Particle Accelerators, 1986, Vol. 19, pp. 25–35
0031-2460/86/1904–0025/$20.00/0

GALAXIES AND PLANETARY RINGS: GRAVITATIONAL ANALOGUES OF NONNEUTRAL PLASMAS

JAMES W-K. MARK

Lawrence Livermore National Laboratory, Livermore, California

(Received March 7, 1985)

Orbit and collective dynamics in disk galaxies and in Saturn's rings are gravitational analogues of those occurring in nonneutral plasmas. Interesting problems for such "gravitational plasmas" include 2D spiral density waves in these disks which are responsible for observable spiral-structure. These waves are analogous to electrostatic waves in nonneutral beam plasmas, particularly analogous to single-disk studies of transverse waves in particle beams. Disk galaxies and planetary rings also exhibit 3D bending eaves with motions which distort (or bend) flat disks perpendicular to their original flat plane but whose propagation vector is largely restricted to directions in the original disk. Various orbit-wave resonance phenomena are frequently of importance in both spiral density and bending waves.

I. INTRODUCTION: SPIRAL DENSITY AND BENDING WAVES

Galaxies and planetary rings (e.g., that of Saturn) are astrophysical systems where some of the most significant phenomena involve orbit and particle dynamics mathematically analogous to Vlasov or slightly collisional beam plasmas (self-fields are of importance). In particular, topics of great interest include wave phenomena analogous to electrostatic plasma waves in nonneutral particle beams as well as resonant interaction between particle orbits and collective waves. In this paper we briefly outline some instances where such phenomena are of interest to astrophysicists. We will concentrate largely on the so called disk galaxies with significant matter in a very thin disk, and will only be able to cite similar phenomena in planetary rings, which are weakly-collisional systems with wave dynamics similar to disk-galaxies. In this regard, some of the references given in our earlier review papers[1,2] will not be repeated here.

Galaxies are astrophysical systems where an ensemble of point particles (stars) interact collectively in a near collisionless manner under long range $1/r^2$ forces of mutual gravitational attraction. Relaxation times[3] due to two-body irregular forces are defined in terms of the time necessary for significant net change in kinetic energy or velocity direction of test particles. Galaxies have typically 10^8–10^{12} stars with masses like our sun, and have dynamical time-scales of $<10^9$–10^{10} yrs (the latter number is comparable to the age of galaxies and of the universe), relaxation times $\gg 10^{10}$ yrs and length scales ~ 10–10^3 kpc (kilo-parsecs, 1 parsec = 1 pc $\sim$ 3.3 light years).

When seen edge-on, disk galaxies in optical photographs shows a disk of the combined light of many stars (major component of mass in disk, see figures on p.

25 of Ref. 4, which light is partially obscured by interstellar dust coexisting with hydrogen gas (~10% of mass in disk). Frequently, the disk appears like two "sunny-side up fried eggs" pasted together on their flat sides. The "egg-yolk" like part is the spheroidal (or bulge-nucleus) subsystem (photographs only show the easily observable part of this subsystem).

Internal velocities (cf. Ref. 3) by doppler shifts in spectral lines indicate that the disk is dominated dynamically by a balance of centrifugal forces of circular rotation, with circular velocity $\Theta_C(r)$ versus attractive force of self-gravity $g_r(r, z)$ for the entire system (here r, θ, z is a disk-centered polar coordinate system with z-direction perpendicular to disk). The angular velocities $\Omega = \Theta_C/r$ are not uniform, rather they have length scales comparable to disk size.

The spheroidal subsystem has larger dispersive (random) versus systematic rotation velocities which is also evident in the fact that their vertical z-width is thicker than typical disk stars. Not all galaxies possess the centrally concentrated part of the spheroidal component, but the evidence points to at least an extended spheroidal subsystem of dark-matter with length-scales and masses comparable to or larger than the disk. Theoreticians call this the "halo" and it is of great interest whether the amount of dark-matter in galaxies is sufficiently large to affect the closure of the universe (for some purely dynamical discussions in this regard, cf. Refs. 2, 5, 6 and p. 437 of Ref. 7).

The collective dynamics of the disk stars is governed by a Vlasov-like kinetic equation (cf. Ref. 3) here expressed in terms of Poisson bracket notation

$$\frac{\partial f}{\partial t} + [f, H] = 0, \tag{1}$$

where $f(r, \theta, z, p_r, p_\theta, p_z, t)$ is the mass distribution function, and H is the Hamiltonian per unit mass

$$H = \tfrac{1}{2}p_r^2 + \tfrac{1}{2}(p_\theta/r)^2 + \tfrac{1}{2}p_z^2 + \mathcal{V}(r, \theta, z, t). \tag{2}$$

Here $\mathcal{V}$ is the gravitational potential and (p_r, p_θ, p_z) are momenta per unit mass. A unique feature of this non-neutral plasma-like system is that the "charge" and mass of particles divide out so that the single-species kinetic Eq. (1) describes the result of a continuum of stellar masses in galaxies. Equations (1) and (2) must be complemented by Poisson's equation in the form

$$\nabla^2\mathcal{V} = 4\pi G\rho(r, \theta, z, t) = 4\pi G\int f\frac{1}{r}\, dp_r\, dp_\theta\, dp_z, \tag{3}$$

where G is the gravitational constant and ρ is the volume mass density of matter in the disk.

Small amplitude wave perturbations can always be represented as components like

$$\mathcal{V} = \mathcal{V}_0(r, z) + \mathcal{V}_1(r, \theta, z, t) = \mathcal{V}_0 + \mathrm{Re}\,\{V(r, z)\exp[i(\omega t - m\theta)]\} \tag{4}$$

where ω is wave frequency and m is the wavenumber of a particular azimuthal component. Typical waves in cylindrical systems exhibit spiral form. For example for electromagnetic waves the equivalent of $V(r, z)$ might be proportional to Hankel functions in r so that the real part of $V \exp[i(\omega t - m\theta)]$ would clearly exhibit spiral-like wave fronts in the limit of large radial wavenumbers. The number m indicates number of "spiral arms" radially spiraling out from the center. In astronomy, the spirals are leading if the angle θ of the wave front changes outwards in the same direction as the general rotation and are called trailing in the opposite case. The spiral pattern itself rotates as a whole at a pattern rotation frequency $\Omega_p = \omega/m$.

The presence of mostly trailing waves of this type is generally attributed to be the cause for observable global spiral structure of galaxies. Optical spiral structure observed in the unfiltered photographs[4] is more the effect of a small fraction of massive bright young stars formed in the spiral structure itself together with the $\sim 10^4$ °K hydrogen gas (HII) ionized by the photons from these stars. The radio 21-cm line observations are indicative of the response of the cold ($\sim 10^2$ °K) HI hydrogen gas which also participates in the spiral. These observables are actually all effectively "plasma-probes" of the underlying driving spiral wave in the bulk of the disk stars (described in terms of Eqs. 1–4). Physically, this underlying wave can be seen only when red optical filters are used[8] to isolate the collective effect of these majority of stars with a redder spectrum. This astrophysical theory is the so called "Density Wave Theory of Spiral Structure" (cf. e.g. Refs. 9, 10, 1, 6–8, 11–32).

Returning to the wave potential of Eq. (4), several interesting orbit-wave resonances occur which influence the collective-dynamics. If we take the approximation that the typical star is moving in an infinitesimally thin disk, its motion is an equilibrium rosette orbit in a central field given by the potential $\mathscr{V}_0(r)$. This rosette orbit, in the typical good approximation of moderate radial excursion, can be approximated by the sum of a circular rotation of a guiding center with angular rotation frequency $\Omega(r)[\Omega^2 r \simeq d\mathscr{V}_0/dr]$ about the galaxy center together with an elliptical "gyration" or epicyclic motion about this guiding center with epicyclic frequency $\kappa(r)$. For a specific wave of pattern rotation frequency Ω_p the most important resonances occur at radii where

$$\omega \equiv m\Omega_p = m\Omega + j\kappa \quad \text{and} \quad j = -1, 0, +1, \tag{5}$$

called respectively, inner Lindblad resonance ($j = -1$), the corotation (or particle $j = 0$) resonance, and outer Lindblad resonance ($j = +1$) with typical relative radial positions of the resonances in annular strips near the inner, middle (corotation) and outer parts of the galaxy disks. In the rotating wave frame of a typical spiral wave, the resonances imply periodic and nearly periodic orbits in those annular regions where Eq. (5) is approximately valid. The existence of the inner resonance $\Omega_p = \Omega - \kappa/m$ is guaranteed only for galaxies with a centrally concentrated part of the spheroidal subsystem. In addition, orbit-wave resonances through two-stream instabilities occur[24] between waves in the disk and particles of the halo (spheroidal subsystem).

The above frequencies Ω and κ are related to the transverse betatron frequencies of a beam particle orbit but expressed in cylindrical geometry. Since the equivalent of beam focusing forces is a nonlinear self-field, close beam analogies include the transverse motion of a propagating pinched-beam of electrons or ions. In the above scheme of resonances, astrophysicists interested in galaxy spiral structure are interested in $|j| = 0$, 1 of Eq. (5), while for propagation of high-current beams $j = 0$ and sometimes $|j| = 1$ are most interesting (e.g. hose instabilities). This analogy has been partially explored in several of our papers[33–36]. In particular, for the question of existence of MHD-like hydrodynamic models[35] of pinched-beam dynamics, our equivalent "gravitato-hydrodynamics" has been adapted in straight-forward manner and even partially tested[36] numerically. Preliminary results suggest that these hydrodynamic models are applicable to small amplitude nonlinear waves. Broader applications require more careful study or even further modifications of the model. For example, there may be the need to impose outgoing wave boundary conditions because a particle model does not reflect waves from the outer beam edge as a hydro-model might. (Such boundary conditions were found necessary in a different context in our discussions of spiral waves in the latter part of Section II.)

Galaxies could also exhibit "bending waves". These are intrinsically 3D waves which exhibit motions that distort or bend flat disks perpendicular to their original flat plane but whose propagation vector is largely restricted to directions near the original disk plane. The observed phenomena is called warps[37] in the astronomical literature, and the bent outer disk of the gas distribution is typically seen in 21-cm radio observations. Such bending waves with self-gravity[38–40] also exhibit instabilities due to resonant interaction of particles and waves. A recent review was given in our Ref. 2.

Both spiral density and bending waves exist in Saturn's rings but they are driven by interaction of the ring disk with the moons of Saturn. In Saturn's rings these waves[41–44] have very short radial wavelengths giving the impression of a phonograph record to the distant observer.

II. SPIRAL MODES IN GALAXIES INVOLVING INTERACTION OF PARTICLE ORBITS WITH DENSITY WAVES

An example of spiral modes in the underlying disk of galaxies is shown in Fig. 1 which is a controur diagram showing curves of constant wave perturbation density

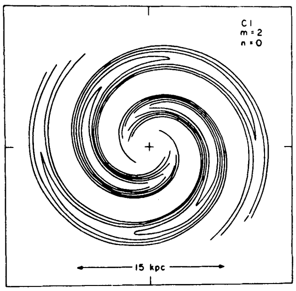

FIGURE 1 Contour curves of constant perturbation density for a spiral mode.

in the mode. Mode number m gives the number of spiral arms, while the number n involves radial structure. This mode e-folds in $\simeq 3 \times 10^8$ yr. and is the fastest growing in a certain galaxy model.[12]

In the disk of stars (galaxies), spiral wave amplitudes frequently tend to be a few percent up to 10–20%. In these cases, linear theory give estimates on the phenomena (nonlinear problems will be referred to later). The mathematical method of characteristics can be applied to the solution of the linearized form of Eqs. (1)–(2). We obtain an integral equation[21–24,27] for the perturbed distribution function f_1 in terms of the potential $\mathscr{V}_1$. The solution of Poisson's Eq. (3) for a thin disk is normally also an integral relation. The orbit-wave resonances of Eq. (5) contribute as singular resonant demoninators in the kernals so that care must be taken in analytical-continuation of these kernals from initial-value problems. This process is generalized from the analytical continuations of dispersion relations in the Landau theory of plasma waves.

The density wave mode of Fig. 1 is evaluated numerically using a differential equation derived after considerable asymptotic analysis from the reasonably involved integral equation outlined in the last paragraph. This differential equation has the form

$$d^2u/dr^2 + k_3^2 u = 0 \tag{6}$$

where $u(r)$ is related to the wave gravitational potential $\mathscr{V}_1$ and $k_3^2(r, \Omega_p)$ is a function of the radial distance r and pattern frequency Ω_p through various equilibrium quantities. The asymptotics[21–24] are based on the small parameters

$$\varepsilon \sim (\delta r/r) \sim (rp_r/p_\theta) < 1 \tag{7}$$

$$\varepsilon \sim |d \ln \mathscr{V}_1/d \ln r|^{-1} < 1 \tag{8}$$

typical of density waves in galaxies. The assumptions (7) relates to the fact that the rosette orbit of particles is not too eccentric; Eq. (8) relates to the short radial wavelength (WKB) of these modes.

The structure of Eq. (6) is familiar, we know from this type of differential equation that the modes involve the superposition of two spiral waves propagating radially in opposite directions. In the following paragraphs we will eludicate the wave structures shown in Fig. 1. A standing wave is generated because these propagating waves reflect into one another near the annular inner and corotation regions of galaxy disks (cf. Ref. 25). At these reflection regions, the waves have similar wavelengths; but in between these regions they split into "long" and "short" wavelength branches. The boundary condition near galaxy center is that of vanishing wave amplitude due to total wave reflection. At the corotation region there is some leakage of waves so the proper outerboundary condition is that of outgoing waves because the outgoing trailing waves are absorbed by resonant interaction with stars (at the outer Lindblad resonance) and possibly also by dissipation in the gas, which is a more important component in the outer disk.

The growth rate of the modes can be approximated[25] by the formula $\gamma = \ln \Gamma_r/2\tau_g$ where τ_g is the cycle time for group propagation from inner reflection to corotation and back. Also Γ_r is the wave angular momentum amplification factor per cycle. This factor Γ_r can be illustrated by discussing the processes at the corotation region where $\Omega_p \simeq \Omega(r)$. (There are additional contributions[24,25] due to resonant interactions with stars in the spheroidal subsystem.)

The behavior of density waves at corotation is dominated by a three-wave interaction amplifier which exists even in linear theory. Mark[23] called it "Wave Amplification by Stimulated Emission" (abbrev. "WASER") by analogy to Laser. In this spiral-wave case the trailing wave propagating outwards from galactic center is the signal incident upon the annular corotation region (Fig. 3); this wave stimulates the emission of two other trailing waves which leave the corotation region in opposite directions as indicated by the direction of the arrows. The number of arrows in each wave is proportional to its "luminosity." The accompanying number indicate the algebraic ratio of these "luminosities" in terms of the angular momentum carried radially into and out of the corotation region per unit time by each wave in its direction of group propagation.

This "stimulated emission" process is like interaction of positive and negative energy waves in plasma theory, but more directly expressible here as wave angular momentum instead of energy with also a limited radial extent of interaction region. Since spiral waves are non-axisymmetric disturbances which exert torques, it is not surprising that they carry wave angular momentum. The waves inside corotation represent density disturbances which rotate slower than the disk; therefore wave amplification necessitates a deceleration or decrease in angular momentum in that region of the disk inside corotation. The converse holds for waves outside corotation because they rotate more rapidly than the local disk. The incident signal provides a coupling across corotation which "stimulates" a favorable exchange of angular momentum between the emitted signals, resulting in the amplification of the emitted waves by stimulated emission. Because all three waves possess similar wavelengths near corotation, this WASER process exists in galaxies even in the linear regime. However, away from wave interaction regions, these waves split into "long" and "short" wavelength branches. The square of relative amplitudes give the above Γ_r which contributes to growth rate γ as well as the relative ratios of wave angular momenta. In Fig. 3, a local disk parameter[23] μ_d has been chosen so that the ratios of wave angular momenta are integers.

This three-wave process at corotation appears mathematically as a singular turning-point problem for a WKB analysis of Eq. (6). There is outward—leakage of a very strong wave signal as seen in Fig. 3. The satisfiaction of the boundary conditions at the outer disk must involve dissipation of the outgoing wave at radii larger than the corotation annulus. For example, pure mirror-like reflection of waves from the outer-disk (e.g. as from an edge with sharp density fall-off) would convert an outgoing trailing wave into an incoming leading wave; and the superposition of trailing and leading waves of equal strength in a normal mode would lead to a disturbance appearing like a straight rotating bar having no sense of spiralling at all (neither leading nor trailing)! This potential theoretical dilemma called[45] "antispiral theorem" has been shown by Mark[18,21] to be

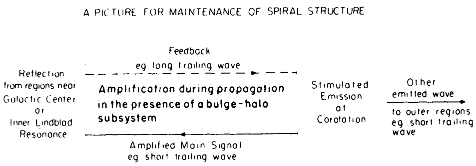

FIGURE 2 Conceptual diagram of a spiral mode in galaxies.

circumvented in galaxies because the outgoing wave beyond corotation (Figs. 2–3) is dissipated at the outer Lindblad resonance region where $\Omega_p \simeq \Omega + \kappa/m$.

In the absence of an inner Lindblad resoance, the reflection process in the inner regions arise[25] because the inner regions are mostly dominated by the spheroidal subsystems whose high dispersive speeds of internal motion make them unfavorable to the propagation of density waves. Thus an incoming wave incident upon this region is reflected. This inner reflection process of Fig. 2 can reflect trailing waves into trailing waves because it also occurs as a consequence of the "long and short" wavelength branches reaching the same wavelength in the inner regions. Sometimes, an inner resonance also occurs in this region and the observable phenomena might be[18,21] the formation of an inner ring-like feature in galaxies.

Some of the nonlinear problems have been discussed with the help of numerical simulations. In particular, N-body particle codes is one means used to further study the nonlinear saturation of wave growth and the long term persistence of spiral structure in a disk of stars. Although even more sophisticated techniques might be needed for final resolution of the nonlinear issues, Fig. 4 illustrates the fact that quasistationary spiral structure obtains in models similar to those described in previous paragraphs. In particular this N-body numerical simulation *in three dimensions*[6] has 30% of its mass in a nuclear bulge (central part of spheroidal subsystem) which replaces the central disk. Thus the central hole in the $t = 0$ frame for this model represents a disk of 20,000 model stars tapered-off at the center to accommodate the bulge. The extended spiral structure is detected by Fourier analysis and illustrated by the solid curves. After initial transients have decayed away, the quasistationary spiral persists until the time (4×10^9 yr) when we ended our experiment for reasons of computer expense. A 3D simulation was considered necessary because the simulation was designed actually to cover as

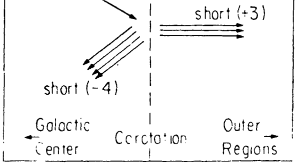

FIGURE 3 Schematic diagram illustrating the effect of density Wave Amplification by Stimulated Emission ("WASER", see text) at the corotation region.

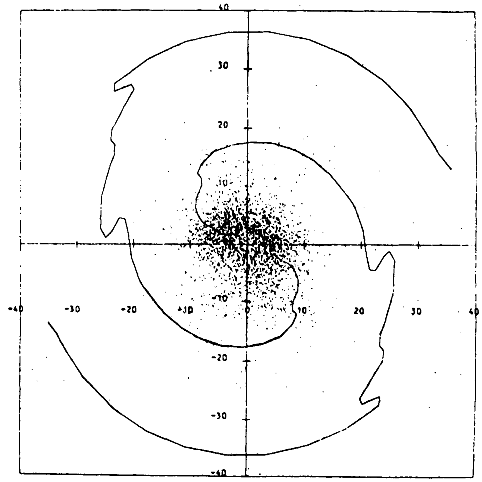

FIGURE 4 This illustrates the quasi-stationary spiral structure detected by Fourier analysis at time 3.3×10^9 yr in the nonlinear 3D particle code simulations of Berman and Mark.[9]

first priority the cosmological consequences of halo matter.[6] For other such simulations please refer to Dr. Miller's talk in this Conference.

III. A HEURISTIC DISCUSSION OF THE BENDING WAVE THEORY

As mentioned in the introduction, disk galaxies and planetary rings also exhibit bending waves which distort flat disks in the direction perpendicular to the original disk-plane (cf. Ref. 2). The right frame of Fig. 5 shows an originally flat circular disk bent at the edges so the cross section is now like an integral sign rather than the original straight line. (The additional influence of spheroidal Halo matter is discussed later.) Particles displaced vertically out of the disk see restoring forces resulting in vertical oscillation frequencies as well as allowing horizontal propagation of wave information, as modeled by the vertical and horizontal springs of the oscillator analogy in Fig. 5, left frame. Even if the entire disk were not bent and we just vertically displace a single test particle, this particle exhibits vertical oscillation (among other reasons, the flattened nature of disk galaxies give a vertical restoring force). The left frame of Fig. 6 shows both

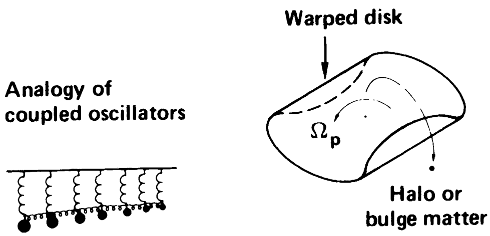

FIGURE 5 A simple mechanical analogue of bending waves in self-gravitating disks.

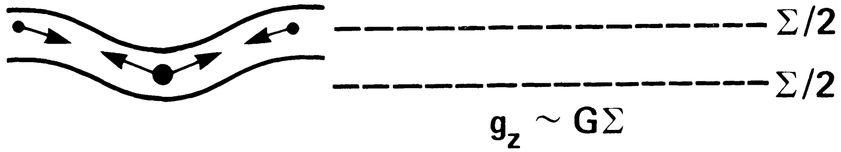

FIGURE 6 Physical picture for the restoring force of self-gravity in a bent disk.

the vertical and horizontal forces due to self-gravity in a wavy bent disk. The vertical restoring self-gravity can be estimated by first analyzing the wave bent disk into two planes with densities $\Sigma/2$ and attraction $g_z \sim G\Sigma$ (G = gravitational constant, see right frame of Fig. 6). Clearly, the real situation has an additional angular projection factor (h/wavelength) or kh. Thus local self-gravity gives $g_z \sim -G\,|k|\,h$ for the bent disk. If we denote height of bending $\Delta z = h(r, \theta, t) = \mathrm{Re}\,\{\exp[i(\omega t - m\theta + \int k\,dr)]\}$, where $k(r)$ can be a complex-valued wavenumber containing both phase and amplitude information, then h is governed by,

$$\left[\frac{\partial}{\partial t} + \Omega(r)\frac{\partial}{\partial \theta}\right]^2 h = \nu_z^2(r)h + g_z. \tag{9}$$

Here $\Omega(r)$ is the angular frequency of rotation of the disk, and $\nu_z^2(r)$ represents the vertical oscillation frequency due to restoring force of entire galaxy. Using above form for $h(r, \theta, t)$ gives the dispersion relation

$$[\omega - m\Omega(r)]^2 = \nu_z^2(r) + 2\pi G\Sigma(r)k(r) \tag{10}$$

where the extra 2π factor can be known only by detailed calculations.[40,46] This wave has radial group velocity $c_g = -d\omega/dk$. Additional effects such as disk thickness, response of stellar disk including vertical mode structure and resonances were discussed by Mark.[40] This paper also gives further details of the "fire-hose" instability mentioned already in the Kulsrud *et al.* paper.[39]

In the above discussion, the halo or spheroidal out of plane component (see right frame of Fig. 7) only provides an inactive basic gravitational field which contributes to ν_Z. On the other hand, the warped disk has an azimuthal component of the gravitational force which exerts a torque on the the particles of the halo that pass close enough to the disk (cf., right frame of Fig. 5 or left frame of Fig. 7). This resonant interaction allows a two stream instability[40,46] which amplifies wave disturbances in the disk. We emphasize that these bending waves also have spiral wave-fronts and propagate in the azimuthal direction with

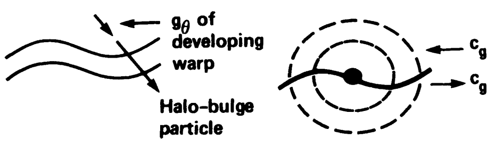

FIGURE 7 As the bending wave propagates through the disk (solid curves in right frame) with group velocity c_g, it interacts with the particles of the spheroidal subsystem (dashed curves in right frame) through gravitational fields. These fields are illustrated in the left frame where we see the wave in azimuthal angle θ at fixed r.

rotation frequency $\Omega_p = R_e(\omega)/m$. Considerations[2] are suggestive that the theoretical picture of self-gravitating warps is consistent with substantial haloes inside the region where the disk warps exist as a diagnostic. This is related to our discussion in Section I regarding the possibility of a closed universe. More discussions on this and comparisons with observations are given in Ref. 2.

We appreciate the comments and encouragements of Drs. D. Keefe, J. Marx, T. Tajima and L. Smith.

REFERENCES

(Since part of this paper refer to our reviews 1–2 below for further clarification, some important references quoted there are not repeated here.)

1. J. W-K. Mark, in *Photometry, Kinematics and Dynamics of Galaxies,* edited by David S. Evans (Publ. Univ. of Texas, Austin, Texas, 1979), p. 393.
2. J. W-K. Mark, in *Kinematics, Dynamics and Structure of the Milky Way,* edited by W. L. H. Shuter (D. Reidel, 1983), p. 289.
3. S. Chandrasekhar, *Principles of Stellar Dynamics* (Dover, New York, N.Y., 1960).
4. A. Sandage, *The Hubble Atlas of Galaxies* (Carnegie Institution of Washington, Washington, D.C. 1961).
5. J. P. Ostriker and P. J. E. Peebles, *Astrophysical J.,* **186** 467 (1973).
6. R. H. Berman and J. W-K. Mark, *Astronomy and Astrophysics,* **77,** 31 (1979).
7. R. W. Hockney and J. W. Eastwood, *Computer Simulation Using Particles,* Hockney & Eastwood, McGraw Hill, New York, N.Y., 1981).
8. F. Schweizer, *Astrophysical J. Suppl. Series,* **31,** 313 (1976).
9. C. C. Lin and F. H. Shu, *Astrophysical J.,* **140,** 646 (1964).
10. C. C. Lin and F. H. Shu, *Proc. Nat. Acad. Sci. USA,* **55,** 229 (1966).
11. R. H. Berman and J. W-K. Mark, *Astrophysical J.,* **231,** 388 (1979).
12. G. Bertin, Y. Y. Lau, C. C. Lin, J. W-K. Mark and L. Sugiyama, *Proc. Nat. Acad. Sci. USA,* **74,** 4726 (1977).
13. G. Contopoulos, *Astrophysical J.,* **201,** 566 (1975); *Astronomy and Astrophysics,* **71,** 221 (1979).
14. C. Hunter, *Monthly Notices Royal Astronomical Soc. England,* **126,** 299 (1963).
15. C. Hunter, *Astrophysical J.,* **227,** 73 (1979).
16. Y. Y. Lau, C. C. Lin and J. W-K. Mark, *Proc. Nat. Acad. Sci. USA,* **73,** 1379 (1976).
17. D. Lynden-Bell and A. J. Kalnajs, *Monthly Notices Royal Astronomical Soc. England,* **157,** 1 (1972).
18. J. W-K. Mark, *Proc. Nat. Acad. Sci. USA,* **68,** 2095 (1971).
19. J. W-K. Mark, in *The Formation and Dynamics of Galaxies,* edited by J. R. Shakeshaft (D. Reidel, 1974), p. 417.
20. J. W-K. Mark, in *La Dynamique des Galaxies Spirales,* edited by L. Weliachew (Centre National Recherche Scientifique, Paris, 1975), p. 67.
21. J. W-K. Mark, *Astrophysical J.,* **193,** 539 (1974).
22. J. W-K. Mark, *Astrophysical J.,* **203,** 81 (1976).
23. J. W-K. Mark, *Astrophysical J.,* **205,** 363 (1976).
24. J. W-K. Mark, *Astrophysical J.,* **206,** 418 (1976).
25. J. W-K. Mark, *Astrophysical J.,* **212,** 645 (1977).
26. W. W. Roberts, *Astrophysical J.,* **158,** 123 (1969).
27. F. H. Shu, *Astrophysical J.,* **160,** 99 (1970).
28. F. H. Shu, R. V. Stachnik and J. C. Yost, *Astrophysical J.,* **166,** 465 (1971).
29. W. W. Roberts, M. S. Roberts and F. H. Shu, *Astrophysical J.,* **196,** 381 (1975).
30. A. Toomre, *Astrophysical J.,* **139,** 1217 (1964).
31. A. Toomre, *Astrophysical J.,* **158,** 899 (1969).

32. R. Wielen, *Publ. Astronomical Soc. Pacific,* **86,** 341 (1974).
33. J. W-K. Mark and L. Smith, *Soc. Ind. Appl. Math. J. Appl. Math.,* **42,** 903 (1982).
34. J. W-K. Mark, *Soc. Ind. Appl. Math. J. Appl. Math.,* **42,** 914 (1982).
35. J. W-K. Mark and S. S. Yu, *Phys. Lett,* **92A,** 179 (1982).
36. J. K. Boyd, J. W-K. Mark, W. M. Sharp and S. S. Yu, in *Heavy Ion Accelerators and Their Applications to Inertial Fusion,* edited by Y. Hirao, T. Katayama and N. Tokuda (Institute for Nuclear Study, Univ. Tokyo, Tokyo, 1984), p. 485.
37. R. Sancisi, *Astronomy and Astrophysics,* **53,** 159 (1976).
38. C. Hunter and A. Toomre, *Astrophysical J.,* **155,** 747 (1969).
39. R. M. Kulsrud, J. W-K. Mark and A. Caruso, *Astrophysics and Space Sci.,* **14,** 52 (1971).
40. J. W-K. Mark, *Astrophysical J.,* **169,** 455 (1971).
41. P. Goldreich and S. Tremaine, *Icarus,* **34,** 240 (1978).
42. J. N. Cuzzi, J. L. Lissauer and F. H. Shu, *Nature,* **292,** 703 (1981).
43. F. H. Shu, J. N. Cuzzi and J. L. Lissauer, *Icarus,* **53,** 185 (1983).
44. F. H. Shu, C. Yuan and J. J. Lissauer, *Astrophysical J.,* in press (1985).
45. D. Lynden-Bell and J. P. Ostriker, *Monthly Notices Royal Astronomical Soc. England,* **136,** 293 (1967).
46. G. Bertin and J. W-K. Mark, *Astronomy and Astrophysics,* **88,** 289 (1980).