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Entanglement of purification and projection operator in conformal field theories

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ABSTRACT

We investigate entanglement of purification (EoP) in conformal field theory. By using Reeh-Schlieder theorem, we construct a set of the purification states for ρ_{AB}, where ρ_{AB} is reduced density matrix for subregion AB of a global state ρ. The set can be approximated by acting all the unitary observables, located in the complement of subregion AB, on the global state ρ, as long as the global state ρ is *cyclic* for every local algebra, e.g., the vacuum state. Combining with the gravity explanation of unitary operations in the context of the so-called surface/state correspondence, we give an explanation of holographic EoP formula. We also explore the projection operator with the conformal basis in conformal field theory. In some limit we may produce the holographic EoP results by using the projection operator. Finally, we discuss the similarity and difference between the projection operator and unitary operations for calculating EoP.

1. Introduction

Recent studies on the gravity dual of some information-theoretical quantities have provided us more insights on the nature of gravity and AdS/CFT correspondence [1]. Quantum entanglement in the field theory has a mysterious relation to the definition of geometry in the bulk. In AdS/CFT, the entanglement entropy is given by the area of a minimal surface in AdS [2] [3].

The entanglement in quantum field theory (QFT) has a deep relation with the structure and symmetry of the theory. In the framework of algebraic QFT, the constructions of the theory are by observables rather than the states [4] [5]. Along with this aspect the celebrated Reeh-Schlieder theorem give a strong constraint on the local properties of QFT. In fact this theorem characterizes the strong entanglement between different subregions.

In this paper we will use Reeh-Schlieder theorem to investigate a quantity called entanglement of purification (EoP), which is another good entanglement measurement even for mixed state [6]. Similar as entanglement entropy this quantity is also proposed to have geometric interpretation in the context of AdS/CFT. The holographic EoP is proposed in [7] [8].

EoP is a quantity to characterize the correlation between different subsystems A and B for a given state ρ. For a subsystem A the reduced density matrix ρ_A is defined as $\rho_A = tr_{\bar{A}}\rho$, where $\bar{A}$ is the complement of A. The entanglement entropy S_A is given by the von Neumann entropy

$$S(\rho_A) := -tr\rho_A \log\rho_A. \tag{1}$$

The entanglement of purification is defined as

$$E_P(\rho_{AB}) = \min_{\rho_{AB}=tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi|} S(\rho_{A\tilde{A}}), \tag{2}$$

where the states $|\psi\rangle$ are called purifications of ρ_{AB} by introducing $\tilde{A}$ and $\tilde{B}$, and $\rho_{A\tilde{A}} := tr_{B\tilde{B}}|\psi\rangle\langle\psi|$. The minimization is taken over all the possible purifications $|\psi\rangle$.

The holographic EoP is given by the area of the minimal cross of entanglement wedge, denoted by Σ_{AB},

$$E_W(\rho_{AB}) = \frac{\min\{\text{area}(\Sigma_{AB})\}}{4G}, \tag{3}$$

where G is the Newton constant. In this construction the entanglement wedge is the region surrounded by AB and the minimal surface homologous to them, which is expected to be dual to reduced density matrix ρ_{AB} [9–12]. The holographic EoP conjecture provides us a new way to understand the geometry in AdS and entanglement structure in the dual CFT. The generalization to the multipartite cases are discussed in [13–16]. There are also other proposals to extract the entanglement wedge cross section [17] [18] [19]. One may see [20–26] for more recent studies.

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The calculation of EoP in QFT is very hard [27], for some simple models we may rely on numerical calculations [28] [29] [30]. In this paper we construct a set of the purification states $|\psi\rangle$ by using Reeh-Schlieder theorem. The set is obtained by unitary transformations in the complement of AB. Using this result we give an explanation of holographic EoP (3) with the help of the surface/state correspondence [31] [32]. In the end we also discuss the similarity and difference between projection operator [33] [34] and the minimization procedure of EoP.

The paper is organized as follows. In Sec. 2 we use the Reeh-Schlieder theorem to construct the set of purifications, which is associated with unitary operations on the complementary part of AB. In Sec. 3 we explain holographic EoP formula (Heop) in the context of surface/state correspondence. In Sec. 4 we find the projection operators with conformal basis may produce the holographic EoP results in some limit of the intervals A and B. In Sec. 5 we show the similarity and difference between the projection operators and the unitary operation for calculating EoP. Sec. 6 is the conclusion and comments on the calculations of EoP in CFTs.

2. Reeh-Schlieder theorem and purification

The minimization procedure (2) makes the calculation of EoP become a very difficult task in QFT since, in principle, one have to deal with infinite states. Actually there is no method to systematically construct the states $|\psi\rangle$.

But this problem will be much easier if the state of the entire system ρ is *cyclic*. To explain what is meant by a cyclic state we need some basic elements of algebraic QFT [4], see also [35]. In the framework of algebraic QFT any open region O can be associated with a von Neumann algebra of local observables, denoted by $\mathscr{R}(O)$. For O being the entire space region, we have a global algebra $\mathscr{U}$. We denote the Hilbert space of QFT by $\mathcal{H}$. A state $|\Psi\rangle$ is said to be cyclic for $\mathscr{R}(O)$ with respect to the Hilbert space $\mathcal{H}$, if the set $\mathcal{H}_O := \{\mathcal{O}|\Psi\rangle, \mathcal{O} \in \mathscr{R}(O)\}$ is dense in $\mathcal{H}$. In other words, any state $|\Psi'\rangle \in \mathcal{H}$ can be approximated by the elements in set $\mathcal{H}_O$ as closely as we like. For example, the vacuum state $|0\rangle$ is a cyclic state for the global algebra $\mathscr{U}$. But the Reeh-Schlieder theorem gives a much stronger conclusion than that, it shows the vacuum state $|0\rangle$ is also a cyclic state for *every* local algebra $\mathscr{R}$. More precisely,

Reeh-Schlieder Theorem. *Suppose O to be any bounded open region, then the vacuum state $|0\rangle$ is cyclic for $\mathscr{R}(O)$.*

One may refer to [5] for the proof of this theorem, see also a more modern treatment [36]. The reason for vacuum state being cyclic for local algebra is that different regions are highly entangled in vacuum state. Now we come back to our discussion of purification and its relation to Reeh-Schlieder theorem. ρ_{AB} is the reduced density matrix of the cyclic state $\rho = |0\rangle\langle 0|$. Firstly, one could show the set of the purification states $|\psi\rangle$ of ρ_{AB} can be approximated by the elements in

$$\mathcal{H}_{\overline{AB}} := \{\mathcal{O}_{\overline{AB}}|0\rangle, \mathcal{O}_{\overline{AB}} \in \mathscr{R}(\overline{AB})\}, \tag{4}$$

where $\mathcal{O}_{\overline{AB}}$ is the operator located in the region $\overline{AB}$. The Reeh-Schlieder theorem guarantees the set $\mathcal{H}_{\overline{AB}}$ is dense in $\mathcal{H}$. This means any $|\psi\rangle$ can be approximated by a state $\mathcal{O}_{\overline{AB}}(\psi)|0\rangle$ in $\mathcal{H}_{\overline{AB}}$. We simply write it as[1]

$$|\psi\rangle = \mathcal{O}_{\overline{AB}}(\psi)|0\rangle. \tag{5}$$

We may choose the auxiliary parts $\tilde{A}\tilde{B}$ as $\overline{AB}$. The constraint $tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi| = \rho_{AB}$ is equal to

$$tr_{AB}(O_{AB} tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi|) = tr_{AB}(O_{AB}\rho_{AB}), \tag{6}$$

for arbitrary operator $O_{AB} \in \mathscr{R}(AB)$. This leads to

$$\begin{aligned} &\langle 0|(\mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) - \mathbf{1})\mathcal{O}_{AB}|0\rangle = 0, \\ &\langle 0|(\mathcal{O}^{\dagger}_{\overline{AB}}(\psi)\mathcal{O}_{\overline{AB}}(\psi) - \mathbf{1})\mathcal{O}_{AB}|0\rangle = 0, \end{aligned} \tag{7}$$

where we have used the cyclic property of trace and the microcausality condition for local algebra, i.e., $[\mathcal{O}(x), \mathcal{O}(y)] = 0$ when x and y are spacelike separated [4]. Since (7) is true for any operator $\mathcal{O}_{AB}$, using the Reeh-Schlieder theorem again, there should exist an operator $\mathcal{O}_{AB}(\psi)$ such that $\mathcal{O}_{AB}(\psi)|0\rangle = (\mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) - \mathbf{1})|0\rangle$ and $\mathcal{O}_{AB}|0\rangle = (\mathcal{O}^{\dagger}_{\overline{AB}}(\psi)\mathcal{O}_{\overline{AB}}(\psi) - \mathbf{1})|0\rangle$. Therefore, by using (7), the norm of states $(\mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) - \mathbf{1})|0\rangle$ and $(\mathcal{O}^{\dagger}_{\overline{AB}}(\psi)\mathcal{O}_{\overline{AB}}(\psi) - \mathbf{1})|0\rangle$ are vanishing.[2] Finally, we have

$$\mathcal{O}^{\dagger}_{\overline{AB}}(\psi)\mathcal{O}_{\overline{AB}}(\psi) = \mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) = \mathbf{1}. \tag{8}$$

Now we arrive at our main result in this section.

Corollary 1. *The set of the purifications of reduced density matrix ρ_{AB} can be approximated by the Hilbert space $\mathcal{H}_{\psi}$ constructed by acting unitary local operators $\mathcal{U}_{\overline{AB}}$ on the vacuum, i.e.,*

$$\mathcal{H}_{\psi} = \{\mathcal{U}_{\overline{AB}}|0\rangle, \quad unitary \quad \mathcal{U}_{\overline{AB}} \in \mathscr{R}(\overline{AB})\}. \tag{9}$$

3. Surface/state correspondence and holographic EoP

Even though we have constrained the set of the purifications to be $\mathcal{H}_{\psi}$, it is still hard to find the minimization of $S_{A\tilde{A}}$ by directly calculating in field theory. But to show (9) is a useful approach to the EoP, we would turn to using some arguments based on surface/state correspondence [31]. In the original paper of holographic EoP [7] the authors have given some arguments to explain the holographic EoP formula by surface/state correspondence. Here we would like to make the argument more precise and complete.

In the paper [31] the authors proposed a new duality relation between a bulk codimension-2 spacelike surface and quantum states in the dual field theory based on the similarity between tensor network and AdS/CFT [37], which is expected to be a generalization of original AdS/CFT. In the context of surface/state correspondence, the gravity lives on a manifold M_{d+2}, any codimension-2 *convex* surface Σ corresponds to state in the total Hilbert space $\mathcal{H}$. In this paper we would work in AdS$_3$, the states are represented by curves σ in AdS space. We would like to summarize the three important points of this correspondence:

1. A pure state $|\Phi(\sigma)\rangle$ corresponds to topologically trivial curve, i.e., homologous to a point, in the bulk.
2. If two curves σ_1 and σ_2 are connected by a smooth deformation that preserves *convexity*, the corresponding states of them are related by a unitary transformation, that is

   $$|\Phi(\sigma_1)\rangle = U(1,2)|\Phi(\sigma_2)\rangle, \tag{10}$$

   where $U(1,2)$ is a unitary operator associated with deformation paths.

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[1] More precisely, for any state $|\psi\rangle$ one always could find an operator $\mathcal{O}_{\overline{AB}}(\psi)$ such that the $||\psi\rangle - \mathcal{O}_{\overline{AB}}(\psi)|0\rangle| < \epsilon$, where ϵ is an arbitrary positive constant.

[2] In fact this follows by another important property of vacuum state, that is the separating property. A state $|\Omega\rangle$ is said to be separating for the local algebra $\mathscr{R}(O)$ if $\mathcal{O}|\Omega\rangle = 0 \Rightarrow \mathcal{O} = 0$.

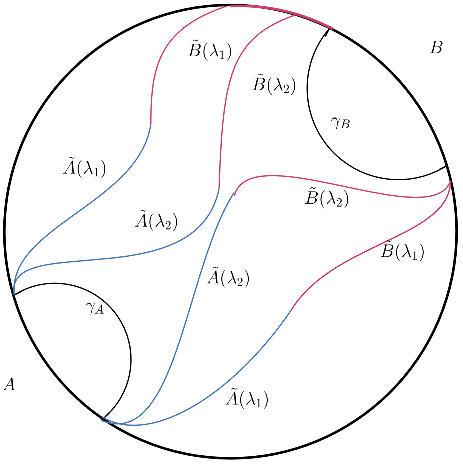

Fig. 1. A series of deformation of $\tilde{A}$ (blue) and $\tilde{B}$ (red) for the disconnected entanglement wedge.

3. The entanglement entropy for a subregion σ_A of the curve σ is conjectured to be given by the area formula,

$$S_{\sigma,A} = \frac{\min\{\text{area}(\gamma_{\sigma,A})\}}{4G}, \tag{11}$$

where G is the Newton constant.

If taking the curve σ to be AdS boundary, these would be the AdS/CFT correspondence, specially the entanglement entropy is RT formula. According to **Corollary 1**, we are interested in the unitary transformation $\mathcal{U}_{\overline{AB}}$ that act on subregion $\overline{AB}$. In the bulk these transformations are dual to deformations of curve on the AdS boundary while keeping the boundary of $\overline{AB}$ invariant. Note that for a unitary operator $\mathcal{U}_A$ located in a subregion A acting on the state $\Phi(\sigma)$, the corresponding deformation of surface σ cannot transcend the extremal surface $\gamma_{\sigma,A}$. Only in this way one could keep the *convexity* of the deformed curves, and this also guarantees the holographic entanglement entropy of subregion A is invariant under unitary transformation $\mathcal{U}_A$ [31].

Now we are ready to give an explanation of holographic EoP formula based on the surface/state correspondence. For simplicity we choose A and B to be two disconnected intervals as shown in Fig. 1.

If A and B are far away from each other, the entanglement wedge W_{AB}, defined by a region surrounded by A, B and the minimal surface $\gamma_{A,B}$ homologous to them, would become disconnected, see Fig. 1. In this case we may choose a series of deformations of the curve $\tilde{A}(\lambda)$ and $\tilde{B}(\lambda)$. Since these deformations correspond to unitary transformations $\mathcal{U}_{\overline{AB}}$, they just need to keep the boundary of $\tilde{A}(\lambda)\tilde{B}(\lambda)$ invariant. As shown in the Fig. 1 we always could choose a series of deformations $\tilde{A}(\lambda_n)$ and $\tilde{B}(\lambda_n)$ such that $\tilde{A}(\lambda_\infty) = \lim_{n\to\infty} \tilde{A}(\lambda_n)$ becomes connected in the bulk. Recall the definition of EoP (2), it is equal to the minimal value of entanglement entropy $S_{A\tilde{A}}$. The holographic entanglement entropy for $A\tilde{A}(\lambda_n)$ is given by (11). Therefore, we get $S_{A\tilde{A}(\lambda_\infty)} = 0$.[3] This means the holographic EoP is zero. Note that in the Fig. 1 we only draw a special example for the deformations. In principle, there exist infinite ways to make $S_{A\tilde{A}(\lambda_\infty)} = 0$. For example the deformations corresponding to $\mathcal{U}_{\tilde{A}}$ or $\mathcal{U}_{\tilde{B}}$ would never effect the value of $S_{A\tilde{A}(\lambda_n)}$.

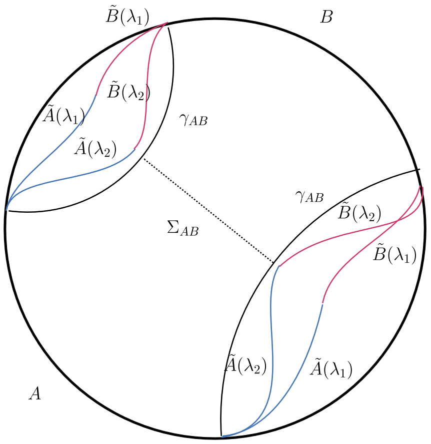

Fig. 2. A series of deformation of $\tilde{A}$ (blue) and $\tilde{B}$ (red) for the connected entanglement wedge.

If the entanglement wedge W_{AB} becomes connected as shown in Fig. 2, a series of deformations $\tilde{A}(\lambda)$ and $\tilde{B}(\lambda)$ corresponding to the unitary transformations $\mathcal{U}_{\overline{AB}}$ still keep the boundary of $\tilde{A}(\lambda)\tilde{B}(\lambda)$ invariant. One of the examples is shown in Fig. 2. In this case the curve of $\tilde{A}(\lambda_n)$ would never possible become connected, since the deformation should never transcend the extremal surface γ_{AB}. Suppose Σ_{AB} is the extremal surface as well as minimal area with the end points on the extremal surface γ_{AB}.

Therefore, to get the minimal value of $S_{A\tilde{A}}$ one could construct a series of deformations such that the end points of $\tilde{A}(\lambda_\infty) = \lim_{n\to\infty} \tilde{A}(\lambda_n)$ coincide with the ones of Σ_{AB}. In this limit we would have the minimal value of $S_{A\tilde{A}}$ which is given by

$$S_{A\tilde{A}(\lambda_\infty)} = \frac{\Sigma_{AB}}{4G}. \tag{12}$$

In above discussion we only focus on two intervals case, but it is straightforward to generalize the argument to more complicated cases. We have given an explanation of the holographic EoP in the context of surface/state correspondence. Full proof of holographic EoP formula relies on the explicit construction of the unitary operations associated with the bulk surface deformations.

Recently the authors in [38] give a derivation of holographic EoP in the context of the bit threads formulation [39]. The bit threads formulation reforms the RT formula by introducing the vector field v in the bulk. The different bulk geometry would have different field v that produces the RT formula. In the approaches by surface/state correspondence the purifications are associated with some bulk surfaces by smooth deformation of the boundary. In the bit threads formulation the purifications are related to the bulk vector field v [38]. The basic ideas underlying the two approaches are very different. But both the bulk surfaces and the constrained bulk vector field v catch some essential features of the bulk geometric states. It would be an interesting direction to see whether these two presentations of CFT states are affinitive.

4. Projection operator and EoP in CFT

Another interesting question is what kinds of unitary operators $\mathcal{U}_{\overline{AB}}(\psi)$ would produce the minimal value of $S_{A\tilde{A}}$. We should note that the unitary operator is not unique, if $\mathcal{U}_{\overline{AB}}(\psi_M)$ is one, so are the operators $\mathcal{U}_{\tilde{A}}\mathcal{U}_{\overline{AB}}(\psi_M)$ and $\mathcal{U}_{\tilde{B}}\mathcal{U}_{\overline{AB}}(\psi_M)$. It is still an open question whether the operator is unique up to the above gauge.

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[3] The original state corresponding to $A\tilde{A}$ is a mixed state, since it is a part of a closed surface, i.e., AdS boundary. Under a series of unitary transformations, the state would approach to a pure state. This is consistent with the intuition of purification process.

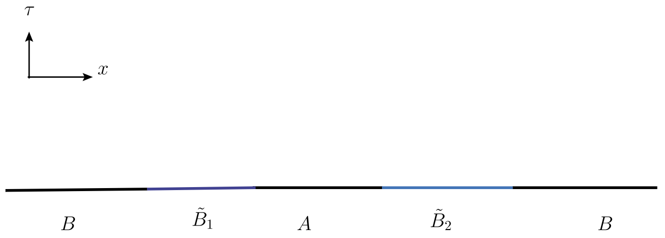

Fig. 3. The post measurement state in region $\tilde{B} := \tilde{B}_1\tilde{B}_2$ (blue line) can be represented by path integral on the lower half plane with two slits.

In this section we will discuss one special operator belonging to the algebra $\mathscr{R}(\overline{AB})$, that is the projection operator in CFT. The projection operator in 1+1 dimensional CFT was discussed in [33] [34] [40], its holographic explanation and applications can be found in [41]. We focus on the projection operator $\mathcal{P}^{\alpha}_{\overline{AB}}$, which makes the states in the region $\overline{AB}$ fixed by some conformal bases α. For example, for free boson theory, a projection operator with fixed ϕ in region A corresponds to Dirichlet boundary condition on A, which is a conformal boundary.

Note that the projection operators $\mathcal{P}^{\alpha}_{\overline{AB}}$ are not unitary. But as we will show soon the entanglement entropy $S_{A\tilde{A}}$ with $\tilde{B} = \overline{AB}$ is very close to the holographic EoP.

We would follow the results in [40], the author considered the projection operators for two intervals as shown in Fig. 3. The post measurement state $\mathcal{P}_{\tilde{B}}|0\rangle$ can be represented by path integral on the lower half plane with two slits on $\tilde{B}_1$ and $\tilde{B}_2$. Assume the length of the intervals $l_{\tilde{B}_1} = s_1$, $l_{\tilde{B}_2} = s_2$ and $l_A = l$. To calculate Rényi entropy S^n_A for subsystem A in the state $\mathcal{P}_{\tilde{B}}|0\rangle$ we need to evaluate the path integral on the n-sheet surface Σ_n with two slits $\tilde{B}_1$, $\tilde{B}_2$ and branch cut on A. The Rényi entropy is given by

$$S^n_A = \frac{1}{1-n}\log\frac{\mathcal{Z}_{\Sigma_n}}{\mathcal{Z}^n_{\Sigma_1}}, \tag{13}$$

where Σ_1 is the surface with two slits. The entanglement entropy is just $S_A = \lim_{n\to 1} S^n_A$. The partition function $\mathcal{Z}_{\Sigma_n}$ can be calculated throw a conformal mapping $w_n(z)$ from Σ_n to annulus, see Appendix of [34] for the detail of the mapping. For general s_1, s_2, l there are no analytical results. In the limit $l \gg s_1 = s_2 = s$ the result is

$$S_A = \frac{c}{3}\log\frac{l}{s} + ..., \tag{14}$$

where ... denote the contributions from the boundary, which are not related to central charge c. In the limit $l \ll s_1 = s_2 = s$, $S_A = 0$ up to some boundary contributions.

In the limit $s_2 \ll s_1, l$,

$$S_A = \frac{c}{6}\log\frac{l(l+s_1)}{s_1 s_2} + ..., \tag{15}$$

with ... being the boundary contributions.

Now we would like to compare the post measurement entanglement entropy with holographic EoP for AB. In the limit $l \ll s_1 = s_2 = s$, the entanglement wedge of AB becomes disconnected, the holographic EoP is vanishing. The post measurement entanglement entropy is also vanishing up to some boundary contributions.

In the limit $l \gg s_1 = s_2 = s$ or $s_2 \ll s_1, l$ the entanglement wedge of AB should be connected. In the appendix we calculate the holographic EoP for the interval A and B, the result is

$$E_{AB} = \frac{c}{6}\log\Big[\frac{s_1(2l+s_2)}{s_1 s_2} + \frac{2\left(l^2 + l s_2 + \sqrt{l(l+s_1)(l+s_2)(l+s_1+s_2)}\right)}{s_1 s_2}\Big]. \tag{16}$$

We have

$$E_{AB} = \begin{cases} \dfrac{c}{3}\log\dfrac{2l}{s}, \text{ in the limit } l \gg s_1 = s_2 = s \\ \dfrac{c}{6}\log\dfrac{4l(l+s_1)}{s_1 s_2}, \text{ in the limit } s_2 \ll s_1, l. \end{cases}$$

Comparing with the results after measurement (14) and (15) in the same limit, we find that they are same. Here we ignore the difference $\frac{c}{3}\log 2$, since for the cases we consider the constant is quite small, e.g., $\frac{l}{s} \gg 2$ in the limit $l \gg s_1 = s_2 = s$.

5. Disentanglement and EoP

If the region $\tilde{A}$ is near A as shown in Fig. 1 or Fig. 2, the process of minimal process of EoP can be seen as a unitary disentanglement operations working on the region $\tilde{A}\tilde{B}$. The role of the projection operator on $\tilde{B}$ is to reduce the entanglement between $A\tilde{A}$ and $B\tilde{B}$. The projection operators $\mathcal{P}^{\alpha}_{\overline{AB}}$ totally disentangling $\tilde{A}$ from $\tilde{B}$. One may guess the post measurement state may be very near to the state that produce the EoP, i.e., there exists an unitary operation $\mathcal{U}_{\overline{AB}}(\psi_M)$ such that in the state

$$|\psi\rangle_M := \mathcal{U}_{\overline{AB}}(\psi_M)|0\rangle, \tag{17}$$

$S_{A\tilde{A}}$ is equal to the holographic EoP result (16). But it is shown in a recent paper written by the author that this is not right [43] for the case that the intervals A and B are close to each other. It is straightforward to generalize the argument in [43] to our case. Further, we can consider the perturbation state $e^{i\delta\mathcal{H}_{\overline{AB}}}|\psi\rangle_M$, where $\mathcal{H}_{\overline{AB}}$ is any hermitian operators in the region $\overline{AB}$, δ is some dimensionless parameter. The condition that $S_{A\tilde{A}}$ is minimal in the state $|\psi\rangle_M$ would give a constraint on the modular Hamiltonian $K_{A\tilde{A},M}$ in the state $|\psi\rangle_M$ [43],

$$[K_{A\tilde{A},M}, \mathcal{O}_{\tilde{A}}] = 0, \tag{18}$$

for *any* operators $\mathcal{O}_{\tilde{A}} \in \mathcal{R}(\tilde{A})$. This means the operator $K_{A\tilde{A},M} \in \mathcal{R}'(\tilde{A})$, where $\mathcal{R}'$ denotes the algebra commutating with $\mathcal{R}$. By using the Haag duality relation [4]

$$\mathcal{R}'(\tilde{A}) = \mathcal{R}(\tilde{A}'), \tag{19}$$

where $\tilde{A}'$ denotes the complementary part of $\tilde{A}$, we conclude that the modular Hamiltonian $K_{A\tilde{A},M}$ can only be expanded as operators located in the region A.

The modular Hamiltonian $K_{A,\mathcal{P}}$ of A in the state $\mathcal{P}^{\alpha}_{\overline{AB}}|0\rangle$ can be written as an integral over the local stress energy tensor in the region A by using the method in [44]. But we can't see whether the two modular Hamiltonian have some relations.

To see more difference between the post measurement state $\mathcal{P}^{\alpha}_{\overline{AB}}|0\rangle$ and $|\psi\rangle_M$ we turn to the holographic explanation of these states. It can be expected that not all of the states in the set of purifications $\mathcal{H}_{\psi}$ (9) can be described by some classical geometry. The arguments involved of holography implicitly assume we only deal with the states with geometry dual. For AdS$_3$/CFT$_2$ the bulk geometry can be described by the Bañados geometry,

$$ds^2 = \frac{dz^2 + dwd\bar{w}}{z^2} + L(w)dw^2 + \bar{L}(\bar{w})d\bar{w}^2 + z^2 L(w)\bar{L}(\bar{w})dwd\bar{w}, \tag{20}$$

where $L(w)$ and $\bar{L}(\bar{w})$ are holomorphic and anti-holomorphic functions. The functions $L(w)$ and $\bar{L}(\bar{w})$ are expected to be associated with the expectation value of boundary stress energy tensor in some states ρ by $L(w) = -\frac{6}{c}tr(\rho T(w))$ and $\bar{L}(\bar{w}) = -\frac{6}{c}tr(\rho \bar{T}(\bar{w}))$, where the central charge c is related to Newton constant G by $c = \frac{3}{2G}$.

For the states $|\psi\rangle$ in the set $\mathcal{H}_\psi$ we have

$$L(x) \propto \langle\psi|T(x)|\psi\rangle = 0, \tag{21}$$

if $x \in A$ or B. But $\langle\psi|T(x)|\psi\rangle$ is non-vanishing for $x \in \tilde{A}$ or $\tilde{B}$ depending on the details of the unitary operations $\mathcal{U}(\psi)$. This at least should determine the metric near the AdS boundary $z \sim 0$.

In the Appendix B we review the geometric dual of the post measurement state $\mathcal{P}^{\alpha}_{\overline{AB}}|0\rangle$ for the special case $\tilde{B}_1 = [-q, q]$ and $A = [q, q+l]$. The metric on the time slice $\tau = 0$ is given by

$$ds^2|_{\tau=0} = \frac{dz^2}{z^2} + \frac{\left[4\left(q^2-x^2\right)^2 - 3q^2z^2\right]^2 dx^2}{16z^2\left(q^2-x^2\right)^4}. \tag{22}$$

Note that the above metric is divergent at the point $x = q$ or $-q$. It of course doesn't satisfy the expected form of dual geometry of $|\psi\rangle_M$. When we calculate the holographic entanglement entropy of the subsystem A in the metric (22) by using RT formula. It seems the minimal surface doesn't have some direct relations to the minimal cross of entanglement wedge Σ_{AB} in the holographic EoP formula.

In conclusion the post measurement state *cannot* be a candidate of $|\psi\rangle_M$. We cannot see the direct relation between projection operators and EoP calculations. It seems the only similarity is the projection operators do reduce the entanglement between $\tilde{A}$ and $\tilde{B}$. The minimization procedure of EoP can also be seen as a process to find a series of unitary disentanglement operations such that $S_{A\tilde{A}}$ reaches its minimal value. In this respect the projection operators can be used as a way to estimate the amount of disentanglement between $\tilde{A}$ and $\tilde{B}$. But as shown in [43] one could also find other states that may produce the holographic EoP result such as the joint local quench state [45]. In this sense the projection operation seems to be not too relevant to the calculation of EoP.

6. Discussions

In this paper we discuss on the calculation of EoP in CFTs. The first step for the calculation is to construct the set of purifications. By using the Reeh-Schlieder theorem we show that the set of purifications can be associated with some unitary operations on the region $\overline{AB}$ as long as the state of total system is *cyclic*. Further, the unitary operations in CFTs can be related to some bulk surface deformations in the context of surface/state correspondence. With this we explain the holographic EoP formula. Of course, since the surface/state correspondence is itself an conjecture based on the tensor network, this is not a solid result. But this can be seen as consistent check between surface/state correspondence and holographic EoP. It may help us understand more on the geometric dual of CFT states. We also find under some projection one could produce the holographic EoP results in some special case. Even though this seems to suggest the post measurement state is associated with the states $|\psi\rangle_M$, we show that their similarity is only the disentanglement between $\tilde{A}$ and $\tilde{B}$. Their difference is obvious, we show these by some holographic argument.

Our discussions are mainly for vacuum state, but it is straightforward to generalize to other cyclic state, such as the states on which the translation group acts holomorphically [36]. For mixed state in 1+1 dimensional CFT the thermal state is conformal equal to the vacuum state, our discussions may be generalized to that case. This may fail for non-entangled state, such as the boundary state in CFT [46] [47].

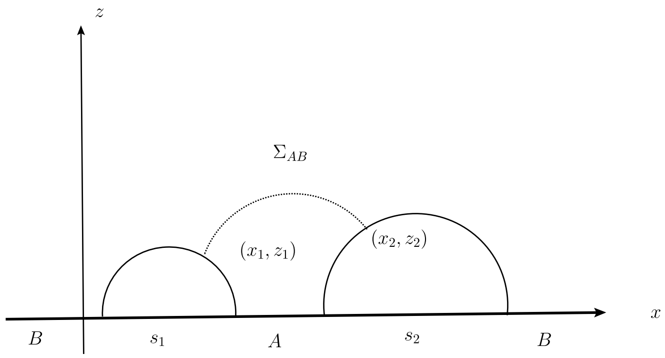

Fig. 4. Calculations of holographic EoP for two interval A and B with $l_A = l$ and the distances between A, B are s_1 and s_2.

Our explanation of holographic EoP only includes the states that can be described by geometry in the bulk. At least in 2D CFT it is expected there are many states that cannot be dual to a classical geometry [48]. So the explanation is only true for the class of geometric states.

Even though in this paper we haven't successfully calculate the EoP in 1+1 dimensional CFTs, our **Corollary 1** is an interesting approach to this problem. At present the difficulty to make the calculation is how to calculate the EE of $A\tilde{A}$ in the state belonging to $\mathcal{H}_\psi$. On the one hand, it is still not clear what kinds of unitary operations we should choose and why they are important. On the other hand we should find a method to calculate the EE of such excited states. These two problems are both very difficult. It seems to be impractical for directly calculating all the possible states in $\mathcal{H}_\psi$. But we prefer the relation (18) may be a break-through point on this problem. It actually gives a very strong constraint on the state $|\psi\rangle_M$. It is possible that one may prove the holographic EoP formula by using (18) and other relevant tools in AdS/CFT. We will explore this in the near future.

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Appendix A. Holographic EoP of two intervals in 1+1 dimensional CFT

We will derive the holographic EoP of two intervals in 1+1 dimensional CFT in this section. To compare with the result in the main text we choose A and B as shown in Fig. 4. We only plot the case when AB has connected entanglement wedge in Fig. 4. To calculate holographic EoP we need to compute the length of the entanglement wedge cross section, i.e., Σ_{AB} in Fig. 4. The minimal length condition leads to the curve Σ_{AB} is perpendicular to the extremal surface of entanglement wedge at the points (x_1, z_1) and (x_2, z_2). With some simple calculations we get

$$z_1 = \frac{s_1\sqrt{l\,(l+s_1)\,(l+s_2)\,(l+s_1+s_2)}}{s_1^2+2l\,(l+s_2)+s_1\,(2l+s_2)}$$
$$z_2 = \frac{s_2\sqrt{l\,(l+s_1)\,(l+s_2)\,(l+s_1+s_2)}}{2l^2+2l\,(s_1+s_2)+s_2\,(s_1+s_2)}, \tag{23}$$

and the equation of the curve Σ_{AB}, $(x-x_0)^2+z^2=z_*^2$ with

$$x_0 = \frac{l(s_2-s_2)}{2(2l+s_1+s_2)}$$
$$z_* = \frac{\sqrt{l\,(l+s_1)\,(l+s_2)\,(l+s_1+s_2)}}{2l+s_1+s_2}. \tag{24}$$

The length of Σ_{AB} is

$$L = \log\left[\frac{s_1\,(2l+s_2)}{s_1s_2} + \frac{2\left(l^2+ls_2+\sqrt{l\,(l+s_1)\,(l+s_2)\,(l+s_1+s_2)}\right)}{s_1s_2}\right]. \tag{25}$$

Appendix B. Holographic explanation of the post measurement state

According to the results of last section one may guess the post measurement state can be a candidate of the purifications that produce the minimal value of $S_{A\tilde{A}}$. To better understand the possible relation between post measurement state and holographic EoP in this section we will use holographic method to explain the post measurement state. For simplicity, we will set $s_2=0$, $s_1=2q$ and take the midpoint of $\tilde{B}_1$ as the origin of the coordinate. Therefore, we have $\tilde{B}_1=[-q,q]$ and $A=[q,q+l]$.

With the conformal transformation

$$\xi = f(w) := \sqrt{\frac{q+w}{q-w}}, \quad \bar{\xi} = \bar{f}(\bar{w}) := \sqrt{\frac{q+\bar{w}}{q-\bar{w}}}, \tag{26}$$

the Riemann surface with a slice $\tilde{B}_1$ is mapped to the upper half plane (UHP) $Im(\xi)\geq 0$, where $w := x+i\tau$ and $\bar{w} := x-i\tau$. The UHP has a holographic dual by using the prescription of $AdS/BCFT$ proposed in [42]. In this approach the boundary of the BCFT is extended to a bulk boundary Q, which satisfies the condition,

$$K_{ab}-Kh_{ab}+Th_{ab}=0, \tag{27}$$

where h_{ab} and K_{ab} are the induced metric and the extrinsic curvature of the bulk surface Q. Here we will only consider $T=0$. The bulk surface Q is given by $Im(\xi)=0$. The dual spacetime exists in the region $Im(\xi)>0$, the metric in the bulk is

$$ds^2 = \frac{d\eta^2+d\xi d\bar{\xi}}{\eta^2}, \tag{28}$$

where we set the AdS radius $R=1$, η is the holographic direction. There is a bulk coordinate transformation associated with the conformal transformation (26) on the boundary,

$$\xi = f(w) - \frac{2z^2 f'^2 \bar{f}''}{4f'\bar{f}'+z^2 f''\bar{f}''},$$
$$\bar{\xi} = \bar{f}(w) - \frac{2z^2 \bar{f}'^2 f''}{4f'\bar{f}'+z^2 f''\bar{f}''},$$
$$\eta = \frac{4z(f'\bar{f}')^{3/2}}{4f'\bar{f}'+z^2 f''\bar{f}''}. \tag{29}$$

Using the above coordinate transformation we could get the metric of the spacetime dual to the post measurement state. Let's see the metric on the time slice $\tau=0$,

$$ds^2|_{\tau=0} = \frac{dz^2}{z^2} + \frac{\left[4\left(q^2-x^2\right)^2-3q^2z^2\right]^2 dx^2}{16z^2\left(q^2-x^2\right)^4}. \tag{30}$$

Note that the metric is divergent at the point $x=q$ or $-q$. This divergence may be attributed to the "sharp" projection operator. One needs some regularization to make the operator well-defined. One could read the energy density from the metric (30), it is non-vanishing for the region $x\in A$. But for the purifications (9) it is expected the energy density in the region A would be zero.

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