\}$ but rather yields a new generator, which is called a level-two generator. Denoting $\mathrm{J} = \mathrm{J}^{(0)}$ and $\widehat{\mathrm{J}} = \mathrm{J}^{(1)}$, we define an element of grade two by the relation

$$\mathrm{J}^{\delta\,(2)} = \frac{1}{2\mathfrak{c}} f^{\delta}{}_{\rho\kappa}\big[\mathrm{J}^{\kappa\,(1)}, \mathrm{J}^{\rho\,(1)}\big]\,. \tag{2.55}$$

Here, $\mathfrak{c}$ is the dual Coxeter number, which is defined as

$$f^{\delta}{}_{\rho\kappa} f^{\kappa\rho}{}_{\sigma} = 2\mathfrak{c}\delta^\delta_\sigma\,. \tag{2.56}$$

[3]The brackets denote symmetrization or antisymmetrization of the enclosed indices. More precisely, we define $X_{(i_1\dots i_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} X_{i_{\sigma(1)}\dots i_{\sigma(n)}}$ and $X_{[i_1\dots i_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma) X_{i_{\sigma(1)}\dots i_{\sigma(n)}}$.

The commutator between two level-one generators can then be expressed as

$$\left[\mathrm{J}^{\delta\,(1)}, \mathrm{J}^{\rho\,(1)}\right] = f^{\delta\rho}{}_{\kappa}\mathrm{J}^{\kappa\,(2)} + X^{\delta\rho}\,, \tag{2.57}$$

with $f^{\delta}{}_{\rho\kappa}X^{\kappa\rho} = 0$. Plugging the above form of the commutator into the Serre relation (2.54) yields

$$f^{[\rho\kappa}{}_{\sigma}X^{\delta]\sigma} = \tfrac{1}{12}\, f^{\rho\sigma}{}_{\omega}f^{\kappa\alpha}{}_{\beta}f^{\delta\gamma}{}_{\varepsilon}f_{\sigma\alpha\gamma}\,\mathrm{J}^{(\omega}\mathrm{J}^{\beta}\mathrm{J}^{\varepsilon)}\,, \tag{2.58}$$

where we have used that the term which includes the level-two generator vanishes due to the Jacobi identity. It may happen that for some level-zero representations the right-hand side of the above equation vanishes. In these cases the commutator closes with $X = 0$. Conversely, if the right-hand side does not vanish, X has to be chosen such that the Serre relation (2.58) is fulfilled. The form of the commutator of two level-one generators is thus non-universal but nevertheless completely determined by the Serre relation [73]. The procedure described above can be iterated leading to an infinite tower of Yangian generators which are organized into levels. The structure of the Yangian is in fact very reminiscent of the structure of a loop algebra $\mathfrak{g}[u]$, which is spanned by the generators $\mathrm{J}^{(n)} = u^n\mathrm{J}$ obeying

$$\left[\mathrm{J}^{\delta\,(n)}, \mathrm{J}^{\rho\,(m)}\right] = f^{\delta\rho}{}_{\kappa}\mathrm{J}^{\kappa\,(n+m)}\,. \tag{2.59}$$

And indeed, the Yangian can be understood as a deformation of this algebra.

An important aspect of the Yangian is that it possesses the structure of a Hopf algebra. In particular, this means that it can be equipped with a comultiplication operation $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$. On basis elements, the coproduct acts as

$$\begin{aligned}\Delta(\mathrm{J}^{\delta}) &= \mathrm{J}^{\delta}\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{J}^{\delta}\,,\\ \Delta(\widehat{\mathrm{J}}^{\delta}) &= \widehat{\mathrm{J}}^{\delta}\otimes\mathbb{1} + \mathbb{1}\otimes\widehat{\mathrm{J}}^{\delta} + \tfrac{1}{2}\, f^{\delta}{}_{\rho\kappa}\,\mathrm{J}^{\kappa}\otimes\mathrm{J}^{\rho}\,.\end{aligned} \tag{2.60}$$

The coproduct is consistent with the commutation relations (2.52) as well as with the Serre relation (2.53). In fact, the latter follows from demanding that the coproduct furnishes an algebra morphism, see [75]. The main purpose of the coproduct is to define tensor products of representations, i.e. given a one-site representation, the coproduct determines the action of the generators on a multi-site space. More precisely, the action of the generators on the tensor product of n fields is determined by $\Delta^{n-1}(\mathrm{J})$,

$$\Delta^{n-1}(\mathrm{J}^{\delta}) = \sum_{k=1}^{n}\mathrm{J}^{\delta}_{k}\,, \qquad \Delta^{n-1}(\widehat{\mathrm{J}}^{\delta}) = \sum_{k=1}^{n}\widehat{\mathrm{J}}^{\delta}_{k} + \tfrac{1}{2}f^{\delta}{}_{\rho\kappa}\sum_{i<j=1}^{n}\mathrm{J}^{\kappa}_{i}\mathrm{J}^{\rho}_{j}\,. \tag{2.61}$$

Note that there is no ambiguity in the above relations as the coproduct is coassociative, i.e. $(\mathbb{1}\otimes\Delta)\Delta = (\Delta\otimes\mathbb{1})\Delta$. This concludes our discussion on the definition of the Yangian algebra.

Representations of the Yangian. A natural question that arises is how the level-one generators can be represented. More precisely, we need to define how the level-one generators act on a single site. In general, finding representations of the Yangian is a rather difficult task. In fact, we will only cover the simplest case here and that is when a representation of a Lie algebra can be lifted to a representation of the corresponding Yangian. This representation is called evaluation representation and it is given by

$$\widehat{\mathrm{J}}^{\delta} = v\mathrm{J}^{\delta} \,. \tag{2.62}$$

The level-one generators are chosen to be proportional to the corresponding Lie algebra generators with v being the spectral parameter or evaluation parameter. The so-defined level-one generators clearly satisfy the commutation relation (2.52). Moreover, we observe that the left-hand side of the Serre relation (2.54) vanishes trivially due to the Jacobi identity. However, whether the right-hand side of the Serre relation vanishes depends on the considered Lie algebra representation. Hence, we conclude that the above-described prescription leads to a valid representation of the Yangian if and only if the right-hand side of equation (2.54) vanishes in the representation under consideration. This criterion was investigated in [65] for certain representations of $\mathfrak{su}(N)$, $\mathfrak{u}(N|M)$ as well as $\mathfrak{psu}(2,2|4)$. Given the one-site representation (2.62), the multi-site action of the generators follows immediately by repeatedly applying the coproduct rule (2.60),

$$\mathrm{J}^{\delta} = \sum_{k=1}^{n} \mathrm{J}_k^{\delta} \,, \qquad \widehat{\mathrm{J}}^{\delta} = \sum_{k=1}^{n} v_k \mathrm{J}_k^{\delta} + \tfrac{1}{2} f^{\delta}{}_{\rho\kappa} \sum_{i<j=1}^{n} \mathrm{J}_i^{\kappa} \mathrm{J}_j^{\rho} \,. \tag{2.63}$$

Here, we have dropped the coproduct symbol as is frequently done in the literature. Although the evaluation representation is quite special, it has a wide range of applications in $\mathcal{N}=4$ SYM theory. In particular, the planar one-loop dilatation operator [23] as well as color-ordered scattering amplitudes [25] transform in the (trivial) evaluation representation of the Yangian.

2.2. $\mathcal{N} = 4$ Super Yang–Mills Theory

In this section, we introduce the $\mathcal{N} = 4$ SYM model, which is the unique maximally supersymmetric gauge theory in four dimensions. A convenient way to introduce this theory is to perform a dimensional reduction procedure on the ten-dimensional $\mathcal{N} = 1$ SYM model, as was first described in reference [9]. In this section, we will review this procedure and comment briefly on the quantum aspects of the model, such as its β-function.

2.2.1. $\mathcal{N} = 1$ SYM Theory in Ten Dimensions

Let us start by introducing the $\mathcal{N} = 1$ SYM model in ten-dimensional Minkowski space with metric $\eta^{\hat{\mu}\hat{\nu}} = \mathrm{diag}(1,-1,\ldots,-1)$. The field content consists of a ten-dimensional gauge field $A_{\hat{\mu}}$ and a real chiral spinor $\psi^{\hat{\alpha}}$. All fields transform in the adjoint representation of $\mathfrak{su}(N)$ and it is thus convenient to introduce the following matrix-valued fields

$$A_{\hat{\mu}} = A^{\mathfrak{m}}_{\hat{\mu}}(x) T_{\mathfrak{m}}\,, \qquad \psi^{\hat{\alpha}} = \psi^{\hat{\alpha}\mathfrak{m}}(x) T_{\mathfrak{m}}\,, \qquad \text{with} \quad \begin{array}{l} \hat{\mu} = 0, \ldots, 9\,, \\ \hat{\alpha} = 1, \ldots, 16\,, \end{array} \tag{2.64}$$

where $T_{\mathfrak{m}}$ are the antihermitian SU(N) generators obeying

$$T^{\dagger}_{\mathfrak{m}} = -T_{\mathfrak{m}}\,, \qquad \operatorname{tr}(T_{\mathfrak{m}}) = 0\,, \qquad \operatorname{tr}(T_{\mathfrak{m}} T_{\mathfrak{n}}) = -\frac{\delta_{\mathfrak{m}\mathfrak{n}}}{2}\,. \tag{2.65}$$

Under a finite gauge transformation parametrized by

$$\Omega = \exp(-\omega(x))\,, \qquad \text{with} \qquad \omega(x) = \omega^{\mathfrak{m}}(x) T_{\mathfrak{m}}\,, \tag{2.66}$$

the fields transform as

$$A_{\hat{\mu}} \to \Omega(A_{\hat{\mu}} + \partial_{\hat{\mu}})\Omega^{\dagger}\,, \qquad \psi^{\hat{\alpha}} \to \Omega \psi^{\hat{\alpha}} \Omega^{\dagger}\,. \tag{2.67}$$

For later convenience, let us also note the infinitesimal form of the above transformation laws. Expanding the equations (2.67) in ω and denoting the generator of infinitesimal gauge transformations by $\mathbb{G}[\omega]$ yields

$$\mathbb{G}[\omega] A_{\hat{\mu}} = D_{\hat{\mu}}\, \omega(x)\,, \qquad \mathbb{G}[\omega] \psi^{\hat{\alpha}} = [\psi^{\hat{\alpha}}, \omega(x)]\,, \tag{2.68}$$

where $D_{\hat{\mu}}$ is the gauge-covariant derivative, which in our conventions reads

$$D_{\hat{\mu}} = \partial_{\hat{\mu}} + A_{\hat{\mu}}\,, \qquad D_{\hat{\mu}} = \partial_{\hat{\mu}} + [A_{\hat{\mu}},\ \ ]\,. \tag{2.69}$$

All the fields in this thesis transform in the adjoint representation of the gauge group, so that the second expression for the gauge-covariant derivative is the appropriate one. However, sometimes we shall make the commutator explicit in which case the first definition has to be used. Given the gauge-covariant derivative, the field strength tensor follows immediately

$$F_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A_{\hat{\nu}} - \partial_{\hat{\nu}} A_{\hat{\mu}} + [A_{\hat{\mu}}, A_{\hat{\nu}}]\,. \tag{2.70}$$

Note that we use conventions in which the gauge field has mass dimension $[A] = 1$, the spinor field has mass dimension $[\psi] = 3/2$ and the coupling constant has dimension $[g_{10}] = -3$. Compared to standard conventions we have thus rescaled the fields by the dimensionful coupling constant g_{10}. In this conventions, the ten-dimensional $\mathcal{N} = 1$ action takes the form

$$S_{\mathcal{N}=1} = \frac{1}{g_{10}^2} \int \mathrm{d}^{10}x \ \operatorname{tr}\Big[\tfrac{1}{2} F_{\hat{\mu}\hat{\nu}}\, F^{\hat{\mu}\hat{\nu}} + \psi \Gamma_{\hat{\mu}} D^{\hat{\mu}} \psi\Big]\,. \tag{2.71}$$

Here, $\Gamma_{\hat{\mu}}$ are the ten-dimensional Pauli matrices

$$\Gamma^{\hat{\mu}} \bar{\Gamma}^{\hat{\nu}} + \Gamma^{\hat{\nu}} \bar{\Gamma}^{\hat{\mu}} = 2\eta^{\hat{\mu}\hat{\nu}}\,, \tag{2.72}$$

which furnish the chiral representation of the ten-dimensional gamma matrices, see appendix A.3 for our conventions. Finally, let us note that the above action is invariant under the following supersymmetry transformations of the fields

$$\delta_{\zeta} A_{\hat{\mu}} = [\zeta \mathbb{Q}, A_{\hat{\mu}}] = -\zeta \Gamma_{\hat{\mu}} \psi\,, \qquad \delta_{\zeta} \psi = [\zeta \mathbb{Q}, \psi] = \tfrac{1}{2}(\bar{\Gamma}^{\hat{\mu}\hat{\nu}} \zeta) F_{\hat{\mu}\hat{\nu}}\,, \tag{2.73}$$

where $\bar{\Gamma}^{\hat{\mu}\hat{\nu}} := \frac{1}{2}(\bar{\Gamma}^{\hat{\mu}} \Gamma^{\hat{\nu}} - \bar{\Gamma}^{\hat{\mu}} \Gamma^{\hat{\nu}})$. For a proof of this statement see, for example, [76].

2.2.2. $\mathcal{N} = 4$ SYM Theory in Four Dimensions

Having introduced the ten-dimensional $\mathcal{N} = 1$ SYM model, we are now ready to perform the dimensional reduction down to four dimensions. For this, we split the ten-dimensional spacetime coordinates according to

$$x^{\hat{\mu}} = (x^\mu, y^i)\,, \qquad \text{with} \qquad \begin{matrix} \mu = 0, \dots, 3\,, \\ i = 1, \dots, 6\,, \end{matrix} \tag{2.74}$$

where x^μ are coordinates on $\mathbb{R}^{1,3}$, while the y^i parametrize six internal directions. The fields are assumed to be independent of the internal coordinates, i.e.

$$\partial_i A_{\hat{\mu}}(x) = 0\,, \qquad \partial_i \psi^{\hat{\alpha}}(x) = 0\,, \qquad \partial_i \Omega(x) = 0\,, \tag{2.75}$$

and the same applies to local gauge transformations. Therefore, six of the ten components of the gauge field transform as scalars. We thus decompose the ten-dimensional gauge field as

$$A_{\hat{\mu}} \to (A_\mu, \phi_i)\,. \tag{2.76}$$

Finally, it remains to specify how the sixteen-component spinor field can be expressed in terms of four-dimensional spinor fields. For this, we first need to express the ten-dimensional Pauli matrices in terms of four- and six-dimensional sigma matrices. This is discussed in appendix A. The decomposition of $\psi^{\hat{\alpha}}$ which is adapted to our choice of basis reads

$$\psi^{\hat{\alpha}} \to (\psi^{a\alpha}, \bar{\psi}^{\dot{\alpha}}{}_a)\,. \tag{2.77}$$

Here, we have split the original ten-dimensional spinor index into two pairs, each consisting of an $\mathfrak{su}(2)$ index as well as an $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ R-symmetry index.

With the decomposition rules described above and the Pauli matrix identities of appendix A, it becomes a straightforward exercise to dimensionally reduce the ten-dimensional action (2.71). One obtains

$$\begin{aligned} S_{\mathcal{N}=4} = \frac{1}{g^2} \int \mathrm{d}^4 x \; \mathrm{tr}\Big[& \tfrac{1}{2} F_{\mu\nu}^2 + \big(D_\mu \phi_i\big)^2 + 2\, \bar{\psi}^{\dot{\alpha}}{}_a \sigma_{\mu\, \alpha\dot{\alpha}} D^\mu \psi^{a\alpha} + \tfrac{1}{2} \big[\phi^i, \phi^j\big]\big[\phi_i, \phi_j\big] \\ & - \psi^{a\alpha} \Sigma^i_{ab} \varepsilon_{\alpha\beta} \big[\phi_i, \psi^{b\beta}\big] - \bar{\psi}^{\dot{\alpha}}{}_a \bar{\Sigma}^{i\,ab} \varepsilon_{\dot{\alpha}\dot{\beta}} \big[\phi_i, \bar{\psi}^{\dot{\beta}}{}_b\big]\Big]\,, \end{aligned} \tag{2.78}$$

where we have absorbed the volume integral over the internal space into a redefinition of the coupling constant g_{10}, i.e. $g = V^{-1/2} g_{10}$. The new coupling constant is thus dimensionless. In the further course of this thesis, we will often interpret the fundamental fields as matrices in spinor space. In order to treat the bosonic and fermionic fields on the same footing, we define

$$\phi_{ab} = \Sigma^i_{ab} \phi_i\,, \qquad A_{\alpha\dot{\alpha}} = A_\mu \sigma^\mu_{\alpha\dot{\alpha}}\,, \tag{2.79}$$

where Σ and σ are the Pauli matrices in six and in four dimensions, respectively, see appendix A. For later convenience, let us also state the four-dimensional supersymmetry transformations. Reducing the expressions in equation (2.73) yields

$$\begin{aligned} \delta_{\zeta,\bar\zeta} A_\mu &= -\zeta^{a\alpha}\sigma_{\mu\,\alpha\dot\alpha}\bar\psi^{\dot\alpha}{}_a - \bar\zeta^{\dot\alpha}{}_a\sigma_{\mu\,\alpha\dot\alpha}\psi^{a\alpha}\,, \\ \delta_{\zeta,\bar\zeta}\phi_i &= \zeta^{a\alpha}\varepsilon_{\alpha\beta}\Sigma_{i\,ab}\psi^{b\beta} + \bar\zeta^{\dot\alpha}{}_a\varepsilon_{\dot\alpha\dot\beta}\bar\Sigma_i^{ab}\bar\psi^{\dot\beta}{}_b\,, \\ \delta_{\zeta,\bar\zeta}\psi^{a\alpha} &= -\tfrac{1}{2}\zeta^{a\beta}(\sigma^{\mu\nu})_\beta{}^\alpha F_{\mu\nu} + \tfrac{1}{2}\zeta^{b\alpha}(\bar\Sigma^{ij})^a{}_b[\phi_i,\phi_j] + \bar\zeta^{\dot\beta}{}_b\varepsilon_{\dot\beta\dot\gamma}\bar\sigma^{\mu\,\dot\gamma\alpha}\bar\Sigma^{i\,ab}D_\mu\phi_i\,, \\ \delta_{\zeta,\bar\zeta}\bar\psi^{\dot\alpha}{}_a &= \tfrac{1}{2}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\zeta^{\dot\beta}{}_a F_{\mu\nu} + \tfrac{1}{2}\bar\zeta^{\dot\alpha}{}_b(\Sigma^{ij})_a{}^b[\phi_i,\phi_j] + \zeta^{b\beta}\varepsilon^{\dot\gamma\dot\alpha}\sigma^\mu_{\beta\dot\gamma}\Sigma^i_{ab}D_\mu\phi_i\,, \end{aligned} \tag{2.80}$$

where

$$\delta_{\zeta,\bar\zeta}X = [\zeta\mathbb{Q} + \bar\zeta\bar{\mathbb{Q}}, X]\,. \tag{2.81}$$

The four-dimensional action (2.78) inherits the invariance under supersymmetry transformations from the ten-dimensional theory, which now appears as $\mathcal{N} = 4$ supersymmetry. It is furthermore easy to see that the classical action is not only super Poincaré invariant but also scale invariant as well as invariant under R-symmetry transformations. While the latter symmetry is manifest due to properly contracted R-symmetry indices, the former can easily be proven by checking that all summands in the Lagrangian indeed scale uniformly with a scaling weight exactly opposite to that of the measure. The classical action is thus invariant under the superconformal group $\mathrm{PSU}(2,2|4)$.

An important point to note is that the superconformal invariance of the action does not automatically imply that the theory is superconformal at the quantum level. In fact, conformal invariance is usually anomalous. This is related to the fact that while quantizing the theory, one typically needs to introduce a regulator in order to control UV divergences. However, this inevitably requires the introduction of a mass scale μ which typically breaks conformal symmetry for which scale invariance is indispensable. The quantity which signals whether conformal symmetry is preserved is the so-called β-function, which encodes the scale dependence of a coupling constant. In the case at hand, there is only one coupling constant and the associated β-function reads

$$\beta = \mu\frac{\mathrm{d}g}{\mathrm{d}\mu}\,. \tag{2.82}$$

For $\mathcal{N} = 4$ SYM theory this function is believed to vanish to all orders in perturbation theory as well as non-perturbatively [77–82],

$$\beta = 0\,. \tag{2.83}$$

The vanishing of the β-function implies that the conformal symmetry of the classical theory continues to hold at the quantum level. The quantum theory thus still has the full $\mathfrak{psu}(2,2|4)$ invariance.

Since the coupling constant is not running with scale, the $\mathcal{N} = 4$ model has two freely tunable parameters, namely the coupling constant g and the rank of the gauge group N. The two parameters are usually combined into the 't Hooft coupling λ, being defined as

$$\lambda := g^2 N \,. \tag{2.84}$$

When the effective coupling constant λ is introduced in perturbative results for physical observables, these admit a double expansion in powers of λ and N^{-1}. The integrability of the $\mathcal{N} = 4$ model sits in the λ-expansion, while the $1/N$ corrections are typically not integrable. For this reason, we shall consider all quantities in the large-N limit [14], in which the number of colors is taken to infinity while the 't Hooft coupling is held fixed. This limit is also called planar limit because it suppresses all Feynman diagrams[4] which cannot be drawn on a plane without crossings. Therefore, the planar limit implements a two-dimensional characteristics within the four-dimensional theory and it is thus no accident that the integrability of $\mathcal{N} = 4$ SYM theory arises in this exact limit.

2.3. Wilson Loops

In this section, we introduce one of the central observables that we shall study in this thesis which is the Maldacena–Wilson loop operator. We begin by briefly reviewing the definition of Wilson loops in non-abelian gauge theories and proceed by discussing an important application of Wilson loops: the computation of the quark-antiquark potential. We then turn to the Maldacena–Wilson loop operator, which we will derive by considering the dimensional reduction of a light-like Wilson loop in ten-dimensional $\mathcal{N} = 1$ SYM theory. Finally, we shall discuss the BPS property of the Maldacena–Wilson loop operator and comment on the fundamental role that this operator plays within the context of the AdS/CFT correspondence.

2.3.1. Wilson Loops in Gauge Theories

One of the most important observables in a non-abelian gauge theory is the Wilson loop operator. In fact, in pure gauge theories these operators form an (over)complete basis for gauge-invariant functions and are thus as fundamental as the gauge connection itself [83]. However, Wilson loops also play an important role in gauge theories that contain matter. In particular, they capture the potential between two static probe particles and they can thus serve as an order parameter for confinement. In this section, we introduce the Wilson line operator based on general considerations of gauge covariance following references [45, 84, 85].

Let us start by considering two quark fields $\psi(z)$ and $\psi(y)$ which transform in the fundamental representation of the gauge group $\mathrm{SU}(N)$. Under a gauge transformation,

[4]Here, by Feynman diagrams we mean Feynman diagrams in double-line notation, which graphically encode the gauge structure, see reference [14] for more details.

these fields transform as

$$\psi(z) \to \Omega(z)\psi(z)\,, \qquad\qquad \psi(y) \to \Omega(y)\psi(y)\,. \tag{2.85}$$

Since the gauge transformation is local, the two fields cannot directly be compared to each other. This problem is very similar to that of comparing two tangent vectors which live at different points on a manifold. Fortunately, there exists a natural solution to this problem, which is called parallel transport. The construction is borrowed from differential geometry and works as follows: Let γ be a curve that goes from y to z with a parametrization denoted by $x(\tau)$ such that we have $x(0) = y$ and $x(1) = z$. The parallel transport of the spinor field $\psi(y)$ along the curve γ is then defined as the solution to the following initial value problem:

$$\dot{x}^\mu D_\mu V_\gamma(x(\tau), y)\psi(y) = 0\,, \qquad \text{with} \qquad V_\gamma(y, y)\psi(y) = \psi(y)\,, \tag{2.86}$$

where we have written the transported spinor field as an operator acting on the original spinor $\psi(y)$. In words, the above equation states that the field $V_\gamma(x(\tau), y)\psi(y)$ is covariantly constant along the curve γ. Stripping of the spinor field $\psi(y)$ and plugging in the definition of the covariant derivative yields

$$\partial_\tau V_\gamma(x(\tau), y) = -\dot{x}^\mu(\tau) A_\mu(x(\tau)) V_\gamma(x(\tau), y)\,, \qquad \text{with} \qquad V_\gamma(y, y) = \mathbb{1}\,. \tag{2.87}$$

This equation is obviously an ordinary first order differential equation and the existence and uniqueness of the solution is thus guaranteed by the Picard–Lindelöf theorem. In the context of gauge theories, the operator V_γ is called a Wilson line operator. As a small consistency check, let us prove that under a gauge transformation the operator V_γ indeed transforms as

$$V'_\gamma(x, y) = \Omega(x) V_\gamma(x, y) \Omega^\dagger(y)\,. \tag{2.88}$$

For this, we first note that independent of whether or not the above equation states the correct relation between V_γ and V'_γ, the abstract operator V'_γ satisfies by definition the defining equation (2.87) with A replaced by A'. Let us now show that the right-hand side of equation (2.88) satisfies this defining relation as well. Indeed, a short computation reveals that

$$\begin{aligned} \partial_\tau V'_\gamma(x, y) &= -\dot{x}^\mu \Big(\Omega(x)(\partial_\mu + A_\mu)\Omega^\dagger(x)\Big)\Omega(x) V_\gamma(x, y)\Omega^\dagger(y) \\ &= -\dot{x}^\mu A'_\mu(x) V'_\gamma(x, y)\,, \end{aligned} \tag{2.89}$$

where we have abbreviated $x = x(\tau)$. Furthermore, we note that the right-hand side of equation (2.88) also reproduces the correct boundary condition $V'_\gamma(y, y) = \mathbb{1}$. Under these conditions, the solution to equation (2.87) is unique and we thus conclude that equation (2.88) indeed states the correct relation between V_γ and V'_γ. Having established the Wilson line operator, one can now compare $\psi(z)$ and $\psi(y)$ by multiplying $\psi(y)$ with $V_\gamma(z, y)$. Both fields then transform in the same way under a gauge transformation.

In order to obtain a concrete expression for the Wilson line operator V_γ, it is useful to replace the differential equation (2.87) by an equivalent integral equation. The latter is obtained by integrating equation (2.87) with respect to τ,

$$V_\gamma(x(1), y) = \mathbb{1} - \int_0^1 \mathrm{d}\tau_1\, \dot{x}_1^\mu A_\mu(x_1) V_\gamma(x_1, y)\,. \tag{2.90}$$

By iterating this recursion, we find that the Wilson line operator is formally given by

$$V_\gamma(z, y) = \overleftarrow{\mathrm{P}} \exp\left(-\int_0^1 \mathrm{d}\tau_1\, \dot{x}_1^\mu A_\mu(x_1)\right)\,. \tag{2.91}$$

Here, $\overleftarrow{\mathrm{P}}$ denotes path ordering and the arrow indicates that greater values of τ are ordered to the left. However, for later convenience, we will reverse the path and trade the minus sign for a path-ordering prescription exactly opposite to the one described above, namely greater values of τ are ordered to the right. The Wilson line operator then reads

$$V_\gamma(y, z) = \overrightarrow{\mathrm{P}} \exp\left(\int_0^1 \mathrm{d}\tau_1\, \dot{x}_1^\mu A_\mu(x_1)\right)\,. \tag{2.92}$$

As we will exclusively use this convention throughout this thesis, we will omit the arrow over the path-ordering symbol from now on. Technically speaking, the expression in the exponent of e is the integral over a Lie algebra-valued one-form and $\mathrm{d}\tau_1\, \dot{x}_1^\mu A_\mu(x_1)$ is the pullback of the one-form $A = \mathrm{d}x^\mu A_\mu$ on the path. Without relying on a particular parametrization, the Wilson line can thus be written as

$$V(\gamma) = \mathrm{P} \exp\left(\int_\gamma A\right)\,. \tag{2.93}$$

With the Wilson line operator established, we can now also define a gauge-invariant quantity, which is known as the Wilson loop. Let us note that for a closed curve γ the Wilson line transforms as

$$V'(\gamma) = \Omega(x) V(\gamma) \Omega^\dagger(x)\,. \tag{2.94}$$

Based on this observation, we define the gauge-invariant Wilson loop operator as

$$W(\gamma) = \frac{1}{N} \operatorname{tr} V(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp\left(\oint_\gamma A\right)\,. \tag{2.95}$$

The normalization factor N^{-1} has been included to ensure that the zeroth order term in the expansion of $W(\gamma)$ is one.

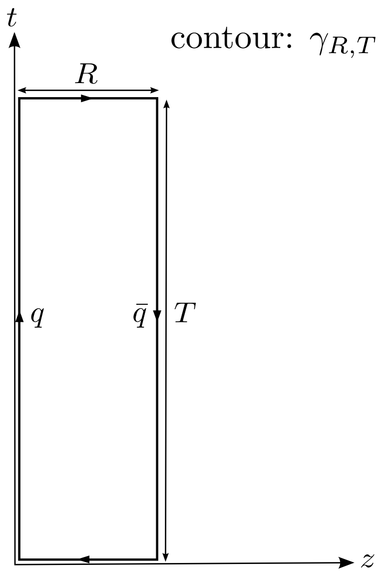


Figure 2.1.: A rectangular Wilson loop of size $R \times T$.

2.3.1.1. The Rectangular Wilson Loop

Wilson loops can be defined for arbitrary closed curves γ, but there exist several classes of contours which lead to particularly interesting results. One of these classes is furnished by the rectangular contours with one time-like and one space-like direction, see figure 2.1. Wilson loops defined over such contours have a very physical interpretation as they carry information on the potential between two heavy probe charges which are separated by a distance R. In what follows, we shall briefly sketch how this relation arises following reference [86].

Let us consider two infinitely heavy charges which are separated by a distance R. Due to their infinite mass, these charges are non-dynamical, so that it suffices to study the problem in a pure gauge theory. In this picture, the charges simply act as sources for the electromagnetic field. For simplicity, we will choose the gauge group to be U(1). The underlying action is thus the plain QED action without matter terms, i.e.

$$S_E = -\frac{1}{4} \int \mathrm{d}^4 x F_{\mu\nu} F^{\mu\nu} \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{2.96}$$

where the subscript E indicates that we have Wick-rotated the theory. The partition function including a source term is then given by

$$Z[J] = \int [\mathcal{D}A] \exp\left(-S_E + \int \mathrm{d}^4 x J^\mu(x) A_\mu(x)\right) \,. \tag{2.97}$$

The charges are assumed to be static and rest at the origin and at position R along the z-direction, respectively, cf. figure 2.1. The appropriate source term for such a configuration is given by

$$J^\mu(y) = \delta^{(3)}(\vec{y} - \vec{0})\delta_0^\mu - \delta^{(3)}(\vec{y} - \vec{R})\delta_0^\mu\,, \tag{2.98}$$

where we have assumed that the charges have unit strength. To make contact to Wilson loops, we rewrite the above source term in the following way

$$J^\mu(y) = \int \mathrm{d}\tau\, \dot{x}^\mu(\tau)\delta^{(4)}(y - x(\tau))\,, \tag{2.99}$$

with the left edge of the rectangle being parametrized by $x_\mu^l(\tau) = (\tau, 0, 0, 0)$, while for the right edge we choose $x_\mu^r(\tau) = (-\tau, 0, 0, R)$ and τ runs from $-\infty$ to ∞. The two short sides of the rectangle do not contribute as they are located at infinity. For large Euclidean times T, the partition function $Z[0]$ behaves as $\exp(-E_0 T)$ with E_0 being the ground state energy. The quotient of $Z[J]$ and $Z[0]$ thus yields the potential between the two charges

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln\left(\frac{Z[J]}{Z[0]}\right)\,. \tag{2.100}$$

However, substituting equation (2.99) into equation (2.97) shows that this quotient is nothing more than the expectation value of a rectangular Wilson loop and thus

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln\Big(\langle W(\gamma_{R,T})\rangle\Big)\,. \tag{2.101}$$

Although we have motivated the above result only for a U(1) gauge theory, it also holds in non-abelian Yang–Mills theories, cf., e.g. reference [87]. When computed non-perturbatively, for example by using methods of lattice gauge theory, the Wilson loop can thus serve as an order parameter to distinguish confining and non-confining phases. In order to decide whether a gauge theory is confining or not for a certain value of the coupling constant, it is convenient to study the exponent of the Wilson loop while the size of the loop is increased. The expectation value then typically behaves as

$$\begin{aligned} \langle W(\gamma_{R,T})\rangle &\sim e^{-\kappa P} && \text{Coulomb phase}\,, \\ \langle W(\gamma_{R,T})\rangle &\sim e^{-\sigma A} && \text{confining phase}\,, \end{aligned} \tag{2.102}$$

where $P = 2(R+T)$ is the perimeter and $A = RT$ is the area of the rectangle. If the exponent satisfies an area law, the potential grows linearly

$$V(R) \approx \sigma R\,, \tag{2.103}$$

and it is thus not possible to separate the two charges.

Let us now go back to the $\mathcal{N} = 4$ SYM model. $\mathcal{N} = 4$ SYM theory is a conformal theory and for this reason the exponent can only be proportional to the dimensionless ratio T/R. The potential is thus the Coulomb potential. The coefficient, i.e. the effective Coulomb charge, is, however, a highly non-trivial function of g^2 and N. In the planar limit, this function reduces to a function of the 't Hooft coupling λ, i.e.

$$V(R) = -\frac{f(\lambda)}{R} \,. \tag{2.104}$$

Over the last two decades, much effort has been put into computing the function $f(\lambda)$ both at weak coupling and at strong coupling. Thanks to this effort, the function is currently known up to seven loops at weak coupling [88–95] and through one loop at strong coupling [27, 28, 96, 97].

2.3.2. The Maldacena–Wilson Loop

In a pure Yang–Mills theory, the path-ordered exponential of the gauge field is one of the most natural objects to consider. The $\mathcal{N} = 4$ SYM model is, however, not a pure gauge theory as it also contains scalars as well as fermions. The enlarged field content of the model opens the window for the construction of generalized loop operators which are better in tune with the concept of a supersymmetric field theory. In this section, we introduce the Maldacena–Wilson loop operator, which locally preserves half of the Poincaré supercharges.

The Maldacena–Wilson loop operator, as it was introduced by Juan Maldacena in [27], is a Wilson loop operator that in addition to the gauge field couples to the six real scalars of the $\mathcal{N} = 4$ SYM model. More precisely, it is defined as

$$W_M(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp\left(\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu(x) + n^i(\tau) \sqrt{\dot{x}^2} \phi_i(x) \right) \right) \,, \tag{2.105}$$

where $n^i(\tau)$ is a unit six-vector which describes a path on S^5. The operator can be motivated by studying the dimensional reduction of a ten-dimensional Wilson loop while using supersymmetry as a guiding principle for fixing the internal couplings. As we will later on generalize this reasoning to construct the full supersymmetric Wilson loop of $\mathcal{N} = 4$ SYM theory, let us explain the procedure in detail. We begin by considering an ordinary Wilson loop in ten-dimensional $\mathcal{N} = 1$ SYM theory, which is given by

$$W(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp\left(\oint_\gamma \mathrm{d}x^{\hat{\mu}} A_{\hat{\mu}} \right) \,. \tag{2.106}$$

Before we perform the dimensional reduction, let us consider the supersymmetry variation of the above operator. Using the field variations as stated in equation (2.73), we find

$$\delta_\zeta W(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \left[\int \mathrm{d}\tau \, (-\zeta \dot{x}^{\hat{\mu}} \Gamma_{\hat{\mu}} \psi) \exp\left(\oint_\gamma \mathrm{d}x^{\hat{\mu}} A_{\hat{\mu}} \right) \right] \,, \tag{2.107}$$

which shows that the supersymmetry variation is in general non-vanishing. However, it is not too hard to see that if we set

$$\zeta = \kappa \dot{x}^{\hat{\mu}} \bar{\Gamma}_{\hat{\mu}} \,, \tag{2.108}$$

and assume the contour to be light-like in ten dimensions, the above variation vanishes due to the algebra relation satisfied by the ten-dimensional Pauli matrices

$$\kappa \dot{x}^{\hat{\nu}} \dot{x}^{\hat{\mu}} \bar{\Gamma}_{(\hat{\nu}} \Gamma_{\hat{\mu})} = \kappa \dot{x}^{\hat{\mu}} \dot{x}_{\hat{\mu}} = 0\,. \tag{2.109}$$

It is important to note that this supersymmetry is local because the spinor ζ is in general not constant along the loop. Moreover, only eight out of the sixteen supersymmetries are preserved at each point along the loop because the matrix $\dot{x}^{\hat{\mu}} \bar{\Gamma}_{\hat{\mu}}$ has an eight-dimensional kernel. The ten-dimensional light-like Wilson loop is thus locally a 1/2 BPS object, where the number refers to the fact that half of the super Poincaré charges are preserved.

Let us now proceed with the dimensional reduction of the ten-dimensional Wilson loop operator (2.106). Decomposing the gauge field and the coordinates as described in section 2.2.2 yields

$$W_M(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp\left(\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu(x) + \dot{y}^i \phi_i(x) \right) \right)\,, \tag{2.110}$$

where $y^i(\tau)$ describes a path in the internal space. Above, we have argued that the ten-dimensional Wilson loop is locally 1/2 BPS provided that the underlying curve is light-like. We want to preserve this feature and thus demand that the combined contour parametrized by $(x^\mu(\tau), y^i(\tau))$ satisfies a ten-dimensional light-likeness constraint, i.e.

$$\dot{x}^\mu \dot{x}_\mu + \dot{y}^i \dot{y}_i = 0\,. \tag{2.111}$$

Indeed, this constraint guarantees that the property of being locally supersymmetric carries over to the four-dimensional counterpart of the light-like Wilson loop in ten dimensions, which is the Maldacena–Wilson loop. To prove this statement, we vary the operator in equation (2.110) and obtain

$$\delta_\zeta W_M(\gamma) = -\frac{1}{N} \operatorname{tr} \mathrm{P}\left[\int \mathrm{d}\tau\, \zeta (\dot{x}^\mu \Gamma_\mu + \dot{y}^i \Gamma_i) \psi \, \exp\left(\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu(x) + \dot{y}^i \phi_i(x) \right) \right) \right], \tag{2.112}$$

Since the Pauli matrices $\bar{\Gamma}_\mu$ and Γ_i anticommute, it is clear that the argument goes through as before. Hence, the four-dimensional Maldacena–Wilson loop is locally a 1/2 BPS object. The Maldacena–Wilson loop is typically stated with the light-likeness constraint (2.111) explicitly solved, i.e.

$$\dot{y}^i(\tau) = n^i(\tau) \sqrt{\dot{x}^2}\,, \tag{2.113}$$

where $n^i(\tau)$ describes a path on S^5. Often, the unit vector n^i is chosen to be constant, so that the Maldacena–Wilson loop operator becomes

$$W_M(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp\left(\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu(x) + n^i \sqrt{\dot{x}^2} \phi_i(x) \right) \right)\,, \tag{2.114}$$

in agreement with equation (2.105). Before we move on, let us elaborate on the supersymmetry of the Maldacena–Wilson loop and its consequences. As explained above, the Maldacena–Wilson loop is in general only locally supersymmetric. Conversely, the action only has global supersymmetry. Global supersymmetry typically severely constrains the expectation value of supersymmetric observables, while the implications of local supersymmetry are in general rather mild. For this reason, it is interesting to look for contours which lead to constant supersymmetry parameters. One very transparent case is that of a straight line. In this case, $\dot{x}^\mu$ is constant and the Maldacena–Wilson loop becomes a true 1/2 BPS object. Due to the high number of globally conserved supercharges, the expectation value of such a loop operator does not receive quantum corrections at all, implying that

$$\langle W_M(|)\rangle = 1\,. \tag{2.115}$$

Although this result looks very particular, it has far reaching consequences for all smooth Maldacena–Wilson loops. This is related to the fact that divergences in expectation values of smooth Wilson loop operators are typically of the UV type arising from integration regions where two or more points come close to each other. However, as every smooth curve looks locally like a straight line, the expectation value of smooth Maldacena–Wilson loops is finite. Below, we shall verify this result explicitly at the one loop level.

The straight line is, however, not the only geometry which globally preserves some amount of supersymmetry. In fact, in reference [98] it was noted that one can construct more general 1/4, 1/8 and 1/16 BPS Wilson loops by letting the S^5 vector $n^i(\tau)$ follow the spacetime path $x^\mu(\tau)$ in a certain way. Further classes of contours were obtained in [99] by also considering superconformal transformations of the Maldacena–Wilson loop. A classification of Wilson loops which globally preserve at least one supercharge was given in [100, 101].

Having defined the Maldacena–Wilson loop operator, let us now compute its one-loop expectation value and prove that the corresponding integral is finite. To obtain this expectation value, we expand the operator in equation (2.105) to two fields which are consequently joined by a propagator. The relevant propagators read (in Feynman gauge)

$$\langle A_\mu^{\mathfrak{m}}(x_1)A_\nu^{\mathfrak{n}}(x_2)\rangle = \frac{g^2}{4\pi^2}\frac{\eta_{\mu\nu}\delta^{\mathfrak{mn}}}{(x_1-x_2)^2}\,, \qquad \langle \phi_i^{\mathfrak{m}}(x_1)\phi_j^{\mathfrak{n}}(x_2)\rangle = \frac{g^2}{4\pi^2}\frac{\eta_{ij}\delta^{\mathfrak{mn}}}{(x_1-x_2)^2}\,. \tag{2.116}$$

Using these expressions, we obtain

$$\langle W_M(\gamma)\rangle = 1 - \frac{\lambda}{16\pi^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\frac{\dot{x}_1\cdot\dot{x}_2 + n_1\cdot n_2\,|\dot{x}_1||\dot{x}_2|}{(x_1-x_2)^2} + \mathcal{O}(\lambda^2)\,, \tag{2.117}$$

where we have employed equation (2.65) as well as the fact that the algebra $\mathfrak{su}(N)$ has dimension $N^2-1\approx N^2$. Note that we use metric tensors which have mostly minus signature, so that $n_1\cdot n_1$ is equal to minus one. To prove that the one-loop expectation value is indeed finite, we set $\tau_2=\tau_1+\varepsilon$ and expand the integrand I_{τ_1,τ_2} of equation

(2.117) in powers of ε,

$$I_{\tau,\tau+\varepsilon} = \frac{1}{\varepsilon^2 \dot{x}^2}\Big(\big(\dot{x}^2 + \dot{x}^2\, n\cdot n\big) + \varepsilon\, \partial_\tau \big(\dot{x}^2 + \dot{x}^2\, n\cdot n\big) + \mathcal{O}(\varepsilon^2)\Big)\,. \tag{2.118}$$

Obviously, the coefficient of both poles vanishes due to the light-likeness condition (2.111). The same constraint which leads to local supersymmetry thus also ensures that the one-loop expectation value is finite. Furthermore, we observe that neither the scalar part nor the gauge part is free of divergences. In fact, both propagators are UV divergent but the divergences cancel out exactly. Hence, only Maldacena–Wilson loops have a finite expectation value, while ordinary Wilson loops typically suffer from UV divergences.

Finally, let us remark that the Maldacena–Wilson loop couples only to the bosonic degrees of freedom of $\mathcal{N}=4$ SYM theory. It is therefore clear that the operator (2.105) is merely the bottom component of a manifestly supersymmetric Wilson loop operator which couples to a path in superspace. We shall construct this operator in chapter 4 and investigate it further in chapter 5.

2.3.2.1. The Maldacena–Wilson Loop at Strong Coupling

One of the most remarkable features of $\mathcal{N}=4$ SYM theory is that the model has a dual description in terms of (super)strings on $\mathrm{AdS}_5\times\mathrm{S}^5$. All gauge theory observables therefore have a completely equivalent string description, which can typically be used to gain insights into the strong-coupling behavior of the observables. Below, we shall briefly review the strong-coupling description of the Maldacena–Wilson loop, mainly following reference [45].

The strong-coupling description of the Maldacena–Wilson loop was obtained in [27], where is was argued that the expectation value of a loop operator $W_M(\gamma)$ is given by the action of a string bounded by the curve γ at the conformal boundary of space. In the limit of large λ, the string sigma model becomes weakly coupled and the partition function can be calculated using the method of steepest descent. Minima of the string action correspond to minimal surfaces whose boundary is the curve γ. Consequently, the Wilson loop expectation value can be approximated as

$$\langle W_M(\gamma)\rangle \overset{\lambda\gg 1}{=} \exp\left(-\tfrac{\sqrt{\lambda}}{2\pi}A_{\mathrm{ren}}(\gamma)\right)\,, \tag{2.119}$$

where $A_{\mathrm{ren}}(\gamma)$ is the area of the minimal surface ending on the contour γ at the conformal boundary. Let us precise the above formula by providing the necessary mathematical background for calculating this area. The underlying boundary value problem is most conveniently formulated in Poincaré coordinates (X^μ, z), in which the AdS_5 metric reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^\mu\,\mathrm{d}X_\mu + \mathrm{d}z\,\mathrm{d}z}{z^2}\,. \tag{2.120}$$

The conformal boundary of AdS_5 sits at $z = 0$ and that is where the contour γ is located. Thus, we impose the following boundary conditions

$$X^\mu(\tau = 0, \sigma) = x^\mu(\sigma)\,, \qquad\qquad z(\tau = 0, \sigma) = 0\,. \tag{2.121}$$

Note that in the above relations we have denoted the curve parameter by σ instead of τ to ensure consistency with standard string conventions. The area of the minimal surface is obtained from the area functional A, which in the context of string theory is called Nambu-Goto action

$$A = \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{\det(\gamma_{ij})}\,, \qquad \text{with} \qquad i, j \in \{\tau, \sigma\}\,. \tag{2.122}$$

Here, γ_{ij} is the induced metric on the surface, i.e.

$$\gamma_{ij} = \frac{\partial_i X^\mu\, \partial_j X_\mu + \partial_i z\, \partial_j z}{z^2}\,. \tag{2.123}$$

Obviously, the metric becomes divergent when the surface approaches the conformal boundary of AdS_5, which is located at $z = 0$. For this reason, the area needs to be renormalized. This is conveniently done by imposing a cut-off in the z-direction so that the integration is only over the region satisfying $z \geqslant \varepsilon$. Since the minimal surface leaves the conformal boundary perpendicular, the pole is proportional to the length of the contour γ and can thus be removed in the following way:

$$A_{\mathrm{ren}}(\gamma) = \lim_{\varepsilon \to 0} \left\{ A(\gamma)\Big|_{z \geqslant \varepsilon} - \frac{L(\gamma)}{\varepsilon} \right\}\,. \tag{2.124}$$

Let us point out that this divergence does not mean that the expectation value of a Maldacena–Wilson loop is divergent at strong coupling. In fact, it was noted in [102] that the divergence is merely an artifact stemming from a slightly incorrect definition of the partition function. The correct definition is somewhat cumbersome and involves the Legendre transform of the area functional. Fortunately, considering the Legendre transform effectively amounts to dropping $1/\varepsilon$ divergences whenever they appear. This is implemented via the above renormalization prescription. Finally, let us remark that the above considerations only apply to the case of a constant vector n^i. If the vector depends on σ, one needs to lift the discussion to $AdS_5 \times S^5$. The minimal surface is then bound by the curve $x^\mu(\sigma)$ in AdS_5 and by $n^i(\sigma)$ in S^5.

3. $\mathcal{N} = 4$ SYM Theory in Superspace

Supersymmetric field theories are often most efficiently formulated in terms of superfields defined on superspace, which package together all the component fields of the theory in a nice and elegant fashion. In searching for these descriptions, one always aims for one which makes the supersymmetry of the model manifest, i.e. for a formulation in which the supersymmetry algebra closes off shell. Unfortunately, such a formulation of the $\mathcal{N} = 4$ SYM model is not known. Fortunately, however, one can write down an on-shell non-chiral superspace formulation of $\mathcal{N} = 4$ SYM theory and we begin by reviewing this construction. Later on, we will make extensive use of this formalism when we discuss the super Maldacena–Wilson loop operator. The superspace description can be formulated using ten-dimensional language as well as four-dimensional language. In what follows, we shall review it covering both perspectives, thereby taking the opportunity to lay out our conventions. In the further course of this section, we will then start deriving new results, namely we shall derive expressions for the superfield propagators. These will play an important role later on.

3.1. The Ten-Dimensional Perspective

In this section, we shall give a brief introduction to the on-shell superspace formulation of the ten-dimensional $\mathcal{N} = 1$ SYM model. We begin by discussing various geometric aspects of the appropriate superspace and continue by introducing the superfields as well as the constraints that have to be imposed on them. Taking into account the Bianchi identities of the gauge theory, we then analyze in detail the implications of the constraint equations and show that they are equivalent to the equations of motion of the $\mathcal{N} = 1$ SYM model. The second part is devoted to establishing the precise relation between the superconnection and the component fields of $\mathcal{N} = 1$ SYM theory. We establish this relation by picking a particular type of transversal gauge which eliminates all the fermionic gauge degrees of freedom. Finally, we use the component expansion of the superconnection to compute its propagator through quartic order in an expansion in the anticommuting variables.

3.1.1. Superspace Geometry and the Constraints

We begin our introduction to the superfield formulation of the ten-dimensional $\mathcal{N} = 1$ SYM model by describing the geometry of the underlying superspace. The material presented here is standard and can be found in many places including [103–105]. The superspace is parametrized by

$$(x^{\hat{\mu}}, \theta^{\hat{\alpha}}) , \qquad \text{with} \qquad \hat{\mu} = 0, .., 9, \;\; \hat{\alpha} = 1, .., 16 , \tag{3.1}$$

where $x^{\hat{\mu}}$ are coordinates on $\mathbb{R}^{1,9}$ and $\theta^{\hat{\alpha}}$ are sixteen anticommuting Graßmann coordinates, which form a sixteen-component Majorana–Weyl spinor. On this superspace, the supercharges can be represented as

$$\mathrm{Q}_{\hat{\alpha}} = (\Gamma^{\hat{\mu}}\theta)_{\hat{\alpha}}\partial_{\hat{\mu}} - \partial_{\hat{\alpha}} , \qquad \{\mathrm{Q}_{\hat{\alpha}}, \mathrm{Q}_{\hat{\beta}}\} = -2\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\partial_{\hat{\mu}} , \tag{3.2}$$

where $\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}$ are the ten-dimensional Pauli matrices as defined in appendix A.3. The associated susy-covariant derivatives[1] read

$$D_{\hat{\alpha}} = (\Gamma^{\hat{\mu}}\theta)_{\hat{\alpha}}\partial_{\hat{\mu}} + \partial_{\hat{\alpha}} . \tag{3.3}$$

They obey the relations

$$\{D_{\hat{\alpha}}, \mathrm{Q}_{\hat{\beta}}\} = 0 , \qquad \{D_{\hat{\alpha}}, D_{\hat{\beta}}\} = 2\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\partial_{\hat{\mu}} . \tag{3.4}$$

In what follows, we will often make use of methods of differential geometry. In differential geometry, the set of derivatives $(\partial_{\hat{\mu}}, \partial_{\hat{\alpha}})$ is interpreted as a basis for the space of vector fields. However, one can as well expand vector fields, such as Lie algebra generators, in the susy-covariant basis spanned by the set $(\partial_{\hat{\mu}}, D_{\hat{\alpha}})$. The latter is in fact often much more convenient than the plain basis as susy-covariant derivatives preserve the transformation properties superfields.

The natural habitat of gauge theories is the vector space which is dual to the space of vector fields. The dual space is the space of one-forms, which is defined as the span of the differentials $(\mathrm{d}x^{\hat{\mu}}, \mathrm{d}\theta^{\hat{\alpha}})$. By definition, the following equation holds true

$$\mathrm{d}X^{\mathcal{A}}(\partial_{\mathcal{B}}) = (-1)^{|\mathcal{A}|}\delta^{\mathcal{A}}_{\mathcal{B}} , \tag{3.5}$$

where under the name $X^{\mathcal{A}} = (x^{\hat{\mu}}, \theta^{\hat{\alpha}})$ we lump together the complete set of superspace coordinates, i.e. $\mathcal{A}$ is now a multi-index. A general element of the space of one-forms can be decomposed as

$$\omega = \mathrm{d}x^{\hat{\mu}}\omega_{\hat{\mu}}(x,\theta) + \mathrm{d}\theta^{\hat{\alpha}}\omega_{\hat{\alpha}}(x,\theta) . \tag{3.6}$$

However, as before, it will be much more convenient to work in a slightly different basis in which the different components of a superform do not mix under supersymmetry transformations. This basis is the one which is dual to the basis spanned by the set $(\partial_{\hat{\mu}}, D_{\hat{\alpha}})$ and can therefore be obtained by demanding the following equation to hold

$$e^{\mathcal{A}}(D_{\mathcal{B}}) = (-1)^{|\mathcal{A}|}\delta^{\mathcal{A}}_{\mathcal{B}} , \tag{3.7}$$

[1] The name susy covariant refers to the fact that given a superfield Φ, $D_{\hat{\alpha}}\Phi$ is again a superfield in the sense that it transforms in the same way under a supersymmetry transformation as the original field. This follows immediately from the fact that the susy-covariant derivatives anticommute with the supercharges.

where $D_{\mathcal{B}} = (\partial_{\hat{\mu}}, D_{\hat{\alpha}})$ and $e^{\mathcal{A}}$ are linear combinations of the plain basis one-forms. One easily convinces oneself that the solution to equation (3.7) is given by

$$e^{\hat{\mu}} = \mathrm{d}x^{\hat{\mu}} + \theta\Gamma^{\hat{\mu}}\mathrm{d}\theta\,, \qquad\qquad e^{\hat{\alpha}} = \mathrm{d}\theta^{\hat{\alpha}}\,. \tag{3.8}$$

Note that these expressions indeed provide a susy-invariant basis for one-forms, but there is a price that has to be paid for this which is that the exterior derivative, being defined as

$$\mathrm{d} = \mathrm{d}x^{\hat{\mu}}\partial_{\hat{\mu}} + \mathrm{d}\theta^{\hat{\alpha}}\partial_{\hat{\alpha}} = e^{\hat{\mu}}\partial_{\hat{\mu}} + \mathrm{d}\theta^{\hat{\alpha}}D_{\hat{\alpha}}\,, \tag{3.9}$$

of these basis one-forms does no longer vanish. More precisely, we have

$$\mathrm{d}e^{\hat{\mu}} = \mathrm{d}\theta^{\hat{\alpha}} \wedge \mathrm{d}\theta^{\hat{\beta}}\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\,, \qquad\qquad \mathrm{d}e^{\hat{\alpha}} = 0\,. \tag{3.10}$$

Note that the expression $\mathrm{d}\theta^{\hat{\alpha}} \wedge \mathrm{d}\theta^{\hat{\beta}}$ has both form grading and Graßmann grading. The wedge product of two basis one-forms is thus no longer antisymmetric but rather graded antisymmetric. As the exterior product is the only product that can be used to multiply forms, we will typically omit the wedge symbol in what follows.

Having set up the basics, we can now start defining $\mathcal{N} = 1$ SYM theory in superspace. We begin by introducing an $\mathfrak{su}(N)$-valued gauge connection one-form $\mathcal{A} = \mathcal{A}^{\mathfrak{m}}T_{\mathfrak{m}}$. We label the expansion coefficients with respect to the susy-invariant basis by $\mathcal{A}_{\hat{\mu}}$ and $\mathcal{A}_{\hat{\alpha}}$, i.e.

$$\mathcal{A} = e^{\hat{\mu}}\mathcal{A}_{\hat{\mu}}(x,\theta) + \mathrm{d}\theta^{\hat{\alpha}}\mathcal{A}_{\hat{\alpha}}(x,\theta)\,. \tag{3.11}$$

Given this connection one-form, we define the susy- and gauge-covariant derivative as

$$\mathcal{D} = \mathrm{d} + \mathcal{A} = e^{\hat{\mu}}\mathcal{D}_{\hat{\mu}} + \mathrm{d}\theta^{\hat{\alpha}}\mathcal{D}_{\hat{\alpha}}\,, \tag{3.12}$$

where

$$\mathcal{D}_{\hat{\mu}} = \partial_{\hat{\mu}} + \mathcal{A}_{\hat{\mu}}(x,\theta)\,, \qquad\qquad \mathcal{D}_{\hat{\alpha}} = D_{\hat{\alpha}} + \mathcal{A}_{\hat{\alpha}}(x,\theta)\,. \tag{3.13}$$

As explained in section 2.2.1, the (graded) commutator with the gauge field is implied unless it is stated explicitly. The superconnection $\mathcal{A}$ is defined up to gauge transformations of the form

$$\mathcal{A} \to e^{-\Lambda}(\mathrm{d} + \mathcal{A})e^{\Lambda}\,, \tag{3.14}$$

where $\Lambda = \Lambda(x,\theta)$ is a Lie algebra-valued function that depends on the superspace variables x and θ. Expanding equation (3.14) to first order in Λ leads to the following infinitesimal transformation law:

$$\mathbb{G}[\Lambda]\mathcal{A} = [\mathcal{D},\Lambda] = \mathrm{d}\Lambda + [\mathcal{A},\Lambda]\,. \tag{3.15}$$

Having introduced the superconnection $\mathcal{A}$, we now want to make contact to the ordinary formulation of $\mathcal{N}=1$ SYM theory. This is most transparently done by establishing the relation between the coefficient fields appearing in the Graßmann expansion of $\mathcal{A}$ and the component fields of $\mathcal{N}=1$ SYM theory. However, so far the Graßmann expansion of $\mathcal{A}$ produces many more fields than appropriate. This statement continues to hold even after taking into account that a certain number of components can be gauged away. To reproduce the expected spectrum, we need to impose constraints on the superfields. The appropriate constraints leading to $\mathcal{N}=1$ SYM in ten dimensions were introduced by Sohnius in [105] and involve the fermionic components of the field strength two-form, which is defined as

$$\mathcal{F}=\mathrm{d}\mathcal{A}+\mathcal{A}\wedge\mathcal{A}\,. \tag{3.16}$$

For the components of this two-form with respect to the natural basis given by $\{e^{\hat\nu}\wedge e^{\hat\mu},\mathrm{d}\theta^{\hat\alpha}\wedge e^{\hat\mu},\mathrm{d}\theta^{\hat\alpha}\wedge\mathrm{d}\theta^{\hat\beta}\}$, one finds

$$\mathcal{F}=-\tfrac{1}{2}e^{\hat\nu}e^{\hat\mu}\mathcal{F}_{\hat\mu\hat\nu}-\mathrm{d}\theta^{\hat\alpha}e^{\hat\mu}\mathcal{F}_{\hat\mu\hat\alpha}-\tfrac{1}{2}\mathrm{d}\theta^{\hat\beta}\mathrm{d}\theta^{\hat\alpha}\mathcal{F}_{\hat\alpha\hat\beta}\,, \tag{3.17}$$

where

$$\begin{aligned}
\mathcal{F}_{\hat\mu\hat\nu}&=[\mathcal{D}_{\hat\mu},\mathcal{D}_{\hat\nu}]=\partial_{\hat\mu}\mathcal{A}_{\hat\nu}-\partial_{\hat\nu}\mathcal{A}_{\hat\mu}+[\mathcal{A}_{\hat\mu},\mathcal{A}_{\hat\nu}]\,,\\
\mathcal{F}_{\hat\mu\hat\alpha}&=[\mathcal{D}_{\hat\mu},\mathcal{D}_{\hat\alpha}]=\partial_{\hat\mu}\mathcal{A}_{\hat\alpha}-D_{\hat\alpha}\mathcal{A}_{\hat\mu}+[\mathcal{A}_{\hat\mu},\mathcal{A}_{\hat\alpha}]\,,\\
\mathcal{F}_{\hat\alpha\hat\beta}&=\{\mathcal{D}_{\hat\alpha},\mathcal{D}_{\hat\beta}\}-2\Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\mathcal{D}_{\hat\mu}=D_{\hat\alpha}\mathcal{A}_{\hat\beta}+D_{\hat\beta}\mathcal{A}_{\hat\alpha}+\{\mathcal{A}_{\hat\alpha},\mathcal{A}_{\hat\beta}\}-2\Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\mathcal{A}_{\hat\mu}\;\;.
\end{aligned} \tag{3.18}$$

Formulated in terms of the components of $\mathcal{F}$ as given above, the constraints presented in [105] read

$$\mathcal{F}_{\hat\alpha\hat\beta}=0\,. \tag{3.19}$$

Note that due to the covariant transformation law of the field strength $\mathcal{F}$,

$$\mathcal{F}\rightarrow e^{-\Lambda}\mathcal{F}e^{\Lambda}\,, \tag{3.20}$$

the constraints (3.19) are gauge invariant as expected.

It is instructive to study the implications of the constraint equations by looking at the Bianchi identities for the gauge-covariant derivatives. The Bianchi identity involving three fermionic derivatives reads

$$[\mathcal{D}_{\hat\alpha},\{\mathcal{D}_{\hat\beta},\mathcal{D}_{\hat\gamma}\}]+[\mathcal{D}_{\hat\beta},\{\mathcal{D}_{\hat\gamma},\mathcal{D}_{\hat\alpha}\}]+[\mathcal{D}_{\hat\gamma},\{\mathcal{D}_{\hat\alpha},\mathcal{D}_{\hat\beta}\}]=0\,. \tag{3.21}$$

We can now use the equations (3.18) and the constraint (3.19) to rewrite the former equation as

$$[\mathcal{D}_{(\hat\alpha},\{\mathcal{D}_{\hat\beta},\mathcal{D}_{\hat\gamma)}\}]=2\Gamma^{\hat\mu}_{(\hat\beta\hat\gamma}\mathcal{F}_{\hat\alpha)\hat\mu}=0\,, \tag{3.22}$$

where the parentheses indicate total symmetrization of the enclosed indices as defined in section 2.1.3. An important aspect of the ten-dimensional Pauli matrices is that they satisfy the so-called magic identity (A.31). With this relation in mind, it becomes obvious that equation (3.22) is solved by

$$\mathcal{F}_{\hat{\mu}\hat{\alpha}} = \Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\beta}}\Psi^{\hat{\beta}}\,. \tag{3.23}$$

Thus, equation (3.22) allows us to identify the mixed component of the field strength two-form with the fermionic superfield $\Psi^{\hat{\beta}}$. We continue with our analysis by studying the two mixed Bianchi identities. The Bianchi identity involving one bosonic and two fermionic gauge-covariant derivatives reads

$$[\mathcal{D}_{\hat{\mu}}, \{\mathcal{D}_{\hat{\alpha}}, \mathcal{D}_{\hat{\beta}}\}] = \{\mathcal{D}_{\hat{\alpha}}, [\mathcal{D}_{\hat{\mu}}, \mathcal{D}_{\hat{\beta}}]\} + \{\mathcal{D}_{\hat{\beta}}, [\mathcal{D}_{\hat{\mu}}, \mathcal{D}_{\hat{\alpha}}]\}\,. \tag{3.24}$$

Using the constraint (3.19) as well as the relation (3.23), we deduce that

$$\Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\beta}}\,\mathcal{F}_{\hat{\mu}\hat{\nu}} = \tfrac{1}{2}\Gamma_{\hat{\mu}\,\hat{\beta}\hat{\gamma}}\Big\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\gamma}}\Big\} + \tfrac{1}{2}\Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\gamma}}\Big\{\mathcal{D}_{\hat{\beta}}, \Psi^{\hat{\gamma}}\Big\}\,. \tag{3.25}$$

By multiplying this equation with $\bar{\Gamma}_{\hat{\rho}}$ and taking the trace, we obtain the following set of equations:

$$\mathcal{F}_{\hat{\mu}\hat{\nu}} = -\tfrac{1}{16}(\bar{\Gamma}_{\hat{\mu}\hat{\nu}})^{\hat{\alpha}}{}_{\hat{\gamma}}\Big\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\gamma}}\Big\}\,, \qquad \Big\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\alpha}}\Big\} = 0\,. \tag{3.26}$$

We proceed by expanding the expression $\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\beta}}\}$ in a basis of ten-dimensional Pauli matrices. This yields

$$\Big\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\beta}}\Big\} = -\tfrac{1}{2}(\Gamma^{\hat{\mu}\hat{\nu}})_{\hat{\alpha}}{}^{\hat{\beta}}\mathcal{F}_{\hat{\mu}\hat{\nu}} + \tfrac{1}{16}\tfrac{1}{4!}\Big[(\Gamma_{\hat{\sigma}\hat{\rho}\hat{\nu}\hat{\mu}})_{\hat{\gamma}}{}^{\hat{\epsilon}}\Big\{\mathcal{D}_{\hat{\epsilon}}, \Psi^{\hat{\gamma}}\Big\}\Big](\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}})_{\hat{\alpha}}{}^{\hat{\beta}}\,. \tag{3.27}$$

By plugging this expression back into equation (3.25) and making repeated use of the identity (A.29), we learn that the coefficient multiplying $\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ vanishes. Thus, the Bianchi identity (3.24) implies that

$$\Big\{\mathcal{D}_{\hat{\alpha}}, \Psi^{\hat{\beta}}\Big\} = -\tfrac{1}{2}(\Gamma^{\hat{\mu}\hat{\nu}})_{\hat{\alpha}}{}^{\hat{\beta}}\mathcal{F}_{\hat{\mu}\hat{\nu}}\,. \tag{3.28}$$

The second mixed Bianchi identity reads

$$[\mathcal{D}_{\hat{\alpha}}, [\mathcal{D}_{\hat{\mu}}, \mathcal{D}_{\hat{\nu}}]] = -[\mathcal{D}_{\hat{\nu}}, [\mathcal{D}_{\hat{\alpha}}, \mathcal{D}_{\hat{\mu}}]] - [\mathcal{D}_{\hat{\mu}}, [\mathcal{D}_{\hat{\nu}}, \mathcal{D}_{\hat{\alpha}}]]\,. \tag{3.29}$$

By using equation (3.23), we find

$$[\mathcal{D}_{\hat{\alpha}}, \mathcal{F}_{\hat{\mu}\hat{\nu}}] = \Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\nu}}, \Psi^{\hat{\beta}}\Big] - \Gamma_{\hat{\nu}\,\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}}, \Psi^{\hat{\beta}}\Big]\,. \tag{3.30}$$

Importantly, the constraints can only be realized on shell, i.e. the equations $\mathcal{F}_{\hat{\alpha}\hat{\beta}} = 0$ ultimately imply the equations of motion for the fields. To see this, we look at the expression

$$\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}}, \Psi^{\hat{\beta}}\Big] = -\tfrac{1}{2}(\Gamma^{\hat{\rho}\hat{\sigma}})_{(\hat{\alpha}}{}^{\hat{\beta}}\,[\mathcal{D}_{\hat{\beta})}, \mathcal{F}_{\hat{\rho}\hat{\sigma}}]\,, \tag{3.31}$$

which can be derived by exploiting the constraint (3.19) to rewrite the bosonic gauge- and susy-covariant derivative in terms of fermionic ones and using equation (3.28) to express the derivative of the fermionic superfield $\Psi^{\hat{\beta}}$ in terms of the super field strength. In the next step, we use equation (3.30) to rewrite the right-hand side of (3.31) as

$$\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}},\Psi^{\hat{\beta}}\Big]=-\tfrac{1}{4}(\Gamma^{\hat{\rho}\hat{\sigma}})_{\hat{\alpha}}{}^{\hat{\beta}}\Big(\Gamma_{\hat{\rho}\,\hat{\beta}\hat{\gamma}}[\mathcal{D}_{\hat{\sigma}},\Psi^{\hat{\gamma}}]-\Gamma_{\hat{\sigma}\,\hat{\beta}\hat{\gamma}}[\mathcal{D}_{\hat{\rho}},\Psi^{\hat{\gamma}}]\Big)\,, \tag{3.32}$$

where we have already taken into account that the trace of $\Gamma^{\hat{\rho}\hat{\sigma}}$ vanishes. The Dirac equation

$$\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}},\Psi^{\hat{\beta}}\Big]=0\,, \tag{3.33}$$

now follows from (3.32) by noting that the right-hand side vanishes due to the Pauli matrix identity

$$(\Gamma^{\hat{\rho}\hat{\sigma}})_{\hat{\alpha}}{}^{\hat{\beta}}\Gamma_{\hat{\rho}\,\hat{\beta}\hat{\gamma}}=9\Gamma^{\hat{\sigma}}_{\hat{\alpha}\hat{\gamma}}\,. \tag{3.34}$$

Finally, let us derive the Yang–Mills equation for the super field strength from the constraints. To do so, we multiply the Dirac equation (3.33) by $\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}\mathcal{D}_{\hat{\gamma}}$ from the right and obtain

$$\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big\{\mathcal{D}_{\hat{\gamma}},\Big[\mathcal{D}_{\hat{\mu}},\Psi^{\hat{\beta}}\Big]\Big\}=0\,. \tag{3.35}$$

Using the definition of the mixed components of the field strength two-form (3.18), the above equation can be rewritten as

$$\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big(\Big\{\mathcal{F}_{\hat{\gamma}\hat{\mu}},\Psi^{\hat{\beta}}\Big\}+\Big[\mathcal{D}_{\hat{\mu}},\Big\{\mathcal{D}_{\hat{\gamma}},\Psi^{\hat{\beta}}\Big\}\Big]\Big)=0\,. \tag{3.36}$$

Inserting equation (3.23) and (3.28) into the above relation yields

$$-\tfrac{1}{2}\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}\Big(\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Gamma_{\hat{\mu}\,\hat{\gamma}\hat{\delta}}+\Gamma^{\hat{\mu}}_{\hat{\gamma}\hat{\beta}}\Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\delta}}\Big)\Big\{\Psi^{\hat{\delta}},\Psi^{\hat{\beta}}\Big\}-\tfrac{1}{2}\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}(\Gamma_{\hat{\rho}\hat{\sigma}})_{\hat{\gamma}}{}^{\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}},\mathcal{F}^{\hat{\rho}\hat{\sigma}}\Big]=0\,, \tag{3.37}$$

where we have made use of the fact that the ten-dimensional Pauli matrices are symmetric. With the help of the magic identity (A.31) as well as relation (A.30), we can rewrite the above equation as

$$\tfrac{1}{2}\bar{\Gamma}^{\hat{\alpha}\hat{\gamma}}_{\hat{\nu}}\Gamma^{\hat{\mu}}_{\hat{\gamma}\hat{\alpha}}\Gamma_{\hat{\mu}\,\hat{\beta}\hat{\delta}}\Big\{\Psi^{\hat{\delta}},\Psi^{\hat{\beta}}\Big\}-\tfrac{1}{2}\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\Big(\Gamma_{\hat{\nu}\hat{\rho}\hat{\sigma}}+\eta_{\hat{\nu}\hat{\rho}}\Gamma_{\hat{\sigma}}-\eta_{\hat{\nu}\hat{\sigma}}\Gamma_{\hat{\rho}}\Big)^{\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}},\mathcal{F}^{\hat{\rho}\hat{\sigma}}\Big]=0\,. \tag{3.38}$$

By noting that the $\Gamma_{\hat{\nu}\hat{\rho}\hat{\sigma}}$-term does not contribute as it is antisymmetric under exchange of $\hat{\alpha}$ and $\hat{\beta}$ and using the trace identity (A.26), we deduce that

$$\Big[\mathcal{D}_{\hat{\mu}},\mathcal{F}^{\hat{\mu}\hat{\nu}}\Big]+\tfrac{1}{2}\Gamma^{\hat{\nu}}_{\hat{\beta}\hat{\delta}}\Big\{\Psi^{\hat{\delta}},\Psi^{\hat{\beta}}\Big\}=0\,, \tag{3.39}$$

which is the desired result, namely the Yang–Mills equation for the super field strength. This completes our proof that the constraints (3.19) imply the equations of motion. In fact, it can be shown that the converse statement is also true. Thus, the constraints are equivalent to the equations of motion. For a detailed proof of this fact see [103].

3.1.2. Component Expansion of the Superfields

Having set the stage, we now come back to the question of how the superconnection can be expressed in terms of the physical fields of $\mathcal{N}=1$ SYM theory. To establish this relation, we shall use a method due to Harnad and Shnider, which was introduced in [103, 106]. The key element in their procedure is to introduce a suitable type of transversal gauge which completely eliminates the fermionic gauge freedom. In the following, we review this construction in detail.

In the previous section, we derived the following set of equations [104]:

$$\begin{aligned}
\mathcal{F}_{\hat{\alpha}\hat{\beta}} &= 0\,, \\
\mathcal{F}_{\hat{\mu}\hat{\alpha}} &= \Gamma_{\hat{\mu},\hat{\alpha}\hat{\beta}}\,\Psi^{\hat{\beta}}\,, \\
\left\{\mathcal{D}_{\hat{\alpha}},\Psi^{\hat{\beta}}\right\} &= -\tfrac{1}{2}\left(\Gamma^{\hat{\mu}\hat{\nu}}\right)_{\hat{\alpha}}{}^{\hat{\beta}}\mathcal{F}_{\hat{\mu}\hat{\nu}}\,, \\
[\mathcal{D}_{\hat{\alpha}},\mathcal{F}_{\hat{\mu}\hat{\nu}}] &= \Gamma_{\hat{\mu},\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\nu}},\Psi^{\hat{\beta}}\Big] - \Gamma_{\hat{\nu},\hat{\alpha}\hat{\beta}}\Big[\mathcal{D}_{\hat{\mu}},\Psi^{\hat{\beta}}\Big]\,.
\end{aligned} \tag{3.40}$$

The method of Harnad and Shnider makes use of the fact that by imposing a particular transversal gauge condition, the above equations can be converted into recursion relations which then allow for the reconstruction of the superfields in terms of their lowest order components. Explicitly, the gauge condition reads

$$\theta^{\hat{\alpha}}\mathcal{A}_{\hat{\alpha}}(x,\theta) = 0\,. \tag{3.41}$$

Next, one defines the operator

$$\mathbf{D} := \theta^{\hat{\alpha}}\mathcal{D}_{\hat{\alpha}} = \theta^{\hat{\alpha}}\frac{\partial}{\partial\theta^{\hat{\alpha}}}\,, \tag{3.42}$$

where the rightmost statement is a consequence of equation (3.41) and the fact that the ten-dimensional Pauli matrices are symmetric. Combining the gauge condition (3.41) with the equations (3.40) immediately yields the desired set of recursion relations

$$\begin{aligned}
(1+\mathbf{D})\,\mathcal{A}_{\hat{\alpha}} &= 2(\theta\Gamma^{\hat{\mu}})_{\hat{\alpha}}\,\mathcal{A}_{\hat{\mu}}\,, \\
\mathbf{D}\,\mathcal{A}_{\hat{\mu}} &= -(\theta\,\Gamma_{\hat{\mu}}\Psi)\,, \\
\mathbf{D}\,\Psi^{\hat{\alpha}} &= -\tfrac{1}{2}\,(\theta\Gamma^{\hat{\mu}\hat{\nu}})^{\hat{\alpha}}\mathcal{F}_{\hat{\mu}\hat{\nu}}\,, \\
\mathbf{D}\,\mathcal{F}_{\hat{\mu}\hat{\nu}} &= (\theta\Gamma_{\hat{\mu}}\mathcal{D}_{\hat{\nu}}\Psi) - (\theta\Gamma_{\hat{\nu}}\mathcal{D}_{\hat{\mu}}\Psi)\,.
\end{aligned} \tag{3.43}$$

Using these relations, we can now reconstruct the superfields entirely from the lowest-order data

$$\mathcal{A}_{\hat{\mu}}(x,\theta) = A_{\hat{\mu}}(x) + \mathcal{O}(\theta)\,, \qquad \mathcal{A}_{\hat{\alpha}}(x,\theta) = \mathcal{O}(\theta)\,, \qquad \Psi^{\hat{\alpha}}(x,\theta) = \psi^{\hat{\alpha}}(x) + \mathcal{O}(\theta)\,.$$

The result reads

$$
\begin{aligned}
\mathcal{A}_{\hat\mu}(x,\theta) &= A_{\hat\mu}(x) - \Big(\theta\Gamma_{\hat\mu}\psi(x)\Big) - \tfrac{1}{4}\Big(\theta\Gamma_{\hat\mu}\bar\Gamma^{\hat\rho\hat\sigma}\theta\Big)\Big(F_{\hat\rho\hat\sigma}(x) + \tfrac{2}{3}\Big(\theta\Gamma_{\hat\rho}D_{\hat\sigma}\psi(x)\Big) \\
&\quad + \tfrac{1}{12}\Big(\theta\Gamma_{\hat\rho}\bar\Gamma^{\hat\nu\hat\kappa}\theta\Big)D_{\hat\sigma}F_{\hat\nu\hat\kappa}(x) + \tfrac{1}{6}\Big[\Big(\theta\Gamma_{\hat\rho}\psi(x)\Big),\Big(\theta\Gamma_{\hat\sigma}\psi(x)\Big)\Big]\Big) + \mathcal{O}(\theta^5)\,, \\
\Psi^{\hat\alpha}(x,\theta) &= \psi^{\hat\alpha}(x) + \tfrac{1}{2}\Big(\bar\Gamma^{\hat\mu\hat\nu}\theta\Big)^{\hat\alpha}\Big(F_{\hat\mu\hat\nu}(x) + \Big(\theta\Gamma_{\hat\mu}D_{\hat\nu}\psi(x)\Big) \\
&\quad + \tfrac{1}{6}\Big(\theta\Gamma_{\hat\mu}\bar\Gamma^{\hat\rho\hat\sigma}\theta\Big)D_{\hat\nu}F_{\hat\rho\hat\sigma}(x) + \tfrac{1}{3}\Big[\Big(\theta\Gamma_{\hat\mu}\psi(x)\Big),\Big(\theta\Gamma_{\hat\nu}\psi(x)\Big)\Big]\Big) + \mathcal{O}(\theta^4)\,, \\
\mathcal{A}_{\hat\alpha}(x,\theta) &= \Big(\theta\Gamma^{\hat\mu}\Big)_{\hat\alpha}\Big(A_{\hat\mu}(x) - \tfrac{2}{3}\Big(\theta\Gamma_{\hat\mu}\psi(x)\Big) - \tfrac{1}{8}\Big(\theta\Gamma_{\hat\mu}\bar\Gamma^{\hat\rho\hat\sigma}\theta\Big)F_{\hat\rho\hat\sigma}(x)\Big) + \mathcal{O}(\theta^4)\,,
\end{aligned}
\tag{3.44}
$$

where $D_{\hat\nu} = \partial_{\hat\nu} + [A_{\hat\nu},\ \ ]$ is the usual bosonic gauge-covariant derivative.

In the linearized theory, one can in fact easily find the complete θ-expansion of $\mathcal{A}_{\hat\mu}$ and $\mathcal{A}_{\hat\alpha}$. For this, note that at the linear level the recursion relations (3.43) imply

$$
\mathbf{D}(\mathbf{D}-1)\mathcal{A}^{\text{lin}}_{\hat\mu} = -\Sigma_{\hat\mu}{}^{\hat\nu}\mathcal{A}^{\text{lin}}_{\hat\nu}, \tag{3.45}
$$

where we have introduced the operator

$$
\Sigma_{\hat\mu}{}^{\hat\nu} := \Big(\theta\Gamma_{\hat\mu}{}^{\hat\rho\hat\nu}\theta\Big)\,\partial_{\hat\rho}\,. \tag{3.46}
$$

Given the second-order relation (3.45) and the two lowest-order components of the bosonic superfield $\mathcal{A}_{\hat\mu}$, it is not too hard to see that in the linearized theory the superfield $\mathcal{A}_{\hat\mu}$ is to all orders given by

$$
\mathcal{A}^{\text{lin}}_{\hat\mu}(x,\theta) = \left(\sum_{n=0}^{8}\frac{(-1)^n}{(2n)!}\,(\Sigma^n)_{\hat\mu}{}^{\hat\nu}\right)A_{\hat\nu}(x) - \left(\sum_{n=0}^{7}\frac{(-1)^n}{(2n+1)!}\,(\Sigma^n)_{\hat\mu}{}^{\hat\nu}\right)\Big(\theta\Gamma_{\hat\nu}\psi(x)\Big)\,, \tag{3.47}
$$

where

$$
(\Sigma^n)_{\hat\mu}{}^{\hat\nu} := \Sigma_{\hat\mu}{}^{\hat\sigma}\Sigma_{\hat\sigma}{}^{\hat\rho}\ldots\Sigma_{\hat\lambda}{}^{\hat\kappa}\Sigma_{\hat\kappa}{}^{\hat\nu}\,, \qquad (\Sigma^0)_{\hat\mu}{}^{\hat\nu} := \delta^{\hat\nu}_{\hat\mu}\,. \tag{3.48}
$$

Using the first relation of equation (3.43), we can now also write down an all-order expression for the linearized fermionic superfield $\mathcal{A}^{\text{lin}}_{\hat\alpha}$,

$$
\begin{aligned}
\mathcal{A}^{\text{lin}}_{\hat\alpha}(x,\theta) = 2\Big(\theta\Gamma^{\hat\mu}\Big)_{\hat\alpha}\Bigg\{&\left(\sum_{n=0}^{7}\frac{(-1)^n}{(2n)!(2n+2)}\,(\Sigma^n)_{\hat\mu}{}^{\hat\nu}\right)A_{\hat\nu}(x) \\
&-\left(\sum_{n=0}^{7}\frac{(-1)^n}{(2n+1)!(2n+3)}\,(\Sigma^n)_{\hat\mu}{}^{\hat\nu}\right)\Big(\theta\Gamma_{\hat\nu}\psi(x)\Big)\Bigg\}\,.
\end{aligned}
\tag{3.49}
$$

The generalizations of these component field expansions to the full non-linear case can be found in [107].

3.1.3. Superfield Propagators in Harnad–Shnider Gauge

With the component expansions of the fields established, we now address the construction of propagators. Conventionally, the propagators are derived by inverting the kinetic terms in the action. Unfortunately, a manifestly supersymmetric action for the $\mathcal{N} = 4$ SYM model is not known. Nevertheless, we can construct propagators and perform quantum calculations at the one-loop level. In this section, we shall sketch how the free two-point functions can be derived by using the formerly established component expansions of the fields. Later on, we will also discuss another approach where we exploit the conformal symmetry to determine various (gauge-invariant) two-point functions.

Our strategy to compute the superfield propagators is simple: Given the component expansions of the superfields (3.44), we want to compute the superfield propagators by relating them to ordinary propagators. However, let us recall that the constraints force the fields on shell, see section 3.1.1. Naively, this puts us in a slightly inconvenient position. On the one hand, the superspace constraints demand the fields to be on shell. On the other hand, we want to compute propagators which are typically of the form $1/(p^2 - i\varepsilon)$ and therefore clearly off shell. This apparent clash was discussed in [40], where is was shown that the position space propagator can be defined in a way that is compatible with the fields being on shell. Let us briefly sketch their argument. For simplicity, we consider a field theory with just one scalar field which we will quantize using canonical quantization. One advantage of the canonical formalism lies in the fact that we can consider the VEV of two fields which are not time ordered

$$\Delta(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle \, . \tag{3.50}$$

Such an expression has no analog in the path integral formalism because the path integral automatically yields the time-ordered expectation value, i.e. the Feynman propagator

$$\Delta_F(x - y) = \langle 0|\mathrm{T}\{\phi(x)\phi(y)\}|0\rangle = \langle \phi(x)\phi(y)\rangle \, . \tag{3.51}$$

In contrast to the Feynman propagator, the non-time-ordered expectation value is a perfectly well-defined on-shell quantity. It can be evaluated by using the standard mode expansion of the Klein–Gordon field. For massless fields, one finds

$$\Delta(x) = \langle 0|\phi(x)\phi(0)|0\rangle = -\frac{g^2}{4\pi^2}\left(\frac{1}{x^2} + i\pi\,\mathrm{sign}(x^0)\delta(x^2)\right) , \tag{3.52}$$

see [40] for more details. Curiously, the non-time-ordered VEV differs from the usual Feynman propagator only by a distributional amount. The latter follows immediately by evaluating

$$\begin{aligned}\Delta_F(x) =&\Theta(x^0)\Delta(x) + \Theta(-x^0)\Delta(-x) = -\frac{g^2}{4\pi^2}\left(\frac{1}{x^2} + i\pi\delta(x^2)\right)\\ =&-\frac{g^2}{4\pi^2}\frac{1}{x^2 - i\varepsilon} \, .\end{aligned} \tag{3.53}$$

The above considerations show that the Feynman propagator in position space can indeed be derived while fully respecting the on-shell condition. In what follows, we will thus use the standard position space propagators of the component fields to construct the desired superfield propagators. We will determine these through quartic order in an expansion in the anticommuting variables. Later on, we will use these propagators in order to compute the one-loop expectation value of the super Maldacena–Wilson loop. In what follows, we shall focus on the $\mathcal{N}=4$ SYM model in four dimensions. However, instead of decomposing all the expressions into their four-dimensional building blocks, we will keep the ten-dimensional notation for the vectors and spinors and simply neglect all partial derivatives with respect to the six extra coordinates, i.e. $\partial_i A_{\hat{\mu}}(x) = \partial_i \psi(x) = 0$. To get started, let us once more list the component expansions of the linearized superfields $\mathcal{A}^{\rm lin}_{\hat{\mu}}$ and $\mathcal{A}^{\rm lin}_{\hat{\alpha}}$. Spelling out the first few summands of (3.47) and (3.49) while discarding derivative terms with respect to the six extra coordinates yields

$$\begin{aligned}\mathcal{A}^{\rm lin}_{\hat{\mu}}(x,\theta) =& A_{\hat{\mu}}(x) - \Big(\theta\Gamma_{\hat{\mu}}\psi(x)\Big) - \tfrac{1}{2}\Big(\theta\Gamma_{\hat{\mu}}{}^{\rho\hat{\nu}}\theta\Big)\partial_\rho A_{\hat{\nu}}(x) + \tfrac{1}{6}\Big(\theta\Gamma_{\hat{\mu}}{}^{\rho\hat{\nu}}\theta\Big)\Big(\theta\Gamma_{\hat{\nu}}\partial_\rho\psi(x)\Big) \\ &+ \tfrac{1}{24}\Big(\theta\Gamma_{\hat{\mu}}{}^{\rho\hat{\nu}}\theta\Big)\Big(\theta\Gamma_{\hat{\nu}}{}^{\sigma\hat{\kappa}}\theta\Big)\partial_\rho\partial_\sigma A_{\hat{\kappa}}(x) + \mathcal{O}(\theta^5)\,, \\ \mathcal{A}^{\rm lin}_{\hat{\alpha}}(x,\theta) =& \Big(\theta\Gamma^{\hat{\mu}}\Big)_{\hat{\alpha}}\Big(A_{\hat{\mu}}(x) - \tfrac{2}{3}\Big(\theta\Gamma_{\hat{\mu}}\psi(x)\Big) - \tfrac{1}{4}\Big(\theta\Gamma_{\hat{\mu}}{}^{\rho\hat{\nu}}\theta\Big)\partial_\rho A_{\hat{\nu}}(x)\Big) + \mathcal{O}(\theta^4)\,. \end{aligned} \tag{3.54}$$

Here, as before, hatted vector indices run from zero to nine, while unhatted ones run from zero to three. The final ingredients needed to compute the superfield propagators are the propagators of the component fields. These follow directly from the SYM action (2.78) and take the form (in Feynman gauge)

$$\langle A_{\hat{\mu}}(x_1)\, A_{\hat{\nu}}(x_2)\rangle = \frac{g^2}{4\pi^2}\frac{\eta_{\hat{\mu}\hat{\nu}}}{x_{12}^2}\,, \qquad \Big\langle \psi^{\hat{\alpha}}(x_1)\,\psi^{\hat{\beta}}(x_2)\Big\rangle = \frac{g^2}{2\pi^2}\frac{\bar{\Gamma}_\rho^{\hat{\alpha}\hat{\beta}}x_{12}^\rho}{x_{12}^4}\,, \tag{3.55}$$

where we abbreviated $x_{12}^\rho := x_1^\rho - x_2^\rho$ and stripped off the color dependence. Using the expressions given above, it is now a straightforward exercise to compute the leading order terms in the Graßmann expansion of the superfield propagators. Through quartic order in the anticommuting variables, we find the following expression for the propagator of the bosonic component of the superconnection:

$$\begin{aligned}\Big\langle \mathcal{A}_{\hat{\mu}}(1)\,\mathcal{A}_{\hat{\nu}}(2)\Big\rangle = \frac{g^2}{4\pi^2}\Big(&\eta_{\hat{\mu}\hat{\nu}} + \eta_{\hat{\mu}\hat{\nu}}\Big(\theta_1\Gamma_\rho\theta_2\Big)\partial_2^\rho - 2\eta_{\rho(\hat{\nu}}\Big(\theta_1\Gamma_{\hat{\mu})}\theta_2\Big)\partial_2^\rho + \tfrac{1}{2}\Big(\theta_{12}\Gamma_{\hat{\mu}\rho\hat{\nu}}\theta_{12}\Big)\partial_2^\rho \\ &+ \tfrac{1}{24}\Big(\theta_{12}\Gamma_{\hat{\mu}\sigma}{}^{\hat{\kappa}}\theta_{12}\Big)\Big(\theta_{12}\Gamma_{\hat{\kappa}\rho\hat{\nu}}\theta_{12}\Big)\partial_2^\rho\partial_2^\sigma - \tfrac{1}{6}\eta_{\sigma[\hat{\mu}}\Big(\theta_1\Gamma_{\hat{\nu}]\rho\hat{\kappa}}\theta_2\Big)\Big(\theta_1\Gamma^{\hat{\kappa}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma \\ &+ \tfrac{1}{6}\eta_{\sigma[\hat{\mu}}\Big(\theta_{12}\Gamma_{\hat{\nu}]\rho\hat{\kappa}}\theta_{12}\Big)\Big(\theta_1\Gamma^{\hat{\kappa}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma + \tfrac{1}{6}\eta_{\sigma\hat{\mu}}\Big(\theta_2\Gamma_{\hat{\nu}\rho\hat{\kappa}}\theta_2\Big)\Big(\theta_1\Gamma^{\hat{\kappa}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma \\ &- \tfrac{1}{6}\eta_{\sigma\hat{\nu}}\Big(\theta_1\Gamma_{\hat{\mu}\rho\hat{\kappa}}\theta_1\Big)\Big(\theta_1\Gamma^{\hat{\kappa}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma + \tfrac{1}{2}\Big(\theta_{12}\Gamma_{\hat{\mu}\sigma\hat{\nu}}\theta_{12}\Big)\Big(\theta_1\Gamma_\rho\theta_2\Big)\partial_2^\rho\partial_2^\sigma \\ &+ \tfrac{1}{6}\eta_{\hat{\mu}\rho}\eta_{\hat{\nu}\sigma}\Big(\theta_1\Gamma_{\hat{\kappa}}\theta_2\Big)\Big(\theta_1\Gamma^{\hat{\kappa}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma + \tfrac{1}{2}\eta_{\hat{\mu}\hat{\nu}}\Big(\theta_1\Gamma_\rho\theta_2\Big)\Big(\theta_1\Gamma_\sigma\theta_2\Big)\partial_2^\rho\partial_2^\sigma \\ &- \tfrac{1}{2}\eta_{\hat{\mu}\rho}\Big(\theta_1\Gamma_{\hat{\nu}}\theta_2\Big)\Big(\theta_1\Gamma_\sigma\theta_2\Big)\partial_2^\rho\partial_2^\sigma - \tfrac{1}{2}\eta_{\sigma\hat{\nu}}\Big(\theta_1\Gamma_\rho\theta_2\Big)\Big(\theta_1\Gamma_{\hat{\mu}}\theta_2\Big)\partial_2^\rho\partial_2^\sigma \\ &+ \mathcal{O}(\theta^6)\Big)x_{12}^{-2}\,. \end{aligned} \tag{3.56}$$

In order to bring the propagator to this particular form, we made repeated use of the magic identity (A.31) as well as of the reduction formula (A.29). For the mixed superfield propagator, we obtain

$$\begin{aligned}\Big\langle \mathcal{A}_{\hat\mu}(1)\,\mathcal{A}_{\hat\alpha}(2)\Big\rangle = \frac{g^2}{4\pi^2}\Big(\theta_2\Gamma^{\hat\nu}\Big)_{\hat\alpha}\Big(&\eta_{\hat\mu\hat\nu}+\tfrac{1}{4}\Big(\theta_2\Gamma_{\hat\mu\rho\hat\nu}\theta_2\Big)\partial_2^\rho+\tfrac{1}{2}\Big(\theta_1\Gamma_{\hat\mu\rho\hat\nu}\theta_1\Big)\partial_2^\rho \\ &-\tfrac{4}{6}\Big(\theta_1\Gamma_{\hat\mu\rho\hat\nu}\theta_2\Big)\partial_2^\rho-\tfrac{8}{6}\eta_{\rho(\hat\nu}\Big(\theta_1\Gamma_{\hat\mu)}\theta_2\Big)\partial_2^\rho+\tfrac{4}{6}\eta_{\hat\mu\hat\nu}\Big(\theta_1\Gamma_\rho\theta_2\Big)\partial_2^\rho+\mathcal{O}(\theta^4)\Big)x_{12}^{-2}\,.\end{aligned} \tag{3.57}$$

The simplest propagator is that of the fermionic component of the superconnection, which is at leading order given by

$$\Big\langle \mathcal{A}_{\hat\alpha}(1)\,\mathcal{A}_{\hat\beta}(2)\Big\rangle = \frac{g^2}{4\pi^2}\Big(\theta_1\Gamma^{\hat\kappa}\Big)_{\hat\alpha}\Big(\theta_2\Gamma_{\hat\kappa}\Big)_{\hat\beta}\,x_{12}^{-2}+\mathcal{O}(\theta^4)\,. \tag{3.58}$$

Note that in order to determine the propagator of the full superconnection through quartic order in an expansions in the anticommuting variables we do not need to compute the higher-order terms in (3.58) as the fermionic vielbeine are themselves linear in the Graßmann coordinates. We will compute the full propagator in section 4.3.1 when we discuss the one-loop VEV of the super Maldacena–Wilson loop.

3.2. The Four-Dimensional Perspective

In the last section, we introduced the non-chiral superspace formulation of the ten-dimensional $\mathcal{N}=1$ SYM model. While the ten-dimensional formalism is in general quite useful when dealing with $\mathcal{N}=4$ SYM theory, some computations are more conveniently performed in four dimensions. This typically applies to situations where the conformal boost symmetries are involved as those are not contained in the ten-dimensional Poincaré group $\mathrm{SO}(1,9)$. To set the stage for these computations, we start this section by reviewing the on-shell superspace construction adopting the four-dimensional point of view. We then continue by discussing the action of the conformal inversion element on the superfields. Subsequently, we will use the newly gained insights to derive the scalar propagator as well as the field strength correlator.

3.2.1. Superspace Geometry and the Constraints

Let us start by introducing the non-chiral superspace suitable for describing $\mathcal{N}=4$ SYM theory in four dimensions. The superspace has four bosonic and sixteen fermionic directions and is parametrized by

$$(x^{\dot\alpha\alpha},\theta^{a\alpha},\bar\theta^{\dot\alpha}{}_a)\,. \tag{3.59}$$

Here, α and $\dot\alpha$ are $\mathfrak{su}(2)$ indices taking values in $\{1,2\}$, while $a=1,..,4$ is an $\mathfrak{su}(4)$ R-symmetry index. In what follows, we will frequently use a matrix notation similar to the one in reference [39]. The bosonic coordinate $x^{\dot\alpha\alpha}=x^\mu\bar\sigma_\mu^{\dot\alpha\alpha}$ (see Appendix A.1

for our conventions) is represented by a 2×2 matrix, while $\theta^{a\alpha}$ is a 4×2 matrix and $\bar{\theta}^{\dot{\alpha}}{}_a$ is a 2×4 matrix. The superspace coordinates are subject to the following reality conditions

$$x^\dagger = x \,, \qquad \theta^\dagger = \bar{\theta} \,, \qquad \bar{\theta}^\dagger = \theta \,. \tag{3.60}$$

Note that for Graßmann variables χ, ψ we use the convention that $(\chi\psi)^\dagger = -\psi^\dagger\chi^\dagger$, while for transpositions we use $(\chi\psi)^\intercal = -\psi^\intercal\chi^\intercal$. We continue by introducing the supercharges of the SYM model. On the non-chiral superspace, they can be represented as

$$\mathrm{Q}_{\alpha a} = \bar{\theta}^{\dot{\alpha}}{}_a \partial_{\alpha\dot{\alpha}} - \partial_{\alpha a} \,, \qquad \bar{\mathrm{Q}}^a{}_{\dot{\alpha}} = \theta^{a\alpha}\partial_{\alpha\dot{\alpha}} - \bar{\partial}^a{}_{\dot{\alpha}} \,. \tag{3.61}$$

These expressions follow directly by dimensionally reducing the ten-dimensional supercharges $\mathrm{Q}_{\hat{\alpha}}$, see equation (3.2). More precisely, they are obtained by splitting the fermionic index $\hat{\alpha}$ according to equation (3.59) into the two pairs (αa) and $\binom{a}{\dot{\alpha}}$ and using the Pauli matrix expressions listed in appendix A.3 while neglecting terms which vanish upon acting on functions being independent of the six internal bosonic coordinates. In a completely similar manner, one obtains the following expressions for the susy-covariant derivatives

$$D_{\alpha a} = \bar{\theta}^{\dot{\alpha}}{}_a \partial_{\alpha\dot{\alpha}} + \partial_{\alpha a} \,, \qquad \bar{D}^a{}_{\dot{\alpha}} = \theta^{a\alpha}\partial_{\alpha\dot{\alpha}} + \bar{\partial}^a{}_{\dot{\alpha}} \,. \tag{3.62}$$

Given these derivatives, it is convenient to define the chiral coordinates $x^\pm$

$$x^\pm := x \mp 2\bar{\theta}\theta \,. \tag{3.63}$$

Note that the chiral derivative D annihilates functions depending on $(x^-, \bar{\theta})$, while the antichiral derivative $\bar{D}$ annihilates functions depending on (x^+, θ). Hence, the two pairs of coordinates (x^+, θ) and $(x^-, \bar{\theta})$ define chiral and antichiral superspace. Inverting the equations (3.63) yields

$$x = \tfrac{1}{2}(x^+ + x^-) \,, \qquad \bar{\theta}\theta = \tfrac{1}{4}(x^- - x^+) \,. \tag{3.64}$$

These relations will prove useful later on when it comes to performing explicit superspace computations. Having introduced the supercharges Q and $\bar{\mathrm{Q}}$, we take the opportunity and introduce the supertranslation-invariant distances on which susy-invariant quantities can depend. The minimal supertranslation-invariant interval reads

$$x_{12} := x_1 - x_2 + 2\bar{\theta}_2\theta_1 - 2\bar{\theta}_1\theta_2 \,, \tag{3.65}$$

where minimal refers to the Graßmann tail of the expression. For later convenience, we also introduce shorthand notation for the following two non-minimal supertranslation-invariant intervals

$$\begin{aligned} x_{12}^{+-} &:= x_{12} - 2\bar{\theta}_{12}\theta_{12} = x_1^+ - x_2^- + 4\bar{\theta}_2\theta_1 \,, \\ x_{12}^{-+} &:= x_{12} + 2\bar{\theta}_{12}\theta_{12} = x_1^- - x_2^+ - 4\bar{\theta}_1\theta_2 \,. \end{aligned} \tag{3.66}$$

Note that these distances have no definite overall chirality. Instead, as indicated by the signs in the superscript, they have mixed chirality. For instance, x_{12}^{+-} is chiral in the coordinates of point one and antichiral in the coordinates of point two.

We proceed by defining the dual space on which our gauge theory lives. The dual space is the space of one-forms and a basis of it is given by the differentials $(\mathrm{d}x^{\dot{\alpha}\alpha}, \mathrm{d}\theta^{a\alpha}, \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a)$. However, as before, it is convenient to use another basis in which the different components of a general one-form do not mix under a general supersymmetry transformation. This basis is dual to the covariantized basis $(\partial_{\alpha\dot{\alpha}}, D_{\alpha a}, \bar{D}^a{}_{\dot{\alpha}})$ and is explicitly given by

$$e = \mathrm{d}x - 2\mathrm{d}\bar{\theta}\theta + 2\bar{\theta}\mathrm{d}\theta\,, \qquad \mathrm{d}\theta\,, \qquad \mathrm{d}\bar{\theta}\,. \tag{3.67}$$

While working in this basis is in general very convenient, the prize to be paid is that the basis is not torsion-free anymore, i.e.[2]

$$\mathrm{d}e = 4\mathrm{d}\bar{\theta}\mathrm{d}\theta\,, \qquad \mathrm{d}\mathrm{d}\theta = 0\,, \qquad \mathrm{d}\mathrm{d}\bar{\theta} = 0\,. \tag{3.68}$$

Here, d is the total differential in superspace and can be expressed in terms of plain or susy-covariant derivatives

$$\mathrm{d} = \tfrac{1}{2}\mathrm{d}x^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\alpha}} + \mathrm{d}\theta^{a\alpha}\partial_{\alpha a} + \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\bar{\partial}^a{}_{\dot{\alpha}} = \tfrac{1}{2}e^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\alpha}} + \mathrm{d}\theta^{a\alpha}D_{\alpha a} + \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\bar{D}^a{}_{\dot{\alpha}}. \tag{3.69}$$

Having introduced the appropriate vector space, we are now ready to define the four-dimensional $\mathcal{N} = 4$ SYM model. We start by introducing an $\mathfrak{su}(N)$-valued gauge connection one-form $\mathcal{A} = \mathcal{A}^{\mathfrak{m}}T_{\mathfrak{m}}$, which may be decomposed on the vielbein basis as

$$\mathcal{A} = \tfrac{1}{2}e^{\dot{\alpha}\alpha}\mathcal{A}_{\alpha\dot{\alpha}} + \mathrm{d}\theta^{a\alpha}\mathcal{A}_{\alpha a} + \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\bar{\mathcal{A}}^a{}_{\dot{\alpha}}\,. \tag{3.70}$$

Given the gauge connection one-form, we define the gauge-covariant derivative as

$$\mathcal{D} = \mathrm{d} + \mathcal{A} = \tfrac{1}{2}e^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}} + \mathrm{d}\theta^{a\alpha}\mathcal{D}_{\alpha a} + \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\bar{\mathcal{D}}^a{}_{\dot{\alpha}}\,, \tag{3.71}$$

where

$$\mathcal{D}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}}\,, \qquad \mathcal{D}_{\alpha a} = D_{\alpha a} + \mathcal{A}_{\alpha a}\,, \qquad \bar{\mathcal{D}}^a{}_{\dot{\alpha}} = \bar{D}^a{}_{\dot{\alpha}} + \bar{\mathcal{A}}^a{}_{\dot{\alpha}}\,. \tag{3.72}$$

When written in terms of differential forms, the gauge transformations of $\mathcal{A}$ take the same form as in the ten-dimensional case, see equation (3.14) and (3.15). Finally, we need to discuss the constraints that have to be imposed on the field strength two-form in order to reduce the number of degrees of freedom of the superconnection. The field strength two-form itself is given by the expression

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\,, \tag{3.73}$$

[2]Note that we will typically omit the wedge symbol in the product of differential forms.

and its components with respect to the susy-invariant vielbein basis read

$$\begin{aligned}\mathcal{F} = &-\tfrac{1}{8}e^{\dot\alpha\alpha}e^{\dot\beta\beta}\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} - \tfrac{1}{2}e^{\dot\alpha\alpha}\mathrm{d}\theta^{b\beta}\mathcal{F}_{\beta b\alpha\dot\alpha} - \tfrac{1}{2}e^{\dot\alpha\alpha}\mathrm{d}\bar\theta^{\dot\beta}{}_b\,\mathcal{F}^b{}_{\dot\beta\alpha\dot\alpha} - \tfrac{1}{2}\mathrm{d}\theta^{a\alpha}\mathrm{d}\theta^{b\beta}\mathcal{F}_{\beta b\alpha a}\\ &-\tfrac{1}{2}\mathrm{d}\bar\theta^{\dot\alpha}{}_a\mathrm{d}\bar\theta^{\dot\beta}{}_b\,\mathcal{F}^b{}_{\dot\beta}{}^a{}_{\dot\alpha} - \mathrm{d}\theta^{a\alpha}\mathrm{d}\bar\theta^{\dot\beta}{}_b\,\mathcal{F}^b{}_{\dot\beta\alpha a}\,,\end{aligned} \tag{3.74}$$

where

$$\begin{aligned}\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} &= [\mathcal{D}_{\beta\dot\beta},\mathcal{D}_{\alpha\dot\alpha}] = \partial_{\beta\dot\beta}\mathcal{A}_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}\mathcal{A}_{\beta\dot\beta} + [\mathcal{A}_{\beta\dot\beta},\mathcal{A}_{\alpha\dot\alpha}]\,,\\ \mathcal{F}_{\beta b\alpha\dot\alpha} &= [\mathcal{D}_{\beta b},\mathcal{D}_{\alpha\dot\alpha}] = D_{\beta b}\mathcal{A}_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}\mathcal{A}_{\beta b} + [\mathcal{A}_{\beta b},\mathcal{A}_{\alpha\dot\alpha}]\,,\\ \mathcal{F}^b{}_{\dot\beta\alpha\dot\alpha} &= [\bar{\mathcal{D}}^b{}_{\dot\beta},\mathcal{D}_{\alpha\dot\alpha}] = \bar D^b{}_{\dot\beta}\mathcal{A}_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}\bar{\mathcal{A}}^b{}_{\dot\beta} + [\bar{\mathcal{A}}^b{}_{\dot\beta},\mathcal{A}_{\alpha\dot\alpha}]\,,\\ \mathcal{F}_{\beta b\alpha a} &= \{\mathcal{D}_{\beta b},\mathcal{D}_{\alpha a}\} = D_{\beta b}\mathcal{A}_{\alpha a} + D_{\alpha a}\mathcal{A}_{\beta b} + \{\mathcal{A}_{\beta b},\mathcal{A}_{\alpha a}\}\,,\\ \mathcal{F}^b{}_{\dot\beta}{}^a{}_{\dot\alpha} &= \{\bar{\mathcal{D}}^b{}_{\dot\beta},\bar{\mathcal{D}}^a{}_{\dot\alpha}\} = \bar D^b{}_{\dot\beta}\bar{\mathcal{A}}^a{}_{\dot\alpha} + \bar D^a{}_{\dot\alpha}\bar{\mathcal{A}}^b{}_{\dot\beta} + \{\bar{\mathcal{A}}^b{}_{\dot\beta},\bar{\mathcal{A}}^a{}_{\dot\alpha}\}\,,\\ \mathcal{F}^b{}_{\dot\beta\alpha a} &= \{\bar{\mathcal{D}}^b{}_{\dot\beta},\mathcal{D}_{\alpha a}\} - 2\delta^b_a\mathcal{D}_{\alpha\dot\beta} = \bar D^b{}_{\dot\beta}\mathcal{A}_{\alpha a} + D_{\alpha a}\bar{\mathcal{A}}^b{}_{\dot\beta} + \{\bar{\mathcal{A}}^b{}_{\dot\beta},\mathcal{A}_{\alpha a}\} - 2\delta^b_a\mathcal{A}_{\alpha\dot\beta}\,.\end{aligned} \tag{3.75}$$

The appropriate set of constraints that have to be imposed on the four-dimensional field strength two-form can be obtained by dimensionally reducing the ten-dimensional constraints $\mathcal{F}_{\hat\alpha\hat\beta} = 0$. For this, one splits the ten-dimensional index $\hat\alpha$ into two pairs (αa) and $\binom{a}{\dot\alpha}$ and decomposes the bosonic component of the ten-dimensional gauge connection as

$$\mathcal{A}_{\hat\mu} \to (\mathcal{A}_\mu, \Phi_i)\,, \tag{3.76}$$

where Φ is the superfield corresponding to the component field ϕ. The gauge-covariant derivatives then become

$$\mathcal{D}_{\hat\mu} \to (\partial_\mu + \mathcal{A}_\mu, \Phi_i)\,. \tag{3.77}$$

Using this as well as the Pauli matrix conventions laid out in appendix A.3, one finds

$$\begin{aligned}\{\mathcal{D}_{\alpha a},\mathcal{D}_{\beta b}\} &= -2\varepsilon_{\alpha\beta}\Phi_{ab}\,,\\ \{\mathcal{D}_{\alpha a},\bar{\mathcal{D}}^b{}_{\dot\beta}\} &= 2\delta^b_a\mathcal{D}_{\alpha\dot\beta}\,,\\ \{\bar{\mathcal{D}}^a{}_{\dot\alpha},\bar{\mathcal{D}}^b{}_{\dot\beta}\} &= -\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{abcd}\Phi_{cd}\,.\end{aligned} \tag{3.78}$$

Finally, we discuss how the Bianchi identities and the constraints can be used to express the components of the field strength two-form (3.75) in terms of the superscalar Φ_{ab} and derivatives of this field. We begin by taking a fermionic covariant derivative of the first constraint equation in (3.78). This yields

$$[\bar{\mathcal{D}}^c{}_{\dot\gamma},\{\mathcal{D}_{\alpha a},\mathcal{D}_{\beta b}\}] = -2\varepsilon_{\alpha\beta}[\bar{\mathcal{D}}^c{}_{\dot\gamma},\Phi_{ab}]\,. \tag{3.79}$$

Using the Bianchi identity for fermionic covariant derivatives, we rewrite the left-hand side of the former equation as

$$-[\mathcal{D}_{\alpha a},\{\mathcal{D}_{\beta b},\bar{\mathcal{D}}^c{}_{\dot\gamma}\}] - [\mathcal{D}_{\beta b},,\{\bar{\mathcal{D}}^c{}_{\dot\gamma},\mathcal{D}_{\alpha a}\}] = -2\varepsilon_{\alpha\beta}[\bar{\mathcal{D}}^c{}_{\dot\gamma},\Phi_{ab}]\,. \tag{3.80}$$

In the next step, we plug in the mixed chiral constraint (3.78) and contract the complete equation with δ^b_c ,

$$4[\mathcal{D}_{\alpha a}, \mathcal{D}_{\beta\dot{\gamma}}] + [\mathcal{D}_{\beta a}, \mathcal{D}_{\alpha\dot{\gamma}}] = \varepsilon_{\alpha\beta}[\bar{\mathcal{D}}^c{}_{\dot{\gamma}}, \Phi_{ac}]\,. \tag{3.81}$$

Symmetrizing this equation in α and β yields

$$[\mathcal{D}_{\alpha a}, \mathcal{D}_{\beta\dot{\gamma}}] = -[\mathcal{D}_{\beta a}, \mathcal{D}_{\alpha\dot{\gamma}}]\,. \tag{3.82}$$

By plugging this back into equation (3.81), we obtain

$$\mathcal{F}_{\alpha a\beta\dot{\beta}} = [\mathcal{D}_{\alpha a}, \mathcal{D}_{\beta\dot{\beta}}] = \tfrac{1}{3}\varepsilon_{\alpha\beta}[\bar{\mathcal{D}}^b{}_{\dot{\beta}}, \Phi_{ab}]\,, \tag{3.83}$$

which is the desired result. In pretty much the same manner one can show that the other two remaining coefficients, which are not directly covered by the constraints, can be expressed as follows

$$\begin{aligned} \mathcal{F}^a{}_{\dot{\alpha}\beta\dot{\beta}} &= [\bar{\mathcal{D}}^a{}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \tfrac{1}{3}\varepsilon_{\dot{\alpha}\dot{\beta}}[\mathcal{D}_{\beta b}, \bar{\Phi}^{ab}]\,, \\ \mathcal{F}_{\alpha\dot{\alpha}\beta\dot{\beta}} &= [\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \tfrac{1}{24}\varepsilon_{\dot{\alpha}\dot{\beta}}\{\mathcal{D}_{\alpha a}, [\mathcal{D}_{\beta b}, \bar{\Phi}^{ab}]\} + \tfrac{1}{24}\varepsilon_{\alpha\beta}\{\bar{\mathcal{D}}^a{}_{\dot{\alpha}}, [\bar{\mathcal{D}}^b{}_{\dot{\beta}}, \Phi_{ab}]\}\,. \end{aligned} \tag{3.84}$$

Finally, we use the equations (3.78), (3.83) and (3.84) to rewrite the field strength two-form as

$$\begin{aligned} \mathcal{F} = \;& (\mathrm{d}\theta\varepsilon\mathrm{d}\theta^{\intercal})^{ab}\Phi_{ab} + \tfrac{1}{6}(e\varepsilon\mathrm{d}\theta^{\intercal})^{\dot{\alpha}a}[\bar{\mathcal{D}}^b{}_{\dot{\alpha}}, \Phi_{ab}] + \tfrac{1}{192}(e\varepsilon e^{\intercal})^{\dot{\alpha}\dot{\beta}}\{\bar{\mathcal{D}}^a{}_{\dot{\alpha}}, [\bar{\mathcal{D}}^b{}_{\dot{\beta}}, \Phi_{ab}]\} \\ &+ (\mathrm{d}\bar{\theta}^{\intercal}\varepsilon\mathrm{d}\bar{\theta})_{ab}\bar{\Phi}^{ab} + \tfrac{1}{6}(e^{\intercal}\varepsilon\mathrm{d}\bar{\theta})^{\alpha}{}_a[\mathcal{D}_{\alpha b}, \bar{\Phi}^{ab}] + \tfrac{1}{192}(e^{\intercal}\varepsilon e)^{\alpha\beta}\{\mathcal{D}_{\alpha a}, [\mathcal{D}_{\beta b}, \bar{\Phi}^{ab}]\}\,. \end{aligned} \tag{3.85}$$

When dealing with the quantum theory, this particular form of $\mathcal{F}$ will turn out to be useful as it can be used to express the field strength correlator in terms of scalar correlators.

3.2.2. Superfield Propagators

In section 2.1.1, we have shown that conformal symmetry heavily constrains the form of two- and three-point functions of local gauge-invariant operators. In fact, the functional form of the two-point correlation function of two such operators is completely fixed by conformal symmetry. In the present section, we want to use these insights to determine expressions for certain propagators. Importantly, conformal symmetry only constrains the physical degrees of freedom, while gauge degrees of freedom are typically not constraint. To identify the propagators which can possibly be determined by using conformal symmetry, let us note that for computing propagators it is sufficient to focus on the linearized theory, which is obtained by replacing the gauge group $\mathrm{SU}(N)$ by $\mathrm{U}(1)^{N^2-1}$. Under a linearized gauge transformation, the fields transform as

$$\mathbb{G}_{\mathrm{L}}\Phi^{\mathfrak{m}}_i = 0\,, \qquad \mathbb{G}_{\mathrm{L}}\mathcal{A}^{\mathfrak{m}} = \mathrm{d}\Lambda^{\mathfrak{m}}\,. \tag{3.86}$$

The following propagators are thus gauge invariant:

$$\langle \Phi(1)\Phi(2)\rangle\,, \qquad \langle \mathcal{F}(1)\Phi(2)\rangle\,, \qquad \langle \mathcal{F}(1)\mathcal{F}(2)\rangle\,. \tag{3.87}$$

In what follows, we shall determine the all-order form[3] of these propagators by using considerations of symmetry as well as the superspace constraints. We begin by discussing the action of the conformal inversion on the superspace coordinates and derive the corresponding transformation laws of the superfields. With the conformal variations of the superscalars established, we will then determine the scalar propagator via elementary CFT considerations. Finally, we compute concrete expressions for the field strength correlators by exploiting the implications of the superspace constraints. The all-order form of the above-listed two-point functions is at the heart of our proof of the one-loop Yangian symmetry of super Maldacena–Wilson loops, which we will present in chapter 5.

3.2.2.1. The Scalar Propagator

As outlined above, we begin by focusing on the propagator of the superscalars. Although the scalar propagator represents the simplest correlator that we wish to compute, determining its form is neither a trivial task nor a simple one. Our strategy for computing this two-point function is the following: First, we discuss the action of the conformal inversion on the superspace coordinates as well as on the fields. Note that we will not yet discuss the full superspace representation of the superconformal algebra. We postpone this discussion to chapter 4 as we ultimately need a representation on the S^5-extended non-chiral superspace, which we have not yet introduced. In the second step, we will then use the translation and inversion symmetry to completely fix the functional form of the scalar propagator.

Conformal inversion symmetry. In this paragraph, we determine the transformation laws of the superfields under conformal inversion. This amounts to lifting the bosonic discussion given in section 2.1.1 to full-fledged superspace. We begin by focusing on the components of the superconnection and subsequently derive the transformation laws of the superscalars by using the constraint equations (3.78).

Before we attack the problem in full-fledged superspace, let us briefly demonstrate our strategy in a bosonic example. Recall from section 2.1.1 that under inversion the point x^μ gets mapped to

$$\mathrm{I}_b[x^\mu] = \frac{x^\mu}{x^2}\,. \tag{3.88}$$

A convenient way to derive the inversion law of a dimension one vector field is to consider the derivative of a dimensionless scalar field. According to equation (2.14), a dimensionless scalar field transforms trivially under inversion, i.e.

$$\varphi'(x') = \varphi(x)\,. \tag{3.89}$$

[3]Here, by all order we mean to all orders in the fermionic coordinates.

In contrast to all the other formulas presented in section 2.1.1, this equation does not get modified in superspace. To derive the inversion law of the gauge field, we consider the derivative of the above scalar field

$$\mathrm{d}\varphi = \mathrm{d}x^\mu \partial_\mu \varphi(x)\,, \tag{3.90}$$

which can be identified with a connection that is pure gauge, i.e. $A = \mathrm{d}\varphi$. The desired transformation law is now easily established by using the fact that

$$\mathrm{d}x^\mu \partial_\mu \varphi(x) = \mathrm{d}x'^\mu \partial'_\mu \varphi'(x')\,. \tag{3.91}$$

Expressing the differential on the left-hand side in terms of the primed coordinates yields

$$\mathrm{d}x^\mu = \mathrm{d}x'^\rho \Big(x^2 \delta^\mu_\rho - 2x^\mu x_\rho \Big)\,. \tag{3.92}$$

By plugging this back, we obtain

$$\mathrm{d}x'^\mu \Big(x^2 \delta^\nu_\mu - 2x^\nu x_\mu \Big) \partial_\nu \varphi(x) = \mathrm{d}x'^\mu \partial'_\mu \varphi'(x')\,, \tag{3.93}$$

from which we infer

$$A'_\mu(x') = x^2 \left(\delta^\nu_\mu - \frac{2x^\nu x_\mu}{x^2} \right) A_\nu(x)\,. \tag{3.94}$$

This formula is in complete agreement with equation (2.14) of section 2.1.1.

Let us now proceed and generalize the analysis to fields defined on superspace. The action of the (generalized) conformal inversion on the superspace coordinates was discussed in [39] and is given by

$$\begin{aligned} \mathrm{I}[x^\pm] &= \varepsilon x^{\mp,\intercal,-1} \varepsilon\,, \\ \mathrm{I}[\theta] &= -M \bar\theta^\intercal x^{-,\intercal,-1} \varepsilon\,, \\ \mathrm{I}[\bar\theta] &= \varepsilon x^{+,\intercal,-1} \theta^\intercal M^{-1}\,, \end{aligned} \tag{3.95}$$

where M is a symmetric unitary position-independent matrix ($M = M^\intercal$, $M^{-1} = M^\dagger$) which is needed for correct transformations under R-symmetry. The constraint ($M = M^\intercal$) follows from demanding that the inversion map is an involution

$$\begin{aligned} \mathrm{I}[\mathrm{I}[\theta]] &= -M \mathrm{I}[\bar\theta]^\intercal \mathrm{I}[x^-]^{\intercal,-1} \varepsilon \\ &= M M^{T,-1} \theta \\ &= \theta\,. \end{aligned} \tag{3.96}$$

It can be checked that the constraint $x^- - x^+ = 4\bar\theta\theta$ is preserved by inversion. The reality conditions $\theta^\ddagger = \bar\theta$ and $(x^+)^\ddagger = x^-$ are preserved as well. The inversion of the matrix x can easily be obtained by using the identity $x = \frac{1}{2}(x^+ + x^-)$,

$$\mathrm{I}[x] = \varepsilon x^{-,\intercal,-1} x^\intercal x^{+,\intercal,-1} \varepsilon\,. \tag{3.97}$$

To obtain the transformations of the components of the superconnection, we again consider the exterior derivative. As a first step, we invert the susy-covariant one-form basis

$$e^{\dot{\alpha}\alpha} = \mathrm{d}x^{\dot{\alpha}\alpha} - 2\mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\theta^{a\alpha} + 2\bar{\theta}^{\dot{\alpha}}{}_a\mathrm{d}\theta^{a\alpha}\,, \qquad \mathrm{d}\theta^{a\alpha}\,, \qquad \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\,. \tag{3.98}$$

The computation is a bit lengthy but straightforward and we will therefore only state the final result, which reads

$$\begin{aligned} e' &= \mathrm{dI}[x] - 2\mathrm{dI}[\bar{\theta}]\mathrm{I}[\theta] + 2\mathrm{I}[\bar{\theta}]\mathrm{dI}[\theta] = -\varepsilon x^{+,\intercal,-1}e^\intercal x^{-,\intercal,-1}\varepsilon\,, \\ \mathrm{d}\theta' &= \mathrm{dI}[\theta] = -U^\intercal \mathrm{d}\bar{\theta}^\intercal x^{-,\intercal,-1}\varepsilon + M\bar{\theta}^\intercal x^{-,\intercal,-1}e^\intercal x^{-,\intercal,-1}\varepsilon\,, \\ \mathrm{d}\bar{\theta}' &= \mathrm{dI}[\bar{\theta}] = \varepsilon x^{+,\intercal,-1}\mathrm{d}\theta^\intercal U^{\dagger,\intercal} - \varepsilon x^{+,\intercal,-1}e^\intercal x^{+,\intercal,-1}\theta^\intercal M^{-1}\,, \end{aligned} \tag{3.99}$$

where we have introduced shorthand notation for the following unitary matrix:

$$U = M - 4\theta x^{-,-1}\bar{\theta}M\,, \qquad U^\dagger = M^{-1} + 4M^{-1}\theta x^{+,-1}\bar{\theta}\,, \qquad UU^\dagger = 1\,. \tag{3.100}$$

Before we continue, let us compute the determinant of U as we will need it later on. To do so, we use the following identity

$$\det(1 + AB) = \det(1 + BA)^{-1}\,, \tag{3.101}$$

which is valid for A and B being odd matrices of dimension $n \times m$ and $m \times n$. Equation (3.101) can be shown to hold true by comparing two equivalent expressions for the superdeterminant of a special supermatrix K,

$$K = \begin{pmatrix} 1 & -A \\ B & 1 \end{pmatrix}, \qquad \mathrm{sdet}(K) = \det(1 + AB) = \det(1 + BA)^{-1}\,. \tag{3.102}$$

Using identity (3.101) as well as the fact that M has unit determinant, we find

$$\det(U) = \det(1 - 4\bar{\theta}\theta x^{-,-1})^{-1} = \det(1 - (x^- - x^+)x^{-,-1})^{-1} = \frac{\det(x^-)}{\det(x^+)}\,. \tag{3.103}$$

We now express the old vielbeine in terms of the new ones. We find

$$\begin{aligned} e &= -x^-\varepsilon e'^\intercal \varepsilon x^+\,, \\ \mathrm{d}\theta &= U\mathrm{d}\bar{\theta}'^\intercal \varepsilon x^+ - \theta\varepsilon e'^\intercal \varepsilon x^+\,, \\ \mathrm{d}\bar{\theta} &= -x^-\varepsilon \mathrm{d}\theta'^\intercal U^\dagger - x^-\varepsilon e'^\intercal \varepsilon\bar{\theta}\,. \end{aligned} \tag{3.104}$$

We proceed by plugging these expressions into the formula for the total differential

$$\begin{aligned} \mathrm{d} =& \tfrac{1}{2}e^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\alpha}} + \mathrm{d}\theta^{a\alpha}D_{\alpha a} + \mathrm{d}\bar{\theta}^{\dot{\alpha}}{}_a\bar{D}^a{}_{\dot{\alpha}} \\ =& \tfrac{1}{2}e'^{\dot{\beta}\beta}\Big[-(x^-\varepsilon)^{\dot{\alpha}}{}_\beta(\varepsilon x^+)_{\dot{\beta}}{}^\alpha\partial_{\alpha\dot{\alpha}} - 2(\varepsilon x^+)_{\dot{\beta}}{}^\alpha(\theta\varepsilon)^a{}_\beta D_{\alpha a} - 2(x^-\varepsilon)^{\dot{\alpha}}{}_\beta(\varepsilon\bar{\theta})_{\dot{\beta}a}\bar{D}^a{}_{\dot{\alpha}}\Big] \\ &+ \mathrm{d}\theta'^{b\beta}\Big[-(x^-\varepsilon)^{\dot{\alpha}}{}_\beta(U^\dagger)_{ba}\bar{D}^a{}_{\dot{\alpha}}\Big] \\ &+ \mathrm{d}\bar{\theta}'^{\dot{\beta}}{}_b\Big[(\varepsilon x^+)_{\dot{\beta}}{}^\alpha U^{ab}D_{\alpha a}\Big]\,. \end{aligned} \tag{3.105}$$

From this expression, we can directly read off how the different components of the superconnection transform under inversion. We find

$$\begin{aligned}
\mathcal{A}_{\alpha\dot\alpha} &\to -\Big(\varepsilon x^{-,\intercal}\mathcal{A}_p^\intercal x^{+,\intercal}\varepsilon + 2\varepsilon\theta^\intercal\mathcal{A}_\theta^\intercal x^{+,\intercal}\varepsilon - 2\varepsilon x^{-,\intercal}\bar{\mathcal{A}}_{\bar\theta}^\intercal\bar\theta^\intercal\varepsilon\Big)_{\alpha\dot\alpha}\,,\\
\mathcal{A}_{\alpha a} &\to \Big(\varepsilon x^{-,\intercal}\bar{\mathcal{A}}_{\bar\theta}^\intercal U^{\dagger,\intercal}\Big)_{\alpha a}\,,\\
\bar{\mathcal{A}}^a{}_{\dot\alpha} &\to -\Big(U^\intercal\mathcal{A}_\theta^\intercal x^{+,\intercal}\varepsilon\Big)^a{}_{\dot\alpha}\,.
\end{aligned} \tag{3.106}$$

Using the above transformation formulas, we can show that the constraints (alias the equations of motion)

$$\begin{aligned}
\{\mathcal{D}_{\alpha a},\mathcal{D}_{\beta b}\} &= -2\varepsilon_{\alpha\beta}\Phi_{ab}\,,\\
\{\bar{\mathcal{D}}^a{}_{\dot\alpha},\bar{\mathcal{D}}^b{}_{\dot\beta}\} &= -\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{abcd}\Phi_{cd}\,,\\
\{\mathcal{D}_{\alpha a},\bar{\mathcal{D}}^b{}_{\dot\beta}\} &= 2\delta_a^b\mathcal{D}_{\alpha\dot\alpha}\,,
\end{aligned} \tag{3.107}$$

are indeed invariant for a certain transformation behavior of the superscalars which we will now determine. Useful formulas to perform this calculation are

$$\begin{aligned}
D_{\alpha a}\Big((\varepsilon x^+)_{\dot\alpha}{}^\gamma U^{cb}\Big) &= 4\delta_\alpha^\gamma(\varepsilon\bar\theta)_{\dot\alpha a}U^{cb} - 4\delta_a^c(\varepsilon x^+)_{\dot\alpha}{}^\gamma(x^{-,-1}\bar\theta M)_\alpha{}^b\,,\\
\bar{D}^b{}_{\dot\alpha}\Big(-(x^-\varepsilon)^{\dot\gamma}{}_\alpha(U^\dagger)_{ad}\Big) &= -4\delta_{\dot\alpha}^{\dot\gamma}(\theta\varepsilon)^b{}_\alpha(U^\dagger)_{ad} + 4\delta_d^b(x^-\varepsilon)^{\dot\gamma}{}_\alpha(M^{-1}\theta x^{+,-1})_{a\dot\alpha}\,,\\
\bar{D}^a{}_{\dot\alpha}\Big((\varepsilon x^+)_{\dot\gamma}{}^\gamma U^{cb}\Big) &= 4(\varepsilon x^+)_{\dot\gamma}{}^\gamma(\theta x^{-,-1})^c{}_{\dot\alpha}U^{ab}\,,\\
D_{\alpha a}\Big(-(x^-\varepsilon)^{\dot\gamma}{}_\gamma(U^\dagger)_{cd}\Big) &= -4(x^-\varepsilon)^{\dot\gamma}{}_\gamma(x^{+,-1}\bar\theta)_{\alpha d}(U^\dagger)_{ca}\,.
\end{aligned} \tag{3.108}$$

We begin by investigating the first constraint of (3.107). Evaluating the anticommutator of the two transformed susy- and gauge-covariant derivatives yields

$$\{\mathcal{D}'_{\alpha a},\mathcal{D}'_{\beta b}\}=(x^-\varepsilon)^{\dot\alpha}{}_\alpha(x^-\varepsilon)^{\dot\beta}{}_\beta(U^\dagger)_{ac}(U^\dagger)_{bd}\{\bar{\mathcal{D}}^c{}_{\dot\alpha},\bar{\mathcal{D}}^d{}_{\dot\beta}\}\,. \tag{3.109}$$

Note that the two contributions involving a susy-covariant derivative of the matrix $U^\dagger$ cancel out. Using the constraints (3.107) as well as the identity (A.15), the last expression can be rewritten as

$$\{\mathcal{D}'_{\alpha a},\mathcal{D}'_{\beta b}\}=-\,2\varepsilon_{\alpha\beta}\det(x^-)(U^\dagger)_{ac}(U^\dagger)_{bd}\bar\Phi^{cd}\,. \tag{3.110}$$

Obviously, the constraint is invariant if under inversion the superscalars transforms as

$$\Phi_{ab}\to\Phi'_{ab}=\det(x^-)(U^\dagger)_{ac}(U^\dagger)_{bd}\bar\Phi^{cd}\,. \tag{3.111}$$

Let us now check whether this transformation is compatible with the second constraint of (3.107). Evaluating the anticommutator of two transformed susy- and gauge-covariant derivatives $\bar{\mathcal{D}}'^a{}_{\dot\alpha}$ and inserting the constraint yields

$$\{\bar{\mathcal{D}}'^a{}_{\dot\alpha},\bar{\mathcal{D}}'^b{}_{\dot\beta}\} = -\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{abgk}\Big(\tfrac{1}{2}\varepsilon_{gkef}\det(x^+)U^{ce}U^{df}\Phi_{cd}\Big)\,, \tag{3.112}$$

which implies

$$\Phi_{ab} \to \Phi'_{ab} = \tfrac{1}{2}\varepsilon_{abef}\det(x^+)U^{ce}U^{df}\Phi_{cd}\,. \tag{3.113}$$

In order to show that this is equivalent to (3.111), we use the following identity, which is valid for invertible 4×4 matrices T:[4]

$$T^{[c|e}T^{|d]f} = \tfrac{1}{4}\varepsilon^{gkef}\varepsilon^{mncd}\det(T)T^{-1}_{gm}T^{-1}_{kn}\,. \tag{3.114}$$

Using this identity as well as equation (3.103), the equivalence between (3.111) and (3.113) is easily established. Finally, we need to verify that an inversion also leaves invariant the last constraint of (3.107). However, as this works out straightforwardly we will not discuss it here in detail.

Determining the propagator. Before we discuss the derivation of the propagator of the superscalars, let us again consider a simple example to demonstrate our strategy. In section 2.1.1, we have seen that conformal symmetry implies that the correlator of n scalar primary fields satisfies the following identity

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = \left|\frac{\partial x'_1}{\partial x_1}\right|^{1/4}\cdots\left|\frac{\partial x'_n}{\partial x_n}\right|^{1/4}\langle\phi(x'_1)\ldots\phi(x'_n)\rangle\,, \tag{3.115}$$

where $|\partial x'/\partial x|$ denotes the Jacobian of the transformation. Here, we have set $d=4$ and $\Delta=1$ as we are interested in the free two-point function, i.e. the propagator. Specializing the above equation to the case of translations yields

$$\langle\phi(x)\phi(y)\rangle = \langle\phi(x-y)\phi(0)\rangle\,, \tag{3.116}$$

simply stating that the propagator does only depend on the difference between $x-y$. To fix the functional form of the correlator $\langle\phi(x-y)\phi(0)\rangle$, we now study the implications of inversion symmetry. The Jacobian of this transformation is given by $|\partial x'/\partial x| = 1/(x^2)^4$. Using this in equation (3.115) yields

$$\langle\phi(x-y)\phi(0)\rangle = \frac{1}{(x-y)^2}\lim_{z\to\infty} z^2\langle\phi((x-y)')\phi(z)\rangle\,. \tag{3.117}$$

Importantly, this limit does not depend on coordinates and just yields the normalization of the propagator. Thus, we find the unsurprising result that

$$\langle\phi(x)\phi(y)\rangle = -\frac{g^2}{4\pi^2}\frac{1}{(x-y)^2}\,. \tag{3.118}$$

[4] This formula can be shown to hold true by using $(T^{-1})_{gm} = \mathrm{adj}(T)_{gm}/\det(T)$ on the right-hand side of the equation with $\det(T) = \frac{1}{4!}\varepsilon_{i_1i_2i_3i_4}\varepsilon_{j_1j_2j_3j_4}T^{j_1i_1}T^{j_2i_2}T^{j_3i_3}T^{j_4i_4}$ and $\mathrm{adj}(T)_{gm} = \frac{1}{3!}\varepsilon_{gi_1i_2i_3}\varepsilon_{mj_1j_2j_3}T^{j_1i_1}T^{j_2i_2}T^{j_3i_3}$.

We will now apply the same logic to derive the propagator of the superscalars. In the first step, we use the supertranslation invariance to place the second superscalar at the origin, i.e.

$$\begin{aligned}\langle\bar{\Phi}^{ab}(x_1,\theta_1,\bar{\theta}_1)\Phi_{cd}(x_2,\theta_2,\bar{\theta}_2)\rangle &= \langle\bar{\Phi}^{ab}(x_{12},\theta_{12},\bar{\theta}_{12})\Phi_{cd}(0,0,0)\rangle \\ &= \langle\bar{\Phi}^{ab}(x_{12},\theta_{12},\bar{\theta}_{12})\phi_{cd}(0)\rangle\,, \end{aligned} \tag{3.119}$$

where x_{12}, θ_{12} and $\bar{\theta}_{12}$ are the translated superspace coordinates as defined in (3.65), i.e.

$$x_{12} = x_1 - x_2 + 2\bar{\theta}_2\theta_1 - 2\bar{\theta}_1\theta_2\,, \qquad \theta_{12} = \theta_1 - \theta_2\,, \qquad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2\,. \tag{3.120}$$

Note that at the origin of superspace, the superscalar reduces to its bottom component, which is the ordinary scalar field of $\mathcal{N} = 4$ SYM. In the second step, we need to take into account that the superscalars have a non-trivial $\mathfrak{su}(4)$ index structure. The transformation of the indices can be incorporated by replacing the Jacobian in equation (3.115) by the appropriate transformation matrix which can be read off from equation (3.111), see also section 2.1.1. Using this more general version of the conformal Ward identity, we can express the shifted correlator (3.119) in terms of the same correlator evaluated at the inverted points

$$\begin{aligned}\langle\bar{\Phi}^{ab}(1)\Phi_{cd}(2)\rangle =& \frac{\left(M - 4\theta_{12}x_{12}^{-+,-1}\bar{\theta}_{12}M\right)^{ae}\left(M - 4\theta_{12}x_{12}^{-+,-1}\bar{\theta}_{12}M\right)^{bf}}{(x_{12}^{-+})^2} \\ &\times M^{-1}_{kc}M^{-1}_{jd}\lim_{z\to\infty} z^2\langle\Phi_{ef}((1-2)')\bar{\phi}^{kj}(z)\rangle\,. \end{aligned} \tag{3.121}$$

As in the bosonic example, the limit does not depend on coordinates and evaluates to

$$\lim_{z\to\infty} z^2\langle\Phi_{ef}((1-2)')\bar{\phi}^{kj}(z)\rangle = -\tfrac{g^2}{\pi^2}\,\delta^k_{[e}\delta^j_{f]}\,. \tag{3.122}$$

Plugging this back into (3.121) yields the desired propagator

$$\langle\bar{\Phi}^{ab}(1)\Phi_{cd}(2)\rangle = -\frac{g^2}{\pi^2}\frac{\left(1 - 4\theta_{12}x_{12}^{-+,-1}\bar{\theta}_{12}\right)^a{}_{[c}\left(1 - 4\theta_{12}x_{12}^{-+,-1}\bar{\theta}_{12}\right)^b{}_{d]}}{(x_{12}^{-+})^2}\,. \tag{3.123}$$

Note that here we have again dropped the trivial color dependence of the propagator. We will stick to this convention throughout this chapter. By interchanging the points one and two and using that $\theta_{12} = -\theta_{21}$, $\bar{\theta}_{12} = -\bar{\theta}_{21}$ as well as $x_{12}^{+-} = -x_{21}^{-+}$, we obtain

$$\langle\Phi_{ab}(1)\bar{\Phi}^{cd}(2)\rangle = -\frac{g^2}{\pi^2}\frac{\left(1 + 4\theta_{12}x_{12}^{+-,-1}\bar{\theta}_{12}\right)^c{}_{[a}\left(1 + 4\theta_{12}x_{12}^{+-,-1}\bar{\theta}_{12}\right)^d{}_{b]}}{(x_{12}^{+-})^2}\,. \tag{3.124}$$

This expression can be shown to be compatible with the reality constraint $\bar{\Phi}^{ab} = \frac{1}{2}\varepsilon^{abcd}\Phi_{cd}$ by using the analogue of equation (3.114) with one index up and one index down.

3.2.2.2. The Field Strength Propagator

Let us continue by computing the free two-point function of the field strength two-form $\mathcal{F}$. In principle, we could compute this correlator by employing the same logic as used above. However, since the situation for the field strength is more complicated due to its non-scalar nature, we shall use a different approach, namely we express it in terms of the scalar propagator and its derivatives. To see how this works, we recall the following identity from section 3.2.1,

$$\begin{aligned}\mathcal{F} = \quad &(\mathrm{d}\theta\varepsilon\mathrm{d}\theta^{\intercal})^{ab}\Phi_{ab} + \tfrac{1}{6}(e\varepsilon\mathrm{d}\theta^{\intercal})^{\dot\alpha a}[\bar{\mathcal{D}}^{b}{}_{\dot\alpha}, \Phi_{ab}] + \tfrac{1}{192}(e\varepsilon e^{\intercal})^{\dot\alpha\dot\beta}\{\bar{\mathcal{D}}^{a}{}_{\dot\alpha}, [\bar{\mathcal{D}}^{b}{}_{\dot\beta}, \Phi_{ab}]\} \\ &+ (\mathrm{d}\bar\theta^{\intercal}\varepsilon\mathrm{d}\bar\theta)_{ab}\bar\Phi^{ab} + \tfrac{1}{6}(e^{\intercal}\varepsilon\mathrm{d}\bar\theta)^{\alpha}{}_{a}[\mathcal{D}_{\alpha b}, \bar\Phi^{ab}] + \tfrac{1}{192}(e^{\intercal}\varepsilon e)^{\alpha\beta}\{\mathcal{D}_{\alpha a}, [\mathcal{D}_{\beta b}, \bar\Phi^{ab}]\}\,, \quad (3.125)\end{aligned}$$

which relates the field strength two-form to the superscalar field and its gauge-covariant derivatives. Note that for computing the field strength propagator we only need to keep terms which are linear in the fields as all the non-linear terms do not contribute at this order in perturbation theory. Thus, we can safely replace all the gauge-covariant derivatives in equation (3.85) by susy-covariant derivatives. This allows us to compute the $\langle\mathcal{F}(1)\mathcal{F}(2)\rangle$ propagator by differentiating the scalar correlator (3.123). For later convenience, we split the linearized field strength into its chiral and antichiral components $\mathcal{F}_{\mathrm{lin}} = \mathcal{F}^{+} + \mathcal{F}^{-}$. These take the following form

$$\begin{aligned}\mathcal{F}^{-} &= \Big[(\mathrm{d}\theta\varepsilon\mathrm{d}\theta^{\intercal})^{ab} + \tfrac{1}{6}(e\varepsilon\mathrm{d}\theta^{\intercal})^{\dot\alpha a}\bar{D}^{b}{}_{\dot\alpha} + \tfrac{1}{192}(e\varepsilon e^{\intercal})^{\dot\alpha\dot\beta}\bar{D}^{a}{}_{\dot\alpha}\bar{D}^{b}{}_{\dot\beta}\Big]\Phi_{ab}\,, \\ \mathcal{F}^{+} &= \Big[(\mathrm{d}\bar\theta^{\intercal}\varepsilon\mathrm{d}\bar\theta)_{ab} + \tfrac{1}{6}(e^{\intercal}\varepsilon\mathrm{d}\bar\theta)^{\alpha}{}_{a}D_{\alpha b} + \tfrac{1}{192}(e^{\intercal}\varepsilon e)^{\alpha\beta}D_{\alpha a}D_{\beta b}\Big]\bar\Phi^{ab}. \qquad (3.126)\end{aligned}$$

Let us start by computing the basic building block for computing the whole field strength correlator, namely the two-point function of the (anti)chiral component of the field strength and the superscalar. To keep equations as short as possible, we introduce the abbreviations

$$C_{12}{}^{a}{}_{b} := (1 - 4\theta_{12}x_{12}^{-+,-1}\bar\theta_{12})^{a}{}_{b}\,, \qquad H_{12\,\alpha a} := (x_{12}^{-+,-1}\bar\theta_{12})_{\alpha a}\,. \qquad (3.127)$$

Using the result (3.123) as well as the identity

$$D_{1\alpha a}C_{12}{}^{b}{}_{c} = -4\delta^{b}_{a}H_{12\,\alpha c}\,, \qquad (3.128)$$

we find after some manipulations the following expression for the $\langle\mathcal{F}^{+}(1)\Phi_{cd}(2)\rangle$ propagator:

$$\Big\langle\mathcal{F}^{+}(1)\Phi_{cd}(2)\Big\rangle = -\frac{g^2}{\pi^2}\frac{\varepsilon_{\dot\alpha\dot\beta}}{(x_{12}^{-+})^2}(e_1H_{12} - \mathrm{d}\bar\theta_1C_{12})^{\dot\alpha}{}_{c}\,(e_1H_{12} - \mathrm{d}\bar\theta_1C_{12})^{\dot\beta}{}_{d}\,. \qquad (3.129)$$

To simplify this expression further, we note that

$$\mathrm{d}_1H_{12\,\alpha a} = (x_{12}^{-+,-1})_{\alpha\dot\beta}(\mathrm{d}\bar\theta_1C_{12} - e_1H_{12})^{\dot\beta}{}_{a}\,. \qquad (3.130)$$

The last identity in combination with (A.15) allows us to rewrite the correlator (3.129) as

$$\left\langle \mathcal{F}^+(1)\Phi_{cd}(2)\right\rangle = -\tfrac{g^2}{\pi^2}\,\mathrm{d}_1(x_{12}^{-+,-1}\bar\theta_{12})_{\gamma c}\,\varepsilon^{\gamma\delta}\,\mathrm{d}_1(x_{12}^{-+,-1}\bar\theta_{12})_{\delta d}$$
$$= -\tfrac{g^2}{\pi^2}\,(\mathrm{d}_1 H_{12}^{\mathsf{T}}\,\varepsilon\,\mathrm{d}_1 H_{12})_{cd}\,. \tag{3.131}$$

Note that in what follows, we will always be aiming at rewriting the correlators in a form like (3.131), which makes the exactness of the wedged one-forms manifest. This will turn out to be useful when proving the Yangian invariance of the one-loop VEV of the super Maldacena–Wilson loop.

Next, we compute the mixed chiral two-point function $\langle\mathcal{F}^+(1)\mathcal{F}^-(2)\rangle$. For this, we need to evaluate the expression

$$\left\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\right\rangle = \Big[(\mathrm{d}\theta_2\varepsilon\mathrm{d}\theta_2^{\mathsf{T}})^{ab} + \tfrac{1}{6}(e_2\varepsilon\mathrm{d}\theta_2^{\mathsf{T}})^{\dot\alpha a}\bar D^b_{2\,\dot\alpha} + \tfrac{1}{192}(e_2\varepsilon e_2^{\mathsf{T}})^{\dot\alpha\dot\beta}\bar D^a_{2\,\dot\alpha}\bar D^b_{2\,\dot\beta}\Big]$$
$$\times\left\langle \mathcal{F}^+(1)\Phi_{ab}(2)\right\rangle. \tag{3.132}$$

Inserting the expression (3.131) and applying the susy-covariant derivatives, which act on H_{12} as

$$\bar D^a_{2\,\dot\alpha} H_{12\,\beta b} = -\delta^a_b(x_{12}^{-+,-1})_{\beta\dot\alpha}\,, \tag{3.133}$$

leads after some rearrangements to

$$\left\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\right\rangle = -\frac{g^2}{16\pi^2}\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta}(\mathrm{d}_1 x_{12}^{-+,-1}e_2 + 4\mathrm{d}_1 H_{12}\mathrm{d}\theta_2)_\gamma{}^\alpha$$
$$\times(\mathrm{d}_1 x_{12}^{-+,-1}e_2 + 4\mathrm{d}_1 H_{12}\mathrm{d}\theta_2)_\delta{}^\beta. \tag{3.134}$$

To rewrite the former expression in a more compact fashion, we use the identity

$$\mathrm{d}_1(x_{12}^{-+,-1}\mathrm{d}_2 x_{12}^{-+})_\gamma{}^\alpha = -(\mathrm{d}_1 x_{12}^{-+,-1}e_2 + 4\mathrm{d}_1 H_{12}\mathrm{d}\theta_2)_\gamma{}^\alpha\,, \tag{3.135}$$

which can easily be proven by direct computation. For the color-stripped mixed correlator, we thus obtain the following final result:

$$\left\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\right\rangle = -\frac{g^2}{16\pi^2}\,\mathrm{d}_1(x_{12}^{-+,-1}\mathrm{d}_2 x_{12}^{-+}\varepsilon)_{\gamma\beta}\,\mathrm{d}_1(\varepsilon x_{12}^{-+,-1}\mathrm{d}_2 x_{12}^{-+})^{\gamma\beta}\,. \tag{3.136}$$

Last but not least, let us turn to the computation of the chiral - chiral correlator $\langle\mathcal{F}^+(1)\mathcal{F}^+(2)\rangle$. As before, our goal is to rewrite this two-point function in a nice and compact fashion as exterior derivatives acting on some coordinate expression. For the correlator of homogeneous chirality, such a rewriting requires quite some algebraic effort. However, for the sake of readability we will not provide too much detail on this and instead only present the important steps one has to take in order to reach the desired goal. To get started, we need to evaluate the following expression

$$\left\langle \mathcal{F}^+(1)\mathcal{F}^+(2)\right\rangle = \Big[(\mathrm{d}\bar\theta_2^{\mathsf{T}}\varepsilon\mathrm{d}\bar\theta_2)_{cd} + \tfrac{1}{6}(e_2^{\mathsf{T}}\varepsilon\mathrm{d}\bar\theta_2)^\alpha{}_c D_{2\alpha d} + \tfrac{1}{192}(e_2^{\mathsf{T}}\varepsilon e_2)^{\alpha\beta}D_{2\alpha c}D_{2\beta d}\Big]$$
$$\times\left\langle \mathcal{F}^+(1)\bar\Phi^{cd}(2)\right\rangle. \tag{3.137}$$

For the action of the susy-covariant derivative $D_{2\alpha a}$ on the building block H_{12} of the chiral - scalar correlator, one finds

$$D_{2\alpha a}H_{12\,\gamma c}=-4H_{12\,\gamma a}H_{12\,\alpha c}\,. \tag{3.138}$$

Using this identity as well as the decomposition rule (A.12), it can be shown that equation (3.137) can be written as

$$\Big\langle\mathcal{F}^+(1)\mathcal{F}^+(2)\Big\rangle=\tfrac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{cdef}(e_2H_{12}-\mathrm{d}\bar\theta_2)^{\dot\alpha}{}_c(e_2H_{12}-\mathrm{d}\bar\theta_2)^{\dot\beta}{}_d\Big\langle\mathcal{F}^+(1)\Phi_{ef}(2)\Big\rangle\,. \tag{3.139}$$

It remains to express the first two factors as exterior derivatives of some function depending on the supertranslation-invariant superspace intervals. To achieve this, we note that from equation (3.130) it follows that

$$\mathrm{d}_2H_{21\,\alpha a}=(x_{21}^{-+,-1})_{\alpha\dot\beta}(\mathrm{d}\bar\theta_2C_{21}-e_2H_{21})^{\dot\beta}{}_a\,. \tag{3.140}$$

Multiplying this equation by x_{12}^{+-} from the left and C_{12} from the right yields

$$(x_{12}^{+-})^{\dot\alpha\alpha}\mathrm{d}_2H_{21\,\alpha b}C_{12}{}^b{}_a=(e_2H_{12}-\mathrm{d}\bar\theta_2)^{\dot\alpha}{}_a\,, \tag{3.141}$$

where we have used that

$$C_{21}=1+4\theta_{12}x_{12}^{+-,-1}\bar\theta_{12}=C_{12}^{-1}\,. \qquad\qquad H_{21}C_{12}=H_{12}\,. \tag{3.142}$$

The equations (3.142) can easily be shown to hold true by exploiting the relation $4\bar\theta_{12}\theta_{12}=x_{12}^{-+}-x_{12}^{+-}$. By combining equation (3.141) with equation (3.139), we obtain our final expression for the chiral - chiral correlator which reads

$$\Big\langle\mathcal{F}^+(1)\mathcal{F}^+(2)\Big\rangle=-\frac{g^2}{2\pi^2}\,\Xi^{abcd}(1,2)(\mathrm{d}_2H_{21}^{\intercal}\,\varepsilon\,\mathrm{d}_2H_{21})_{ab}(\mathrm{d}_1H_{12}^{\intercal}\,\varepsilon\,\mathrm{d}_1H_{12})_{cd}\,, \tag{3.143}$$

where

$$\Xi^{abcd}(1,2)=(x_{12}^{+-})^2\varepsilon^{efcd}C_{12}{}^a{}_eC_{12}{}^b{}_f\,. \tag{3.144}$$

Note that the $\langle\mathcal{F}^-(1)\mathcal{F}^-(2)\rangle$ correlator as well as the $\langle\mathcal{F}^-(1)\mathcal{F}^+(2)\rangle$ correlator can be obtained from the ones derived above by conjugating equation (3.136) and equation (3.143), respectively. However, since we will not need these expressions, we do not state them here explicitly.

4. The Super Maldacena–Wilson Loop

The Maldacena–Wilson loop operator, as it was introduced above, couples only to the bosonic degrees of freedom of $\mathcal{N} = 4$ SYM theory. It is therefore clear that this operator is merely the bottom component of a manifestly supersymmetric loop operator. In reference [1], the super Maldacena–Wilson loop operator was constructed through quadratic order in an expansion in the anticommuting variables using supersymmetry as a guiding principle. In this chapter, we aim to complete this construction. However, instead of pushing further the analysis initiated in [1], we shall define the full operator by making use of the superspace formalism that has been introduced above. Our approach is inspired by the paper of Ooguri et al. [108], in which the authors study super Wilson loops from the ten-dimensional point of view. Having established the operator, we shall then turn to an investigation of its symmetries. Besides discussing the superconformal invariance of the operator, we shall prove that the four-dimensional super Maldacena–Wilson loop also enjoys local kappa symmetry. The latter will turn out to be intimately related to the 1/2 BPS property of the bosonic Maldacena–Wilson loop operator. The last section of this chapter is devoted to computing the one-loop VEV of the super Maldacena–Wilson loop and proving that it is finite. We will perform this calculation in Harnad–Shnider gauge and work consistently up to fourth order in an expansion in the Graßmann variables. Finally, we shall then verify that the one-loop expectation value is fully superconformal.

4.1. Definition

In section 2.3.2, we argued that the Maldacena–Wilson loop [27] can be derived by considering the dimensional reduction of a light-like Wilson loop in ten-dimensional $\mathcal{N} = 1$ SYM theory. Our strategy to derive the supersymmetric analog of the Maldacena–Wilson loop is to lift this discussion to superspace. For this reason, let us briefly recall the important steps of the bosonic derivation. The ordinary Wilson loop operator in $\mathcal{N} = 1$ SYM theory is given by

$$W(\gamma) = \frac{1}{N}\,\mathrm{tr}\,\mathrm{P}\exp\left(\oint_\gamma A\right) = \frac{1}{N}\,\mathrm{tr}\,\mathrm{P}\exp\left(\int \mathrm{d}\tau\, \dot{x}^{\hat{\mu}} A_{\hat{\mu}}(x)\right) . \tag{4.1}$$

Here, γ denotes a closed path in a ten-dimensional spacetime which satisfies a light-likeness constraint at every point along the loop

$$\dot{x}^{\hat{\mu}}\dot{x}_{\hat{\mu}} = 0\,. \tag{4.2}$$

The counterpart of the operator (4.1) in the four-dimensional $\mathcal{N} = 4$ SYM theory is the Maldacena–Wilson loop operator and can be obtained by dimensionally reducing the expression in equation (4.1). Decomposing the ten-dimensional gauge field in equation (4.1) as $A_{\hat{\mu}} \to (A_\mu, \phi_i)$ yields

$$W_M(\gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left(\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu(x) + \dot{y}^i \phi_i(x) \right) \right) , \tag{4.3}$$

where we have relabeled the coordinates characterizing the path in the internal space as $y^i(\tau)$. An important point to emphasize is that the coordinates are still subject to the light-likeness constraint (4.2), i.e.

$$\dot{x}^\mu \dot{x}_\mu + \dot{y}^i \dot{y}_i = 0 \, . \tag{4.4}$$

Only if the path is light-like in a ten-dimensional sense, the Wilson loop operator (4.3) falls into the class of Maldacena–Wilson loops. Typically, the Maldacena–Wilson loop is stated with the constraint explicitly solved

$$\dot{y}^i(\tau) = n^i(\tau) \sqrt{\dot{x}^2} \, , \tag{4.5}$$

where $n^i(\tau)$ is a unit six-vector that characterizes a path on S^5. However, one can as well work with the operator (4.3) and handle the constraint separately. In fact, in what follows we will frequently adopt this point of view.

Let us now supersymmetrize the above-given Wilson loop operators. With the superspace formalism at our disposal, we can immediately write down the supersymmetric analog of the plain bosonic Wilson loop operator in equation (4.1). Replacing the gauge connection by the superconnection yields

$$\mathcal{W}(\Gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left(\oint_\Gamma \mathcal{A} \right) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left(\int \mathrm{d}\tau \left(p^{\hat{\mu}} \mathcal{A}_{\hat{\mu}} + \dot{\theta}^{\hat{\alpha}} \mathcal{A}_{\hat{\alpha}} \right) \right) , \tag{4.6}$$

where $\Gamma = (x^{\hat{\mu}}(\tau), \theta^{\hat{\alpha}}(\tau))$ is a path in superspace and $p^{\hat{\mu}}$ denotes the pullback of the supertranslation-invariant one-form (3.8), i.e.

$$p^{\hat{\mu}} = \dot{x}^{\hat{\mu}} + \theta \Gamma^{\hat{\mu}} \dot{\theta} \, . \tag{4.7}$$

From now on, we shall refer to $p^{\hat{\mu}}$ as the supermomentum. The so-defined operator is not only gauge and reparametrization invariant but also supersymmetric by construction. More precisely, it is supersymmetric in the sense that the susy transformation of the fields can be rewritten as minus[1] the action on the superpath variables

$$[\mathbb{Q}_{\hat{\alpha}}, \mathcal{W}] = -\mathrm{Q}_{\hat{\alpha}} \mathcal{W} \, . \tag{4.8}$$

[1] The minus is necessary in order to ensure the consistency of the two algebras. For a more detailed discussion on this point see section 5.1.1.

Here, the generator $\mathbb{Q}_{\hat{\alpha}}$ acts on the fields (2.73), while $\mathrm{Q}_{\hat{\alpha}}$ acts on the superspace coordinates (3.2). From this equation, it follows immediately that the VEV $\langle \mathcal{W}(\Gamma) \rangle$ is annihilated by the supercharges $\mathrm{Q}_{\hat{\alpha}}$.[2]

We are now ready to define the full super Maldacena–Wilson loop operator. As in the purely bosonic case, we construct it using the technique of dimensional reduction. For this, we decompose the ten-dimensional gauge field in equation (4.6) as $\mathcal{A}_{\hat{\mu}} \to (\mathcal{A}_\mu, \Phi_i)$ and demand that the reduced fields are independent of the coordinates parametrizing the internal directions. Here, $\mathcal{A}_\mu$ represents the bosonic component of the four-dimensional superconnection, while by Φ_i we refer to the six superscalars of $\mathcal{N} = 4$ SYM theory, see also section 3.2.1. Performing this step yields

$$\mathcal{W}_M(\Gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left(\int \mathrm{d}\tau \left(p^\mu \mathcal{A}_\mu + \dot{\theta}^{\hat{\alpha}} \mathcal{A}_{\hat{\alpha}} + q^i \, \Phi_i \right) \right) , \tag{4.9}$$

where the contour integral is over a superpath Γ parametrized by $(x^\mu(\tau), \theta^{\hat{\alpha}}(\tau), q^i(\tau))$. In what follows, we shall refer to the superspace spanned by the set $(x^\mu, \theta^{\hat{\alpha}}, q^i)$ as extended non-chiral superspace. Importantly, the superpath Γ is subject to a generalized ten-dimensional light-likeness constraint, which explicitly reads

$$p^\mu p_\mu + q^i q_i = 0 \qquad \longleftrightarrow \qquad q^2 = q^i q^i = p^\mu p_\mu \, . \tag{4.10}$$

Recall that we are working with a metric which has mostly minus signature, therefore explaining the extra minus sign in the rightmost equation. As in the bosonic case, the constraint can be solved explicitly by setting

$$q^i = n^i(\tau) \sqrt{p^\mu p_\mu} \, , \tag{4.11}$$

with $n^i(\tau)$ being a unit six-vector as in equation (4.5). The above equations also make it clear that q^i cannot be interpreted as $\dot{y}^i + \theta \Gamma^i \dot{\theta}$ with y^i describing a purely bosonic path. The dimensional reduction procedure employed is thus of a slightly generalized nature. For completeness, let us also state the super Maldacena–Wilson loop using purely four-dimensional notation. It reads

$$\mathcal{W}_M(\Gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left(\int \mathrm{d}\tau \left(p^\mu \mathcal{A}_\mu + q^i \, \Phi_i \, + \dot{\theta}^{a\alpha} \mathcal{A}_{\alpha a} + \dot{\bar{\theta}}^{\dot{\alpha}}{}_a \, \mathcal{A}^a{}_{\dot{\alpha}} \right) \right) , \tag{4.12}$$

with p^μ being given by

$$p^\mu = \dot{x}^\mu + \theta \sigma^\mu \dot{\bar{\theta}} - \dot{\theta} \sigma^\mu \bar{\theta} \, . \tag{4.13}$$

As its ten-dimensional ancestor (4.6), this operator is gauge and reparametrization invariant as well as supersymmetric by construction. Inserting the component field expansions (3.44) furthermore shows that it limits to the bosonic Wilson loop operator (4.3) and exactly reproduces the result found in reference [1], where the present

[2]Here, we are assuming that the vacuum state is invariant under supersymmetry transformations.

author and his collaborators constructed the super Maldacena–Wilson loop through quadratic order in an expansion in the anticommuting variables. We thus believe that the operator (4.12) represents the correct supersymmetric extension of the ordinary Maldacena–Wilson loop operator and is most likely the object which, at strong coupling, is described by the minimal surface of the full-fledged $AdS_5 \times S^5$ superstring that ends on the curve Γ [43, 108] at the boundary.

Finally, let us comment on the existing literature. To the authors' knowledge, the supersymmetric generalization of the Maldacena–Wilson loop operator has first been considered in the appendix of reference [102]. In this paper, the authors gave a formal definition of the object but constructed it only through linear order in an expansion in the anticommuting coordinates. From the superspace point of view, super (Maldacena–)Wilson loops were discussed in reference [108]. Our derivation here is in fact very much inspired by the one presented in [108] but corrects a technical detail concerning the dimensional reduction of the ten-dimensional super Wilson loop operator (4.6). Although the operator has been (almost) available for more than fifteen years, it has never been studied in detail for arbitrary smooth contours. For polygonal contours which are null in four dimensions, the operator was investigated in [39, 40], but in this case the couplings to the scalars becomes irrelevant due to the light-likeness of the contour. In this thesis, we want to fill this gap and study the super Maldacena–Wilson loop for arbitrary smooth supercontours.

4.2. Symmetries

An important aspect of all physical observables is symmetry. Symmetries can often be used to simplify explicit computations drastically. For this reason, it is crucial to gain a complete understanding of them. The super Wilson loop, as introduced above, enjoys local kappa symmetry and is expected have global superconformal symmetry. In the first part of this section, we shall review the kappa symmetry of the ten-dimensional super Wilson loop [108] and show that it also holds for the four-dimensional super Maldacena–Wilson loop. In the second part, we turn to the question of superconformal symmetry. As it is unknown how the superconformal algebra acts on extended non-chiral superspace, we shall start our discussion on this topic by deriving a consistent representation acting on this space. Subsequently, we will demonstrate that superconformal symmetry is compatible with kappa symmetry. Verifying the superconformal invariance of the Wilson loop expectation value will be postponed to the next section.

4.2.1. Kappa Symmetry

The ten-dimensional super Wilson loop (4.6) enjoys local kappa symmetry. This statement has been proven in reference [108]. The kappa symmetry is in fact closely related to the 1/2 BPS property of the bosonic Wilson loop operator, see section 2.3.2. To see how this works in detail, let us first consider the simpler example of a local BPS operator.

BPS property of local operators. Let $O(0)$ be a local gauge-invariant operator that is located at the origin. To define the operator at an arbitrary point in spacetime, we can use the familiar translation property of local operators

$$O(x) = e^{x\mathbb{P}} O(0) e^{-x\mathbb{P}} . \tag{4.14}$$

Here, the generator $\mathbb{P}$ acts on the fields and has to be distinguished from the differential operator P, see section 2.1.1. If the underlying theory is furthermore supersymmetric, we can use a similar relation to construct the supersymmetric completion of the local operator $O(x)$. Explicitly, we define

$$\mathcal{O}(x,\theta) = e^{\theta\mathbb{Q}} O(x) e^{-\theta\mathbb{Q}} , \tag{4.15}$$

where $\mathbb{Q}$ are the supercharges acting on the fields. However, an important point to note is that unlike the generators of translations, the supercharges do not commute in general. In order to prove that the operator $\mathcal{O}(x,\theta)$ really represents the supersymmetric completion of the local operator $O(x)$, let us consider how it transforms under a supersymmetry transformation. Under a finite supersymmetry transformation, the operator $\mathcal{O}(x,\theta)$ gets mapped to

$$e^{\zeta\mathbb{Q}} \mathcal{O}(x,\theta) e^{-\zeta\mathbb{Q}} = \mathcal{O}(x - \zeta\Gamma\theta, \theta + \zeta) , \tag{4.16}$$

where ζ is a parameter that specifies the transformation. Expanding the above equation in ζ and looking at the terms being linear in ζ yields the following relation

$$[\mathbb{Q}_{\hat{\alpha}}, \mathcal{O}(x,\theta)] = -\mathrm{Q}_{\hat{\alpha}} \mathcal{O}(x,\theta) , \tag{4.17}$$

where $\mathrm{Q}_{\hat{\alpha}}$ is the representation of the supercharges that acts on the coordinates, see equation (3.2). The local operator $\mathcal{O}(x,\theta)$ is thus supersymmetric in the same sense in which the super Wilson loop operator is supersymmetric, cf. equation (4.8).

In complete analogy to the discussion in section 2.3.2, it may happen that that a certain linear combination of Poincaré supercharges annihilates the operator $O(x)$, i.e.

$$[\zeta\mathbb{Q}, O(x)] = 0 , \tag{4.18}$$

for some fixed spinor ζ. In this case, the supersymmetric version $\mathcal{O}(x,\theta)$ depends only on some θ's and therefore the Graßmann expansion is shorter. These are BPS operators. An interesting question to answer is what the BPS property of the bottom component $O(x)$ implies for the full-fledged operator $\mathcal{O}(x,\theta)$. To clarify this question, we consider the equation

$$\mathcal{O}(x,\theta) = e^{\theta\mathbb{Q}} e^{\zeta\mathbb{Q}} O(x) e^{-\zeta\mathbb{Q}} e^{-\theta\mathbb{Q}} . \tag{4.19}$$

Note that the inner conjugation does not harm the relation as $\zeta\mathbb{Q}$ commutes with $O(x)$ by assumption. Using the algebra relation of the supercharges $\{\mathbb{Q}_{\hat{\alpha}}, \mathbb{Q}_{\hat{\beta}}\} = 2\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\mathbb{P}_{\hat{\mu}}$, the above equation can be rewritten as

$$\mathcal{O}(x,\theta) = e^{(\theta+\zeta)\mathbb{Q}+(\zeta\Gamma^{\hat{\mu}}\theta)\mathbb{P}_{\hat{\mu}}} O(x) e^{-(\theta+\zeta)\mathbb{Q}-(\zeta\Gamma^{\hat{\mu}}\theta)\mathbb{P}_{\hat{\mu}}} = \mathcal{O}(x+\zeta\Gamma\theta, \theta+\zeta)\,. \tag{4.20}$$

Expanding this relation in ζ yields

$$\mathcal{O}(x,\theta) = \mathcal{O}(x,\theta) + (\zeta D)\mathcal{O}(x,\theta) + \mathcal{O}(\zeta^2)\,, \tag{4.21}$$

where $D_{\hat{\alpha}}$ is the susy-covariant derivative (3.3). The conclusion of this computation is that the BPS property of the bottom component implies that the operator ζD annihilates the supersymmetric operator $\mathcal{O}(x,\theta)$,

$$(\zeta D)\mathcal{O}(x,\theta) = 0\,. \tag{4.22}$$

This argument generalizes straightforwardly to the case of Wilson loops. This is what we will discuss next.

Kappa symmetry. In section 2.3.2, we showed that the bosonic Maldacena–Wilson loop operator is a 1/2 BPS object, meaning that it commutes locally with half of the supercharges. As a preparation for the discussion of kappa symmetry, let us briefly recall the important formulas using ten-dimensional notation. The ten-dimensional gauge connection can be written as $A(\tau) = \mathrm{d}\tau\, \dot{x}^{\hat{\mu}} A_{\hat{\mu}}$ and we assume the path to be null in ten dimensions, i.e. $\dot{x}^2 = 0$. Under a supersymmetry transformation (2.73), the gauge connection transforms as

$$[\zeta\mathbb{Q}, A] = -\mathrm{d}\tau(\zeta\dot{x}^{\hat{\mu}}\Gamma_{\hat{\mu}}\psi)\,. \tag{4.23}$$

If $\zeta = \kappa\dot{x}^{\hat{\mu}}\bar{\Gamma}_{\hat{\mu}}$, the above variation vanishes due to the light-likeness of the vector $\dot{x}^{\hat{\mu}}$ and the algebra relations fulfilled by the Pauli matrices, see appendix A.3. As $\dot{x}^{\hat{\mu}}$ is in general not constant, different supersymmetries are preserved at different points along the contour. Moreover, it is a 1/2 BPS condition because the matrix $\dot{x}^{\hat{\mu}}\bar{\Gamma}_{\hat{\mu}}$ has an eight-dimensional kernel and thus projects out eight of the sixteen degrees of freedom of the spinor κ.

In analogy to the case of local operators, the invariance of the bosonic Wilson loop operator under some local worldline supersymmetry lifts to the invariance of the super Wilson loop under some local superdiffeomorphisms. These diffeomorphisms are generated by the vector field

$$\kappa_{\hat{\alpha}} p^{\hat{\mu}} \bar{\Gamma}^{\hat{\alpha}\hat{\beta}}_{\hat{\mu}} D_{\hat{\beta}} \equiv \zeta(\kappa)^{\hat{\alpha}} D_{\hat{\alpha}}\,, \tag{4.24}$$

which is obtained by replacing $\zeta \to \kappa p^{\hat{\mu}}\bar{\Gamma}_{\hat{\mu}}$ and $\mathbb{Q} \to D$, where $p^{\hat{\mu}}$ is the supermomentum as introduced in equation (4.7). For the variation of the superspace coordinates, we find

$$\delta_\kappa p^{\hat{\mu}} = \zeta(\kappa)^{\hat{\alpha}} D_{\hat{\alpha}}\, p^{\hat{\mu}} = 2\zeta(\kappa)\Gamma^{\hat{\mu}}\dot{\theta}\,, \qquad \delta_\kappa \theta^{\hat{\alpha}} = \zeta(\kappa)^{\hat{\beta}} D_{\hat{\beta}}\, \theta^{\hat{\alpha}} = \zeta(\kappa)^{\hat{\alpha}}\,. \tag{4.25}$$

Let us emphasize again that while κ has sixteen degrees of freedom, the spinor ζ has only eight degrees of freedom because eight are projected out by the matrix $p^{\hat{\mu}}\bar{\Gamma}_{\hat{\mu}}$. Using the light-likeness condition on p, it is in fact easy to show that there is an equivalence relation $\kappa \sim \kappa + p \cdot \Gamma\xi$ on the spinors κ. Hence, the spinors κ and $\kappa + p \cdot \Gamma\xi$ produce the same kappa-symmetry transformation.

Let us now explicitly show that the ten-dimensional super Wilson loop (4.6) is invariant under kappa-symmetry transformations. Varying the superconnection with respect to the coordinates yields

$$\delta_\kappa \mathcal{A} = \mathrm{d}\tau \Big(p^{\mathcal{B}} \zeta(\kappa)^{\mathcal{A}} \mathcal{F}_{\mathcal{A}\mathcal{B}} + \Big[p^{\mathcal{A}} \mathcal{D}_{\mathcal{A}}, \zeta(\kappa)^{\mathcal{B}} \mathcal{A}_{\mathcal{B}} \Big] \Big) \,, \tag{4.26}$$

where $\mathcal{A} = (\hat{\mu}, \hat{\alpha})$ is a collective superspace index and we have introduced the abbreviation $p^{\mathcal{A}} = (p^{\hat{\mu}}, \dot{\theta}^{\hat{\alpha}})$. In the above formula, $\zeta(\kappa)^{\mathcal{A}}$ is the coefficient of the kappa-symmetry-generating vector field when written in the basis of susy-covariant derivatives $(\partial_{\hat{\mu}}, D_{\hat{\alpha}})$. From equation (4.24), we read off the following explicit form of $\zeta(\kappa)^{\mathcal{A}}$,

$$\zeta(\kappa)^{\mathcal{A}} = (0, \zeta(\kappa)^{\hat{\alpha}}) = (0, \delta_\kappa \theta^{\hat{\alpha}}) \,. \tag{4.27}$$

Note that the latter term in equation (4.26) represents an infinitesimal field-dependent gauge transformation and can therefore be neglected if δ_κ is applied to a gauge-invariant operator. For more details on this see section 5.1.1. For the variation of the ten-dimensional super Wilson loop operator, we thus find

$$\begin{aligned}
\delta_\kappa \mathcal{W}(\Gamma) &= \frac{1}{N} \operatorname{tr} \mathrm{P} \left\{ \exp\left(\oint_\Gamma \mathcal{A} \right) \oint_\Gamma \delta_\kappa \mathcal{A} \right\} \\
&= \frac{1}{N} \operatorname{tr} \mathrm{P} \left\{ \exp\left(\oint_\Gamma \mathcal{A} \right) \int \mathrm{d}\tau \Big(p^{\hat{\mu}}\, \delta_\kappa \theta^{\hat{\alpha}}\, \mathcal{F}_{\hat{\alpha}\hat{\mu}} + \dot{\theta}^{\hat{\beta}}\, \delta_\kappa \theta^{\hat{\alpha}}\, \mathcal{F}_{\hat{\alpha}\hat{\beta}} \Big) \right\} \,.
\end{aligned} \tag{4.28}$$

Fortunately, both terms are zero in ten-dimensional $\mathcal{N} = 1$ SYM theory. While the latter one is zero due to the constraint $\mathcal{F}_{\hat{\alpha}\hat{\beta}} = 0$ (3.19), the former one vanishes only if the contour satisfies a generalized light-likeness constraint, namely $p^{\hat{\mu}} p_{\hat{\mu}} = 0$. To see this, let us recall that the constraints allowed us to identify the mixed component of the field strength tensor with the fermionic superfield Ψ,

$$\mathcal{F}_{\hat{\mu}\hat{\alpha}} = \Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\beta}} \Psi^{\hat{\beta}} \,, \tag{4.29}$$

, cf. equation (3.23). Using this relation, the first term in equation (4.28) can be rewritten as

$$\delta_\kappa \theta^{\hat{\alpha}} p^{\hat{\mu}}\, \Gamma_{\hat{\mu}\,\hat{\alpha}\hat{\beta}} \Psi^{\hat{\beta}} = \Big(\kappa \bar{\Gamma}_{\hat{\nu}} \Gamma_{\hat{\mu}} \Psi \Big) p^{\hat{\nu}} p^{\hat{\mu}} \propto p^{\hat{\mu}} p_{\hat{\mu}} = 0 \,. \tag{4.30}$$

For a light-like contour $p^{\hat{\mu}}(\tau)$ this expression vanishes. This completes the proof that the ten-dimensional super Wilson loop is kappa symmetric. Note that a similar proof has been given in reference [108]. However, there the authors did not elaborate on the relation between kappa symmetry and the 1/2 BPS condition.

Let us now discuss the situation for the dimensionally-reduced super Maldacena–Wilson loop operator (4.9). The kappa-symmetry transformations of the variables p^μ and $\theta^{\hat\alpha}$ follow immediately from the ten-dimensional equations presented above. Explicitly, we find

$$\delta_\kappa \theta^{\hat\alpha} = \kappa_{\hat\beta}(p^\mu \bar\Gamma_\mu + q^i \bar\Gamma_i)^{\hat\beta\hat\alpha}\,, \qquad \delta_\kappa p^\mu = 2\delta_\kappa\theta\Gamma^\mu\dot\theta\,, \tag{4.31}$$

where we have decomposed the ten-dimensional supermomentum as $p^{\hat\mu} \to (p^\mu, q^i)$. However, how does the coordinate q^i transform? To answer this question, we observe that the proof of kappa symmetry goes through without further ado if the coordinate q^i transforms in the same way as the extra components of ten-dimensional supermomentum $p^{\hat\mu}$. Therefore, we demand that

$$\delta_\kappa q^i = 2\delta_\kappa\theta\Gamma^i\dot\theta\,. \tag{4.32}$$

This can be achieved by letting the S^5 vector n^i transform as

$$\delta_\kappa n^i = 2(\delta^i_j + n^i n_j)\frac{\delta_\kappa\theta\Gamma^j\dot\theta}{\sqrt{p^\mu p_\mu}}\,. \tag{4.33}$$

To prove this, we consider

$$\delta_\kappa q^i = \delta_\kappa n^i\sqrt{p^\mu p_\mu} + n^i\,\frac{\delta_\kappa p^\mu p_\mu}{\sqrt{p^\mu p_\mu}} = 2\delta_\kappa\theta\Gamma^i\dot\theta + \frac{2n^i}{\sqrt{p^\mu p_\mu}}\delta_\kappa\theta\Gamma^{\hat\mu}\dot\theta p_{\hat\mu}\,, \tag{4.34}$$

where $p_{\hat\mu}$ in the rightmost term is defined as $p_{\hat\mu} = (p_\mu, q_i)$. The last term in the above equation represents the variation of the ten-dimensional light-likeness constraint. By inserting the explicit expression for $\delta_\kappa\theta$, one can easily show that the light-likeness constraint is preserved by a kappa-symmetry transformation and therefore the extra term vanishes.

For completeness and later convenience, let us also state the transformation of the superspace coordinates using purely four-dimensional notation. Using the Pauli matrix conventions as laid out in appendix A.3, we obtain the following expressions for the transformations of the odd variables

$$\begin{aligned}\delta_\kappa\theta &= \bar\kappa p - \bar q\kappa^\intercal\varepsilon\,,\\ \delta_\kappa\bar\theta &= p\kappa - \varepsilon\bar\kappa^\intercal q\,,\end{aligned} \tag{4.35}$$

where $p = \dot x + 2\bar\theta\dot\theta - 2\dot{\bar\theta}\theta$. Here, we haven again taken $\kappa_{\alpha a}$ to be a 2×4 matrix and $\bar\kappa^a{}_{\dot\alpha}$ to be a 4×2 matrix. Also, $\bar q^{ab}$ is the vector q^i written in spinor form and $q_{ab} = \frac{1}{2}\varepsilon_{abcd}\bar q^{cd}$, see appendix A.2 for our conventions. The kappa transformations of the remaining quantities read

$$\begin{aligned}\delta_\kappa x &= 2(p\kappa - \varepsilon\bar\kappa^\intercal q)\theta - 2\bar\theta(\bar\kappa p - \bar q\kappa^\intercal\varepsilon)\,,\\ \delta_\kappa p &= 4(p\kappa - \varepsilon\bar\kappa^\intercal q)\dot\theta - 4\dot{\bar\theta}(\bar\kappa p - \bar q\kappa^\intercal\varepsilon)\,,\\ \delta_\kappa\bar q^{ab} &= -8(\dot\theta\varepsilon\delta_\kappa\theta^\intercal)^{[ab]} - 4\varepsilon^{abcd}(\dot{\bar\theta}^\intercal\varepsilon\delta_\kappa\bar\theta)_{cd}\,.\end{aligned} \tag{4.36}$$

This concludes our discussion on the kappa symmetry of the super Maldacena–Wilson loop operator.

Consistency. Consistency requires that kappa-symmetry transformations and loop reparametrizations from a closed algebra. Loop reparametrizations represent a closely related class of transformations which leave the super Wilson loop invariant. On the superspace coordinates, the infinitesimal reparametrization transformations act as

$$\delta_\sigma \theta(\tau) = \sigma \dot{\theta}(\tau)\,, \qquad \delta_\sigma x^{\hat{\mu}}(\tau) = \sigma \dot{x}^{\hat{\mu}}(\tau)\,. \tag{4.37}$$

Using these transformations as well as the formulas for kappa transformations, it is straightforward to verify that

$$\begin{aligned} [\delta_{\kappa_1}, \delta_{\kappa_2}] &= \delta_{\kappa(\kappa_1,\kappa_2)} + \delta_{\sigma(\kappa_1,\kappa_2)}\,, \\ [\delta_{\sigma_1}, \delta_{\sigma_2}] &= \delta_{\sigma(\sigma_1,\sigma_2)}\,, \\ [\delta_{\sigma_1}, \delta_{\kappa_2}] &= \delta_{\kappa(\sigma_1,\kappa_2)}\,, \end{aligned} \tag{4.38}$$

where

$$\begin{aligned} \kappa(\kappa_1,\kappa_2) &= 4(\dot{\theta}\kappa_2)\kappa_1 - 4(\dot{\theta}\kappa_1)\kappa_2 + 2(\kappa_1 \bar{\Gamma}_{\hat{\mu}} \kappa_2)(\Gamma^{\hat{\mu}}\dot{\theta})\,, \quad & \kappa(\sigma_1,\kappa_2) &= \dot{\sigma}_1 \kappa_2 - \sigma_1 \dot{\kappa}_2\,, \\ \sigma(\kappa_1,\kappa_2) &= -4(\kappa_1 p^{\hat{\mu}} \bar{\Gamma}_{\hat{\mu}} \kappa_2)\,, & \sigma(\sigma_1,\sigma_2) &= \sigma_2 \dot{\sigma}_1 - \sigma_1 \dot{\sigma}_2\,. \end{aligned} \tag{4.39}$$

The kappa-symmetry transformations thus form a consistent algebra. We can conceive of kappa-symmetry transformations and reparametrizations as transformations mapping points along some path to nearby points in superspace. Since these transformations close under the Lie bracket, they locally generate some manifold of dimension $1|8$ with one bosonic and eight fermionic directions. Any two paths within this manifold are related by kappa transformations and reparametrizations and the corresponding Wilson loops are equivalent.

4.2.2. Superconformal Symmetry in Extended Superspace

In chapter 3, we did not really touch upon the topic of a superspace representation of the superconformal algebra $\mathfrak{psu}(2,2|4)$. We postponed this discussion to the present chapter because ultimately we do not only need a representation on the superspace spanned by the coordinates $(x,\theta,\bar{\theta})$ but also on the extended superspace, which is parametrized by $(x,\theta,\bar{\theta},q)$. Constructing a representation of the superconformal algebra that acts on (extended) non-chiral superspace is neither a trivial task nor a simple one. In fact, reference [1] failed to do this correctly. The challenge lies in representing the conformal boosts, which do not simply follow from a dimensional reduction procedure as they are not contained in the symmetry algebra of ten-dimensional $\mathcal{N}=1$ SYM theory. Fortunately, we can use the conformal inversion (3.95) to construct these generators. Discussing this in detail will be the subject of the next paragraphs.

Conformal boosts in non-chiral superspace. Let us start by constructing a representation of the conformal boost generators S and $\bar{\mathrm{S}}$ that acts on the non-extended superspace parametrized by $(x,\theta,\bar{\theta})$. Such a representation has been obtained in [39] and we will follow their approach here. The key idea is to consider the conformal inversion instead of the more complicated conformal boosts. Under conformal inversion, the superspace coordinates get mapped to

$$\begin{aligned} \mathrm{I}[x^{\pm}] &= \varepsilon x^{\mp,\intercal,-1}\varepsilon\,, \\ \mathrm{I}[\theta] &= -M\bar{\theta}^{\intercal}x^{-,\intercal,-1}\varepsilon\,, \\ \mathrm{I}[\bar{\theta}] &= \varepsilon x^{+,\intercal,-1}\theta^{\intercal}M^{-1}\,, \end{aligned} \tag{4.40}$$

where M is some symmetric unitary matrix whose explicit form we will not need, see also our discussion in section 3.2.2.1. A convenient feature of the conformal group $\mathrm{SO}(2,4)$ is that the inversion relates the generator of translations and the generator of special conformal transformations, see section 2.1.1. Fortunately, similar relations also exist in the superconformal case. In fact, we can determine the transformations of the superspace coordinates under superconformal boosts by considering the following sequence of transformations:

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\,X = (\mathrm{I}\circ\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\circ\mathrm{I})[X]\,. \tag{4.41}$$

Here, $\delta_{\mathrm{Q},\bar{\mathrm{Q}}}$ denotes the variation generated by

$$\delta_{\mathrm{Q},\bar{\mathrm{Q}}} = \zeta^{a\alpha}\mathrm{Q}_{\alpha a} + \bar{\zeta}^{\dot{\alpha}}{}_{a}\bar{\mathrm{Q}}^{a}{}_{\dot{\alpha}}\,, \tag{4.42}$$

where Q and $\bar{\mathrm{Q}}$ are the Poincaré supercharges as given in equation (3.61). Under this variation, the coordinates transform as

$$\begin{aligned} &\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,\theta = -\zeta\,, && \delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,\bar{\theta} = -\bar{\zeta}\,, \\ &\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,x^{+} = 4\bar{\zeta}\theta\,, && \delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,x^{-} = -4\bar{\theta}\zeta\,, \\ &\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,x = 2\bar{\zeta}\theta - 2\bar{\theta}\zeta\,, && \delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,p = 0\,. \end{aligned} \tag{4.43}$$

Evaluating equation (4.41) is somewhat lengthy and we shall merely give the final result, which reads

$$\begin{aligned} &\delta_{\mathrm{S},\bar{\mathrm{S}}}\,\theta = \bar{\xi}x^{+} - 4\theta\xi\theta\,, && \delta_{\mathrm{S},\bar{\mathrm{S}}}\,\bar{\theta} = x^{-}\xi + 4\bar{\theta}\bar{\xi}\bar{\theta}\,, \\ &\delta_{\mathrm{S},\bar{\mathrm{S}}}\,x^{+} = -4x^{+}\xi\theta\,, && \delta_{\mathrm{S},\bar{\mathrm{S}}}\,x^{-} = 4\bar{\theta}\bar{\xi}x^{-}\,, \\ &\delta_{\mathrm{S},\bar{\mathrm{S}}}\,x = 2\bar{\theta}\bar{\xi}x^{-} - 2x^{+}\xi\theta\,, && \delta_{\mathrm{S},\bar{\mathrm{S}}}\,p = 4\bar{\theta}\bar{\xi}p - 4p\xi\theta\,, \end{aligned} \tag{4.44}$$

where we have identified the transformation parameters as

$$\xi := \varepsilon\zeta^{\intercal}M^{-1}\,, \qquad \bar{\xi} = M\bar{\zeta}^{\intercal}\varepsilon^{\intercal}\,. \tag{4.45}$$

Given the variation of the coordinates, it is a straightforward exercise to reconstruct the generators S and $\bar{\mathrm{S}}$. However, we shall postpone this task until we have determined the transformations of the coordinates q^{i}. Finally, let us note that given the Poincaré supercharges as well as the superconformal boost generators, all the other generators follow from algebra considerations. We shall present an exhaustive list of them momentarily.

4.2.2.1. Conformal boosts in extended superspace.

While the fields live in non-chiral superspace, the super Maldacena–Wilson loop is defined on extended non-chiral superspace. The extra coordinates enter, however, only explicitly through the couplings of the superscalars, cf. (4.12). Nevertheless, we need to know how the superconformal algebra acts on this bigger space. To find the transformations of the coordinates $\bar{q}$, we again study the conformal inversion. The key property that we will use to fix the transformation law of the coordinates $\bar{q}$ is that the Wilson loop operator must be invariant under a combined transformation of the fields and the coordinates. In section 3.2.2.1, we employed this logic to find the transformations of the gauge components. Here, we use it to find the transformation law of the coordinates $\bar{q}$.

We begin by demonstrating the above statement for the bosonic Maldacena–Wilson loop operator. As discussed in section 2.3.2, the scalar part of this operator is of the following form

$$\mathrm{d}\tau\sqrt{\dot{x}^2}\phi_i(x)n^i\,. \tag{4.46}$$

Under inversion, the dimension-one field $\phi_i(x)$ transforms as

$$\phi_i(x) \to \phi_i'(x') = x^2\phi_i(x)\,. \tag{4.47}$$

On the other hand, using $x^\mu \to x^\mu/x^2$, we find that the square root factor transforms as

$$\sqrt{\dot{x}^2} \to \frac{\sqrt{\dot{x}^2}}{x^2}\,. \tag{4.48}$$

Hence, the combination of terms in equation (4.46) stays invariant.

Let us now fix the inversion law of $\bar{q}$ by demanding that the same holds true for the super Maldacena–Wilson loop. Recall from section 3.2.2.1 that the superscalars transform under inversion as

$$\Phi \to \Phi' = \det(x^-)\, U^\dagger \bar{\Phi}\, U^{\dagger,\intercal}\,, \tag{4.49}$$

where

$$U = M - 4\theta x^{-,-1}\bar{\theta}M\,, \qquad U^\dagger = M^{-1} + 4M^{-1}\theta x^{+,-1}\bar{\theta}\,, \qquad UU^\dagger = 1\,. \tag{4.50}$$

If we want the super Wilson loop to be invariant under inversion, the coordinates $\bar{q}^{ab}$ must transform in such a way that it compensates the transformation of the fields Φ_{ab}. We obtain

$$\mathrm{I}[\bar{q}] = \det(x^-)^{-1}U^\intercal q\, U\,, \qquad \mathrm{I}[q] = \det(x^+)^{-1}U^\dagger \bar{q}\, U^{\dagger,\intercal}\,. \tag{4.51}$$

To find the action of the superconformal boosts, we proceed as described above, see equation (4.41). For the susy variation of the inverted coordinate, we obtain

$$\begin{aligned}(\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\circ \mathrm{I})[\bar{q}] =& -2(x^-)^{-4}\,\mathrm{tr}\Big(\varepsilon x^{-,\intercal}\varepsilon\bar{\theta}\zeta - \varepsilon\zeta^\intercal\bar{\theta}^\intercal\varepsilon x^-\Big)U^\intercal q U \\ & - 4(x^-)^{-2}\Big(M\bar{\zeta}^\intercal x^{-,\intercal,-1}\theta^\intercal + M\bar{\theta}^\intercal x^{-,\intercal,-1}\zeta^\intercal M^{-1}U^\intercal\Big)qU \\ & + 4(x^-)^{-2}U^\intercal q\Big(UM^{-1}\zeta x^{-,-1}\bar{\theta}M + \theta x^{-,-1}\bar{\zeta}M\Big)\,. \end{aligned} \tag{4.52}$$

Note that q is supertranslation invariant, i.e. $\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\,q = 0$. The final step consists of applying another inversion to the above equation. The following identities come in handy when performing this computation

$$\mathrm{I}[(x^-)^{-2}] = (x^+)^2\,, \qquad\qquad \mathrm{I}[U] = U^\intercal\,. \tag{4.53}$$

Using these, we obtain the following result for the superconformal variation of $\bar{q}$:

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\,\bar{q} = 4\bar{q}\Big(\xi^\intercal\theta^\intercal + \bar{\theta}^\intercal\bar{\xi}^\intercal\Big) - 4\Big(\theta\xi + \bar{\xi}\bar{\theta}\Big)\bar{q} - 4\mathrm{tr}\Big(\xi\theta\Big)\bar{q}\,. \tag{4.54}$$

The transformation law of q with lower indices can easily be obtained by complex conjugating the above equation

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\,q = 4q\Big(\bar{\xi}\bar{\theta} + \theta\xi\Big) - 4\Big(\bar{\theta}^\intercal\bar{\xi}^\intercal + \xi^\intercal\theta^\intercal\Big)q + 4\mathrm{tr}\Big(\bar{\theta}\bar{\xi}\Big)q\,. \tag{4.55}$$

Note that superconformal boosts preserve the duality relation $\bar{q}^{ab} = \frac{1}{2}\varepsilon^{abcd}q_{cd}$, which can be proven by using Schouten's identity in five indices.

4.2.2.2. Superspace representation of the superconformal algebra.

Given the variations of the coordinates under supertranslations and superconformal boosts, we can now construct a complete representation of the superconformal algebra. The first step consists of translating the variations (4.44) and (4.54) into differential operator expressions for S and $\bar{\mathrm{S}}$. The other generators can then be found by algebra considerations, see the commutation relations in appendix B. We refrain from presenting any intermediate steps and merely state the final result, which reads

$$\begin{aligned}
\mathrm{P}_{\alpha\dot{\alpha}} &= -\partial_{\alpha\dot{\alpha}}\,,\\
\mathrm{Q}_{\alpha a} &= \bar{\theta}^{\dot{\alpha}}{}_a\partial_{\alpha\dot{\alpha}} - \partial_{\alpha a}\,,\\
\bar{\mathrm{Q}}^a{}_{\dot{\alpha}} &= \theta^{a\alpha}\partial_{\alpha\dot{\alpha}} - \bar{\partial}^a{}_{\dot{\alpha}}\,,\\
\mathrm{L}^\alpha{}_\beta &= -x^{\dot{\gamma}\alpha}\partial_{\beta\dot{\gamma}} - 2\theta^{c\alpha}\partial_{\beta c} + \tfrac{1}{2}\delta^\alpha_\beta\Big(x^{\dot{\gamma}\gamma}\partial_{\gamma\dot{\gamma}} + 2\theta^{c\gamma}\partial_{\gamma c}\Big)\,,\\
\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}} &= -x^{\dot{\alpha}\gamma}\partial_{\gamma\dot{\beta}} - 2\bar{\theta}^{\dot{\alpha}}{}_c\bar{\partial}^c{}_{\dot{\beta}} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\Big(x^{\dot{\gamma}\gamma}\partial_{\gamma\dot{\gamma}} + 2\bar{\theta}^{\dot{\gamma}}{}_c\bar{\partial}^c{}_{\dot{\gamma}}\Big)\,,\\
\mathrm{D} &= -\tfrac{1}{2}\Big(x^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\alpha}} + \theta^{a\alpha}\partial_{\alpha a} + \bar{\theta}^{\dot{\alpha}}{}_a\bar{\partial}^a{}_{\dot{\alpha}} + \tfrac{1}{2}q^{ab}\partial_{ab}\Big)\,,\\
\mathrm{S}^{a\alpha} &= -(x^+)^{\dot{\delta}\alpha}\theta^{a\delta}\partial_{\delta\dot{\delta}} + 4\theta^{c\alpha}\theta^{a\gamma}\partial_{\gamma c} + (x^-)^{\dot{\gamma}\alpha}\bar{\partial}^a{}_{\dot{\gamma}} + 2\theta^{c\alpha}q^{ad}\partial_{cd} - \theta^{a\alpha}q^{cd}\partial_{cd}\,,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathrm{S}}^{\dot\alpha}{}_a &= -(x^-)^{\dot\alpha\gamma}\bar\theta^{\dot\gamma}{}_a\partial_{\gamma\dot\gamma} - 4\bar\theta^{\dot\gamma}{}_a\bar\theta^{\dot\alpha}{}_c\bar\partial^c{}_{\dot\gamma} + (x^+)^{\dot\alpha\gamma}\partial_{\gamma a} - 2\bar\theta^{\dot\alpha}{}_b q^{be}\partial_{ae}\,,\\
\mathrm{K}^{\dot\alpha\alpha} &= 2\theta^{c\alpha}(x^+)^{\dot\alpha\gamma}\partial_{\gamma c} + 2(x^-)^{\dot\gamma\alpha}\bar\theta^{\dot\alpha}{}_c\bar\partial^c{}_{\dot\gamma} + \tfrac{1}{2}(x^+)^{\dot\gamma\alpha}(x^+)^{\dot\alpha\gamma}\partial_{\gamma\dot\gamma}\\
&\quad + \tfrac{1}{2}(x^-)^{\dot\gamma\alpha}(x^-)^{\dot\alpha\gamma}\partial_{\gamma\dot\gamma} - 4\theta^{c\alpha}\bar\theta^{\dot\alpha}{}_a q^{ad}\partial_{cd} + \tfrac{1}{2}(x^+)^{\dot\alpha\alpha}q^{cd}\partial_{cd}\,,\\
\mathrm{R}^a{}_b &= 2\bar\theta^{\dot\gamma}{}_b\bar\partial^a{}_{\dot\gamma} - 2\theta^{a\gamma}\partial_{\gamma b} - q^{ad}\partial_{bd} - \tfrac{1}{4}\delta^a_b\Big(2\bar\theta^{\dot\gamma}{}_c\bar\partial^c{}_{\dot\gamma} - 2\theta^{c\gamma}\partial_{\gamma c} - q^{cd}\partial_{cd}\Big)\,.
\end{aligned} \tag{4.56}$$

Here, ∂_{cd} denotes the derivative[3] with respect to the coordinates $\bar{q}^{cd}$, which enter the super Maldacena–Wilson loop via the couplings to the superscalars. This derivative acts on $\bar{q}^{ab}$ as

$$\partial_{cd}\bar{q}^{ab} = 2\Big(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c\Big)\,. \tag{4.57}$$

A complete list of all the commutation relations can be found in appendix B. To obtain a representation acting on the smaller non-chiral superspace, one can simply set to zero all the q-terms in the generators (4.56). The closure of the algebra is not affected by this modification.

Finally, let us extend the superconformal algebra by the hypercharge generator B as well as the central charge C. Together with the generators of $\mathfrak{psu}(2,2|4)$ these generators span the algebra $\mathfrak{u}(2,2|4)$. We will need this extension later on because we want to discuss the level-one hypercharge symmetry [109, 110] of super Maldacena–Wilson loops. On our superspace, the generators B and C are represented by

$$\mathrm{B} = \tfrac{1}{2}\theta^{a\alpha}\partial_{\alpha a} - \tfrac{1}{2}\bar\theta^{\dot\alpha}{}_a\bar\partial^a{}_{\dot\alpha}\,, \qquad \mathrm{C} = 0\,. \tag{4.58}$$

This concludes our discussion on the superspace representation of the superconformal algebra.

4.2.3. Consistency

In this section, we shall perform two consistency checks. The first consists of proving that the generalized ten-dimensional light-likeness constraint is preserved by all superconformal transformations. The second one concerns the algebra of kappa transformations and superconformal transformations. Consistency requires that the commutator of a kappa transformation and a superconformal transformation closes onto another kappa transformation. Verifying this property will the subject of the second paragraph.

The constraint and superconformal symmetry. The question that we want to address in this paragraph is whether the ten-dimensional light-likeness condition

$$p^\mu p_\mu + q^i q_i = 0 \qquad \leftrightarrow \qquad \mathrm{tr}(\bar{q}q) - 2\mathrm{tr}(\varepsilon p\varepsilon p^{\mathsf{T}}) = 0\,, \tag{4.59}$$

[3] As usual, we define the derivative with spinor indices as $\partial_{cd} = \Sigma^i_{cd}\partial_i$ with Σ^i_{cd} being the six-dimensional Pauli matrices, see appendix A.2 for our conventions.

is preserved by superconformal transformations. Obviously, it is preserved by Lorentz transformations and R-symmetry transformations because they descend from ten-dimensional $\mathfrak{so}(1,9)$ Lorentz transformations. Furthermore, (4.59) is clearly susy invariant as

$$\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\, p = \delta_{\mathrm{Q},\bar{\mathrm{Q}}}\, q = 0\,. \tag{4.60}$$

However, for the remaining elements of $\mathfrak{psu}(2,2|4)$, namely for the generators of conformal and superconformal boosts, the situation is less clear. Fortunately, it suffices to analyze the situation for the conformal inversion as the invariance of the constraint under all types of boosts follows if (4.59) is preserved by I. Recall that inverting the coordinates p, q and $\bar{q}$ yields

$$\begin{aligned}
\mathrm{I}[p] &= -\varepsilon x^{+,\intercal,-1} p^{\intercal} x^{-,\intercal,-1}\varepsilon\,,\\
\mathrm{I}[\bar{q}] &= \det(x^{-})^{-1} U^{\intercal} q U\,,\\
\mathrm{I}[q] &= \det(x^{+})^{-1} U^{\dagger}\bar{q} U^{\dagger,\intercal}\,.
\end{aligned} \tag{4.61}$$

Using these formulas as well as the identity (A.15), one derives the following equation

$$\mathrm{I}[\mathrm{tr}(\bar{q}q) - 2\mathrm{tr}(\varepsilon p \varepsilon p^{\intercal})] = \frac{1}{(x^{-})^2(x^{+})^2}\Big(\mathrm{tr}(\bar{q}q) - 2\mathrm{tr}(\varepsilon p\varepsilon p^{\intercal})\Big)\,, \tag{4.62}$$

from which we conclude that the light-likeness condition is preserved by inversion. Since (4.59) is also (super)translation invariant and

$$\delta_{\mathrm{K}} = \mathrm{I}\circ\delta_{\mathrm{P}}\circ\mathrm{I}\,, \qquad \delta_{\mathrm{S},\bar{\mathrm{S}}} = \mathrm{I}\circ\delta_{\mathrm{Q},\bar{\mathrm{Q}}}\circ\mathrm{I}\,, \tag{4.63}$$

the constraint is also preserved by conformal and superconformal boosts.

The algebra of kappa transformations and superconformal transformations. In this paragraph, we investigate the algebraic relation between superconformal transformations and kappa transformations. We do this using the language of variations, being defined as

$$\begin{aligned}
&\delta_{\mathrm{P}} = \tfrac{1}{2}a^{\dot{\alpha}\alpha}\mathrm{P}_{\alpha\dot{\alpha}}\,, && \delta_{\mathrm{L},\bar{\mathrm{L}}} = \omega_{\alpha}{}^{\beta}\mathrm{L}_{\beta}{}^{\alpha} + \bar{\omega}^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}\,, && \delta_{\mathrm{D}} = \lambda\mathrm{D}\,, && \delta_{\mathrm{P}} = \tfrac{1}{2}b_{\alpha\dot{\alpha}}\mathrm{K}^{\dot{\alpha}\alpha}\,,\\
&\delta_{\mathrm{R}} = \chi^{b}{}_{a}\mathrm{R}^{a}{}_{b}\,, && \delta_{\mathrm{Q},\bar{\mathrm{Q}}} = \zeta^{a\alpha}\mathrm{Q}_{\alpha a} + \bar{\zeta}^{\dot{\alpha}}{}_{a}\bar{\mathrm{Q}}^{a}{}_{\dot{\alpha}}\,, && \delta_{\mathrm{S},\bar{\mathrm{S}}} = \xi_{\alpha a}\mathrm{S}^{a\alpha} + \bar{\xi}^{a}{}_{\dot{\alpha}}\bar{\mathrm{S}}^{\dot{\alpha}}{}_{a}\,,
\end{aligned} \tag{4.64}$$

where $\omega, \bar{\omega}$ and χ are traceless matrices and $\{\mathrm{P},\mathrm{L},\bar{\mathrm{L}},\mathrm{D},\mathrm{K},\mathrm{Q},\bar{\mathrm{Q}},\mathrm{S},\bar{\mathrm{S}},\mathrm{R}\}$ are the superconformal generators as listed in equation (4.56). To study the algebra, we need to compute the following commutators

$$[\delta_{\kappa},\delta_{\mathrm{s.c.}}]\,p\,, \qquad [\delta_{\kappa},\delta_{\mathrm{s.c.}}]\,\theta\,, \qquad [\delta_{\kappa},\delta_{\mathrm{s.c.}}]\,\bar{\theta}\,, \qquad [\delta_{\kappa},\delta_{\mathrm{s.c.}}]\,\bar{q}\,, \tag{4.65}$$

where $\delta_{\mathrm{s.c.}}$ represents a general superconformal transformation as defined in equation (4.64). Let us start by focusing on the easiest commutators which are those involving translations and supertranslations. By recalling that the two coordinates p and q are translation as well as supertranslation invariant, one sees that

$$[\delta_\kappa, \delta_{\mathrm{P}}] = [\delta_\kappa, \delta_{\mathrm{Q},\bar{\mathrm{Q}}}] = 0\,, \tag{4.66}$$

holds on all the superpath variables. Next, we consider the generators of rotations, namely $\delta_{\mathrm{L},\bar{\mathrm{L}}}$ and δ_{R}. These computations are a bit more involved but still straightforward and we shall merely give the final results. One finds

$$\begin{aligned} [\delta_\kappa, \delta_{\mathrm{L},\bar{\mathrm{L}}}] &= \delta_{\kappa'}\,, &\qquad \text{with} \qquad & \kappa' = 2\omega\kappa\,, \\ [\delta_\kappa, \delta_{\mathrm{R}}] &= \delta_{\kappa'}\,, &\qquad \text{with} \qquad & \kappa' = 2\kappa\chi\,. \end{aligned} \tag{4.67}$$

Furthermore, for dilatations we obtain

$$[\delta_\kappa, \delta_{\mathrm{D}}] = \delta_{\kappa'}\,, \qquad \text{with} \qquad \kappa' = \tfrac{\lambda}{2}\kappa\,. \tag{4.68}$$

Let us now come to the most involved commutator that we shall investigate explicitly, which is the one involving a kappa transformation and a superboost. Evaluating the action of this commutator on the coordinates p, θ and $\bar\theta$ requires quite some effort but can still be done in a straightforward manner. Going carefully through the calculation while using the formulas of section 4.2.2.1 and 4.2.1 yields the following result

$$[\delta_\kappa, \delta_{\mathrm{S},\bar{\mathrm{S}}}] \begin{pmatrix} p \\ \theta \\ \bar\theta \end{pmatrix} = \begin{pmatrix} \delta_{\kappa'} p \\ \delta_{\kappa'}\theta \\ \delta_{\kappa'}\bar\theta \end{pmatrix}\,, \tag{4.69}$$

with

$$\kappa' = 4\xi\theta\kappa + 4\kappa\bar\xi\bar\theta + 4\kappa\theta\xi\,. \tag{4.70}$$

Hence, on p, θ and $\bar\theta$ the commutator closes onto a kappa transformation as expected. However, for q the situation is slightly different. To see this, we calculate

$$\begin{aligned} [\delta_\kappa, \delta_{\mathrm{S},\bar{\mathrm{S}}}]\, \bar q^{cd} = \Big[&- 32\dot\theta\varepsilon p^\intercal \Big(\bar\kappa^\intercal \xi^\intercal \theta^\intercal + \bar\kappa^\intercal \bar\theta^\intercal \bar\xi^\intercal - \bar\kappa^\intercal \operatorname{tr}(\xi\theta) + \bar\xi^\intercal \bar\theta^\intercal \bar\kappa^\intercal\Big) - 32\dot\theta\Big(\xi\theta\varepsilon + \varepsilon^\intercal\theta^\intercal\xi^\intercal\Big)p^\intercal\bar\kappa^\intercal \\ &- 32\dot\theta\Big(\xi\theta\kappa + \kappa\theta\xi + \kappa\bar\xi\bar\theta\Big)\bar q\Big]^{[cd]} \\ &+ 16\varepsilon^{abcd}\Big[\dot{\bar\theta}^\intercal\bar\kappa^\intercal\Big(q(\bar\xi\bar\theta + \theta\xi) - (\bar\theta^\intercal\bar\xi^\intercal + \xi^\intercal\theta^\intercal)q + \operatorname{tr}(\bar\theta\bar\xi)q\Big) + \dot{\bar\theta}^\intercal\varepsilon(\bar\theta\bar\xi p - p\xi\theta)\kappa \\ &\qquad + \Big(\operatorname{tr}(\xi\theta)\dot{\bar\theta}^\intercal - \xi^\intercal\theta^\intercal\dot{\bar\theta}^\intercal - \bar\theta^\intercal\bar\xi^\intercal\dot{\bar\theta}^\intercal - \dot{\bar\theta}^\intercal\bar\xi^\intercal\bar\theta^\intercal\Big)\varepsilon(p\kappa - \varepsilon\bar\kappa^\intercal q)\Big]_{ab} \\ &+ 16\Big(\dot{\bar\theta}^\intercal\varepsilon(p\kappa - \varepsilon\bar\kappa^\intercal q)\Big)_{ab}\Big[\varepsilon^{abed}(\theta\xi + \bar\xi\bar\theta)^c{}_e - \varepsilon^{abce}(\xi^\intercal\theta^\intercal + \bar\theta^\intercal\bar\xi^\intercal)_e{}^d\Big] \\ &+ 4\bar q\Big(\xi^\intercal p^\intercal\bar\kappa^\intercal + \xi^\intercal\varepsilon\kappa\bar q^\intercal + q^\intercal\bar\kappa\varepsilon\bar\xi^\intercal + \kappa^\intercal\varepsilon\xi\bar q - \operatorname{tr}(\xi\bar\kappa p - \xi\bar q\kappa^\intercal\varepsilon)\bar q\Big) \\ &+ 4\bar\kappa p\Big(\varepsilon^\intercal p^\intercal\bar\xi^\intercal - \xi\bar q\Big) + 4\bar\xi\Big(\varepsilon\bar\kappa^\intercal q\bar q + p\varepsilon p^\intercal\bar\kappa^\intercal\Big) + 4\varepsilon^{abcd}\Big[\xi^\intercal p^\intercal\varepsilon(p\kappa - \varepsilon\bar\kappa^\intercal q)\Big]_{ab}\,. \end{aligned}$$

Now, using standard $\mathfrak{su}(2)$ identities like (A.12) as well as Schouten's identity in five indices for the terms in the fifth and in the last line, one shows that the right-hand side of the former equation can be written as

$$[\delta_\kappa, \delta_{\mathrm{S},\bar{\mathrm{S}}}]\, \bar{q}^{cd} = \delta_{\kappa'} \bar{q}^{cd} + \Big(4\varepsilon^{abcd} (\xi^{\mathsf{T}} \varepsilon \kappa)_{ab} + 4\bar{\xi}\varepsilon\bar{\kappa}^{\mathsf{T}} - 4\bar{\kappa}\varepsilon\bar{\xi}^{\mathsf{T}} \Big) \Big(p^2 + q^2 \Big) \,, \tag{4.71}$$

with κ' as given in equation (4.70). The latter term, however, vanishes if the path we are acting on is light-like in a ten-dimensional sense[4], so that in this case the commutator closes as expected. Last but not least, we could check the commutator between kappa transformations and the generators of special conformal transformations δ_{K}. However, as the Jacobi identity together with the knowledge that superboosts preserve the light-likeness constraint $p^2 + q^2 = 0$ (see section 4.2.3) already guarantees the closure of this commutator, we will not carry out this computation.

4.3. The Super Wilson Loop in Harnad–Shnider Gauge

Having established the operator and a consistent representation of the superconformal algebra, we now turn to computing its one-loop VEV. Using the propagators derived in section 3.1.3, we will determine the expectation value through fourth order in an expansion in the anticommuting variables and subsequently investigate the question of finiteness. We shall conclude by verifying explicitly the superconformal invariance of the one-loop expectation value at the first non-trivial orders in the fermionic coordinates.

4.3.1. The One-Loop Vacuum Expectation Value

Let us start this section by explaining our notation. For reasons of compactness, we prefer to perform the computation in ten dimensions rather than in four dimensions. This means that we will keep the ten-dimensional notation for the spinors and the vectors but assume the component fields to be independent of the six extra coordinates, see also section 3.1.3. Using ten-dimensional language, the super Maldacena–Wilson loop operator can be written as

$$\mathcal{W}_M(\Gamma) = \frac{1}{N} \operatorname{tr} \mathrm{P} \, \exp \left(\int \mathrm{d}\tau \, \Big(p^{\hat{\mu}} \mathcal{A}_{\hat{\mu}} + \dot{\theta}^{\hat{\alpha}} \mathcal{A}_{\hat{\alpha}} \Big) \right) . \tag{4.72}$$

Here, we have reassembled the four and six vector that couple to $\mathcal{A}_\mu$ and Φ_i, respectively, into one ten-dimensional vector, being defined as

$$p^{\hat{\mu}} = \begin{cases} \dot{x}^\mu + \Big(\theta \Gamma^\mu \dot{\theta} \Big) & \text{for} \quad \hat{\mu} = 0, \ldots, 3 \\ q^i = n^i \sqrt{p^\nu p_\nu} & \text{for} \quad \hat{\mu} = 4, \ldots, 9 \,. \end{cases} \tag{4.73}$$

[4]Note that kappa symmetry requires the superpath to be light-like in ten dimensions, see (4.30).

To obtain the one-loop correction to the VEV, the operator (4.72) must be expanded to two fields which are consequently joint by a propagator. This yields

$$\langle \mathcal{W}_M(\Gamma)\rangle = 1 - \frac{N}{4}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\Big(p_1^{\hat\mu}p_2^{\hat\nu}\big\langle \mathcal{A}_{\hat\mu}(1)\,\mathcal{A}_{\hat\nu}(2)\big\rangle + 2p_1^{\hat\mu}\dot\theta_2^{\hat\alpha}\big\langle \mathcal{A}_{\hat\mu}(1)\,\mathcal{A}_{\hat\alpha}(2)\big\rangle \\ - \dot\theta_1^{\hat\alpha}\dot\theta_2^{\hat\beta}\big\langle \mathcal{A}_{\hat\alpha}(1)\,\mathcal{A}_{\hat\beta}(2)\big\rangle\Big) + \ \ldots\ . \tag{4.74}$$

Note that in the above equation we have already used that the propagators are diagonal in color space. Moreover, we have replaced the color factor $(N^2-1)/N$ by N because we are only interested in the large-N behavior of the VEV. After plugging in the superfield propagators as derived in section 3.1.3 and performing a few manipulations using the identity (A.29) as well as integration by parts, we obtain

$$\langle \mathcal{W}_M(\Gamma)\rangle = 1 - \frac{\lambda}{16\pi^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\ \mathrm{F}(\tau_1,\tau_2) + \mathcal{O}(\lambda^2)\,, \tag{4.75}$$

where the integrand is given by

$$\mathrm{F}(\tau_1,\tau_2) = \Bigg\{p_1^{\hat\mu}p_2^{\hat\nu}\Big(\eta_{\hat\mu\hat\nu} + \tfrac{1}{2}\big(\theta_{12}\Gamma_{\hat\mu\rho\hat\nu}\theta_{12}\big)\partial_2^\rho + \tfrac{1}{24}\big(\theta_{12}\Gamma_{\hat\mu\rho}{}^{\hat n}\theta_{12}\big)\big(\theta_{12}\Gamma_{\hat n\sigma\hat\nu}\theta_{12}\big)\partial_2^\rho\partial_2^\sigma\Big) \\ + 2p_1^{\hat\mu}\Big(\eta_{\hat\mu\hat\kappa} + \tfrac{1}{4}\big(\theta_{12}\Gamma_{\hat\mu\rho\hat\kappa}\theta_{12}\big)\partial_2^\rho\Big)\big(\theta_{12}\Gamma^{\hat\kappa}\dot\theta_2\big) \\ - \tfrac{2}{3}\big(\theta_{12}\Gamma^{\hat\mu}\dot\theta_1\big)\big(\theta_{12}\Gamma_{\hat\mu}\dot\theta_2\big)\Bigg\}\frac{1}{(x_{12}-\theta_1\Gamma\theta_2)^2} + \mathcal{O}(\theta^6)\,. \tag{4.76}$$

Note that for the sake of manifest super Poincaré invariance, we have completed the distance x_{12} in the denominator above. Upon Taylor expansion all terms which are of order six or higher in the anticommuting variables need to be neglected. Now, given the integrand (4.76), it is tempting to speculate about the all-order form of the one-loop integrand. In fact, by taking into account the all-order expressions (3.47) and (3.49) for the linearized superfields, one can come up with an educated guess for the complete one-loop integrand

$$\mathrm{F}(\tau_1,\tau_2) = \Bigg\{p_1^{\hat\mu}\Bigg(\sum_{n=0}^{16}\frac{1}{(2n)!}\big(\bar\Sigma_{12}^n\big)_{\hat\mu}{}^{\hat\nu}\Bigg)p_{2\hat\nu} + 4p_1^{\hat\mu}\Bigg(\sum_{n=0}^{15}\frac{1}{(2n)!(2n+2)}\big(\bar\Sigma_{12}^n\big)_{\hat\mu}{}^{\hat\nu}\Bigg)\big(\theta_{12}\Gamma_{\hat\nu}\dot\theta_2\big) \\ - \frac{2}{3}\big(\theta_{12}\Gamma^{\hat\mu}\dot\theta_1\big)\Bigg(\sum_{n=0}^{15}c_n\big(\bar\Sigma_{12}^n\big)_{\hat\mu}{}^{\hat\nu}\Bigg)\big(\theta_{12}\Gamma_{\hat\nu}\dot\theta_2\big)\Bigg\}\frac{1}{(x_{12}-\theta_1\Gamma\theta_2)^2}\,, \tag{4.77}$$

where we have used a similar notation as in (3.47), with $\bar\Sigma_{12}$ being defined as

$$\big(\bar\Sigma_{12}\big)_{\hat\mu}{}^{\hat\nu} := \big(\theta_{12}\Gamma_{\hat\mu}{}^{\rho\hat\nu}\theta_{12}\big)\,\partial_{2\rho}\,, \qquad \text{and} \qquad \big(\bar\Sigma_{12}^0\big)_{\hat\mu}{}^{\hat\nu} := \delta_{\hat\mu}^{\hat\nu}. \tag{4.78}$$

Unfortunately, there seems to be no easy way of determining the coefficients c_n beyond explicit calculation, so that at this point we only know that they are rational and that $c_0 = 1$.

4.3.2. Finiteness

Although it has never been rigorously proven, the Maldacena–Wilson loop operator, as it was introduced in [27], is believed to have a finite expectation value as long as the path on which it depends is smooth and non-intersecting. In this section, we will address the question whether a similar statement holds true for the supersymmetric Maldacena–Wilson loop operator (4.72). Since an all-loop analysis seems to be far beyond reach, we will restrict ourselves to a discussion at the one-loop level. More precisely, we shall only focus on the short-distance behavior of the one-loop integrand (4.76) as the absence of short-distance singularities at the level of the integrand already guarantees the finiteness of the one-loop expectation value.[5]

To investigate the UV behavior of the super Wilson loop (4.72), we will study the short-distance limit of the one-loop integrand (4.76). For this, we introduce a parameter ε being defined as $\varepsilon = \tau_2 - \tau_1$ and expand the integrand (4.76),

$$\mathrm{F}(\tau_1, \tau_2) = \mathrm{F}(\tau, \tau + \varepsilon)\,, \tag{4.79}$$

for small ε. If we can show that the resulting series is free of poles, the integrand stays finite over the whole integration domain, thus leading to a finite VEV. The loop contour that we are going to consider is assumed to be a smooth non-intersecting closed path in superspace which is furthermore super light-like in ten dimensions, i.e.

$$p^{\hat{\mu}} p_{\hat{\mu}} = p^{\mu} p_{\mu} - q^i q^i = 0 \qquad \longrightarrow \qquad p^{\hat{\mu}} \dot{p}_{\hat{\mu}} = 0\,, \tag{4.80}$$

but nowhere null in four dimensions, i.e. $p^{\mu} p_{\mu} \neq 0$ for all τ. Here, we have used the same definition for the ten-dimensional vector $p_{\hat{\mu}}$ as in the previous section, see (4.73). The expansions of the superpath variables $Z(\tau + \varepsilon)$ around $\varepsilon = 0$ are given by

$$\begin{aligned} x_1^{\mu} &= x^{\mu}, \qquad & x_2^{\mu} &= x^{\mu} + \varepsilon \dot{x}^{\mu} + \tfrac{1}{2}\varepsilon^2 \ddot{x}^{\mu} + \mathcal{O}(\varepsilon^3), \\ \theta_1^{\hat{\alpha}} &= \theta^{\hat{\alpha}}, & \theta_2^{\hat{\alpha}} &= \theta^{\hat{\alpha}} + \varepsilon \dot{\theta}^{\hat{\alpha}} + \tfrac{1}{2}\varepsilon^2 \ddot{\theta}^{\hat{\alpha}} + \mathcal{O}(\varepsilon^3), \\ q_1^i &= q^i, & q_2^i &= q^i + \varepsilon \dot{q}^i + \tfrac{1}{2}\varepsilon^2 \ddot{q}^i + \mathcal{O}(\varepsilon^3)\,. \end{aligned} \tag{4.81}$$

We start our investigation of the pole structure of (4.79) by focusing on the denominator term in (4.76) and its derivatives. Using the expansions (4.81), we find

$$\frac{1}{(x_{12} - \theta_1 \Gamma \theta_2)^2} = \frac{1}{\varepsilon^2} \left(\frac{1}{p^{\mu} p_{\mu}} - \varepsilon\, \frac{p^{\mu} \dot{p}_{\mu}}{(p^{\nu} p_{\nu})^2} + \mathcal{O}(\varepsilon^2) \right)\,. \tag{4.82}$$

For the first derivative of the denominator term, we obtain the following Taylor expansion

$$\partial_2^{\rho} \left(\frac{1}{(x_{12} - \theta_1 \Gamma \theta_2)^2} \right) = \frac{1}{\varepsilon^3} \left(\frac{-2p^{\rho}}{(p^{\mu} p_{\mu})^2} + \varepsilon\, \frac{4p^{\rho} p^{\mu} \dot{p}_{\mu} - \dot{p}^{\rho} p^{\mu} p_{\mu}}{(p^{\nu} p_{\nu})^3} + \mathcal{O}(\varepsilon^2) \right)\,. \tag{4.83}$$

[5] Due to the Minkowski structure of our superspace, it could in principal happen that the denominator of (4.76) becomes zero although the two points $(x(\tau_1), \theta(\tau_1))$ and $(x(\tau_2), \theta(\tau_2))$ do not coincide. However, for the time being, let us not consider this case and instead restrict to curves for which the square of the supertranslation-invariant interval does not vanish for any two distinct τ_1 and τ_2.

In general, each partial derivative increases the power of the leading divergence by one unit. Therefore, we can write down the following formal Taylor series for the n-th derivative of the inverse square of the supertranslation-invariant interval

$$(\partial_2)^n\left(\frac{1}{(x_{12}-\theta_1\Gamma\theta_2)^2}\right)=\frac{1}{\varepsilon^{2+n}}\left(S_n(\tau)+\varepsilon T_n(\tau)+\mathcal{O}(\varepsilon^2)\right)\,, \tag{4.84}$$

where S_n and T_n denote two different rank-n tensors. Having discussed the denominator term and its derivatives, we now need to compute the Taylor expansions of the numerator terms. We start by focusing on the sum of all terms which do not involve a partial derivative. For these terms, we find the following result

$$p_1^{\hat\mu}\Big(p_{2\hat\mu}+2\Big(\theta_{12}\Gamma_{\hat\mu}\dot\theta_2\Big)\Big)-\tfrac{2}{3}\Big(\theta_{12}\Gamma^{\hat\mu}\dot\theta_1\Big)\Big(\theta_{12}\Gamma_{\hat\mu}\dot\theta_2\Big)=\tfrac{1}{2}\varepsilon^2p^{\hat\mu}\Big(\ddot p_{\hat\mu}-2\Big(\dot\theta\Gamma_{\hat\mu}\ddot\theta\Big)\Big)+\mathcal{O}(\varepsilon^3), \tag{4.85}$$

where we have used the relations (4.80) as well as the fact that the ten-dimensional Pauli matrices are symmetric. If we multiply the two expansions (4.82) and (4.85), we immediately see that the resulting series starts at order ε^0 and does therefore have a finite limit as $\varepsilon\to 0$. For the sum of all terms which multiply the first derivative of the denominator term (4.83), we obtain the following expansion:

$$\tfrac{1}{2}p_1^{\hat\mu}p_2^{\hat\nu}\Big(\theta_{12}\Gamma_{\hat\mu\rho\hat\nu}\theta_{12}\Big)+\tfrac{1}{2}p_1^{\hat\mu}\Big(\theta_{12}\Gamma_{\hat\mu\rho\hat\kappa}\theta_{12}\Big)\Big(\theta_{12}\Gamma^{\hat\kappa}\dot\theta_2\Big)=\tfrac{1}{2}\varepsilon^3p^{\hat\mu}\dot p^{\hat\nu}\Big(\dot\theta\Gamma_{\hat\mu\rho\hat\nu}\dot\theta\Big)+\mathcal{O}(\varepsilon^4)\,. \tag{4.86}$$

Here, we have used the fact that the term at order ε^2 vanishes due to the antisymmetry of $\Gamma_{\hat\mu\rho\hat\nu}$. Therefore, the product of the two expansions (4.83) and (4.86) is also free of poles. The last term which we will investigate explicitly is the one which multiplies the second derivative of the denominator term. Its Taylor expansion reads

$$\tfrac{1}{24}p_1^{\hat\mu}p_2^{\hat\nu}\Big(\theta_{12}\Gamma_{\hat\mu\rho}{}^{\hat n}\theta_{12}\Big)\Big(\theta_{12}\Gamma_{\hat n\sigma\hat\nu}\theta_{12}\Big)=\tfrac{1}{24}\varepsilon^4p^{\hat\mu}p^{\hat\nu}\Big(\dot\theta\Gamma_{\hat\mu\rho}{}^{\hat n}\dot\theta\Big)\Big(\dot\theta\Gamma_{\hat n\sigma\hat\nu}\dot\theta\Big)+\mathcal{O}(\varepsilon^5)\,. \tag{4.87}$$

If we combine this expression with equation (4.84) (for $n=2$), we see that the resulting series is again UV safe, thus completing our proof that the first three terms in the Graßmann expansion of the complete one-loop integrand are free of short-distance singularities. Given this result, one can now of course raise the question whether this statement also holds for the higher-order terms in the θ-expansion. In fact, provided that our conjecture about the form of the complete one-loop integrand (4.77) is structurally correct, one easily sees that all the higher-order terms are also UV finite since every operator $\bar\Sigma_{12}$ comes with two ε's originating from the expansions of the two θ_{12} but contains only one partial derivative, cf. (4.84).

4.3.3. Superconformal Symmetry

Finally, let us confirm the full superconformal invariance of the one-loop VEV (4.75). The one-loop expectation value (4.75) is manifestly invariant under (super)translations, Lorentz transformations and internal rotations. However, it remains to check the invariance under superboosts. In what follows, we will check this superconformal Ward identity at the first three orders in the Graßmann expansion of the one-loop VEV (4.75).

In ten-dimensional notation, the superconformal transformation laws of the superspace coordinates (4.44) and (4.54) take the following form

$$
\begin{aligned}
\delta_{\mathrm{S},\bar{\mathrm{S}}}\, x_\mu &= x^\nu\Big(\theta\Gamma_\mu\bar{\Gamma}_\nu\xi\Big) - \tfrac{1}{12}\Big(\theta\Gamma^{\nu\rho}\xi\Big)\Big(\theta\Gamma_\mu\bar{\Gamma}_{\nu\rho}\theta\Big) + \tfrac{1}{12}\Big(\theta\Gamma^{ij}\xi\Big)\Big(\theta\Gamma_\mu\bar{\Gamma}_{ij}\theta\Big)\,,\\
\delta_{\mathrm{S},\bar{\mathrm{S}}}\, \theta^{\hat{\alpha}} &= x^\mu\Big(\xi\bar{\Gamma}_\mu\Big)^{\hat{\alpha}} + \tfrac{1}{2}\Big(\theta\xi\Big)\theta^{\hat{\alpha}} - \tfrac{1}{4}\Big(\theta\Gamma^{\mu\nu}\xi\Big)\Big(\theta\Gamma_{\mu\nu}\Big)^{\hat{\alpha}} + \tfrac{1}{4}\Big(\theta\Gamma^{ij}\xi\Big)\Big(\theta\Gamma_{ij}\Big)^{\hat{\alpha}}\,,\\
\delta_{\mathrm{S},\bar{\mathrm{S}}}\, q^i &= 2q_j\Big(\theta\Gamma^j\bar{\Gamma}^i\xi\Big)\,,
\end{aligned}
\tag{4.88}
$$

where ξ is the parameter which specifies the transformation. In order to be able to split the subsequent computation in smaller but self-contained pieces, we will work with the following form of the one-loop expectation value

$$
\langle\mathcal{W}_M(\Gamma)\rangle_{(1)} = \frac{\lambda}{16\pi^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\Big\{I_p(\tau_1,\tau_2) + I_q(\tau_1,\tau_2) + \mathcal{O}(\theta^4)\Big\}\,,
\tag{4.89}
$$

where we have separated the integrand into a part which does not involve the coordinate q at all

$$
I_p(\tau_1,\tau_2) = \frac{p_1^\mu p_2^\nu}{x_{12}^2}\left(\eta_{\mu\nu} + 2\eta_{\mu\nu}\Big(\theta_1\Gamma_\rho\theta_2\Big)\frac{x_{12}^\rho}{x_{12}^2} + \Big(\theta_{12}\Gamma_{\mu\rho\nu}\theta_{12}\Big)\frac{x_{12}^\rho}{x_{12}^2}\right) + \frac{2p_1^\mu}{x_{12}^2}\Big(\theta_{12}\Gamma_\mu\dot{\theta}_2\Big)\,,
\tag{4.90}
$$

and a piece which involves all the remaining terms

$$
\begin{aligned}
I_q(\tau_1,\tau_2) = \frac{q_1^i q_2^j}{x_{12}^2}&\left(\eta_{ij} + 2\eta_{ij}\Big(\theta_1\Gamma_\rho\theta_2\Big)\frac{x_{12}^\rho}{x_{12}^2} + \Big(\theta_{12}\Gamma_{i\rho j}\theta_{12}\Big)\frac{x_{12}^\rho}{x_{12}^2}\right)\\
&+ \frac{2q_1^i}{x_{12}^2}\left(p_2^\mu\Big(\theta_{12}\Gamma_{i\rho\mu}\theta_{12}\Big)\frac{x_{12}^\rho}{x_{12}^2} + \Big(\theta_{12}\Gamma_i\dot{\theta}_2\Big)\right)\,.
\end{aligned}
\tag{4.91}
$$

This particular splitting turns out to be useful since the p-part and the q-part do not mix under superconformal transformations.

Let us start by computing the variation of the integrand $I_p(\tau_1,\tau_2)$. Under a superconformal boost, the supermomentum p^μ transforms as

$$
\delta_{\mathrm{S},\bar{\mathrm{S}}}\, p_\mu = 2\dot{x}^\nu\Big(\theta\Gamma_\mu\bar{\Gamma}_\nu\xi\Big) + \mathcal{O}(\theta^3)\,.
\tag{4.92}
$$

Using this expression, we can now compute the variation of the integrand (4.90). If we only keep terms which are linear in θ after the variation, we find

$$\begin{aligned}\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_p(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=}\; & \frac{2\dot{x}_1^\mu}{x_{12}^2}\Big(2\dot{x}_2^\nu\big(\theta_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) - x_{12}^\nu\big(\dot{\theta}_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) + \dot{x}_2^\nu\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big)\Big) \\ & - \frac{2\dot{x}_1^\mu\dot{x}_2^\nu}{x_{12}^4}\Big(\eta_{\mu\nu}x_{12}^\rho x_{12}^\sigma\big(\theta_1\Gamma_\rho\bar{\Gamma}_\sigma\xi\big) + \eta_{\mu\nu}x_{12}^\rho x_{12}^\sigma\big(\theta_2\Gamma_\rho\bar{\Gamma}_\sigma\xi\big) \\ & - x_{12}^\rho x_{12}^\sigma\big(\theta_{12}\Gamma_{\mu\rho\nu}\bar{\Gamma}_\sigma\xi\big)\Big)\,, \end{aligned} \tag{4.93}$$

where by $\hat{=}$ we mean that the expression on the right-hand side gives the same result when integrated over τ_1 and τ_2, i.e. we already made use of the freedom to relabel the integration variables. Using the Clifford algebra relation for the Pauli matrices as well as the identity (A.29), the former expression can be rewritten as

$$\begin{aligned}\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_p(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=}\; & \frac{2\dot{x}_1^\mu}{x_{12}^2}\Big(2\dot{x}_2^\nu\big(\theta_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) - x_{12}^\nu\big(\dot{\theta}_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) + \dot{x}_2^\nu\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big) - \dot{x}_{2\mu}\big(\theta_1\xi\big) \\ & - \dot{x}_{2\mu}\big(\theta_2\xi\big) - \dot{x}_2^\nu\big(\theta_{12}\Gamma_{\mu\nu}\xi\big) + \frac{2\dot{x}_2^\rho x_{12\rho}}{x_{12}^2}\, x_{12}^\nu\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big)\Big)\,. \end{aligned} \tag{4.94}$$

By use of the Clifford algebra relation as well as integration by parts, the variation of $I_p(\tau_1,\tau_2)$ can be shown to be equivalent to

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_p(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=} 2\dot{x}_1^\mu\Big(-\frac{\dot{x}_2^\nu}{x_{12}^2}\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big) - \frac{x_{12}^\nu}{x_{12}^2}\big(\dot{\theta}_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) + \frac{2\dot{x}_2^\rho x_{12\rho}}{x_{12}^4}\, x_{12}^\nu\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big)\Big). \tag{4.95}$$

This expression can, however, easily be seen to be a total derivative with respect to τ_2

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_p(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=} 2\dot{x}_1^\mu\,\frac{\partial}{\partial\tau_2}\left(\frac{x_{12}^\nu}{x_{12}^2}\big(\theta_{12}\Gamma_\mu\bar{\Gamma}_\nu\xi\big)\right)\,, \tag{4.96}$$

and does therefore vanish when integrated over. Let us note that the expression is also non-singular in the limit $1 \to 2$.

We now turn to the variation of $I_q(\tau_1,\tau_2)$. If we again discard terms which are of higher order than linear in θ, we find

$$\begin{aligned}\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_q(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=}\; & \frac{2q_1^i q_2^j}{x_{12}^2}\Big(\big(\theta_{12}\Gamma_{ij}\xi\big) + \eta_{ij}\big(\theta_1\xi\big) + \eta_{ij}\big(\theta_2\xi\big) - \frac{1}{x_{12}^2}\Big(\eta_{ij}x_{12}^\mu x_{12}^\nu\big(\theta_1\Gamma_\mu\bar{\Gamma}_\nu\xi\big) \\ & + \eta_{ij}x_{12}^\mu x_{12}^\nu\big(\theta_2\Gamma_\mu\bar{\Gamma}_\nu\xi\big) - x_{12}^\mu x_{12}^\rho\big(\theta_{12}\Gamma_{i\rho j}\bar{\Gamma}_\mu\xi\big)\Big)\Big) \\ & + \frac{2q_1^i}{x_{12}^2}\Big(-x_{12}^\mu\big(\dot{\theta}_2\Gamma_i\bar{\Gamma}_\mu\xi\big) + \dot{x}_2^\mu\big(\theta_{12}\Gamma_i\bar{\Gamma}_\mu\xi\big) + \frac{2p_2^\nu}{x_{12}^2}\, x_{12}^\mu x_{12}^\rho\big(\theta_{12}\Gamma_{i\rho\nu}\bar{\Gamma}_\mu\xi\big)\Big)\,. \end{aligned} \tag{4.97}$$

Using the Clifford algebra relation as well as the reduction formula (A.29), one easily shows that the part being quadratic in q vanishes without leaving behind a total derivative term. Note that this is expected as the terms being quadratic in q originate from the correlator of two scalar superfields Φ_i and the combination $q^i\Phi_i$ is superconformally invariant, see section 4.2.2.1. The terms being linear in q can be rewritten as

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_q(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=} 2q_1^i\left(-\frac{x_{12}^\mu}{x_{12}^2}\Big(\dot{\theta}_2\Gamma_i\bar{\Gamma}_\mu\xi\Big) - \frac{\dot{x}_2^\mu}{x_{12}^2}\Big(\theta_{12}\Gamma_i\bar{\Gamma}_\mu\xi\Big) + \frac{2\dot{x}_2^\rho x_{12\rho}}{x_{12}^4}\, x_{12}^\mu\Big(\theta_{12}\Gamma_i\bar{\Gamma}_\mu\xi\Big)\right),$$

which is pretty much the same total derivative that we encountered when varying the integrand $I_p(\tau_1,\tau_2)$,

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\, I_q(\tau_1,\tau_2)\Big|_{\mathcal{O}(\theta)} \hat{=} 2q_1^i\,\frac{\partial}{\partial\tau_2}\left(\frac{x_{12}^\nu}{x_{12}^2}\Big(\theta_{12}\Gamma_i\bar{\Gamma}_\nu\xi\Big)\right). \tag{4.98}$$

Putting both results together yields

$$\delta_{\mathrm{S},\bar{\mathrm{S}}}\langle\mathcal{W}_M(\Gamma)\rangle_{(1)} = \frac{\lambda}{8\pi^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\left\{\frac{\partial}{\partial\tau_2}\left(\frac{x_{12}^\nu}{x_{12}^2}\Big(\theta_{12}\Big(\dot{x}_1^\mu\Gamma_\mu + q_1^i\Gamma_i\Big)\bar{\Gamma}_\nu\xi\Big)\right) + \mathcal{O}(\theta^3)\right\}, \tag{4.99}$$

which shows that the one-loop expectation value of the super Maldacena–Wilson loop operator (4.72) is annihilated by the generators of superboosts modulo terms which are at least cubic in the anticommuting variables.

5. Integrability of Smooth Super Wilson Loops

We now turn to the discussion of the non-local symmetries of super Maldacena–Wilson loops. In reference [1], it was shown that the one-loop VEV of the super Maldacena–Wilson loop is Yangian invariant at the leading order in the Graßmann expansion. In this paper, the authors formulated the Yangian symmetry in terms of second order variational derivative operators acting on the (super)path of the Wilson loop operator. However, it was noted that such a definition is subtle as these operators easily lead to ill-defined distributional terms when applied to the VEV of the super Maldacena–Wilson loop. Here, we will now reconsider the definition of the Yangian generators and argue for an approach based on gauge-covariant field insertions. We begin by explaining our formalism in the context of level-zero symmetry and compare it to the path-based approach used in reference [1]. Subsequently, we shall define the level-one generators and discuss various algebraic consistency conditions. Finally, we will apply our formalism to the case of smooth super Wilson loops and prove the Yangian symmetry of the full one-loop VEV.

5.1. Yangian Action on Wilson Lines

In the following, we will establish the action of the Yangian generators on a generic Wilson line operator. In the beginning, we will keep the discussion fairly general in the sense that we will neither fix the underlying algebra nor the specific form of the Wilson line operator. This will not only streamline the discussion but also allow us to see more clearly which of the many exceptional features of the $\mathcal{N} = 4$ SYM model are responsible for its integrability. For simplicity, we shall, however, assume that the underlying Lie algebra is purely bosonic. This will make algebra expressions a lot more readable, while the generalization to the case of a superalgebra is straightforward. The Wilson line operator that we are going to consider is of the following form

$$\mathcal{V} = \mathrm{P}\exp\int_\Gamma \mathcal{A}\,, \tag{5.1}$$

where we have abbreviated $\mathcal{V} := \mathcal{V}(\Gamma)$ for later convenience. Here, Γ denotes an open path in a general spacetime $\Gamma : \tau \mapsto X^{\mathcal{A}}(\tau)$ and for concreteness we will use a parametrization that runs from $\tau = 0$ to $\tau = 1$. The index $\mathcal{A}$ is to be interpreted as a multi-index running over the bosonic and fermionic degrees of freedom. In what follows, we will often expand the Lie algebra-valued gauge connection $\mathcal{A}$ in a plain basis of one-forms, i.e. $\mathcal{A} = \mathrm{d}X^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}$. Note that this can be done without loss of generality as one can easily translate between a supertranslation-invariant basis and the plain basis spanned by the $\mathrm{d}X^{\mathcal{A}}$, see section 5.1.3. In order to distinguish the expansion coefficients with respect to the plain basis from the expansion coefficients with respect to the susy-covariant basis, we have added a superscript p to the components of the gauge connection. For the sake of simplicity, we have refrained from making explicit any coupling to the scalars in equation (5.1). However, the Maldacena extension can be introduced straightforwardly as will be discussed in section 5.1.3.

5.1.1. Level-Zero Action on the Wilson Line

The Lie algebra constitutes the level-zero algebra upon which the Yangian is built. In fact, the bi-local part of the level-one generators is completely determined once the level-zero algebra has been fixed and a representation for these generators has been chosen, see section 2.1.3. There are, however, different ways of representing the level-zero algebra, which are equivalent up to boundary terms. While these boundary terms are irrelevant for the level-zero symmetry, they can easily spoil the level-one symmetry. In what follows, we will discuss and compare the different level-zero representations in detail, thereby laying the foundations for formulating the action of the level-one generators.

Useful definitions and gauge transformations. Let us start our discussion by collecting the relevant definitions that will be used throughout the next sections. As we will often consider small variations of the Wilson line operator (5.1), the following notation suggests itself:

$$\mathcal{V}[Q] := \int \mathcal{V}_{[0,\tau]}\, Q(\tau)\, \mathcal{V}_{[\tau,1]}\,. \tag{5.2}$$

Here, we have inserted a local operator, like, for example, the variation of the gauge connection $\mathcal{A}$, into the path-ordered exponential at the position $X(\tau)$. Note that for tactical reasons we have omitted the integration measure $\mathrm{d}\tau$ in the above formula. In practice, $Q(\tau)$ will always be a one-form field which has to be understood as pulled back to the path. For example, the pullback of the gauge connection one-form $\mathcal{A}$ reads $\mathcal{A} = \mathrm{d}\tau \dot{X}^{\mathcal{A}}(\tau)\mathcal{A}^p_{\mathcal{A}}(X(\tau))$.[1] For later convenience, let us also define a shorthand notation for a Wilson loop with two ordered insertions:

$$\mathcal{V}[Q;Q'] := \int_{\tau<\tau'} \mathcal{V}_{[0,\tau]}\, Q(\tau)\, \mathcal{V}_{[\tau,\tau']}\, Q'(\tau')\, \mathcal{V}_{[\tau',1]}\,. \tag{5.3}$$

Here, as above, $Q(\tau)$ and $Q'(\tau')$ denote the pullback of two one-forms.

[1] In order to distinguish between the pullback of the form and the one-form itself, we should in principal introduce a different notation for the pullback, like, for example, $\Gamma^*\mathcal{A}$. However, in this thesis we will refrain from doing so as it will be clear from the context which object is meant.

One of our main guiding principles in the following will be gauge covariance (gauge invariance). Let us therefore consider gauge transformations of Wilson lines/loops in detail, see also section 2.3.1. As discussed in section 3.1.1, a gauge connection one-form transforms under an infinitesimal gauge transformation $\mathbb{G}[\Lambda]$ as

$$\mathbb{G}[\Lambda]\mathcal{A} = \mathcal{D}\Lambda = \mathrm{d}\Lambda + [\mathcal{A}, \Lambda] \,. \tag{5.4}$$

When pulled back to a path parametrized by $X(\tau)$, the former equation becomes

$$(\mathbb{G}[\Lambda]\mathcal{A})(X(\tau)) = \mathrm{d}\tau\,(\partial_\tau\Lambda + [\dot{X}^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}, \Lambda]) =: \mathrm{d}\tau\,\mathcal{D}_\tau\,\Lambda \,. \tag{5.5}$$

Here, we have introduced a new notation namely the covariant derivative along the path $\mathcal{D}_\tau = \partial_\tau + [\dot{X}^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}, \;\;]$. In order to obtain the transformation behavior of a generic Wilson line operator, we vary the operator (5.1) with respect to the gauge field $\mathcal{A}$ and plug in equation (5.5) for the variation $\delta\mathcal{A}$. This yields

$$\mathbb{G}[\Lambda]\mathcal{V} = \mathcal{V}[\mathcal{D}\Lambda] = \int \mathrm{d}\tau\;\mathcal{V}_{[0,\tau]}\,\mathcal{D}_\tau\,\Lambda(X(\tau))\,\mathcal{V}_{[\tau,1]} \,. \tag{5.6}$$

Note that we can integrate the first term of the covariant derivative along the path by parts. The action of the partial derivative ∂_τ on the two Wilson line operators $\mathcal{V}_{[0,\tau]}$ and $\mathcal{V}_{[\tau,1]}$ in fact cancels the commutator term, so that the only remaining term is a total derivative term which can be integrated and yields two boundary terms, i.e.

$$\mathbb{G}[\Lambda]\mathcal{V} = -\Lambda(0)\mathcal{V} + \mathcal{V}\Lambda(1) \,. \tag{5.7}$$

This is in agreement with the finite gauge transformation of a Wilson line, see equation (2.88). Consequently, a closed Wilson loop is a gauge-invariant quantity

$$\mathbb{G}[\Lambda]\,\mathrm{tr}\,\mathcal{V} = \mathrm{tr}[\mathcal{V}, \Lambda] = 0 \,. \tag{5.8}$$

Next, let us consider the gauge transformation of a Wilson line operator with one insertion. The variation can either hit the first Wilson loop operator in (5.2), the insertion itself or the second Wilson loop operator in (5.2). Consequently, we get

$$\mathbb{G}[\Lambda]\mathcal{V}[Q] = \mathcal{V}[\mathcal{D}\Lambda; Q] + \mathcal{V}\,\Big[\mathbb{G}[\Lambda]Q\Big] + \mathcal{V}[Q; \mathcal{D}\Lambda]. \tag{5.9}$$

Integrating the covariant derivatives yields

$$\begin{aligned}\mathbb{G}[\Lambda]\mathcal{V}[Q] &= -\Lambda(0)\mathcal{V}[Q] + \mathcal{V}[\Lambda Q] + \mathcal{V}\Big[\mathbb{G}[\Lambda]Q\Big] - \mathcal{V}[Q\Lambda] + \mathcal{V}[Q]\Lambda(1) \\ &= -\Lambda(0)\mathcal{V}[Q] + \mathcal{V}[Q]\Lambda(1) + \mathcal{V}\Big[\mathbb{G}[\Lambda]Q - [Q, \Lambda]\Big] \,.\end{aligned} \tag{5.10}$$

From the last equation, we conclude that if the insertion transforms covariantly, i.e. $\mathbb{G}[\Lambda]Q = [Q, \Lambda]$, the Wilson line with one insertion transforms in the same way as the Wilson line itself, see equation (5.6). In particular, we note that a Wilson loop with a covariant insertion is still gauge invariant. This completes our preliminary considerations. We proceed by discussing the different level-zero representations.

Action on the path. A convenient way of representing the level-zero generators is by specifying their action on the path $X^{\mathcal{A}}(\tau)$. Under a small variation of the contour $\delta X^{\mathcal{A}}$, the Wilson line operator changes as

$$\delta\mathcal{V} = \int \mathrm{d}\tau\ \mathcal{V}_{[0,\tau]}\Big(\partial_\tau(\delta X^{\mathcal{A}})\mathcal{A}^p_{\mathcal{A}} + \dot{X}^{\mathcal{A}}\delta X^{\mathcal{B}}\partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}}\Big)\mathcal{V}_{[\tau,1]}\,. \tag{5.11}$$

As explained above, we will not yet fix the underlying Lie algebra and instead just assume some general expansion of the symmetry-generating vector fields

$$\mathrm{J} = \mathrm{J}X^{\mathcal{A}}_p(X)\partial_{\mathcal{A}} = \mathrm{J}X^{\mathcal{A}}(X)D_{\mathcal{A}}\,, \tag{5.12}$$

where the $\partial_{\mathcal{A}}$ furnish the plain basis of vector fields, while by $D_{\mathcal{A}}$ we refer to the susy-covariant derivatives, see also chapter 3. The subscript p helps to distinguish the two sets of expansion coefficients. Acting with a level-zero transformation on the point $X^{\mathcal{A}}$ obviously yields $\mathrm{J}X^{\mathcal{A}}_p(X)$ and we will therefore denote (with abuse of notation) the infinitesimal displacement of the path $X^{\mathcal{A}}(\tau)$ by $\mathrm{J}X^{\mathcal{A}}_p(\tau)$. Note that for the moment, we omit the Lie algebra index α by conceiving of the Lie algebra generators as being contracted with a parameter ε. In order to distinguish the path variation of the Wilson line operator (5.11) from, for example, the field variation, we choose to denote the path variation of $\mathcal{V}$ by $\mathrm{J}\mathcal{V}$. Using this notation, equation (5.11) becomes

$$\mathrm{J}\mathcal{V} = \int \mathrm{d}\tau\ \mathcal{V}_{[0,\tau]}\Big(\mathcal{D}_\tau(\mathrm{J}X^{\mathcal{A}}_p\mathcal{A}^p_{\mathcal{A}}) + \dot{X}^{\mathcal{A}}\mathrm{J}X^{\mathcal{B}}_p\mathcal{F}^p_{\mathcal{B}\mathcal{A}}\Big)\mathcal{V}_{[\tau,1]}\,, \tag{5.13}$$

where $\mathcal{F}^p_{\mathcal{B}\mathcal{A}}$ is the field strength tensor, being defined as

$$\mathcal{F}^p_{\mathcal{B}\mathcal{A}} = \partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}} - (-1)^{|\mathcal{A}||\mathcal{B}|}\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}} + [\mathcal{A}^p_{\mathcal{B}}, \mathcal{A}^p_{\mathcal{A}}\}\,, \tag{5.14}$$

where

$$[\mathcal{A}^p_{\mathcal{B}}, \mathcal{A}^p_{\mathcal{A}}\} := \mathcal{A}^p_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}} - (-1)^{|\mathcal{A}||\mathcal{B}|}\mathcal{A}^p_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}\,. \tag{5.15}$$

Note that in the above equation we have rewritten the ∂_τ-term as a covariant derivative along the path acting on $\mathrm{J}X^{\mathcal{A}}_p\mathcal{A}^p_{\mathcal{A}}$, see also equation (5.5). The extra terms needed for this cancel exactly against those terms that were used to rewrite the second term in equation (5.11) as an insertion of the full field strength tensor. Therefore, the equations (5.11) and (5.13) are completely equivalent. By comparing equation (5.13) to equation (5.6), we see that the first term is merely a field-dependent gauge transformation, which can be integrated to the boundary. For Wilson loops, these boundary terms cancel out due to the presence of the trace. Hence, the gauge-covariant derivative term in equation (5.13) can be safely disregarded when discussing the Lie algebra symmetry of Wilson loops. However, while being irrelevant for level-zero symmetry, such terms have to be treated with care when dealing with level-one symmetry as they can easily lead to a breakdown of gauge invariance. This is one of the reasons why we refrain from formulating the Yangian using this path-based approach. Instead, we shall use generators which act on the fields.

Action on the fields. An alternative approach to formulate the level-zero symmetry is by using generators which act on the fields rather than on the path. Varying the Wilson line operator (5.1) with respect to the gauge connection $\mathcal{A}$ yields

$$\mathbb{J}\mathcal{V} = \int \mathrm{d}\tau\, \mathcal{V}_{[0,\tau]}\, (\mathbb{J}\mathcal{A})(\tau)\, \mathcal{V}_{[\tau,1]} = \mathcal{V}[\mathbb{J}\mathcal{A}]\,. \tag{5.16}$$

Here, we have employed a similar notation as in the paragraph above, namely we denote the variation of the gauge connection by $\mathbb{J}\mathcal{A}$ and the variation of the Wilson line operator by $\mathbb{J}\mathcal{V}$. The field variation of the one-form field $\mathcal{A}$ is defined as minus the path variation, i.e.

$$\begin{aligned}\mathbb{J}\mathcal{A} &= -\mathrm{d}X^{\mathcal{A}}\, \mathrm{J}X_p^{\mathcal{B}}\, \partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}} - \mathrm{d}(\mathrm{J}X_p^{\mathcal{B}})\, \mathcal{A}^p_{\mathcal{B}} \\ &= -\mathrm{d}X^{\mathcal{A}}\mathrm{J}X_p^{\mathcal{B}}\mathcal{F}^p_{\mathcal{B}\mathcal{A}} - \mathcal{D}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\,.\end{aligned} \tag{5.17}$$

Note that the minus sign in the definition of $\mathbb{J}$ is required to match the Lie algebra of the field generators $\mathbb{J}$ to the algebra of derivative operators J, which act on coordinates. To see this, let us consider the product of two generators $\mathbb{J}$ acting on a field Ψ,

$$\mathbb{J}_1\mathbb{J}_2\Psi = -\mathbb{J}_1\mathrm{J}_2\Psi = -\mathrm{J}_2\mathbb{J}_1\Psi = \mathrm{J}_2\mathrm{J}_1\Psi\,. \tag{5.18}$$

Since the order of generators is effectively reversed, we need the extra minus sign in the definition of the generators $\mathbb{J}$ to absorb the minus sign resulting from reordering the indices of the structure constants. This argument makes it clear that by construction the level-zero actions on the fields satisfy the same algebra relations as the generators J^δ, i.e.

$$[\mathbb{J}^\delta, \mathbb{J}^\rho] = f^{\delta\rho}{}_{\kappa}\mathbb{J}^\kappa. \tag{5.19}$$

For later convenience, let us note that the transformation law (5.17) also has a nice mathematical interpretation. From the point of view of differential geometry, the generators of the Lie algebra $\mathrm{J} = \mathrm{J}X_p^{\mathcal{A}}\partial_{\mathcal{A}}$ form vector fields which we chose to write in the local coordinate basis spanned by the $\partial_{\mathcal{A}}$. In this framework, the field variation of the gauge connection is interpreted as minus the Lie derivative of the one-form field $\mathcal{A}$ with respect to the vector field $\mathrm{J} = \mathrm{J}X_p^{\mathcal{A}}\partial_{\mathcal{A}}$, meaning that

$$\mathbb{J}\mathcal{A} = -\mathcal{L}_{\mathrm{J}}\mathcal{A} = -i[\mathrm{J}]\mathrm{d}\mathcal{A} - \mathrm{d}(i[\mathrm{J}]\mathcal{A})\,. \tag{5.20}$$

Here, we have introduced a new operation denoted by $i[\mathrm{J}]$, which is called the interior product. The interior product takes a vector field, such as $\mathrm{J} = \mathrm{J}X_p^{\mathcal{A}}\partial_{\mathcal{A}}$, as an argument and acts on a generic p-form ω,

$$\omega = \tfrac{1}{r!}\, \mathrm{d}X^{\mathcal{A}_1} \wedge \mathrm{d}X^{\mathcal{A}_2} \wedge \ldots \wedge \mathrm{d}X^{\mathcal{A}_r}\omega_{\mathcal{A}_r\ldots\mathcal{A}_2\mathcal{A}_1}\,, \tag{5.21}$$

in the following way:

$$i[\mathrm{J}]\,\omega = \tfrac{1}{(r-1)!}\, \mathrm{J}X_p^{\mathcal{A}}\mathrm{d}X^{\mathcal{A}_2} \wedge \ldots \wedge \mathrm{d}X^{\mathcal{A}_r}\omega_{\mathcal{A}_r\ldots\mathcal{A}_2\mathcal{A}}\,. \tag{5.22}$$

Obviously, the operator $i[\mathrm{J}]$ acts on a differential form by subsequently replacing the basis differentials by the vector $\mathrm{J}X_p^{\mathcal{A}}$. Note, however, that the interior product acts as an antiderivation, meaning that

$$i[X](\omega \wedge \eta) = (i[X]\omega) \wedge \eta + (-1)^p \omega \wedge (i[X]\eta)\,, \tag{5.23}$$

where ω is a general p-form and η is a general q-form. For this reason, one picks up a minus sign when commuting the operator $i[X]$ past a basis differential. In the further course of this thesis, we shall call the object $i[\mathrm{J}]\,\omega$ the contraction of J with ω. Given these explanations, we can now easily check that the expressions (5.17) and (5.20) indeed agree. To get familiar with the formalism, let us carry out this computation in detail. Applying the exterior derivative and the interior product operator to the gauge connection $\mathcal{A}$ yields

$$\mathbb{J}\mathcal{A} = i[\mathrm{J}]\Big(\mathrm{d}X^{\mathcal{A}} \wedge \mathrm{d}X^{\mathcal{B}}\tfrac{1}{2}(\partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}} - (-1)^{|\mathcal{A}||\mathcal{B}|}\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}})\Big) - \mathrm{d}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\,. \tag{5.24}$$

Carrying out the contraction in the above equation leads to the expression

$$\begin{aligned}\mathbb{J}\mathcal{A} &= \mathrm{J}X_p^{\mathcal{A}}\mathrm{d}X^{\mathcal{B}}(\partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}} - (-1)^{|\mathcal{A}||\mathcal{B}|}\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}) - \mathrm{d}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\\ &= -\mathrm{d}X^{\mathcal{B}}\mathrm{J}X_p^{\mathcal{A}}\mathcal{F}^p_{\mathcal{AB}} - \mathcal{D}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\,,\end{aligned} \tag{5.25}$$

where in the second line we have added and subtracted the commutator term in order to match the right-hand side of the above equation to equation (5.17). This completes our short excursion into differential geometry.

The above definition of the level-zero action makes direct reference to the gauge potential $\mathcal{A}$, which is not gauge covariant by itself. This will later lead to problems in the definition of the Yangian. For this reason, let us consider gauge transformations in more detail. Under an infinitesimal gauge transformation, the field insertion $\mathbb{J}\mathcal{A}$ changes as

$$\mathbb{G}[\Lambda](\mathbb{J}\mathcal{A}) = [\mathbb{J}\mathcal{A}, \Lambda] - \mathcal{D}(\mathrm{J}\Lambda)\,. \tag{5.26}$$

Consequently, the Wilson line operator with a field insertion of the form $\mathbb{J}\mathcal{A}$ transforms as

$$\mathbb{G}[\Lambda]\mathcal{V}[\mathbb{J}\mathcal{A}] = -\Lambda(0)\mathcal{V}[\mathbb{J}\mathcal{A}] + \mathcal{V}[\mathbb{J}\mathcal{A}]\Lambda(1) + (\mathrm{J}\Lambda)(0)\mathcal{V} - \mathcal{V}(\mathrm{J}\Lambda)(1)\,. \tag{5.27}$$

To obtain this result, we have used equation (5.10). The insertion of $\mathbb{J}\mathcal{A}$ obviously leads to two extra boundary terms which can be interpreted as follows: Under a gauge transformation, the Wilson line operator changes by the gauge-parameter field Λ evaluated at its two ends. Under a subsequent level-zero transformation, the Wilson line transforms as described above but the generator can also hit the gauge-parameter field explaining the two extra terms in equation (5.27). However, note that for a Wilson loop operator all the terms on the right-hand side of equation (5.27) conveniently cancel out

$$\mathbb{G}[\Lambda]\,\mathrm{tr}\,\mathcal{V}[\mathbb{J}\mathcal{A}] = \mathrm{tr}\Big[\mathcal{V}[\mathbb{J}\mathcal{A}], \Lambda\Big] - \mathrm{tr}[\mathcal{V}, \mathrm{J}\Lambda] = 0\,. \tag{5.28}$$

Consequently, the level-zero variation of a gauge-invariant Wilson loop operator is still gauge invariant.

Covariantized field action. Let us begin this paragraph by recalling the level-zero representation on the fields as introduced above. We found that the generator $\mathbb{J}$ acts on the gauge connection $\mathcal{A}$ in the following way:

$$\mathbb{J}\mathcal{A} = -\mathrm{d}X^{\mathcal{A}}\mathrm{J}X_p^{\mathcal{B}}\mathcal{F}^p_{\mathcal{B}\mathcal{A}} - \mathcal{D}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\,. \tag{5.29}$$

Taking into account our discussion on gauge transformations in the first paragraph, we observe that the second term in the above equation is just a field-dependent gauge transformation. While the field strength tensor transforms covariantly under a gauge transformation, the latter term does not and is therefore the source of the additional terms in equations (5.26) and (5.27). Furthermore, it is exactly this term which will later lead to problems when it comes to defining the level-one generators. For this reason, we will now introduce the gauge-covariant field representation. As gauge transformations do not play a role when dealing with gauge-invariant quantities, such as Wilson loops, we can redefine the level-zero actions of the last paragraph by supplementing each level-zero generator by a compensating gauge transformation of the form $\mathbb{G}[\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}]$. On the gauge connection $\mathcal{A}$, such a gauge transformation acts as

$$\mathbb{G}[\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}]\mathcal{A} = \mathcal{D}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\,. \tag{5.30}$$

It thus makes sense to define the covariantized field actions in the following way:

$$\mathbb{J}_* := \mathbb{J} + \mathbb{G}[\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}]\,. \tag{5.31}$$

When applied to the gauge potential $\mathcal{A}$, the compensating gauge transformation cancels the covariant derivative term being present in equation (5.29), so that only the field strength term survives

$$\mathbb{J}_*\mathcal{A} = -\mathrm{d}X^{\mathcal{A}}\mathrm{J}X_p^{\mathcal{B}}\mathcal{F}^p_{\mathcal{B}\mathcal{A}}\,. \tag{5.32}$$

The gauge-covariant level-zero representation thus acts on the Wilson line by the insertion of the manifestly gauge-covariant field strength tensor

$$\mathbb{J}_*\mathcal{V} = \mathcal{V}[\mathbb{J}_*\mathcal{A}] = \mathcal{V}\Big[-\mathrm{d}X^{\mathcal{B}}\mathrm{J}X_p^{\mathcal{A}}\mathcal{F}^p_{\mathcal{A}\mathcal{B}}\Big]\,. \tag{5.33}$$

Let us compare this to the straight level-zero field action $\mathbb{J}\mathcal{V} := \mathcal{V}[\mathbb{J}\mathcal{A}]$ defined in equation (5.16). The difference between the two prescriptions is a gauge transformation, which acts as the insertion of a covariant derivative operator into the Wilson line

$$\begin{aligned}\mathbb{J}_*\mathcal{V} - \mathbb{J}\mathcal{V} &= \mathbb{G}[\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}}]\mathcal{V} = \mathcal{V}\Big[\mathcal{D}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})\Big]\\ &= -(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})(0)\mathcal{V} + \mathcal{V}(\mathrm{J}X_p^{\mathcal{A}}\mathcal{A}^p_{\mathcal{A}})(1)\,.\end{aligned} \tag{5.34}$$

Ultimately, the difference is thus given by the gauge-parameter field evaluated at the two ends of the path. However, for a Wilson loop the terms on the right-hand side cancel out as expected

$$\mathbb{J}\,\mathrm{tr}\,\mathcal{V} = \mathbb{J}_*\,\mathrm{tr}\,\mathcal{V}\,. \tag{5.35}$$

For the purpose of studying the Lie algebra symmetry, it therefore makes no difference whether we use the straight field action (5.17) or the covariantized field action (5.31). However, the local actions are different and this difference is very important for the Yangian action to be discussed below.

Comparison. As already pointed out in the beginning of this section, the level-one symmetry of super Wilson loops has been investigated before in reference [1], where the present author and his collaborators proved the Yangian symmetry to leading order in the Graßmann expansion. In this paragraph, we want to compare the three different level-zero representations introduced above to the representation used in reference [1]. The approach that the authors pursued in their paper is actually equivalent to what we call the path action and only differs from it in small technical aspects. Let us explain this in more detail. In reference [1], the authors represented the level-zero generators in terms of variational derivative operators of the form

$$\mathbf{J} = \int \mathrm{d}\tau\, \mathbf{J}(\tau)\,, \qquad \mathbf{J}(\tau) = \mathrm{J}X_p^{\mathcal{A}}(X(\tau))\,\frac{\delta}{\delta X^{\mathcal{A}}(\tau)}\,. \tag{5.36}$$

When applied to the path variables $X^{\mathcal{A}}(\sigma)$, such variational derivative operators produce delta functions, for example

$$\frac{\delta X^{\mathcal{A}}(\sigma)}{\delta X^{\mathcal{B}}(\tau)} = \delta^{\mathcal{A}}_{\mathcal{B}}\delta(\sigma-\tau)\,, \qquad \frac{\delta \dot{X}^{\mathcal{A}}(\sigma)}{\delta X^{\mathcal{B}}(\tau)} = \delta^{\mathcal{A}}_{\mathcal{B}}\partial_\sigma\delta(\sigma-\tau)\,. \tag{5.37}$$

Applying the generator (5.36) to the Wilson line operator (5.1) thus yields

$$\begin{aligned}\mathbf{J}\mathcal{V} = \int \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{J}X_p^{\mathcal{A}}(\tau)\mathcal{V}_{[0,\sigma]}\Big(&\mathcal{A}^p_{\mathcal{A}}(\sigma)\partial_\sigma\delta(\sigma-\tau) \\ &+(-1)^{|\mathcal{A}||\mathcal{B}|}\dot{X}^{\mathcal{B}}(\sigma)\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}(\sigma)\delta(\sigma-\tau)\Big)\mathcal{V}_{[\sigma,1]}\,.\end{aligned} \tag{5.38}$$

In order to get rid of the derivative of the delta function, we have two options. We can either perform integration by parts directly on σ or we can use the translation invariance of the delta function to write $\partial_\sigma\delta(\sigma-\tau) = -\partial_\tau\delta(\sigma-\tau)$ and perform integration by parts on τ. Pursuing the first prescription yields

$$\begin{aligned}\mathbf{J}\mathcal{V} &\simeq -\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{J}X_p^{\mathcal{A}}(\tau)\delta(\sigma-\tau)\partial_\sigma\left(\mathcal{V}_{[0,\sigma]}\mathcal{A}^p_{\mathcal{A}}(\sigma)\mathcal{V}_{[\sigma,1]}\right) \\ &\quad+\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{J}X_p^{\mathcal{A}}(\tau)\delta(\sigma-\tau)\mathcal{V}_{[0,\sigma]}(-1)^{|\mathcal{A}||\mathcal{B}|}\dot{X}^{\mathcal{B}}(\sigma)\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}(\sigma)\mathcal{V}_{[\sigma,1]} \\ &= -\int \mathrm{d}\tau\,\mathcal{V}_{[0,\tau]}\mathrm{J}X_p^{\mathcal{A}}\dot{X}^{\mathcal{B}}\Big(\mathcal{A}^p_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}}+\partial_{\mathcal{B}}\mathcal{A}^p_{\mathcal{A}}-(-1)^{|\mathcal{A}||\mathcal{B}|}(\mathcal{A}^p_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}+\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}})\Big)\mathcal{V}_{[\tau,1]} \\ &= \mathcal{V}\Big[-\mathrm{J}X_p^{\mathcal{A}}\mathrm{d}X^{\mathcal{B}}\mathcal{F}^p_{\mathcal{B}\mathcal{A}}\Big] = -\mathcal{V}[\mathbb{J}_*\mathcal{A}] = -\mathbb{J}_*\mathcal{V}\,.\end{aligned} \tag{5.39}$$

Alternatively, we can perform integration by parts on τ and obtain

$$\begin{aligned}\mathbf{J}\mathcal{V} &\simeq \int \mathrm{d}\tau\,\mathrm{d}\sigma\,\partial_\tau\Big(\mathrm{J}X^{\mathcal{A}}_p(\tau)\Big)\delta(\sigma-\tau)\mathcal{V}_{[0,\sigma]}\mathcal{A}^p_{\mathcal{A}}(\sigma)\mathcal{V}_{[\sigma,1]}\\ &\quad+\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{J}X^{\mathcal{A}}_p(\tau)\delta(\sigma-\tau)\mathcal{V}_{[0,\sigma]}\dot{X}^{\mathcal{B}}(\sigma)\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}(\sigma)(-1)^{|\mathcal{A}||\mathcal{B}|}\mathcal{V}_{[\sigma,1]}\\ &=\int \mathrm{d}\tau\,\mathcal{V}_{[0,\tau]}\left(\partial_\tau(\mathrm{J}X^{\mathcal{A}}_p)\mathcal{A}^p_{\mathcal{A}}+\dot{X}^{\mathcal{B}}\mathrm{J}X^{\mathcal{A}}_p\partial_{\mathcal{A}}\mathcal{A}^p_{\mathcal{B}}\right)\mathcal{V}_{[\tau,1]}\\ &=-\mathcal{V}[\mathbb{J}\mathcal{A}]=-\mathbb{J}\mathcal{V}\,.\end{aligned}\tag{5.40}$$

Obviously, the two expressions are different as one corresponds to an insertion of the gauge-covariant object $\mathbb{J}_*\mathcal{A}$, while the other one corresponds to an insertion of the straight field action $\mathbb{J}\mathcal{A}$. While this difference is irrelevant for the level-zero symmetry of Wilson loops, the Yangian crucially depends on it. In fact, as will be explained below, gauge covariance (gauge invariance) requires the use of the $\mathbb{J}_*\mathcal{A}$ action in the level-one generators. The attentive reader might object that the difference in the two equations given above is merely due to the different boundary terms that have been dropped while performing integration by parts. However, note that the boundary terms are not even well-defined as they contain delta functions which are located at the boundaries of the integration domain. In any case, the equations (5.39) and (5.40) hopefully make it clear that the path derivative approach to Yangian symmetry has to be used with care as one might easily loose gauge invariance, which would render the computation meaningless. For this reason, we refrain from using this framework and instead define the level-one generators based on the gauge-covariant field action that has been introduced above. This will be the subject of the next section.

Gauge-covariant algebra. Before we turn to the construction of the level-one generators, let us study the algebra of the different level-zero representations. As already explained above, the generators of path transformations J^δ and the generators of straight field transformations $\mathbb{J}^\delta$ satisfy the same algebra relations by construction, i.e.

$$[\mathrm{J}^\delta,\mathrm{J}^\rho]=f^{\delta\rho}{}_\kappa\,\mathrm{J}^\kappa\,,\qquad\qquad [\mathbb{J}^\delta,\mathbb{J}^\rho]=f^{\delta\rho}{}_\kappa\,\mathbb{J}^\kappa\,.\tag{5.41}$$

However, for the generators in the gauge-covariant field representation $\mathbb{J}^\delta_*$ the algebra relations (5.41) are no longer guaranteed. For this reason, let us investigate them in more detail. Making one of the generators on the left-hand side of equation (5.41) gauge covariant yields the following mixed algebra relation:

$$[\mathbb{J}^\delta,\mathbb{J}^\rho_*]=f^{\delta\rho}{}_\kappa\,\mathbb{J}^\kappa_*\,.\tag{5.42}$$

This equation can easily be verified by acting with the generators on the fundamental field $\mathcal{A}$. Commuting two gauge-covariant field actions, however, yields an extra term on the right-hand side. Explicitly, we find

$$[\mathbb{J}^\delta_*,\mathbb{J}^\rho_*]\mathcal{A}=f^{\delta\rho}{}_\kappa\mathbb{J}^\kappa_*\mathcal{A}-\mathcal{D}\mathcal{G}^{\delta\rho}\,,\tag{5.43}$$

where

$$\mathcal{G}^{\delta\rho} = -\mathrm{J}^\delta X_p^{\mathcal{A}}\, \mathrm{J}^\rho X_p^{\mathcal{B}}\, \mathcal{F}_{\mathcal{B}\mathcal{A}}^p\,. \tag{5.44}$$

The first term in equation (5.43) represents a covariantized level-zero transformation, while the second term corresponds to a gauge transformation with gauge-parameter field $-\mathcal{G}^{\delta\rho}$,

$$[\mathbb{J}_*^\delta, \mathbb{J}_*^\rho] = f^{\delta\rho}{}_\kappa \mathbb{J}_*^\kappa + \mathbb{G}[-\mathcal{G}^{\delta\rho}]\,. \tag{5.45}$$

This result is not surprising as the generators $\mathbb{J}_*^\delta$ are composite objects, which combine a level-zero transformation with a gauge transformation. On gauge-invariant objects, such as Wilson loops, the algebra reduces to the plain level-zero algebra. The appearance of gauge transformations in the algebra of spacetime transformations is in fact a standard feature of gauge theories with extended supersymmetry, see, for example, [111] for a textbook treatment of this topic. For this reason, it seems to be natural to use the composite level-zero actions $\mathbb{J}_*$ instead of the plain ones $\mathbb{J}$.

5.1.2. Yangian Action on the Wilson Line

We now turn to one of the central aspects of this chapter: the construction of the level-one generators. For simplicity, let us start by considering a discrete multi-site space, which, for example, could be spanned by the product of n matrix-valued fields

$$\mathcal{A}(X_{\tilde{1}})\mathcal{A}(X_{\tilde{2}})\mathcal{A}(X_{\tilde{3}})\mathcal{A}(X_{\tilde{4}})\mathcal{A}(X_{\tilde{5}})\mathcal{A}(X_{\tilde{6}})\,. \tag{5.46}$$

In fact, expanding the Wilson line operator exactly yields expressions of this form, but in this case the coordinates $X_{\tilde{i}}$ were to read as $X_{\tilde{i}} := X(\tau_{\tilde{i}})$ with the $\tau_{\tilde{i}}$'s being the curve parameters, which would also determine the positions of the fields within the matrix product due to path ordering. For the moment, let us, however, stay general and interpret the coordinates $X_{\tilde{i}}$'s just as spacetime coordinates. At each site i we have a representation of the underlying Lie algebra in terms of generators $\mathbb{J}_i^\kappa$ acting on the fields. Here, i labels the position of the field within the matrix product, while $\tilde{i}$ enumerates the external coordinates. It is important to note that, in the first place, the number of sites is determined by the number of fields in the matrix product and not by the number of external coordinates. Each field represents one site. On the multi-site tensor product space, the level-zero generators thus act as

$$\mathbb{J}^\kappa = \sum_i \mathbb{J}_i^\kappa\,. \tag{5.47}$$

Let us note that having specified the level-zero representation, the bi-local part of the level-one generators is fully determined thanks to the coproduct rule (2.60). Explicitly, one finds

$$\widehat{\mathbb{J}}_{\mathrm{bi}}^\kappa = f^\kappa{}_{\rho\delta} \sum_{i<j} \mathbb{J}_i^\delta\, \mathbb{J}_j^\rho\,, \tag{5.48}$$

where we have dropped the prefactor for later convenience. A few comments on this bi-local part of the Yangian generators are in order. Let us emphasize again that the index i enumerates fields and not coordinates. The dimension with respect to which the ordering in the level-one generator is defined is thus a dimension in color space, being defined by the matrix product of matrix-valued fields, see equation (5.46). In fact, it is the only sensible interpretation in the context of four-dimensional theories as there is in general no ordering prescription which is related to the underlying spacetime. Furthermore, it also explains the crucial role that the planar limit plays for integrability in $\mathcal{N} = 4$ SYM theory: Only for large N can we expect single-trace structures to be dominant, leading to a single dimension in color space with respect to which the ordering can be defined. As soon as double-trace terms enter the game the prescription breaks down. Obviously, there exists a trivial bijection between the two sets of indices i and $\tilde{i}$ in case that all the coordinates are different. However, this mapping fails to be injective if fields are located at the same point in spacetime. For example, let us assume that the first two fields in equation (5.46) are located at the same point in spacetime. Applying the level-one generator (5.48) to this expression yields

$$\widehat{\mathbb{J}}^{\kappa}_{\text{bi}}(\mathcal{A}(X_{\tilde{1}})\mathcal{A}(X_{\tilde{1}})\mathcal{A}(X_{\tilde{2}})\ldots) = f^{\kappa}{}_{\rho\delta}((\mathbb{J}^{\delta}_{1}\mathcal{A}(X_{\tilde{1}}))(\mathbb{J}^{\rho}_{2}\mathcal{A}(X_{\tilde{1}}))\mathcal{A}(X_{\tilde{2}})\ldots) + \ldots\,, \tag{5.49}$$

where the dots represent all the other terms which we have omitted for brevity. Now, suppose we had written the level-one generator directly in terms of differential operators $\mathrm{J}^{\delta}_{\tilde{i}}$ with indices $\tilde{i}$ enumerating the external coordinates, i.e.

$$\widehat{\mathrm{J}}^{\kappa} = f^{\kappa}{}_{\rho\delta} \sum_{\tilde{i}<\tilde{j}} \mathrm{J}^{\delta}_{\tilde{i}}\, \mathrm{J}^{\rho}_{\tilde{j}}\,. \tag{5.50}$$

Quite obviously we would miss the first term in equation (5.49) and exclusively get those terms which are represented by dots in the very same equation. This reasoning makes it clear that it is not always possible to rewrite the higher-level generators of the Yangian in terms of differential operators which act from the outside on the object under consideration. For this reason, we consider the representation (5.48) with level-zero generators acting on the fields and an index i specifying the position of the field within the matrix product the most fundamental one. Quite often, like, for example, in the case of color-ordered tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM theory with generic momenta, one can directly go over to a generator written in term of differential operators (5.50). However, as explained above, this is not always possible. Finally, let us point the reader to the paper [26], where the (covariantized) field action approach to Yangian symmetry was used to show the invariance of the equations of motion of $\mathcal{N} = 4$ SYM theory as well as of the classical action.

Let us now generalize the above definitions to the case of Wilson lines/loops. In analogy to the straight level-zero action (5.16), we define the straight level-one action as

$$\widehat{\mathbb{J}}^{\kappa}\mathcal{V} = \mathcal{V}\Big[\widehat{\mathbb{J}}^{\kappa}\mathcal{A}\Big] + f^{\kappa}{}_{\rho\delta}\mathcal{V}\Big[\mathbb{J}^{\delta}\mathcal{A};\mathbb{J}^{\rho}\mathcal{A}\Big]\,. \tag{5.51}$$

Here, the first term represents a local term $\widehat{\mathbb{J}}^\kappa\mathcal{A}$, which we have included for completeness. However, since we do not yet know its form, we shall mainly disregard it in what follows. The second term denotes an ordered bi-local insertion of two level-zero actions combined with the structure constants and represents the continuous version of (5.48), see also equation (5.3).

As discussed before, there are two alternative definitions $\mathbb{J}$ and $\mathbb{J}_*$ for the local level-zero action. While in the case of level-zero symmetry it does not matter which action we choose, the Yangian crucially depends on it. To see this, let us investigate how the bi-local part in equation (5.51) transforms under an infinitesimal gauge transformation. We find

$$\begin{aligned}
\mathbb{G}[\Lambda]\widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V} &= f^\kappa{}_{\rho\delta}\mathbb{G}[\Lambda]\mathcal{V}\Big[\mathbb{J}^\delta\mathcal{A};\mathbb{J}^\rho\mathcal{A}\Big] \\
&= -\Lambda(0)\widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V} + \widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V}\Lambda(1) \\
&\quad + f^\kappa{}_{\rho\delta}\mathcal{V}\Big[\mathcal{D}(\mathrm{J}^\delta\Lambda);\mathrm{J}^\rho\mathcal{A}\Big] + f^\kappa{}_{\rho\delta}\mathcal{V}\Big[\mathrm{J}^\delta\mathcal{A};\mathcal{D}(\mathrm{J}^\rho\Lambda)\Big] \\
&= -\Lambda(0)\widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V} + \widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V}\Lambda(1) \\
&\quad + f^\kappa{}_{\rho\delta}\mathcal{V}\Big[\mathrm{J}^\delta\Lambda\,\mathrm{J}^\rho\mathcal{A} - \mathrm{J}^\delta\mathcal{A}\,\mathrm{J}^\rho\Lambda\Big] \\
&\quad - f^\kappa{}_{\rho\delta}\mathrm{J}^\delta\Lambda(0)\mathcal{V}[\mathrm{J}^\rho\mathcal{A}] + f^\kappa{}_{\rho\delta}\mathcal{V}[\mathrm{J}^\delta\mathcal{A}]\mathrm{J}^\rho\Lambda(1)\,.
\end{aligned} \tag{5.52}$$

Note that in the second line we have rewritten the straight field action in terms of a generator that acts on coordinates for the cost of an extra minus sign. Beyond the expected terms, we obviously find extra local terms as well as terms which involve simultaneous insertions in the bulk and at the boundary. Unfortunately, these terms do not cancel out for a closed Wilson loop

$$\mathbb{G}[\Lambda]\,\mathrm{tr}(\widehat{\mathbb{J}}^\kappa_{\mathrm{bi}}\mathcal{V}) = f^\kappa{}_{\rho\delta}\,\mathrm{tr}\,\mathcal{V}\Big[\{\mathrm{J}^\delta\Lambda,\mathrm{J}^\rho\mathcal{A}\}\Big] - 2f^\kappa{}_{\rho\delta}\,\mathrm{tr}\Big(\mathrm{J}^\delta\Lambda\,\mathcal{V}[\mathrm{J}^\rho\mathcal{A}]\Big)\,. \tag{5.53}$$

Note that to obtain this equation we have used the antisymmetry of the structure constants. While the first term could well be canceled by the local term in equation (5.51), there is no obvious resolution for the second term in equation (5.53). The above computation thus shows that the definition of the Yangian action (5.51) does not respect gauge symmetry. It maps a gauge-invariant quantity to something which is not gauge invariant anymore and can thus not serve as an observable in a gauge theory.

Here is where the gauge-covariant level-zero representation comes into play. Instead of using the straight level-zero action to define the level-one generators, we can work with the gauge-covariant representation, i.e.

$$\begin{aligned}
\widehat{\mathbb{J}}^\kappa_*\mathcal{V} &= \mathcal{V}\Big[\widehat{\mathbb{J}}^\kappa_*\mathcal{A}\Big] + f^\kappa{}_{\rho\delta}\mathcal{V}\Big[\mathbb{J}^\delta_*\mathcal{A};\mathbb{J}^\rho_*\mathcal{A}\Big] \\
&= \mathcal{V}\Big[\widehat{\mathbb{J}}^\kappa_*\mathcal{A}\Big] + f^\kappa{}_{\rho\delta}\mathcal{V}\Big[\mathrm{J}^\delta X^\mathcal{A}_p\mathrm{d}X^\mathcal{B}\mathcal{F}^p_{\mathcal{B}\mathcal{A}};\mathrm{J}^\rho X^\mathcal{C}_p\mathrm{d}X^\mathcal{D}\mathcal{F}^p_{\mathcal{D}\mathcal{C}}\Big]\,.
\end{aligned} \tag{5.54}$$

As all the insertions transform covariantly under a gauge transformation, this definition respects gauge symmetry by construction. In particular, one can easily show that

$$\mathbb{G}[\Lambda]\,\mathrm{tr}\Big(\widehat{\mathbb{J}}^\kappa_{*,\mathrm{bi}}\mathcal{V}\Big) = 0\,. \tag{5.55}$$

This discussion makes it clear that while the level-zero symmetry can be studied using any of the three representations, the Yangian can only be defined meaningfully in terms of the covariantized representation. In what follows, we shall thus exclusively focus on the definition (5.54).

An important point to note is that the bi-locally transformed Wilson line, i.e. the second term in equation (5.54), is not a plain Wilson line anymore but a Wilson line with two operator insertions. Unlike a level-zero transformation, which can be interpreted as a deformation of the path, the bi-local level-one transformation cannot be interpreted in such a way. This will become clear later on when we show that the bi-locally transformed Wilson loop has a different divergence structure than the Wilson loop itself. The potentially different divergence structure of the bi-locally transformed Wilson line, however, immediately raises the question how these divergences can be absorbed in the final result. This is where the local piece of the level-one action comes into play. Let us see how this works in detail. In the further course of this chapter when it comes to concrete computations in section 5.3, we shall introduce a local cut-off that keeps the two insertions at a minimum distance

$$f^{\kappa}{}_{\rho\delta}\mathcal{V}\Big[\mathbb{J}^{\delta}_{*}\mathcal{A};\mathbb{J}^{\rho}_{*}\mathcal{A}\Big]_{\varepsilon} := f^{\kappa}{}_{\rho\delta}\int_{\tau'>\tau+\varepsilon}\mathcal{V}_{[0,\tau]}\,(\mathbb{J}^{\delta}_{*}\mathcal{A})(\tau)\,\mathcal{V}_{[\tau,\tau']}\,(\mathbb{J}^{\rho}_{*}\mathcal{A})(\tau')\,\mathcal{V}_{[\tau,1]}\,. \tag{5.56}$$

Since the divergence originates from the region where the two insertions approach each other, this prescription will regularize the UV divergence, which now manifests itself in terms of local $1/\varepsilon^n$ poles. To renormalize these divergent terms, we will adjust the local level-one action in such a way that both terms, i.e. the local as well as the bi-local term, are finite at finite ε and that there sum is independent of ε in the limit $\varepsilon \to 0$,

$$\widehat{\mathbb{J}}^{\kappa}_{*}\mathcal{V} = \mathcal{V}\Big[(\widehat{\mathbb{J}}^{\kappa}_{*}\mathcal{A})_{\varepsilon}\Big] + f^{\kappa}{}_{\rho\delta}\mathcal{V}\Big[\mathbb{J}^{\delta}_{*}\mathcal{A};\mathbb{J}^{\rho}_{*}\mathcal{A}\Big]_{\varepsilon}\,. \tag{5.57}$$

Fortunately, the bi-local action applied to the super Maldacena–Wilson loop produces only local terms even at the finite orders. This is totally non-generic and a clear sign of Yangian symmetry as it will allow us to promote the bi-local level-one actions to true symmetry generators by appropriately adjusting the local term. Proving this will be the subject of section 5.3.

5.1.3. Superspace and Scalar Couplings

The above discussion was based on slightly simplified assumptions. For this reason, let us now discuss how to generalize the results to superspace and to Wilson loops with scalar couplings.

Superspace coordinates. The generalization to superspace consists of two steps. The first step is to introduce graded coordinates parametrizing a spacetime with bosonic and fermionic directions. In the above discussion, we treated the index $\mathcal{A}$ as a true multi-index, meaning that we kept track of all the factors of $(-1)^{|\mathcal{A}|}$. For this reason, the $\mathcal{N}=4$ superspace coordinates $(x,\theta,\bar{\theta})$ can be introduced straightforwardly by just splitting apart the index $\mathcal{A}$ into bosonic and fermionic regions.

Superspace torsion. The second step is to introduce superspace torsion. As discussed in section 3.1.1, it is often more convenient to work in a supertranslation-invariant basis of one-forms instead of the plain basis spanned by the $\mathrm{d}X^{\mathcal{A}}$. However, both are valid bases and one can easily translate between them. This is what we shall discuss below.

In our discussion, we will not immediately specify to the case of the $\mathcal{N}=4$ non-chiral superspace but rather stay a bit more general. More precisely, we refer to the susy-covariant derivatives collectively by

$$D_{\mathcal{A}} = \partial_{\mathcal{A}} + T^{\mathcal{C}}_{\mathcal{A}\mathcal{B}} X^{\mathcal{B}} \partial_{\mathcal{C}} \,. \tag{5.58}$$

Here, $T^{\mathcal{C}}_{\mathcal{A}\mathcal{B}}$ is the torsion tensor, which is proportional to the ten-dimensional Pauli matrices in the $\mathcal{N}=4$ non-chiral superspace. However, the only assumptions that we make on this tensor are (a) that it is graded antisymmetric in the lower two indices and (b) that it is non-zero only if the upper index is bosonic and the lower indices are both fermionic. This certainly applies to the Pauli matrices. The basis of one-forms which is dual to the susy-covariant derivatives reads

$$\mathcal{E}^{\mathcal{A}} = \mathrm{d}X^{\mathcal{A}} + T^{\mathcal{A}}_{\mathcal{B}\mathcal{C}} X^{\mathcal{B}} \mathrm{d}X^{\mathcal{C}} \,. \tag{5.59}$$

The exterior derivative d can either be expressed using the supertranslation-invariant basis of one-forms or the plain basis, i.e.

$$\mathrm{d} = \mathcal{E}^{\mathcal{A}} D_{\mathcal{A}} = \mathrm{d}X^{\mathcal{A}} \partial_{\mathcal{A}} \,. \tag{5.60}$$

Furthermore, let us note that the quantities $\mathcal{A}$ and $\mathcal{F}$ are basis independent as well. In superspace, they are normally expanded in terms of the one-form basis $\mathcal{E}^{\mathcal{A}}$,

$$\mathcal{A} = \mathcal{E}^{\mathcal{A}} \mathcal{A}_{\mathcal{A}}, \qquad \mathcal{F} = -\tfrac{1}{2} \mathcal{E}^{\mathcal{A}} \wedge \mathcal{E}^{\mathcal{B}} \mathcal{F}_{\mathcal{B}\mathcal{A}} \,. \tag{5.61}$$

The advantage of expanding in the $\mathcal{E}^{\mathcal{A}}$-basis lies in the fact that in this basis the components of the gauge connection and of the field strength tensor do not mix under a supersymmetry transformation. Furthermore, some of the components $\mathcal{F}_{\mathcal{B}\mathcal{A}}$ are forced to zero by the constraints of the superspace gauge theory, see section 3.1.1. However, one can as well expand the quantities $\mathcal{A}$ and $\mathcal{F}$ in terms of the plain basis of one-forms, i.e.

$$\mathcal{A} = \mathrm{d}X^{\mathcal{A}} \mathcal{A}^{p}_{\mathcal{A}}, \qquad \mathcal{F} = -\tfrac{1}{2} \mathrm{d}X^{\mathcal{A}} \wedge \mathrm{d}X^{\mathcal{B}} \mathcal{F}^{p}_{\mathcal{B}\mathcal{A}} \,. \tag{5.62}$$

In both cases $\mathcal{F}$ and $\mathcal{A}$ are related via the equation $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. In the previous discussion of level-zero and level-one symmetry we often made use of the plain components $\mathcal{A}^{p}_{\mathcal{A}}$ and $\mathcal{F}^{p}_{\mathcal{B}\mathcal{A}}$. Given the equations (5.59), (5.61) and (5.62), it is, however, an easy exercise to show that the plain components and the susy-covariant components are related in the following way:

$$\begin{aligned} \mathcal{A}^{p}_{\mathcal{A}} &= \mathcal{A}_{\mathcal{A}} - T^{\mathcal{C}}_{\mathcal{A}\mathcal{B}} X^{\mathcal{B}} \mathcal{A}_{\mathcal{C}} \,, \\ \mathcal{F}^{p}_{\mathcal{B}\mathcal{A}} &= \mathcal{F}_{\mathcal{B}\mathcal{A}} - T^{\mathcal{D}}_{\mathcal{B}\mathcal{C}} X^{\mathcal{C}} \mathcal{F}_{\mathcal{D}\mathcal{A}} - T^{\mathcal{C}}_{\mathcal{A}\mathcal{D}} \mathcal{F}_{\mathcal{B}\mathcal{C}} X^{\mathcal{D}} - T^{\mathcal{D}}_{\mathcal{A}\mathcal{C}} X^{\mathcal{C}} T^{\mathcal{F}}_{\mathcal{B}\mathcal{E}} X^{\mathcal{E}} \mathcal{F}_{\mathcal{F}\mathcal{D}} \,. \end{aligned} \tag{5.63}$$

Similarly, the generators of level-zero transformations can be expressed using either of the two bases described above

$$\mathrm{J} = \mathrm{J}X_p^{\mathcal{A}}\partial_{\mathcal{A}} = \mathrm{J}X^{\mathcal{A}}D_{\mathcal{A}}\,, \tag{5.64}$$

where the infinitesimal displacements are related as

$$\mathrm{J}X_p^{\mathcal{A}} = \mathrm{J}X^{\mathcal{A}} + T^{\mathcal{A}}_{\mathcal{C}\mathcal{D}}\,\mathrm{J}X^{\mathcal{C}}X^{\mathcal{D}}\,. \tag{5.65}$$

It is straightforward to verify that the following two forms of the gauge-covariantized level-zero action are completely equivalent:

$$\mathbb{J}_*\mathcal{A} = -\mathrm{d}X^{\mathcal{B}}\mathrm{J}X_p^{\mathcal{A}}\mathcal{F}^p_{\mathcal{A}\mathcal{B}} = -\mathcal{E}^{\mathcal{B}}\mathrm{J}X^{\mathcal{A}}\mathcal{F}_{\mathcal{A}\mathcal{B}}. \tag{5.66}$$

In fact, by noting that the transformation law (5.66) can be written as the contraction of the vector field J with the field strength two-form $\mathcal{F}$, it becomes clear that the expression cannot depend on the basis chosen. Since this applies to all the equations we have written so far, one can effectively perform the following replacements in the above relations:

$$\begin{aligned} \mathcal{A}^p_A &\to \mathcal{A}_{\mathcal{A}}, \qquad & \mathrm{d}X^{\mathcal{A}} &\to \mathcal{E}^{\mathcal{A}}, \\ \mathcal{F}^p_{\mathcal{A}\mathcal{B}} &\to \mathcal{F}_{\mathcal{A}\mathcal{B}}, \qquad & \mathrm{J}X_p^{\mathcal{A}} &\to \mathrm{J}X^{\mathcal{A}}. \end{aligned} \tag{5.67}$$

The replacements make torsion more evident, but they do not change the content of the relations.

Scalar couplings. In order to make contact to the super Maldacena–Wilson loop operator, we need to generalize the above relations to Wilson lines with scalar couplings. We can write such a Wilson line as

$$\mathcal{V} = \mathrm{P}\exp\int_\Gamma(\mathcal{A}+\Phi)\,. \tag{5.68}$$

Here, $\mathcal{A}$ denotes the pullback of the ordinary gauge connection one-form, while by Φ we refer to a one-form on the path of the form

$$\Phi = \mathrm{d}\tau\,Y^{\mathcal{M}}(\tau)\Phi_{\mathcal{M}}(X(\tau))\,, \tag{5.69}$$

where $Y^{\mathcal{M}}(\tau)$ describes a path in the internal space. Note that we assume $Y^{\mathcal{M}}(\tau)$ to be independent of $X^{\mathcal{A}}(\tau)$, so that the generators of level-zero transformations act on it exclusively via derivatives with respect to $Y^{\mathcal{M}}$, see also section 4.2.2.2. For the super Maldacena–Wilson loop operator, where $Y^{\mathcal{M}}(\tau)$ corresponds to $q^i(\tau)$, this means that we treat $q^i(\tau)$ as an independent coordinate which together with the supermomentum satisfies a ten-dimensional light-likeness constraint. We do not solve this constraint explicitly.

Let us collect some relevant identities for the scalar term Φ. First of all, Φ transforms covariantly under gauge transformations, i.e.

$$\mathbb{G}[\Lambda]\Phi = [\Phi, \Lambda]\,. \tag{5.70}$$

The level-zero field actions on Φ are defined in an analogous way as the field actions on $\mathcal{A}$. More precisely, we define the plain field action and the covariantized field action as minus the path action

$$\begin{aligned} \mathbb{J}\Phi &:= -\mathrm{J}X_p^{\mathcal{A}}\partial_{\mathcal{A}}\Phi - (\mathrm{J}Y^{\mathcal{M}})\Phi_{\mathcal{M}}\,, \\ \mathbb{J}_*\Phi &:= -(\mathrm{J}X_p^{\mathcal{A}})\mathcal{D}^p_{\mathcal{A}}\Phi - (\mathrm{J}Y^{\mathcal{M}})\Phi_{\mathcal{M}}\,, \end{aligned} \tag{5.71}$$

where $\mathcal{D}^p_{\mathcal{A}}$ denotes the plain gauge-covariant derivative, i.e. $\mathcal{D}^p_{\mathcal{A}} = \partial_{\mathcal{A}} + \mathcal{A}^p_{\mathcal{A}}$. Note that the path now also includes the internal directions, which are parametrized by $Y^{\mathcal{M}}(\tau)$. The internal path enters the expressions, however, only via the coupling to the scalars as all the fields are assumed to be independent of this coordinate. We shall denote its infinitesimal displacement by $\mathrm{J}Y^{\mathcal{M}}$. As explained above, these relations can also be expressed using the susy-covariant basis of vector fields

$$\begin{aligned} \mathbb{J}\Phi &:= -\mathrm{J}X^{\mathcal{A}}D_{\mathcal{A}}\Phi - (\mathrm{J}Y^{\mathcal{M}})\Phi_{\mathcal{M}}\,, \\ \mathbb{J}_*\Phi &:= -(\mathrm{J}X^{\mathcal{A}})\mathcal{D}_{\mathcal{A}}\Phi - (\mathrm{J}Y^{\mathcal{M}})\Phi_{\mathcal{M}}\,. \end{aligned} \tag{5.72}$$

Here, $\mathcal{D}_{\mathcal{A}}$ is assumed to be both gauge and superspace covariant.

Finally, we note that the scalar term changes the relation between gauge-covariant derivatives and boundary terms. An insertion of a plain gauge-covariant derivative is no longer sufficient to produce a boundary term. Instead, we need to take into account the coupling to the scalars

$$\mathcal{V}\Big[\mathcal{D}\Lambda + [\Phi, \Lambda]\Big] = -\Lambda(0)\mathcal{V} + \mathcal{V}\Lambda(1)\,. \tag{5.73}$$

This relation, however, confirms the rule that the scalar term can typically be introduced by replacing all instances of $\mathcal{A}$ by

$$\mathcal{A} \to \mathcal{A} + \Phi\,. \tag{5.74}$$

This can be done even within gauge-covariant derivatives. For the remainder of this general part, we shall largely disregard the scalar term in the Wilson loop operator because it clutters the relations somewhat, while it can be reintroduced straightforwardly.

5.2. Consistency and the Yangian Algebra

An important question to answer is whether our formulation of Yangian symmetry is actually consistent. There are several aspects to be discussed in this context. The first one concerns the cyclicity of the Wilson loop operator, which is not manifestly respected by the level-one generators. Moreover, we need to verify the Yangian algebra relations as we have formulated the Yangian generators in terms of covariantized level-zero actions, which have an impact on the algebra. Subsequently, we briefly discuss the relation between kappa symmetry and Yangian symmetry and comment on the compatibility of the Yangian and the superspace constraints. During our investigations, we will see that the consistency of the Yangian heavily relies on a novel type of identity that involves the level-zero vector fields and the field strength two-form. Proving this identity for the superconformal algebra will be the subject of the last section.

5.2.1. Cyclicity

An important feature of closed Wilson loops is cyclicity. By cyclicity, we refer to the fact that the base point of the line integral can be chosen arbitrarily, i.e.

$$\operatorname{tr} \mathcal{V}_{[0,1]} = \operatorname{tr} \mathcal{V}_{[\tau,1+\tau]} \,. \tag{5.75}$$

Since the level-one generators are typically in tension with this feature, we need to verify the consistency explicitly. Our arguments towards cyclicity are similar to the ones used in [25] with some additions concerning gauge symmetry.

In order to check that the level-one action on two equivalent objects yields two equivalent results, we consider

$$\begin{aligned}
\widehat{\mathbb{J}}^\delta_* \operatorname{tr} \mathcal{V}_{[\tau,1+\tau]} - \widehat{\mathbb{J}}^\delta_* \operatorname{tr} \mathcal{V}_{[0,1]} &= -2 f^\delta{}_{\kappa\rho} \operatorname{tr} \mathbb{J}^\rho_* \mathcal{V}_{[0,\tau]} \mathbb{J}^\kappa_* \mathcal{V}_{[\tau,1]} \\
&= -2 f^\delta{}_{\kappa\rho}\, \mathbb{J}^\rho_* \Big[\operatorname{tr} \mathcal{V}_{[0,\tau]} \mathbb{J}^\kappa_* \mathcal{V}_{[\tau,1]}\Big] + 2 f^\delta{}_{\kappa\rho} \operatorname{tr} \mathcal{V}_{[0,\tau]} \mathbb{J}^\rho_* \mathbb{J}^\kappa_* \mathcal{V}_{[\tau,1]} \\
&= -2 f^\delta{}_{\kappa\rho}\, \mathbb{J}^\rho \Big[\operatorname{tr} \mathcal{V}_{[0,\tau]} \mathbb{J}^\kappa_* \mathcal{V}_{[\tau,1]}\Big] \\
&\quad + f^\delta{}_{\kappa\rho} f^{\rho\kappa}{}_\sigma \operatorname{tr} \mathcal{V}_{[0,\tau]} \mathbb{J}^\sigma_* \mathcal{V}_{[\tau,1]} - f^\delta{}_{\kappa\rho} \operatorname{tr} \mathcal{V}_{[0,\tau]} \mathcal{V}_{[\tau,1]} [\mathcal{D}\mathcal{G}^{\rho\kappa}] \,.
\end{aligned} \tag{5.76}$$

Here, we wrote the difference as the product of two level-zero generators acting on a part of the Wilson loop. We then let the first level-zero generator act on everything and compensated for that by the second term in the third line. Subsequently, we used the antisymmetry of the structure constants to rewrite the second term as a commutator and used the level-zero algebra relation (5.43) to obtain the third line. The result is clearly not zero in general, but fortunately all the terms have special properties. The first term represents a level-zero transformation of a gauge-invariant object and does therefore vanish within an expectation value. The second term is proportional to the dual Coxeter number of the underlying Lie algebra

$$f^\delta{}_{\kappa\rho} f^{\rho\kappa}{}_\sigma = 2\mathfrak{c}\, \delta^\delta_\sigma \,. \tag{5.77}$$

For the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ this number is zero, $\mathfrak{c} = 0$. The final term contains the combination

$$f^\delta{}_{\kappa\rho}\, \mathcal{G}^{\rho\kappa} \simeq 0 \,. \tag{5.78}$$

This combination is zero for $\mathcal{N}=4$ SYM theory as will be shown in section 5.2.6. In what follows, we shall refer to it as the $\mathcal{G}$-identity. This concludes our discussion on the cyclicity of the Wilson loop operator.

5.2.2. Mixed Level-One Algebra

The second point to be addressed is the algebra of the Yangian generators. As discussed in section 2.1.3, the Yangian generators need to transform in the adjoint representation of the underlying Lie algebra, i.e.

$$[\mathrm{J}^\delta, \widehat{\mathrm{J}}^\rho] = f^{\delta\rho}{}_\kappa \widehat{\mathrm{J}}^\kappa\,. \tag{5.79}$$

In the case at hand, we are forced to represent the level-one generators by the gauge-covariant level-one actions $\widehat{\mathbb{J}}_*$. However, for the level-zero generators there is a choice. We can either work with the straight field representation $\mathbb{J}$ or with the covariant representation $\mathbb{J}_*$. Let us consider the mixed algebra relation first. In what follows, we shall disregard the local terms of the level-one generators as we do not yet know their form

$$\widehat{\mathbb{J}}^\delta_* \mathcal{A} = 0\,. \tag{5.80}$$

In fact, in [26] it was shown that the level-one generators map the gauge connection to terms which are non-linear in the fields. However, these are not relevant for the Wilson loop expectation value at one loop and for this reason we shall disregard them here.

Let us start by evaluating the first half of the commutator. The level-zero generator can either hit one of the three Wilson line operators in equation (5.3) or one of the two insertions

$$\begin{aligned}
\mathbb{J}^\delta \widehat{\mathbb{J}}^\rho_* \mathcal{V} &= f^\rho{}_{\sigma\kappa}\, \mathbb{J}^\delta \mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}] \\
&= f^\rho{}_{\sigma\kappa}\Big(\mathcal{V}[\mathbb{J}^\delta\mathcal{A}; \mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}] + \mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\delta\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}] + \mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}; \mathbb{J}^\delta\mathcal{A}]\Big) \\
&\quad + f^\rho{}_{\sigma\kappa}\Big(\mathcal{V}[\mathbb{J}^\delta\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}] + \mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\delta\mathbb{J}^\sigma_*\mathcal{A}]\Big)\,.
\end{aligned} \tag{5.81}$$

Acting with the level-one generator as described in section 5.1.2 yields the following expression for the second half of the commutator:

$$\begin{aligned}
\widehat{\mathbb{J}}^\rho_* \mathbb{J}^\delta \mathcal{V} &= \widehat{\mathbb{J}}^\rho_* \mathcal{V}[\mathbb{J}^\delta\mathcal{A}] \\
&= f^\rho{}_{\sigma\kappa}\Big(\mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}; \mathbb{J}^\delta\mathcal{A}] + \mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\delta\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}] + \mathcal{V}[\mathbb{J}^\delta\mathcal{A}; \mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}]\Big) \\
&\quad + f^\rho{}_{\sigma\kappa}\Big(\mathcal{V}[\mathbb{J}^\kappa_*\mathcal{A}; \mathbb{J}^\sigma_*\mathbb{J}^\delta\mathcal{A}] + \mathcal{V}[\mathbb{J}^\kappa_*\mathbb{J}^\delta\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}]\Big)\,.
\end{aligned} \tag{5.82}$$

The first three terms are obviously present in both equations and therefore cancel out immediately. The remaining terms can be further simplified by using the mixed algebra relation (5.42) as well as the Jacobi identity for the structure constants. We thus find

$$\begin{aligned}\Big[\mathbb{J}^\delta,\widehat{\mathbb{J}}^\rho_*\Big]\mathcal{V} &= f^\rho{}_{\sigma\kappa}\Big(\mathcal{V}\Big[[\mathbb{J}^\delta,\mathbb{J}^\kappa_*]\mathcal{A};\mathbb{J}^\sigma_*\mathcal{A}\Big]+\mathcal{V}\Big[\mathbb{J}^\kappa_*\mathcal{A};[\mathbb{J}^\delta,\mathbb{J}^\sigma_*]\mathcal{A}\Big]\Big)\\ &= f^\rho{}_{\sigma\kappa}\Big(f^{\delta\kappa}{}_\omega\mathcal{V}\Big[\mathbb{J}^\omega_*\mathcal{A};\mathbb{J}^\sigma_*\mathcal{A}\Big]+f^{\delta\sigma}{}_\omega\mathcal{V}\Big[\mathbb{J}^\kappa_*\mathcal{A};\mathbb{J}^\omega_*\mathcal{A}\Big]\Big)\\ &= f^{\delta\rho}{}_\kappa f^\kappa{}_{\omega\sigma}\mathcal{V}\Big[\mathbb{J}^\sigma_*\mathcal{A};\mathbb{J}^\omega_*\mathcal{A}\Big] = f^{\delta\rho}{}_\kappa\,\widehat{\mathbb{J}}^\kappa_*\mathcal{V}\,. \end{aligned} \tag{5.83}$$

Fortunately, all violations of gauge covariance have canceled out in the commutator.

5.2.3. Gauge-Covariant Level-One Algebra

Let us now study the algebra of purely gauge-covariant actions. Naively, one might think that it suffices to decorate all the level-zero generators in equation (5.83) with a star and subsequently use the commutation relation of gauge-covariant level-zero generators (5.43). However, this is in fact not true. The derivation is more subtle and moreover relies on a special feature of $\mathcal{N}=4$ SYM theory. Let us therefore go through it in detail.

Let us start by considering the algebra on a single gauge connection $\mathcal{A}$. At first sight, this appears to be trivial because we have set to zero the local terms of the Yangian generators, see equation (5.80). However, there is a subtlety hiding which resolves an issue later on. Obviously, only one half of the commutator contributes non-trivially

$$\Big[\mathbb{J}^\delta_*,\widehat{\mathbb{J}}^\rho_*\Big]\mathcal{A} = -\widehat{\mathbb{J}}^\rho_*\mathbb{J}^\delta_*\mathcal{A} = \mathrm{d}X^\mathcal{A}\mathrm{J}^\delta X^\mathcal{B}_p\,\widehat{\mathbb{J}}^\rho_*\mathcal{F}^p_{\mathcal{B}\mathcal{A}}\,. \tag{5.84}$$

Here, it is tempting to disregard the action of $\widehat{\mathbb{J}}^\rho_*$ on $\mathcal{F}^p_{\mathcal{B}\mathcal{A}}$ as the field strength consists of $\mathcal{A}$ only on which the action is trivial. However, note that $\mathcal{F}$ contains products of fields which are located at the same point. As explained is section 5.1.2, we need to act on this contribution with the bi-local piece of the level-one generator. Doing so yields

$$\begin{aligned}\widehat{\mathbb{J}}^\rho_*\mathcal{F}^p_{\mathcal{B}\mathcal{A}} &= \widehat{\mathbb{J}}^\rho_*(\mathcal{A}^p_\mathcal{B}\mathcal{A}^p_\mathcal{A}-(-1)^{|\mathcal{A}||\mathcal{B}|}\mathcal{A}^p_\mathcal{A}\mathcal{A}^p_\mathcal{B})\\ &= f^\rho{}_{\sigma\kappa}\,\mathbb{J}^\kappa_*\mathcal{A}^p_\mathcal{B}\,\mathbb{J}^\sigma_*\mathcal{A}^p_\mathcal{A}+(-1)^{|\mathcal{A}||\mathcal{B}|}f^\rho{}_{\sigma\kappa}\,\mathbb{J}^\sigma_*\mathcal{A}^p_\mathcal{A}\,\mathbb{J}^\kappa_*\mathcal{A}^p_\mathcal{B}\,.\end{aligned} \tag{5.85}$$

Here, we have used the antisymmetry of the structure constants to swap the Lie algebra indices in the second term. Inserting the above relation back into equation (5.84) yields

$$\Big[\mathbb{J}^\delta_*,\widehat{\mathbb{J}}^\rho_*\Big]\mathcal{A} = f^\rho{}_{\sigma\kappa}\Big\{\mathrm{J}^\delta X^\mathcal{B}_p\mathbb{J}^\kappa_*\mathcal{A}^p_\mathcal{B},\mathbb{J}^\sigma_*\mathcal{A}\Big\} = f^\rho{}_{\sigma\kappa}\Big\{\mathcal{G}^{\delta\kappa},\mathbb{J}^\sigma_*\mathcal{A}\Big\}\,. \tag{5.86}$$

The term on the right-hand side can be further rewritten by letting the gauge-covariant level-zero generator act on everything and correcting for the discrepancy

$$\Big[\mathbb{J}^\delta_*,\widehat{\mathbb{J}}^\rho_*\Big]\mathcal{A} = f^\rho{}_{\sigma\kappa}\,\mathbb{J}^\sigma_*\Big\{\mathcal{G}^{\delta\kappa},\mathcal{A}\Big\} - f^\rho{}_{\sigma\kappa}\Big\{\mathbb{J}^\sigma_*\mathcal{G}^{\delta\kappa},\mathcal{A}\Big\}\,. \tag{5.87}$$

Using the gauge-covariant level-zero algebra relation (5.43), we can show that the discrepancy term is actually zero for $\mathcal{N}=4$ SYM theory

$$\begin{aligned}f^\rho{}_{\sigma\kappa}\mathbb{J}^\sigma_*\mathcal{G}^{\delta\kappa} &= f^\rho{}_{\sigma\kappa}\mathrm{J}^\delta X^\mathcal{A}_p\mathbb{J}^\sigma_*\mathbb{J}^\kappa_*\mathcal{A}^p_\mathcal{A} = \tfrac{1}{2}f^\rho{}_{\sigma\kappa}\mathrm{J}^\delta X^\mathcal{A}_p(f^{\sigma\kappa}{}_\omega\mathbb{J}^\omega_*\mathcal{A}^p_\mathcal{A}-\mathcal{D}^p_\mathcal{A}\mathcal{G}^{\sigma\kappa})\\ &= -\tfrac{1}{2}f^\rho{}_{\sigma\kappa}f^{\kappa\sigma}{}_\omega\mathcal{G}^{\delta\omega}+\tfrac{1}{2}f^\rho{}_{\sigma\kappa}\mathrm{J}^\delta X^\mathcal{A}_p\mathcal{D}^p_\mathcal{A}\mathcal{G}^{\kappa\sigma}\simeq 0\,.\end{aligned} \tag{5.88}$$

The first term vanishes because the dual Coxeter number is zero. The second term contains the combination $f^{\rho}{}_{\sigma\kappa}\mathcal{G}^{\kappa\sigma}$ which will be proven to be zero in section 5.2.6. Thus, we can write the algebra on $\mathcal{A}$ as

$$\Big[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}\Big]\mathcal{A}\simeq f^{\rho}{}_{\sigma\kappa}\,\mathbb{J}^{\sigma}_{*}\Big\{\mathcal{G}^{\delta\kappa},\mathcal{A}\Big\}\,. \tag{5.89}$$

Obviously, the result does not agree with the expected algebra as the commutation relation closes only up to a level-zero transformation. However, there are two things to keep in mind. First of all, let us recall that the above algebra relation was derived under the assumption that the level-one generators act trivially on the gauge connection, cf. equation (5.80). In reference [26], it was, however, shown that the level-one generators map the gauge connection to a term which is non-linear in the fields. Such a term would of course have an impact on the algebra even though it is unlikely that the above level-zero action term would get canceled. In fact, if the above term were not present, the algebra on the Wilson line would not close. This fact, which we will prove momentarily, actually makes another explanation more likely: The algebra does not need to close on the gauge connection alone, what matters is the algebra on the Wilson line. Let us demonstrate that this algebra indeed works out fine. First, we note that the first line in equation (5.83) has not only to be decorated by stars but also to be supplemented by the action of the commutator on a single gauge connection

$$\Big[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}\Big]\mathcal{V}=f^{\rho}{}_{\sigma\kappa}\Big(\mathcal{V}\Big[[\mathbb{J}^{\delta}_{*},\mathbb{J}^{\kappa}_{*}]\mathcal{A};\mathbb{J}^{\sigma}_{*}\mathcal{A}\Big]+\mathcal{V}\Big[\mathbb{J}^{\kappa}_{*}\mathcal{A};[\mathbb{J}^{\delta}_{*},\mathbb{J}^{\sigma}_{*}]\mathcal{A}\Big]+\mathcal{V}\Big[[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}]\mathcal{A}\Big]\Big)\,. \tag{5.90}$$

The reason why the last term could be dropped in equation (5.83) is that the straight field action is linear in the fields, while the gauge-covariant level-zero action is not. Using the gauge-covariant level-zero algebra relation (5.43), the Jacobi identity for the structure constants as well as equation (5.86), we can bring the algebra relation to the following form:

$$\begin{aligned}\Big[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}\Big]\mathcal{V}=&f^{\delta\rho}{}_{\kappa}\,\widehat{\mathbb{J}}^{\kappa}_{*}\mathcal{V}+f^{\rho}{}_{\sigma\kappa}\mathcal{V}\Big[\{\mathcal{G}^{\delta\kappa},\mathbb{J}^{\sigma}_{*}\mathcal{A}\}\Big]-f^{\rho}{}_{\sigma\kappa}\mathcal{V}\Big[\mathbb{J}^{\kappa}_{*}\mathcal{A};\mathcal{D}\mathcal{G}^{\delta\sigma}\Big]\\&-f^{\rho}{}_{\sigma\kappa}\mathcal{V}\Big[\mathcal{D}\mathcal{G}^{\delta\kappa};\mathbb{J}^{\sigma}_{*}\mathcal{A}\Big]\,.\end{aligned} \tag{5.91}$$

The covariant derivatives can now be integrated, leading to two local terms and two bulk-boundary terms. The former ones are, however, precisely canceled by the algebra acting on the gauge connection. We are thus left with

$$\Big[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}\Big]\mathcal{V}=f^{\delta\rho}{}_{\kappa}\,\widehat{\mathbb{J}}^{\kappa}_{*}\mathcal{V}+f^{\rho}{}_{\sigma\kappa}\Big(\mathcal{V}[\mathbb{J}^{\sigma}_{*}\mathcal{A}]\mathcal{G}^{\delta\kappa}(1)+\mathcal{G}^{\delta\kappa}(0)\mathcal{V}[\mathbb{J}^{\sigma}_{*}\mathcal{A}]\Big)\,. \tag{5.92}$$

As explained above, we can extend the gauge-covariant level-zero action in the second term such that it acts on the whole expression for the cost of a term which is zero in $\mathcal{N}=4$ SYM theory. Consequently, we obtain the following final result:

$$\Big[\mathbb{J}^{\delta}_{*},\widehat{\mathbb{J}}^{\rho}_{*}\Big]\mathcal{V}\simeq f^{\delta\rho}{}_{\kappa}\,\widehat{\mathbb{J}}^{\kappa}_{*}\mathcal{V}+f^{\rho}{}_{\sigma\kappa}\,\mathbb{J}^{\sigma}_{*}\Big(\mathcal{V}\mathcal{G}^{\delta\kappa}(1)+\mathcal{G}^{\delta\kappa}(0)\mathcal{V}\Big)\,. \tag{5.93}$$

This shows that the gauge-covariant level-one algebra relation closes but only on shell in $\mathcal{N}=4$ SYM theory.[2] Furthermore, as the commutator produces an extra level-zero transformation on the right-hand side, the algebra relation only holds at the level of the expectation value

$$\Big\langle \Big[\mathbb{J}_*^\delta, \widehat{\mathbb{J}}_*^\rho\Big] \operatorname{tr}\mathcal{V}\Big\rangle \simeq f^{\delta\rho}{}_\kappa \Big\langle \widehat{\mathbb{J}}_*^\kappa \operatorname{tr}\mathcal{V}\Big\rangle\,. \tag{5.94}$$

Here, we have used the level-zero symmetry which guarantees that $\langle \mathbb{J}_* H\rangle = 0$ for any gauge-invariant observable H,

$$f^\rho{}_{\sigma\kappa}\Big\langle \mathbb{J}_*^\sigma \operatorname{tr}\Big(\mathcal{V}\mathcal{G}^{\delta\kappa}(1) + \mathcal{G}^{\delta\kappa}(0)\mathcal{V}\Big)\Big\rangle = 0\,. \tag{5.95}$$

This concludes our discussion of the gauge-covariant level-one commutation relations. Finally, let us emphasize that verifying the adjoint transformation law of the level-one generators is strictly speaking not sufficient to prove that an algebra is a Yangian algebra. In fact, in a Yangian algebra the Serre relations (2.53) need to hold as well. In [65], it was investigated under which conditions the Serre relations are guaranteed to hold. Unfortunately, the arguments presented there cannot directly be applied to the case at hand. However, as the Serre relations are much more elaborate than the level-one commutation relations (5.93), we will make no attempt here to prove these relations. Clarifying whether and how they are satisfied is left for future work. If for some reason the Serre relations should not hold, the symmetry algebra of Wilson loops would not be a Yangian, but it would still be an infinite-dimensional algebra.

5.2.4. Yangian Symmetry and Kappa Symmetry

In section 4.2.1, we have shown that super Wilson loops which are null in a ten-dimensional sense enjoy eight additional fermionic translation symmetries which are in close relation to kappa symmetry of string theory. An important question to answer is whether kappa symmetry is compatible with Yangian symmetry. For the sake of consistency, we will discuss this question using the formalism developed in this chapter. The expressions that we will write are to be interpreted in a ten-dimensional sense and we use the susy-covariant basis to make the relation to the ten-dimensional equations given in section 4.2.1 completely obvious.

Definition. In analogy to the level-zero actions, we can formulate kappa symmetry as the covariantized action of a variation δ_κ^* on the fields

$$\delta_\kappa^* \mathcal{A}_\mathcal{A} := -\delta_\kappa X^\mathcal{B}\mathcal{F}_{\mathcal{B}\mathcal{A}}\,. \tag{5.96}$$

Here, $\delta_\kappa X^\mathcal{B}$ denotes the derivative coefficients of the kappa-symmetry-generating vector field when written in the basis of susy-covariant derivatives

$$\delta_\kappa = \delta_\kappa X^\mathcal{A} D_\mathcal{A}\,, \tag{5.97}$$

[2]The on-shell condition enters via the $\mathcal{G}$-identity, which only holds on shell.

see also equation (4.24). Note that relation (5.96) is in complete accordance with equation (4.26) up to the necessary minus sign and the compensating gauge transformation. Under a kappa-symmetry transformation, the Wilson line transforms as

$$\delta^*_\kappa \mathcal{V} = \mathcal{V}[\delta^*_\kappa \mathcal{A}] \,. \tag{5.98}$$

For a kappa-symmetric Wilson line the above variation vanishes exactly. This is due to the fact that the variation of the gauge connection vanishes when pulled back to a light-like path

$$\delta^*_\kappa \mathcal{A} = \delta_\kappa X^{\mathcal{A}}\, \mathcal{E}^{\mathcal{B}} \mathcal{F}_{\mathcal{B}\mathcal{A}} \to 0 \,. \tag{5.99}$$

Algebra. Let us now consider the algebra of covariantized kappa-symmetry transformations and level-zero as well as level-one transformations. For the level-zero commutator, we find

$$[\mathbb{J}^\rho_*, \delta^*_\kappa]\mathcal{A}_{\mathcal{A}} = -\delta_{\kappa'^\rho} X^{\mathcal{B}} \mathcal{F}_{\mathcal{B}\mathcal{A}} + \mathcal{D}_{\mathcal{A}}(\mathrm{J}^\rho X^{\mathcal{B}}\, \delta_\kappa X^{\mathcal{C}}\, \mathcal{F}_{\mathcal{C}\mathcal{B}}) \,. \tag{5.100}$$

In deriving this relation, we have assumed that at the level of coordinates the algebra of level-zero transformations and kappa-symmetry transformations closes onto another kappa-symmetry transformation. In section 4.2.3, we have verified this for the case at hand, i.e. for the superconformal algebra $\mathfrak{psu}(2,2|4)$. The first term in the above relation represents another kappa-symmetry transformation, while the second term corresponds to a gauge transformation. The general algebra thus reads

$$[\mathbb{J}^\rho_*, \delta^*_\kappa] = \delta^*_{\kappa'^\rho} + \mathbb{G}[\delta_\kappa X^{\mathcal{C}}\, \mathbb{J}^\rho_* \mathcal{A}_{\mathcal{C}}] \,. \tag{5.101}$$

Let us now consider the algebra of kappa-symmetry transformations and level-one transformations. For the action of the commutator on the gauge connection, we find

$$[\widehat{\mathbb{J}}^\rho_*, \delta^*_\kappa]\mathcal{A}_{\mathcal{A}} = -f^\rho{}_{\delta\sigma}\Big\{\delta_\kappa X^{\mathcal{C}} \mathbb{J}^\sigma_*\, \mathcal{A}_{\mathcal{C}}, \mathbb{J}^\delta_* \mathcal{A}_{\mathcal{A}}\Big\} \,. \tag{5.102}$$

Given this result, we can now straightforwardly derive the action of the commutator on the Wilson line. For a kappa-symmetric Wilson line, we obtain

$$\begin{aligned}
[\widehat{\mathbb{J}}^\delta_*, \delta^*_\kappa]\mathcal{V} =& \mathcal{V}\Big[[\widehat{\mathbb{J}}^\delta_*, \delta^*_\kappa]\mathcal{A}\Big] + f^\delta{}_{\sigma\rho}\mathcal{V}\Big[[\mathbb{J}^\rho_*, \delta^*_\kappa]\mathcal{A}; \mathbb{J}^\sigma_*\mathcal{A}\Big] + f^\delta{}_{\sigma\rho}\mathcal{V}\Big[\mathbb{J}^\rho_*\mathcal{A}; [\mathbb{J}^\sigma_*, \delta^*_\kappa]\mathcal{A}\Big] \\
=& - f^\rho{}_{\delta\sigma}\mathcal{V}\Big[\{\delta_\kappa X^{\mathcal{C}} \mathbb{J}^\sigma_*\, \mathcal{A}_{\mathcal{C}}, \mathbb{J}^\delta_*\mathcal{A}_{\mathcal{A}}\}\Big] + f^\delta{}_{\sigma\rho}\mathcal{V}\Big[\mathcal{D}(\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\rho_*\mathcal{A}_{\mathcal{B}}); \mathbb{J}^\sigma_*\mathcal{A}\Big] \\
& + f^\delta{}_{\sigma\rho}\mathcal{V}\Big[\mathbb{J}^\rho_*\mathcal{A}; \mathcal{D}(\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\sigma_*\mathcal{A}_{\mathcal{B}})\Big] \\
=& - f^\delta{}_{\sigma\rho}(\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\rho_*\mathcal{A}_{\mathcal{B}})(0)\mathcal{V}\Big[\mathbb{J}^\sigma_*\mathcal{A}\Big] - f^\delta{}_{\sigma\rho}\mathcal{V}\Big[\mathbb{J}^\sigma_*\mathcal{A}\Big](\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\rho_*\mathcal{A}_{\mathcal{B}})(1) \\
\simeq& - f^\delta{}_{\sigma\rho}\, \mathbb{J}^\sigma_*\, \Big((\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\rho_*\mathcal{A}_{\mathcal{B}})(0)\mathcal{V} + \mathcal{V}(\delta_\kappa X^{\mathcal{B}}\, \mathbb{J}^\rho_*\mathcal{A}_{\mathcal{B}})(1)\Big) \,.
\end{aligned} \tag{5.103}$$

This result is very reminiscent of (5.93). The extra gauge transformation on the right-hand side of equation (5.101) again leads to the insertion of two gauge-covariant derivatives, which can be integrated and lead to two bulk-boundary terms. However, as before, these terms can be rewritten as a complete level-zero transformation of a gauge-covariant (gauge-invariant) object because the discrepancy term is zero for $\mathcal{N} = 4$ SYM theory, see section 5.2.6. Therefore, the right-hand side of equation (5.103) vanishes at the level of the expectation value, hence proving that the Yangian algebra respects kappa symmetry.

5.2.5. Yangian Symmetry and the Constraints

A further aspect of consistency concerns the superspace formalism that we used to define the gauge theory. In chapter 3, we have shown that the formulation of $\mathcal{N} = 4$ SYM theory in non-chiral superspace requires constraints. For the Yangian to be consistent with the underlying gauge theory, the constraints need to be respected by the level-one actions. However, in this thesis we will not attempt to verify this and instead refer the reader to reference [26]. In this paper, the authors proved that the equations of motion of $\mathcal{N} = 4$ SYM theory close onto themselves under the action of the covariantized level-one generators. Since the equations of motion are completely equivalent to the constraints (see section 3.1.1), the consistency follows immediately from the considerations in this paper.

5.2.6. The $\mathcal{G}$-Identity

Let us close this section by proving an identity which played a prominent role in this section, namely the so-called $\mathcal{G}$-identity. For this purpose, we now specify the underlying Lie algebra to the case of interest for us, namely the superconformal algebra of $\mathcal{N} = 4$ SYM theory. Explicitly, the $\mathcal{G}$-identity reads

$$f^{\omega}{}_{\rho\kappa}\, \mathcal{G}^{\kappa\rho} \simeq 0\,. \tag{5.104}$$

Here, $f^{\omega}{}_{\rho\kappa}$ are the structure constants of $\mathfrak{u}(2,2|4)$ and $\mathcal{G}^{\kappa\rho}$ is given by

$$\begin{aligned} \mathcal{G}^{\kappa\rho} &= -(-1)^{|\mathcal{A}||\rho|}\, \mathrm{J}^{\kappa}X^{\mathcal{A}}\, \mathrm{J}^{\rho}X^{\mathcal{B}}\, \mathcal{F}_{\mathcal{B}\mathcal{A}} \\ &= -i[\mathrm{J}^{\kappa}]\, i[\mathrm{J}^{\rho}] \left(-\tfrac{1}{2}\mathcal{E}^{\mathcal{A}} \wedge \mathcal{E}^{\mathcal{B}}\mathcal{F}_{\mathcal{B}\mathcal{A}}\right) \\ &= -i[\mathrm{J}^{\kappa}]\, i[\mathrm{J}^{\rho}]\, \mathcal{F}\,. \end{aligned} \tag{5.105}$$

The extra minus sign compared to equation (5.44) is a consequence of the fact that we have now generalized the expression for $\mathcal{G}$ to the case of a superalgebra. As before, $i[\mathrm{J}^{\rho}]\,\mathcal{F}$ denotes the contraction (see section 5.1.1) of the vector field $\mathrm{J}^{\kappa} = \mathrm{J}^{\kappa}X^{\mathcal{A}}\, D_{\mathcal{A}}$ with the field strength two-form. Note that in relation (5.104) we have tactically extended the underlying algebra from $\mathfrak{psu}(2,2|4)$ to $\mathfrak{u}(2,2|4)$. The reason for this is that we want to discuss the invariance of our Wilson loop under level-one hypercharge transformations. An important point to note, however, is that once we have proven the $\mathcal{G}$-identity for $\mathfrak{u}(2,2|4)$, the $\mathfrak{psu}(2,2|4)$ version follows immediately. This becomes obvious by noting that the index range of the summed indices in (5.104) can actually be restricted to $\mathfrak{psu}(2,2|4)$ as the differential operator corresponding to C vanishes exactly, see section 4.2.2.2.

Our strategy to prove (5.104) is the following: First, we will show that the $\mathcal{G}$-identity holds for ω corresponding to the hypercharge generator B. In the next step, we will then use the superconformal transformation properties of $\mathcal{G}^{\kappa\rho}$ to argue that (5.104) holds for any $\mathfrak{u}(2,2|4)$ index ω. Note that for ω corresponding to C, the $\mathcal{G}$-identity is trivial since the structure constants vanish in this case.

If we fix ω to be equal to B, equation (5.104) becomes

$$f^{\omega}{}_{\rho\kappa}\,\mathcal{G}^{\kappa\rho} \propto \Big(i\big[\mathrm{S}^{b\beta}\big]i\big[\mathrm{Q}_{\beta b}\big] - i\big[\bar{\mathrm{S}}^{\dot\beta}{}_{b}\big]i\big[\bar{\mathrm{Q}}^{b}{}_{\dot\beta}\big]\Big)\,\mathcal{F}\,. \tag{5.106}$$

In order to evaluate this expression, it is useful to first compute the contractions between the vector fields $\mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{S}, \bar{\mathrm{S}}$ and the basis one-forms. We find

$$\begin{aligned}
i\big[\mathrm{Q}_{\beta b}\big]e^{\dot\alpha\alpha} &= 4\delta^{\alpha}_{\beta}\bar\theta^{\dot\alpha}{}_{b}\,, & i\big[\mathrm{Q}_{\beta b}\big]\mathrm{d}\theta^{a\alpha} &= -\delta^{a}_{b}\delta^{\alpha}_{\beta}\,,\\
i\big[\bar{\mathrm{Q}}^{b}{}_{\dot\beta}\big]e^{\dot\alpha\alpha} &= 4\delta^{\dot\alpha}_{\dot\beta}\theta^{b\alpha}\,, & i\big[\bar{\mathrm{Q}}^{b}{}_{\dot\beta}\big]\mathrm{d}\theta^{a\alpha} &= 0\,,\\
i\big[\mathrm{S}^{b\beta}\big]e^{\dot\alpha\alpha} &= -4(x^-)^{\dot\alpha\beta}\theta^{b\alpha}\,, & i\big[\mathrm{S}^{b\beta}\big]\mathrm{d}\theta^{a\alpha} &= 4\theta^{a\beta}\theta^{b\alpha}\,,\\
i\big[\bar{\mathrm{S}}^{\dot\beta}{}_{b}\big]e^{\dot\alpha\alpha} &= -4\bar\theta^{\dot\alpha}{}_{b}(x^+)^{\dot\beta\alpha}\,, & i\big[\bar{\mathrm{S}}^{\dot\beta}{}_{b}\big]\mathrm{d}\theta^{a\alpha} &= \delta^{a}_{b}(x^+)^{\dot\beta\alpha}\,,\\
i\big[\mathrm{Q}_{\beta b}\big]\mathrm{d}\bar\theta^{\dot\alpha}{}_{a} &= 0\,, & i\big[\bar{\mathrm{Q}}^{b}{}_{\dot\beta}\big]\mathrm{d}\bar\theta^{\dot\alpha}{}_{a} &= -\delta^{b}_{a}\delta^{\dot\alpha}_{\dot\beta}\,,\\
i\big[\mathrm{S}^{b\beta}\big]\mathrm{d}\bar\theta^{\dot\alpha}{}_{a} &= \delta^{b}_{a}(x^-)^{\dot\alpha\beta}\,, & i\big[\bar{\mathrm{S}}^{\dot\beta}{}_{b}\big]\mathrm{d}\bar\theta^{\dot\alpha}{}_{a} &= -4\bar\theta^{\dot\alpha}{}_{b}\bar\theta^{\dot\beta}{}_{a}\,.
\end{aligned} \tag{5.107}$$

Given these formulas, we can now apply the double-contraction expression of (5.106) to all the basis two-forms appearing in the decomposition of the super field strength (3.85). Note that due to the constraints (3.78), the field strength contains only a restricted set of basis two-forms. In particular, the coefficient of the mixed fermionic two-form is set to zero by (3.78). A useful formula for computing (5.106) is $i[X]i[Y](\alpha\wedge\beta) = -(i[X]\alpha)(i[Y]\beta) + (i[Y]\alpha)(i[X]\beta)$, where X, Y are vector fields and α, β are one-forms (this holds when all the quantities are Graßmann even; if not, extra signs are needed). As an example of a computation where fermionic signs appear, we consider

$$\begin{aligned}
&\Big(i\big[\mathrm{S}^{c\gamma}\big]i\big[\mathrm{Q}_{\gamma c}\big] - i\big[\bar{\mathrm{S}}^{\dot\gamma}{}_{c}\big]i\big[\bar{\mathrm{Q}}^{c}{}_{\dot\gamma}\big]\Big)\,\Big(\mathrm{d}\theta^{a\alpha}\,\varepsilon_{\alpha\beta}\,\mathrm{d}\theta^{b\beta}\Big)\\
&= i\big[\mathrm{S}^{c\gamma}\big]\Big[\Big(i\big[\mathrm{Q}_{\gamma c}\big]\mathrm{d}\theta^{a\alpha}\Big)\varepsilon_{\alpha\beta}\,\mathrm{d}\theta^{b\beta} + \mathrm{d}\theta^{a\alpha}\,\varepsilon_{\alpha\beta}\Big(i\big[\mathrm{Q}_{\gamma c}\big]\mathrm{d}\theta^{b\beta}\Big)\Big]\\
&= \varepsilon_{\alpha\beta}\Big[4\theta^{a\beta}\theta^{b\alpha} - 4\theta^{a\beta}\theta^{b\alpha}\Big]\\
&= 0\,.
\end{aligned} \tag{5.108}$$

In a similar way, one can show that the double-contraction expression in (5.106) also annihilates all the other basis two-forms appearing in (3.85).

Having proven the $\mathcal{G}$-identity for ω corresponding to B, we now proceed and argue that it actually holds for any $\mathfrak{u}(2,2|4)$ index ω. To do so, we use the following identity

$$\mathrm{J}^{\delta}_{*}\mathcal{G}^{\kappa\rho} = -\mathbb{J}^{\delta}_{*}\mathcal{G}^{\kappa\rho} + f^{\delta\kappa}{}_{\sigma}\mathcal{G}^{\sigma\rho} + (-1)^{|\delta||\kappa|}f^{\delta\rho}{}_{\sigma}\mathcal{G}^{\kappa\sigma}\,, \tag{5.109}$$

where the differential operator J^δ_* is defined as $\mathrm{J}^\delta_* = \mathrm{J}^\delta X^{\mathcal{A}}\mathcal{D}_{\mathcal{A}}$ and $\mathbb{J}^\delta_*$ is a composite superconformal generator of field transformations, which acts on $\mathcal{A}_{\mathcal{A}}$ as

$$\mathbb{J}^\delta_* \mathcal{A}_{\mathcal{A}} = -\mathrm{J}^\delta X^{\mathcal{B}}\,\mathcal{F}_{\mathcal{B}\mathcal{A}}\,. \tag{5.110}$$

Equation (5.109), simply stating that J^δ_* acts on $\mathcal{G}^{\kappa\rho}$ by transforming the field as well as the conformal indices, can easily be proven by acting on $\mathcal{G}^{\kappa\rho}$ with $\mathbb{J}^\delta_*$ and using the Bianchi identity as well as the superconformal commutation relation. If we contract equation (5.109) with $f^\omega{}_{\rho\kappa}$ and rewrite the last two terms using a supersymmetric version of the Jacobi identity

$$f^\omega{}_{\rho\kappa} f^{\delta\kappa}{}_\sigma + (-1)^{|\delta||\sigma|} f^\omega{}_{\kappa\sigma} f^{\delta\kappa}{}_\rho = f^{\delta\omega}{}_\kappa f^\kappa{}_{\rho\sigma}\,, \tag{5.111}$$

we get

$$\left(\mathrm{J}^\delta_* + \mathbb{J}^\delta_*\right) f^\omega{}_{\rho\kappa}\,\mathcal{G}^{\kappa\rho} = f^{\delta\omega}{}_\kappa f^\kappa{}_{\rho\sigma}\mathcal{G}^{\sigma\rho}\,. \tag{5.112}$$

Now, the important thing that we need to recall is that the $\mathcal{N} = 4$ SYM constraints (3.78) are preserved by a $\mathfrak{psu}(2,2|4)$ transformation, see section 3.2.2.1. Thus, if ω is fixed to B and we choose δ to be equal to a $\mathfrak{psu}(2,2|4)$ index, the left-hand side of the above equation vanishes, leaving us with

$$f^{\delta\omega}{}_\kappa f^\kappa{}_{\rho\sigma}\,\mathcal{G}^{\sigma\rho} = 0\,. \tag{5.113}$$

The last equation immediately allows for the conclusion that the $\mathcal{G}$-identity (5.104) also holds for ω corresponding to Q, $\bar{\mathrm{Q}}$, S and $\bar{\mathrm{S}}$. Having proven the $\mathcal{G}$-identity for B as well as for the four above-mentioned indices, one can now argue that it holds for any $\mathfrak{u}(2,2|4)$ index ω by iterating the argument with the free indices properly chosen.

5.3. Yangian Invariance at One Loop

In this section, we demonstrate the Yangian invariance of the super Maldacena–Wilson loop operator (4.12) at the leading perturbative order. At the one-loop level, there is only one term contributing: The Wilson loop operator has to be expanded to two fields which are consequently joined by a propagator

$$\left\langle \mathcal{W}_M \right\rangle_{(1)} = -\frac{N}{4}\int_{1,2}\left\langle (\mathcal{A}+\Phi)(1)\,(\mathcal{A}+\Phi)(2)\right\rangle. \tag{5.114}$$

Note that here we have again stripped off the color generators and used that the propagators are diagonal in color space. In fact, at this perturbative order all effects of a non-abelian gauge group are irrelevant. For this reason, we shall throughout this section assume that the gauge group is effectively abelian. In particular, we will drop all the non-linear terms in the field strength, meaning that we replace $\mathcal{F} \to \mathcal{F}_{\text{lin}} = \mathrm{d}\mathcal{A}$. However, for brevity we shall refer to the linearized field strength still by $\mathcal{F}$. All expressions in this section are to be understood as one-loop expressions. Higher-order terms will always be neglected.

Under the action of the bi-local part of a covariantized level-one generator (5.54), the above one-loop expectation value gets mapped to

$$\left\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \mathcal{W}_M \right\rangle_{(1)} = -\frac{N}{2}\, f^{\kappa}{}_{\rho\delta} \int_{1<2} \left\langle \mathbb{J}^{\delta}_{*}(\mathcal{A}+\Phi)(1)\, \mathbb{J}^{\rho}_{*}(\mathcal{A}+\Phi)(2) \right\rangle . \tag{5.115}$$

In what follows, we shall now analyze this expression in detail.

5.3.1. Symmetry of the Gauge Propagator

Let us start by considering the action of the Yangian on the gauge propagator

$$C^{\kappa}_{12} = \left\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\mathcal{A}(1)\, \mathcal{A}(2) \Big) \right\rangle = f^{\kappa}{}_{\rho\delta} \left\langle \mathbb{J}^{\delta}_{*} \mathcal{A}(1)\, \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \right\rangle . \tag{5.116}$$

We now extend the first generator over the whole expression and correct for the discrepancy. This yields

$$f^{\kappa}{}_{\rho\delta} \left\langle \mathbb{J}^{\delta}_{*} \mathcal{A}(1)\, \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \right\rangle = f^{\kappa}{}_{\rho\delta} \left\langle \mathbb{J}^{\delta}_{*} \Big(\mathcal{A}(1)\, \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \Big) \right\rangle - f^{\kappa}{}_{\rho\delta} \left\langle \mathcal{A}(1)\, \mathbb{J}^{\delta}_{*} \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \right\rangle . \tag{5.117}$$

To simplify the second term, we use the graded antisymmetry of the structure constants to rewrite the product as a graded commutator and subsequently apply the algebra relation (5.43),

$$\begin{aligned} \tfrac{1}{2} f^{\kappa}{}_{\rho\delta} \left\langle \mathcal{A}(1)\, [\mathbb{J}^{\delta}_{*}, \mathbb{J}^{\rho}_{*}\} \mathcal{A}(2) \right\rangle =& \tfrac{1}{2} f^{\kappa}{}_{\rho\delta} f^{\delta\rho}{}_{\sigma} \left\langle \mathcal{A}(1)\, \mathbb{J}^{\sigma}_{*} \mathcal{A}(2) \right\rangle \\ &- \tfrac{1}{2} f^{\kappa}{}_{\rho\delta} \left\langle \mathcal{A}(1)\, \mathcal{D}\mathcal{G}^{\delta\rho}(2) \right\rangle . \end{aligned} \tag{5.118}$$

Both terms are actually zero; the former because the dual Coxeter number of the superconformal algebra is zero[3], the latter because of the $\mathcal{G}$-identity, see section 5.2.6. Hence, we are left with

$$C^{\kappa}_{12} = f^{\kappa}{}_{\rho\delta} \left\langle \mathbb{J}^{\delta}_{*} \Big(\mathcal{A}(1)\, \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \Big) \right\rangle . \tag{5.119}$$

As the first generator acts on both fields, the whole expression obviously corresponds to a symmetry variation of an expectation value. If the object inside the brackets were gauge invariant, we would immediately conclude that C^{κ}_{12} vanishes. However, as one of the fields is a gauge potential, this statement does not hold. In fact, the symmetry only forces to zero the physical degrees of freedom. The unphysical gauge degrees of freedom are not constrained by the symmetry and we thus conclude that the result is zero only up to a total derivative term, representing an (effectively) abelian gauge transformation

$$f^{\kappa}{}_{\rho\delta} \left\langle \mathbb{J}^{\delta}_{*} \Big(\mathcal{A}(1)\, \mathbb{J}^{\rho}_{*} \mathcal{A}(2) \Big) \right\rangle = \mathrm{d}_1 B^{\kappa}_{12} . \tag{5.120}$$

[3]In fact, for $\mathfrak{u}(2,2|4)$ the combination $f^{\kappa}{}_{\rho\delta} f^{\delta\rho}{}_{\sigma}$ is not zero but rather proportional to $\delta^{\kappa}_{\mathrm{B}} \delta^{\mathrm{C}}_{\sigma}$. However, note that since C is always represented by zero (see 4.2.2.2), this expression effectively vanishes as well.

Here, d_1 is an exterior derivative in superspace and B_{12} is some function of the superspace coordinates. By performing the "partial integration" on the second generator instead of the first one, we can show that $C^{\kappa}_{12} = -\mathrm{d}_2 B^{\kappa}_{21}$. Altogether this implies that C_{12} is in fact the double derivative of some function R_{12},

$$C^{\kappa}_{12} = \Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\mathcal{A}(1)\, \mathcal{A}(2) \Big) \Big\rangle = \mathrm{d}_1 \mathrm{d}_2 R^{\kappa}_{12}\,. \tag{5.121}$$

In what follows, we shall refer to the function R_{12} as the remainder function. It should be added for the benefit of the doubtful reader that we will verify the relation (5.121) later on explicitly for the level-one momentum generator and also comment on an easy way of proving that it holds in general.

5.3.2. Symmetry of the Wilson Loop

Apart from the gauge connection, the super Maldacena–Wilson loop also couples to the superscalars of $\mathcal{N}=4$ SYM theory. For this reason, we need to investigate the bi-local action of the level-one generators on the propagators $\langle \mathcal{A}\Phi \rangle$ and $\langle \Phi\Phi \rangle$ as well. However, their contributions to the VEV of the Yangian transformed Wilson loop follow easily from the above considerations. To see how this works, let us recall that the superscalar is actually a particular component of the super field strength two-from $\mathcal{F}$, cf. equation (3.85). For the bi-local level-one action on the $\langle \mathcal{F}\mathcal{A} \rangle$ propagator, we obtain

$$\Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\mathcal{F}(1)\, \mathcal{A}(2) \Big) \Big\rangle = \Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\mathrm{d}\mathcal{A}(1)\, \mathcal{A}(2) \Big) \Big\rangle = \mathrm{d}_1 \mathrm{d}_1 \mathrm{d}_2 R^{\kappa}_{12} = 0\,. \tag{5.122}$$

The same of course applies to the propagator $\langle \mathcal{F}\mathcal{F} \rangle$,

$$\Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\mathcal{F}(1)\, \mathcal{F}(2) \Big) \Big\rangle = 0\,. \tag{5.123}$$

Note that $\widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}}$ acts exclusively on the fields and does therefore not change the structure of the basis two-forms of $\mathcal{F}$. Since all the basis two-forms are linearly independent by definition, the zero actually holds for all the constituents individually. For this reason, the two above relations imply that

$$\begin{aligned} \Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\Phi(1)\, \mathcal{A}(2) \Big) \Big\rangle &= 0\,, \\ \Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \Big(\Phi(1)\, \Phi(2) \Big) \Big\rangle &= 0\,. \end{aligned} \tag{5.124}$$

Obviously, the mixed propagator as well as the scalar propagator do not contribute at the one-loop level.

In principle, we can now use the above results towards Yangian symmetry of the Wilson loop expectation value

$$\begin{aligned} \Big\langle \widehat{\mathbb{J}}^{\kappa}_{*,\mathrm{bi}} \mathcal{W}_M \Big\rangle &\sim \int_{1<2} \mathrm{d}_1 \mathrm{d}_2 R^{\kappa}_{12} = \int \mathrm{d}_2 R^{\kappa}_{12}|_{1=2} - \int \mathrm{d}_2 R^{\kappa}_{02} \\ &= \int \mathrm{d}_2 R^{\kappa}_{12}|_{1=2} - R^{\kappa}_{00} + R^{\kappa}_{00} = \int \mathrm{d}_2 R^{\kappa}_{12}|_{1=2}\,. \end{aligned} \tag{5.125}$$

The result is obviously not zero, but fortunately all the bi-local and bulk-boundary terms have gone away. What remains is a local term for which we can compensate by properly adjusting the local action of the level-one generators. However, in the above computation we have been careless with respect to divergences. The function R^{κ}_{12} is in fact UV divergent when the two points approach each other. For this reason, we have to repeat the computation in the presence of a regulator. However, before we do this, let us compute the remainder functions R^{κ}_{12}.

5.3.3. Remainder Functions

In the previous two sections, we have argued that the only contribution to the remainder term R^{σ}_{12} comes from the correlator of the product of two gauge fields to which the respective bi-local level-one generator $\widehat{\mathbb{J}}^{\sigma}_{*,\mathrm{bi}}$ has been applied

$$\Big\langle\widehat{\mathbb{J}}^{\sigma}_{*,\mathrm{bi}}\Big(\mathcal{A}(1)\mathcal{A}(2)\Big)\Big\rangle = f^{\sigma}{}_{\rho\kappa}\Big\langle\mathbb{J}^{\kappa}_{*}\mathcal{A}(1)\,\mathbb{J}^{\rho}_{*}\mathcal{A}(2)\Big\rangle = \mathrm{d}_1\mathrm{d}_2 R^{\sigma}_{12}\,. \tag{5.126}$$

Here, we will determine all the remainder functions R^{σ}_{12}. For this, we shall first explicitly compute the level-one momentum and the level-one hypercharge remainder function and then determine the other remainder functions by algebra considerations. In principle, it would be sufficient to only compute the level-one hypercharge remainder function explicitly as all the other remainder terms can be derived from this one.[4] However, since $\widehat{\mathbb{P}}_{*}$ is the more generic Yangian generator, we will mainly focus on this one in the subsequent discussion and in return shorten the level-one hypercharge discussion.

We start by rewriting the left-hand side of (5.126) in such a way that we can take advantage of the fact that we already know the two-point function of the field strength two-form, see section 3.2.2.2. By plugging in the definition of $\mathbb{J}^{\kappa}_{*}\mathcal{A}$ (5.110), the left-hand side of equation (5.126) becomes

$$\Big\langle\widehat{\mathbb{J}}^{\sigma}_{*,\mathrm{bi}}\Big(\mathcal{A}(1)\mathcal{A}(2)\Big)\Big\rangle = f^{\sigma}{}_{\rho\kappa}\,\Big\langle \mathrm{J}^{\kappa} X_1^{\mathcal{A}}\,\mathcal{E}_1^{\mathcal{B}}\,\mathcal{F}_{1\mathcal{B}\mathcal{A}}\,\mathrm{J}^{\rho} X_2^{\mathcal{C}}\,\mathcal{E}_2^{\mathcal{D}}\,\mathcal{F}_{2\,\mathcal{D}\mathcal{C}}\Big\rangle. \tag{5.127}$$

Using the interior product notation that we have introduced in section 5.1.1, the former expression can be rewritten as

$$\widehat{\mathbb{J}}^{\sigma}_{*,\mathrm{bi}}\Big\langle\,\mathcal{A}(1)\mathcal{A}(2)\Big\rangle = f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\,\Big\langle\mathcal{F}(1)\mathcal{F}(2)\Big\rangle\,. \tag{5.128}$$

By taking into account the decomposition of the field strength $\mathcal{F} = \mathcal{F}^{+} + \mathcal{F}^{-}$, we see that computing the remainder term for $\widehat{\mathbb{P}}_{*,\mathrm{bi}}$ effectively amounts to calculating the following two contraction expressions ($\sigma \sim \mathrm{P}$):

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\,\Big\langle\mathcal{F}^{+}(1)\mathcal{F}^{+}(2)\Big\rangle \qquad \Big(\text{chiral contributions}\Big)\,, \tag{5.129}$$

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\,\Big\langle\mathcal{F}^{+}(1)\mathcal{F}^{-}(2)\Big\rangle \qquad \Big(\text{mixed-chiral contributions}\Big)\,. \tag{5.130}$$

Having computed these, the full result can be constructed by symmetry considerations.

[4] Note that this statement does not hold for the level-one momentum remainder function as the level-one hypercharge remainder function cannot be derived from $R_{12}[\mathrm{P}]$.

Level-one momentum remainder function. The bi-local part of the level-one momentum generator $\widehat{\mathbb{P}}_*$ can be written as combinations of the superconformal generators $\mathbb{P}_*$ (bosonic translations), $\mathbb{Q}_*$, $\bar{\mathbb{Q}}_*$ (fermionic translations), $\mathbb{L}_*$, $\bar{\mathbb{L}}_*$ (rotations) and $\mathbb{D}_*$ (dilatations)

$$\widehat{\mathbb{P}}_{*,\mathrm{bi},\alpha\dot{\alpha}} = \mathbb{P}_{*,\beta\dot{\alpha}} \wedge \mathbb{L}^{\beta}_{*\,\alpha} + \mathbb{P}_{*,\alpha\dot{\beta}} \wedge \bar{\mathbb{L}}^{\dot{\beta}}_{*\,\dot{\alpha}} + 2\mathbb{P}_{*,\alpha\dot{\alpha}} \wedge \mathbb{D}_* + \mathbb{Q}_{*,\alpha a} \wedge \bar{\mathbb{Q}}^{a}_{*\dot{\alpha}}\,. \tag{5.131}$$

Here, the action of a bi-local term $(\mathbb{J}_* \wedge \mathbb{J}'_*)$ is defined in analogy to (5.54) as

$$\begin{aligned}\Big(\mathbb{J}_* \wedge \mathbb{J}'_*\Big)\mathcal{W} =&\mathcal{W}\Big[\mathbb{J}_*\Big(\mathcal{A}+\Phi\Big); \mathbb{J}'_*\Big(\mathcal{A}+\Phi\Big)\Big] \\ &- (-1)^{|\mathbb{J}_*||\mathbb{J}'_*|}\mathcal{W}\Big[\mathbb{J}'_*\Big(\mathcal{A}+\Phi\Big); \mathbb{J}_*\Big(\mathcal{A}+\Phi\Big)\Big].\end{aligned} \tag{5.132}$$

It turns out to be convenient to rewrite the expression for $\widehat{\mathbb{P}}_{*,\mathrm{bi}}$ by introducing the following modified rotation generators:

$$\begin{aligned}\mathbb{L}'^{\alpha}_{*\;\beta} &= \mathbb{L}^{\alpha}_{*\,\beta} + \delta^{\alpha}_{\beta}\Big(\mathbb{D}_* - \mathbb{B}_*\Big), \\ \bar{\mathbb{L}}'^{\dot{\alpha}}_{*\;\dot{\beta}} &= \bar{\mathbb{L}}^{\dot{\alpha}}_{*\,\dot{\beta}} + \delta^{\dot{\alpha}}_{\dot{\beta}}\Big(\mathbb{D}_* + \mathbb{B}_*\Big).\end{aligned} \tag{5.133}$$

Using these definitions, the Yangian generator $\widehat{\mathbb{P}}_{*,\mathrm{bi}}$ can be rewritten as

$$\widehat{\mathbb{P}}_{*,\mathrm{bi},\alpha\dot{\alpha}} = \mathbb{P}_{*,\beta\dot{\alpha}} \wedge \mathbb{L}'^{\beta}_{*\;\alpha} + \mathbb{P}_{*,\alpha\dot{\beta}} \wedge \bar{\mathbb{L}}'^{\dot{\beta}}_{*\;\dot{\alpha}} + \mathbb{Q}_{*,\alpha a} \wedge \bar{\mathbb{Q}}^{a}_{*\dot{\alpha}}\,. \tag{5.134}$$

From this equation, we can now directly read off the double-contraction operator (c.f. (5.128)) that we want to use for the computation of the remainder term $R_{12}[\mathrm{P}]$. It reads

$$\begin{aligned}f^{\sigma}{}_{\rho\kappa}\, i\Big[\mathrm{J}^{\kappa}_1\Big]\, i\Big[\mathrm{J}^{\rho}_2\Big] =& + i\Big[\mathrm{P}_{1,\beta\dot{\alpha}}\Big]\, i\Big[\mathrm{L}'^{\beta}_{2\;\alpha}\Big] + i\Big[\mathrm{P}_{1,\alpha\dot{\beta}}\Big]\, i\Big[\bar{\mathrm{L}}'^{\dot{\beta}}_{2\;\dot{\alpha}}\Big] + i\Big[\mathrm{Q}_{1,\alpha a}\Big]\, i\Big[\bar{\mathrm{Q}}^{a}_{2\;\dot{\alpha}}\Big] \\ &- i\Big[\mathrm{L}'^{\beta}_{1\;\alpha}\Big]\, i\Big[\mathrm{P}_{2,\beta\dot{\alpha}}\Big] - i\Big[\bar{\mathrm{L}}'^{\dot{\beta}}_{1\;\dot{\alpha}}\Big]\, i\Big[\mathrm{P}_{2,\alpha\dot{\beta}}\Big] + i\Big[\bar{\mathrm{Q}}^{a}_{1\;\dot{\alpha}}\Big]\, i\Big[\mathrm{Q}_{2,\alpha a}\Big]\,,\end{aligned} \tag{5.135}$$

where here and in the following the index σ corresponds to the generator $\mathrm{P}_{\alpha\dot{\alpha}}$.

We start with the computation of the remainder term $R_{12}[\mathrm{P}]$ by investigating the chiral contributions, see (5.129). The relevant correlation function was computed in section 3.2.2.2 and is given by

$$\begin{aligned}\Big\langle \mathcal{F}^+(1)\mathcal{F}^+(2)\Big\rangle =& -\frac{g^2}{2\pi^2}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\Xi^{abcd}(1,2)\, \mathrm{d}_1\Big(x^{-+,-1}_{12}\bar{\theta}_{12}\Big)_{\alpha c}\, \mathrm{d}_1\Big(x^{-+,-1}_{12}\bar{\theta}_{12}\Big)_{\beta d} \\ &\cdot \mathrm{d}_2\Big(x^{+-,-1}_{12}\bar{\theta}_{12}\Big)_{\gamma a}\, \mathrm{d}_2\Big(x^{+-,-1}_{12}\bar{\theta}_{12}\Big)_{\delta b}\,.\end{aligned} \tag{5.136}$$

By symmetry, it is sufficient to compute the action of the double-contraction operator (5.135) on $W_{\gamma c,\delta d} = \mathrm{d}_1(x^{-+,-1}_{12}\bar{\theta}_{12})_{\gamma c}\, \mathrm{d}_2(x^{+-,-1}_{12}\bar{\theta}_{12})_{\delta d}$. Note that the contraction operators $i[\ldots]$ effectively just replace (modulo a sign) the corresponding exterior derivative d by the differential operator specified in the square brackets. For example, we have

$$\begin{aligned}i\Big[\mathrm{P}_{1,\beta\dot{\alpha}}\Big]\, i\Big[\mathrm{L}'^{\beta}_{2\;\alpha}\Big]\, W_{\gamma c,\delta d} &= -\mathrm{P}_{1,\beta\dot{\alpha}}\Big(x^{-+,-1}_{12}\bar{\theta}_{12}\Big)_{\gamma c}\, \mathrm{L}'^{\beta}_{2\;\alpha}\Big(x^{+-,-1}_{12}\bar{\theta}_{12}\Big)_{\delta d} \\ &= 4\Big(x^{-+,-1}_{12}\Big)_{\gamma\dot{\alpha}}\Big(x^{+-,-1}_{12}x^{-}_{2}x^{-+,-1}_{12}\bar{\theta}_{12}\Big)_{\delta c}\Big(x^{+-,-1}_{12}\bar{\theta}_{12}\Big)_{\alpha d}.\end{aligned} \tag{5.137}$$

Note the extra minus sign in the first line coming from commuting the contraction operator $i\Big[\mathrm{L}'^{\beta}_{2\ \alpha}\Big]$ past the d_1. Carrying out the remaining contractions and adding all the contributions up shows that the chiral contributions vanish

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\, \Big\langle \mathcal{F}^+(1)\mathcal{F}^+(2)\Big\rangle = 0\,. \tag{5.138}$$

Let us now focus on the mixed-chiral contributions. The relevant two-point function was also computed in section 3.2.2.2 and is given by

$$\Big\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\Big\rangle = -\frac{g^2}{16\pi^2}\mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2 x_{12}^{-+}\varepsilon\Big)_{\beta\gamma}\mathrm{d}_1\Big(\varepsilon x_{12}^{-+,-1}\mathrm{d}_2 x_{12}^{-+}\Big)^{\beta\gamma}. \tag{5.139}$$

Applying the double-contraction expression of (5.135) to this correlator yields

$$\begin{aligned} i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\, \Big\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\Big\rangle = \frac{g^2}{8\pi^2}\,\varepsilon_{\lambda\gamma}\,\varepsilon^{\beta\rho}\Big\{ & \mathrm{J}^{\kappa}_1\Big(x_{12}^{-+,-1}\mathrm{J}^{\rho}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\,\mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma} \\ & - (-1)^{|\mathrm{J}^{\kappa}_1||\mathrm{J}^{\rho}_2|}\,\mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{J}^{\rho}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\mathrm{J}^{\kappa}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma}\Big\}\,, \end{aligned} \tag{5.140}$$

where we have left out the structure constant for brevity. Some useful formulas for computing the former expression are:

$$\begin{aligned} \mathrm{L}'^{\alpha}_{1\ \beta}(x_{12}^{-+})^{\dot\gamma\gamma} &= -2\delta^{\gamma}_{\beta}(x_1^{-})^{\dot\gamma\alpha}, \\ \mathrm{L}'^{\alpha}_{2\ \beta}(x_{12}^{-+})^{\dot\gamma\gamma} &= 2\delta^{\gamma}_{\beta}(x_1^{-} - x_{12}^{-+})^{\dot\gamma\alpha}, \\ \bar{\mathrm{L}}'^{\dot\alpha}_{1\ \dot\beta}(x_{12}^{-+})^{\dot\gamma\gamma} &= -2\delta^{\dot\gamma}_{\dot\beta}(x_2^{+} + x_{12}^{-+})^{\dot\alpha\gamma}, \\ \bar{\mathrm{L}}'^{\dot\alpha}_{2\ \dot\beta}(x_{12}^{-+})^{\dot\gamma\gamma} &= 2\delta^{\dot\gamma}_{\dot\beta}(x_2^{+})^{\dot\alpha\gamma}. \end{aligned} \tag{5.141}$$

The expression (5.140) obviously consists of two different terms, one term where both generators act on the same piece of the mixed-chiral two-point function and another term where the two generators act on different pieces. First, we compute the term where both generators act on a single term. After some computation, we find that

$$f^{\sigma}{}_{\rho\kappa}\,\mathrm{J}^{\kappa}_1\Big(x_{12}^{-+,-1}\mathrm{J}^{\rho}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda} = 8\delta^{\lambda}_{\alpha}\Big(x_{12}^{-+,-1}\Big)_{\beta\dot\alpha} - 4\delta^{\lambda}_{\beta}\Big(x_{12}^{-+,-1}\Big)_{\alpha\dot\alpha}. \tag{5.142}$$

For the second term of (5.140),

$$T^{\lambda\gamma}_{\beta\rho} = f^{\sigma}{}_{\rho\kappa}(-1)^{|\mathrm{J}^{\kappa}_1||\mathrm{J}^{\rho}_2|}\,\mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{J}^{\rho}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\mathrm{J}^{\kappa}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma}\,, \tag{5.143}$$

we obtain

$$T^{\lambda\gamma}_{\beta\rho} = -4\Big[\delta^{\lambda}_{\alpha}\Big(x_{12}^{-+,-1}\Big)_{\rho\dot\alpha}\mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\gamma} + \delta^{\lambda}_{\rho}\mathrm{d}_1\Big(x_{12}^{-+,-1}\Big)_{\beta\dot\alpha}\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\alpha}{}^{\gamma}\Big].$$

Plugging these two expressions back into equation (5.140) and using the identity $\varepsilon_{\lambda\gamma}\varepsilon^{\beta\rho} = \delta^{\beta}_{\lambda}\delta^{\rho}_{\gamma} - \delta^{\beta}_{\gamma}\delta^{\rho}_{\lambda}$ yields

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}^{\kappa}_1]\, i[\mathrm{J}^{\rho}_2]\, \Big\langle \mathcal{F}^+(1)\mathcal{F}^-(2)\Big\rangle = \frac{g^2}{2\pi^2}\,\mathrm{d}_1\mathrm{d}_2\,\Big(x_{12}^{-+,-1}\Big)_{\alpha\dot\alpha}\,, \tag{5.144}$$

which is a total derivative with respect to points one and two.

Given the results (5.138) and (5.144), the full result for the action of $\widehat{\mathbb{P}}_{*,\mathrm{bi}}$ on the correlator of two gauge fields can now be constructed by symmetry considerations. For this, note that (5.144) implies

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}_2^{\kappa}]\, i[\mathrm{J}_1^{\rho}] \left\langle \mathcal{F}^+(2)\mathcal{F}^-(1)\right\rangle = \frac{g^2}{2\pi^2}\, \mathrm{d}_2\mathrm{d}_1 \left(x_{21}^{-+,-1}\right)_{\alpha\dot{\alpha}}. \tag{5.145}$$

Now, using the property that the two exterior derivatives anticommute as well as the fact that the double-contraction expression on the left-hand side of the above equation is symmetric under exchange of points one and two, we find the following final result for the level-one momentum remainder term

$$R_{12}[\mathrm{P}_{\alpha\dot{\alpha}}] = R_{12}^{-+}[\mathrm{P}_{\alpha\dot{\alpha}}] - R_{21}^{-+}[\mathrm{P}_{\alpha\dot{\alpha}}]\,, \tag{5.146}$$

where

$$R_{12}^{-+}[\mathrm{P}_{\alpha\dot{\alpha}}] = \frac{g^2}{2\pi^2}\left(x_{12}^{-+,-1}\right)_{\alpha\dot{\alpha}}. \tag{5.147}$$

A common feature of all the remainder functions is that they can be cast into the form (5.146). For this reason, we will from now on omit the second piece of (5.146) and only work with the building blocks $R_{12}^{-+}[\mathrm{J}^{\sigma}]$.

Level-one hypercharge remainder function. In this paragraph, we compute the remainder function R_{12}^{σ} for the Yangian bonus symmetry generator $\widehat{\mathbb{B}}_*$. The bi-local part of this generator takes the following form

$$\widehat{\mathbb{B}}_{*,\mathrm{bi}} = \tfrac{1}{4}\Big(\mathbb{Q}_{*,\alpha a} \wedge \mathbb{S}_*^{a\alpha} - \bar{\mathbb{Q}}^{a}_{*\dot{\alpha}} \wedge \bar{\mathbb{S}}^{\dot{\alpha}}_{*\,a}\Big)\,, \tag{5.148}$$

and acts on the Wilson loop operator as defined in equation (5.132). From the last equation, we can again directly read off the relevant double-contraction operator. It reads

$$\begin{aligned} f^{\sigma}{}_{\rho\kappa}\, i\Big[\mathrm{J}_1^{\kappa}\Big]\, i\Big[\mathrm{J}_2^{\rho}\Big] = \frac{1}{4}\Big(& i\Big[\mathrm{Q}_{1,\alpha a}\Big]\, i\Big[\mathrm{S}_2^{a\alpha}\Big] - i\Big[\bar{\mathrm{Q}}^{a}_{1\,\dot{\alpha}}\Big]\, i\Big[\bar{\mathrm{S}}^{\dot{\alpha}}_{2\,a}\Big] \\ &+ i\Big[\mathrm{S}_1^{a\alpha}\Big]\, i\Big[\mathrm{Q}_{2,\alpha a}\Big] - i\Big[\bar{\mathrm{S}}^{\dot{\alpha}}_{1\,a}\Big]\, i\Big[\bar{\mathrm{Q}}^{a}_{2\,\dot{\alpha}}\Big]\Big)\,, \end{aligned} \tag{5.149}$$

where here and throughout this paragraph $\sigma \sim \mathrm{B}$. As in the case of the level-one momentum remainder function, we will now apply this double-contraction operator to the chiral and mixed-chiral part of the field strength two-point function.

Again the computation splits up into the chiral and mixed-chiral contributions. In order to determine the chiral contributions to the level-one hypercharge remainder function, it is again sufficient to compute the action of the contraction operator (5.149) on the form $\mathrm{d}_1(x_{12}^{-+,-1}\bar{\theta}_{12})_{\gamma c}\, \mathrm{d}_2(x_{12}^{+-,-1}\bar{\theta}_{12})_{\delta d}$. Using the representation of the generators in terms of differential operators (see section 4.2.2.2), one shows that this action vanishes and thus

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}_1^{\kappa}]\, i[\mathrm{J}_2^{\rho}]\, \langle \mathcal{F}^{+}(1)\mathcal{F}^{+}(2)\rangle = 0\,. \tag{5.150}$$

Turning to the mixed-chiral contributions, we recall the formula for the action of a general double-contraction operator on the mixed-chiral two-point function:

$$\begin{aligned} i[\mathrm{J}_1^{\kappa}]\, i[\mathrm{J}_2^{\rho}] \left\langle \mathcal{F}^{+}(1)\mathcal{F}^{-}(2)\right\rangle = \frac{g^2}{8\pi^2}\,\varepsilon_{\lambda\gamma}\,\varepsilon^{\beta\rho}\Big\{ & \mathrm{J}_1^{\kappa}\Big(x_{12}^{-+,-1}\mathrm{J}_2^{\rho}\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\, \mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma} \\ & - (-1)^{|\mathrm{J}_1^{\kappa}||\mathrm{J}_2^{\rho}|}\, \mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{J}_2^{\rho}\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\mathrm{J}_1^{\kappa}\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma}\Big\}. \end{aligned} \tag{5.151}$$

As before, we will compute the two different terms of the former expression individually. For the contribution where both generators act on a single term, we find

$$f^{\sigma}{}_{\rho\kappa}\, \mathrm{J}_1^{\kappa}\Big(x_{12}^{-+,-1}\mathrm{J}_2^{\rho}\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda} = 4\delta_{\beta}^{\lambda}\Big\{\mathrm{tr}\Big(x_{12}^{-+,-1}\bar\theta_1\theta_2\Big) - 1\Big\} - 8\Big(x_{12}^{-+,-1}\bar\theta_1\theta_2\Big)_{\beta}{}^{\lambda}. \tag{5.152}$$

For the second contribution, we obtain

$$\begin{aligned} & f^{\sigma}{}_{\rho\kappa}\, \mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{J}_2^{\rho}\, x_{12}^{-+}\Big)_{\beta}{}^{\lambda}\mathrm{J}_1^{\kappa}\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\rho}{}^{\gamma} \\ &= -4\Big(x_{12}^{-+,-1}\bar\theta_1\theta_2\Big)_{\rho}{}^{\lambda}\, \mathrm{d}_1\Big(x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)_{\beta}{}^{\gamma} \\ &\quad + 4\delta_{\rho}^{\lambda}\mathrm{d}_1\Big(x_{12}^{-+,-1}\bar\theta_1\Big)_{\beta a}\Big\{\mathrm{d}_2\theta_2^{a\gamma} - \Big(\theta_2\, x_{12}^{-+,-1}\mathrm{d}_2\, x_{12}^{-+}\Big)^{a\gamma}\Big\}\,. \end{aligned} \tag{5.153}$$

Combining both expressions and adding the correct prefactor yields

$$f^{\sigma}{}_{\rho\kappa}\, i[\mathrm{J}_1^{\kappa}]\, i[\mathrm{J}_2^{\rho}]\, \langle \mathcal{F}^{+}(1)\mathcal{F}^{-}(2)\rangle = -\frac{g^2}{2\pi^2}\mathrm{d}_1\mathrm{d}_2\Big(\ln\Big(\Big(x_{12}^{-+}\Big)^2\Big) + \mathrm{tr}\Big(x_{12}^{-+,-1}\bar\theta_1\theta_2\Big)\Big)\,. \tag{5.154}$$

From this result, we can now read off the form of the essential building block $R_{12}^{-+}[\mathrm{B}]$ of the level-one hypercharge remainder function, see (5.146). It reads

$$R_{12}^{-+}[\mathrm{B}] = -\frac{g^2}{2\pi^2}\Big(\ln\Big(\Big(x_{12}^{-+}\Big)^2\Big) + \mathrm{tr}\Big(x_{12}^{-+,-1}\bar\theta_1\theta_2\Big)\Big)\,. \tag{5.155}$$

Other level-one remainder functions. Having computed the remainder functions $R_{12}[\mathrm{P}]$ and $R_{12}[\mathrm{B}]$, we can now determine all the other remainder functions by applying the appropriate superconformal transformations to one of them. This can be seen as follows. Under the action of a superconformal generator $\mathrm{J}^{\delta} = \mathrm{J}^{\delta}X_p^{\mathcal{A}}\partial_{\mathcal{A}} = \mathrm{J}^{\delta}X^{\mathcal{A}}D_{\mathcal{A}}$, the one-form $\mathbb{J}_*^{\kappa}\mathcal{A}$ transforms by the Lie derivative. A short computation shows that J^{δ} acts on $\mathbb{J}_*^{\kappa}\mathcal{A}$ by transforming the field as well as the conformal index

$$\mathrm{J}^{\delta}\mathbb{J}_*^{\kappa}\mathcal{A} = -\mathbb{J}^{\delta}\mathbb{J}_*^{\kappa}\mathcal{A} + f^{\delta\kappa}{}_{\omega}\,\mathbb{J}_*^{\omega}\mathcal{A}\,. \tag{5.156}$$

Using this identity, one easily shows that the following equation holds true:

$$\Big(\mathrm{J}_1^\delta + \mathrm{J}_2^\delta\Big) f^\sigma{}_{\rho\kappa} \Big\langle \mathbb{J}_*^\kappa \mathcal{A}(1)\, \mathbb{J}_*^\rho \mathcal{A}(2) \Big\rangle = \Big[f^\sigma{}_{\rho\kappa} f^{\delta\kappa}{}_\omega + (-1)^{|\delta||\omega|} f^\sigma{}_{\kappa\omega} f^{\delta\kappa}{}_\rho \Big] \Big\langle \mathbb{J}_*^\omega \mathcal{A}(1)\, \mathbb{J}_*^\rho \mathcal{A}(2) \Big\rangle \\ - f^\sigma{}_{\rho\kappa} \Big\langle \Big(\mathbb{J}_1^\delta + \mathbb{J}_2^\delta\Big) \Big(\mathbb{J}_*^\kappa \mathcal{A}(1)\, \mathbb{J}_*^\rho \mathcal{A}(2)\Big) \Big\rangle\,. \tag{5.157}$$

Importantly, the expression in the second line vanishes. This is because the term in brackets represents a conformal variation of an object which is invariant under linearized gauge transformations. Thus, using the supersymmetric Jacobi identity (5.111), we find

$$\Big(\mathrm{J}_1^\delta + \mathrm{J}_2^\delta\Big) f^\sigma{}_{\rho\kappa} \Big\langle \mathbb{J}_*^\kappa \mathcal{A}(1)\, \mathbb{J}_*^\rho \mathcal{A}(2) \Big\rangle = f^{\delta\sigma}{}_\kappa f^\kappa{}_{\rho\omega} \Big\langle \mathbb{J}_*^\omega \mathcal{A}(1)\, \mathbb{J}_*^\rho \mathcal{A}(2) \Big\rangle\,. \tag{5.158}$$

By combining this expression[5] with equation (5.121), we obtain

$$\Big[\mathrm{J}_1^\delta + \mathrm{J}_2^\delta, \mathrm{d}_1\mathrm{d}_2 R_{12}^\sigma\Big\} = f^{\delta\sigma}{}_\kappa \mathrm{d}_1\mathrm{d}_2 R_{12}^\kappa\,. \tag{5.159}$$

In fact, it turns out that if one excludes the level-one hypercharge remainder function, the R_{12}^σ satisfy the adjoint transformation law on their own, i.e.

$$\Big[\mathrm{J}_1^\delta + \mathrm{J}_2^\delta, R_{12}^\sigma\Big\} = f^{\delta\sigma}{}_\kappa R_{12}^\kappa\,. \tag{5.160}$$

Using these equations, one can now iteratively compute all the yet undetermined remainder functions. One finds

$$\begin{aligned}
R_{12}^{-+}[\bar{\mathrm{L}}'] &= \frac{g^2}{2\pi^2} x_2^+ x_{12}^{-+,-1} - \frac{g^2}{2\pi^2} \ln(x_{12}^{-+})^2\,,\\
R_{12}^{-+}[\bar{\mathrm{S}}] &= \frac{g^2}{\pi^2} x_2^+ x_{12}^{-+,-1} \bar{\theta}_1\,,\\
R_{12}^{-+}[\mathrm{K}] &= -\frac{g^2}{2\pi^2} x_2^+ x_{12}^{-+,-1} x_1^-\,,\\
R_{12}^{-+}[\bar{\mathrm{Q}}] &= -\frac{g^2}{\pi^2} \theta_2 x_{12}^{-+,-1}\,,\\
R_{12}^{-+}[\mathrm{R}'] &= -\frac{2g^2}{\pi^2} \theta_2 x_{12}^{-+,-1} \bar{\theta}_1 + \frac{g^2}{2\pi^2} \ln(x_{12}^{-+})^2\,,\\
R_{12}^{-+}[\mathrm{S}] &= \frac{g^2}{\pi^2} \theta_2 x_{12}^{-+,-1} x_1^-\,,\\
R_{12}^{-+}[\mathrm{Q}] &= -\frac{g^2}{\pi^2} x_{12}^{-+,-1} \bar{\theta}_1\,,\\
R_{12}^{-+}[\mathrm{L}'] &= \frac{g^2}{2\pi^2} x_{12}^{-+,-1} x_1^- + \frac{g^2}{2\pi^2} \ln(x_{12}^{-+})^2\,.
\end{aligned} \tag{5.161}$$

This concludes our discussion on the remainder functions.

[5] Note that equation (5.158) can also be used to confirm the result of section 5.3.1, namely, that all the level-one generators $\widehat{\mathbb{J}}_{*,\mathrm{bi}}^\sigma$ exclusively produce a double-derivative term. For example, choosing $\sigma \sim \mathrm{P}$ and $\delta \sim \mathrm{S}$ in (5.158) and taking into account the result of 5.3.3 as well as the fact that $\mathrm{J}_i^\delta \mathrm{d}_i = \mathrm{d}_i \mathrm{J}_i^\delta$, we learn that $\widehat{\bar{\mathbb{Q}}}_{*,\mathrm{bi}}$ produces a double-derivative term as well. By continuing this analysis we can thus confirm that (5.121) indeed holds true.

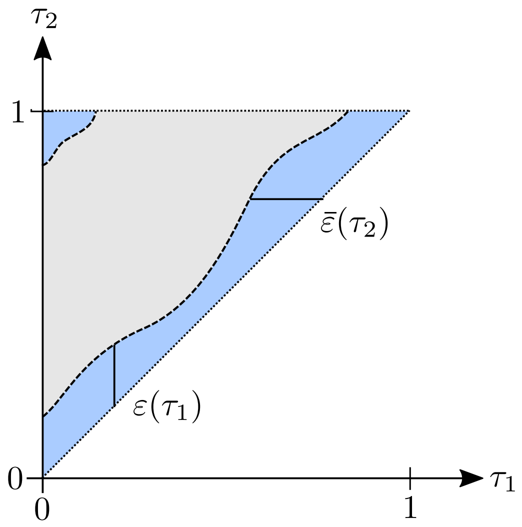


Figure 5.1.: Integration domain for two ordered insertions in point-splitting regularization. The colored stripe and corner are removed by the regularization.

5.3.4. Regularization and Local Terms

The above remainder functions are obviously UV divergent when the two points approach each other. For this reasons, we need to repeat the computation in equation (5.125) in the presence of a regulator. Using point-splitting regularization, we will determine the residual terms that are left behind by the bi-local actions and finally define the full Yangian generators which annihilate the one-loop VEV of the super Maldacena–Wilson loop.

Regularization. In order to regularize the short distance singularities of the remainder functions, we introduce a point-splitting regulator in the following way: While the remainder function is integrated over the upper triangle as depicted in figure 5.1 we always keep the two points at a minimum distance which is determined by some general function $\varepsilon(\tau)$, i.e.

$$(\tau_2 - \tau_1) > \varepsilon(\tau_1) = \bar{\varepsilon}(\tau_2)\,, \tag{5.162}$$

for a generic configuration with τ_2 being greater than τ_1. Here, $\varepsilon(\tau_1)$ specifies this minimum parameter space distance in the forward direction, while $\bar{\varepsilon}(\tau_2)$ specifies the very same distance in the backward direction. By construction, the functions $\varepsilon(\tau)$ and $\bar{\varepsilon}(\tau)$ are thus related to each other as follows

$$\varepsilon(\tau) = \bar{\varepsilon}(\tau + \varepsilon(\tau))\,, \qquad \bar{\varepsilon}(\tau) = \varepsilon(\tau - \bar{\varepsilon}(\tau))\,. \tag{5.163}$$

For later convenience, let us solve this relation perturbatively. We obtain

$$\bar{\varepsilon} = \varepsilon - \varepsilon\dot{\varepsilon} + \varepsilon\dot{\varepsilon}^2 + \tfrac{1}{2}\varepsilon^2\ddot{\varepsilon} + \mathcal{O}(\varepsilon^4)\,. \tag{5.164}$$

Later on, we will use this equation in order to express everything in terms $\varepsilon(\tau)$. Note that we could also use a scheme in which the coordinate τ is proportional to the arc length of the curve and the function ε is a constant cut-off parameter $\varepsilon = \bar{\varepsilon}$. However, as this choice would obscure reparametrization invariance, we prefer to work with a fairly general scheme where $\dot{x}^2$ and $\varepsilon, \bar{\varepsilon}$ are not assumed to be constant.

We proceed by evaluating the regularized version of the integral (5.125). Note that the full integration domain $0 < \tau_1 < \tau_2 < 1$ needs to be adjusted not only where $\tau_1 \approx \tau_2$ but also at $\tau_1 \approx 0$, $\tau_2 \approx 1$ where both points approach each other due to the periodicity of the Wilson loop, see figure 5.1. For the regularized integral of $\mathrm{d}_1\mathrm{d}_2 R_{12}$, we obtain

$$
\begin{aligned}
\int_\varepsilon \mathrm{d}_1\mathrm{d}_2 R_{12} &= \int_{\varepsilon(0)}^{1-\bar{\varepsilon}(1)} \mathrm{d}\tau_2 \int_0^{\tau_2-\bar{\varepsilon}(\tau_2)} \mathrm{d}\tau_1\, \partial_1\partial_2 R_{12} + \int_{1-\bar{\varepsilon}(1)}^{1} \mathrm{d}\tau_2 \int_{\tau_2-1+\varepsilon(\tau_2)}^{\tau_2-\bar{\varepsilon}(\tau_2)} \mathrm{d}\tau_1\, \partial_1\partial_2 R_{12} \\
&= \int_\varepsilon^{1-\bar{\varepsilon}} \mathrm{d}\tau\, (\partial_2 R)(\tau-\bar{\varepsilon},\tau) - \int_\varepsilon^{1-\bar{\varepsilon}} \mathrm{d}\tau\, (\partial_2 R)(0,\tau) \\
&\quad + \int_{1-\bar{\varepsilon}}^{1} \mathrm{d}\tau\, (\partial_2 R)(\tau-\bar{\varepsilon},\tau) - \int_{1-\bar{\varepsilon}}^{1} \mathrm{d}\tau\, (\partial_2 R)(\tau-1+\varepsilon,\tau) \\
&= \int_0^1 \mathrm{d}\tau\, (\partial_2 R)(\tau-\bar{\varepsilon},\tau) + R(0,\varepsilon) - R(0,-\bar{\varepsilon}) \\
&\quad - \int_0^\varepsilon \mathrm{d}\tau\, (\partial_2 R)(\tau-\bar{\varepsilon},\tau) - \int_{-\bar{\varepsilon}}^{0} \mathrm{d}\tau\, (\partial_2 R)(\tau+\varepsilon,\tau)\,,
\end{aligned}
\tag{5.165}
$$

where we have abbreviated $\tau + \varepsilon := \tau + \varepsilon(\tau)$ and similarly for $\bar{\varepsilon}(\tau)$. Here, we have integrated the bi-local double-derivative term to a local term and some terms located at the boundary. The next step is to expand the above relation in ε. For the UV expansion of the contributing terms, we obtain

$$
\begin{aligned}
R(\tau,\tau+\varepsilon) &= \frac{1}{\varepsilon} R_{-1}(\tau) + \tfrac{1}{2}\dot{R}_{-1}(\tau) + R_1(\tau)\varepsilon + \mathcal{O}(\varepsilon^2)\,, \\
(\partial_2 R)(\tau,\tau+\varepsilon) &= -\frac{1}{\varepsilon^2} R_{-1}(\tau) + R_1(\tau) + \mathcal{O}(\varepsilon)\,.
\end{aligned}
\tag{5.166}
$$

Note that the constant term $\frac{1}{2}\dot{R}_{-1}(\tau)$ in the ε-expansion of $R(\tau,\tau+\varepsilon)$ follows from the antisymmetry $R(\tau_1,\tau_2) = -R(\tau_2,\tau_1)$. Inserting these expressions into equation (5.165) and using equation (5.164) in order to express everything in terms of $\varepsilon(\tau)$ yields

$$
\int_\varepsilon \mathrm{d}_1\mathrm{d}_2 R_{12} = \int_0^1 \mathrm{d}\tau \left(-\frac{1+\dot{\varepsilon}}{\varepsilon^2} R_{-1}(\tau) + R_1(\tau) \right) + \frac{4+2\dot{\varepsilon}}{\varepsilon} R_{-1}(0) + \mathcal{O}(\varepsilon)\,. \tag{5.167}
$$

Definition of the full Yangian action. Fortunately, all the remaining terms in equation (5.167) have a local form. The first contribution is a (divergent) local term which is integrated over the whole loop, while the second term is located at the boundary. Fortunately, we can absorb these terms by an appropriate redefinition of the action of the Yangian, see section 5.1.2. The term which is integrated over the whole loop can be absorbed by defining the local action of the Yangian on the gauge fields as follows

$$
(\widehat{\mathbb{J}}_* \mathcal{A})_\varepsilon = \frac{N}{2} \mathrm{d}\tau (\partial_2 R)(\tau-\bar{\varepsilon},\tau)\,. \tag{5.168}
$$

Note that this action maps the gauge field to a plain number times the identity matrix. Therefore, the generator does not preserve the gauge structure of the field itself. However, this issue is of no concern as the action clearly preserves the gauge structure of the Wilson loop. Moreover, the identity matrix appears naturally in the expansion of a Wilson loop operator. On the components of the gauge field, the above local action can be realized as[6]

$$(\widehat{\mathbb{J}}_* \mathcal{A}_\mu)_\varepsilon = \frac{N}{2} \frac{p_\mu}{p^2} \, (\partial_2 R)(\tau - \bar{\varepsilon}, \tau) \,. \tag{5.169}$$

For the remaining boundary term in equation (5.167) we need to adjust the definition of the Yangian action on the Wilson line (5.57) by a boundary term

$$\widehat{\mathbb{J}}^\kappa_* \mathcal{V} = \mathcal{V}\Big[(\widehat{\mathbb{J}}^\kappa_* \mathcal{A})_\varepsilon\Big] + f^\kappa{}_{\rho\delta} \mathcal{V}\Big[\mathbb{J}^\delta_* \mathcal{A}; \mathbb{J}^\rho_* \mathcal{A}\Big]_\varepsilon + N\Big(R^\kappa(0,\varepsilon) - R^\kappa(0,-\bar{\varepsilon})\Big)\mathcal{V} \,. \tag{5.170}$$

Such boundary terms are in fact natural since the boundary is also involved in the regularization, see figure 5.1. Altogether, Yangian symmetry of Wilson loops requires a $1/\varepsilon^2$ divergent local term, a finite local term as well as a $1/\varepsilon$ divergent boundary term. The crucial point in favor of Yangian symmetry is that no bi-local and no bulk-boundary counterterms are needed.

Before moving on, let us note that the above form of the local term (5.168) is almost fully in line with the findings in reference [1]. In this paper, a similar term was found that had to be incorporated in order to make the Yangian a true symmetry. In fact, the local term determined here represents the full supersymmetric extension of the expression computed in reference [1] and reduces to the latter when all θ's are set to zero and ε is chosen to be constant. This will become more obvious in a moment when we expand the level-one momentum remainder function. An important difference compared to the results in reference [1] is, however, the presence of a boundary term in equations (5.167) and (5.170). There are two possible explanations for this difference. First of all, we formulated the bi-local level-one actions in terms covariantized level-zero generators, which differ from the straight field actions (and potentially also from the path actions) by a compensating gauge transformation which could lead to extra boundary terms, see section 5.1.1. Another point to be taken into account is that the authors of reference [1] did not regularize the upper left corner of the integration region (see figure 5.1) where $\tau_1 \approx 0$ and $\tau_2 \approx 1$. However, as the result is divergent there as well, there is no way around regularizing this point. In conclusion, it is probably fair to state that we have been very careful here with respect to both the definition of the Yangian as well as the regularization of the remainder functions and we therefore believe that our result, which includes boundary corrections, is correct.

Finally, it is also interesting to compare our results to the findings at strong coupling, which have been derived using methods of string theory. The bosonic situation was already analyzed in reference [1], while the superspace discussion has been carried out

[6] Other component field realizations which involve the fermionic components of the gauge field are conceivable as well.

more recently in references [43, 44]. At strong coupling, the Yangian generators consist of a standard bi-local piece and a finite local contribution, taking the form of a curve integral. Divergent local terms as well as boundary terms are not present. The curve integral represents the counterpart of the $R_1(\tau)$ term in equation (5.167) and these terms can therefore directly be compared. For the level-one momentum generator the term $R_1(\tau)$ can easily be read off from equation (5.171) in the following paragraph. Close inspection of the corresponding expression at strong coupling [43] shows that both local terms share many common features but do not exactly agree. The most apparent difference is that at strong coupling the local term involves the S^5 coordinate $q^i(\tau)$, while our result does not depend on it. Clarifying the origins of this discrepancy is, however, beyond the scope of this thesis and is therefore left for future work.

Level-one momentum. The above definition of the full Yangian action makes use of the expansion coefficients appearing in the short distance expansion of the remainder functions R^κ_{12}. As an example, let us expand the remainder function of the level-one momentum generator. Using the explicit form as given in equation (5.146), we find

$$\begin{aligned} R^{\mathrm{P}}(\tau,\tau+\varepsilon) =& \frac{2g^2}{\pi^2}\Bigg[-\frac{2}{\varepsilon}p^{-1}+p^{-1}\dot{p}p^{-1} \\ &+\varepsilon p^{-1}\Big(\tfrac{1}{3}\ddot{p}+\tfrac{2}{3}\ddot{\bar{\theta}}\dot{\theta}-\tfrac{2}{3}\dot{\bar{\theta}}\ddot{\theta}-\tfrac{1}{2}\dot{p}p^{-1}\dot{p}-8\dot{\bar{\theta}}\dot{\theta}p^{-1}\dot{\bar{\theta}}\dot{\theta}\Big)p^{-1}+\mathcal{O}(\varepsilon^2)\Bigg], \end{aligned} \tag{5.171}$$

from which we can directly read off the relevant functions R^{P}_{-1} and R^{P}_{1},

$$\begin{aligned} R^{\mathrm{P}}_{-1}(\tau) &= \tfrac{-4g^2}{\pi^2}\,p^{-1}\,, \\ R^{\mathrm{P}}_{1}(\tau) &= \tfrac{2g^2}{\pi^2}\,p^{-1}\Big(\tfrac{1}{3}\ddot{p}+\tfrac{2}{3}\ddot{\bar{\theta}}\dot{\theta}-\tfrac{2}{3}\dot{\bar{\theta}}\ddot{\theta}-\tfrac{1}{2}\dot{p}p^{-1}\dot{p}-8\dot{\bar{\theta}}\dot{\theta}p^{-1}\dot{\bar{\theta}}\dot{\theta}\Big)p^{-1}\,. \end{aligned} \tag{5.172}$$

A curious observation concerns the transformation behavior of these functions under a reparametrization $\tau \mapsto \sigma(\tau)$. The function R^{P}_{-1} transforms covariantly

$$R^{\mathrm{P}}_{-1} \mapsto \dot{\sigma}\, R^{\mathrm{P}}_{-1}\,. \tag{5.173}$$

Conversely, the function R^{P}_{1} is not covariant. Instead, it transforms by the addition of a peculiar Schwarzian derivative which multiplies the divergence term R^{P}_{-1},

$$R^{\mathrm{P}}_{1} \mapsto \frac{1}{\dot{\sigma}}\left(R^{\mathrm{P}}_{1}+\tfrac{1}{6}S(\sigma)R^{\mathrm{P}}_{-1}\right), \tag{5.174}$$

with

$$S(\sigma) := \frac{\dddot{\sigma}}{\dot{\sigma}}-\frac{3}{2}\left(\frac{\ddot{\sigma}}{\dot{\sigma}}\right)^2. \tag{5.175}$$

However, an important point to note is that the regulator $\varepsilon(\tau)$ transforms under a reparametrization as well. In fact, demanding that the cut out distance in target space is the same in both parametrizations yields the following transformation rule for ε,

$$\varepsilon(\tau) \mapsto \sigma(\tau+\varepsilon(\tau))-\sigma(\tau) = \varepsilon\dot{\sigma}+\tfrac{1}{2}\varepsilon^2\ddot{\sigma}+\tfrac{1}{6}\varepsilon^3\dddot{\sigma}+(\varepsilon^4)\,. \tag{5.176}$$

The relevant combination of ε-terms in equation (5.167) therefore happens to transform with a Schwarzian derivative as well

$$\frac{1+\dot{\varepsilon}}{\varepsilon^2} \mapsto \frac{1}{\dot{\sigma}^2}\left(\frac{1+\dot{\varepsilon}}{\varepsilon^2} + \tfrac{1}{6}S(\sigma) + \mathcal{O}(\varepsilon)\right) . \tag{5.177}$$

The extra term multiplying R_{-1} cancels nicely with the transformation of R_1. Finally, by taking into account the transformation of the integral measure $\mathrm{d}\tau \mapsto \mathrm{d}\tau\,\dot{\sigma}$ we see that all the local contributions in equation (5.167) are properly reparametrization invariant. Moreover, since

$$\frac{1+\frac{1}{2}\dot{\varepsilon}}{\varepsilon} \mapsto \frac{1}{\dot{\sigma}}\left(\frac{1+\frac{1}{2}\dot{\varepsilon}}{\varepsilon} + \mathcal{O}(\varepsilon)\right) , \tag{5.178}$$

the boundary term stays invariant as well. This concludes our discussion on the local terms of the Yangian.

6. Yangian Symmetry of Fishnet Feynman Graphs

In the second part of this thesis, we shift our focus to a different class of physical quantities: fishnet Feynman graphs. Fishnet Feynman diagrams are built from four-valent vertices that are joined by scalar propagators and correspond to planar high-loop Feynman integrals. Almost all of these integrals are believed to be elliptic [112, 113] or of even more general type, making them both interesting and hard to evaluate. Surprisingly, they exhibit a number of outstanding properties: First, fishnet integrals are all IR and UV finite, which makes them particularly well-suited for symmetry investigations. Second, they feature a (dual) conformal Lie algebra symmetry. Third, fishnet graphs (with periodic boundary conditions) furnish an integrable lattice system, as was demonstrated by Zamolodchikov already in 1980 [114]. Here, we add a further remarkable property to the above list, namely we demonstrate that fishnet Feynman integrals have conformal Yangian symmetry. We begin by discussing the relation between fishnet Feynman graphs and correlators/amplitudes in the recently proposed bi-scalar quantum field theory [46] that was obtained by studying a particular double-scaling limit of the γ_i-deformed $\mathcal{N} = 4$ SYM model. Subsequently, we expose the Yangian symmetry of off-shell fishnet graphs, which we discuss from the perspective of the first realization of the Yangian as well as from the point of view of the RTT realization. Finally, we consider on-shell limits and comment on the relation between conformal, dual conformal and Yangian symmetry.

6.1. Strongly-Twisted $\mathcal{N} = 4$ SYM Theory

Scalar fishnet Feynman graphs are in one-to-one correspondence with planar correlators in the recently proposed bi-scalar $\chi\mathrm{FT}_4$ model [46]. The latter can be obtained by considering a particular double-scaling limit of the γ_i-deformed $\mathcal{N} = 4$ SYM model. Here, we review this construction based on references [46, 115] and present arguments in favor of the aforementioned conjecture.

6.1.1. The γ_i-Deformed $\mathcal{N} = 4$ SYM Model

We begin by briefly reviewing the γ_i-deformation of the $\mathcal{N} = 4$ SYM model. This deformation was introduced by Frolov in [47,48] as an integrable [47,48,116–118] three-parameter generalization of the β-deformed SYM model [119,120], see [121] for a review

B	A_μ	φ_1	φ_2	φ_3	$\psi_{1\alpha}$	$\psi_{2\alpha}$	$\psi_{3\alpha}$	$\psi_{4\alpha}$
q_B^1	0	1	0	0	$+1/2$	$-1/2$	$-1/2$	$+1/2$
q_B^2	0	0	1	0	$-1/2$	$+1/2$	$-1/2$	$+1/2$
q_B^3	0	0	0	1	$-1/2$	$-1/2$	$+1/2$	$+1/2$

Table 6.1.: Charges of the fields under the Cartan generators of $\mathfrak{su}(4)$. The conjugated fields carry opposite charges.

on deformations of $\mathcal{N}=4$ SYM theory. The γ_i-deformed $\mathcal{N}=4$ SYM model can be obtained by replacing all products of fields in the $\mathcal{N}=4$ SYM Lagrangian (2.78) by a Moyal-like $*$-product based on the $\mathfrak{su}(4)$ Cartan charges of the fields. The $*$-product of two fields is defined as

$$A * B = AB\,\mathrm{e}^{\frac{i}{2}(\mathbf{q}_A \wedge \mathbf{q}_B)}\,, \tag{6.1}$$

where $\mathbf{q}_A = (q_A^1, q_A^2, q_A^3)$ and $\mathbf{q}_B = (q_B^1, q_B^2, q_B^3)$ are the $\mathfrak{su}(4)$ Cartan charge vectors of the fields A and B, see table 6.1. The antisymmetric $\mathbf{C}$-product in equation (6.1) is defined as

$$\mathbf{q}_A \wedge \mathbf{q}_B := (\mathbf{q}_A)^{\mathrm{T}}\mathbf{C}\,\mathbf{q}_B\,, \qquad \mathbf{C} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{6.2}$$

The phase factor of products of more than two fields can simply be obtained by applying the rule (6.1) successively, i.e.

$$A * B * C = \mathrm{e}^{\frac{i}{2}(\mathbf{q}_A \wedge \mathbf{q}_B)} AB * C = \mathrm{e}^{\frac{i}{2}(\mathbf{q}_A \wedge \mathbf{q}_B)}\mathrm{e}^{\frac{i}{2}((\mathbf{q}_A + \mathbf{q}_B)\wedge \mathbf{q}_C)} ABC\,. \tag{6.3}$$

Replacing all the products in the $\mathcal{N}=4$ SYM Lagrangian by $*$-products yields[1]

$$\begin{aligned} S_\gamma = N \int \mathrm{d}^4x\ \mathrm{tr}\Big[& -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \big(D^\mu \bar{\varphi}^i\big)\big(D_\mu \varphi_i\big) + i\,\bar{\psi}^a_{\dot\alpha} D^{\dot\alpha\beta}\psi_{a\beta} \\ & - \sqrt{\frac{\lambda}{2}}\Big(i\varepsilon^{ijk}\psi_i^\alpha \big[\varphi_j, \psi_{k\alpha}\big]_* + 2i\psi_i^\alpha\big[\bar{\varphi}^i, \psi_{4\alpha}\big]_* + i\varepsilon_{ijk}\bar{\psi}^i_{\dot\alpha}\big[\bar{\varphi}^j, \bar{\psi}^{k\dot\alpha}\big]_* + 2i\bar{\psi}^i_{\dot\alpha}\big[\varphi_i, \bar{\psi}^{4\dot\alpha}\big]_*\Big) \\ & + \lambda\Big(\big[\varphi_j, \varphi_k\big]_*\big[\bar{\varphi}^j, \bar{\varphi}^k\big]_* - \tfrac{1}{2}\big[\varphi_j, \bar{\varphi}^j\big]\big[\varphi_k, \bar{\varphi}^k\big]\Big)\Big]\,, \end{aligned} \tag{6.4}$$

where we have dropped the $*$ in cases where the $*$-product trivially reduces to the usual product. Note that in the above, we have adapted the field conventions of reference [115] in order to make contact to the existing literature on the subject. The indices i, j, k run from one to three and ε^{ijk} is the totally antisymmetric tensor in three dimensions. The scalars are thus no longer real but complex. Furthermore, we have rescaled the fields by a factor of $\sqrt{N}$ for later convenience. Using equation (6.1) and

[1]Note that here, the fields are hermitian and the color generators are normalized to $\mathrm{tr}(T^{\mathfrak{m}}T^{\mathfrak{n}}) = \delta^{\mathfrak{mn}}$.

(6.2) as well as the Cartan charges as given in table 6.1, we can now easily construct the phase factors multiplying the interaction terms in the γ_i-deformed Lagrangian (6.4). We find

$$\begin{aligned} S_\gamma =& N \int \mathrm{d}^4x \ \mathrm{tr}\Big[-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \Big(D^\mu\bar{\varphi}^i\Big)\Big(D_\mu\varphi_i\Big) + i\,\bar{\psi}^a_{\dot{\alpha}}D^{\dot{\alpha}\beta}\psi_{a\beta} \\ & - \sqrt{\frac{\lambda}{2}}\Big(2i\varepsilon^{ijk}\psi^\alpha_i\varphi_j\psi_{k\alpha}e^{\frac{i}{2}\varepsilon_{kil}\gamma^+_l} + 2i\varepsilon_{ijk}\bar{\psi}^i_{\dot{\alpha}}\bar{\varphi}^j\bar{\psi}^{k\dot{\alpha}}e^{\frac{i}{2}\varepsilon_{kil}\gamma^+_l} + 2i\psi^\alpha_i\bar{\varphi}^i\psi_{4\alpha}e^{-\frac{i}{2}\gamma^-_i} \\ & - 2i\psi^\alpha_4\bar{\varphi}^i\psi_{i\alpha}e^{\frac{i}{2}\gamma^-_i} + 2i\bar{\psi}^i_{\dot{\alpha}}\varphi_i\bar{\psi}^{4\dot{\alpha}}e^{-\frac{i}{2}\gamma^-_i} - 2i\bar{\psi}^4_{\dot{\alpha}}\varphi_i\bar{\psi}^{i\dot{\alpha}}e^{\frac{i}{2}\gamma^-_i}\Big) \\ & + \lambda\Big(2\varphi_j\varphi_k\bar{\varphi}^j\bar{\varphi}^k e^{-i\varepsilon_{jkl}\gamma_l} - \tfrac{1}{2}\Big\{\varphi_j,\bar{\varphi}^j\Big\}\Big\{\varphi_k,\bar{\varphi}^k\Big\}\Big)\Big], \end{aligned} \tag{6.5}$$

where a summation over all doubly- and triply-repeated indices is implied. The abbreviations γ_i^+ and γ_i^- are defined as

$$\gamma_1^\pm = -\frac{\gamma_3 \pm \gamma_2}{2}, \qquad \gamma_2^\pm = -\frac{\gamma_1 \pm \gamma_3}{2}, \qquad \gamma_3^\pm = -\frac{\gamma_2 \pm \gamma_1}{2}. \tag{6.6}$$

An important point to note is that the γ_i-deformation naively not only breaks the SU(4) R-symmetry group to its Cartan subgroup $\mathrm{U}(1)^{\times 3}$ but also the supersymmetry is completely broken for generic values of the deformation parameters γ_i. However, recent investigations [122–125] in the context of the β-deformed model (corresponding to $\gamma_i = \beta$) have shown that in the β-deformed case, the symmetries are not really broken but rather hidden in the sense that the full $\mathrm{PSU}(2,2|4)$ symmetry can (at least classically) be restored by twisting the plain symmetry generators with an appropriate Drinfel'd–Reshetikhin twist factor. Based on this, one might speculate that the same can be done for the full γ_i-deformed SYM model. However, even if the full $\mathrm{PSU}(2,2|4)$ symmetry can classically be restored, this does not imply that the deformed theories stay (super)conformal at the quantum level. In fact, it was shown in [126] that the action (6.5) does not lead to a consistent quantum CFT as renormalization requires the action to be modified by additional double-trace terms. The couplings of these double-trace terms were shown to be running with scale — even in the planar limit — and it was argued that the corresponding β-functions do not possess a real-valued fixed point.[2] However, since we are only interested in a particular double-scaling limit of (6.5), we will not discuss these terms here in all generality but rather comment on those which are relevant for us. This will be done in the next section.

6.1.2. The Bi-Scalar Double-Scaling Limit

Having introduced the γ_i-deformed $\mathcal{N} = 4$ model, we shall now discuss a particular class of double-scaling limits, which were introduced by V. Kazakov and Ö. Gürdoğan

[2] Fixed points do, however, exist if one allows for complex values of the tree-level double-trace couplings [51, 126, 127].

in [46] and combine a large imaginary twist with the weak coupling limit in the following way:

$$\gamma_i \to i\infty\,, \quad \lambda \to 0\,, \qquad \text{such that} \qquad \xi_i = \sqrt{\lambda}\,\mathrm{e}^{-\frac{i}{2}\gamma_i} = \text{const.}\,. \tag{6.7}$$

Applying this limit to the action (6.5) while keeping the effective coupling constants ξ_i distinguishable yields

$$S_{\text{ds}} = N \int \mathrm{d}^4x \;\operatorname{tr}\Big[-\tfrac{1}{4}F^{\mu\nu}_{\text{lin}}F^{\text{lin}}_{\mu\nu} - (\partial^\mu\bar{\varphi}^i)(\partial_\mu\varphi_i) + i\,\bar{\psi}^a_{\dot{\alpha}}\partial^{\dot{\alpha}\beta}\psi_{a\beta} + \mathcal{L}_{\text{int}}\Big]\,. \tag{6.8}$$

Here, $F^{\mu\nu}_{\text{lin}}$ is the linearized field strength, i.e. $F^{\mu\nu}_{\text{lin}} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\mathcal{L}_{\text{int}}$ is the interaction Lagrangian, which in our conventions reads

$$\begin{aligned}\mathcal{L}_{\text{int}} =& 2\xi_1^2\varphi_2\varphi_3\bar{\varphi}^2\bar{\varphi}^3 + 2\xi_2^2\varphi_3\varphi_1\bar{\varphi}^3\bar{\varphi}^1 + 2\xi_3^2\varphi_1\varphi_2\bar{\varphi}^1\bar{\varphi}^2 - i\sqrt{2\xi_2\xi_3}(\psi_3\varphi_1\psi_2 + \bar{\psi}^3\bar{\varphi}^1\bar{\psi}^2)\\ &- i\sqrt{2\xi_1\xi_3}(\psi_1\varphi_2\psi_3 + \bar{\psi}^1\bar{\varphi}^2\bar{\psi}^3) - i\sqrt{2\xi_1\xi_2}(\psi_2\varphi_3\psi_1 + \bar{\psi}^2\bar{\varphi}^3\bar{\psi}^1)\,.\end{aligned} \tag{6.9}$$

Note that for better readability we have suppressed the $\mathfrak{su}(2)$ indices in the last equation. They can, however, easily be restored by substituting

$$\psi_i\varphi_k\psi_j \to \psi_i^\alpha\varphi_k\psi_{j\alpha}\,, \qquad \bar{\psi}^i\bar{\varphi}^k\bar{\psi}^j \to \bar{\psi}^i_{\dot{\alpha}}\bar{\varphi}^k\bar{\psi}^{j\dot{\alpha}}\,. \tag{6.10}$$

An interesting point to note is that the gauge field as well as the fourth fermion decouples in this limit and we will thus drop their kinetic terms. Furthermore, we note that the limit projects out the hermitian conjugate terms of the interaction vertices, i.e. the resulting Lagrangian is not invariant with respect to hermitian conjugation. The authors of [46] called this behavior of the Lagrangian (6.9) “chiral”[3] and we will stick to this terminology even though it conflicts with the standard meaning of chirality. One consequence of this chirality is immediate: The double-scaled theories are non-unitary as their action is not real. Nevertheless, they lead to very sensible results as we will see in a moment.

In this thesis, we will focus on a very particular limit of the γ_i-deformed Lagrangian (6.5). This limit is obtained by sending the deformation parameters γ_i to imaginary infinity and λ to zero in such a way that two of the three effective couplings ξ_i vanish, while the third one remains finite. In this limit, the fermions and one of the three scalars decouples as well, so that the action contains only two scalars which interact via one chiral quartic vertex. Explicitly, the action reads

$$S_{\text{bs}} = N \int \mathrm{d}^4x \;\operatorname{tr}\Big[-(\partial^\mu\bar{\varphi}^1)(\partial_\mu\varphi_1) - (\partial^\mu\bar{\varphi}^2)(\partial_\mu\varphi_2) + 2\xi^2\varphi_1\varphi_2\bar{\varphi}^1\bar{\varphi}^2\Big]\,. \tag{6.11}$$

The corresponding Feynman rules are shown in figure 6.1. Although the theory de-

[3]The “antichiral” actions are obtained by taking the opposite double-scaling limits. More precisely, these limits are characterized by sending the deformation parameters γ_i to minus imaginary infinity and λ to zero in such a way that the quotient $\tilde{\xi}_i = \mathrm{e}^{-\frac{i}{2}\gamma_i}/\sqrt{\lambda}$ is constant. The resulting theories are, however, physically equivalent to the ones that we are considering.

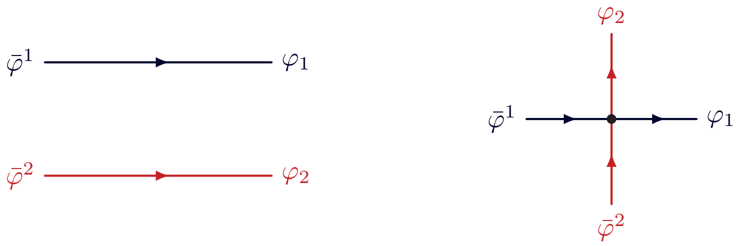


Figure 6.1.: Feynman rules for the bi-scalar $\chi\mathrm{FT}_4$ model. The left figure represents the two scalar propagators. Blue lines correspond to a propagating φ_1 field, while red lines correspond to a propagating φ_2 field. The arrows on the lines indicate the flavor flow of complex scalars. The only interaction vertex is a particular quartic one. Due to the imaginary twist factor, interactions can only happen with the orientation shown above. The vertex with opposite orientation is absent.

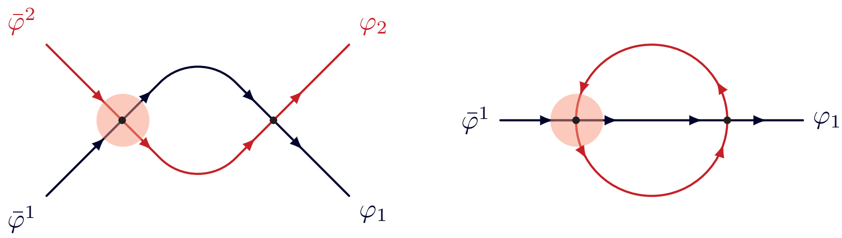


Figure 6.2.: The planar one-loop diagrams (in momentum space) which could potentially renormalize the coupling ξ^2 (left figure) or generate a mass (right figure). Both diagrams are absent since the left vertex is not present in the Lagrangian. This argument can be generalized to any loop order. Hence, neither the coupling nor the mass is renormalized in the planar limit.

scribed by (6.11) looks a bit pathological, its planar limit has a very rich structure and a number of intriguing features, which we will now briefly discuss. Let us start by analyzing the spacetime symmetries of the model. Besides being Poincaré invariant, the action is obviously scale invariant and therefore conformal at the classical level. Naively, conformality continues to hold at the quantum level in the large-N limit as the chirality of the vertex forbids planar diagrams which could renormalize the coupling ξ^2 or generate a mass, see figure 6.2. However, as was argued in [127], renormalizability requires the action (6.11) to be supplemented by the following double-trace terms:

$$\mathcal{L}_{\mathrm{dt}} = -\frac{\xi^2}{N}\left[\sum_{i\leq j=1}^{2} Q_{ij}^{ij}\,\mathrm{tr}(\varphi_i\varphi_j)\,\mathrm{tr}(\bar{\varphi}^i\bar{\varphi}^j) + \tilde{Q}\,\mathrm{tr}(\varphi_1\bar{\varphi}^2)\,\mathrm{tr}(\varphi_2\bar{\varphi}^1)\right]. \tag{6.12}$$

The corresponding couplings have a non-vanishing β-function already in the planar limit and therefore render the considered bi-scalar theory non-conformal.[4] However, one should note that the double-trace terms (6.12) only play a role for special correlation functions [51, 127]. More precisely, these are those which either directly contain operators of length $L = 2$, such as $\mathrm{tr}(\varphi_i\varphi_j)$ and $\mathrm{tr}(\bar{\varphi}^1\varphi_2)$, or which implicitly depend on them via the OPE of two operators. The majority of correlation functions is thus unaltered by the double-trace terms and show a perfectly conformal behavior in the planar limit. Almost quantum conformality is, however, by far not the only nice feature of this theory. Intriguingly, the double-scaling limit leaves not only intact the approximate conformality of the parent theory but also planar integrability seems to be preserved. In fact, the tools of (twisted) AdS/CFT integrability, such as twisted asymptotic Bethe ansätze [116], their generalizations in terms of the twisted Quantum Spectral Curve (QSC) [118] as well as the recently developed tools for computing structure constants [128, 129], seem to remain applicable [49, 50, 130]. This is a very remarkable result as the theory (6.11) has no gauge symmetry or supersymmetry at all — features which have long been believed to be prerequisites for higher-dimensional quantum integrability. Finally, let us remark that the integrability of this theory can not only be used to compute its observables, such as two- and three-point functions, but can rather assist in computing individual Feynman graphs. The reason for this lies in the special structure of the interaction term (6.11). As was noticed in [46], due to the chirality of the vertex, planar observables typically receive contributions from at most one Feynman diagram at any given order in perturbation theory. In the bulk, these Feynman diagrams always look like a regular square lattice, therefore justifying the name fishnet graphs. Due to this (almost) one-to-one correspondence between planar observables and Feynman diagrams, the bi-scalar QFT (6.11) provides us with the unique opportunity to use tools of integrability for computing certain high-loop Feynman graphs and a first few very promising results have already been obtained [46, 49–51, 130].

6.2. Correlators and Amplitudes

6.2.1. Definitions, Diagrammatics and Examples

Having introduced the field-theoretic framework, let us now define the quantities that we will study in the next sections. The main objects that we shall study are off-shell single-trace correlators of the following form

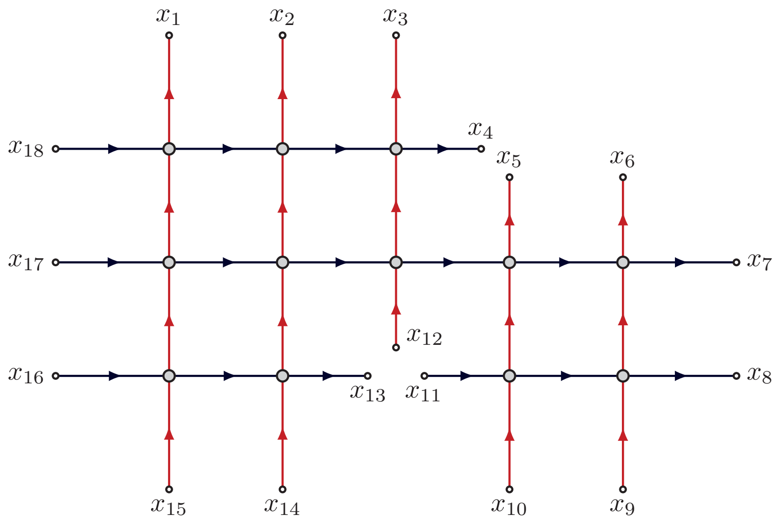
$$G(x_i) = \langle \mathrm{tr}(\chi_1(x_1)\dots\chi_{2M}(x_{2M}))\rangle \,, \tag{6.13}$$

where the $\chi_i(x_i)$ are elements of the set

$$\chi_i(x_i) \in \{\varphi_1(x_i), \varphi_2(x_i), \bar{\varphi}^1(x_i), \bar{\varphi}^2(x_i)\}\,. \tag{6.14}$$

[4]Concerning the question of fixed points of these β-functions see our comment in the previous section.

The word off shell in this context refers to the fact that we take all the position space coordinates to be independent, i.e. there are no constraints on these variables. We frequently use this terminology in order to distinguish the position space coordinates of correlators from the dual variables of amplitudes, where this will no longer be the case. For the time being, let us assume that all the coordinates x_i are different. A typical correlator in our bi-scalar model is then of the following form:



Here, gray-filled blobs denote loop integrations, while the smaller white blobs represent external points. Note that we will always be exclusively interested in the connected and planar contribution to the correlator. All non-planar contributions will be ignored. Obviously, due to charge conservation, the number of fields φ_1, which we call M_1, equals the number of fields $\bar{\varphi}^1$. The same applies to the fields $\bar{\varphi}^2$ and we will denote their number by M_2. The total number of fields is therefore $2M := 2(M_1 + M_2)$. The interaction vertex in (6.11) also conserves each of the two flavors. Hence, lines of a certain flavor run continuously from one external field to another external field. Furthermore, if we leave aside the double-trace terms (6.12), lines of the same flavor never intersect and the interactions happen only with one particular orientation due to the chirality of the vertex in (6.11). In our conventions, this leads to the fact that lines of flavor one will always continuously flow from the left to the right, while lines of flavor two flow from the bottom of the diagram to the top. Given these explanations, we note the following fact: In the planar approximation, a generic single-trace multi-point correlation function receives contributions from exactly one Feynman diagram which looks in the bulk like a regular square fishnet. The correlator therefore lives at a particular order in perturbation theory and does not receive further quantum corrections.

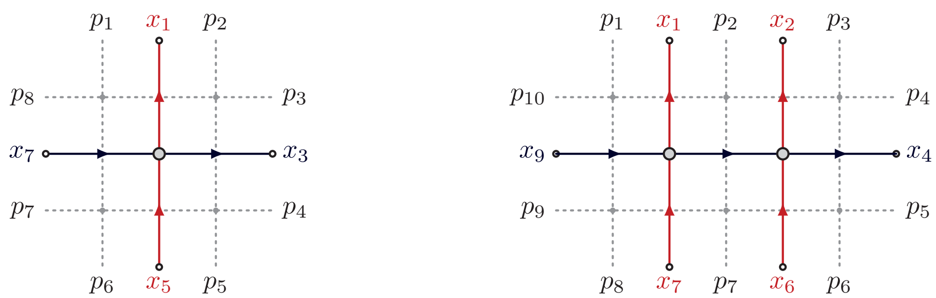
Before moving on, let us pause for a moment and reflect again on the double-trace terms (6.12). Naively, the double-trace terms are suppressed as they enter the Lagrangian with a relative factor of $1/N$ compared to the single-trace interaction term.

However, as was explained in [126] and [131], the overall power of N of a diagram containing these couplings can get enhanced to the planar level in particular cases. This typically happens when the traces of the multi-trace couplings are fully contracted with other traces of the same length. Consider, for example, the first correction to the two-point function of two appropriately chosen single-trace length-two operators. If both operators are fully contracted with one of the traces of the multi-trace coupling, the order in N gets enhanced, so that the diagram, despite being naively suppressed, is of the same order as the corresponding single-trace interaction diagram. However, as we are exclusively considering the expectation value of long single-trace operators, such an enhancement cannot occur.

Let us now move on and discuss the two simplest loop correlators of the form (6.13) in some more detail. In what follows, we will consider these two correlation functions frequently and we will furthermore use them here to explain the relation between single-trace correlators and amplitudes in the bi-scalar model (6.11). Explicitly, they read

$$
\begin{aligned}
G_4(x_i) &= \left\langle \mathrm{tr}(\varphi_2(x_1)\varphi_1(x_3)\bar{\varphi}^2(x_5)\bar{\varphi}^1(x_7)) \right\rangle , \\
G_6(x_i) &= \left\langle \mathrm{tr}(\varphi_2(x_1)\varphi_2(x_2)\varphi_1(x_4)\bar{\varphi}^2(x_6)\bar{\varphi}^2(x_7)\bar{\varphi}^1(x_9)) \right\rangle ,
\end{aligned} \tag{6.15}
$$

where the labeling of the points has been chosen for later convenience. Now, as explained above, both correlators receive contributions from only one Feynman diagram in the planar limit. The two corresponding Feynman diagrams look as follows:



Let us ignore the dual graph (momentum space) drawings for a moment and focus on the correlators themselves. Applying the Feynman rules yields

$$
\begin{aligned}
G_4(x_i) &= \frac{2\xi^2 N}{(4\pi^2)^4} \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{30}^2 x_{50}^2 x_{70}^2} , \\
G_6(x_i) &= \frac{4\xi^4 N}{(4\pi^2)^7} \int \frac{\mathrm{d}^4 x_0 \mathrm{d}^4 x_{0'}}{x_{10}^2 x_{70}^2 x_{90}^2 x_{00'}^2 x_{20'}^2 x_{40'}^2 x_{60'}^2} .
\end{aligned} \tag{6.16}
$$

Obviously, the planar correlators G_4 and G_6 are nothing more than the so-called cross and double-cross integral. While the cross integral is explicitly known to evaluate to [132],

$$G_4(x_i) = \frac{2\xi^2 N}{(4\pi^2)^4} \frac{\pi^2}{x_{15}^2 x_{37}^2} \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log z\bar{z} \log \frac{1-z}{1-\bar{z}}}{z - \bar{z}}\,, \tag{6.17}$$

with z and $\bar{z}$ being related to the conformal cross-ratios as

$$u = z\bar{z} = \frac{x_{13}^2 x_{57}^2}{x_{15}^2 x_{37}^2}\,, \qquad v = (1-z)(1-\bar{z}) = \frac{x_{17}^2 x_{35}^2}{x_{15}^2 x_{37}^2}\,, \tag{6.18}$$

the double-cross integral has not been computed yet and is furthermore believed to be elliptic [112]. However, recently a slightly simpler version of this integral[5] has been evaluated and expressed as an integral over a standardized elliptic measure times a weight-three hyperlogarithm [113]. Still, from the whole family of fishnet integrals obtained by gluing together crosses, the cross itself is the only one which is explicitly known. However, despite being not known, these scalar integrals play an important role in many four-dimensional quantum field theories as they are fairly general quantities.

An important point to note is also their relation to ordinary momentum space integrals [133–135] and therefore to loop amplitudes in $\chi\mathrm{FT}_4$. The corresponding momentum space integrals are easily obtained by interpreting the differences of x's not as distances in position space but rather as so-called region momenta. Let us make this more explicit using the cross integral as an example. We perform the following change of variables:

$$p_1^\mu = x_{12}^\mu\,, \qquad p_2^\mu = x_{23}^\mu\,, \qquad \ldots \qquad p_8^\mu = x_{81}^\mu\,, \qquad k^\mu = x_{10}^\mu\,. \tag{6.19}$$

The cross integral then becomes

$$G_4(p_i) = \frac{2\xi^2 N}{(4\pi^2)^4} \int \frac{\mathrm{d}^4 k}{k^2 (k - p_1 - p_2)^2 (k - p_1 - p_2 - p_3 - p_4)^2 (k + p_7 + p_8)^2}\,. \tag{6.20}$$

Graphically, this transformation corresponds to going over to the dual graph, which we have drawn in light grey in the above figure. The cross integral is obviously equivalent to the well-known (four-mass) box integral. Note, however, that by convention we still consider the x_i's as independent at this point, so that the external momentum legs in above figure are off shell, i.e. $p_i^2 \neq 0$, as $x_{i\,i+1}^2 \neq 0$ generically. In order to make contact to amplitudes, we obviously need to go on shell with the external legs. While for the box integral there is no difference between the on-shell and the off-shell case as the momenta enter the graph always in pairs of two, it will make a difference for all higher-loop graphs. This, however, raises the question which of the considered graphs are actually finite and whether we can always go on shell without creating divergences. Let us postpone this question to the next section and for the moment just assume that

[5] In reference [113], the upper two points and the lower two points of the double cross are assumed to be null separated, which reduces the number of conformal cross-ratios from nine to seven.

all the graphs are finite, off shell as well as on shell. If the on-shell limit is always well-defined, it becomes, however, an easy exercise to extract color-ordered amplitudes from the correlator graphs (6.13).[6] All one needs to do is to introduce momentum variables according to the prescription $p_i = x_{i\,i+1}$ and set to zeros all the squares of individual momenta. The resulting function then describes a color-ordered loop amplitude with its momentum configuration being determined by the dual graph, see the above figure.

6.2.2. Finiteness

An important point which we have not yet discussed is whether the considered graphs are actually finite or not. We will address this question now, adopting the momentum space point of view. Since a rigorous proof of finiteness is somewhat beyond the scope of this thesis, we content ourselves with presenting evidence in favor of their finiteness. Let us start by considering graphs which are made out of boxes all glued together in a fishnet type manner with all the external momenta being off shell. Such a graph represents the dual graph of a generic correlator graph as considered above. In general, Feynman graphs can suffer from two different types of divergences: UV divergences and IR divergences. While the former originate from integration regions where some or all of the loop momenta are large, the latter come from regions where the loop momenta are small or collinear to one of the external (null) momenta, see [138] for a more detailed discussion. However, in what follows, we will argue that our graphs are actually free of both types of divergences. For this, we first note that we do not need to worry about collinear divergences at the moment because so far all the external momenta are considered to be off shell. In order to argue that UV and soft divergences are absent as well, we will employ power counting. In general, the overall UV degree of divergence of a graph Γ in our theory is given by

$$\omega(\Gamma) = 4L - 2P\,, \tag{6.21}$$

where L is the number of loops and P is the number of propagators. This formula can easily be derived by introducing spherical coordinates in $4 \times L$ dimensions. Evaluating (6.21) for the box diagram G_4 and the double-box diagram G_6 yields

$$\omega(G_4) = -4\,, \qquad\qquad \omega(G_6) = -6\,. \tag{6.22}$$

It is easy to convince oneself that ω in general drops further as the number of loops grows. As our fishnet graphs do furthermore not contain any divergent subgraphs, we conclude that they are indeed all UV finite. A similar reasoning can be applied to argue that IR divergences are absent as well. Indeed, noting that the leading divergence of the denominator of each square loop is captured by the respective loop momentum

[6] Note that the considered correlators only allow for the extraction of color-ordered amplitudes. The full amplitudes can, however, easily be reconstructed from those, see [136, 137] for an introduction to color ordering.

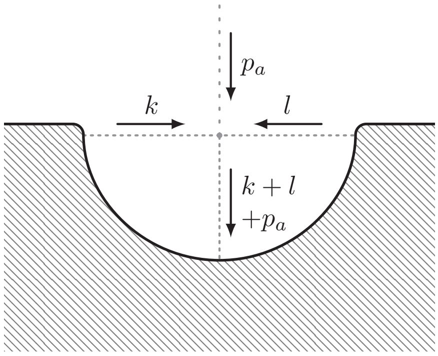


Figure 6.3.: The figure represents a generic vertex in momentum space which is located on the inside of one of the outer edges of a fishnet diagram. This diagram could, for example, represent the upper middle vertex of the double-box diagram. Here, p_a (loosely dotted line) denotes an external on-shell momentum, while k and l (dotted lines) denote loop momenta.

squared,[7] it becomes almost obvious that no IR divergences are present as the integral measure cancels the square of the loop momentum in the denominator. Having settled the off-shell case, let us now turn to the more interesting case of on-shell momenta. While the UV behavior does not depend on whether the inflowing momenta are on or off shell, the IR behavior potentially does. However, the only vertices we need to inspect again are those sitting on the inside of the outer edges as these are the only vertices where a single massless momentum gets injected into the graph, see figure 6.3. At the corners, external momenta always enter in pairs, so that the effective momentum injected there is generically still off shell and no divergences are expected to arise from these particular parts of the diagram. This being said, let us investigate the vertices on the outer edges in a bit more detail. For this, we analyze a generic vertex of this type as shown in figure 6.3. The relevant part of the integral looks as follows:

$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-\mathrm{d}k_\perp^2\,\mathrm{d}l^+\,\mathrm{d}l^-\mathrm{d}l_\perp^2}{(2k^+k^- - k_\perp^2)(2l^+l^- - l_\perp^2)(k^- + l^-)p_a^+}\,. \tag{6.23}$$

Here, we have already employed the so-called eikonal approximation, i.e. we only keep the leading term in the denominator. Furthermore, we have introduced light-cone coordinates in such a way that p_a has only one non-zero component, which is p_a^+. As mentioned before, there are two integration regions which can potentially lead to divergences: the soft region and the collinear region. To investigate the soft region, we set $k^+ \to t$ and rescale all the other momentum components by this factor, i.e. $x \to tx$. The first two terms in the denominator obviously scale homogeneously as t^2, while the last one scales linearly in t. The denominator thus becomes proportional to t^5, but the

[7] All other propagators do not contribute as they stay finite in the zero-momentum limit due to inflowing off-shell momenta.

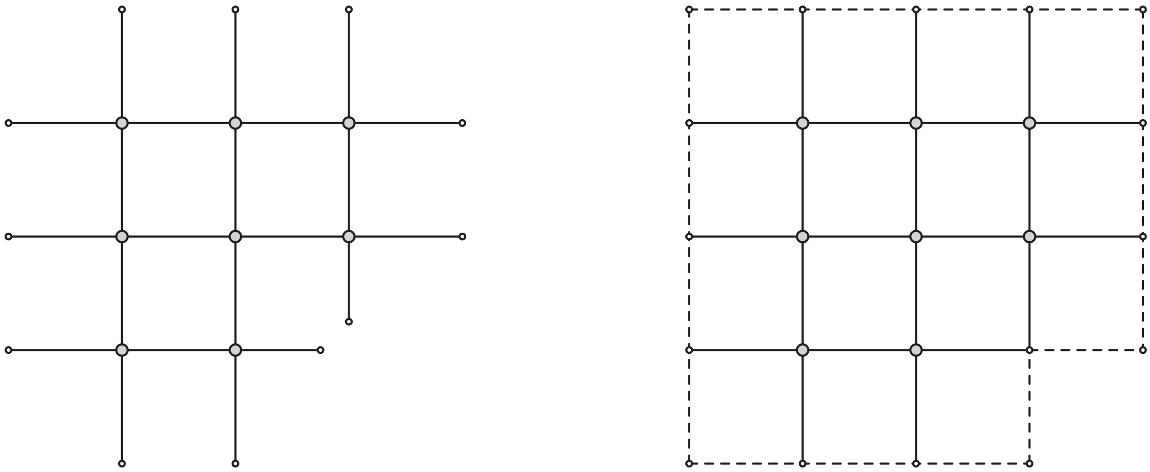
transformation of the measure yields a factor of t^7. As all the other propagators of the two involved boxes stay finite in the zero-momentum limit, the above-considered part of the multi-loop integral does obviously not lead to a soft divergence. Similarly, one can convince oneself that the other parts of the diagram stay finite as well, so that fishnet integrals with on-shell momenta do not suffer from soft divergences. However, to make sure that the whole integral stays finite, we need to argue that collinear singularities are absent as well. For this, we scale the momenta as

$$\begin{aligned} k^+ &\to k^+ , \qquad & k^- &\to t^2 , \qquad & \vec{k}_\perp &\to t\,\vec{k}_\perp , \\ l^+ &\to l^+ , \qquad & l^- &\to t^2 l^- , \qquad & \vec{l}_\perp &\to t\,\vec{l}_\perp , \end{aligned} \tag{6.24}$$

so that in the limit $t \to 0$ the loop momenta become collinear with the external on-shell momentum. In this case, the denominator scales as t^6, while the transformation of the measure again yields a factor of t^7. Hence, collinear singularities are absent as well, so that on-shell fishnet integrals are indeed finite. This concludes our argument towards the finiteness of fishnet Feynman graphs.

6.3. Symmetries of Fishnet Feynman Graphs

Having discussed the relation between regular fishnet diagrams and correlators in the bi-scalar $\chi\mathrm{FT}_4$ model, let us now turn to the main part of this chapter, namely an in-depth investigation of the symmetries of these graphs. In fact, there are many hints pointing towards the fact that scalar fishnet graphs come endowed with a rich algebraic structure. One of the earliest hints was found by Zamolodchikov, who showed in 1980 that scalar fishnet graphs, which are built from four-point vertices and massless propagators, define a completely integrable statistical lattice model [114]. Furthermore, as was argued in the previous sections, these graphs feature a conformal symmetry as they can be identified with planar correlators in a theory which behaves (almost) like a CFT in the large-N limit. The theory, namely the $\chi\mathrm{FT}_4$ model, is furthermore believed to be an integrable field theory and it is thus natural to search for signs of integrability within the sector of single-trace correlators alias fishnet Feynman graphs. A typical sign of integrability is an infinite-dimensional symmetry algebra of Yangian type, see also 2.1.3. In what follows, we will establish a conformal Yangian symmetry for the following two types of diagrams:



Note that as our symmetry considerations will apply to generic fishnet diagrams independent of their origin in a particular theory, we will drop the arrows and flavor distinction from now on. The diagram on the left represents such a generic scalar fishnet graph. It is made out of four-point vertices which are connected by scalar massless propagators. By assumption, we first take the external points x_i to be different. Later on, we will also consider slightly generalized fishnet graphs like the one depicted on the right-hand side of the above figure. Considering such graphs will allow us to prove the Yangian symmetry of on-shell graphs (scattering amplitudes). In this case, the x-variables are no longer independent due to the on-shell constraint $x_{i\,i+1}^2 = 0$. We will take these constraints into account by multiplying the respective correlator graph with a product of delta functions $\delta(x_{i\,i+1}^2)$. Graphically, we depict these delta functions by dashed lines. Without further ado, let us now start with the analysis.

6.3.1. First Realization of the Yangian

In this section, we shall analyze the symmetries of scalar fishnet graphs employing the language of level-zero and level-one generators. This construction is known as Drinfel'd's first realization of the Yangian and has been introduced in detail in section 2.1.3. We will use this language to demonstrate the Yangian symmetry of two pedagogical examples, namely the cross integral and the double-cross integral. In the further course of this thesis, we will then introduce the powerful RTT formalism and use it to prove the Yangian symmetry of generic scalar fishnet Feynman graphs.

6.3.1.1. Lie Algebra Symmetry

In order to lay the foundations for the discussion of Yangian symmetry, let us briefly review the underlying Lie algebra symmetry of fishnet graphs. The algebra on which the construction of the Yangian will be based is the conformal algebra $\mathfrak{so}(2,4)$, which we have already introduced in section 2.1.1. In this section, we use the following representation of the algebra:

$$\begin{aligned}
\mathrm{P}_\mu &= -i\partial_\mu \,, \\
\mathrm{L}_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \,, \\
\mathrm{D}^\Delta &= -i(x^\mu \partial_\mu + \Delta) \,, \\
\mathrm{K}_\mu^\Delta &= 2x^\nu L_{\nu\mu} - ix^2 \partial_\mu - 2i\Delta x_\mu \,.
\end{aligned} \tag{6.25}$$

Note that, for later convenience, we have changed to a hermitian basis of the conformal algebra. Moreover, we have included the conformal dimension Δ, which we encountered in the context of the field representation in section 2.1.1. In principle, we could as well work with a representation acting on the fields as all the fishnet graphs are in one-to-one correspondence with a planar correlator in the $\chi\mathrm{FT}_4$ model. However, we prefer to investigate fishnet graphs from a general point of view without relying on the correlator interpretation. Since all the external coordinates are assumed to be different, there is after all not much difference between the two representations because subtleties as those described in section 5.1.2 do not arise.

To investigate the conformal properties of fishnet Feynman graphs, we again focus on the inversion. As discussed in section 2.1.1, the inversion is a discrete element of the conformal group and acts on the coordinates as

$$\mathrm{I}_b[x^\mu] = \frac{x^\mu}{x^2} \,. \tag{6.26}$$

It relates the generator of translations P_μ and the generator of conformal boosts $\mathrm{K}_\mu^{\Delta=0}$ in the following way:

$$\mathrm{K}_\mu^{\Delta=0} = -\mathrm{I}_b \circ \mathrm{P}_\mu \circ \mathrm{I}_b \,. \tag{6.27}$$

Let us now check the inversion properties of the two simplest fishnet diagrams, namely the cross and the double cross. We have written down theses integrals already while discussing correlators, but for convenience let us state them here again

$$|F_4\rangle = \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \,, \qquad |F_6\rangle = \int \frac{\mathrm{d}^4 x_0 \, \mathrm{d}^4 x_{0'}}{x_{10}^2 x_{50}^2 x_{60}^2 x_{00'}^2 x_{20'}^2 x_{30'}^2 x_{40'}^2} \,. \tag{6.28}$$

The notation as a state has been chosen in anticipation of the subsequent discussion on the RTT realization, where fishnet graphs are interpreted as eigenstates of a monodromy matrix. In order to invert the two expressions (6.28), it is useful to note the following relations:

$$\mathrm{I}_b[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2} \,, \qquad \mathrm{I}_b[\mathrm{d}^4 x_0] = \frac{\mathrm{d}^4 x_0}{x_0^8} \,. \tag{6.29}$$

Using these formulas, we find

$$\mathrm{I}_b[|F_4\rangle] = x_1^2 x_2^2 x_3^2 x_4^2 \, |F_4\rangle \,, \qquad \mathrm{I}_b[|F_6\rangle] = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 \, |F_6\rangle \,. \tag{6.30}$$

The two considered fishnet graphs are obviously not invariant under a conformal inversion but rather transform covariantly. Taking into account the relation between P_μ and $\mathrm{K}_\mu^{\Delta=0}$ (6.27), it is not hard to see that under a special conformal transformation the cross and the double cross transform as

$$\mathrm{K}^\mu_{\vec{\Delta}=0} |F_4\rangle = \sum_{i=1}^{4} 2ix_i^\mu |F_4\rangle , \qquad \mathrm{K}^\mu_{\vec{\Delta}=0} |F_6\rangle = \sum_{i=1}^{6} 2ix_i^\mu |F_6\rangle , \tag{6.31}$$

where we have introduced the shorthand notation $\vec{\Delta}$ for the set of conformal dimensions. By comparing this to equation (6.25), we see that the two graphs are invariant if we choose the conformal dimensions Δ_i to be equal to one at each site, i.e.

$$\mathrm{K}^\mu_{\vec{\Delta}=1} |F_4\rangle = \mathrm{K}^\mu_{\vec{\Delta}=1} |F_6\rangle = 0 . \tag{6.32}$$

Note that due to the commutation relations (2.7), the invariance under $\mathrm{D}_{\vec{\Delta}=1}$ is guaranteed and the graphs are hence conformally invariant as expected.

Given these two examples, it is actually not hard to convince oneself that the pattern (6.30) continues to higher loop orders and thus all the graphs of plain fishnet type feature a conformal symmetry. We will postpone the discussion of the on-shell graphs to the section on the RTT formulation.

6.3.1.2. Yangian Symmetry

Having discussed the Lie algebra symmetry of fishnet graphs, we now turn to the construction of the Yangian. The level-zero algebra on which the construction is based is the conformal algebra $\mathfrak{so}(2,4)$ and, as before, we choose to represent the level-zero generators by the differential operators given in equation (6.25). Based on our discussion in section 2.1.3, we write down the following ansatz for the level-one generators:

$$\widehat{\mathrm{J}}^\delta = \tfrac{1}{2} f^\delta{}_{\rho\kappa} \sum_{j<k} \mathrm{J}_j^\kappa \mathrm{J}_k^\rho + \sum_k v_k \mathrm{J}_k^\delta . \tag{6.33}$$

Here, $f^\delta{}_{\rho\kappa}$ are the inverse structure constants and the v_k's are the evaluation parameters, which can be chosen arbitrarily without spoiling the defining algebra relations of the Yangian, see section 2.1.3. In many cases, like, for example, in the context of superamplitudes in $\mathcal{N}=4$ SYM theory [25], the local contribution is absent and we have $v_k = 0$ for all k. However, in the present case we will use this freedom to choose the v_k's to define a consistent Yangian symmetry algebra of fishnet graphs. Let us illustrate this on the example of the cross integral and the double-cross integral. Note that, as in the case of Wilson loops, it again suffices to show invariance under one level-one generator. The level-zero invariance together with the Yangian commutation relations (2.52) guarantees that all the other level-one generators annihilate the (double-)cross integral as well.

In what follows, we will choose the simplest level-one generator, which is the level-one momentum generator $\widehat{\mathrm{P}}^\mu$. Using the formula (6.33), we find the following expression for the bi-local piece of $\widehat{\mathrm{P}}^\mu$,

$$\widehat{\mathrm{P}}^\mu_{\mathrm{bi}} = -\tfrac{i}{2}\sum_{j<k}\Big[(\mathrm{L}_j^{\mu\nu} + \eta^{\mu\nu}\mathrm{D}_j^{\Delta_j=1})\mathrm{P}_{k,\nu} - (j \leftrightarrow k)\Big]\,, \tag{6.34}$$

where $\mathrm{L}_j^{\mu\nu}$, $\mathrm{D}_j^{\Delta_j=1}$ and P_k^μ are the single-site conformal generators as introduced above. Applying this generator to the box integral (6.28) yields

$$\widehat{\mathrm{P}}^\mu_{\mathrm{bi}}\,|F_4\rangle = \Big(\mathrm{P}_2^\mu + 2\mathrm{P}_3^\mu + 3\mathrm{P}_4^\mu\Big)\,|F_4\rangle\,. \tag{6.35}$$

The form of the right-hand side makes it obvious that we can use the freedom to choose the v_k's in equation (6.33) to construct a true symmetry generator. Explicitly, we define the full level-one momentum generator as

$$\widehat{\mathrm{P}}^\mu_{F_4} := \widehat{\mathrm{P}}^\mu_{\mathrm{bi}} - \mathrm{P}_2^\mu - 2\mathrm{P}_3^\mu - 3\mathrm{P}_4^\mu\,. \tag{6.36}$$

As a second example, let us consider the double-cross diagram. Applying the bi-local generator (6.34) to the double-cross integral (6.28) yields

$$\widehat{\mathrm{P}}^\mu_{\mathrm{bi}}\,|F_6\rangle = \Big(\mathrm{P}_3^\mu + 2\mathrm{P}_4^\mu + 2\mathrm{P}_5^\mu + 3\mathrm{P}_6^\mu\Big)\,|F_6\rangle\,. \tag{6.37}$$

Again, we see that we can define an algebraically consistent level-one momentum generator that annihilates the double-cross integral by choosing the inhomogeneities as follows

$$\widehat{\mathrm{P}}^\mu_{F_6} := \widehat{\mathrm{P}}^\mu_{\mathrm{bi}} - \mathrm{P}_3^\mu - 2\mathrm{P}_4^\mu - 2\mathrm{P}_5^\mu - 3\mathrm{P}_6^\mu\,. \tag{6.38}$$

In principle, this method can be continued to higher and higher loop orders. However, in practice, it cannot because the complexity of the computation grows vastly with the number of loops. Furthermore, one should note that symmetry statements such as

$$\widehat{\mathrm{P}}^\mu_{F_4}|F_4\rangle = \widehat{\mathrm{P}}^\mu_{F_6}|F_6\rangle = 0\,, \tag{6.39}$$

do typically not hold at the level of the integrand due to total derivative terms. Computer algebra systems can therefore only be of partial help in proving these statements. In conclusion, we have to state that this method is probably unsuitable for proving the Yangian symmetry of a generic fishnet Feynman graph. Fortunately, there exists another formulation of the Yangian, which goes under the name of the RTT realization. This formalism will turn out to be highly capable of tackling this problem and we will introduce it and explain its relation to the first realization of the Yangian in section 6.3.2. Although the first realization seems to have its drawbacks, it is well-suited for studying the implications of Yangian symmetry and that is what we shall attempt next.

6.3.1.3. Differential Equations from Yangian Symmetry

An obvious aspect of interest is whether the Yangian symmetry can actually help in computing the fishnet Feynman integrals that we are considering. A natural approach to this question is to examine the differential equations that the Yangian symmetry implies. In general, these differential equations, such as (6.39), are very complicated due to the vast number of terms in the bi-local part of the generator. For this reason, it is crucial to simplify them by introducing an appropriate set of variables. Here, we shall focus on the simplest fishnet integral, which is the cross integral. A clear advantage of focusing on this representative is that the integral is explicitly known and we can therefore directly test our equations. Furthermore, the integral depends only on four external points, which keeps the number of conformal cross-ratios manageable. We begin by simplifying the four-point level-one generator. For convenience, let us define the following abbreviation:

$$\widehat{\mathrm{J}}^{\delta}_{\mathrm{bi}} = \tfrac{1}{2}\sum_{j<k} C^{\delta}_{jk}\,, \qquad \text{where} \qquad C^{\delta}_{jk} = f^{\delta}{}_{\rho\kappa}\mathrm{J}^{\kappa}_{j}\mathrm{J}^{\rho}_{k}\,. \tag{6.40}$$

In the four-point case, we obviously have

$$2\widehat{\mathrm{J}}^{\delta}_{\mathrm{bi}} = C^{\delta}_{12} + C^{\delta}_{34} + C^{\delta}_{13} + C^{\delta}_{14} + C^{\delta}_{23} + C^{\delta}_{24}\,. \tag{6.41}$$

To simplify this equation, we rewrite the last four terms according to

$$\begin{aligned} C^{\delta}_{13} + C^{\delta}_{14} + C^{\delta}_{23} + C^{\delta}_{24} &= f^{\delta}{}_{\rho\kappa}(\mathrm{J}^{\kappa}_{1} + \mathrm{J}^{\kappa}_{2})(\mathrm{J}^{\rho}_{3} + \mathrm{J}^{\rho}_{4}) \\ &= f^{\delta}{}_{\rho\kappa}(\mathrm{J}^{\kappa}_{1} + \mathrm{J}^{\kappa}_{2})\mathrm{J}^{\rho} - \tfrac{1}{2} f^{\delta}{}_{\rho\kappa} f^{\kappa\rho}{}_{\sigma}(\mathrm{J}^{\sigma}_{1} + \mathrm{J}^{\sigma}_{2})\,. \end{aligned} \tag{6.42}$$

The first term obviously annihilates level-zero invariants and we will thus drop it. The second term is proportional to the dual Coxeter number (2.56) of the conformal algebra, which is non-vanishing. For level-zero invariants $|F_n\rangle$, and with $\widehat{\mathrm{J}}^{\delta} = \widehat{\mathrm{J}}^{\delta}_{\mathrm{bi}} + \sum_k v_k \mathrm{J}^{\delta}_{k}$, we thus have

$$\widehat{\mathrm{J}}^{\delta}|F\rangle = 0 \qquad \Leftrightarrow \qquad \Big(C^{\delta}_{12} + C^{\delta}_{34} + \sum_{k=1}^{4}[2v_k - \mathfrak{c}(\delta_{k,1} + \delta_{k,2})]\mathrm{J}^{\delta}_{k}\Big)|F\rangle = 0. \tag{6.43}$$

Let us now specify to the case of the level-one momentum generator. The corresponding C-operator reads

$$\begin{aligned} C^{\mathrm{P}^{\mu}}_{jk} &= -i(\mathrm{L}^{\mu\nu}_{j} + \eta^{\mu\nu}\mathrm{D}^{\Delta_j}_{j})\mathrm{P}_{k,\nu} - (j \leftrightarrow k) \\ &= -i\Big[x^{\mu}_{j}\partial^{\nu}_{j}\partial_{k,\nu} - x^{\nu}_{j}\partial^{\mu}_{j}\partial_{k,\nu} - x_{j,\nu}\partial^{\nu}_{j}\partial^{\mu}_{k} - \Delta_j\partial^{\mu}_{k}\Big] - (j \leftrightarrow k) \\ &= -i\Big(x^{\mu}_{jk}\eta^{\rho\sigma} - x^{\rho}_{jk}\eta^{\mu\sigma} - x^{\sigma}_{jk}\eta^{\mu\rho}\Big)\partial_{j\rho}\partial_{k\sigma} + i\Big(\Delta_j\partial^{\mu}_{k} - \Delta_k\partial^{\mu}_{j}\Big)\,, \end{aligned} \tag{6.44}$$

with Δ_j being equal to one at each site. Taking into account the insights discussed above, we can rephrase the level-one momentum invariance of the cross integral as follows

$$\left[C^{\mathrm{P}^{\mu}}_{12} + C^{\mathrm{P}^{\mu}}_{34} + \sum_{k=1}^{4}\theta_k \mathrm{P}^{\mu}_{k}\right]|F_4\rangle = 0\,. \tag{6.45}$$

By plugging in the result of the former section, namely $v_k = -(k-1)$, and using that $\mathfrak{c} = 4$, we find (modulo an overall shift)

$$\theta_k = 2v_k - \mathfrak{c}(\delta_{k1} + \delta_{k2}) = (+1, -1, +1, -1) \,. \tag{6.46}$$

At this point, we note that we can absorb the Δ-terms in equation (6.44) into a redefinition of the parameters θ_k. Doing so yields the following differential equation

$$\left[C^{\mathrm{P}^\mu}_{12,\Delta=0} + C^{\mathrm{P}^\mu}_{34,\Delta=0} + \sum_{k=1}^{4} \theta'_k \mathrm{P}^\mu_k \right] |F_4\rangle = 0 \,, \tag{6.47}$$

with

$$\theta'_k = (0, -4, 0, -4) \,. \tag{6.48}$$

After having simplified the level-one equation, we shall now perform a change of variables. The new variables are the conformal cross-ratios u and v, which make manifest the level-zero symmetry of the problem. These variables will not only allow us to rephrase the level-one equation (6.47) in a nice and compact fashion but also help in separating the implications of level-zero symmetry and level-one invariance. Our strategy to introduce them is the following: First, we note that we can use the level-zero invariance of the cross integral to argue that the integrated result is of the following form:

$$|F_4\rangle = \frac{1}{x_{13}^2 x_{24}^2}\, \phi(u, v) \,. \tag{6.49}$$

Here, u and v are the conformally-invariant cross-ratios,[8] being defined as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \,, \qquad\qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,, \tag{6.50}$$

and ϕ is an arbitrary function of them. In the second step, we now plug the ansatz (6.49) into the simplified level-one equation (6.47). The computation is in principal straightforward but nevertheless quite lengthy. However, we will refrain from presenting it here in detail and only state the final result. It reads

$$\begin{aligned} &\left[C^{\mathrm{P}^\mu}_{12,\Delta=0} + C^{\mathrm{P}^\mu}_{34,\Delta=0} + \sum_{k=1}^{4} \theta'_k \mathrm{P}^\mu_k \right] |F_4\rangle = 0 = \frac{-4i}{x_{13}^2 x_{24}^2} \times \\ &\times \left[\left(\frac{x_{14}^\mu}{x_{14}^2} - \frac{x_{23}^\mu}{x_{23}^2} \right) \Big(v + 3vu\partial_u + (3v-1)v\partial_v + vu^2\partial_u^2 + (v-1)v^2\partial_v^2 + 2v^2u\partial_u\partial_v \Big)\phi \right. \\ &\left. - \left(\frac{x_{12}^\mu}{x_{12}^2} + \frac{x_{34}^\mu}{x_{34}^2} \right) \Big(u + (3u-1)u\partial_u + 3uv\partial_v + (u-1)u^2\partial_u^2 + uv^2\partial_v^2 + 2u^2v\partial_u\partial_v \Big)\phi \right] . \end{aligned} \tag{6.51}$$

[8]The variables u and v are annihilated by the conformal generators (6.25) with Δ_i uniformly chosen to be equal to zero.

Keeping all squares x_{ij}^2 fixed so that u and v are constant removes six of the sixteen degrees of freedom of four Minkowski vectors. One can check that this leaves enough freedom to vary the two coefficients that multiply the differential equations in (6.51) independently, so that we conclude that the function ϕ must satisfy the following system of differential equations:

$$\begin{aligned}&\Big(1+3u\partial_u+(3v-1)\partial_v+u^2\partial_u^2+(v-1)v\partial_v^2+2vu\partial_u\partial_v\Big)\phi(u,v)=0\,,\\&\Big(1+(3u-1)\partial_u+3v\partial_v+(u-1)u\partial_u^2+v^2\partial_v^2+2uv\partial_u\partial_v\Big)\phi(u,v)=0\,.\end{aligned}\tag{6.52}$$

Obviously, the two equations are related to each other by exchange of u and v. It can be checked that the solution (6.17) indeed satisfies them. An interesting point to investigate is how constraining the above equations are and how they can be solved. However, clarifying these and related questions is left for future work.

6.3.1.4. Yangian Symmetry and Cyclicity

An important point when dealing with Yangian symmetry of quantities which have a cyclic shift symmetry is the interplay between these two symmetries, see also our discussion in section 5.2.1. While the level-zero generators are cyclic, the level-one generators typically map cyclic functions to non-cyclic ones. This is due to the fact that the ordering prescription in the bi-local piece of the level-one generators (6.33) singles out a reference point. In the case of color-ordered scattering amplitudes in $\mathcal{N}=4$ SYM theory, the compatibility between Yangian symmetry and cyclicity is ensured by the vanishing of the dual Coxeter number of the underlying Lie algebra [25]. For the case at hand, i.e. for the conformal algebra $\mathfrak{so}(2,4)$, the dual Coxeter number is, however, non-vanishing. Nevertheless, we have some cyclicity constraints for certain types of fishnet graphs. For instance, an entire n-point fishnet graph (with complete rows only) has an $\frac{n}{2}$-site cyclic shift symmetry of the external legs. One may thus wonder how the consistency is achieved in this case and furthermore which restrictions the cyclicity imposes on the most generic level-one Yangian generator. That is what we are going to discuss next.

We start by considering the bi-local piece of the level-one generator

$$\widehat{\mathrm{J}}^\delta_{\mathrm{bi}}|_{1,n}=\tfrac{1}{2}f^\delta{}_{\rho\kappa}\sum_{j<k=1}^{n}\mathrm{J}^\kappa_j\mathrm{J}^\rho_k\,,\tag{6.53}$$

where $|_{1,n}$ denotes the boundaries of summation. Shifting the summation range in the above equation by m units and subtracting the original bi-local generator yields

$$\widehat{\mathrm{J}}^\delta_{\mathrm{bi}}|_{1+m,n+m}-\widehat{\mathrm{J}}^\delta_{\mathrm{bi}}|_{1,n}=\tfrac{1}{2}f^\delta{}_{\rho\kappa}f^{\kappa\rho}{}_\sigma\sum_{k=1}^{m}\mathrm{J}^\sigma_k-f^\delta{}_{\rho\kappa}\sum_{k=1}^{m}\mathrm{J}^\kappa_k\mathrm{J}^\rho.\tag{6.54}$$

Restricting to the space of level-zero invariants, the second term is trivially zero and we will neglect it from now on. Equation (6.54) can be further simplified by noting that the product of structure constants can be replaced by the dual Coxeter number, see equation (2.56). We thus have

$$\widehat{\mathrm{J}}^{\delta}_{\mathrm{bi}}|_{1+m,n+m} - \widehat{\mathrm{J}}^{\delta}_{\mathrm{bi}}|_{1,n} = \mathfrak{c} \sum_{k=1}^{m} \mathrm{J}^{\delta}_{k} \,. \tag{6.55}$$

From this equation, we conclude that if the underlying level-zero algebra has a non-vanishing dual Coxeter number and we are considering a quantity which has an m-site cyclic shift symmetry, the bi-local piece on its own can never be a symmetry generator as the right-hand side of the equation does not vanish upon acting on the object under consideration.

However, the key observation now is that we can compensate for the term on the right-hand side of equation (6.55) by adding an appropriate local term $\sum_{k=1}^{n} v_k \mathrm{J}^{\delta}_{k}$ to the level-one generator. Under an m-site cyclic shift, such a term transforms as

$$\begin{aligned}\widehat{\mathrm{J}}^{\delta}_{\mathrm{lo}}|_{1+m,n+m} - \widehat{\mathrm{J}}^{\delta}_{\mathrm{lo}}|_{1,n} &= \sum_{i=m+1}^{n+m} v_{i-m}\, \mathrm{J}^{\delta}_{i} - \sum_{i=1}^{n} v_i\, \mathrm{J}^{\delta}_{i} \\ &= \sum_{i=1}^{m} (v_{i-m+n} - v_i) \mathrm{J}^{\delta}_{i} + \sum_{i=m+1}^{n} (v_{i-m} - v_i) \mathrm{J}^{\delta}_{i} \,. \end{aligned} \tag{6.56}$$

In total, we thus have

$$\widehat{\mathrm{J}}^{A}|_{1+m,n+m} - \widehat{\mathrm{J}}^{\delta}|_{1,n} = \sum_{i=1}^{m} (v_{i-m+n} - v_i + \mathfrak{c}) \mathrm{J}^{\delta}_{i} + \sum_{i=m+1}^{n} (v_{i-m} - v_i) \mathrm{J}^{\delta}_{i} \,. \tag{6.57}$$

To ensure consistency between Yangian symmetry and cyclic shift symmetry, we require that

$$\widehat{\mathrm{J}}^{\delta}|_{1+m,n+m} - \widehat{\mathrm{J}}^{\delta}|_{1,n} = a_m \sum_{i=1}^{n} \mathrm{J}^{\delta}_{i} \,, \tag{6.58}$$

where we allow the constant a_m to depend on the length of the shift. This leads to the following set of equations:

$$\begin{aligned} v_{i-m+n} - v_i + \mathfrak{c} &= a_m \,, && \forall i \in \{1, \dots, m\} \,, \\ v_{i-m} - v_i &= a_m \,, && \forall i \in \{m+1, \dots, n\} \,. \end{aligned} \tag{6.59}$$

Setting $m = \frac{n}{2}$, the above system of equations (6.59) has $n+1$ free parameters and the solution reads

$$v_{k>\frac{n}{2}} = v_{k-\frac{n}{2}} - \frac{\mathfrak{c}}{2} \,, \qquad\qquad a_{\frac{n}{2}} = \frac{\mathfrak{c}}{2} \,. \tag{6.60}$$

Hence, in the case of entire fishnet graphs, half of the evaluation parameters v_k are actually fixed by cyclicity.

Unique level-one generator for truly cyclic quantities. We can apply the above arguments to the case when the object we are acting on is truly cyclic in the sense that it is invariant under the shift $x_k \to x_{k+1}$. This, for example, applies to the cross integral $|F_4\rangle$, see (6.28). In this case, the system of equations (6.59) admits the following solution:

$$v_k = v - (k-1)\frac{\mathfrak{c}}{n}\,, \qquad a_1 = \frac{\mathfrak{c}}{n}\,. \tag{6.61}$$

Here, v represents the freedom to shift the level-one generator by a full level-zero generator and we may set this parameter to zero without loss of generality. Hence, we see that in the case at hand the evaluation parameters v_k are completely fixed by cyclicity. This means that there is a unique level-one generator that is consistent with true cyclicity.

6.3.2. RTT Realization

In order to demonstrate the Yangian symmetry of generic fishnet Feynman graphs, it is useful to formulate the invariance discussed above in terms of the powerful RTT formulation of the Yangian algebra. Here, the Yangian generators are packaged into a monodromy matrix $\mathrm{T}(u)$,

$$\mathrm{T}_{\alpha\beta}(u) = \delta_{\alpha\beta} + \sum_{n=0}^{\infty} u^{-n-1}\mathcal{J}^{(n)}_{\alpha\beta}\,, \tag{6.62}$$

where u is the spectral parameter and $\mathcal{J}^{(n)}$ is the infinite set of Yangian generators. The precise relation between the generators $\mathcal{J}^{(n)}$ and the ones introduced above will be established in section 6.3.2.4. In this framework, the defining algebra relations of the Yangian are encoded into a Yang-Baxter equation for the monodromy matrix $\mathrm{T}(u)$, which is called the RTT relation

$$\mathrm{R}(u-v)\mathrm{T}(u)\otimes\mathrm{T}(v) = \mathrm{T}(v)\otimes\mathrm{T}(u)\mathrm{R}(u-v)\,, \tag{6.63}$$

where $\mathrm{R}(u)$ is Yang's R-matrix

$$\mathrm{R}(u) = \mathbb{1} + u\mathbb{P}\,, \tag{6.64}$$

with $\mathbb{P}$ being the permutation operator. The RTT formulation of the Yangian algebra has first appeared implicitly in the context of the QISM [63, 139], much earlier than its general definition by Drinfel'd [60].

The simplest solution to equation (6.63) is provided by the trivial evaluation representation of the Yangian, being characterized by $\mathcal{J}^{(n)} = 0$ for $n > 0$. The corresponding solution is called the Lax operator and explicitly reads

$$\mathrm{L}_{\alpha\beta}(u) := u\mathrm{T}_{\alpha\beta}(u) = u\delta_{\alpha\beta} + \mathcal{J}^{(0)}_{\alpha\beta}\,. \tag{6.65}$$

It satisfies equation (6.63) and will play the role of a basic building block in our construction. We will present a detailed discussion of this operator in the next section. Note that in equation (6.65) we have changed the normalization of the operator by including an extra factor of u. This will ensure that the monodromy matrices are polynomial in the spectral parameter. Given the Lax operator (6.65), we can now define the n-site inhomogeneous monodromy matrix

$$\mathrm{T}(\vec{u}) = \mathrm{L}_n(u_n)\ldots\mathrm{L}_2(u_2)\mathrm{L}_1(u_1)\,, \tag{6.66}$$

where $\vec{u} = (u_1,\ldots,u_n)$. The subscript i of the Lax operator L_i denotes the site on which the operator acts. Each Lax operator acts on its own site and we will identify these sites with the external legs. In the RTT framework, the Yangian invariance of fishnet graphs $|F_n\rangle$ translates into an eigenvalue equation for the n-site monodromy matrix $\mathrm{T}(\vec{u})$,

$$\mathrm{T}(\vec{u})|F_n\rangle = \mathrm{L}_n(u_n)\ldots\mathrm{L}_2(u_2)\mathrm{L}_1(u_1)|F_n\rangle = \lambda(\vec{u})|F_n\rangle\mathbb{1}\,. \tag{6.67}$$

Here, the u_i's are defined as $u_i = u + \delta_i$ where u is the spectral parameter and the δ_i's are the inhomogeneity parameters, which have to be chosen appropriately for each diagram. The eigenvalue λ is a polynomial in u of degree n and its exact form depends on the considered graph. Note that since the right-hand side of equation (6.67) is proportional to the identity matrix, the off-diagonal generators, obtained from the expansion of (6.67), have to annihilate the state $|F_n\rangle$, while the diagonal generators act covariantly, i.e.

$$\mathcal{J}^{(n)}_{\alpha\beta}|F_n\rangle = c_n\delta_{\alpha\beta}|F_n\rangle\,. \tag{6.68}$$

Before moving on to the construction of the appropriate Lax operator, let us point out that monodromy eigenvalue equations of the form (6.67) have been considered before in [69, 70, 140]. They were shown to bridge the gap between the Yangian invariance of scattering amplitudes in $\mathcal{N}=4$ SYM theory and the QISM and thus paved the way for the application of integrability techniques to the scattering problem in $\mathcal{N}=4$ SYM theory [69–71, 141–143] and in ABJM theory [141].

6.3.2.1. The Conformal Lax Operator

Let us now introduce the most fundamental object of this section which is the Lax operator $\mathrm{L}_{\alpha\beta}$. The Lax operator that we are going to use has been considered before in [144] and is of the following form:

$$\mathrm{L}_{\alpha\beta}(u,\Delta) = u\delta_{\alpha\beta} + \tfrac{1}{2}\mathrm{S}^{ab}_{\alpha\beta}\,\mathrm{J}^{\Delta}_{ab}\,. \tag{6.69}$$

Here, $\mathrm{S}^{ab}_{\alpha\beta}$ labels a finite-dimensional matrix representation of the conformal algebra $\mathfrak{so}(2,4)$ acting on so-called auxiliary space, while by J^{Δ}_{ab} we refer to the differential operator representation (6.25) acting on quantum space. Note that we have labeled the fifteen generators by an antisymmetric pair of indices both running from one to six.[9] In the case at hand, the generators S_{ab} form an irreducible spinor representation of the algebra $\mathfrak{so}(2,4)$. To construct this representation, we use a basis of gamma matrices Γ_a for the six-dimensional space $\mathbb{R}^{2,4}$. A spinor representation of the algebra $\mathfrak{so}(2,4)$ can then be constructed by considering the commutator $\frac{i}{4}[\Gamma_a,\Gamma_b]$. This representation is, however, reducible since all matrices take a block-diagonal form. The generators S_{ab} are obtained by projecting onto the Weyl subspace V_+, i.e.

$$\mathrm{S}_{ab} = \tfrac{i}{4}[\Gamma_a,\Gamma_b]\Big|_{\downarrow V_+} . \tag{6.70}$$

By plugging this into equation (6.69), we find the following explicit expression for the Lax operator [145],

$$\mathrm{L}(u_+,u_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_+\cdot\mathbf{1} & \mathbf{p} \\ \mathbf{0} & u_-\cdot\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix}, \tag{6.71}$$

where the block 2×2 matrices are defined as $\mathbf{x} \equiv -i\bar{\boldsymbol{\sigma}}^\mu x_\mu$ and $\mathbf{p} \equiv -\frac{i}{2}\boldsymbol{\sigma}^\mu\partial_\mu$ with the Pauli matrices as given in appendix A.1. Note that, for later convenience, we have introduced the following abbreviations

$$u_+ \equiv u + \tfrac{1}{2}(\Delta - 4)\,, \qquad\qquad u_- \equiv u - \tfrac{1}{2}\Delta\,, \tag{6.72}$$

where u is the spectral parameter and Δ is the conformal dimension, see equation (6.25). Inverting the two equations (6.72) yields

$$u = \tfrac{1}{2}(u_+ + u_- + 2)\,, \qquad\qquad \Delta = u_+ - u_- + 2\,. \tag{6.73}$$

In what follows, we will use both notations, $\mathrm{L}(u_+,u_-)$ and $\mathrm{L}(u;\Delta)$, for the Lax operator (6.71).

Lax operator relations. For later convenience, let us list a number of relations that the Lax operator (6.71) fulfills and that we will need in the next two sections. Most of the relations can easily be checked, either by hand or using Mathematica, and we will therefore refrain from presenting these computations here.

- We denote by L^T the transposed Lax operator in the non-compact physical space (not the auxiliary matrix space), i.e. we have $x_\mu^T = x_\mu$ and $\partial_\mu^T = -\partial_\mu$, which is equivalent to integration by parts. The inverse of the Lax operator coincides with its transposition (up to permutation and shift of parameters)

$$\mathrm{L}^T_{\alpha\beta}(v-2,u-2)\mathrm{L}_{\beta\gamma}(u,v) = uv\,\delta_{\alpha\gamma}\,. \tag{6.74}$$

[9] The similarity transformation relating the two different bases used in equation (6.25) and (6.69) is standard material and can, for example, be found in [144].

- The Lax operator acts diagonally on 1 at $\Delta = 0$, i.e.

$$\mathrm{L}_{\alpha\beta}(u, u+2) \cdot 1 = (u+2)\delta_{\alpha\beta}\,, \quad \mathrm{L}^T_{\alpha\beta}(u+2, u) \cdot 1 = (u+2)\delta_{\alpha\beta}\,. \tag{6.75}$$

We can think of 1 as a local pseudo-vacuum state of the Lax operator.

- The scalar propagator x_{12}^{-2} is an intertwining operator, permuting spectral parameters of the two-site monodromy [145–148]:

$$x_{12}^{-2}\mathrm{L}_1(u, v)\mathrm{L}_2(w, u+1) = \mathrm{L}_1(u+1, v)\mathrm{L}_2(w, u)x_{12}^{-2}. \tag{6.76}$$

- As we are working in Minkowski space, we can consider the unitary cut of the Feynman propagator $1/(x_{12}^2 + i\varepsilon)$, which is $\delta(x_{12}^2)$. Interestingly, this delta function satisfies the same intertwining relation

$$\delta(x_{12}^2)\mathrm{L}_1(u, v)\mathrm{L}_2(w, u+1) = \mathrm{L}_1(u+1, v)\mathrm{L}_2(w, u)\delta(x_{12}^2). \tag{6.77}$$

- The eigenvalue problems for monodromies with the same cyclic ordering of Lax operators are equivalent:

$$\begin{aligned} \mathrm{L}_n(u_n; \Delta_n) \ldots \mathrm{L}_1(u_1; \Delta_1)\,|F_n\rangle &= \lambda\,|F_n\rangle\mathbb{1}\,, \\ &\Updownarrow \\ \mathrm{L}_{n-1}(u_{n-1}; \Delta_{n-1}) \ldots \mathrm{L}_1(u_1; \Delta_1)\mathrm{L}_n(u_n - 4; \Delta_n)\,|F_n\rangle &= \widetilde{\lambda}\,|F_n\rangle\mathbb{1}, \end{aligned} \tag{6.78}$$

where by u_k we mean different spectral parameters for each Lax operator. Here, the eigenvalues λ and $\widetilde{\lambda}$ are related by $u_{n+}u_{n-}\widetilde{\lambda} = (u_{n+} - 2)(u_{n-} - 2)\lambda$. A proof of this statement can be found in reference [140] and [5].

6.3.2.2. Yangian Symmetry of Correlator Fishnet Graphs

We are now ready to prove the Yangian symmetry of a generic fishnet graph, such as depicted in figure 6.6. As explained above, proving the Yangian symmetry of a generic fishnet graph in the language of the RTT realization amounts to showing that the following relation holds true:

$$\mathrm{L}_n[\delta_n^+, \delta_n^-] \ldots \mathrm{L}_2[\delta_2^+, \delta_2^-]\mathrm{L}_1[\delta_1^+, \delta_1^-]|F_n\rangle = \lambda(\delta_i^+, \delta_i^-)|F_n\rangle\mathbb{1}\,. \tag{6.79}$$

Here, we have introduced a new notation, which we will frequently use from now on

$$\mathrm{L}_i[\delta_i^+, \delta_i^-] := \mathrm{L}_i(u + \delta_i^+, u + \delta_i^-)\,, \tag{6.80}$$

where the Lax operator L is to be seen as a function of u_+ and u_-, i.e. $\mathrm{L} = \mathrm{L}(u_+, u_-)$. The inhomogeneity parameters δ_i^+ and δ_i^- as well as the eigenvalue $\lambda(\delta_i^+, \delta_i^-)$ depend on the graph under consideration and we will explain how to choose the δ_i's momentarily.

Before we explain how to prove the Yangian symmetry of a generic fishnet graph, let us consider again our two primary examples, namely the cross integral and the double-cross integral. The eigenvalue relation for the cross integral reads

$$\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2]\,|F_4\rangle = [3][4]^2[5]\,|F_4\rangle\,\mathbb{1}\,, \tag{6.81}$$

where $[\delta_k]$ is shorthand for $(u+\delta_k)$. The strategy to prove equation (6.81) is the following: First, we extend the monodromy matrix on the left-hand side by inserting a sophisticated identity of the form $\mathbb{1} = [2]^{-1}\,\mathrm{L}_0^T[2,0]\cdot 1$ (see equation (6.75)), where 0 labels the point that is integrated over. We then integrate the Lax operator $\mathrm{L}_0^T[2,0]$ by parts so that it acts on the integrand of the cross integral, i.e.

$$\begin{aligned}&[2]^{-1}\int \mathrm{d}^4x_0\,\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2](\mathrm{L}_0^T[2,0]\cdot 1)\,x_{10}^{-2}x_{20}^{-2}x_{30}^{-2}x_{40}^{-2}\\ =&[2]^{-1}\int \mathrm{d}^4x_0\,\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2]\mathrm{L}_0[2,0]x_{10}^{-2}x_{20}^{-2}x_{30}^{-2}x_{40}^{-2}\,.\end{aligned} \tag{6.82}$$

In the next step, we repeatedly use the intertwining relation (6.76) to pull the scalar propagators through the Lax operators. As soon as there is no more coordinate dependence to the right of the external Lax operators, i.e. they act only on the vacuum state, we use equation (6.75) to replace them by a numerical factor times an identity matrix. Let us see how this works in detail. First, we pull through the factor x_{10}^{-2},

$$\mathrm{L}_1[1,2]\mathrm{L}_0[2,0]\,x_{10}^{-2} = x_{10}^{-2}\,\mathrm{L}_1[0,2]\mathrm{L}_0[2,1] = [2]\,x_{10}^{-2}\,\mathrm{L}_0[2,1]\,. \tag{6.83}$$

In the next step, we pull through x_{20}^{-2},

$$\mathrm{L}_2[2,3]\mathrm{L}_0[2,1]\,x_{20}^{-2} = x_{20}^{-2}\,\mathrm{L}_2[1,3]\mathrm{L}_0[2,2] = [3]\,x_{20}^{-2}\,\mathrm{L}_0[2,2]\,. \tag{6.84}$$

This procedure can be continued until the integrand is completely to the left of the product of Lax operators. In the last step, we replace the Lax operators L_4 and L_0, which act on the vacuum, by a numerical factor times the identity matrix

$$\mathrm{L}_4[4,5]\mathrm{L}_0[2,3]x_{40}^{-2} = x_{40}^{-2}\,\mathrm{L}_4[3,5]\mathrm{L}_0[2,4] = [4][5]\,x_{40}^{-2}\,\mathbb{1}\,. \tag{6.85}$$

Collecting all the numerical factors yields the polynomial $[3][4]^2[5]$ and equation (6.81) is hence proven. Already in the case of the cross integral the expressions would have been quite bulky if we had written out the complete integral in each step. For this reason, we need a more efficient notation in order to be able to prove the Yangian symmetry of generic graphs. Fortunately, there exists a nice graphical way to represent the sequence of transformations used to prove the eigenvalue relation for the cross integral, see figure 6.4. In this figure, the monodromy is depicted by the oriented contour that embraces the graph. Lax operators correspond to solid red segments, while dashed orange lines denote summation over matrix indices. The shifts in the arguments of the Lax operators $\mathrm{L}(u+\delta_i^+, u+\delta_i^-)$ are denoted by the two numbers in square brackets next to the nodes. Each picture in figure 6.4 corresponds to one of the transformation steps explained above.

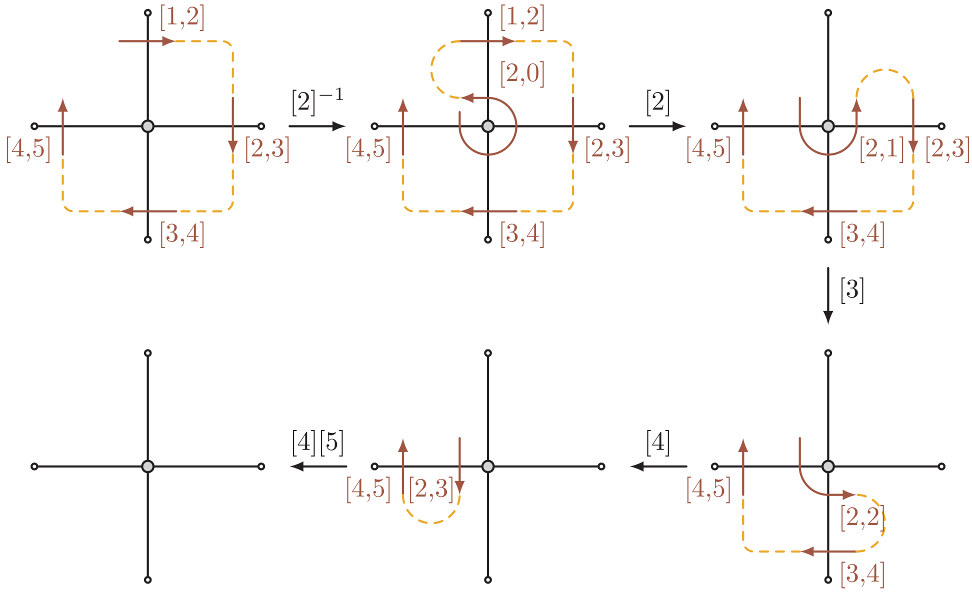


Figure 6.4.: This figure visualizes the sequence of transformations that we use to prove that the cross integral furnishes an eigenstate of the four-point monodromy matrix. The Lax operators of the external part of the monodromy act on fixed external coordinates. The Lax operator introduced around the integrated middle node acts on the coordinate of this node. Numerical factors appearing in the process are indicated above arrows.

Using the same logic, we can now also prove the Yangian invariance of the double-cross integral. The eigenvalue equation for this graph reads

$$\mathrm{L}_6[4,5]\mathrm{L}_5[3,4]\mathrm{L}_4[3,4]\mathrm{L}_3[2,3]\mathrm{L}_2[1,2]\mathrm{L}_1[1,2]\,|F_6\rangle = [3]^2[4]^3[5]\,|F_6\rangle\mathbb{1}\,. \tag{6.86}$$

Again, the logic is to extend the above monodromy by inserting two identities in the form of two additional Lax operators which act on the integration points x_0 and $x_{0'}$. After integrating these by parts, we get

$$[2]^{-2}\,\mathrm{L}_6[4,5]\mathrm{L}_5[3,4]\mathrm{L}_4[3,4]\mathrm{L}_3[2,3]\mathrm{L}_2[1,2]\mathrm{L}_{0'}[2,0]\mathrm{L}_1[1,2]\mathrm{L}_0[2,0]\,. \tag{6.87}$$

Subsequently, we use the intertwining relation (6.76) as well as the vacuum identities (6.75) to pull the monodromy line through the integrand. The whole sequence of transformations is depicted in figure 6.5.

Finally, we now attempt the general case. We will mainly use the graphical methods explained above to prove the Yangian invariance of generic graphs as explicit formulas would be quite bulky. In the first step, we draw the monodromy line and attribute inhomogeneity parameters to the external legs according to the following regular pattern:

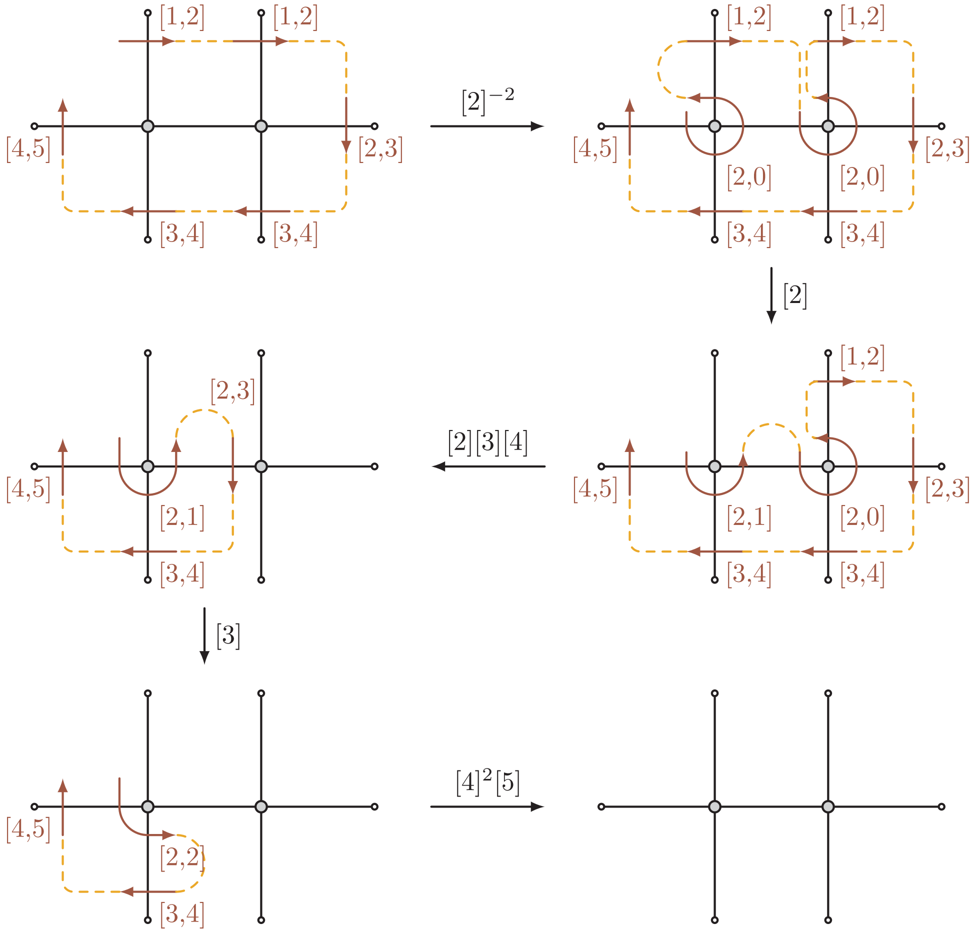


Figure 6.5.: This figure visualizes the sequence of transformations that can be used to prove that the double-cross integral furnishes an eigenstate of the six-point monodromy matrix. Numerical factors appearing in the process are indicated above the arrows.

- At the first leg, we choose $[\delta_1^+, \delta_1^-] = [1, 2]$ (of course the overall spectral parameter u is allowed to be shifted uniformly in all u_{i+} and u_{i-} along the contour).
- We do not change the inhomogeneities of the Lax operators when moving straight along a horizontal or vertical segment of the contour.
- We increase $\delta_{i+1}^{\pm} = \delta_i^{\pm} + 1$ at a convex corner $i \to i+1$ when turning by an angle $\pi/2$.
- We decrease $\delta_{i+1}^{\pm} = \delta_i^{\pm} - 1$ at a concave corner $i \to i+1$ when turning by an angle $-\pi/2$.

A representative graph including the monodromy line with appropriately chosen inhomogeneities is depicted in figure 6.6. In order to show that the claimed eigenvalue

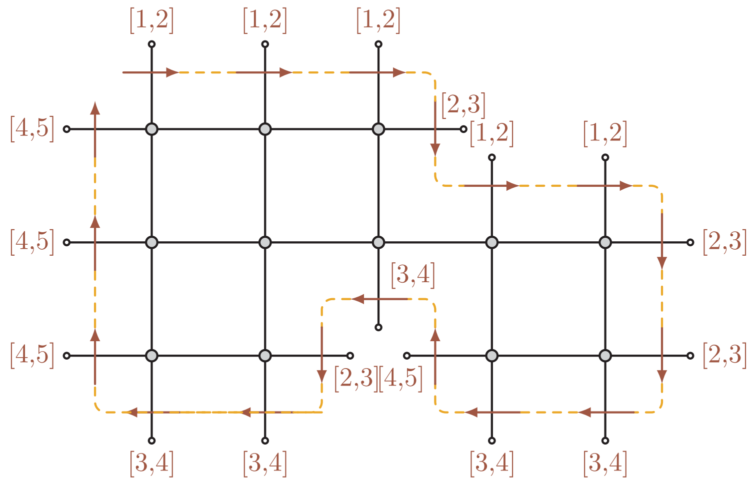


Figure 6.6.: This diagram represents a generic fishnet Feynman graph. The corresponding n-site monodromy is depicted by the oriented contour. As before, Lax operators correspond to solid red segments, while dashed orange lines denote summation over matrix indices.

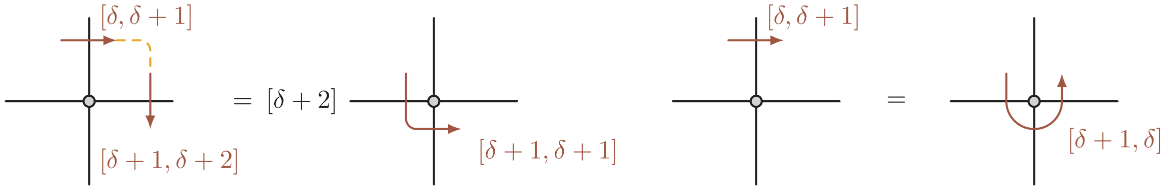


Figure 6.7.: Local transformations employed to prove the Yangian symmetry of a generic fishnet Feynman graph. Pushing the contour inside the graph involves integration by parts. If the initial monodromy acts on an L-loop graph, then, after one local transformation, the new monodromy (transformed contour) acts on an $(L-1)$-loop integral.

relation holds true, we again pull the monodromy through the graph using the local transformations depicted in figure 6.7. We also use the intertwining relation (6.76) and the vacuum relations (6.75). The local transformations depicted in figure 6.7 can be justified using the same manipulations that we employed to prove the eigenvalue relation of the cross integral. In order to obtain the eigenvalue of the monodromy relation (6.79), one just needs to keep track of all the factors while deforming the contour. However, the expression for the eigenvalue can also directly be read off from the respective graph. For this, we split the set of $2M$ external legs, which we will call $\mathcal{C}$, into pairs of antipodes. More precisely, we decompose the set $\mathcal{C}$ according to the following rule: A leg which we encounter first when moving along the monodromy contour (it has a lower number) belongs to the set $\mathcal{C}_{\text{in}}$, and a leg which we encounter last moving along

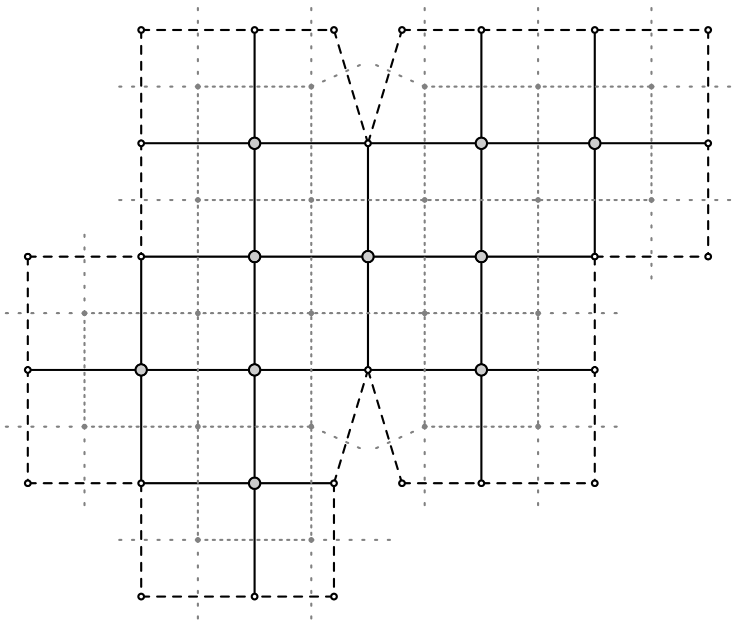


Figure 6.8.: A fishnet amplitude graph representing a generic scattering amplitude in the bi-scalar theory and its dual graph. The amplitude graph is drawn by dotted lines, which denote scalar propagators p_i^{-2}. All inflowing momenta are on shell. They are denoted by loosely dotted lines. The dual graph is formed by solid lines, denoting scalar propagators x_{ij}^{-2}, and dashed lines, which represent delta functions $\delta(x_{ij}^2)$.

the contour (it has a higher number) belongs to the set $\mathcal{C}_{\text{out}}$. Using this prescription, we decompose the set of external legs according to $\mathcal{C} = \mathcal{C}_{\text{in}} \cup \mathcal{C}_{\text{out}}$. The eigenvalue λ in equation (6.79) is then simply given by the following formula:

$$\lambda(\delta_i^+, \delta_i^-) = \prod_{i \in \mathcal{C}_{\text{out}}} [\delta_i^+][\delta_i^-] \,. \tag{6.88}$$

For example, for the graph depicted in figure 6.6, we obtain

$$\lambda(u) = ([3][4])^5([4][5])^4 = (u+3)^5(u+4)^9(u+5)^4 \,. \tag{6.89}$$

This completes our prove of the Yangian symmetry of generic fishnet correlator graphs.

6.3.2.3. Yangian Symmetry of Amplitude Fishnet Graphs

Having discussed the Yangian invariance of correlator fishnet graphs, let us now consider on-shell momentum space graphs and make contact to color-ordered scattering amplitudes in the bi-scalar χFT$_4$ model. A generic on-shell momentum space graph is depicted in figure 6.8 by dotted lines. To show that it satisfies a monodromy relation, we consider the corresponding dual graph. The dual graph is drawn using solid lines and, as before, filled blobs denote integration vertices, while white blobs denote external points. An important point to note is that we now interpret the x_i's as region momenta and no longer as coordinates in position space. The region momenta are related to the external momenta in the following way:

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \,. \tag{6.90}$$

An important difference between the correlator case considered before and the case at hand is that the x_i's are no longer all independent as the external momenta square to zero, i.e. $p_i^2 = 0$. Obviously, the region momenta have to be constrained by $x_{i\,i+1}^2 = 0$. In order to account for these constraints, we have modified the boundary of the dual graph in figure 6.8 in the following way: First, we added an extra external node at each convex corner in order to account for the fact that we always have two inflowing momenta at these corners. Conversely, we identified external coordinates at concave corners, see figure 6.8. Subsequently, we have multiplied the whole graph by a product of delta functions $\delta(x_{i\,i+1}^2)$ to enforce the light-likeness constraints. Graphically, we have depicted the delta functions by dashed lines, see figure 6.8. An important point to note is that the delta functions $\delta(x_{i\,i+1}^2)$ satisfy the same intertwining relation as the propagators, see equation (6.77). To prove that the dual graph in figure 6.8 obeys a monodromy relation, we can thus employ the same techniques that we used to prove the Yangian symmetry of correlator graphs. Let us elaborate on this a bit more. First, we again draw the monodromy line by encircling the whole graph. This is depicted in figure 6.9. The assignment of inhomogeneities for such a graph follows the rules depicted in figure 6.10. It is now easy to convince oneself that the monodromy can again be pulled through the graph by using the local transformations depicted in figure 6.7, the intertwining relations for propagators (6.76) and delta functions (6.77) as well as the vacuum relation (6.75). Explicit expressions for the eigenvalue λ can straightforwardly be worked out for each particular graph by keeping track of all the factors while deforming the contour.

There exists, however, a quicker way to find the eigenvalue λ. By applying n-times the cyclicity argument (6.78) to the initial eigenvalue problem, we obtain an equivalent eigenvalue relation which is related to the original one by a uniform shift of the spectral parameter u by a factor of minus four. By comparing the two eigenvalues λ and $\widetilde{\lambda}$, we learn that the eigenvalue satisfies the following finite-difference equation

$$\frac{\lambda(u)}{\lambda(u-4)} = \frac{P(u)}{P(u-2)} \,, \tag{6.91}$$

where $\lambda(u)$ is a polynomial of degree n and $P(u)$ is given by

$$\lambda(u) = \prod_{i=1}^{n} (u + a_i) \,, \qquad\qquad P(u) = \prod_{i=1}^{n} (u + \delta_i^+)(u + \delta_i^-) \,. \tag{6.92}$$

Equation (6.91) is reminiscent of the Bethe ansatz equations but in comparison much easier to solve. In practice, it can often be solved by just staring at the equation. For

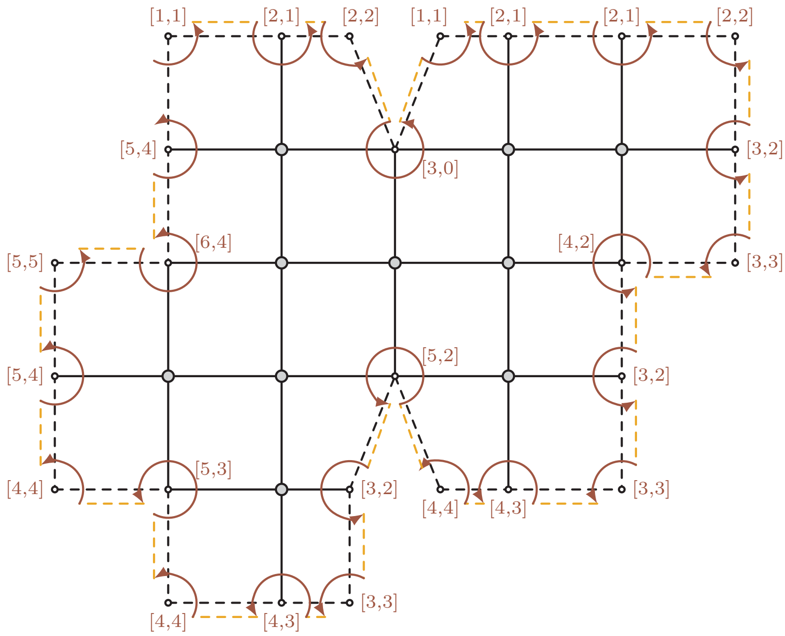


Figure 6.9.: The amplitude graph depicted in figure 6.8 is Yangian invariant, i.e. it is an eigenstate of the monodromy with the indicated inhomogeneities $[\delta_i^+, \delta_i^-]$. The assignment of inhomogeneities follows the rules depicted in Fig. 6.10.

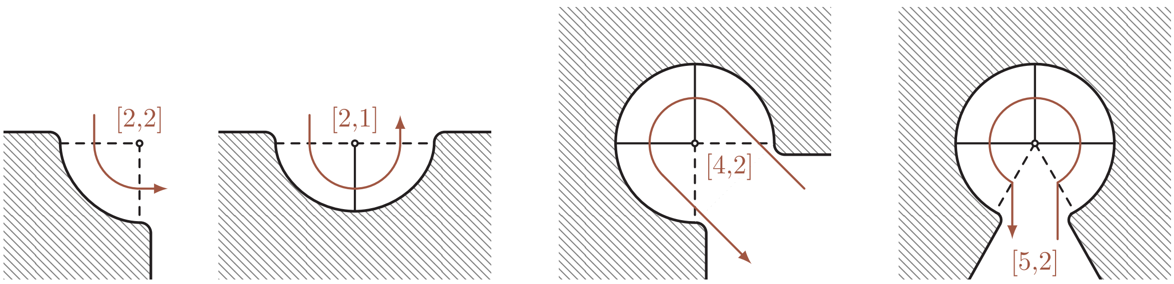


Figure 6.10.: The Lax operator from the monodromy acting on the junction of: (A) two, (B) three, (C) four, (D) five legs. Rotating this picture by $\pm\pi/2$ results in a shift of both inhomogeneities $\delta \to \delta \pm 1$. As before, we use dashed lines to represent delta functions and solid lines to represent scalar propagators.

example, for the graph in figure 6.9, equation (6.91) reads

$$\frac{\lambda(u)}{\lambda(u-4)} = \frac{(3+u)^7(4+u)^{11}(5+u)^6(6+u)}{(u-2)(u-1)^7u^{11}(1+u)^6}\,. \tag{6.93}$$

From this equation, we can more or less directly read off the eigenvalue λ,

$$\lambda(u) = (u+2)(u+3)^7(u+4)^{11}(u+5)^6(u+6)\,. \tag{6.94}$$

For later convenience, it is also useful to solve equation (6.91) perturbatively for the first few coefficients in the u-expansion of the monodromy eigenvalue λ. We find

$$\lambda(\vec{u}) = u^n + \tfrac{1}{2}u^{n-1}\sum_{k=1}^{n}\hat{\delta}_k + \tfrac{1}{4}u^{n-2}\Bigg[\sum_{i<j=1}^{n}\hat{\delta}_i\hat{\delta}_j - \tfrac{1}{2}\sum_{j=1}^{n}\hat{\Delta}_j\Bigg] + \mathcal{O}(u^{n-3})\,, \tag{6.95}$$

Here, we use the shorthand notations $\hat{\delta}_k = \delta_k^+ + \delta_k^- + 2$ and $\hat{\Delta}_i = \Delta_i(\Delta_i - 4)$ with $\Delta_k = \delta_k^+ - \delta_k^- + 2$.

Finally, let us point the reader to the recent paper [149], where it was argued that the momentum-space conformal symmetry of finite on-shell loop integrals of the type considered above is anomalous. The anomaly is of contact type and originates from integration regions where the loop momenta become collinear to one of the external null momenta. The breaking of (super)conformal symmetry of massless scatting amplitudes due to collinear singularities is in fact a well-known phenomenon in $\mathcal{N}=4$ SYM theory [66, 67] and ABJM theory [150]. However, the mechanism described in reference [149] seems to be of a slightly different nature and causes anomalies even though the integral is completely finite. As the level-one symmetry of on-shell fishnet integrals is tightly linked to their conformal symmetry in momentum space (see reference [25] as well as our discussion in section 6.3.3), one may wonder whether the Yangian symmetry is actually anomalous as well. However, we leave this investigation for future work.

6.3.2.4. Monodromy Expansion

In section 6.3.2, we promised to make precise the relation between the expansion coefficients of the monodromy, see equation (6.62), and the level-zero and level-one generators as introduced in section 6.3.1. We will now make good on this promise and expand a general n-site monodromy of the type used above in the spectral parameter u,

$$\mathrm{T}(\vec{u}) = \mathrm{L}_n(u+\delta_n^+, u+\delta_n^-)\mathrm{L}_{n-1}(u+\delta_{n-1}^+, u+\delta_{n-1}^-)\ldots\mathrm{L}_1(u+\delta_1^+, u+\delta_1^-)\,. \tag{6.96}$$

We employ the Lax operator given in (6.69) and (6.71), which yields

$$\begin{aligned}\mathrm{T}(\vec{u}) =& u^n\,\mathbb{1} + \tfrac{1}{2}u^{n-1}\sum_{k=1}^{n}\Big(\hat{\delta}_k\,\mathbb{1} + \mathrm{S}^{ab}\mathrm{J}^{\Delta_k}_{k,ab}\Big) \\ &+ \tfrac{1}{8}u^{n-2}\Bigg[\sum_{k=1}^{n}\sum_{j=1}^{k-1} + \sum_{j=1}^{n}\sum_{k=1}^{j-1}\Bigg]\Big(\hat{\delta}_j\,\mathbb{1} + \mathrm{S}^{ab}\mathrm{J}^{\Delta_j}_{j,ab}\Big)\Big(\hat{\delta}_k\,\mathbb{1} + \mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,cd}\Big) + \ldots\,.\end{aligned} \tag{6.97}$$

Here, we again use the shorthand notation $\hat{\delta}_k = \delta_k^+ + \delta_k^- + 2$. Note that in the above equation, the Δ_k's are no longer free parameters. In fact, as soon as we choose values for $\{\delta_k^+\}$ and $\{\delta_k^-\}$, the conformal dimensions Δ_k are fixed, see equation (6.73),

$$\Delta_k = \delta_k^+ - \delta_k^- + 2\,. \tag{6.98}$$

In order to make contact to the level-zero and level-one generators of section 6.3.1, we also need to take into account the function on the right-hand side of the generic monodromy equation (6.79). For the first few orders in u, we found the following expression, see equation (6.95):

$$\lambda(\vec{u}) = u^n + \tfrac{1}{2}u^{n-1}\sum_{k=1}^{n}\hat{\delta}_k + \tfrac{1}{4}u^{n-2}\Big[\sum_{i<j=1}^{n}\hat{\delta}_i\hat{\delta}_j - \tfrac{1}{2}\sum_{j=1}^{n}\hat{\Delta}_j\Big] + \mathcal{O}(u^{n-3})\,. \tag{6.99}$$

Here, again $\hat{\Delta}_i = \Delta_i(\Delta_i - 4)$ with $\Delta_k = \delta_k^+ - \delta_k^- + 2$. Subtracting the eigenvalue λ from the monodromy matrix yields the following operator, which now annihilates invariants under the Yangian algebra for arbitrary spectral parameter u,

$$\begin{aligned}\mathrm{T}(\vec{u}) - \lambda(\vec{u})\,\mathbb{1} = 0 \times u^n\,\mathbb{1} &+ u^{n-1}\Big[\tfrac{1}{2}\sum_{k=1}^{n}\mathrm{S}^{ab}\mathrm{J}^{\Delta_k}_{k,ab}\Big] \qquad (6.100)\\ &+ u^{n-2}\Big[\tfrac{1}{4}\sum_{j<k=1}^{n}\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_j}_{j,cd} + \tfrac{1}{4}\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\hat{\delta}_j\mathrm{S}^{ab}\mathrm{J}^{\Delta_k}_{k,ab} - \tfrac{1}{8}\sum_{k=1}^{n}(4-\Delta_k)\Delta_k\,\mathbb{1}\Big] + \ldots\,.\end{aligned}$$

As expected, at order u^{n-1} we see the plain level-zero generators appearing. Since the equation is valid for any u, we conclude that they annihilate all graphs on their own. At the next order in u there is, however, a little bit more hiding than the plain level-one generator. To separate the latter one, we rewrite

$$\begin{aligned}\tfrac{1}{4}\sum_{j<k=1}^{n}\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_j}_{j,cd} = &\tfrac{1}{8}\sum_{j<k=1}^{n}[\mathrm{S}^{ab},\mathrm{S}^{cd}]\,\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_j}_{j,cd} + \tfrac{1}{8}\sum_{j,k=1}^{n}\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_j}_{j,cd}\\ &- \tfrac{1}{8}\sum_{k=1}^{n}\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_k}_{k,cd}\,.\end{aligned} \tag{6.101}$$

The first term on the right-hand side reproduces the bi-local piece of the level-one generator. The second term is the product of two level-zero generators, which annihilates the diagrams under consideration and can thus be dropped. Noting that

$$\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_k}_{k,cd} = (\Delta_k - 4)\Delta_k\,\mathbb{1} - 4\,\mathrm{S}^{ab}\mathrm{J}^{\Delta_k}_{k,ab}\,, \tag{6.102}$$

we can rewrite the last term according to

$$-\tfrac{1}{8}\sum_{k=1}^{n}\mathrm{S}^{ab}\,\mathrm{S}^{cd}\mathrm{J}^{\Delta_k}_{k,ab}\,\mathrm{J}^{\Delta_k}_{k,cd} = \tfrac{1}{8}\sum_{k=1}^{n}(4-\Delta_k)\Delta_k\,\mathbb{1} + \tfrac{1}{2}\sum_{k=1}^{n}\mathrm{S}^{ab}\mathrm{J}^{\Delta_k}_{k,ab}\,. \tag{6.103}$$

Here, the first term cancels the piece proportional to the identity in (6.100), while the last term is a level-zero generator and can thus be dropped. Collecting the remaining terms at order u^{n-2} of (6.100), we thus find the level-one generators to be given by

$$\widehat{\mathrm{J}}_{ab} = \tfrac{1}{8}\,f_{ab}{}^{cd,ef}\sum_{j<k=1}^{n}\mathrm{J}^{\Delta_k}_{k,cd}\,\mathrm{J}^{\Delta_j}_{j,ef} + \tfrac{1}{2}\sum_{k=1}^{n}v_k\,\mathrm{J}_{k,ab}\,. \tag{6.104}$$

Here, ${f_{ab}}^{cd,ef}$ denotes the structure constants with $[\mathrm{S}^{ab},\mathrm{S}^{cd}] = {f^{ab,cd}}_{ef}\,\mathrm{S}^{ef}$ and the evaluation parameters v_k take the form

$$v_k = \tfrac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n} \hat{\delta}_j\,. \tag{6.105}$$

Specifying to the case of the (double-)cross integral, it is easy to check that the generator (6.104) agrees with the expression found in section 6.3.1.

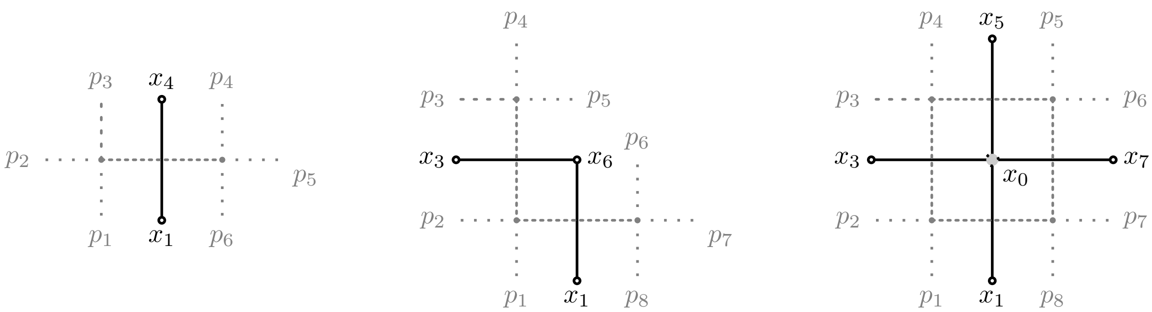
6.3.3. Dual Conformal Symmetry and the Yangian in Momentum Space

In $\mathcal{N}=4$ SYM theory, the Yangian invariance of scattering amplitudes is known to be equivalent to their superconformal and dual superconformal symmetry [24, 25]. It is thus natural to ask whether a similar statement can be made rigorous in the case at hand — namely, for planar scattering amplitudes in $\chi\mathrm{FT}_4$, which enjoy ordinary conformal and dual conformal symmetry, cf. [151] for a discussion of the one-loop box. At the same time, one may wonder whether an analogue of the above construction of the coordinate-space Yangian can also be performed in momentum space. In what follows, we will explicitly demonstrate that the generator of special conformal transformations in the dual (coordinate) space can be rewritten as a Yangian level-one generator acting in on-shell momentum space. We thus derive the Yangian symmetry in momentum space and establish its equivalence to dual conformal symmetry.

Dual conformal symmetry. We begin by briefly discussing the dual conformal symmetry of planar bi-scalar scattering amplitudes. As explained above, this symmetry can be exposed by introducing the dual variables, which are defined through the relation

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,. \tag{6.106}$$

In contrast to the situation in $\mathcal{N}=4$ SYM theory, the dual conformal symmetry of the considered $\chi\mathrm{FT}_4$ model is less universal in the sense that the dual conformal symmetry generators depend, at least in part, not only on the number of external legs but also on the structure of the amplitude itself. To make this statement more clear, let us consider three simple examples of amplitudes:



Using Feynman rules, we write down the following expressions for the amplitudes depicted above:

$$A_6^{\mathrm{t}} = \frac{\delta^{(4)}(x_1 - x_7)}{x_{14}^2}\,, \qquad A_8^{\mathrm{t}} = \frac{\delta^{(4)}(x_1 - x_9)}{x_{16}^2 x_{36}^2}\,, \qquad A_8^{11} = \int \mathrm{d}^4 x_0\, \frac{\delta^{(4)}(x_1 - x_9)}{x_{10}^2 x_{30}^2 x_{50}^2 x_{70}^2}\,, \tag{6.107}$$

A few comments concerning these amplitudes are in order. In equation (6.107), we have already introduced the dual coordinates $x_1, x_2, \ldots, x_n$, which are related to the external momenta as stated in equation (6.106). Obviously, the denominators just represent the region momenta flowing through the different propagators. Furthermore, note that in the above formulas, we have relaxed the cyclicity condition on the x's at the cost of a four-dimensional delta function $\delta^{(4)}(x_1 - x_{n+1})$ reimposing the closure of the light-like polygon. This delta function corresponds to the momentum-conserving delta function $\delta^{(4)}(\mathfrak{P})$, where $\mathfrak{P} = \sum_{j=1}^{n} p_j$, and we have inserted it here for later convenience. On the contrary, we have not included the one-dimensional delta functions ensuring the light-likeness of the edges $x_i - x_{i+1}$. The reason for this is that the dual conformal generators as well as the level-one on-shell momentum-space generators manifestly respect the light-likeness condition, so that we can safely disregard these delta functions. This being said, let us now take a closer look at the dual conformal properties of these amplitudes. The representation of the dual conformal algebra that we will use here was introduced in equation (6.25) and for pedagogical reasons we will start with a representation with conformal dimension $\Delta_i = 0$ at each site. The amplitudes (6.107) are manifestly invariant under translations and rotations and thus they are annihilated by the corresponding generators. Acting with the dilatation generator on the amplitudes in equation (6.107) yields

$$\mathrm{D}^{\vec{\Delta}=0} A_6^{\mathrm{t}} = 6i A_6^{\mathrm{t}}\,, \qquad \mathrm{D}^{\vec{\Delta}=0} A_8^{\mathrm{t}} = 8i A_8^{\mathrm{t}}\,, \qquad \mathrm{D}^{\vec{\Delta}=0} A_8^{11} = 8i A_8^{11}\,, \tag{6.108}$$

where

$$\mathrm{D}^{\vec{\Delta}=0} = -i \sum_{i=1}^{n+1} x_i^{\mu} \partial_{i\mu}\,. \tag{6.109}$$

Note that due to the relaxed cyclicity condition, the dual conformal generators are summed up to $n+1$ instead of n. From equation (6.108), we anticipate that acting with the conformal dilatation generator on an amplitude just yields the number of external legs times i and it is actually not too hard to convince oneself that this statement holds true for all the planar scattering amplitudes in $\chi\mathrm{FT}_4$. In order to make the dilatation generator a true symmetry generator, we will now use the freedom to choose the conformal dimensions Δ_i to compensate for the terms on the right-hand side of equation (6.108). The conformal dimensions Δ_i enter the dilatation generator in the following way:

$$\mathrm{D}^{\vec{\Delta}} = -i \sum_{i=1}^{n+1} \Big(x_i^{\mu} \partial_{i\mu} + \Delta_i \Big)\,. \tag{6.110}$$

There are actually many choices that lead to a generator $\mathrm{D}^{\vec{\Delta}}$ which annihilates the amplitudes (6.107) but the most natural one is

$$\begin{aligned}
\vec{\Delta}_{A_6^{\mathrm{t}}} &= (4+1,0,0,1,0,0,0)\,,\\
\vec{\Delta}_{A_8^{\mathrm{t}}} &= (4+1,0,1,0,0,2,0,0,0)\,,\\
\vec{\Delta}_{A_8^{11}} &= (4+1,0,1,0,1,0,1,0,0)\,.
\end{aligned} \tag{6.111}$$

Note that we can always choose $\Delta_{n+1}=0$ as the delta function allows us to eliminate the coordinate x_{n+1} in favor of x_1. The factors of four in the above equations compensate for the weight that is introduced by the delta function, while all the other numbers are chosen such that the weight coming from the corresponding coordinate is canceled out. Having discussed the dilatation symmetry, let us now focus on special conformal transformations. For this, we again study the inversion and use the identity (6.27) to deduce how the amplitudes transform under special conformal transformations. Using that the region momenta invert as

$$\mathrm{I}_b[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2}\,, \tag{6.112}$$

as well as the formula $\mathrm{I}_b[\delta(x_1-x_{n+1})] = x_1^8\delta(x_1-x_{n+1})$,[10] we find

$$\mathrm{I}_b[A_6^{\mathrm{t}}] = x_1^{10}x_4^2 A_6^{\mathrm{t}}\,,\qquad \mathrm{I}_b[A_8^{\mathrm{t}}] = x_1^{10}x_3^2x_6^4 A_8^{\mathrm{t}}\,,\qquad \mathrm{I}_b[A_8^{11}] = x_1^{10}x_3^2x_5^2x_7^2 A_8^{11}\,, \tag{6.113}$$

see also our discussion in section 6.3.1.1. In contrast to the situation in $\mathcal{N}=4$ SYM theory, the amplitudes obviously do not transform in a completely covariant way. This is on the one hand due to the fact that there is no supermomentum-conserving delta function present which could balance out the inversion weight of the bosonic delta function. On the other hand, also the amplitude functions themselves do not transform in a completely homogeneous way as some of the x's are simply not present, while others come with a power higher than two. Having studied the inversion properties of the amplitudes, we can now easily write down expressions for the action of the generator $\mathrm{K}^{\mu}_{\vec{\Delta}=0}$ on the three amplitudes (6.107). Using equation (6.27), we find

$$\begin{aligned}
\mathrm{K}^{\dot{\alpha}\alpha}_{\vec{\Delta}=0} A_6^{\mathrm{t}} &= i\Big(5x_1^{\dot{\alpha}\alpha} + x_4^{\dot{\alpha}\alpha}\Big) A_6^{\mathrm{t}}\,,\\
\mathrm{K}^{\dot{\alpha}\alpha}_{\vec{\Delta}=0} A_8^{\mathrm{t}} &= i\Big(5x_1^{\dot{\alpha}\alpha} + x_3^{\dot{\alpha}\alpha} + 2x_6^{\dot{\alpha}\alpha}\Big) A_8^{\mathrm{t}}\,,\\
\mathrm{K}^{\dot{\alpha}\alpha}_{\vec{\Delta}=0} A_8^{11} &= i\Big(5x_1^{\dot{\alpha}\alpha} + x_3^{\dot{\alpha}\alpha} + x_5^{\dot{\alpha}\alpha} + x_7^{\dot{\alpha}\alpha}\Big) A_8^{11}\,,
\end{aligned} \tag{6.114}$$

where we employed the standard definitions

$$\begin{aligned}
x^{\dot{\alpha}\alpha} &= \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}\mathrm{x}^{\mu},\\
\mathrm{K}^{\dot{\alpha}\alpha}_{\vec{\Delta}=0} &= \tfrac{1}{2}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}\mathrm{K}^{\mu}_{\vec{\Delta}=0} = -i\sum_{i=1}^{n+1} x_i^{\dot{\alpha}\beta} x_i^{\dot{\beta}\alpha}\partial_{i\beta\dot{\beta}}\,.
\end{aligned} \tag{6.115}$$

[10] This formula can easily be derived by considering the definition of the delta function $\int \mathrm{d}^4x_1\,\delta(x_1-x_{n+1})=1$ and noting that under inversion the measure transforms as $\mathrm{I}_b[\mathrm{d}^4x_1]=\mathrm{d}^4x_1/x_1^8$.

As in the case of dilatation symmetry, we can now adjust the conformal scaling dimensions $\vec{\Delta}$ such that the generator of special conformal transformations annihilates the amplitudes

$$\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha} = \mathrm{K}_{\vec{\Delta}=0}^{\dot{\alpha}\alpha} - i \sum_{i=1}^{n} \Delta_i \, x_i^{\dot{\alpha}\alpha} \,, \tag{6.116}$$

with the Δ_i's as defined in equation (6.111). Our examples and equation (6.111) make it clear that the generators $\mathrm{D}^{\vec{\Delta}}$ and $\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha}$ are no longer universal as the vector $\vec{\Delta}$ does not only depend on the number of external legs but also on the amplitude itself. However, note that the generators $\mathrm{D}^{\vec{\Delta}}$ and $\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha}$ are perfectly consistent with the algebraic restrictions imposed on them by the conformal commutation relations. Hence, the generators P_μ, $\mathrm{L}_{\mu\nu}$, $\mathrm{D}^{\vec{\Delta}}$ and $\mathrm{K}_{\vec{\Delta}}^{\mu}$ still furnish a representation of the conformal algebra $\mathfrak{so}(2,4)$.

Finally, let us comment on the situation for a generic planar amplitude in the bi-scalar $\chi\mathrm{FT}_4$ model. As mentioned above, the plain dilatation generator $\mathrm{D}^{\vec{\Delta}=0}$ acts on an amplitude as follows

$$\mathrm{D}^{\vec{\Delta}=0} \, A_n = i n \, A_n \,. \tag{6.117}$$

For a given planar amplitude, the modified dilatation generator

$$\mathrm{D}^{\vec{\Delta}} = -i \sum_{i=1}^{n+1} \Big(x_i^{\mu} \partial_{i\mu} + \Delta_i \Big) \,, \tag{6.118}$$

annihilating the amplitude can be constructed in the following way: First, we determine which x_i's will be absent in the amplitude by drawing the amplitudes' dual graph and set to zero all the corresponding Δ_i's. For the remaining x_i's, we set the corresponding Δ_i's equal to the number of lines which meet in the point x_i. Finally, we set to zero Δ_{n+1} and add a factor of four to Δ_1 to compensate for the weight of the delta function. The resulting generator $\mathrm{D}^{\vec{\Delta}}$ will then annihilate the considered amplitude. The generator of special conformal transformations annihilating this amplitude follows immediately from the algebra. Explicitly, it reads

$$\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha} = \mathrm{K}_{\vec{\Delta}=0}^{\dot{\alpha}\alpha} - i \sum_{i=1}^{n} \Delta_i \, x_i^{\dot{\alpha}\alpha} \,. \tag{6.119}$$

Finally, note that due to equation (6.117), the Δ_i's satisfy the following relation:

$$\sum_{i=1}^{n} \Delta_i = n \,. \tag{6.120}$$

We will use this equation in the next paragraph, when we establish the connection between the generator $\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha}$ and the level-one momentum generator.

Yangian symmetry in momentum space. In this paragraph, we will now demonstrate that the generator $\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta}$ agrees with the conformal level-one momentum generator up to terms which annihilate the amplitudes by themselves. The discussion follows closely the one presented in [25], where the statement was proven for the case of $\mathfrak{psu}(2,2|4)$. To rewrite $\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta}$ as an operator acting in on-shell spinor-helicity space, we first extend it such that it commutes with the constraint

$$x_i^{\dot\gamma\gamma} - x_{i+1}^{\dot\gamma\gamma} - \lambda_i^{\gamma}\tilde\lambda_i^{\dot\gamma} = 0\,. \tag{6.121}$$

The result reads

$$\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta} = -i\left[\sum_{i=1}^{n+1} x_i^{\dot\alpha\beta}x_i^{\dot\beta\alpha}\partial_{i\beta\dot\beta} + \sum_{i=1}^{n}\Big(x_i^{\dot\alpha\beta}\lambda_i^{\alpha}\partial_{i\beta} + x_{i+1}^{\dot\beta\alpha}\tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\dot\beta} + \Delta_i\, x_i^{\dot\alpha\alpha}\Big)\right]. \tag{6.122}$$

Using the inverse of equation (6.121),

$$x_i^{\dot\alpha\alpha} = x_1^{\dot\alpha\alpha} - \sum_{j=1}^{i-1}\tilde\lambda_j^{\dot\alpha}\lambda_j^{\alpha}\,, \tag{6.123}$$

and dropping the term that includes a derivative with respect to x, we find

$$\begin{aligned}\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta} =& i\sum_{j<i=1}^{n}\Big(\tilde\lambda_j^{\dot\alpha}\lambda_j^{\beta}\lambda_i^{\alpha}\partial_{i\beta} + \Delta_i\,\tilde\lambda_j^{\dot\alpha}\lambda_j^{\alpha}\Big) + i\sum_{j<i+1=1}^{n}\Big(\tilde\lambda_j^{\dot\beta}\lambda_j^{\alpha}\tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\dot\beta}\Big)\\ &- i\sum_{i=1}^{n}\Big(x_1^{\dot\alpha\beta}\lambda_i^{\alpha}\partial_{i\beta} + x_1^{\dot\beta\alpha}\tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\dot\beta} + \Delta_i\,x_1^{\dot\alpha\alpha}\Big)\,.\end{aligned} \tag{6.124}$$

Note that the amplitudes can always be written as distributions depending exclusively on the spinor-helicity variables $\{\lambda_i\}$ and $\{\tilde\lambda_i\}$. For this reason, we could safely disregard all terms containing a derivative with respect to the dual coordinates. Starting from equation (6.124), it is now, however, a straightforward exercise to rewrite $\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta}$ as

$$\begin{aligned}\mathrm{K}^{\dot\alpha\alpha}_{\vec\Delta} =& \tfrac{i}{2}\widehat{\mathfrak{P}}^{\dot\alpha\alpha}_{\mathrm{bi}} + i\sum_{j<i=1}^{n}(\Delta_i-1)\mathfrak{P}_j^{\dot\alpha\alpha} + \tfrac{i}{2}\Big(\mathfrak{P}^{\dot\alpha\beta}\mathfrak{L}^{\alpha}{}_{\beta} + \mathfrak{P}^{\dot\beta\alpha}\bar{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} + \mathfrak{P}^{\dot\alpha\alpha}\mathfrak{D} - \mathfrak{P}^{\dot\alpha\alpha}\Big)\\ &- \tfrac{i}{2}\sum_{i=1}^{n}\mathfrak{P}_i^{\dot\alpha\alpha}\Big(\lambda_i^{\gamma}\partial_{i\gamma} - \tilde\lambda_i^{\dot\gamma}\partial_{i\dot\gamma}\Big) - i\sum_{i=1}^{n}\Big(x_1^{\dot\alpha\beta}\mathfrak{L}_i^{\alpha}{}_{\beta} + x_1^{\dot\beta\alpha}\bar{\mathfrak{L}}_i^{\dot\alpha}{}_{\dot\beta} + x_1^{\dot\alpha\alpha}\mathfrak{D}_i\Big)\,,\end{aligned} \tag{6.125}$$

where we have introduced the conformal generators written in terms of spinor-helicity variables

$$\begin{aligned}\mathfrak{L}_i^{\alpha}{}_{\beta} &= \lambda_i^{\alpha}\partial_{i\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}\lambda_i^{\gamma}\partial_{i\gamma}\,, & \bar{\mathfrak{L}}_i^{\dot\alpha}{}_{\dot\beta} &= \tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde\lambda_i^{\dot\gamma}\tilde\partial_{i\dot\gamma}\,,\\ \mathfrak{P}_i^{\dot\alpha\alpha} &= \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}\,, & \mathfrak{D}_i &= \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\dot\alpha} + 1\,.\end{aligned} \tag{6.126}$$

The generator $\widehat{\mathfrak{P}}^{\dot\alpha\alpha}_{\mathrm{bi}}$ in equation (6.125) is the level-one momentum generator

$$\widehat{\mathfrak{P}}^{\dot\alpha\alpha}_{\mathrm{bi}} = \sum_{j<i=1}^{n}\Big(\mathfrak{P}_j^{\dot\alpha\beta}\mathfrak{L}_i^{\alpha}{}_{\beta} + \mathfrak{P}_j^{\dot\beta\alpha}\bar{\mathfrak{L}}_i^{\dot\alpha}{}_{\dot\beta} + \mathfrak{P}_j^{\dot\alpha\alpha}\mathfrak{D}_i - (i\leftrightarrow j)\Big)\,, \tag{6.127}$$

as it follows from the formula (6.33) with the underlying level-zero algebra being the conformal algebra spanned by the generators (6.126). Note that in order to bring $\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha}$ to the above form, we have also used the constraint equation (6.120), which allowed us to replace the Δ_i's by one in the term that contributes to $x_1^{\dot{\alpha}\alpha}\mathfrak{D}_i$. Finally, using the level-zero invariance of the amplitudes as well as the fact that all the external particles have zero helicity, i.e.

$$\tfrac{1}{2}\Big(\lambda_i^{\gamma}\partial_{i\gamma} - \tilde{\lambda}_i^{\dot{\gamma}}\partial_{i\dot{\gamma}}\Big)A_n = 0\,, \tag{6.128}$$

we see that most of the terms on the right-hand side of equation (6.125) drop out, leaving us with

$$\mathrm{K}_{\vec{\Delta}}^{\dot{\alpha}\alpha} = \tfrac{i}{2}\widehat{\mathfrak{P}}_{\mathrm{bi}}^{\dot{\alpha}\alpha} + i\sum_{j<i=1}^{n}(\Delta_i - 1)\mathfrak{P}_j^{\dot{\alpha}\alpha}\,. \tag{6.129}$$

The local term would obviously vanish if all the Δ_i's were equal to one as they are, for example, in the case of $\mathcal{N}=4$ SYM theory. However, since this is not the case here, we arrive at a purely bosonic Yangian generator with non-vanishing evaluation parameters, which is in complete analogy with the x-space level-one momentum generator that we considered before.

7. Conclusion and Outlook

Quantum integrability has turned out to be an important tool in overcoming the limitations of perturbation theory and reaching a deeper understanding of four-dimensional quantum field theories. The algebraic structure that sits at the heart of integrability is the Yangian, which can be understood as an infinite-dimensional extension of the underlying Lie algebra symmetry. Reaching a profound understanding of its origins, role and implications is still a major desideratum. In this work, we have established and studied the Yangian for Maldacena–Wilson loops and fishnet Feynman graphs and we shall now give a brief summary of the results that have been obtained.

In the first part of this thesis, we studied Maldacena–Wilson loops at weak coupling. Our first goal was to determine the full supersymmetric completion of the Maldacena–Wilson loop operator. In reference [1], this operator was constructed through quadratic order in an expansion in the anticommuting variables by employing the component formulation of the $\mathcal{N} = 4$ SYM model and demanding supersymmetry. Here, we did not pursue this approach but rather made use of the $\mathcal{N} = 4$ non-chiral superspace formulation of the $\mathcal{N} = 4$ SYM model [105], which we introduced in chapter 3. Importantly, we have seen that this space is an on-shell superspace as the superspace constraints imply the equations of motion [103, 104]. Standard quantization techniques, such as the path integral, are therefore clearly not applicable. Nevertheless, we were able to derive propagator expressions for the gauge connection, the superscalars and the field strength tensor. The gauge propagator was established through quartic order in an expansion in the Graßmann variables by using a convenient type of gauge due to Harnad and Shnider [103], while the gauge-invariant propagators were determined to all orders in the anticommuting coordinates by exploiting the superconformal symmetry of the theory.

With the superspace formalism established, we then turned to the construction of the super Maldacena–Wilson loop operator. Based on work done by Ooguri et al. [108], we derived this object by considering the dimensional reduction of the ten-dimensional super Wilson loop. The so-defined loop operator is a highly natural observable from the viewpoint of the AdS/CFT correspondence as it is the object which is dual to a minimal supersurface of the type IIB superstring that is bounded by the Wilson loop path at the boundary of space. This claim was substantiated by the observation that super Maldacena–Wilson loops have a local fermionic symmetry of kappa type. We exposed this symmetry and showed how it is related to the 1/2 BPS property of the bosonic Maldacena–Wilson loop. Additionally, we established the action of the superconformal generators on both the fields and the superspace coordinates. Subsequently, we turned to the one-loop expectation value of the super Maldacena–Wilson loop operator. Using the previously established propagators, we computed the operators vacuum expectation value through quartic order in an expansion in the anticommuting coordinates and put forward an educated guess for the all-order form of the one-loop VEV. Finally, we proved the expectation value to be finite and verified the superconformal symmetry of the object.

One of the key results of this thesis concerns the non-local symmetries of the super Maldacena–Wilson loop operator: We proved the existence of a hidden Yangian symmetry at the leading perturbative order and to all orders in the Graßmann coordinates. To establish this result, we first inspected thoroughly the different possibilities for defining the action of the bi-local part of the level-one generators on the super Wilson loop. More precisely, we compared the path representation of the level-zero generators to the level-zero field actions and the covariantized field actions, which act on the gauge connection through a level-zero transformation plus a compensating gauge transformation. While all three representations yield equivalent results for level-zero transformations, we saw that that the consistency of the Yangian crucially depends on which level-zero representation is chosen. In fact, we found that gauge covariance is only maintained if the bi-local level-one actions are built upon the covariantized level-zero field representation. The claim that the level-one generators are most naturally specified by their action on the fields was further substantiated by realizing that the hidden "space-like" dimension with respect to which the ordering in the bi-local level-one actions is defined is a dimension in color space, being spanned by the matrix product of fields. In the case at hand, this ordering prescription trivially coincides with the ordering along the one-dimensional path but this is obviously specific to Wilson loops. Importantly, we saw that the bi-local level-one actions leave behind a UV-divergent local term when applied to the super Maldacena–Wilson loop operator. No bi-local term remained at the one-loop level, which is a clear sign of Yangian symmetry. We worked out the different local contributions for all level-one generators $\widehat{\mathbb{J}}_* \in Y[\mathfrak{psu}(2,2|4)]$ and showed how the freedom of defining the local action can be used to make the bi-local level-one actions true symmetry generators. Additionally, we found that the super Maldacena–Wilson loop features a level-one bonus symmetry, namely level-one hypercharge symmetry, mirroring the structure found for scattering amplitudes [110].

While we strove to keep the discussion as generic as possible, we noticed that the Yangian symmetry of Wilson loops crucially depends on the following two properties of the underlying gauge theory: first, the vanishing of the dual Coxeter number of the underlying level-zero symmetry algebra; second, a novel identity, which we called $\mathcal{G}$-identity, basically stating that the field strength two-form vanishes when contracted with two level-zero vector fields. Both are fulfilled by $\mathcal{N}=4$ SYM theory and tightly constrain the number of models in which Wilson loops could feature a Yangian symmetry.

Finally, let us compare our findings to the results that have been obtained at strong coupling. In a parallel line of research, the Yangian symmetries of minimal supersurfaces were derived from the integrability of the string sigma model [43, 44]. At strong coupling, the bi-local part of the level-one generators follows the usual pattern, while the local terms that were found are structurally very similar to the ones obtained by us. However, a careful inspection of the local contributions shows that they do not completely agree. Most likely, this indicates that the local terms depend in a non-trivial way on the 't Hooft coupling λ. Nevertheless, the string theory findings strongly support the correctness of the results that have been obtained in this thesis. With the weak coupling results of this work and the cited strong coupling results, the Yangian symmetry of super Wilson loops is now fully established, which completes the analysis that was begun in reference [1].

In the second part of this thesis, we shifted our focus to fishnet Feynman diagrams, which are built from four-valent vertices that are joint by scalar propagators in a regular pattern. In the bulk, these diagrams look like a regular square lattice, a structure that was long ago shown to furnish an integrable lattice system [114]. However, merely the simplest representative (the box integral) of this infinite-dimensional family of Feynman integrals is completely known [132]. Any new insights into the mathematical structure of theses fishnet integrals are therefore highly valuable. Notably, scalar fishnet graphs are in one-to-one correspondence with planar correlators in the recently discovered integrable bi-scalar CFT [46] that can be obtained by taking a specific double-scaling limit of the γ_i-twisted $\mathcal{N} = 4$ SYM model. This correspondence in fact explains in parts the nice integrable structure of these diagrams.

Here, we have shown that scalar fishnet integrals feature a conformal Yangian symmetry with Yangian generators that realize a non-trivial evaluation representation of the Yangian $Y[\mathfrak{so}(2,4)]$.[1] We phrased this symmetry in terms of generators annihilating these graphs as well as in the language of the RTT realization. In the latter formalism, fishnet Feynman graphs are interpreted as eigenstates of an inhomogeneous monodromy matrix in the spirit of the work [69, 70, 140]. Importantly, the considered fishnet Feynman integrals are all free of divergences, which renders the Yangian symmetry an exact loop-level statement. The finiteness even continues to hold when all the external momenta are put on-shell and we were able to prove the Yangian symmetry of on-shell graphs as well. The level-one symmetry implies that fishnet Feynman integrals satisfy a set of (complicated) differential equations. We explicitly worked out the simplest of these equations, which is the differential equation for the box integral. Finally, let us emphasize a result that is of far more general nature: Our analysis shows that cyclic Yangian invariants exist even if the dual Coxeter number of the underlying Lie algebra does not vanish. The non-cyclicity of the bi-local part of the level-one generators can in fact be repaired by choosing appropriate evaluation parameters. A vanishing dual Coxeter number is therefore in general not a necessary condition for integrability.

There are several interesting directions for future investigations. Let us begin by highlighting a couple of interesting questions that concern fishnet Feynman graphs.

A natural way to extend our results is to study more general double-scaling limits of the γ_i-twisted $\mathcal{N} = 4$ SYM model [46, 49]. These limits are believed to preserve

[1] That is to say that the local term is of the form $\sum v_k \mathrm{J}_k$ and therefore proportional to the corresponding level-zero generator.

integrability as well but lead to theories with a richer field content. While my collaborators and I have already made progress in understanding the Yangian symmetry of off-shell correlators in the model with two non-zero couplings, the fully double-scaled model with three effective couplings still poses challenging problems. Also, on-shell limits remain to be completely understood, see our discussion in section 6.3.2.3.

Another interesting question concerns the definition of integrability in four-dimensional field theories. Recently, a criterion for integrability in planar gauge/field theories was put forward in [26]. It would be highly interesting to investigate how the double-scaled models and in particular the bi-scalar theory perform with respect to this criterion. One of the clear advantages in comparison to $\mathcal{N} = 4$ SYM theory is that the bi-scalar model is much simpler, so that open questions potentially become a lot easier to answer. If the criterion holds or can be adapted, this might also allow one to understand the Yangian symmetry of field theory observables on more general grounds, namely in terms of a symmetry of the action.

A further pressing question is whether the Yangian symmetry can actually be used to compute fishnet Feynman diagrams. Fishnet integrals in fact furnish an ideal starting point for attacking the long-standing challenge of turning the Yangian into a computational tool as the symmetry is non-anomalous, while the quantities of interest depend on finitely many variables. A natural strategy is to study the differential equations that the Yangian symmetry implies. However, since the level-one equations are generically very complicated, one first needs to cast them into a usable form. Here, we have shown how to do this for the box integral. It would be interesting to perform the same computation for the (off-shell) double-box integral. This integral belongs to the class of elliptic Feynman integrals [112, 113] about which not much is known.

The question how the Yangian symmetry can be used in practice also applies to the discussed super Wilson loops. In contrast to fishnet graphs, general smooth super Maldacena–Wilson loops depend on infinitely many bosonic and fermionic variables. This clearly renders the task of identifying Yangian invariants highly non-trivial. However, the constraints that are implied by the Yangian should become more transparent when restricting to special types of contours. Two natural classes that come to mind are the highly symmetric curves, such as superspace circles, as well as the null polygonal contours. However, the latter lead to UV-divergent expectation values [39, 40], which renders the symmetries anomalous. For this reason, it could be wise to look at super Wilson loops depending on contours which are almost polygonal in the sense that the corners are smoothed out by splines.

A. Sigma Matrices in Four, Six and Ten Dimensions

In this appendix, we summarize our conventions for the four-, six- and ten-dimensional Pauli matrices and provide the identities which are relevant for the calculations carried out in this thesis.

A.1. Sigma Matrices in Four Dimensions

We consider four-dimensional Minkowski space with metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The four-dimensional Pauli matrices $\sigma_{\mu\,\alpha\dot{\alpha}}$ and $\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}$ are defined as

$$\sigma_\mu = (\mathbf{1}, \vec{\sigma})\,, \qquad \bar{\sigma}_\mu = (\mathbf{1}, -\vec{\sigma})\,, \tag{A.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The sigma matrices transform between the four-dimensional chirality left and right spinors and satisfy the following algebra relations

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\eta_{\mu\nu}\,, \qquad \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\eta_{\mu\nu}\,, \tag{A.3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Furthermore, they obey the following trace identities

$$\begin{aligned} \mathrm{tr}(\bar{\sigma}^\mu \sigma^\nu) &= 2\eta^{\mu\nu}\,, \\ \mathrm{tr}(\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho \sigma^\kappa) &= 2(\eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\nu\rho}\eta^{\mu\kappa} - \eta^{\mu\rho}\eta^{\nu\kappa} - i\varepsilon^{\mu\nu\rho\kappa})\,, \\ \mathrm{tr}(\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\kappa) &= 2(\eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\nu\rho}\eta^{\mu\kappa} - \eta^{\mu\rho}\eta^{\nu\kappa} + i\varepsilon^{\mu\nu\rho\kappa})\,, \end{aligned} \tag{A.4}$$

where $\varepsilon^{\mu\nu\rho\kappa}$ is the totally antisymmetric four-tensor in four dimensions. For convenience, we introduce the following abbreviations for antisymmetrized products of Pauli matrices:

$$\sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)\,, \qquad \bar{\sigma}_{\mu\nu} = \tfrac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)\,. \tag{A.5}$$

The following identities are straightforward to check:

$$\sigma_{\mu,\alpha\dot\alpha}\bar\sigma^{\mu,\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}\,, \qquad (\sigma^{\mu\nu})_\alpha{}^\beta(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = 0\,,$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\gamma{}^\delta = -8\delta_\alpha^\delta\delta_\gamma^\beta + 4\delta_\alpha^\beta\delta_\gamma^\delta\,. \tag{A.6}$$

We proceed by introducing the antisymmetric tensors in two dimensions. There are four types of totally antisymmetric ε-tensors: $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$. In our conventions, these tensors are all numerically equal, i.e.

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{A.7}$$

which implies that

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = -\delta_\alpha^\gamma\,, \qquad \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = -\delta_{\dot\alpha}^{\dot\gamma}\,. \tag{A.8}$$

An important identity that we shall frequently use is the so-called Schouten identity, which reads

$$\varepsilon_{\alpha\beta}\delta_\gamma^\delta + \varepsilon_{\beta\gamma}\delta_\alpha^\delta + \varepsilon_{\gamma\alpha}\delta_\beta^\delta = 0\,, \tag{A.9}$$

plus a similar identity for the ε-symbols with dotted indices. Moreover, we note the following completeness relations:

$$\varepsilon_{\lambda\gamma}\varepsilon^{\beta\alpha} = \delta_\lambda^\beta\delta_\gamma^\alpha - \delta_\gamma^\beta\delta_\lambda^\alpha\,, \qquad \varepsilon_{\dot\lambda\dot\gamma}\varepsilon^{\dot\beta\dot\alpha} = \delta_{\dot\lambda}^{\dot\beta}\delta_{\dot\gamma}^{\dot\alpha} - \delta_{\dot\gamma}^{\dot\beta}\delta_{\dot\lambda}^{\dot\alpha}\,. \tag{A.10}$$

The ε-tensors can be used to raise and lower the spinor indices of the four-dimensional Pauli matrices, i.e.

$$\varepsilon_{\dot\alpha\dot\beta}\bar\sigma_\mu^{\dot\beta\gamma}\varepsilon_{\alpha\gamma} = \sigma_{\mu\,\alpha\dot\alpha}\,, \qquad \varepsilon^{\alpha\beta}\sigma_{\mu\,\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta} = \bar\sigma_\mu^{\dot\beta\beta}\,. \tag{A.11}$$

An important point to note is that the ε-tensor is the only antisymmetric two-tensor in two dimensions. For this reason, each two-tensor can be decomposed as

$$M_{\alpha\delta} = M_{(\alpha\delta)} + \tfrac{1}{2}\varepsilon_{\alpha\delta}\varepsilon^{\beta\gamma}M_{\beta\gamma}\,, \qquad M_{\dot\alpha\dot\delta} = M_{(\dot\alpha\dot\delta)} + \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}M_{\dot\beta\dot\gamma}\,. \tag{A.12}$$

Using the sigma matrices, we assign bi-spinors to four-vectors in the following way:

$$x^{\dot\alpha\alpha} := x^\mu\bar\sigma_\mu^{\dot\alpha\alpha}\,, \qquad \partial_{\alpha\dot\alpha} := \partial^\mu\sigma_{\mu\,\alpha\dot\alpha}\,. \tag{A.13}$$

Note that this implies that

$$\partial_{\beta\dot\beta}x^{\dot\alpha\alpha} = 2\delta_{\dot\beta}^{\dot\alpha}\delta_\beta^\alpha\,. \tag{A.14}$$

Finally, we note the following identities:

$$\varepsilon x^\intercal\varepsilon x = \varepsilon x\varepsilon x^\intercal = -x^2\mathbb{I}\,, \qquad \det(x) = x^2\,. \tag{A.15}$$

A.2. Sigma Matrices in Six Dimensions

We consider Euclidean space $\mathbb{R}^6$ with metric $\eta^{ij} = -\delta^{ij}$. The six-dimensional Pauli matrices are 4×4 matrices, which are defined as

$$(\Sigma^1_{ab}, \dots, \Sigma^6_{ab}) = (\eta_{1ab}, \eta_{2ab}, \eta_{3ab}, -i\bar{\eta}_{1ab}, -i\bar{\eta}_{2ab}, -i\bar{\eta}_{3ab}) \,,$$
$$(\bar{\Sigma}^{1\,ab}, \dots, \bar{\Sigma}^{6\,ab}) = (\eta_{1ab}, \eta_{2ab}, \eta_{3ab}, i\bar{\eta}_{1ab}, i\bar{\eta}_{2ab}, i\bar{\eta}_{3ab}) \,. \tag{A.16}$$

Here, η_{iab} and $\bar{\eta}_{iab}$ are the 't Hooft symbols, which explicitly read

$$\eta_{iab} = \varepsilon_{iab4} + \delta_{ia}\delta_{4b} - \delta_{ib}\delta_{4a} \,, \qquad \bar{\eta}_{iab} = \varepsilon_{iab4} - \delta_{ia}\delta_{4b} + \delta_{ib}\delta_{4a} \,, \tag{A.17}$$

where ε_{ibc4} is the totally antisymmetric four-tensor. The six-dimensional Pauli matrices satisfy the following algebra relations

$$\bar{\Sigma}^i\Sigma^j + \bar{\Sigma}^j\Sigma^i = 2\eta^{ij} \,, \qquad \Sigma^i\bar{\Sigma}^j + \Sigma^j\bar{\Sigma}^i = 2\eta^{ij} \,, \tag{A.18}$$

where η^{ij} is the metric tensor. Further relevant identities are

$$\Sigma^i_{ab} = \tfrac{1}{2}\varepsilon_{abcd}\bar{\Sigma}^{i\,cd} \,, \qquad \bar{\Sigma}^{i\,ab} = \tfrac{1}{2}\varepsilon^{abcd}\Sigma^i_{cd} \,,$$
$$\Sigma^i_{ab}\Sigma^i_{cd} = 2\varepsilon_{abcd} \,, \qquad \bar{\Sigma}^{i\,ab}\Sigma^j_{ab} = 4\delta^{ij} \,, \tag{A.19}$$

with the convention that $\varepsilon_{1234} = \varepsilon^{1234} = 1$. Furthermore, we note that

$$\varepsilon_{dabc}\varepsilon^{dklm} = \delta^{klm}_{abc} + \delta^{mkl}_{abc} + \delta^{lmk}_{abc} - \delta^{lkm}_{abc} - \delta^{mlk}_{abc} - \delta^{kml}_{abc} \,,$$
$$\varepsilon_{abgk}\,\varepsilon^{cdgk} = 2\Big(\delta^{cd}_{ab} - \delta^{dc}_{ab}\Big) \,, \tag{A.20}$$

where $\delta^{a..d}_{e..h} := \delta^a_e..\delta^e_h$. The antisymmetrized products of six-dimensional Pauli matrices are defined as

$$\Sigma^{ij} = \tfrac{1}{2}(\Sigma^i\bar{\Sigma}^j - \Sigma^j\bar{\Sigma}^i) \,, \qquad \bar{\Sigma}^{ij} = \tfrac{1}{2}(\bar{\Sigma}^i\Sigma^j - \bar{\Sigma}^j\Sigma^i) \,. \tag{A.21}$$

As in the four-dimensional case (A.13), we can use the sigma matrices (A.16) to assign antisymmetric 4×4 matrices to a six-dimensional vector X_i,

$$X_{ab} := \Sigma^i_{ab}X_i \,, \qquad \bar{X}^{ab} := \bar{\Sigma}^{i\,ab}X_i \,. \tag{A.22}$$

For the scalar product of two vectors we find the following trace expression:

$$\bar{X}^{ab}Y_{ab} = \bar{\Sigma}^{i\,ab}\Sigma^j_{ab}X_iY_j = 4X_iY_i = -4X^iY_i \,. \tag{A.23}$$

A.3. Sigma Matrices in Ten Dimensions

We consider ten-dimensional Minkowski space with metric $\eta^{\hat{\mu}\hat{\nu}} = \mathrm{diag}(+1, -1, \dots, -1)$. We denote the two types of ten-dimensional Pauli matrices by $\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}$ and $\bar{\Gamma}^{\hat{\mu}\,\hat{\alpha}\hat{\beta}}$. These matrices are real and symmetric

$$\Gamma = \Gamma^{\mathsf{T}} \,, \qquad \bar{\Gamma} = \bar{\Gamma}^{\mathsf{T}} \,, \tag{A.24}$$

and satisfy the following algebra relations:

$$\Gamma^{\hat\mu}\bar\Gamma^{\hat\nu} + \Gamma^{\hat\nu}\bar\Gamma^{\hat\mu} = 2\eta^{\hat\mu\hat\nu}\,, \qquad \bar\Gamma^{\hat\mu}\Gamma^{\hat\nu} + \bar\Gamma^{\hat\nu}\Gamma^{\hat\mu} = 2\eta^{\hat\mu\hat\nu}\,. \tag{A.25}$$

For the trace of two such matrices, we find

$$\mathrm{tr}(\Gamma^{\hat\mu}\bar\Gamma^{\hat\nu}) = 16\eta^{\hat\mu\hat\nu}\,. \tag{A.26}$$

We often use antisymmetrized products of ten-dimensional Γ matrices, such as

$$\Gamma^{\hat\mu\hat\nu} = \tfrac{1}{2}\Big(\Gamma^{\hat\mu}\bar\Gamma^{\hat\nu} - \Gamma^{\hat\nu}\bar\Gamma^{\hat\mu}\Big)\,, \qquad \bar\Gamma^{\hat\mu\hat\nu} = \tfrac{1}{2}\Big(\bar\Gamma^{\hat\mu}\Gamma^{\hat\nu} - \bar\Gamma^{\hat\nu}\Gamma^{\hat\mu}\Big), \tag{A.27}$$

and analogous formulas for higher-rank products

$$\Gamma^{\hat\mu_1\ldots\hat\mu_p} := \frac{1}{p!}\,\Gamma^{[\hat\mu_1}\ldots\Gamma^{\hat\mu_p]}\,. \tag{A.28}$$

As a rule, such a matrix has no bar if the row index is lower and it has a bar if the row index is upper. The product of antisymmetrized Pauli matrices can be rewritten as

$$\Gamma^{\hat\mu_1\ldots\hat\mu_p}\Gamma^{\hat\nu_1\ldots\hat\nu_q} = \sum_{k=0}^{\min\{p,q\}} k!\binom{p}{k}\binom{q}{k}\eta^{[\hat\mu_p|[\hat\nu_1|}\eta^{|\hat\mu_{p-1}|^{|\hat\nu_2|}}\ldots\eta^{|\hat\mu_{p+1-k}|^{|\hat\nu_k|}}\Gamma^{|\hat\mu_1\ldots\hat\mu_{p-k}]^{|\hat\nu_{k+1}\ldots\hat\nu_q]}}\,. \tag{A.29}$$

Note that in the last expression we have for obvious reasons given up the notational distinction between the two types of Pauli matrices. In particular, we will often use the following special case of the above formula:

$$\Gamma^{\hat\mu}\Gamma^{\hat\nu_1\ldots\hat\nu_l} = \Gamma^{\hat\mu\hat\nu_1\ldots\hat\nu_l} + l\eta^{\hat\mu[\hat\nu_1}\Gamma^{\hat\nu_2\ldots\hat\nu_l]}\,. \tag{A.30}$$

Another important identity is the so-called magic identity, which reads

$$\eta_{\hat\mu\hat\nu}\Gamma^{\hat\mu}_{(\hat\alpha\hat\beta}\Gamma^{\hat\nu}_{\hat\gamma)\hat\delta} = \Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\Gamma_{\hat\mu\,\hat\gamma\hat\delta} + \Gamma^{\hat\mu}_{\hat\beta\hat\gamma}\Gamma_{\hat\mu\,\hat\alpha\hat\delta} + \Gamma^{\hat\mu}_{\hat\gamma\hat\alpha}\Gamma_{\hat\mu\,\hat\beta\hat\delta} = 0\,. \tag{A.31}$$

A similar identity of course holds for the Pauli matrices with upper spinor indices.

Finally, we provide the decomposition rules for the ten-dimensional Pauli matrices. If we do not insist on the reality conditions, we can write the ten-dimensional Pauli matrices as

$$\begin{aligned}
\Gamma^{\mu}_{(a\alpha)\binom{b}{\dot\alpha}} &= \delta_a^b\sigma^{\mu}_{\alpha\dot\alpha}\,, & \bar\Gamma^{\mu\,(a\alpha)\binom{b}{\dot\alpha}} &= \delta_b^a\bar\sigma^{\mu\,\dot\alpha\alpha}\,,\\
\Gamma^{i}_{(c\alpha)(d\beta)} &= -\varepsilon_{\alpha\beta}\Sigma^{i}_{cd}\,, & \bar\Gamma^{i\,(c\alpha)(d\beta)} &= \varepsilon^{\alpha\beta}\bar\Sigma^{i\,cd}\,,\\
\Gamma^{i}_{\binom{c}{\dot\alpha}\binom{d}{\dot\beta}} &= -\varepsilon_{\dot\alpha\dot\beta}\bar\Sigma^{i\,cd}\,, & \bar\Gamma^{i\,\binom{c}{\dot\alpha},\binom{d}{\dot\beta}} &= \varepsilon^{\dot\alpha\dot\beta}\Sigma^{i}_{cd}\,.
\end{aligned} \tag{A.32}$$

Note that we have not written all the matrix elements, but the missing ones can either be obtained by symmetry or they vanish.

B. The Algebra $\mathfrak{u}(2,2|4)$

In this appendix, we present details on the algebra $\mathfrak{u}(2,2|4)$, which is a slightly enlarged version of the symmetry algebra of $\mathcal{N}=4$ SYM, namely $\mathfrak{psu}(2,2|4)$.

B.1. The Commutation Relations

We start by listing the commutation relations of the $\mathfrak{su}(2)$, $\mathfrak{su}(4)$, $\mathfrak{su}(2)$ rotation generators $\mathrm{L}^{\alpha}{}_{\beta}$, $\mathrm{R}^{a}{}_{b}$, $\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}$. Under one of these rotations, the indices of any generator J transform canonically according to

$$\begin{aligned}
\Big[\mathrm{L}^{\alpha}{}_{\beta}, \mathrm{J}^{\gamma}\Big] &= -2\delta^{\gamma}_{\beta}\mathrm{J}^{\alpha} + \delta^{\alpha}_{\beta}\mathrm{J}^{\gamma}\,, & \Big[\mathrm{L}^{\alpha}{}_{\beta}, \mathrm{J}_{\gamma}\Big] &= 2\delta^{\alpha}_{\gamma}\mathrm{J}_{\beta} - \delta^{\alpha}_{\beta}\mathrm{J}_{\gamma}\,,\\
\Big[\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathrm{J}^{\dot{\gamma}}\Big] &= -2\delta^{\dot{\gamma}}_{\dot{\beta}}\mathrm{J}^{\dot{\alpha}} + \delta^{\dot{\alpha}}_{\dot{\beta}}\mathrm{J}^{\dot{\gamma}}\,, & \Big[\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathrm{J}_{\dot{\gamma}}\Big] &= 2\delta^{\dot{\alpha}}_{\dot{\gamma}}\mathrm{J}_{\dot{\beta}} - \delta^{\dot{\alpha}}_{\dot{\beta}}\mathrm{J}_{\dot{\gamma}}\,,\\
\Big[\mathrm{R}^{a}{}_{b}, \mathrm{J}^{c}\Big] &= -2\delta^{c}_{b}\mathrm{J}^{a} + \tfrac{1}{2}\delta^{a}_{b}\mathrm{J}^{c}\,, & \Big[\mathrm{R}^{a}{}_{b}, \mathrm{J}_{c}\Big] &= 2\delta^{a}_{c}\mathrm{J}_{b} - \tfrac{1}{2}\delta^{a}_{b}\mathrm{J}_{c}\,.
\end{aligned} \tag{B.1}$$

The commutators involving the conformal dilatation generator D and the hypercharge operator B are given by

$$\Big[\mathrm{D}, \mathrm{J}\Big] = \dim(\mathrm{J})\,\mathrm{J}\,, \qquad \Big[\mathrm{B}, \mathrm{J}\Big] = \mathrm{hyp}(\mathrm{J})\,\mathrm{J}\,, \tag{B.2}$$

with the non-vanishing dimensions and hypercharges

$$\begin{gathered}
\dim(\mathrm{P}) = -\dim(\mathrm{K}) = 1\,, \qquad \dim(\mathrm{Q}) = \dim(\bar{\mathrm{Q}}) = -\dim(\mathrm{S}) = -\dim(\bar{\mathrm{S}}) = \tfrac{1}{2}\,,\\
\mathrm{hyp}(\bar{\mathrm{Q}}) = -\mathrm{hyp}(\mathrm{Q}) = \mathrm{hyp}(\mathrm{S}) = -\mathrm{hyp}(\bar{\mathrm{S}}) = \tfrac{1}{2}\,.
\end{gathered} \tag{B.3}$$

The translation generators $\mathrm{P}_{\alpha\dot{\alpha}}$, the boost generators $\mathrm{K}^{\dot{\alpha}\alpha}$ and their superpartners $\mathrm{Q}_{\alpha a}$, $\bar{\mathrm{Q}}^{a}{}_{\dot{\alpha}}$, $\mathrm{S}^{a\alpha}$ and $\bar{\mathrm{S}}^{\dot{\alpha}}{}_{a}$ satisfy the relations

$$\begin{aligned}
\Big[\mathrm{Q}_{\beta b}, \mathrm{K}^{\dot{\alpha}\alpha}\Big] &= -2\delta^{\alpha}_{\beta}\bar{\mathrm{S}}^{\dot{\alpha}}{}_{b}\,, & \Big[\bar{\mathrm{Q}}^{b}{}_{\dot{\beta}}, \mathrm{K}^{\dot{\alpha}\alpha}\Big] &= -2\delta^{\dot{\alpha}}_{\dot{\beta}}\mathrm{S}^{b\alpha}\,,\\
\Big[\mathrm{P}_{\alpha\dot{\alpha}}, \mathrm{S}^{b\beta}\Big] &= +2\delta^{\beta}_{\alpha}\bar{\mathrm{Q}}^{b}{}_{\dot{\alpha}}\,, & \Big[\mathrm{P}_{\alpha\dot{\alpha}}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\Big] &= +2\delta^{\dot{\beta}}_{\dot{\alpha}}\mathrm{Q}_{\alpha b}\,,\\
\Big\{\mathrm{Q}_{\alpha a}, \bar{\mathrm{Q}}^{b}{}_{\dot{\beta}}\Big\} &= +2\delta^{b}_{a}\mathrm{P}_{\alpha\dot{\beta}}\,, & \Big\{\mathrm{S}^{a\alpha}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\Big\} &= -2\delta^{a}_{b}\mathrm{K}^{\dot{\beta}\alpha}\,.
\end{aligned} \tag{B.4}$$

The remaining non-vanishing commutators are given by

$$\begin{aligned}
\Big[\mathrm{P}_{\beta\dot{\beta}}, \mathrm{K}^{\dot{\alpha}\alpha}\Big] &= 2\delta^{\dot{\alpha}}_{\dot{\beta}}\mathrm{L}^{\alpha}{}_{\beta} + 2\delta^{\alpha}_{\beta}\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}} + 4\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{\alpha}_{\beta}\mathrm{D}\,,\\
\Big\{\mathrm{Q}_{\alpha a}, \mathrm{S}^{b\beta}\Big\} &= -2\delta^{b}_{a}\mathrm{L}^{\beta}{}_{\alpha} + 2\delta^{\beta}_{\alpha}\mathrm{R}^{b}{}_{a} - 2\delta^{b}_{a}\delta^{\beta}_{\alpha}\Big(\mathrm{D} - \mathrm{C}\Big)\,,\\
\Big\{\bar{\mathrm{Q}}^{a}{}_{\dot{\alpha}}, \bar{\mathrm{S}}^{\dot{\beta}}{}_{b}\Big\} &= -2\delta^{a}_{b}\bar{\mathrm{L}}^{\dot{\beta}}{}_{\dot{\alpha}} - 2\delta^{\dot{\beta}}_{\dot{\alpha}}\mathrm{R}^{a}{}_{b} - 2\delta^{a}_{b}\delta^{\dot{\beta}}_{\dot{\alpha}}\Big(\mathrm{D} + \mathrm{C}\Big)\,.
\end{aligned} \tag{B.5}$$

B.2. The Non-Chiral Representation

On the S^5-extended non-chiral superspace spanned by $(x, \theta, \bar{\theta}, q)$ the algebra $\mathfrak{u}(2,2|4)$ can be represented as

$$
\begin{aligned}
\mathrm{P}_{\alpha\dot{\alpha}} &= -\partial_{\alpha\dot{\alpha}}\,, \\
\mathrm{Q}_{\alpha a} &= \bar{\theta}^{\dot{\alpha}}{}_a \partial_{\alpha\dot{\alpha}} - \partial_{\alpha a}\,, \\
\bar{\mathrm{Q}}^a{}_{\dot{\alpha}} &= \theta^{a\alpha}\partial_{\alpha\dot{\alpha}} - \bar{\partial}^a{}_{\dot{\alpha}}\,, \\
\mathrm{L}^\alpha{}_\beta &= -x^{\dot{\gamma}\alpha}\partial_{\beta\dot{\gamma}} - 2\theta^{c\alpha}\partial_{\beta c} + \tfrac{1}{2}\delta^\alpha_\beta\Big(x^{\dot{\gamma}\gamma}\partial_{\gamma\dot{\gamma}} + 2\theta^{c\gamma}\partial_{\gamma c}\Big)\,, \\
\bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}} &= -x^{\dot{\alpha}\gamma}\partial_{\gamma\dot{\beta}} - 2\bar{\theta}^{\dot{\alpha}}{}_c\bar{\partial}^c{}_{\dot{\beta}} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\Big(x^{\dot{\gamma}\gamma}\partial_{\gamma\dot{\gamma}} + 2\bar{\theta}^{\dot{\gamma}}{}_c\bar{\partial}^c{}_{\dot{\gamma}}\Big)\,, \\
\mathrm{D} &= -\tfrac{1}{2}\Big(x^{\dot{\alpha}\alpha}\partial_{\alpha\dot{\alpha}} + \theta^{a\alpha}\partial_{\alpha a} + \bar{\theta}^{\dot{\alpha}}{}_a\bar{\partial}^a{}_{\dot{\alpha}} + \tfrac{1}{2}q^{ab}\partial_{ab}\Big)\,, \\
\mathrm{S}^{a\alpha} &= -(x^+)^{\dot{\delta}\alpha}\theta^{a\delta}\partial_{\delta\dot{\delta}} + 4\theta^{c\alpha}\theta^{a\gamma}\partial_{\gamma c} + (x^-)^{\dot{\gamma}\alpha}\bar{\partial}^a{}_{\dot{\gamma}} + 2\theta^{c\alpha}q^{ad}\partial_{cd} - \theta^{a\alpha}q^{cd}\partial_{cd}\,, \\
\bar{\mathrm{S}}^{\dot{\alpha}}{}_a &= -(x^-)^{\dot{\alpha}\gamma}\bar{\theta}^{\dot{\gamma}}{}_a\partial_{\gamma\dot{\gamma}} - 4\bar{\theta}^{\dot{\gamma}}{}_a\bar{\theta}^{\dot{\alpha}}{}_c\bar{\partial}^c{}_{\dot{\gamma}} + (x^+)^{\dot{\alpha}\gamma}\partial_{\gamma a} - 2\bar{\theta}^{\dot{\alpha}}{}_b q^{be}\partial_{ae}\,, \\
\mathrm{K}^{\dot{\alpha}\alpha} &= 2\theta^{c\alpha}(x^+)^{\dot{\alpha}\gamma}\partial_{\gamma c} + 2(x^-)^{\dot{\gamma}\alpha}\bar{\theta}^{\dot{\alpha}}{}_c\bar{\partial}^c{}_{\dot{\gamma}} + \tfrac{1}{2}(x^+)^{\dot{\gamma}\alpha}(x^+)^{\dot{\alpha}\gamma}\partial_{\gamma\dot{\gamma}} \\
&\qquad + \tfrac{1}{2}(x^-)^{\dot{\gamma}\alpha}(x^-)^{\dot{\alpha}\gamma}\partial_{\gamma\dot{\gamma}} - 4\theta^{c\alpha}\bar{\theta}^{\dot{\alpha}}{}_a q^{ad}\partial_{cd} + \tfrac{1}{2}(x^+)^{\dot{\alpha}\alpha}q^{cd}\partial_{cd}\,, \\
\mathrm{R}^a{}_b &= 2\bar{\theta}^{\dot{\gamma}}{}_b\bar{\partial}^a{}_{\dot{\gamma}} - 2\theta^{a\gamma}\partial_{\gamma b} - q^{ad}\partial_{bd} - \tfrac{1}{4}\delta^a_b\Big(2\bar{\theta}^{\dot{\gamma}}{}_c\bar{\partial}^c{}_{\dot{\gamma}} - 2\theta^{c\gamma}\partial_{\gamma c} - q^{cd}\partial_{cd}\Big)\,, \\
\mathrm{B} &= \tfrac{1}{2}\theta^{a\alpha}\partial_{\alpha a} - \tfrac{1}{2}\bar{\theta}^{\dot{\alpha}}{}_a\bar{\partial}^a{}_{\dot{\alpha}}\,, \qquad\qquad \mathrm{C} = 0\,.
\end{aligned}
\tag{B.6}
$$

The representation of the same algebra on the non-extended superspace is obtained by setting all the q's in the above expressions to zero.

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Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbstständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

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