Holographic renormalization for Lifshitz spacetimes

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This dissertation sets up a foundation for studying physical systems in the vicinity of a strongly-coupled UV fixed point with non-relativistic scaling via holography. The dual gravitational description is defined through imposing asymptotic boundary conditions that are consistent with *Lifshitz*-type scaling symmetry: $t \to b^z t$, $x \to bx$. Special attention is given to the case in which the dynamical exponent is $z = 2$, i.e. when the system shows a quadratic dispersion. Such a system exhibits a Weyl-type anomaly in 3 spacetime dimensions, which is analyzed in detail.

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Contents

Introduction

Many advances in theoretical physics originate from adapting concepts in a different context from where they were initially developed. This is especially true in string theory, starting from the advent of string theory itself as a possible theory of quantum gravity.[1] Prime examples of such 'cross-pollinations', so to speak, arise from what are known as a *dualities*, which have become increasingly ubiquitous in theoretical physics. What we mean by a duality is, in short, having more than one distinct way of describing one physical system; the different descriptions are said to be dual. Theoretical descriptions (or simply 'theories') that are dual in this way may appear vastly different upon first inspection, as is the case for the type of dualities that we consider in this dissertation, namely *holographic dualities*. These dualities admit a translation between a (quantum) theory of gravity and a quantum field theory without gravity in one less dimension. The gravity theory can be viewed as a higher-dimensional *hologram* of its dual quantum field theory.

Holographic principle. The first indication of the holographic nature of spacetime came from the study of black holes. A similarity was observed between the laws of black-hole mechanics and the laws of thermodynamics. In particular, the second law of thermodynamics is reflected by the 'one-way' character of a black hole's event horizon that leads to an ever-increasing black-hole mass (at the classical level) [4]. The area of the horizon seemed to play the role of thermodynamic entropy, which lead to the proposition that $S \propto A$, where S is the black-hole entropy and A is surface area of the horizon. The *area-law* behavior of black-hole entropy was surprising, because the intuition from statistical physics told us that the entropy of a typical thermal system scales as its *volume* rather than some surface area. The identification $S \propto A$ became more serious when the temperature of black holes was computed. Namely, a black hole exhibits black-body radiation with a temperature that can be computed by studying local quantum fluctuations near the horizon [5]. Knowledge of the black-hole temperature allows one to determine the proportionality constant in $S \propto A$ via the first law of thermodynamics. The

[1] String theory was initially developed in the S-matrix formulation of the strong nuclear force.

result is that the black-hole entropy is given by:

$$S = \frac{k_B c^3}{4\hbar G} A \qquad \text{(I.1)}$$

In units where $c = \hbar = k_B = 1$, this reduces to $S = A/4G$. The fact that this *area law* ties together very different topics in physics is reflected by the fact that the proportionality constant involves the speed of light c (relativity), Planck's constant $\hbar$ (quantum theory), Boltzmann's constant k_B (thermal/statistical physics) and Newton's constant G (gravity). Entropy measures the microscopic degeneracy associated with a macroscopic state. Similarly, entropy can be seen as that number of microscopic degrees of freedom that make up a given macroscopic state. The area law has lead to the more general expectation that all microscopic degrees of freedom in a theory of gravity organize themselves in such a way that together they form a macroscopic hologram, as it were [6]. This expectation was dubbed the holographic principle.

Concrete example. The first concrete example where it was possible to actually count the number of microscopic degrees of freedom of a black hole configuration came from the study of D-branes in string theory [7]. The holographic structure of this analysis became clear not long after, due to the advent of the AdS/CFT correspondence. The type of quantum field theories that are dual to gravity theories with anti-De Sitter (AdS) boundary conditions are conformally-invariant field theories (CFT). The first concrete example of AdS/CFT relates string theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background to maximally supersymmetric $SU(N)$ Yang–Mills theory (which is conformally invariant) [8]. The symmetries of the CFT (conformal symmetry and the $SO(6)$ R-symmetry) are translated to isometries of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry.

Simplifying assumptions. The $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality becomes manageable when we take the rank N of the gauge group very large (keeping the 't Hooft coupling $\lambda = N g^2_{\mathrm{YM}}$ finite), which has the effect that the dual gravity theory becomes classical. A further simplification can be made by focusing on situations where the 't Hooft coupling is large, which means we can safely ignore α' corrections on the gravity side. In these limits we can use the AdS/CFT duality to translate a strongly-coupled CFT (at large N) to an ordinary classical (super)gravity theory. This relation is quite exciting, because field-theory computations are notoriously hard to do at strong coupling. Holography thus delivers a new alternative framework for analyzing strongly-coupled systems with relative ease.

Top-down and bottom-up. Inspired by the above example, one might attempt to use holographic dualities to describe strongly-coupled systems that can be realized in a laboratory. A precise gravity dual is in general lacking, but one can nevertheless take a broader perspective and study universal properties instead. The ways one may try to proceed can be divided into two categories: top-down and bottom-up. In top-down approaches one tries to build a model that emulates some general properties of the system under consideration by using the building blocks that are available in string theory (typically brane constructions). In a bottom-up approach one usually starts with Einstein gravity and adds whatever matter is necessary to mimic the desired setup. The advantage of a top-down approach is that there is a better understanding of both sides of the duality, while in bottom-up approaches the field-theory dual is generally unknown. The advantage of a bottom-up approach is that one has more freedom in constructing a model, where in top-down approaches one is restricted to the building blocks available in string theory.

Effective large *N*. Throughout this work we take a bottom-up approach. We continue to consider the regime where the gravity theory is evaluated at a saddle point. The first conceptual issue arises when we ask the question what this classical limit translates to in the dual field theory. Namely, we cannot give a precise meaning to the 'large-N' limit. Nevertheless, we imagine having taken some effective large-N limit. In either the top-down or bottom-up case, we interpret large N as having a 'large number of microscopic degrees of freedom.'

Role of the radial coordinate. Since holographic dualities relate two theories in a different number of dimensions, a natural question to ask is what happens to the gravity theory's additional dimension. The answer is that the radial coordinate has a natural interpretation as the physical mass scale on the field-theory side. Interestingly, the radial coordinate and its dual mass scale are related in such a way that large radii probe short distances in the dual field theory. Thus, besides interchanging strong/weak coupling, holographic dualities also interchange UV/IR. The boundary conditions are imposed on the (conformal) boundary of the geometry at large radius, which means that they correspond to UV boundary conditions in the field theory. The CFT can thus be viewed as associated with a UV fixed point. At the UV fixed point, we only allow relevant or at most marginal operators in order to be able to satisfy the asymptotic boundary conditions.[2] Allowing for marginal operators means that the geometry does not need to be AdS all the way into the interior; it only needs to asymptote to AdS at large radii.

[2] An irrelevant operator would induce a flow away from the UV fixed point when the mass scale is increased, which means that the asymptotic geometry would be altered.

Symmetries. In building an effective model, the first thing one does is to make sure that the symmetries on either side of the purported duality match. As usual, symmetries imply the existence of conserved (Noether) charges, which are 'measured' at infinity. Because the conserved charges are 'measured' at the large radii, the symmetries are put in the system by imposing appropriate UV boundary conditions. A particular symmetry that is part of the conformal algebra is a dilatation, which rescales time and space as $x^a \mapsto bx^a$. From the line element of (Poincaré) AdS,[3]

$$ds^2 \;=\; dr^2 + \mathrm{e}^{2r}\, \eta_{ab}\, dx^a dx^b\,, \tag{I.2}$$

it is obvious that a dilatation can be generated by preforming a radial rescaling: $\mathrm{e}^r \mapsto b\,\mathrm{e}^r$. In this way, dilatations can be promoted to isometries, as $(x^a, \mathrm{e}^r) \mapsto (bx^a, \mathrm{e}^r/b)$.

Precision holography. The quantum field theory can be seen as 'living' on the asymptotic boundary of the bulk (super)gravity field configuration. Holographic dualities provide a *dictionary* that enables translation between bulk fields and QFT sources and operators. In particular, the asymptotic values of the bulk fields are identified with sources in the dual field theory. Let us denote all bulk fields collectively by ϕ. The type of holographic dualities that we consider can then be summarized in the following form [9]:

$$W_{\text{fld.th.}}[J = \phi_{(0)}] \;=\; S_{\text{grav}}[\phi \xrightarrow{r\to\infty} \phi_{(0)}]\,. \tag{I.3}$$

On the left hand side we have the field-theory generating function at strong coupling, while on the right-hand side we have the gravity action evaluated on a solution ϕ that asymptotes to $\phi_{(0)}$ as $r \to \infty$.[4]

Holographic dictionary in AdS/CFT. The *master equation* (I.3) enables us to compute CFT correlation functions by taking ϕ-derivatives of the on-shell gravity action. Of particular interest are the expectation values of conserved quantities, which give us important information about the state of the system under consideration. For instance, the stress tensor T^{ab} contains information about the energy, momentum and stress of the system; its expectation value can be computed in holography as follows:

$$\langle T^{ab} \rangle \;=\; \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g_{ab}} \;=\; \frac{2}{\sqrt{g}}\frac{\delta S_{\text{grav}}}{\delta g_{ab}}\,. \tag{I.4}$$

The boundary metric g_{ab} appears in the AdS bulk metric as

$$ds^2 \;=\; dr^2 + g_{ab}(r,x)\,\mathrm{e}^{2r}\, dx^a dx^b \tag{I.5}$$

[3]We use Gaussian normal coordinates and set the AdS curvature length to one.

[4]The bulk fields can be rescaled in such a way that the leading mode $\phi_{(0)}$ has a finite limit as $r \to \infty$.

More precisely, the boundary metric is the (finite) limit $\lim_{r\to\infty} g_{ab}(r,x)$. This identification is the reason why it is often said that the field theory 'lives' at the (conformal) boundary of the dual geometry. A typical system will not exhibit conformal invariance, so one often wants to move away from the conformal fixed point in the UV. This can be achieved by adding marginal operators, which in holography is done by adding appropriate matter in the dual gravitational picture. For instance, adding a scalar field with mass $m^2 = \Delta(\Delta - d)$ introduces a scalar operator of dimension Δ.[5] Another common example is adding a $U(1)$ gauge field in the bulk, which introduces a global $U(1)$ current in the CFT. We summarize these three most common examples schematically:

$$g_{ab} \leftrightarrow T^{ab}\,, \qquad A_a \leftrightarrow J^a\,, \qquad \varphi \leftrightarrow \mathcal{O}_\varphi\,. \tag{I.6}$$

Divergences and counterterms. The relation (I.3) is quite powerful, because it allows one to compute correlation functions in a strongly-coupled quantum field theory by taking derivatives of the on-shell gravitational action with respect to the leading modes $\phi_{(0)}$. There is, however, a serious issue with (I.3) as it stands now. That is, the on-shell action and the correlation functions it generates are divergent due to the infinite volume of the geometry. Such infinite-volume divergences translate to UV divergences in the dual field theory. Of course, it is to be expected that the (bare) generating functional is UV divergent. We know that in a renormalizable quantum field theory we can add local counterterms to remove such divergences. The goal of holographic renormalization is to find such counterterms on the gravity side of the duality. We discuss holographic renormalization in detail in Chapter 1.

A look ahead. The main goal of this work is to study systems that exhibit scale invariance but are *not* Lorentz invariant. In particular, we focus on a specific class of theories that are invariant under rescalings that treat the time coordinate differently from spatial ones, which is incompatible with Lorentz symmetry. As we mentioned above, we are generally interested in describing universal properties, such as transport properties, of strongly-coupled systems via holography. Before one can hope to perform such studies, however, one needs to make sure that the questions are well posed in the sense that the quantum theory be properly renormalized. This work mainly deals with the latter issue.

Universality is an important concept in the study of second-order phase transitions. We briefly review some of these basic concepts below before we dis-

[5] Δ is the larger of the two roots of $\Delta(\Delta - d) = m^2$.

cuss the Lorentz-incompatible scaling.

Universal scaling. Consider some observable $\mathcal{O}(g_a)$, as a function of the dimensionless coupling constants g_a. For simplicity, we take $a = 1, 2$. The characteristic fall-off of (equal-time) two-point correlators, known as the *correlation length*, diverges at a (quantum) critical point. The removal of the characteristic length scale allows for the system to become scale invariant; it is a renormalization-group (RG) fixed point. In the vicinity of such a fixed point one can study the scaling properties of some generic observable $\mathcal{O}(g_a)$ as follows. Let $b > 1$ be the standard RG scaling parameter[6] and let us denote the scaling of the couplings g_a by λ_a. Also, we parametrize our couplings in such a way that $g_a = 0$ at the fixed point. Suppose that g_1 is a relevant coupling, then we can use scale invariance to reduce the number of arguments of $\mathcal{O}$ by one:

$$\begin{aligned} \mathcal{O}(g_1, g_2) \; &= \; b^{\Delta}\, \mathcal{O}(b^{\lambda_1} g_1, b^{\lambda_2} g_2) \; = \; g_1^{-\Delta/\lambda_1}\, \mathcal{O}(g_{12})\,, \\ g_{12} \; &\equiv \; g_2/g_1^{\lambda_2/\lambda_1}\,. \end{aligned} \tag{I.7}$$

We used scale invariance to set $b = g_1^{-1/\lambda_1}$ and we introduced the notation $\mathcal{O}(g) \equiv \mathcal{O}(1, g)$. The exponent Δ/λ_1 is known as a *critical exponent.* Critical exponents are universal in the sense that they are independent of the precise details of the theory; they only depend on dimensionless parameters such as the number of dimensions or field components. A *universality class* consists of a set of critical exponents, which capture the universal physical properties of a system at a critical point. Many different theories can give rise to the same set of critical exponents, so many different systems can belong to the same universality class.

Dynamical exponent. Many critical exponents are defined as the scaling of observables relative to the scaling of the correlation length. A particular example of such a critical exponent that we come across repeatedly throughout this work is the dynamic critical exponent denoted z. It can be defined as follows. Let us take the above example again. The correlation length ξ satisfies the scaling (I.7) with $\Delta = -1$:

$$\xi(g_1, g_2) \; = \; g_1^{-\nu}\, \xi(g_{12})\,, \tag{I.8}$$

where ν is standard notation for the critical exponent associated with the correlation length (in this case, $\nu = -1/\lambda_1$). A typical observable is the

[6]The rescaling is done after the 'fast modes' are integrated out.

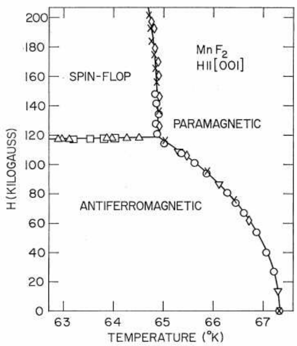
energy E,[7] whose scaling relative to ξ defines the dynamical exponent z:

$$E(g_1, g_2) \;=\; g_1^{z\nu}\, E(g_{12})\,. \tag{I.9}$$

Let t be the time coordinate conjugate to E. At the critical point, the system is invariant under rescaling $(\mathbf{x}, t) \to (b\,\mathbf{x}, b^z t)$ with $z \neq 1$, which is often referred to as *Lifshitz scaling* in the high-energy community. In the condensed matter community, however, Lifshitz scaling typically refers to spatial anisotropy rather than space/time anisotropy. The attribute 'Lifshitz' has its origin in the effective description of a finite-temperature phase diagram that includes a tricritical point called a Lifshitz point. We briefly review this below.

Lifshitz point. A Lifshitz point is a special point in a phase diagram that separates three distinct phases, which typically consist of a high-temperature disordered phase and low-temperature ordered phases that are either spatially uniform or modulated, cf. Figure I.1.

As an example, we consider the phase diagram of MnF_2 [10].[8] The paramagnetic phase is the high-temperature disordered phase, the (anti)ferromagnetic phase is the uniformly ordered phase, and the *spin-flop*[9] phase is the spatially modulated phase.



In the effective description of such a phase diagram, the free energy is given in terms of the order parameter φ (e.g. magnetization):

$$F \;=\; \frac{1}{2}\int d^{d-1}x\, dy\, \left((\partial_y\varphi)^2 + c\,(\partial_{\mathbf{x}}\varphi)^2 + \kappa\,(\Delta_{\mathbf{x}}\varphi)^2 + t\varphi^2\right) \tag{I.10}$$

Here, t is typically the reduced temperature and y and $\mathbf{x}$ are the spatial directions that are respectively parallel and perpendicular to some external (magnetic) field. The associated phase diagram is sketched in Figure I.1. An RG transformation rescales the coupling constants such that:

$$RG: \begin{pmatrix} t \\ c \\ \kappa \end{pmatrix} \;\mapsto\; \begin{pmatrix} b^{2\zeta}\, t \\ b^{2(\zeta-1)}\, c \\ b^{2(\zeta-2)}\, \kappa \end{pmatrix} \tag{I.11}$$

[7]For instance, E could be an effective energy range of low-lying excitations accessible to the system.

[8]See also §4.6 of [11].

[9]In a spin-flop phase transition (from the uniform to the modulated phase), the spins switch from parallel to perpendicular alignment to the external magnetic field. Such a transition happens when the external field is tuned to sufficiently high values.

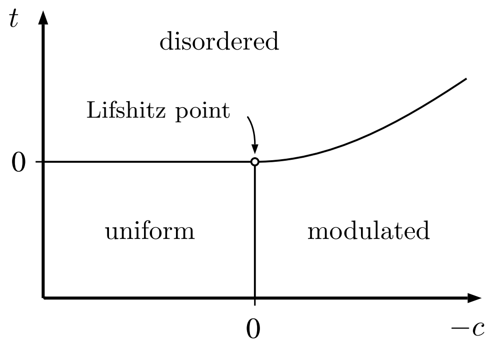


Figure I.1: Generic phase diagram that includes a Lifshitz point. When $c > 0$ the system is in the uniform phase. On the other hand, when $c < 0$ the system is in a modulated phase, because the free energy is minimized when $\varphi \sim \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ with $|\mathbf{k}| > 0$ fixed in terms of the coupling constants.

The Lifshitz point is the fixed point where $c = 0$ and $t = 0$, with relative scaling $\zeta = 2$. The marginal coupling κ parametrizes a line of such fixed points. It is the spatially anisotropic scaling $(\mathbf{x}, y) \to (b\,\mathbf{x}, b^\zeta y)$ that is called Lifshitz scaling. Notice that c and t are relevant couplings, so the Lifshitz point is an unstable fixed point (i.e. related to a second-order phase transition).

Quantum Lifshitz model. To get a feeling for the kind of quantum field theories we attempt to describe holographically we consider the simplest: the *quantum Lifshitz model* [12]. It is defined in analogy to (I.10) at the Lifshitz point. The spatial direction y is now interpreted as the (Euclidean) time direction, which means that ζ is interpreted as the dynamical exponent, $z = \zeta$. The action of the quantum Lifshitz model is given by

$$S \;=\; \frac{1}{2}\int d^2x\,dt\left(-(\partial_t\varphi)^2 + \kappa^2(\Delta\varphi)^2\right) \qquad \text{(I.12)}$$

This action is invariant under Lifshitz-type scaling $(\mathbf{x}, t) \to (b\,\mathbf{x}, b^z t)$ with $z = 2$. The quantum Lifshitz model received considerable interest due to its relation to 2D CFT's [13]. It also serves as an effective description for systems such as the Rokhsar-Kivelson dimer model [12, 14], which lies in the same universality class as (I.12). We discuss this model in more detail in Chapter 2.

Non-Abelian Lifshitz model. A model that is slightly closer to holographic models is the $z = 2$ analogue of $SU(2)$ Yang–Mills theory, presented in [15]:

$$S = \frac{1}{g^2}\int d^2x\,dt\,\mathrm{tr}\Big(E^i\,\partial_t A_i + A_t\,D_i E^i - \frac{1}{2}\,D_i E_j\,D^i E^j + \frac{1}{2}B^2\Big) + S'\,,$$
$$S' = \frac{1}{g^2}\int d^2x\,dt\,\left(g_1\,E_{ij}E^{ij} + g_2(E_i{}^i)^2\right)\,, \tag{I.13}$$

where E^i is the canonical momentum conjugate to A_i, $B = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$ is the magnetic component of the field strength, $E_{ij} \equiv \mathrm{tr}(E_i E_j)$ is just a convenient short-hand notation, and D_i is the gauge-covariant derivative. The above action is invariant under $SU(2)$ gauge transformations and it is conjectured to be in the same universality class as certain spin-$\frac{1}{2}$ systems in which the spins are organized on a honeycomb lattice; these are $z = 2$ scale invariant. At tree level, the action (I.13) describes a $z = 2$ fixed point. However, if one takes loop corrections into account one finds that it is not a proper RG fixed point at weak coupling, because the couplings g_1 and g_2 drive a flow away from the classical fixed point, cf. [15]. It was suggested that the fixed point may be stable at strong coupling, but the analysis could not be performed due to the fact that the perturbative approach becomes unreliable. Systems that are effectively described by a strongly-coupled fixed point at $z = 2$ are typically the type of theories we attempt to describe holographically.

Lifshitz holography. Lifshitz holography aims to describe a class of "large-N" quantum field theories at a Lifshitz-invariant strongly-coupled UV fixed point. Besides the Lifshitz scaling symmetry, Lifshitz theories are also invariant under spatial and time-translations, as well as spatial rotations. These symmetries comprise the Lifshitz algebra. Lifshitz spacetime [16] is constructed in such a way that its isometry algebra is the Lifshitz algebra, in much the same way that the isometry algebra of AdS is the conformal algebra (modulo some subtleties in AdS_3). The line element in Gaussian normal coordinates is an obvious generalization of Poincaré AdS:

$$ds^2 = dr^2 - \mathrm{e}^{2zr}dt^2 + \mathrm{e}^{2r}d\mathbf{x}^2 \tag{I.14}$$

The Lifshitz symmetry algebra is realized by the isometries that are generated by $D = zt\partial_t + x^i\partial_i - \partial_r$, $P_i = \partial_i$, $H = \partial_t$ and $R_{ij} = x_i\partial_j - x_j\partial_i$.

Lifshitz needs additional structure. Lifshitz spacetime is not a solution to the vacuum Einstein equation, so it must be supported by additional structure. The simplest model that admits a Lifshitz solution involves the addition of a massive vector field to the Einstein–Hilbert action, which is known as the Einstein–Proca theory or massive-vector theory. This theory was proposed in the context of holography in [17] and it can be derived from the two-form/three-form theory initially used in [16] by integrating out one of the form fields. We discuss the massive-vector theory in detail in Chapter 2. For other models based on dimensional reduction of the well-established AdS_5 holography, see e.g. [18–20].

IR instabilities. Some issues have been raised concerning the stability of Lifshitz spacetime (I.14). These are mainly related to short-distance instabilities in the interior of the geometry. Although Lifshitz spacetime has no curvature singularities, it was noticed that the geometry suffers from diverging tidal forces as felt by a local observer near $\mathrm{e}^r \approx 0$ [21, 22]. In one holographic model known as the Einstein–Maxwell–dilaton model, a resolution of this issue was proposed by the emergence of an AdS_2 throat in the deep interior [23]. Our stance is that we do not expect these types of singularities to play a crucial role, as the Lifshitz geometry is only presumed sensible at large radii. In other words, we assume that we can impose physically sensible IR boundary conditions that are free of these singularities. In Chapter 3 we see that the Lifshitz geometry naturally flows to AdS, which resolves this issue in the massive-vector model in three boundary dimensions with $z = 2$.

Trapped modes. Another issue that was raised more recently suggests that constructibility of the geometry from the boundary conditions imposed at infinity is hindered due the existence of 'trapped modes' [24]. These modes are are related to null geodesics that cannot reach infinity if they carry some momentum in the transverse spatial directions. In [24] it was also suggested that a possible resolution of this issue might be to introduce a cut-off for the transverse momenta. An explicit satisfactory resolution (if it exists) of this issue is unknown at this stage.

The Lifshitz-type Weyl anomaly. It is well known that Weyl symmetry is anomalously broken in even-dimensional CFT's. In Chapter 1 we compute these Weyl anomalies holographically. It turns out that there is a similar kind of anomaly in Lifshitz-type theories when the dynamical exponent is equal to the number of spatial dimensions, $z = d_s$. We compute this anomaly for the case $z = d_s = 2$ in Chapter 2. In that case, the anomaly is given in terms of two central charges. Knowing these central charges gives us some insight into the universal properties of the theory. For instance, in the relativistic case the central charge completely fixes the free energy at high temperatures in two dimensions. Also, in both two and four dimensions, the central charges control the universal terms in the entanglement entropy. We compute the central charges both for the quantum Lifshitz model (I.12) as well as for the purported class of Lifshitz-type theories that are holographically dual to the massive vector theory. Moreover, a first step is made in identifying the universal contribution to the entanglement entropy for the $z \neq 1$ case in Section 3.1.

Lifshitz-to-AdS holographic RG flow. Consider again the effective description of a Lifshitz point (I.10). Let us consider moving slightly away from the Lifshitz point directly to the left, i.e. $t = 0$ and $c > 0$. The system will then flow to an isotropic $\zeta = 1$ fixed point, where $SO(d)$ symmetry emerges. In Chapter 3 we discuss a similar flow in holography. It so happens that the massive-vector theory contains a marginally relevant operator that induces a flow from the UV fixed point with dynamical exponent $z = 2$ to an IR fixed point with $z = 1$ (AdS). We discuss the role of the marginally relevant operator in the context of holographic renormalization. In the process we resolve some issues that were present in the literature concerning this marginally relevant operator.

Entanglement *c*-function. Besides properly renormalizing the massive-vector theory, we also check that the effective number of degrees of freedom decreases along the Lifshitz-to-AdS flow, as expected on general grounds. As a measure for the effective number of degrees of freedom we take the universal part of the entanglement entropy known as the entanglement c-function, which is easily computed in holography.

Structure of this thesis

This dissertation is organized as follows. In Chapter 1 we review the different methods of holographic renormalization in chronological order, after which we present a hybrid method that combines all the virtues of the previous models. In Chapter 2 we apply this model of holographic renormalization to spacetimes with Lifshitz-type asymptotics. We also compute the Lifshitz-type Weyl anomaly for $z = d_s = 2$, first for in the quantum Lifshitz model and then in a holographic model. Finally, in Chapter 3 we discuss the marginally relevant operator that naturally appears in the holographic massive-vector model.

Chapter 1

Holographic Renormalization

As we mentioned before, the type of holographic dualities that we consider follow from the *master equation* that relates the generating functional of the QFT at strong coupling (and 'large N') to the on-shell value of the dual gravitational action [9]:

$$W_{\text{fld.th.}}[J = \phi_{(0)}] \;=\; S_{\text{grav.}}[\phi \xrightarrow{r\to\infty} \phi_{(0)}]\,, \tag{1.1}$$

where we denote all bulk fields collectively by ϕ. In the Introduction we mentioned that the generating functional is UV divergent. The UV divergences translate to infinite-volume divergences in the dual gravitational on-shell action. To get a feeling for how these infinite-volume divergences appear, let us consider a probe scalar field in a fixed AdS background, whose line element is:

$$ds^2 \;=\; dr^2 + \mathrm{e}^{2r}\left(-dt^2 + d\vec{x}^{\,2}\right)\,. \tag{1.2}$$

The action for the probe scalar is:

$$S \;=\; \int_M dr\, d^dx \sqrt{g}\, \left(\partial_a\varphi\, \partial^a\varphi + m^2\varphi^2\right)\,. \tag{1.3}$$

The on-shell action can be written as a surface term by using the field equations:[1]

$$S \;=\; \oint_{\partial M} d\sigma^a\, \varphi\, \partial_a\varphi \;=\; \lim_{r\to\infty} \int_{\Sigma_r} d^dx\, \sqrt{h}\; \varphi\, \partial_r\varphi\,. \tag{1.4}$$

The solution to the scalar field equation $(\Box - m^2)\varphi = 0$ is spanned by two modes: $\varphi(r,x) \;=\; f_\pm(x)\,\mathrm{e}^{\lambda_\pm r}$, where $\lambda_\pm \;=\; -\frac{d}{2} \pm \sqrt{(\frac{d}{2})^2 + m^2}$. Although the Breitenlohner–Freedman stability bound allows for negative values of the mass-squared, $m^2 \;\geq\; -(\frac{d}{2})^2$, we consider $m^2 \;>\; 0$ for simplicity, such that

[1] In fact, it is easy to show that, for free theories, the on-shell action is always of the form $S = \frac{1}{2}\int d^dx\sqrt{h}\,\pi\varphi$ (just take the variation of the on-shell action with respect to φ and set then $\delta\varphi \to \frac{1}{2}\varphi$).

$\lambda_+ > \lambda_- > 0$. It is now clear that it is the *non-normalizable mode* λ_+ is responsible for the divergence ($\sqrt{h} = \mathrm{e}^{d\,r}$):

$$S \;=\; \lim_{r\to\infty} \int_{\Sigma_r} d^dx \; f_+(x)^2 \left(\lambda_+ \mathrm{e}^{(d+2\lambda_+)r} + \ldots\right) \;=\; \infty \tag{1.5}$$

where the ellipses denote terms that are suppressed by powers of e^{-r}.

A counterterm. Of course, it is to be expected that the (bare) generating functional is UV divergent. We know that in a (renormalizable) quantum field theory we can add local counterterms to remove such divergences. The goal of holographic renormalization is to find such counterterms on the gravity side of the duality. For instance, in the above example, we can add the following surface term to the action before taking the $r \to \infty$ limit:

$$S_{\text{c.t.}} = -\int_{\Sigma_r} d^dx\,\sqrt{h}\,\lambda_+\varphi^2 \tag{1.6}$$

The on-shell action then becomes

$$\begin{aligned} S \;&=\; \lim_{r\to\infty} \int_{\Sigma_r} d^dx\,\sqrt{h}\;\varphi\,(\partial_r - \lambda_+)\varphi \\ &=\; (\lambda_- - \lambda_+) \int_{\Sigma_\infty} d^dx\; f_+(x) f_-(x)\,, \end{aligned} \tag{1.7}$$

which allows us to compute the expectation value of the operator $\mathcal{O}$ that is sourced by the scalar field φ. More precisely, the source is the non-normalizable mode $J = f_+$, so

$$\langle\mathcal{O}\rangle \;=\; \frac{\delta W_{\text{fld.th.}}}{\delta J} \;=\; \frac{\delta S}{\delta f_+} \;=\; (\lambda_- - \lambda_+)\,f_-(x)\,, \tag{1.8}$$

Thus, we see that the expectation value of the operator sourced by the non-normalizable mode f_+ is given by the *normalizable mode* f_-.

Asymptotic radial scaling. Notice that the divergence in the on-shell action is removed in (1.7) by the fact that $\partial_r \approx \lambda_+$ asymptotically. It turns out that this property applies more generally, which we explain in §1.2.

Variational principle. The counterterm (1.6) is needed in order to have a well-defined variational principle. Although we implicitly ignored the surface term when we derived the scalar field equation, it is not so clear that we were allowed to do so. The surface term is generated upon varying the action as follows:

$$\begin{aligned} \delta S \;&=\; \int_M dr\,d^dx\,\sqrt{g}\;(\dot\varphi\,\delta\pi + \pi\,\delta\dot\varphi - \delta H) \\ &=\; \int_M dr\,d^dx\,\sqrt{g}\;(\dot\varphi\,\delta\pi - \dot\pi\,\delta\varphi - \delta H) + \oint_{\partial M} d^dx\,\sqrt{h}\;\pi\,\delta\varphi \end{aligned} \tag{1.9}$$

where dots denote differentiation with respect to r and $\pi = 2\dot{\varphi}$ is the canonical momentum conjugate to φ. The surface term diverges as $r \to \infty$, so it is not clear how one can dispose of this term. However, if we add the counterterm to the action, we find that the surface term becomes finite in much the same way as the on-shell action itself became finite:

$$\begin{aligned}\oint_{\partial M} d^d x \sqrt{h}\, \pi\, \delta\varphi \;\to\; & \oint_{\partial M} d^d x \sqrt{h}\, (\pi - 2\lambda_+ \varphi)\, \delta\varphi \\ = & \lim_{r\to\infty} \int_{\Sigma_r} d^d x \sqrt{h}\, (\pi - 2\lambda_+ \varphi)\, \delta\varphi \qquad (1.10)\end{aligned}$$

One then imposes boundary conditions on the non-normalizable mode such that $\delta f_+ = 0$ on Σ_r before taking the $r \to \infty$ limit, thereby removing the surface term in a consistent way. This viewpoint is explained in much greater detail in [25].

Holographic renormalization. Holographic renormalization provides a systematic framework for finding the counterterms like the one in (1.6) to make the on-shell gravitational action finite. Over the years, there have been roughly three different approaches to holographic renormalization. In chronological order, these are the Fefferman–Graham method [26], the Hamilton–Jacobi method [27], and the Hamiltonian method [28]. In this work we shall employ a hybrid method based on the latter two, which we present in §1.2. The main advantage of this new method is that it uses only one canonical formalism (the HJ formalism), while it does not require the use of an Ansatz. The way the boundary conditions are imposed in the literature typically makes use of the Euler–Lagrange formalism. In order to formulate the boundary conditions for fields that are switched off in the background geometry, this typically requires one to solve the linearized field equations about the background solution. This step is eliminated in our hybrid method. Before we present this hybrid method, however, we quickly review the previous methods in order to motivate the use of the new method.

1.1 Review of Previous Methods

In this section we review the Fefferman–Graham method [26], the Hamilton–Jacobi method and the Hamiltonian method. It is useful to pick a specific model that we wish to renormalize, for which we take Einstein gravity with asymptotically locally AdS boundary conditions. The (off-shell) action is given by:

$$16\pi G\, S \;=\; -\int_M dr\, d^d x\, \sqrt{g}\, \left(\tilde{R} - 2\Lambda\right) - \oint_{\partial M} d^d x \sqrt{g}\, 2K\,. \tag{1.11}$$

The AdS curvature length is related to the cosmological constant as $2\Lambda = -d(d-1)/\ell^2$. We set Newton's constant $16\pi G = 1$ and the AdS length $\ell = 1$ henceforth (they can be restored in the final answer). We give the $(d+1)$-dimensional Ricci scalar $\tilde{R}$ a twiddle to distinguish it from the d-dimensional one R, which we encounter below. The line element of asymptotically locally AdS is given by

$$ds^2 \;=\; dr^2 + g_{ab}\, dx^a dx^b\,, \tag{1.12}$$

where $\partial_r g_{ab} \approx 2 g_{ab}$, where '$\approx$' means equality up to terms that are suppressed by higher powers of e^{-r}.

1.1.1 Fefferman–Graham method

In the Fefferman–Graham (FG) method [26], the metric is expanded in powers of the radial coordinate [29]:

$$g_{ab}(r,x) \;=\; \sum_{n\geq 0} \mathrm{e}^{(2-n)r} g_{(n)ab}(x)\,, \tag{1.13}$$

The first coefficient is fixed by the asymptotic boundary conditions. For instance, imposing asymptotically AdS boundary conditions means setting $g_{(0)ab} = \eta_{ab}$. The higher-order coefficients can be expressed in terms of $g_{(0)ab}$ by using the Euler–Lagrange equations and their radial derivatives, e.g. (for $d > 2$):

$$g_{(2)ab} \;=\; \frac{1}{d-2} R_{(0)ab} - \frac{1}{2(d-1)(d-2)} R_{(0)}\, g_{(0)ab} \tag{1.14}$$

where $R_{(0)} \equiv \mathbf{Ric}[g_{(0)}]$. This recursion relation breaks down when one reaches $n = d$, which means that $g_{(d)ab}$ is not determined by the asymptotic boundary conditions; it is the normalizable mode. The expanded form of the metric can then be plugged into the action (1.11), which gives an expression in terms of $g_{(n)ab}$ and powers of e^r, making the divergences explicit. The on-shell action will then be of the form

$$S \;=\; \int_{\Sigma_r} d^d x\, \mathrm{e}^{d\,r} \left(a_{(0)} + \mathrm{e}^{-2r} a_{(2)} + \ldots\right) \tag{1.15}$$

where $a_{(n)}$ are local expressions in terms of $g_{(n)ab}$ with $n < d$. The term $a_{(d)}$, which depends on the normalizable mode $g_{(d)ab}$, gives a finite contribution to the on-shell action; it is essentially be the renormalized on-shell action.[1]

Inverting the FG series. The next step in the FG method is to covariantize the counterterm action by inverting the FG expansion (1.13), expressing $g_{(n)ab}$ in terms of g_{ab} and its derivatives. After this is done, the FG coefficients will look like

$$\begin{aligned} g_{(0)ab} &= \mathrm{e}^{-2r} g_{ab} - \left(\frac{1}{d-2} R_{ab} - \frac{1}{2(d-1)(d-2)} R\, g_{ab} \right) + \ldots \\ g_{(2)ab} &= \mathrm{e}^{2r} \left(\frac{1}{d-2} R_{ab} - \frac{1}{2(d-1)(d-2)} R\, g_{ab} \right) + \ldots \end{aligned} \tag{1.16}$$

which are then plugged into the counterterm coefficients $a_{(n)}$. This finally gives a covariant expression for the divergent part of the on-shell action, which is (minus) the counterterm action:

$$\begin{aligned} S_{\text{c.t.}} &= - \sum_{0 \le n < d} \mathrm{e}^{(d-n)r} \int d^d x\, a_{(n)} \\ &= \int d^d x \sqrt{g} \left\{ 2(d-1) + \frac{1}{d-2} R \right. \\ &\quad \left. + \frac{1}{(d-4)(d-2)^2} \left(R_{ab} R^{ab} - \frac{d}{4(d-1)} R^2 \right) + \ldots \right\} \end{aligned} \tag{1.17}$$

Logarithmic modes. We mentioned above that the recursion relation that fixes $g_{(n)ab}$ in terms of $g_{(0)ab}$ breaks down at $n = d$. This is completely natural; it just means that the field equations cannot not be used to express the independent integration constant $g_{(d)ab}$ in terms of $g_{(0)ab}$; additional input is needed (e.g. regularity in the interior). It turns out that there is yet another kind of break-down when d is even, which requires we add a logarithmic term $\sim \log(\mathrm{e}^r)$ in the FG expansion in order to solve the equations of motion:

$$\begin{aligned} g_{ab} &= \mathrm{e}^{2r} \left(g_{(0)ab} + \mathrm{e}^{-2r} \left(g_{(2)ab} + r\, \tilde{g}_{(2)ab} \right) + \ldots \right) && (d=2) \\ g_{ab} &= \mathrm{e}^{2r} \left(g_{(0)ab} + \mathrm{e}^{-2r} g_{(2)ab} + \mathrm{e}^{-3r} g_{(3)ab} + \ldots \right) && (d=3) \\ g_{ab} &= \mathrm{e}^{2r} \left(g_{(0)ab} + \mathrm{e}^{-2r} g_{(2)ab} + \mathrm{e}^{-4r} \left(g_{(4)ab} + r\, \tilde{g}_{(4)ab} \right) + \ldots \right) && (d=4) \\ g_{ab} &= \mathrm{e}^{2r} \left(g_{(0)ab} + \mathrm{e}^{-2r} g_{(2)ab} + \mathrm{e}^{-4r} g_{(4)ab} + \mathrm{e}^{-5r} g_{(5)ab} + \ldots \right) && (d=5) \end{aligned} \tag{1.18}$$

[1] It is not the renormalized action on the nose, because there are typically finite contributions coming from the counterterms after they are made covariant by inverting the FG series.

The additional term $\tilde{g}_{(d)ab}$ is a local expression in terms of the source $g_{(0)ab}$. For instance, when $d = 2$:

$$\tilde{g}_{(2)ab} = R_{(0)ab} - \frac{1}{2} R_{(0)}\, g_{(0)ab} \tag{1.19}$$

The appearance of such an additional term is responsible for having a logarithmic divergence in the on-shell action:

$$\begin{aligned}
S_{\text{c.t.}} &= \int d^d x \sqrt{g}\,\big(2 + R\, r\big) && (d=2)\\
S_{\text{c.t.}} &= \int d^d x \sqrt{g}\,\big(4 + R\big) && (d=3)\\
S_{\text{c.t.}} &= \int d^d x \sqrt{g}\,\Big(6 + \frac{1}{2}\,R + \frac{1}{4}\Big(R_{ab}R^{ab} - \frac{1}{3}\,R^2\Big)\,r\Big) && (d=4)\\
S_{\text{c.t.}} &= \int d^d x \sqrt{g}\,\Big(8 + \frac{1}{3}\,R + \frac{1}{9}\Big(R_{ab}R^{ab} - \frac{5}{16}\,R^2\Big)\Big) && (d=5)
\end{aligned} \tag{1.20}$$

A logarithmically divergent term signals the presence of a Weyl anomaly [26]. We elaborate on this below.

1.1.2 Hamilton–Jacobi method

The second method we review is the Hamilton–Jacobi (HJ) method [27]. In the Hamilton–Jacobi canonical formalism, the equations of motions are solved directly in terms of the on-shell action, also known as Hamilton's principle function. The HJ formalism is reviewed in Appendix II. The HJ method makes use of a radial ADM decomposition [30], cf. Appendix I:

$$ds^2 = N^2 dr^2 + g_{ab}\,(dx^a + N^a dr)(dx^b + N^b dr) \tag{1.21}$$

This allows one to construct the radial Hamiltonian H:

$$H = \int_{\Sigma_r} d^d x \sqrt{g}\, \left(N\mathcal{H} + N^a \mathcal{H}_a\right), \tag{1.22}$$

with

$$\mathcal{H} = R - 2\Lambda + \pi^{ab}\pi_{ab} - \frac{1}{d-1}\,\pi^2\,, \qquad \mathcal{H}^a = 2D_b\pi^{ab}\,, \tag{1.23}$$

where $\sqrt{g}\,\pi^{ab}$ is the canonical momentum conjugate to the induced metric g_{ab} on the equal-r slice Σ_r. The HJ equation is a first-order non-linear PDE for the on-shell action:

$$\partial_r S + H = 0\,, \qquad \pi^{ab} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{ab}}\,, \tag{1.24}$$

The first term in the HJ equation is the explicit r-derivative of the on-shell action, holding g_{ab} fixed. Since S describes a theory that is generally covariant, the first term must vanish, such that solving the HJ equation reduces to solving the constraints $\mathcal{H} = 0$ and $\mathcal{H}_a = 0$. The HJ momentum is directly related to the (bare) stress tensor defined on Σ_r, i.e.

$$T^{ab} \equiv \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}} = 2\pi^{ab} . \tag{1.25}$$

This stress tensor is known as the Brown–York stress tensor. The way the counterterm action is found in the HJ method is by choosing a local covariant Ansatz, which in the case of Einstein gravity looks like:

$$S_{\text{c.t.}} = c_0 + c_1 R + c_2 R^2 + c_3 R_{ab} R^{ab} + c_4 R_{abcd} R^{abcd} + \dots \tag{1.26}$$

The renormalized on-shell action $S_{\text{ren}} = S + S_{\text{c.t.}}$ is presumed finite, which can be checked afterwards. The momentum depends linearly on the on-shell action, which means that the bare momentum splits up into

$$\pi^{ab} = \pi^{ab}_{\text{ren.}} - \pi^{ab}_{\text{c.t.}} \tag{1.27}$$

Because we have chosen a covariant Ansatz for the counterterm action, we immediately find that the associated momentum constraint is satisfied due to conservation of the Brown–York stress tensor:

$$2D_b \pi^{ab}_{\text{c.t.}} = D_b T^{ab}_{\text{c.t.}} = 0 \tag{1.28}$$

The 'bare' Hamiltonian constraint thus reduces to the conservation of the renormalized stress tensor:

$$0 = \mathcal{H}^a = D_b T^{ab}_{\text{ren.}} \tag{1.29}$$

Boundary conditions. The next step in the HJ method is to impose the asymptotic boundary conditions. We would like to impose the boundary condition $\partial_r g_{ab} \approx 2 g_{ab}$. By '$\approx$' we mean equality up to terms that are suppressed by powers of e^{-r}. This is done by asymptotically solving the following Hamilton equation:

$$\partial_r g_{ab} = \frac{1}{\sqrt{g}} \frac{\delta H}{\delta \pi^{ab}} = 2\pi_{ab} - \frac{2}{d-1} \pi \, g_{ab} \approx -\frac{c_0}{d-1} g_{ab} . \tag{1.30}$$

So, the boundary condition $\partial_r g_{ab} \approx 2 g_{ab}$ fixes $c_0 = -2(d-1)$.

Solving the Hamiltonian constraint. Now that we fixed the asymptotic boundary condition, we are ready to solve the Hamiltonian constraint. It is useful to express the kinetic term of the Hamiltonian constraint as

$$\{S, S\} \equiv G_{abcd} \frac{\delta S}{\delta g_{ab}} \frac{\delta S}{\delta g_{cd}} \tag{1.31}$$

where $G_{abcd} = g_{a(c}\, g_{d)b} - \frac{1}{d-1}\, g_{ab}\, g_{cd}$ is known as the DeWitt metric. This bracket is symmetric and bilinear (it is *not* a Poisson bracket). The Hamiltonian constraint can then be written as

$$\mathcal{H} \;=\; \{S, S\} + R - 2\Lambda \tag{1.32}$$

The split of the bare on-shell action $S = S_{\text{ren.}} - S_{\text{c.t.}}$ induces a split in the Hamiltonian constraint:

$$\{S_{\text{ren.}}, S_{\text{ren.}}\} - 2\{S_{\text{ren.}}, S_{\text{c.t.}}\} + \{S_{\text{c.t.}}, S_{\text{c.t.}}\} + R - 2\Lambda \tag{1.33}$$

The computation becomes more tractable when we expand the Hamiltonian constraint in terms of the number of derivatives, $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(4)} + \ldots$, where each term must vanish by itself. In AdS, there is a simple relation between the number of derivatives and the radial scaling: $S^{(n)}_{\text{c.t.}} \sim \mathrm{e}^{(d-n)r}$. On the other hand, the renormalized action has a finite limit as $r \to \infty$ by assumption. The bracket terms in the Hamiltonian constraint thus scale as follows:

$$\begin{aligned} \sqrt{g}\,\{S^{(m)}_{\text{c.t.}}, S^{(n)}_{\text{c.t.}}\} \;&\sim\; \mathrm{e}^{(d-m-n)r} \\ \sqrt{g}\,\{S^{(n)}_{\text{c.t.}}, S_{\text{ren}}\} \;&\sim\; \mathrm{e}^{-nr} \\ \sqrt{g}\,\{S_{\text{ren}}, S_{\text{ren}}\} \;&\sim\; \mathrm{e}^{-dr} \end{aligned} \tag{1.34}$$

The constant term in the Hamiltonian constraint $\mathcal{H}^{(0)}$ vanishes automatically when $c_0 = -2(d-1)$, which was fixed by the asymptotic boundary condition $\partial_r g_{ab} \approx 2 g_{ab}$. The next term is

$$0 \;=\; \mathcal{H}^{(2)} \;=\; 2\{S^{(0)}_{\text{c.t.}}, S^{(2)}_{\text{c.t.}}\} + R\,. \tag{1.35}$$

It is easy to find that $c_1 = \frac{1}{d-2}$. Similarly,

$$0 \;=\; \mathcal{H}^{(4)} \;=\; 2\{S^{(0)}_{\text{c.t.}}, S^{(4)}_{\text{c.t.}}\} + \{S^{(2)}_{\text{c.t.}}, S^{(2)}_{\text{c.t.}}\}\,. \tag{1.36}$$

which yields $c_2 = \frac{d}{4(d-4)(d-1)(d-2)^2}$, $c_3 = -\frac{1}{(d-4)(d-2)^2}$ and $c_4 = 0$. Thus, the counterterm action that is generated in this way is the same one as we found using the FG method, cf. (1.17).

Holographic Weyl anomalies. The way anomalies appear in the HJ method is when $\{S^{(0)}_{\text{c.t.}}, S^{(d)}_{\text{c.t.}}\}$ vanishes identically, which is the case when d is even. We notice, however, that $\{S^{(0)}_{\text{c.t.}}, S_{\text{ren.}}\} \sim \mathrm{e}^{-dr}$, which means that this term might 'talk' to other terms of order e^{-dr}. So even if $\{S^{(0)}_{\text{c.t.}}, S^{(d)}_{\text{c.t.}}\}$

vanishes by itself, the Hamiltonian constraint can still be satisfied because of the appearance of $\{S^{(0)}_{\text{c.t.}}, S_{\text{ren.}}\}$:

$$\begin{aligned} 0 \;&=\; \mathcal{H}^{(2)} \;=\; -2\big\{S^{(0)}_{\text{c.t.}}, S_{\text{ren}}\big\} + R && (d=2) \\ 0 \;&=\; \mathcal{H}^{(4)} \;=\; -2\big\{S^{(0)}_{\text{c.t.}}, S_{\text{ren}}\big\} + \big\{S^{(2)}_{\text{c.t.}}, S^{(2)}_{\text{c.t.}}\big\} && (d=4) \end{aligned} \tag{1.37}$$

The relation of such counterterms to the Weyl anomaly is more direct in the HJ method. Namely, it is straightforward to check that

$$-2\big\{S^{(0)}_{\text{c.t.}}, S_{\text{ren}}\big\} \;=\; \int d^dx\; 2g_{ab}\,\frac{\delta S_{\text{ren.}}}{\delta g_{ab}} \;=\; \int d^dx\,\sqrt{g}\,g_{ab}\,T^{ab}_{\text{ren.}}\,. \tag{1.38}$$

So, the Weyl anomaly in $d=2$ and $d=4$ dimensions is directly computed from (1.37):

$$\begin{aligned} g_{ab}\,T^{ab}_{\text{ren.}} \;&=\; -R && (d=2) \\ g_{ab}\,T^{ab}_{\text{ren.}} \;&=\; -\big\{S^{(2)}_{\text{c.t.}}, S^{(2)}_{\text{c.t.}}\big\} \;=\; -\frac{1}{4}\Big(R_{ab}R^{ab} - \frac{1}{3}\,R^2\Big) && (d=4) \end{aligned} \tag{1.39}$$

This allows us to determine the central charges associated with this holographic model. We discuss this at the end of Section 1.2 below.

1.1.3 Hamiltonian Method

The third method we review is the Hamiltonian method.[2] This method essentially a streamlined version of the FG method that uses some key ingredients from the HJ formalism. An important improvement of the Hamiltonian method to the HJ method is that is does not require writing down an Ansatz. The radial expansion is replaced by a covariant expansion that uses the dilatation operator:

$$\delta_D \;=\; \int d^dx\; 2g_{ab}\,\frac{\delta}{\delta g_{ab}}\,. \tag{1.40}$$

The on-shell Lagrangian $\mathcal{L}$, defined through $S = \int d^dx\sqrt{g}\,\mathcal{L}$, is then expanded in terms of scaling weights:

$$\mathcal{L} \;=\; \sum_n \mathcal{L}^{(n)}\,, \qquad\qquad \delta_D\mathcal{L}^{(n)} \;=\; -n\mathcal{L}^{(n)}\,. \tag{1.41}$$

[2]The *Hamiltonian* method is somewhat of a misnomer, as we will shortly see that this method actually uses the Hamilton–Jacobi formalism rather than a true Hamiltonian formalism.

Another ingredient that is used is that the (implicit) radial derivative δ_r, defined as

$$\delta_r \; = \; \int d^d x \; \partial_r g_{ab} \, \frac{\delta}{\delta g_{ab}} \tag{1.42}$$

asymptotes to the dilatation operator $\delta_r \approx \delta_D$, given that the asymptotic boundary conditions are in place. Like before, '$\approx$' denotes equality up to terms that are suppressed by powers of e^{-r}. Using the Gauss–Codazzi equations together with the equations of motion, one can write the on-shell action as a surface term:

$$S \; = \; -2 \int d^d x \sqrt{g} \; (K - \lambda) \tag{1.43}$$

where λ is the on-shell value of the bulk piece of the action; it is defined implicitly as the solution of the following first-order equation:

$$\partial_r \lambda + K\lambda \; = \; d \, . \tag{1.44}$$

The counterterm action is defined as (minus) the divergent part of the on-shell action. Using the expansion defined above, this becomes

$$S_{\text{c.t.}} \; = \sum_{0 \leq n < d} \frac{2(d-1)}{d-n} \int d^d x \sqrt{g} \, K^{(n)} \, , \tag{1.45}$$

where we used the fact that $\lambda^{(n)}$ and $K^{(n)}$ are related via the HJ momentum (1.24), and $K_{ab} = \pi_{ab} - \pi \, g_{ab}$, which gives:

$$(1 + \delta_D) K \; = \; (d + \delta_D)\lambda \qquad \Rightarrow \qquad \lambda^{(n)} \; = \; \frac{1-n}{d-n} \, K^{(n)} \, . \tag{1.46}$$

This is the identity that removes the need for writing down an Ansatz for λ. The terms $K^{(n)}$ are then determined recursively by solving the Hamiltonian constraint:

$$K^2 - K_{ab} K^{ab} \; = \; R + d(d-1) \, , \tag{1.47}$$

which yields

$$\begin{aligned} K^{(2)} \; &= \; \frac{1}{2(d-1)} \, R \, , \\ K^{(n)} \; &= \; \frac{1}{4(d-1)} \sum_{0 < m < n} \left(K^{(m)}_{ab} \, K^{(n-m)ab} - K^{(m)} K^{(n-m)} \right) \, , \end{aligned} \tag{1.48}$$

for $n > 2$. The extrinsic curvature that is not traced over is computed via $K^{(n)}_{ab} = \pi^{(n)}_{ab} - \pi^{(n)} g_{ab}$ again, where $\sqrt{g} \, \pi^{(n)ab} \equiv \delta S^{(n)} / \delta g_{ab}$.

Just as we saw in the FG method, we need to add a logarithmically divergent counterterm when d is even, so

$$S_{\text{c.t.}} \;=\; 2\int d^dx\,\sqrt{g}\,\left(\sum_{0\leq n<d}\frac{d-1}{d-n}\,K^{(n)} - r\,\tilde{\lambda}^{(d)}\right)\,, \tag{1.49}$$

The coefficient of the logarithmic counterterm can be computed easily by first computing $K^{(n)}$ for general d and then taking the limit $d\to n$:

$$\tilde{\lambda}^{(n)} \;=\; \lim_{d\to n}\,(d-n)\,\lambda^{(n)} = (1-d)\,\lim_{d\to n}\,K^{(n)} \tag{1.50}$$

For instance, consider the case where $d=2$. We first compute $K^{(2)}$ for general d, cf. (1.48), after which we take

$$\tilde{\lambda}^{(2)} \;=\; -K^{(2)} \;=\; -\frac{1}{2}\,R\,. \tag{1.51}$$

1.1.4 Complications

It should be clear from the above discussion that the FG method is hands down the most laborious one. On the other hand, the apparent simplicity of the HJ and Hamiltonian method is mostly due to their being tailored to AdS space. We are eventually interested in holographic renormalization for geometries that are asymptotically (locally) Lifshitz rather than AdS, which severely complicates these procedures.

First of all, the FG expansion for asymptotically Lifshitz configurations is a huge mess because of the anisotropy between time and space. Especially inverting the FG series is very difficult (if not impossible) to do.

The derivative expansion in the HJ method can still work, although the radial scaling is no longer simply related to the number of derivatives (as was the case in AdS). The main problem of using the HJ method for Lifshitz spacetimes is that the Ansatz that one needs to write down consists of many many independent terms. For instance, if one needs to find the counterterms up to 4 derivatives, the most general covariant local Ansatz consists of about 30 terms, each of which has a coefficient that is obtained by solving a non-linear differential equation. Although this can be done straightforwardly, it is a rather daunting task.[3]

[3]We actually did this calculation in [2], which is reviewed in Appendix V.

The Hamiltonian method has the great advantage that it removes the need for writing down an Ansatz; it computes the counterterms directly from the asymptotic boundary conditions and the off-shell form of the action. The simple form that the counterterms take are mainly due to the fact that we are working in pure Einstein gravity. For instance, the simple relation between K and λ no longer holds for Lifshitz. The method that we shall use to renormalize Lifshitz spacetimes is based on the Hamiltonian method, but it makes the underlying HJ formalism explicit.

1.2 A Hybrid Method

We will now present the method of holographic renormalization that we will use in this work. This method combines the formal clarity of the Hamilton–Jacobi method whilst removing the need for writing down an Ansatz (inspired by the Hamiltonian method). We still use the example of Einstein gravity with AdS boundary conditions to illustrate the method, but we present it in such a way that generalization to other setups is most clear. Apart from the way the asymptotic boundary conditions are implemented, this method was used in [31].

Boundary conditions. Let us start by imposing the asymptotic boundary conditions. Consider the variations:

$$\delta_r \;=\; \int d^d x\, \partial_r g_{ab}\, \frac{\delta}{\delta g_{ab}}\,, \qquad\qquad \delta_D \;=\; \int d^d x\, 2 g_{ab}\, \frac{\delta}{\delta g_{ab}}\,. \tag{1.52}$$

Quantities (generically denoted 'X') are expanded according to their scaling weights using the dilation operator δ_D:

$$X \;=\; \sum_n X^{(n)}\,, \qquad\qquad \delta_D X^{(n)} \;=\; -n X^{(n)}\,. \tag{1.53}$$

To make life easier we will introduce frame fields e_a^A, with $g_{ab} = \eta_{AB}\, e_a^A e_b^B$. This allows us to give all tensorial quantities flat indices such that we can freely raise and lower indices without changing radial scaling.

The boundary conditions are set by requiring that dilatations are asymptotically generated by (implicit) radial derivatives:

$$\boxed{\delta_r \;\approx\; \delta_D} \tag{1.54}$$

As before, '≈' denotes equality up to terms that become negligible at $r \sim \infty$. The boundary condition (1.54) fixes the leading term of the extrinsic curvature $K_{ab} = \frac{1}{2}\partial_r g_{ab}$, such that ${K_{AB}}^{(0)} = \eta_{AB}$.

A useful relation. The relation that removes the need for using an Ansatz to solve the HJ equation is

$$\int d^d x \; 2{K_{AB}}^{(0)} \, \pi^{AB} \;=\; \delta_D S \,. \tag{1.55}$$

In fact, more specifically, we will use

$$\boxed{2{K_{AB}}^{(0)} \, \pi^{AB\,(n)} \;=\; (d-n)\,\mathcal{L}^{(n)}} \tag{1.56}$$

It is this identity (together with the δ_D-expansion) that is the main ingredient that is taken from the Hamiltonian method.

The counterterms. Next we set out to solve the Hamilton–Jacobi equation, which comes down to solving the Hamiltonian constraint $\mathcal{H} = 0$, where

$$\mathcal{H} \;=\; K_{AB}\,\pi^{AB} + \mathcal{V}\,, \tag{1.57}$$

where the 'potential' $\mathcal{V}$ is whatever is not part of the kinetic term; in this case it is just $\mathcal{V} = R - 2\Lambda$. We use the extrinsic curvature simply as a short-hand notation for $K_{AB} = \pi_{AB} - \frac{1}{d-1}\,\pi\,\eta_{AB}$. The Hamiltonian constraint is expanded in terms of dilatation weights:[1]

$$\mathcal{H}^{(n)} \;=\; \sum_{i+j=n} {K_{AB}}^{(i)} \, \pi^{AB\,(j)} + \mathcal{V}^{(n)}\,. \tag{1.58}$$

The lowest term $\mathcal{H}^{(0)}$ vanishes automatically (unless we had imposed an inconsistent dilatation weight for the metric). The level-0 contribution to the on-shell action is given via (1.56):

$$\mathcal{L}^{(0)} \;=\; \frac{2}{d}\,{K_{AB}}^{(0)}\,\pi^{AB\,(0)} \;=\; -2(d-1)\,. \tag{1.59}$$

The next term in the expanded Hamiltonian constraint is:

$$0 \;=\; \mathcal{H}^{(2)} \;=\; 2{K_{AB}}^{(0)} \, \pi^{AB\,(2)} + \mathcal{V}^{(2)} \;=\; (d-2)\,\mathcal{L}^{(2)} + R\,, \tag{1.60}$$

which immediately gives:

$$\mathcal{L}^{(2)} = -\frac{1}{d-2}\,R\,. \tag{1.61}$$

[1] It is useful to note the symmetry property: ${K_{AB}}^{(i)} \, \pi^{AB\,(j)} = {K_{AB}}^{(j)} \, \pi^{AB\,(i)}$.

The next order in the Hamiltonian constraint is:

$$
\begin{aligned}
0 \;=\; \mathcal{H}^{(4)} \;&=\; 2K_{AB}{}^{(0)}\,\pi^{AB\,(4)} + K_{AB}{}^{(2)}\,\pi^{AB\,(2)} \\
&=\; (d-4)\,\mathcal{L}^{(4)} + K_{AB}{}^{(2)}\,\pi^{AB\,(2)}\,,
\end{aligned}
\tag{1.62}
$$

where $\pi^{AB\,(2)}$ and $K_{AB}{}^{(2)}$ are computed from $\mathcal{L}^{(2)}$:

$$
\begin{aligned}
\pi^{AB\,(2)} \;&=\; \frac{1}{d-2}\left(R_{AB} - \frac{1}{2}R\,\eta_{AB}\right)\,, \\
K_{AB}{}^{(2)} \;&=\; \frac{1}{d-2}\left(R_{AB} - \frac{1}{2(d-1)}R\,\eta_{AB}\,.\right)
\end{aligned}
\tag{1.63}
$$

We thus see that $\pi^{AB\,(2)}$ is proportional to the Einstein tensor G^{AB}, while $K_{AB}{}^{(2)}$ is given by the Schouten tensor S_{AB}. When we plug this back into $\mathcal{H}^{(4)}$, we find

$$
\begin{aligned}
\mathcal{L}^{(4)} \;&=\; -\frac{1}{(d-4)(d-2)}\,S_{ab}\,G^{ab} \\
&=\; -\frac{1}{(d-4)(d-2)^2}\left(R_{ab}R^{ab} - \frac{d}{4(d-1)}\,R^2\right)
\end{aligned}
\tag{1.64}
$$

The counterterm action is simply (minus) the divergent terms thus computed, so

$$
S_{\text{c.t.}} \;=\; -\sum_{0\le n<d}\int d^dx\,\sqrt{g}\;\mathcal{L}^{(n)}
\tag{1.65}
$$

Of course, this is not yet the full story, because there are logarithmically divergent counterterms that one must add when d is even.

Weyl anomalies. In this formulation it is most clear where the logarithmic counterterms come from and how they relate to Weyl anomalies. For instance, when $d=2$ we see that the first term in $\mathcal{H}^{(2)}$ vanishes identically, leaving a non-vanishing remainder. In order to still satisfy the Hamiltonian constraint, one must add a counterterm that depends explicitly on r. In that case there is an extra contribution to the Hamiltonian constraint equation coming from $\partial_r S_{\text{c.t.}}$ in the HJ equation, such that

$$
\partial_r S_{\text{c.t.}} \;=\; \int d^dx\sqrt{g}\;\mathcal{H}^{(d)}\,. \qquad\qquad (d\ \text{even})
\tag{1.66}
$$

Thus, the Hamilton–Jacobi equation is satisfied so far. It is important to note, however, that the bare action $S = S_{\text{ren.}} - S_{\text{c.t.}}$ still satisfies $\partial_r S = 0$.

This means that $\partial_r S_{\rm ren.} = \partial_r S_{\rm c.t.} = \int d^dx\sqrt{g}\,\mathcal{H}^{(d)}$. Now, the renormalized on-shell action must have a finite limit as $r \to \infty$, which means that

$$0 \;\approx\; \frac{dS_{\rm ren.}}{dr} \;=\; \partial_r S_{\rm ren.} + \delta_r S_{\rm ren.}\,. \tag{1.67}$$

We just argued that the first term on the right-hand side is given in terms of $\mathcal{H}^{(d)}$. The second term asymptotes to the Weyl transformation of the renormalized on-shell action due to the boundary conditions $\delta_r \approx \delta_D$. When we put all of this together, we obtain the following chain of relations:

$$\delta_D S_{\rm ren.} \;\approx\; \delta_r S_{\rm ren.} \;\approx\; -\partial_r S_{\rm ren.} \;=\; -\partial_r S_{\rm c.t.} \;=\; -\int d^dx\sqrt{g}\,\mathcal{H}^{(d)}\,. \tag{1.68}$$

We thus find that the Weyl anomaly is given as the non-vanishing remainder in the Hamiltonian constraint $\mathcal{H}^{(d)}$:

$$\boxed{\delta_D S_{\rm ren.} \;=\; -\int d^dx\sqrt{g}\,\mathcal{H}^{(d)}} \tag{1.69}$$

when d is even. Let us write $\delta_D S_{\rm ren.} = \int d^dx\sqrt{g}\,T_{\rm ren.}$, then we find for $d = 2, 4$ the following anomalous traces:

$$\begin{aligned} T_{\rm ren.} &= -R\,. && (d=2)\\ T_{\rm ren.} &= -K^{(2)}_{AB}\,\pi^{(2)AB} \;=\; -\frac{1}{2}\,P_{ab}\,G^{ab} \;=\; -\frac{1}{4}\Big(R_{ab}R^{ab} - \frac{1}{3}\,R^2\Big) && (d=4) \end{aligned} \tag{1.70}$$

The curvature length ℓ and the Newton constant can be restored by replacing:

$$\mathcal{L}^{(n)} \to \frac{\ell^{n-1}}{16\pi G}\,\mathcal{L}^{(n)} \tag{1.71}$$

The standard form of the $d = 2$ and $d = 4$ anomalies are:

$$\begin{aligned} T_{\rm ren.} &= -\frac{c}{24\pi}\,R && (d=2)\\ T_{\rm ren.} &= aE_4 - cI_4 && (d=4) \end{aligned} \tag{1.72}$$

where $E_4 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the 4D Euler density and $I_4 = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2$ is the square of the 4D Weyl tensor. Comparing with (1.70), we find the well-known Brown–Henneaux central charge for $d = 2$, $c = 3\ell/2G$, while for $d = 4$ we find that the two central charges are equal, $a = c$. This means that the class of CFT's that are dual to the Einstein–Hilbert action is restricted in that the two central charges must be equal $a = c$. Naturally, this is the case for maximally supersymmetric Yang–Mills.

In order to lift the $a = c$ degeneracy, one can add higher-derivative corrections to the Einstein–Hilbert action, thereby opening up the possibility of describing a much larger class of CFT's in holography. To illustrate the power of the hybrid method we just presented, we compute this higher-derivative correction in Appendix III. Let us quickly review the main idea. One starts with the Einstein–Hilbert action and adds a Gauss–Bonnet term, which is a special higher-curvature term that is free of ghosts. The action is then (S_{GH} is the Gibbons–Hawking term)

$$S = \int d^d x \sqrt{g} \left(2\Lambda - R - \alpha \left(R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right) \right) + S_{\mathrm{GH}} \quad (1.73)$$

If we then repeat the same steps as before, we find that the 4D anomaly is again given by (1.72), but this time we have $a - c \propto \alpha$. We included this computation mainly to illustrate the power of this method, but since it is not crucial for the remainder of the text we have moved the computation to the appendix.

Conclusion and Look Ahead

Throughout this chapter, the emphasis has mostly been on reviewing different computational techniques and not so much on physical issues. In the next chapter, however, we will actually put the techniques we just developed to good use. The main thing that was established in this chapter was that the on-shell action can be renormalized simply by imposing the boundary condition $\delta_r \approx \delta_D$, i.e. by imposing that dilatations are asymptotically generated by a radial shift. Phrasing the boundary conditions in this manner makes it very easy to generalize asymptotically locally AdS boundary conditions to other types of asymptotics.

The main difficulty one faces when trying to do holography with Lifshitz asymptotics is that it is not so straightforward to figure out how to parametrize the bulk degrees of freedom. A particularly useful parametrization in terms of frame fields was proposed in [31], which works quite well for $z \neq d_s$. Difficulty arises when $z = d_s$ due to logarithmic modes that appear at the leading (non-normalizable) modes. Because of such leading log modes, it becomes tricky to impose the boundary conditions in analogy to how asymptotically locally AdS boundary conditions are imposed. Rephrasing the asymptotic boundary conditions as $\delta_r \approx \delta_D$ removes all such complications. It is by far the easiest way to generalize asymptotically locally AdS boundary conditions to other types of asymptotics. In particular, we soon see that the $z \neq 1$ extension of these boundary conditions are very easily implemented.

Chapter 2

Lifshitz Holography

In this chapter we discuss holography for theories with a Lifshitz-type UV fixed point. At such a fixed point these theories exhibit Lifshitz-type scaling, where time scales differently from space,

$$D : (t, \mathbf{x}) \;\mapsto\; (b^z t, b\mathbf{x})\,. \tag{2.1}$$

Besides this scaling symmetry, Lifshitz theories are also invariant under spatial and time-translations, $P : \mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$ and $H : t \mapsto t + a$ and spatial rotations $R \in SO(d_s)$. These symmetries make up the Lifshitz algebra. As we mentioned before, Lifshitz spacetime is constructed in such a way that its isometry algebra is the Lifshitz algebra. The line element is given by:

$$ds^2 \;=\; dr^2 - \mathrm{e}^{2zr} dt^2 + \mathrm{e}^{2r} d\mathbf{x}^2 \tag{2.2}$$

The Lifshitz symmetry algebra is realized by the isometries that are generated by $D = zt\partial_t + x^i\partial_i - \partial_r$, $P_i = \partial_i$, $H = \partial_t$ and $R_{ij} = x_i\partial_j - x_j\partial_i$. Before moving on, let us discuss two generalizations of the Lifshitz geometry that have received considerable attention in the literature.

Generalizations of Lifshitz spacetime. The Lifshitz algebra is a subalgebra of the Galilean algebra, which also includes Galilean boosts, which map $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$. A geometry that realizes the full Galilean algebra as its isometry algebra is known as *Schrödinger spacetime* [32, 33]:

$$ds^2 \;=\; dr^2 - \mathrm{e}^{2zr} dt^2 + \mathrm{e}^{2r}\left(d\mathbf{x}^2 + 2d\xi\, dt\right) \tag{2.3}$$

Unlike Lifshitz spacetime, Schrödinger spacetime is not invariant under time reversal. The Galilean boosts are generated by $G_i = t\partial_i - x_i\partial_\xi$. In a quantum theory, the Galilean algebra acquires a central charge $[G_i, P_j] = M\delta_{ij}$, where M is the mass/particle-number operator. The mass operator is realized geometrically in the above line element by $M = \partial_\xi$. Moreover, when the dynamical exponent is $z = 2$, the Galilean algebra is enhanced to the Schrödinger algebra, which includes the $z = 2$ equivalent of special conformal transformations. Schrödinger holography is interesting because of its rich symmetry properties. However, attempts to perform holographic renormalization for these geometries have been plagued with issues related to the inclusion of the (null) direction ξ, cf. [34].

A more recent generalization of Lifshitz spacetime is known as a *hyperscaling violating geometry*, which is *co*variant rather than *in*variant under Lifshitz scalings (see e.g. [35] and references therein). The line element can be written as:[1]

$$ds^2 \;=\; \mathrm{e}^{-\frac{2\theta}{d_s}\,r}\left(dr^2 - \mathrm{e}^{2zr}dt^2 + \mathrm{e}^{2r}d\mathbf{x}^2\right) \tag{2.4}$$

This geometry is conformal to Lifshitz spacetime and the conformal factor is responsible for the hyperscaling violation:

$$t \to b^z t\,, \qquad \mathbf{x} \to b\mathbf{x}, \qquad r \to r - \ln b \qquad \Rightarrow \qquad ds^2 \;\to\; b^{2\theta/d_s} ds^2\,. \tag{2.5}$$

These geometries were studied in connection to Fermi surfaces. Since Fermi surfaces arise as an effective weakly-coupled description, while the field theory dual to such geometries is necessarily strongly coupled, it is not clear that it is sensible to think in terms of Fermi surfaces. Nevertheless, these geometries were found to have some characteristic qualitative features in common with Fermi surfaces via holography; they were dubbed *hidden* Fermi surfaces [36].

UV Lifshitz vs. IR Lifshitz. There is an important distinction to be made when one discusses Lifshitz spacetime in the context of holography. The conventional wisdom is that Lorentz symmetry is usually found at relatively high energy scales (UV), while Lorentz symmetry is typically broken at low energy scales (IR). A holographic setup that reflects such a situation will have an asymptotically AdS geometry in the bulk that tends to, say a Lifshitz geometry in the interior, see e.g. [37].

This is not the setup we consider in this work. As we mentioned in the Introduction, we consider situations where one has a Lifshitz-type fixed point in the UV. The IR is kept arbitrary and in some cases one finds that Lorentz symmetry emerges in the IR. An example of a weakly-coupled theory that exhibits this kind of behavior is a BCS superconductor. The theory one starts out with has a quadratic dispersion relation ($z = 2$) and one finds that as the temperature is lowered below a critical value, a new description (Landau–Ginzburg) emerges that is Lorentz invariant ($z = 1$).

Structure of this chapter. In this chapter we apply the techniques developed in Chapter 1 to Lifshitz-type theories. As it turns out, these theories

[1] We choose not to use Gaussian normal coordinates, because it would seriously obfuscate the Lifshitz-covariant nature of this geometry.

generically exhibit a Weyl-type quantum anomaly when the dynamical exponent is set to the critical value $z = d_s$, where d_s is the number of spatial dimensions. We dub this anomaly the *Lifshitz anomaly.* In the first section, we analyze which terms are to be expected in this Lifshitz anomaly on general grounds when $z = d_s = 2$. We find that the general form of the anomaly is fixed up to two independent *central charges.* We then move on to the calculation of these central charges in a free Lifshitz scalar field theory in §2.2 and in a holographic model in §2.3.

2.1 The Lifshitz-type Weyl Anomaly

We now classify the terms that can appear in the Lifshitz anomaly for $z = d_s = 2$. Consider a field theory in 2+1 dimensions coupled to the following Euclidean background geometry:

$$ds^2 \;=\; N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.6}$$

Lifshitz theories are not generally covariant, but they are covariant with regard to *foliation-preserving* diffeomorphisms $(t, \mathbf{x}) \mapsto (s(t), \mathbf{y}(\mathbf{x}, t))$. The shift $N^i(\mathbf{x}, t)$ can be 'gauged away' by a foliating-preserving diffeomorphism, while the lapse $N(\mathbf{x}, t)$ cannot. Henceforth we set $N^i = 0$; it may be restored in our final answer. The classical theory is invariant under local Lifshitz scalings, $\delta_\omega S = 0$, where δ_ω acts on the geometry as:

$$\delta_\omega N \;=\; zN\,\delta\omega\,, \qquad\qquad \delta_\omega h_{ij} \;=\; 2h_{ij}\,\delta\omega\,. \tag{2.7}$$

When we quantize our theory, the classical action S is replaced by the renormalized generating functional $W = -\log Z[N, h]$. The anomaly $\mathcal{A}$ is defined as

$$\delta_\omega W \;=\; \int dt\, d^2x\, N\sqrt{h}\;\mathcal{A}\,\delta\omega \tag{2.8}$$

The anomaly $\mathcal{A} = \delta W/\delta\omega$ is defined up to exact terms. The Wess–Zumino (WZ) consistency condition puts a constraint on the possible form of $\mathcal{A}$. In particular, when acting twice on the effective action with a Weyl transformation we must have:

$$0 \;=\; \delta_2\delta_1 W - \delta_1\delta_2 W \;=\; \int dt\, d^2x \left(\delta_2\big(N\sqrt{h}\;\mathcal{A}\big)\,\delta\omega_1 - \delta_1\big(N\sqrt{h}\;\mathcal{A}\big)\,\delta\omega_2\right) \tag{2.9}$$

We thus look for the most general $\mathcal{A}$ that is consistent with the WZ condition. As a first step, notice that the WZ condition is satisfied if the anomaly itself is Weyl invariant, by which we mean that $\delta_\omega\big(N\sqrt{h}\ \mathcal{A}\big) = 0$. So, we first write down all possible Weyl invariant terms and then we check which of those is consistent in the more restrictive sense of (2.9). The WZ condition also requires that the anomaly be covariant under foliation-preserving diffeomorphisms. Terms that are both covariant and scale invariant are:

$$\mathcal{A} \sim \quad K_{ij}K^{ij} - \frac{1}{2}K^2 \ , \quad \left(R + \frac{\Delta N}{N} - \frac{\partial_i N \partial^i N}{N^2}\right)^2 \ , \quad \frac{\nabla_i J^i_a}{N} \tag{2.10}$$

where the label a runs over $a = 1, \ldots, 6$. The details of this computation is included in Appendix IV; the J^i_a are given in (A.61). Five of the six J^i_a can be removed by adding finite local covariant counterterms to the generating functional W. The one J^i_a that is left after the other five have been removed turns out to be incompatible with the WZ condition (2.9), so it cannot appear in the anomaly. The generic form of the anomaly is thus fixed up to two *central charges*:

$$\mathcal{A} = \frac{C_1}{8\pi}\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + \frac{C_2}{8\pi}\left(R + \frac{\Delta N}{N} - \frac{\partial_i N \partial^i N}{N^2}\right)^2 \tag{2.11}$$

The central charges C_1 and C_2 are determined by the specific details of the Lifshitz theory in question; we included the factors $1/8\pi$ for later convenience. The remainder of this chapter deals with computing these central charges in two different models.

2.2 Lifshitz Scalar Field Theory

The first model we consider is a free Lifshitz-invariant scalar field theory in 2+1 dimensions, also known as the Quantum Lifshitz Model [12]. Its action is given by:

$$S = \frac{1}{2}\int dt\, d^2x\, \left((\partial_t\varphi)^2 - \kappa^2(\Delta\varphi)^2\right) , \tag{2.12}$$

where $\Delta = \delta^{ij}\partial_i\partial_j$ is the flat Laplacian. This theory is invariant under (global) Lifshitz scalings with $z = d_s = 2$. The (classical) Hamiltonian associated with the above action is given by $H = \int d^2x\, \mathcal{H}$, with

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + \kappa^2(\Delta\varphi)^2\right) . \tag{2.13}$$

Relation to 2D CFT. The quantum Lifshitz model has received some attention in the literature due to its connection to two-dimensional conformal fields theories [12, 13]. The underlying structure that is responsible for the connection to 2D CFTs is known as detailed balance. It is easiest to explain this using the model (2.12). The quantum Hamiltonian (density) can be written as a complete square:

$$\mathcal{H} \;=\; Q^\dagger Q\,, \qquad\qquad Q \;=\; \frac{1}{\sqrt{2}}\left(\kappa\,\Delta\varphi - i\pi\right) \tag{2.14}$$

Let us work in the φ-representation, such that $\pi = -i\frac{\delta}{\delta\varphi}$. The ground state $|0\rangle$ is annihilated by Q, so the ground-state wave functional $\Psi_0[\varphi] \equiv \langle\varphi|0\rangle$ can be found by solving

$$\left(\frac{\delta}{\delta\varphi} - \kappa\,\Delta\varphi\right)\Psi_0 \;=\; 0\,. \tag{2.15}$$

Now, the trick is to view the second term in the parentheses as the field equations of a 2D Euclidean action S_{2D}, such that the solution is found immediately:

$$\Psi_0[\varphi] \;=\; \frac{\mathrm{e}^{-\frac{1}{2}S_{2D}}}{\sqrt{Z}}\,, \qquad\qquad S_{2D}[\varphi] \;=\; \kappa\int d^2x\,\partial_i\varphi\,\partial^i\varphi\,. \tag{2.16}$$

Notice that S_{2D} is just a (1D) bosonic string with tension 2κ. The normalization constant Z is the 2D partition function:

$$Z \;=\; \int D\varphi\;\mathrm{e}^{-S_{2D}}\,. \tag{2.17}$$

This is quite powerful, because we can then compute equal-time correlation functions using the 2D CFT machinery [12, 13]:

$$\langle 0|\,\mathcal{O}_1\cdots\mathcal{O}_n\,|0\rangle \;=\; \frac{1}{Z}\int D\varphi\;\left(\mathcal{O}_1\cdots\mathcal{O}_n\right)\,e^{-S_{2D}}\,, \tag{2.18}$$

where $\mathcal{O}_k \equiv \mathcal{O}[\varphi](t, x_k)$. The power of this relation was used in [38, 39] to compute the (ground-state) entanglement entropy.

Detailed balance. The underlying principle that allows for the connection to S_{2D} is due to the fact that the (d_s+1)-dimensional Hamiltonian can be written as a sum of squares, $H = \sum_j Q_j^\dagger Q_j$, together with Q_j being of the form

$$Q_j \;\sim\; \frac{\delta S_{d_s}}{\delta\phi^j} - i\pi_j\,, \tag{2.19}$$

where S_{d_s} is some d_s-dimensional Euclidean action. When the Hamiltonian has this form the theory is said to satisfy the *detailed balance* condition.

What to expect? Because of its connection to a 2D CFT, one may naively expect that the Lifshitz anomaly is somehow related to the 'square' of the two-dimensional Weyl anomaly $\langle T^i{}_i \rangle \propto R$. This would mean that $C_1 = 0$ while $C_2 \neq 0$. We find, however, that this is *not* the case.

2.2.1 Lifshitz-type Weyl Anomaly

Suppose we minimally couple the theory (2.12) to the background geometry (2.6), such that the Euclidean action becomes:

$$S \;=\; \frac{1}{2}\int dt\, d^2x\, N\sqrt{h}\,\left((\partial_n\varphi)^2 + \kappa^2(\Delta\varphi)^2\right)\,, \tag{2.20}$$

where $\partial_n = N^{-1}\partial_t$ is the 'normal' derivative and Δ is the spatial Laplacian compatible with h_{ij}. The above theory is then invariant under *local* Lifshitz scalings (2.7). The action can be rewritten as

$$S \;=\; \frac{1}{2}\int dt\, d^2x\, N\sqrt{h}\;\varphi\, D\varphi\,, \tag{2.21}$$

where

$$D\varphi \;\equiv\; -\frac{1}{\sqrt{h}}\,\partial_n\left(\sqrt{h}\,\partial_n\varphi\right) + \frac{1}{N}\,\Delta\,(N\,\Delta\varphi) \tag{2.22}$$

The scalar is invariant under Weyl transformations (2.7), while the operator D transforms covariantly, $\delta_\omega D\varphi = -4D\varphi\,\delta\omega$. On the other hand, we know that the volume form scales as $\delta_\omega(N\sqrt{h}) = 4N\sqrt{h}\,\delta\omega$, we immediately see that $\delta_\omega S = 0$. In the quantum theory S is replaced by the generating functional W. The anomaly $\mathcal{A}$ is defined through:

$$\delta_\omega W \;=\; \int dt\, d^2x\, N\sqrt{h}\;\mathcal{A}\,\delta\omega\,. \tag{2.23}$$

The generic form of $\mathcal{A}$ was given in (2.11).

Heat-kernel expansion

We now compute the anomaly using a heat-kernel expansion. The quantum effective action W can be computed explicitly; it is given by the formal expression:

$$W \;=\; \frac{1}{2}\ln\det(D)\,, \tag{2.24}$$

where $\det(D)$ is the determinant of the operator D defined in equation (2.22). As usual, this determinant is not well-defined and must be regularized. We will employ ζ-function regularization. We define the generalized zeta function as

$$\zeta(s, f, D) \;=\; \mathrm{Tr}_{L^2}(f D^{-s})\,, \tag{2.25}$$

where s is an arbitrary positive number, $f(t, \mathbf{x})$ is an arbitrary function and L^2 an appropriate function space on which D^{-s} is trace-class. The regularized effective action is given by [40]:

$$W \;=\; -\frac{1}{2}\zeta'(0,1,D) - \frac{1}{2}\ln(\mu^2)\zeta(0,1,D)\,, \tag{2.26}$$

where $\zeta'(0,f,D) = \partial_s\zeta(s,f,D)|_{s=0}$ and μ is the usual arbitrary renormalization scale. The zeta function $\zeta(s,f,D)$ is related via a Mellin transformation to the heat kernel:

$$K(\epsilon, f, D) \;=\; \mathrm{Tr}_{L^2}(f\,\mathrm{e}^{-\epsilon D})\,, \tag{2.27}$$

where ϵ is an arbitrary positive parameter. To be more specific, the relation between the generalized zeta function and the heat kernel is

$$\begin{aligned} \zeta(s,f,D) &= \Gamma(s)^{-1}\int_0^\infty d\epsilon\,\epsilon^{s-1}\,K(\epsilon,f,D)\,,\\ K(\epsilon,f,D) &= \frac{1}{2\pi i}\oint ds\,\epsilon^{-s}\,\Gamma(s)\,\zeta(s,f,D)\,. \end{aligned} \tag{2.28}$$

In principle K depends on the global behavior of the operator D (the trace can be written as a sum over the spectrum of the operator, which is determined by global properties); however there is an asymptotic series of the form:

$$K(\epsilon,f,D) \;=\; \sum_{k=0}^{\infty}\epsilon^{\frac{k}{2}-1}\,\tilde{a}_k(f,D)\,, \tag{2.29}$$

where $\tilde{a}_k(f,D)$ can be computed *locally* from N and h_{ij}. By repeating the analysis of [40] section 7.1, one can show that the variation of the renormalized effective action under an infinitesimal anisotropic local scale transformation $h_{ij} \to (1+2\,\delta\omega)h_{ij}$, $N \to (1+2\,\delta\omega)N$, is given by:[1]

$$\delta W \;=\; -2\tilde{a}_2(\delta\omega, D)\,. \tag{2.30}$$

[1]The factor 2 comes from the factor 4 in $D \to e^{-4\omega}D$ under scale transformations.

In other words, the anomaly is given by the ϵ^0 term in the heat-kernel expansion. As explained above, this will be a local functional of N and h; we will therefore write:

$$\tilde{a}_2(f, D) \;=\; \int dt\, d^2x\, N\sqrt{h}\; f\, a_2(N, h_{ij})\,, \tag{2.31}$$

where $a_2(N, h_{ij})$ is a local function that depends on N and h_{ij}. The spectral function thus becomes:

$$K(\epsilon, f, D) \;=\; \sum_{k\geq 0} \epsilon^{\frac{k}{2}-1} \int dt\, d^2x\, N\sqrt{h}\; f(t,x)\, a_k(N, h_{ij})\,, \tag{2.32}$$

where $a_k(N, h_{ij})$ is a local function of N and h_{ij}. To evaluate this we need a suitable basis; it is customary to use the rescaled Fourier modes so that they are orthonormal with respect to the measure that includes the $N\sqrt{h}$ factor. Nevertheless, as pointed out in [41], the cyclicity of the trace allows us to use the usual flat Fourier modes. We thus find

$$K \;=\; \int \frac{d\omega d^2k}{(2\pi)^3} \int dt\, d^2x\, \mathrm{e}^{-i\omega t - ikx} f \mathrm{e}^{-\epsilon D} \mathrm{e}^{i\omega t + ikx}\,. \tag{2.33}$$

We can conjugate the Fourier mode to the left to get the expression

$$K \;=\; \int \frac{d\omega\, d^2k}{(2\pi)^3} \int dt\, d^2x\, f \mathrm{e}^{-\epsilon\tilde{D}}\,, \tag{2.34}$$

where $\tilde{D}$ is obtained from D by shifting the derivatives as follows:

$$\partial_t \;\to\; \partial_t + i\omega\,, \qquad\qquad \partial_i \;\to\; \partial_i + ik_i\,. \tag{2.35}$$

The most singular term in the heat kernel is the one where we keep only the terms in $\tilde{D}$ without derivatives, leading to

$$\frac{1}{\epsilon}\,\tilde{a}_0(f, D) \;=\; \int \frac{d\omega\, d^2k}{(2\pi)^3} \int dt\, d^2x\;\, f\, \mathrm{e}^{-\epsilon(N^{-2}\omega^2 + (k^2)^2)}\,, \tag{2.36}$$

where $k^2 \equiv h^{ij} k_i k_j$. This expression is easily evaluated to yield the first term in the heat kernel expansion:

$$\tilde{a}_0(f, D) \;=\; \frac{1}{16\pi} \int dt\, d^2x\, N\sqrt{h}\; f(t,x)\,. \tag{2.37}$$

Computing the subleading terms in the heat kernel expansion is now straightforward though somewhat involved. We shall write

$$\tilde{D} \;=\; \tilde{D}_0 + \tilde{D}_{\mathrm{int}}\,, \tag{2.38}$$

where $\tilde{D}_0$ is the piece we isolated above that contains ω^2 and k^4, while $\tilde{D}_{\rm int}$ consists of whatever remains. We then expand the exponential of $\tilde{D}_{\rm int}$. It contains a factor of ϵ, but ω counts as $\epsilon^{-1/2}$ and k as $\epsilon^{-1/4}$ in the Gaussian integral, therefore $\tilde{D}_{\rm int}$ has a term which scales as $\epsilon^{-1/4}$, and to get to the finite term one needs to expand $\tilde{D}_{\rm int}$ up to fourth order so that we get terms up to k^{12}. However, the problem becomes tractable if we consider the time-derivative and space-derivative sectors separately. This is consistent because the anomaly can only have structures involving either *two* time derivatives or *four* spatial derivatives.

The two-derivative anomaly. In order to compute the two-derivative contribution to the anomaly, and in turn C_1, it is sufficient to consider metrics that only depend on t, and not on x^i. Thus we can drop all the terms with spatial derivatives ∂_i in $\tilde{D}_{\rm int}$. This also allows us to redefine t to set the lapse $N = 1$. With these assumptions, we have:

$$\begin{aligned} \tilde{D}_0\varphi &= \left(\omega^2 + k^4\right)\varphi, \\ \tilde{D}_{\rm int}\varphi &= -i\omega\,\partial_t\varphi - \frac{i\omega}{\sqrt{h}}\,\partial_t\big(\sqrt{h}\,\varphi\big) - \frac{1}{\sqrt{h}}\,\partial_t\big(\sqrt{h}\,\partial_t\varphi\big). \end{aligned} \tag{2.39}$$

We have to expand (2.34) to second order in $\tilde{D}_{\rm int}$, for which we use the following formula:

$$\begin{aligned} {\rm e}^{A+B} &= {\rm e}^{A} + \int_{0\le\alpha\le1} d\alpha\,{\rm e}^{\alpha A} B\,{\rm e}^{(1-\alpha)A} \\ &\quad + \int_{0\le\alpha+\beta\le1} d\alpha\,d\beta\,{\rm e}^{\alpha A} B\,{\rm e}^{\beta A} B\,{\rm e}^{(1-\alpha-\beta)A} + \mathcal{O}(B^3). \end{aligned} \tag{2.40}$$

We find the following two-derivative contribution to a_2:[2]

$$2\tilde{a}_2(f,D) = \frac{-1}{768\pi}\int dtd^2x\sqrt{h}f\left(16h^{ij}\ddot{h}_{ij} + 5(h^{ij}\dot{h}_{ij})^2 - 10h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li}\right). \tag{2.41}$$

To reinstate N, we simply replace $dt \to dtN$ and $\partial_t \to \partial_n = N^{-1}\partial_t$. We can remove the term that involves $\ddot{h}_{ij}$ by adding a local counterterm as explained in Appendix IV. We thus obtain the anomaly by comparing with (2.8) and (2.30):

$$\mathcal{A} = \frac{1}{32\pi}\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + (\text{four-deriv. terms}). \tag{2.42}$$

[2]This computation is in principle quite lengthy. However, since there are only few terms that can appear, one can work this out for a diagonal h_{ij} and then reconstruct the full answer.

Using (2.11), we see that

$$\boxed{C_1 = \frac{1}{4}} \tag{2.43}$$

The four-derivative anomaly. We now determine the four derivative contribution, and in turn C_2. As explained in Appendix IV, there are 6 possible terms that can appear, 5 of which are total derivatives. These structures are distinguished by a metric of the form $h_{ij} = e^{2f(x)}\delta_{ij}$ and $N = e^{g(x)}$, which can be used to simplify considerably the computation. The four-derivative contribution to the anomaly is then:

$$\begin{aligned}\mathcal{A} = \frac{1}{480\pi}\frac{1}{N}\nabla_i\Big(& -5(\partial^i N)R + 3(\partial^i N)(\frac{1}{N}\Delta N) \\ & + 2(\partial^j N)(\frac{1}{N}\nabla_j\partial^i N) - 5\partial^i\Delta N\Big)\,. \end{aligned} \tag{2.44}$$

It is interesting to note that this result is a total derivative, and as predicted by the Wess-Zumino consistency condition, it is orthogonal[3] to the non-trivial total derivative $\mathcal{J}$ defined in equation (A.65). As a consequence, this term can be removed by a local counterterm and we conclude that

$$\boxed{C_2 = 0} \tag{2.45}$$

Conclusion and a look ahead

In summary, the Lifshitz model (2.20) exhibits an anomaly under Lifshitz-type Weyl transformations, which after the addition of appropriate counter-terms is given by:

$$\mathcal{A} \;=\; \frac{1}{32\pi}\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right)\,. \tag{2.46}$$

It is striking that the anomaly involves only time derivatives. It is also in contrast to the naive expectation that the anomaly is somehow related to the 'square' of the trace anomaly of a two-dimensional conformal field theory, as we mentioned at the beginning of this section. Below we compute the same anomaly in a holographic model. Let us stress that the holographic model should not be seen as dual to the scalar model we just considered. We can nevertheless see if there are some similarities between the two models.

[3] To see what we mean by 'orthogonal', we refer to the appendix IV.

2.3 Holographic Renormalization of Lifshitz Spacetime

In this section we perform holographic renormalization for Lifshitz spacetime. Our focus will lie on computing the central charges C_1 and C_2 when $z = d_s = 2$, although we keep z and d_s arbitrary for the most part. The general ideas of this analysis appeared in [31, 42]. The same calculation was done using the Hamilton–Jacobi method with an Ansatz for the counterterms in [2]. Since the latter approach is rather long and little illuminating, we discuss the former instead. For completeness, we have included the HJ computation in Appendix V.

The line element of Lifshitz spacetime is given by:

$$ds^2 \;=\; dr^2 + g_{ab}\, dx^a dx^b \;=\; dr^2 - \mathrm{e}^{2zr}\, dt^2 + \mathrm{e}^{2r}\, d\mathbf{x}^{\,2} \tag{2.47}$$

As our bottom-up holographic model, we take Einstein gravity minimally coupled to a massive vector field:[1]

$$S \;=\; \int d^4x\sqrt{g}\,\left(-\tilde{R} + 2\Lambda + \frac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab} + \frac{m^2}{2}\,\tilde{A}_a\tilde{A}^b\right) - \int d^3x\sqrt{\gamma}\,2K\,. \tag{2.48}$$

We gave the $(d+1)$-dimensional quantities twiddles to distinguish them from the d-dimensional ones we encounter below. This action was introduced in the context of Lifshitz holography in [17]. The Lifshitz geometry is a solution to the field equations derived from this action, provided we also turn on the vector,

$$\tilde{A} \;=\; \alpha\,\mathrm{e}^{zr}\,dt\,, \tag{2.49}$$

where α is a constant that is fixed by the field equations. The parameters of the theory are related to the parameters of the geometry as $m^2 = d_s z/\ell^2$ and $\Lambda = (z^2 + (d_s-1)z + d_s^2)/2\ell^2$. We picked our coordinates such that the Lifshitz length scale ℓ is set to one.

The Hamiltonian

The Hamiltonian $H = \int d^3x\sqrt{g}\,\mathcal{H}$ was derived in Appendix I. We use radial gauge: $N = 1$, $N^a = 0$. We write the Hamiltonian constraint as a direct

[1] In our notation, $\sqrt{g} \equiv \sqrt{|\det(g)|}$.

generalization of (1.57):

$$\mathcal{H} \;=\; K_{ab}\,\pi^{ab} + \frac{1}{2}\,E_a E^a + \frac{m^2}{2}\,\Phi^2 + \mathcal{V} \tag{2.50}$$

where the generalized potential is

$$\mathcal{V} \;=\; R - 2\Lambda - \frac{1}{4}F_{ab}F^{ab} - \frac{m^2}{2}\,A_a A^a\,, \tag{2.51}$$

while Φ is the radial component of the vector field, $\tilde{A} = \Phi\,dr + A_a\,dx^a$; it is constrained by the so-called Proca constraint:

$$\Phi \;=\; -\frac{1}{m^2}\,\nabla_a E^a \tag{2.52}$$

We shall only consider situations where the vector A_a is time-like throughout the entire geometry, such that we can pick out coordinates such that the time direction is aligned along the vector: $A = A_t\,dt \equiv (\alpha + \psi)\,dt$. The degree of freedom ψ constitutes a deformation of the vector field away from its background value $A_t = \alpha$, cf. (2.49). Before we start solving the Hamiltonian constraint, we must impose our (asymptotic) boundary conditions first. The HJ momenta are given by:

$$\pi^{ab} \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta S}{\delta g_{ab}}\,, \qquad\qquad E^a \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta S}{\delta A_a}\,. \tag{2.53}$$

For future reference, we also define the following quantities:

$$T^a{}_A \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta S}{\delta e^A_a} \qquad\qquad \pi_\psi \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta S}{\delta \psi} \tag{2.54}$$

They are related to the momenta as:

$$2\pi^{AB} \;=\; T^{AB} + (\alpha+\psi)\,\pi_\psi\,\delta^A_0\delta^B_0 \qquad\qquad E^A \;=\; \pi_\psi\,\delta^A_0 \tag{2.55}$$

$$T^A{}_B \;=\; 2\pi^A{}_B + E^A A_B \qquad\qquad \pi_\psi \;=\; E^0 \tag{2.56}$$

Henceforth we shall mostly use flat indices so that the radial scaling stays the same as we raise and lower indices. The field ψ corresponds to the (unregulated) source of an operator that is relevant for $1 < z < d_s$. When $z = d_s$ this operator becomes marginally relevant, cf. [3, 43]. We discuss this in detail in Chapter 3.

2.3.1 Boundary conditions

Let us introduce the following two variations:

$$\begin{aligned} \delta_r &= \int d^d x \left(\dot{e}^0_a \frac{\delta}{\delta e^0_a} + \dot{e}^I_\mu \frac{\delta}{\delta e^I_a} + \dot{\psi} \frac{\delta}{\delta \psi} \right) , \\ \delta_D &= \int d^d x \left(z e^0_\mu \frac{\delta}{\delta e^0_\mu} + e^I_a \frac{\delta}{\delta e^I_a} - \lambda \psi \frac{\delta}{\delta \psi} \right) . \end{aligned} \tag{2.57}$$

So, δ_r is an (implicit) radial variation, while δ_D is an anisotropic scale transformation. We leave λ (the scaling of ψ) undetermined for now; it will be fixed by the HJ equation below.[2] Then, the above variations can be rewritten as[3]

$$\begin{aligned} \delta_r &= \int d^d x \sqrt{g} \left(2K_{AB}\, \hat{\pi}^{AB} + E_A\, \hat{E}^A \right) , \\ &= \int d^d x\, N\sqrt{h} \left(K_{AB}\, \hat{T}^{AB} + \left(E_0 + (\alpha + \psi)\, K_{00} \right) \hat{\pi}_\psi \right) , \\ \delta_D &= \int d^d x\, N\sqrt{h} \left(z\hat{T}^0{}_0 + \hat{T}^I{}_I - \lambda\psi\, \hat{\pi}_\psi \right) , \end{aligned} \tag{2.58}$$

where the differential operators $\hat{\pi}^{AB}$ and $\hat{E}^A$ (and similarly $\hat{T}^A{}_B$ and $\hat{\pi}_\psi$) are defined in analogy to the canonical momenta,

$$\hat{\pi}^{AB} \equiv \frac{e^A_a e^B_b}{\sqrt{g}} \frac{\delta}{\delta g_{ab}} , \qquad \hat{E}^A \equiv \frac{e^A_a}{\sqrt{g}} \frac{\delta}{\delta A_a} , \tag{2.59}$$

such that they reduce to the canonical momenta when acting on the on-shell action, i.e. $\hat{\pi}^{AB} S = \pi^{AB}$ and $\hat{E}^A S = E^A$. Later on, we will make use of a scaling-weight expansion, e.g. $X = \sum_n X^{(n)}$, where the index (n) denotes the scaling weight:

$$\delta_D X^{(n)} = -n X^{(n)} \tag{2.60}$$

Now, we would like to impose the asymptotic boundary conditions:

$$\boxed{\delta_r \approx \delta_D} \tag{2.61}$$

By '$\approx$' we mean equality up to terms that vanish as the radial cut-off is removed. Because of general covariance, the on-shell action must be of the form:

$$S = \int d^d x \sqrt{g}\, U(\psi) + \text{derivative terms} . \tag{2.62}$$

[2] In fact, we could have left the scaling of e^A_a arbitrary as well and have it be determined by the Hamiltonian constraint.

[3] The "middle-of-the-alphabet" indices $I, J, \ldots$ run over the spatial indices alone.

So, at the non-derivative level, the extrinsic curvature and vector-momentum are given by

$$\begin{aligned} K_{AB} &= \frac{U+(d_s-1)(\alpha+\psi)U'}{2d_s}\,\delta^0_A\delta^0_B + \frac{-U+(\alpha+\psi)U'}{2d_s}\,\delta_{IJ}\,\delta^I_A\delta^J_B\,, \\ E_A &= -U'(\psi)\,\delta^0_A\,. \end{aligned} \tag{2.63}$$

Imposing the boundary conditions (2.61) then yields

$$\begin{aligned} K^{(0)}_{AB} &= -z\,\delta^0_A\delta^0_B + \delta_{IJ}\,\delta^I_A\delta^J_B\,, \\ {E_0}^{(0)} + \alpha {K_{00}}^{(0)} &= 0 \\ {E_0}^{(\lambda)} + \alpha {K_{00}}^{(\lambda)} + \psi {K_{00}}^{(0)} &= -\lambda\psi \end{aligned} \tag{2.64}$$

Let us expand $U(\psi) = u_0 + u_1\psi + u_2\psi^2 + \ldots$, such that we can express the boundary conditions as conditions on the coefficients u_k.

$$\begin{aligned} {K_{00}}^{(0)} &= \frac{u_0 + \alpha\,(d_s-1)\,u_1}{2d_s} & {E_A}^{(0)} &= -u_1\,\delta^0_A \\ {K_{00}}^{(\lambda)} &= \frac{d_s\,u_1 + 2\alpha\,(d_s-1)\,u_2}{2d_s}\,\psi & {E_A}^{(\lambda)} &= -2u_2\psi\,\delta^0_A \\ {K_{IJ}}^{(0)} &= \frac{-u_0+\alpha\,u_1}{2d_s}\,\delta_{IJ} \end{aligned} \tag{2.65}$$

When we plug these into the boundary conditions (2.64), we get

$$\begin{aligned} u_0 &= -2(z+d_s-1)\,, & u_1 &= -z\alpha\,, \\ u_2 &= -\frac{zd_s(2z-1-\lambda)}{2(z+d_s-1)}\,, & \alpha &= \sqrt{\frac{2(z-1)}{z}}\,. \end{aligned} \tag{2.66}$$

We have thus fixed our asymptotic boundary conditions by imposing that dilatations δ_D are asymptotically generated by radial variations δ_r. This immediately gives us the first few counterterms:[4]

$$\mathcal{L} = -2(z+d_s-1) - z\alpha\,\psi - \frac{zd_s(2z-1-\lambda)}{2(z+d_s-1)}\,\psi^2 + \ldots\,. \tag{2.67}$$

Up to this point λ was left undetermined, but we will now see that it is fixed by the HJ equation of motion. The Hamiltonian constraint (2.50) can be expanded in terms of scaling weights using δ_D, $\mathcal{H} = \sum_n \mathcal{H}^{(n)}$, where[5]

$$\mathcal{H}^{(n)} = \sum_{p+q=n}\left({K_{AB}}^{(p)}\,\pi^{AB\,(q)} + \frac{1}{2}{E_A}^{(p)}\,E^{A\,(q)} - \frac{m^2}{2}\,\Phi^{(p)}\Phi^{(q)}\right) + \mathcal{V}^{(n)} \tag{2.68}$$

[4] $\mathcal{L}_{\text{c.t.}}$ is just (minus) the divergent part of the on-shell action.

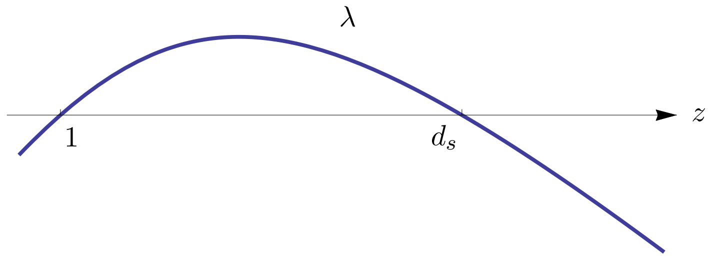
[5] Note that the term ${K_{AB}}^{(p)}\,\pi^{AB\,(q)}$ is symmetric under $p \leftrightarrow q$.

This expansion is useful, because each term must vanish individually so that the HJ equation can be solved recursively.

Let us see if the counterterms that we found by imposing the boundary condition $\delta_r \approx \delta_D$ are consistent with the Hamiltonian constraint. When we plug in (2.67) into the Hamiltonian constraint, we find that $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\lambda)}$ automatically vanish. The next term $\mathcal{H}^{(2\lambda)}$, however, only vanishes if we fix $\lambda = \lambda_\pm$, with

$$\lambda_\pm \; = \; \frac{1}{2}\left(z + d_s \pm \sqrt{(z+d_s)^2 + 8(z-1)(z-d_s)}\right) \, . \tag{2.69}$$

Remember that λ dictates the radial scaling of the non-normalizable mode, $\psi \sim \mathrm{e}^{-\lambda r}$. We must pick the root that corresponds to the non-normalizable mode [1], which is $\max(-\lambda_\pm)$. We thus set $\lambda = \lambda_-$. Eventually we are



interested in the case where $z = d_s = 2$, but for now we take d_s arbitrary, while for the dynamical exponent we restrict our attention to the range for which ψ sources a *relevant* operator ($\lambda > 0$), i.e. $1 < z < d_s$. In Chapter 3 we show that the operator dual to ψ is marginally relevant when $z = d_s$, which at this stage just means that we can safely take the limit $z \to d_s$ when the dust settles.

Useful relation. Now that we have fixed our boundary conditions we are almost ready to start solving the Hamilton–Jacobi equation, thereby generating the counterterm action. Before we do so, however, consider the following identity that will be very useful further down the line:

$$\begin{aligned} & 2K_{AB}{}^{(0)}\,\pi^{AB(n)} + E_A{}^{(0)}\,E^{A(n)} + 2K_{AB}{}^{(\lambda)}\,\pi^{AB(n-\lambda)} + E_A{}^{(\lambda)}\,E^{A(n-\lambda)} \\ & = \; (z + d_s - n)\,\mathcal{L}^{(n)} + K_{AB}{}^{(\lambda)}\,T^{AB(n-\lambda)} + \psi\,K_{0A}{}^{(\lambda)}\,E^{A(n-2\lambda)} \end{aligned} \tag{2.70}$$

where we used the boundary conditions specified above as well as

$$\delta_D \int d^d x \sqrt{g}\, \mathcal{L}^{(n)} \;=\; (z + d_s - n) \int d^d x \sqrt{g}\, \mathcal{L}^{(n)} \,. \tag{2.71}$$

The identity (2.70) will be useful, because the left-hand side appears naturally in the Hamiltonian constraint $\mathcal{H}^{(n)}$, while the right-hand side features the quantity of interest $\mathcal{L}^{(n)}$ directly as well terms that are known from lower orders. More importantly, though, it removes the need for writing down an Ansatz for the on-shell action.

Breakdown of recursion: IR data vs. anomalies

This method only determines the counterterms, which is the part of the on-shell action that depends locally on the UV data set by $\delta_r \approx \delta_D$. The way we know that we have reached the end of the line is when the scaling of the term in the on-shell action is equal to the scaling of the volume form $n = z + d_s$, such that the prefactor of $\mathcal{L}^{(n)}$ in (2.70) vanishes. This means that we have reached the renormalized on-shell action, which depends on IR data. Such a breakdown is therefore to be expected and is completely natural.

There can be another type of breakdown of the recursion relations as well. For instance, in our present set-up we have counterterms with scaling weight $n = 2z$ and $n = 4$. Now consider the case $z = d_s = 2$, such that both of these counterterms scale like the volume form $z + d_s = 4$. Another way of saying that the prefactor of $\mathcal{L}^{(n)}$ in (2.70) vanishes is that the implicit radial derivative vanishes asymptotically, $\delta_r \approx 0$. The terms in the expanded Hamiltonian constraint where this occurs are

$$\mathcal{H}^{(2z)} \;=\; (z + d_s - 2z)\, \mathcal{L}^{(2z)} + \mathcal{H}^{(2z)}_{\text{rem.}} \tag{2.72}$$

$$\mathcal{H}^{(4)} \;=\; (z + d_s - 4)\, \mathcal{L}^{(4)} + \mathcal{H}^{(4)}_{\text{rem.}} \tag{2.73}$$

where $\mathcal{H}^{(n)}_{\text{rem.}}$ is just whatever remains of the Hamiltonian constraint when $(z + d_s - n)\, \mathcal{L}^{(n)}$ is subtracted.

In order to still solve the HJ equation, we must add a counterterm[6] that depends explicitly on the radial cut-off, $r\tilde{\mathcal{L}}^{(4)}$. Remember that the full HJ equation was $\partial_r S + H = 0$, so when we introduce a counterterm that depends

[6] Remember that the counterterm action is just (minus) the divergent part of the on-shell action.

explicitly on r we get a contribution from

$$\partial_r S \;=\; \partial_r S_{\text{ren.}} - \partial_r S_{\text{c.t.}}\,, \qquad \text{where} \qquad \partial_r S_{\text{c.t.}} \;=\; -\int d^d x \sqrt{g}\, \tilde{\mathcal{L}}^{(4)}\,. \tag{2.74}$$

The new counterterm can then be computed from the HJ equation at weight $n = 4$:

$$0 \;=\; \tilde{\mathcal{L}}^{(4)} + \left(\mathcal{H}^{(2z)}_{\text{rem.}} + \mathcal{H}^{(4)}_{\text{rem.}}\right)\Big|_{z=d_s=2} \tag{2.75}$$

This be expressed equivalently in terms of $\mathcal{L}^{(2z)}$ and $\mathcal{L}^{(4)}$ as

$$\tilde{\mathcal{L}}^{(4)} \;=\; \Big((z + d_s - 2z)\,\mathcal{L}^{(2z)} + (z + d_s - 4)\,\mathcal{L}^{(4)}\Big)\Big|_{z=d_s=2} \tag{2.76}$$

The need for introducing the counterterm $r\tilde{\mathcal{L}}^{(4)}$ is related to the presence of a Weyl-type anomaly. The (integrated) anomaly $\mathcal{A}$ is given by the total r-derivative of the renormalized action

$$\int d^d x \sqrt{g}\, \mathcal{A} \;=\; \frac{dS_{\text{ren.}}}{dr} \;=\; \partial_r S_{\text{ren.}} + \delta_r S_{\text{ren.}}\,. \tag{2.77}$$

We know that the renormalized action has a finite limit as $r \to \infty$ (by definition), so

$$\frac{dS_{\text{ren.}}}{dr} \;\approx\; 0 \qquad \Rightarrow \qquad \delta_r S_{\text{ren.}} \;\approx\; -\partial_r S_{\text{ren.}} \;=\; -\partial_r S_{\text{c.t.}} \tag{2.78}$$

In the last equality we used $\partial_r S_{\text{ren.}} = \partial_r S_{\text{c.t.}}$, due to $\partial_r S = 0$ (general covariance in the bulk). On the other hand, the anomaly $\mathcal{A}$ can be defined as

$$\delta_D S_{\text{ren.}} \;=\; \int dt\, d^{d_s}x\, N\sqrt{h}\; \mathcal{A} \tag{2.79}$$

Then, when we use the boundary condition $\delta_D \approx \delta_r$, we finally find that the holographic Lifshitz anomaly is given by

$$\boxed{\mathcal{A} \;=\; -\lim_{r\to\infty} \left(\mathcal{H}^{(2z)}_{\text{rem.}} + \mathcal{H}^{(4)}_{\text{rem.}}\right)\Big|_{z=d_s=2}} \tag{2.80}$$

2.3.2 The Counterterms

All that is left now is to start solving the higher-order Hamiltonian constraints. A detailed computation of the non-derivative counterterms is done in Chapter 3. The focus of this chapter, however, is on the Lifshitz anomaly, which we know does not involve non-derivative terms. So for the purpose of computing the central charges C_1 and C_2, it suffices to just compute the non-derivative counterterms up to $n = 2\lambda$, cf. (2.67). It is useful to pick a specific set of coordinates (t, x^i) that reflects the anisotropy of the Lifshitz geometry, such that the metric and vector become

$$\begin{aligned} ds^2 &= dr^2 + g_{ab}\, dx^a dx^b = dr^2 + N^2 dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt)\,, \\ \tilde{A} &= \Phi\, dr + A_a dx^a = \Phi\, dr + (\alpha + \psi)\, N dt\,, \end{aligned} \tag{2.81}$$

where N, N^i and h_{ij} are the lapse, shift and induced metric associated to the time-like foliation of the radial slices. Equivalently, we can express the frame fields in terms of these decomposed fields. Let us define spatial frame fields $\hat{e}^I_i$, such that $h_{ij} = \delta_{IJ}\,\hat{e}^I_i \hat{e}^J_j$. The frame fields $e^A = e^A_a\, dx^a$ are then decomposed as

$$e^0 = N dt\,, \qquad e^I = \hat{e}^I_i \left(dx^i + N^i dt\right)\,. \tag{2.82}$$

The radial scaling of these fields are set by the boundary conditions (2.61), where

$$\delta_D N = zN\,, \quad \delta_D N^i = 0\,, \quad \delta_D h_{ij} = 2h_{ij}\,, \quad \delta_D \psi = -\lambda\psi\,. \tag{2.83}$$

The dilatation operator can thus be written as

$$\delta_D = \int d^d x \left(zN\frac{\delta}{\delta N} + 2h_{ij}\frac{\delta}{\delta h_{ij}} - \lambda\psi\frac{\delta}{\delta\psi} \right) \tag{2.84}$$

The 'potential' $\mathcal{V}$, cf. (A.26), can then be split up into terms with given scaling weight. The non-derivative terms are:

$$\mathcal{V}^{(0)} = (z + d_s)(z + d_s - 1)\,, \quad \mathcal{V}^{(\lambda)} = z d_s \alpha\,\psi\,, \quad \mathcal{V}^{(2\lambda)} = \frac{z d_s}{2}\,\psi^2\,, \tag{2.85}$$

while the derivative terms are:[7]

$$\begin{aligned} \mathcal{V}^{(2)} &= \hat{R} + \frac{z-1}{z}\,\frac{\partial_i N\,\partial^i N}{N^2}\,, \\ \mathcal{V}^{(2+\lambda)} &= \alpha\psi\left(\frac{\partial_i N\,\partial^i N}{N^2} - \frac{\Delta N}{N}\right)\,, \\ \mathcal{V}^{(2+2\lambda)} &= \frac{1}{2}\partial_i\psi\,\partial^i\psi + \frac{1}{2}\psi^2\left(\frac{\partial_i N\,\partial^i N}{N^2} - \frac{\Delta N}{N}\right)\,, \\ \mathcal{V}^{(2z)} &= \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2\,. \end{aligned} \tag{2.86}$$

[7]We discarded exact terms.

We introduced the intrinsic and extrinsic curvatures on the equal-time slice, $\hat{R}_{ij}$ and $\hat{K}_{ij} = \frac{1}{2N}(\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i)$, respectively. The radial component of the vector field is computed recursively:

$$\Phi = -\frac{1}{m^2}\nabla_a E^a = -\frac{1}{z d_s}\nabla_a\left(e^{0a}E_0\right) = -\frac{1}{z d_s N}\left(N\hat{K} - \partial_t + N^i\partial_i\right)E_0\,, \tag{2.87}$$

where we used $\nabla_a e^{0a} = \hat{K}$ as well as $e^{0a}\partial_a = N^{-1}(-\partial_t + N^i\partial_i)$. The term with scaling weight n is given by

$$\Phi^{(n)} = -\frac{1}{z d_s N}\left(N\hat{K} - \partial_t + N^i\partial_i\right){E_0}^{(n-z)} \tag{2.88}$$

The stress tensor can be computed directly in terms of the ADM variables:

$$\begin{aligned} T^{00} &= -\frac{1}{\sqrt{h}}\frac{\delta S}{\delta N}\,, \\ T^{0I} &= \frac{1}{N\sqrt{h}}\hat{e}^I_i\frac{\delta S}{\delta N_i}\,, \\ T^{IJ} &= \frac{1}{N\sqrt{h}}\left(N^I\hat{e}^J_j\frac{\delta S}{\delta N_j} + 2\hat{e}^I_i\hat{e}^J_j\frac{\delta S}{\delta h_{ij}}\right)\,. \end{aligned} \tag{2.89}$$

Solving the Hamiltonian constraint. Let us start by computing the counterterms at weight $n = 2$. The Hamiltonian constraint at $n = 2$ is

$$\begin{aligned} \mathcal{H}^{(2)} &= 2{K_{AB}}^{(0)}\,\pi^{AB\,(2)} + {E_A}^{(0)}\,E^{A\,(2)} + \mathcal{V}^{(2)} \\ &= (z + d_s - 2)\,\mathcal{L}^{(2)} + \hat{R} + \frac{z-1}{z}\frac{\partial_i N\,\partial^i N}{N^2} \end{aligned} \tag{2.90}$$

where we used the identity (2.70) as well as ${K_{ab}}^{(2-\lambda)} = {E_a}^{(2-\lambda)} = {E_a}^{(2-2\lambda)} = 0$. So, the $n = 2$ Hamiltonian constraint is solved by

$$\mathcal{L}^{(2)} = \frac{-1}{z + d_s - 2}\left(\hat{R} + \frac{z-1}{z}\frac{\partial_i N\,\partial^i N}{N^2}\right) \tag{2.91}$$

We see that this counterterm has a pole at $z = d_s = 1$, which is related the the 2D Weyl anomaly. Moving on, at $n = 2 + \lambda$ we get:

$$\mathcal{L}^{(2+\lambda)} = \frac{1}{z + d_s - 2 - \lambda}\left(-{K_{AB}}^{(\lambda)}\,T^{AB\,(2)} + \mathcal{V}^{(2+\lambda)}\right)\,, \tag{2.92}$$

where $T^{AB\,(2)}$ is directly computed from $\mathcal{L}^{(2)}$ using (2.89):

$$\begin{aligned} T^{00\,(2)} &= \frac{1}{z+d_s-2}\left(\hat{R}+\frac{z-1}{z}\,\frac{\partial_i N\,\partial^i N}{N^2}-\frac{2(z-1)}{z}\,\frac{\Delta N}{N}\right) \\ T^{IJ\,(2)} &= \frac{-1}{z+d_s-2}\Bigg[\left(\hat{R}+\frac{z-1}{z}\,\frac{\partial_i N\,\partial^i N}{N^2}-2\,\frac{\Delta N}{N}\right)\delta^{IJ} \\ &\qquad -2\hat{R}^{IJ}-\frac{2(z-1)}{z}\,\frac{\partial^I N\,\partial^J N}{N^2}+2\frac{D^I\partial^J N}{N}\Bigg] \end{aligned} \tag{2.93}$$

while $K_{ab}{}^{(\lambda)}$ was fixed by the boundary conditions:

$$\begin{aligned} K_{00}{}^{(\lambda)} &= \frac{(d_s-1)\lambda-(2d_s-1)z}{2\,(z+d_s-1)}\,z\alpha\psi \\ K_{IJ}{}^{(\lambda)} &= \frac{\lambda+1-2z}{2\,(z+d_s-1)}\,z\alpha\psi\,\delta_{IJ} \end{aligned} \tag{2.94}$$

The counterterm at this order is thus computed from (2.92). The reason why we go through the trouble of computing the $n=2+\lambda$ counterterm is that we shall need to know $E_0^{(2)}$ in order to compute the $n=4$ counterterm. So, for future reference, we computed:

$$\begin{aligned} E^{0\,(2)} &= \frac{1}{\sqrt{g}}\,\frac{\delta S^{(2+\lambda)}}{\delta\psi} \\ &= \frac{\alpha z\,(d_s-\lambda+3z-2)}{2\,(d_s+z-2)\,(d_s+z-1)\,(d_s-\lambda+z-2)}\,\hat{R} \\ &\quad +\frac{\alpha\left((5-3z)d_s-2d_s^2+(z-1)(-\lambda+z+2)\right)}{2N^2\,(d_s+z-2)\,(d_s+z-1)\,(d_s-\lambda+z-2)}\,\frac{\partial_i N\,\partial^i N}{N^2} \\ &\quad +\frac{\alpha\,(d_s-1)\,(d_s-\lambda+3z-2)}{N\,(d_s+z-2)\,(d_s+z-1)\,(d_s-\lambda+z-2)}\,\frac{\Delta N}{N} \end{aligned} \tag{2.95}$$

The orders $n=2+2\lambda$ etc. are computed straightforwardly. We will not compute such terms explicitly, though, for they are not needed to compute the $z=d_s=2$ anomaly. Moving on, let us consider $n=2z$

$$\begin{aligned} \mathcal{H}^{(2z)} &= 2K_{AB}{}^{(0)}\,\pi^{AB\,(2z)}+E_A{}^{(0)}\,E^{A\,(2z)}+\frac{zd_s}{2}\Phi^{(z)}\Phi^{(z)}+\mathcal{V}^{(2)} \\ &= (z+d_s-2z)\,\mathcal{L}^{(2z)}+\hat{K}_{ij}\hat{K}^{ij}+\frac{z-d_s-1}{d_s}\hat{K}^2 \end{aligned} \tag{2.96}$$

Thus, we find

$$\mathcal{L}^{(2z)} = \frac{-1}{d_s-z}\left(\hat{K}_{ij}\hat{K}^{ij}+\frac{z-d_s-1}{d_s}\hat{K}^2\right) \tag{2.97}$$

Again, we could easily compute $n = 2z + \lambda$ etc., but we focus on the terms relevant for find the anomaly. For this, we move on to $n = 4$:

$$\mathcal{H}^{(4)} = 2K_{AB}{}^{(0)}\,\pi^{AB(4)} + E_A{}^{(0)}\,E^{A(4)} + K_{AB}{}^{(2)}\,\pi^{AB(2)} + \frac{1}{2}E_A{}^{(2)}\,E^{A(2)} \tag{2.98}$$

$$= (z + d_s - 4)\,\mathcal{L}^{(4)} + K_{AB}{}^{(2)}\,\pi^{AB(2)} + \frac{1}{2}E_A{}^{(2)}\,E^{A(2)} \tag{2.99}$$

Interestingly, we find that when $z = d_s = 2$:

$$K_{AB}{}^{(2)}\,\pi^{AB(2)} + \frac{1}{2}E_A{}^{(2)}\,E^{A(2)} = 0\,, \tag{2.100}$$

which means that even though there is an "anomalous" breakdown of the recursion relations at $n = 4$ when $z = d_s = 2$, it does *not* generate a contribution to the Weyl-type anomaly.

2.3.3 Holographic Lifshitz Anomaly

All that is left now is to apply (2.80). We find that the holographic Lifshitz anomaly is given by[8]

$$\mathcal{A} = \frac{\ell^2}{16\pi G}\left(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2\right) \tag{2.101}$$

Comparing with the generic form of the anomaly (2.11) gives us the values for the central charges:

$$\boxed{C_1 = \frac{\ell^2}{2G}\,, \qquad C_2 = 0\,.} \tag{2.102}$$

It is striking that the second central charge, which is associated to spatial curvature, turns out to vanish for both the minimally-coupled scalar model and the holographic massive-vector model. At this point it remains unclear whether there is a deeper reason behind the vanishing of C_2.

Notice that this situation is quite similar to the $a = c$ degeneracy that we find for 4D CFT's dual to Einstein gravity with AdS_4 boundary conditions. In [44] it was argued that a finite value for C_2 can be obtained in the Lifshitz scalar model be introducing a non-minimal coupling:

$$\varphi^2\left(R + \frac{\Delta N}{N} - \frac{\partial_i N \partial^i N}{N^2}\right)^2 \tag{2.103}$$

[8]We reinstate Newton's constant and the Lifshitz length scale, which comes down to replacing $\mathcal{L}^{(n)} \to \frac{\ell^{n-1}}{16\pi G}\,\mathcal{L}^{(n)}$ and $r \to r/\ell$.

Similarly, one can argue that adding the term $\left(R + \frac{\Delta N}{N} - \frac{\partial_i N \partial^i N}{N^2}\right)^2$ to the massive-vector theory will yield a non-zero C_2.[9] One will, however, break general covariance explicitly. A different approach where general covariance is reduced to invariance under foliation-preserving diffeomorphisms was proposed in [44], where the low-energy limit of Hořava–Lifshitz gravity was put forward as a holographic model. Although a precise calculation was never made public, one can imagine that this model would naturally yield a non-zero C_2.

2.4 Casimir energy for $z = 2$ Lifshitz theories

One thing that the knowledge of the two central charges will give us is the Casimir energy and spatial stress tensor. This would be a first step towards a possible comparison of the results obtained in this chapter with experiments. Of course, it is not possible to measure fluctuations of the background geometry directly, but one could imagine that there might be a way to relate the central charges to e.g. finite-size effects.

Knowledge of the Lifshitz anomaly will enable us to compute the Casimir energy in a Lifshitz theory on a conformally flat background, by which we mean

$$ds^2 = -N^2 dt^2 + h_{ij}\, dx^i dx^j\,, \qquad N = \mathrm{e}^{z\sigma}\,, \qquad h_{ij} = \mathrm{e}^{2\sigma}\,\delta_{ij}\,. \tag{2.104}$$

Here, σ is an arbitrary function of time and space. Before we dive into the calculation, let us first state the result. The Casimir energy for any three-dimensional $z = 2$ Lifshitz theory on a conformally flat background is given by:

$$\langle \mathcal{E} \rangle = \frac{1}{8\pi}\,\mathrm{e}^{-4\sigma}\Big[C_1\,\dot{\sigma}^2 + C_2\Big(-4\sigma_{,kkll} + 4\sigma_{,k}\sigma_{,kll} + 4\sigma_{,kl}\sigma_{,kl}\Big)\Big]\,, \tag{2.105}$$

Similarly, we find that the Casimir stress tensor is:

$$\begin{aligned}\langle \Pi^i{}_j \rangle = \frac{1}{8\pi}\,\mathrm{e}^{-4\sigma}\Big\{&C_1\left(\ddot{\sigma} - \dot{\sigma}^2\right)\delta^i_j \\ &+ C_2\Big[2\delta^{im}\Big(-4\sigma_{,mjkk} + 4\sigma_{,mjk}\sigma_{,k} + 4\sigma_{,mk}\sigma_{,jk}\Big) \\ &\qquad - 2\delta^i_j\Big(-4\sigma_{,kkll} + 4\sigma_{,k}\sigma_{,kll} + 4\sigma_{,kl}\sigma_{,kl}\Big)\Big]\Big\}\end{aligned} \tag{2.106}$$

[9] It was claimed that $C_2 = \beta\ell^2/48G$ for Hořava–Lifshitz gravity. The parameter β is the gravitational equivalent of the coupling c in (I.10). In other words, it appears in the effective IR Hořava–Lifshitz Lagrangian as $\mathcal{L} \sim K^2 + \beta R$.

This result does not yet appear in the literature, so we will show the calculation in a little more detail below. It is based on the work done in [58, 59] for the relativistic case.

Casimir Energy for $z = d_s$ Lifshitz theories

In this section we extend the work of [58, 59] in order to obtain the Casimir energy and the Casimir stress tensor for theories with anisotropic scaling $(x^i, t) \to (bx^i, b^z t)$ with $z = d_s$. After finding the generic expressions for any number of spatial dimensions d_s, we apply it to case of $z = d_s = 2$. In the spirit of the conformal anomaly, we wish to put our theory on a curved background. We pick some background (N, h_{ij}), whose corresponding line element is given by:

$$ds^2 = -N^2 dt^2 + h_{ij}\, dx^i dx^j \,. \tag{2.107}$$

The definition of the anisotropic Weyl anomaly $\mathcal{A}$ is

$$\mathcal{A} \equiv z\langle \mathcal{E} \rangle + \langle {\Pi^i}_i \rangle = \frac{1}{N\sqrt{h}} \left(zN \frac{\delta W}{\delta N} + 2h_{ij} \frac{\delta W}{\delta h_{ij}} \right) , \tag{2.108}$$

where $W[N, h] \equiv -\log Z[N, h]$ is the generating functional. We view (N, h_{ij}) as Weyl deformations of some background $(\bar{N}, \bar{h}_{ij})$, such that

$$N(x) = \mathrm{e}^{z\sigma(x)} \bar{N}(x) \,, \qquad h_{ij}(x) = \mathrm{e}^{2\sigma(x)} \bar{h}_{ij}(x) \,. \tag{2.109}$$

This allows us to relate the scale transformation in the definition of the anomaly (2.108) to a variation with respect to $\sigma(x)$:

$$zN \frac{\delta W}{\delta N} + 2h_{ij} \frac{\delta W}{\delta h_{ij}} = \frac{\delta W}{\delta \sigma} \,. \tag{2.110}$$

The σ-variation of the expectation value of the stress-tensor complex $(\mathcal{E}, \Pi^{ij})$ can in turn be written as

$$\begin{aligned} &\frac{\delta}{\delta\sigma(y)} \left(N\sqrt{h}\, \langle \mathcal{E} \rangle \right)(x) \\ &\qquad = \left(zN(y) \frac{\delta}{\delta N(y)} + 2h_{ij}(y) \frac{\delta}{\delta h_{ij}(y)} \right) N(x) \frac{\delta W}{\delta N(x)} \,, \end{aligned} \tag{2.111}$$

$$\begin{aligned} &\frac{\delta}{\delta\sigma(y)} \left(N\sqrt{h}\, \langle {\Pi^i}_j \rangle \right)(x) \\ &\qquad = \left(zN(y) \frac{\delta}{\delta N(y)} + 2h_{kl}(y) \frac{\delta}{\delta h_{kl}(y)} \right) 2h_{jm}(x) \frac{\delta W}{\delta h_{im}(x)} \,. \end{aligned} \tag{2.112}$$

The next ingredient we will use is the commutation relations

$$\left[\left(zN(y)\frac{\delta}{\delta N(y)} + 2h_{ij}(y)\frac{\delta}{\delta h_{ij}(y)}\right),\ N(x)\frac{\delta}{\delta N(x)}\right] = 0\,, \tag{2.113}$$

$$\left[\left(zN(y)\frac{\delta}{\delta N(y)} + 2h_{kl}(y)\frac{\delta}{\delta h_{kl}(y)}\right),\ h_{jm}(x)\frac{\delta}{\delta h_{im}(x)}\right] = 0\,, \tag{2.114}$$

which yields

$$\frac{\delta}{\delta\sigma(y)}\Big(N\sqrt{h}\,\langle\mathcal{E}\rangle\Big)(x) = N(x)\frac{\delta}{\delta N(x)}\Big(N\sqrt{h}\,\mathcal{A}\Big)(y)\,, \tag{2.115}$$

$$\frac{\delta}{\delta\sigma(y)}\Big(N\sqrt{h}\,\langle\Pi^i{}_j\rangle\Big)(x) = 2h_{jk}(x)\frac{\delta}{\delta h_{ik}(x)}\Big(N\sqrt{h}\,\mathcal{A}\Big)(y)\,. \tag{2.116}$$

Following the reasoning of [58], we employ dimensional regularization, i.e. we analytically continue $d_s \to n = d_s + \varepsilon$ (whilst leaving $z = d_s$ unaltered). We introduce a shifted anomaly $\mathcal{A}_\varepsilon$ defined in $n+1$ spacetime dimensions in such a way that $\lim_{\varepsilon\to 0}\mathcal{A}_\varepsilon = \mathcal{A}$ while at the same time preserving the original (anisotropic) scaling relation $\delta_\sigma\mathcal{A}_\varepsilon = -2d_s\mathcal{A}_\varepsilon$, such that

$$\begin{aligned}\frac{\delta\big(N\sqrt{h}\mathcal{A}\big)_\varepsilon}{\delta\sigma} &= (z+n-2d_s)\big(N\sqrt{h}\mathcal{A}\big)_\varepsilon = (n-d_s)\big(N\sqrt{h}\mathcal{A}\big)_\varepsilon \\ &= \varepsilon\big(N\sqrt{h}\mathcal{A}\big)_\varepsilon\end{aligned} \tag{2.117}$$

This can typically be achieved by leaving the scaling behavior of the background fields unaltered, i.e. $\delta_\sigma N = zN = d_sN$ and $\delta_\sigma h_{ij} = 2h_{ij}$. This prescription allows us rewrite the right-hand sides of (2.115) and (2.116) by using the shifted scaling relation:

$$\Big(N\sqrt{h}\,\mathcal{A}\Big)(y) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{\delta}{\delta\sigma(y)}\int d^nx\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon\,, \tag{2.118}$$

such that e.g.

$$\frac{\delta}{\delta\sigma(y)}\Big(N\sqrt{h}\,\langle\mathcal{E}\rangle\Big)(x) = N(x)\frac{\delta}{\delta N(x)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{\delta}{\delta\sigma(y)}\int d^ny'\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon\,. \tag{2.119}$$

We wish to integrate this differential equation with respect to σ. This is still difficult, though, since the σ-derivative on the right-hand side can generally *not* be pulled to the left consistently.

Conformally flat backgrounds

Let us focus on the specific case of a *conformally flat* background, by which we mean that

$$N(x) \;=\; \mathrm{e}^{z\sigma(x)}\,, \qquad\qquad h_{ij}(x) \;=\; \mathrm{e}^{2\sigma(x)}\delta_{ij}\,, \tag{2.120}$$

with $z = d_s$ and $x = (\vec{x}, t)$. Restricting ourselves to such backgrounds allows us to integrate (2.119). To be more specific, we can interchange the σ-variation and the N-variation, such that

$$\frac{\delta}{\delta\sigma(y)}\Big(N\sqrt{h}\,\langle\mathcal{E}\rangle\Big)(x) \;=\; N(x)\frac{\delta}{\delta N(x)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{\delta}{\delta\sigma(y)}\int d^n y'\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon \tag{2.121}$$

$$\;=\; \frac{\delta}{\delta\sigma(y)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}N(x)\frac{\delta}{\delta N(x)}\int d^n y'\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon\,. \tag{2.122}$$

We fixed the background to be conformally flat (2.120) to ensure that

$$\frac{\delta}{\delta N(x)}\int d^n y'\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon \;=\; O(\varepsilon)\,, \tag{2.123}$$

$$\frac{\delta}{\delta h_{ij}(x)}\int d^n y'\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon \;=\; O(\varepsilon)\,. \tag{2.124}$$

Namely, it would instead be of order $O(1)$ if we were to use a generic background. We now integrate this differential equation with the boundary condition that the stress-tensor complex vanishes on a flat background $\sigma = \text{constant}$. This gives our final result for the *Casimir energy*:

$$\langle\mathcal{E}\rangle \;=\; \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{1}{\sqrt{h}}\frac{\delta}{\delta N}\int d^n x\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon\,, \tag{2.125}$$

and similarly, the *Casimir stress tensor*:

$$\langle{\Pi^i}_j\rangle \;=\; \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{2h_{jk}}{N\sqrt{h}}\frac{\delta}{\delta h_{ik}}\int d^n x\,\big(N\sqrt{h}\,\mathcal{A}\big)_\varepsilon\,. \tag{2.126}$$

The specific case of $z = d_s = 2$

In this chapter we found that, for $z = d_s = 2$, the anomaly is of the form

$$\mathcal{A} \;=\; \frac{C_1}{8\pi}G^{ijkl}\,K_{ij}K_{kl} + \frac{C_2}{8\pi}\left(R + \frac{\nabla_i\nabla^i N}{N} - \frac{\nabla_i N\,\nabla^i N}{N^2}\right)^2\,, \tag{2.127}$$

where $K_{ij} = \dot{h}_{ij}/2N$ is the extrinsic curvature on a constant-time surface, ∇_i is the covariant derivative compatible with the spatial metric h_{ij} and $G_{ijkl} \equiv h_{i(k}h_{l)j} - \frac{1}{2}h_{ij}h_{kl}$. The form of the anomaly is universal, i.e. the only way the details of a theory can enter in the anomaly is through the central charges C_1 and C_2. The same must be true of the vacuum expectation value of the stress-tensor complex, since it can be derived from the anomaly directly.

We define our shifted anomaly $\mathcal{A}_\varepsilon$ by setting $d_s \to n = d_s + \varepsilon = 2 + \varepsilon$, whilst keeping $z = d_s = 2$. In other words, we keep the scaling of the fields unchanged, i.e. $\delta_\sigma N = zN = 2N$ and $\delta_\sigma h_{ij} = 2h_{ij}$. Doing so gives us the following expression for the Casimir energy and stress tensor by simply applying . We have put the intermediate steps in this calculation Appendix VI. The result is:

$$\langle \mathcal{E} \rangle = \frac{1}{8\pi}\,\mathrm{e}^{-4\sigma}\Big\{C_1\,\dot{\sigma}^2 + C_2\Big(-4\sigma_{,kkll} + 4\sigma_{,k}\sigma_{,kll} + 4\sigma_{,kl}\sigma_{,kl}\Big)\Big\}\,, \tag{2.128}$$

$$\begin{aligned}\langle \Pi^i{}_j \rangle = \frac{1}{8\pi}\,\mathrm{e}^{-4\sigma}\Big\{ & C_1\left(\ddot{\sigma} - \dot{\sigma}^2\right)\delta^i_j \\ & + C_2\Big[2\delta^{im}\Big(-4\sigma_{,mjkk} + 4\sigma_{,mjk}\sigma_{,k} + 4\sigma_{,mk}\sigma_{,jk}\Big) \\ & - 2\delta^i_j\Big(-4\sigma_{,kkll} + 4\sigma_{,k}\sigma_{,kll} + 4\sigma_{,kl}\sigma_{,kl}\Big)\Big]\Big\}\,.\end{aligned} \tag{2.129}$$

Notice that the trace is not identically zero,

$$z\langle \mathcal{E} \rangle + \langle \Pi^i{}_i \rangle = 4C_1\,\ddot{\sigma}\,\mathrm{e}^{-4\sigma}\,. \tag{2.130}$$

This term is, however, trivial in the sense that it can be removed by adding a finite local counterterm to the action. The leftover term can be seen to come from the covariant term ($\partial_n = N^{-1}\partial_t$ as before):

$$z\langle \mathcal{E} \rangle + \langle \Pi^i{}_i \rangle = 2C_1\,h^{ij}\partial_n K_{ij}\,. \tag{2.131}$$

We can add a counterterm that relates this term to the one we had initially, $G^{ijkl}K_{ij}K_{kl}$. The counterterm that does this is:

$$W \to W + c\int dt\,d^2x\,N\sqrt{h}\,K^2\,, \tag{2.132}$$

where c is some appropriately chosen real number and $K = h^{ij}K_{ij}$ is the trace of the extrinsic curvature, see Appendix IV for a detailed discussion. We are then left with a leftover term proportional to $G^{ijkl}K_{ij}K_{kl}$, which vanishes on a conformally flat background. Thus we see that the trace vanishes indeed:

$$z\langle \mathcal{E} \rangle + \langle \Pi^i{}_i \rangle = 0\,. \qquad \checkmark \tag{2.133}$$

It is interesting to see that the analysis done in [58, 59] can be extended to the case of $z = d_s$. The Casimir energy thus computed could be seen as a first step towards comparing the values of the central charges with experiments. For instance, one could imagine that the central charge C_2 that is related to spatial fluctuations might be related to finite-size effects in systems effective described by $z = 2$ Lifshitz-type theories.

2.5 Two-dimensional central charge and C_1

As an aside, we would like to speculate on a possible relation between the central charge C_1 and the charge c of a two-dimensional CFT. This relation comes from the study of entanglement entropy. The notion of entanglement entropy will be explained in somewhat more detail shortly. For now, we just state some specific results. This is an example where a calculation on both sides of the correspondence is possible. On the field-theory side, one can explicitly compute entanglement entropy in the ground state [38, 39]. This is possible because the detailed balance condition can be used to find that the ground-state wave functional has the structure of a 2D CFT [12], cf. discussion on page 38. On the other hand, the holographic computation makes no distinction between AdS and Lifshitz, given that the (generalized) Wald charge [45] reduces to the area law for the specific holographic model. The entanglement entropy associated to an entangling surface with a cusp was computed in field theory [39] and in holography [46]. The general form of the entanglement entropy is

$$S_{\text{ent}} \;=\; \frac{\ell}{\varepsilon} + f(\Omega)\,\ln\frac{\varepsilon}{\ell} + \text{finite}\,, \tag{2.134}$$

where ℓ is the typical size of the subdomain and ε is a short-distance cut-off. The coefficient $f(\Omega)$ of the universal log has a simple pole at the point where the cusp angle $\Omega \to 0$. At small values of the cusp angle the entanglement entropy probes only short distances, so it is generally expected (but not proved) that universal information in the entanglement entropy of a cusp is contained in the residue. The two different models give the following residues:

$$\operatorname*{Res}_{\Omega=0} f(\Omega) \;=\; \begin{cases} \frac{\pi}{24}\,c & \text{(field theory)} \\[2ex] \frac{4\pi^3}{\Gamma(1/4)^4}\,\frac{\ell^2}{2G} & \text{(holography)} \end{cases} \tag{2.135}$$

It is tempting to say that the central charges may be related as

$$\frac{C_1}{c} \;=\; \frac{\Gamma\left(\frac{1}{4}\right)^4}{96\pi^2} \tag{2.136}$$

This would mean, however, that the 2D central charge is fixed to a non-standard value $c = 24\pi^2/\Gamma(1/4)^4 \approx 1.37$, because $C_1 = 1/4$ for the Lifshitz scalar. The central charge for a single free scalar is usually chosen such that $c = 1$. It should be noted, though, that we compare two different regimes in the above regio C_1/c. Namely, C_1 is computed at strong coupling, while c is computed in a free theory. The ratio might be corrected in a way similar to how the free energy of $\mathcal{N} = 4$ Yang–Mills differs by a factor of 3/4 between weak and strong coupling. It would be very interesting if a relation between the two different central charges would exist, though at this stage it is not much more than a hunch (so please do take these comments with pinch of salt).

Conclusion

In this chapter we computed the anisotropic Weyl anomaly of two Lifshitz-type theories. The first theory was just a simple scalar field theory known as the quantum Lifshitz model (2.20), while the second was a holographic model known as the massive-vector model. Although it is unclear whether it is sensible to compare the two anomalies, we did notice that the anomalies seem quite similar in that one of the two central charges vanishes. The ratio of the non-zero central charges is $2\ell^2/G$.

An aspect that remains unclear in this analysis is that the dynamical critical exponent is in general renormalized [47], while the conformal anomaly is only present when $z = d_s$ exactly. It is tempting to argue that there might be a mechanism that protects the value of $z = 2$, although such a mechanism remains unknown at present.

It would also be interesting to repeat the analysis in Appendix IV, allowing for time-reversal symmetry to be broken. This could possibly lead to contributions in the anomaly with an odd number of time derivatives, which would lead to terms that mix spatial and time derivatives. Another generalization to the anomaly in the Lifshitz scalar theory might be come from coupling the scalar to a gauge field, which may yield an additional term in the anomaly, see e.g. [40] for the $z = 1$ case.

As mentioned before, one of the main uses of the conformal anomaly is that it is a relatively simple property of a field theory which sometimes gives rise to certain universal properties. For instance, in the relativistic case, in $d = 2$ the conformal anomaly completely fixes the free energy at high temperatures, and the central charges also control the universal terms in the entanglement entropy in $d = 2, 4$. In Section 3.1 a first step is made in identifying these universal terms in the $z \neq 1$ case.

We concluded this chapter with two possible extensions of this work. In Section 2.4 we took a first step towards identifying the central charges to physical quantities that could in principle be measured. Although more work is needed to phrase a concrete proposal for measurement, it is still interesting that the knowledge of the Lifshitz anomaly allowed us to compute the Casimir energy and stress tensor. In Section 2.5 we speculated about a possible connection between the central charge C_1 and the two-dimensional central charge c, inspired by the connection between $z = 2$ Lifshitz theories and two-dimensional CFT's (cf. discussion on page 38).

Chapter 3

Marginally Relevant Operator in z=2 Lifshitz Holography

In this chapter we continue our holographic analysis from Chapter 2. It turns out that the solutions of the Einstein–Proca field equations contain a logarithmic branch when $z = d_s = 2$. In particular, the leading behavior of the metric is no longer simply a power of the radial coordinate; it contains leading logarithms.[1] In [43] it was argued that these leading logs are related to having a marginally relevant operator in the system. We will build on this observation, although our way of renormalizing the on-shell action will be radically different. In particular, we show that one can renormalize the on-shell action by adding only local counterterms without the need to introduce explicit dependence on the radial cutoff, contrary to what was claimed in [43, 48].

In order to perform holographic renormalization, one needs to specify boundary conditions for the fields. In Chapter 1 we saw that there is a natural way to fix these boundary conditions by fixing the radial scaling of the fields. This input is restrictive enough to renormalize on-shell action, while at the same time it is lenient enough to allow for the presence of these leading logarithms. For simplicity, we assume translational invariance in the boundary directions. Our method for renormalizing the on-shell action is essentially based on the results of [1], though the special case of $z = 2$ will bring some interesting new features. In particular, we find that the renormalized on-shell action will be a non-analytic function of the only Lorentz scalar one can construct at the non-derivative level: the square of the massive vector. To be precise, we use the scalar perturbation ψ defined as the shift in the vector away from the constant background value: $A_a = (1 + \psi)\, e^0_a$. Conversely, ψ can be expressed in terms of the square of the Proca field as $\psi = \sqrt{-A^2} - 1$.

Before we discuss holographic renormalization we first construct a holo-

[1] As before, we call $\mathrm{e}^{\#r}$ power-law and $r^{\#}$ logarithms.

graphic RG flow that interpolates between a Lifshitz-like fixed point in the UV and a conformally invariant fixed point in the IR. This Lifshitz-to-AdS flow can be seen as a result of turning on the marginally relevant operator. We check that it makes sense to view AdS as an IR solution by studying the universal part of the entanglement entropy in which the entangling surface is a strip. This was proposed as a promising candidate c-function in [49] for $z = 1$ and $d \geq 3$ boundary dimensions. We check whether this same proposed c-function decreases monotonically as one follows the RG flow from the UV $z = 2$ fixed point to the IR $z = 1$ fixed point.

Structure of the chapter. This chapter is organized as follows. Below, we give a brief overview of some interesting properties of the massive vector model. We explain why the case where the dynamical exponent is equal to the number of (boundary) spatial dimensions ($z = d_s$) is special. In Section 3.1 we study the Lifshitz-to-AdS flow using the entanglement entropy of a strip, which we compute holographically. In the process, we derive a nice and simple expression for Myers' entanglement c-function. Finally, Section 3.2 contains the main discussion of this chapter, which is holographic renormalization in the presence of the marginally relevant operator.

Special nature of $z = d_s$ in the massive-vector model

Before we continue our discussion, let us mention some interesting facts about Lifshitz systems with critical values of the dynamical exponent, $z = d_s$, where d_s is the number of spatial dimensions on the field theory side. Namely, besides the Lifshitz-type Weyl anomaly, there are some more incarnations of the special nature of $z = d_s$.

Fixed point of a duality transformation. A first hint that $z = d_s$ is special in the massive-vector model comes from the following argument. Consider the values of the mass and cosmological constant that give rise to a Lifshitz geometry with dynamical exponent z in $d_s + 2$ bulk dimensions,

$$m(z, \ell) = \frac{\sqrt{d_s z}}{\ell}\,, \qquad \Lambda(z, \ell) = -\frac{z^2 + (d_s - 1)z + d_s^2}{2\ell^2}\,. \tag{3.1}$$

In [50] it was noticed that there is a dual pair (z', ℓ') that gives rise to the same m and Λ, because the above relation is quadratic. Solving $m(z, \ell) = m(z', \ell')$ together with $\Lambda(z, \ell) = \Lambda(z', \ell')$ yields

$$z' = \frac{d_s^2}{z}\,, \qquad \ell' = \frac{d_s \ell}{z}\,. \tag{3.2}$$

The map $(z, \ell) \to (z', \ell')$ is a solution-generating technique. The critical value $z = d_s$ is the unique fixed point for this duality map.

A logarithmic branch. If we focus on $d_s = 2$ for the moment, we know from the perturbative analysis [51] that a basis for the independent modes can be chosen as follows:

$$1\,, \qquad \mathrm{e}^{-(z+2)r}\,, \qquad \mathrm{e}^{-\frac{1}{2}(z+2-\beta)r}\,, \qquad \mathrm{e}^{-\frac{1}{2}(z+2+\beta)r}\,, \tag{3.3}$$

where $\beta^2 = (z+2)^2 + 8(z-2)(z-1)$. For $z = 2$ we see that $\beta = z+2 = 4$, which means that two pairs of modes will coincide and so a logarithmic branch will emerge. We will now see what this logarithmic branch looks like when we consider solutions to the equations of motion derived from (2.48).

Seemingly bad logs. At present, we do not have a closed-form solution of the equations of motion that exhibits the expected logarithmic behavior, so we study two approximate solutions instead. For one, we can look at linearized perturbations around the Lifshitz background (2.47) and (2.49). The other approximate solution is an asymptotic expansion, where one expands in powers of e^{-r} (and r^{-1}) for large values of the radius r.

We start with the linearized solution. The linearized field equations of the massive-vector theory were solved some time ago in [51] for $d_s = 2$. For $z = d_s = 2$, it was found that a logarithmic mode emerges that seemed to grow quicker than the background mode. For this reason, it was generally expected to be an irrelevant perturbation of the (pure) Lifshitz solution.[2] The mode (proportional to c) appears in the linearized solution as

$$-g_{tt} = \mathrm{e}^{4r}(1 - 2c\,r + \ldots)\,, \tag{3.4}$$

$$g_{ij} = \mathrm{e}^{2r}(1 + c\,r + \ldots)\,, \tag{3.5}$$

$$A_t = \mathrm{e}^{2r}(1 - c\,(\tfrac{1}{2} + r) + \ldots)\,. \tag{3.6}$$

This looks pretty bad, because it looks like the asymptotics are destroyed by this mode. One can see, however, that the Lorentz scalar A^2 and the volume form constructed from these fields do behave nicely, e.g.

$$A^2 = -1 + c + \ldots\,, \qquad \sqrt{g} = \mathrm{e}^{4r}\big(1 + \ldots\big)\,. \tag{3.7}$$

The ellipses denote other linearized modes that are suppressed by powers of e^{-4r}. The mode proportional to c shifts the background value of A^2.

[2] It was not phrased in this precise way. It was said that the mode proportional to c should be switched off so as to satisfy the asymptotically Lifshitz boundary conditions. The latter depends on what specific boundary conditions one has in mind.

Now let us turn to the asymptotic solution, which was first obtained in [43]. The leading behavior of the asymptotic solution is:

$$-g_{tt} = \frac{e^{4r}}{r^4}\left(1+O(r^{-1})\right) , \tag{3.8}$$

$$g_{xx} = r^2 e^{2r}\left(1+O(r^{-1})\right) , \tag{3.9}$$

$$A_t = \frac{e^{2r}}{r^2}\left(1+O(r^{-1})\right) . \tag{3.10}$$

In this case, the logarithmic modes look even worse. However, just as in the linearized solution, one sees that A^2 and the volume form do behave nicely. In this case they only receive sub-leading logarithmic corrections:

$$A^2 = -1+\frac{2}{r}+O(r^{-2}), \qquad \sqrt{|g|} = e^{4r}\big(1+\frac{1}{r}+O(r^{-2})\big) . \tag{3.11}$$

The shift in A^2 can thus be seen as a logarithmic correction, which vanishes when $r \to \infty$. It was argued in [43] that the logarithms $\sim r$ that appear in the asymptotic solution comprise a marginally *relevant* perturbation of the "pure" Lifshitz solution (2.47). Our results agree with this statement, although our method of renormalizing the on-shell action is inherently different in that we do not allow for explicitly r-dependent counterterms.

In conclusion, we see that a logarithmic branch opened up when $z = d_s = 2$. This logarithmic branch looks problematic if one considers quantities that are *not covariant*. However, everything appears fine again once we consider only covariant quantities (and the volume density). In particular, we checked explicitly that the curvature invariants and the geodesic deviation behave in this same way. In light of this, it seems appropriate to call the configuration (3.8)–(3.10) asymptotically Lifshitz, even though the metric looks quite different from the pure Lifshitz geometry (2.47). Since the asymptotics are not changed in this covariant sense, one can expect that the logarithmic branch is related to a marginally *relevant* perturbation of the pure-Lifshitz solution. We will make this more precise in the context of holographic renormalization.

Tidal forces in the infra-red. It was previously argued that the Lifshitz geometry (2.47) is singular in the infra-red. Even though the curvature invariants are finite everywhere, one finds that the tidal forces that a local observer experiences diverge as $e^r \to 0$ whenever $z \neq 1$, cf. [21, 22]. The logarithmic Lifshitz solution is free of such singularities for the obvious reason that the dynamical exponent flows to $z = 1$ in the infra-red (cf. Figure 3.1).[3] This is in contrast to what was expected in [21], where it was argued that a

[3] See Appendix VII for the numerical setup.

sensible IR geometry that is free of these pathologies was unlikely to exist. The reason why the analysis from [21] does not apply to this particular flow is that we allow for the presence of leading logs.

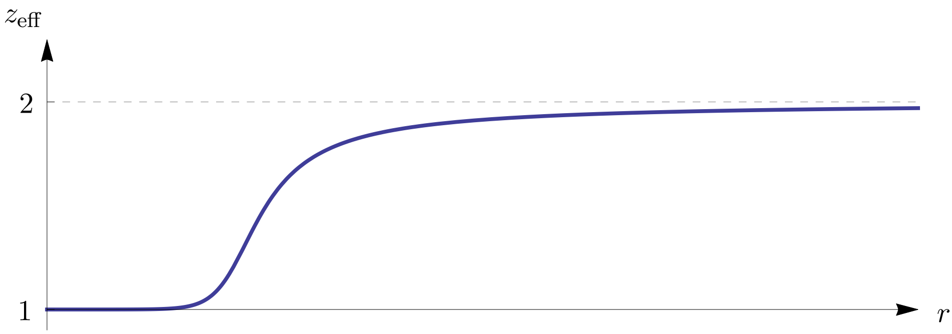
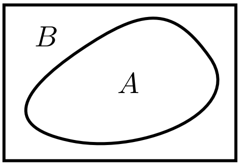


Figure 3.1: The dynamical exponent is evaluated on a numerical background that interpolates between AdS_4 in the interior (left) and Lifshitz spacetime in the asymptotic region (right). The dynamical exponent flows from $z_{\text{eff}} = 1$ in the IR to $z_{\text{eff}} = 2$ in the UV.

3.1 Holographic Entanglement Entropy



Entanglement entropy in quantum field theory is a measure for correlation between subsystems and it provides a powerful diagnostic tool for the theory in question. The most common subsystems one typically considers are obtained by splitting up the full Hilbert space in terms of the Hilbert spaces associated to disjoint spatial subdomains. For instance, for two spatial subdomains labeled A and B, we take $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Suppose we start out with a pure state described by the density matrix ρ associated to the full Hilbert space $\mathcal{H}$. One can then define the reduced density matrix ρ_A by tracing out the part of the state associated to $\mathcal{H}_B$, $\rho_A = \text{Tr}_B \rho$. The entanglement entropy can then be defined as the Von Neumann entropy computed from the reduced density matrix:

$$S_{\text{ent}} \;=\; -\text{Tr}_A \left(\rho_A \ln \rho_A\right) \tag{3.12}$$

This quantity measures the amount of entanglement between the degrees of freedom in the subdomains A and B. It sometimes used as an order parameter when a local order parameter is lacking. Entanglement entropy is notoriously hard to compute in quantum field theory. There is, however, a surprisingly

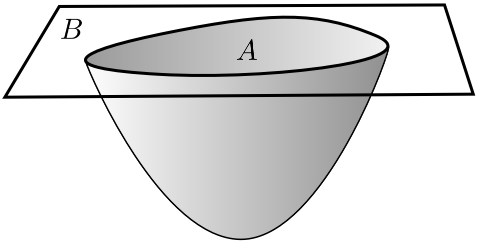


Figure 3.2: The minimal surface is suspended from ∂A and extends into the bulk. The entanglement entropy between subdomain A and B is proportional to the area of the minimal surface.

simple way to compute it in theories with a holographic dual [52]. Namely, the entanglement entropy can be computed in holography by computing the surface area of a minimal surface that extends into the bulk while suspended from the spatial boundary between the two subdomains, see Figure 3.2. For our purposes, we are interested in entanglement entropy because it provides us with an effective measure for the number of degrees of freedom. We discuss this below.

3.1.1 Entanglement c-function

We would like to have a measure for the effective number of degrees of freedom. For this purpose, we shall look at the renormalized entanglement entropy, which was proposed as a candidate c-function in [49] for RG flows that interpolate between conformally invariant fixed points in $d \geq 3$ dimensions. We will follow [49] and study the entanglement entropy associated to a strip-

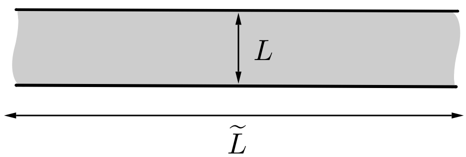


Figure 3.3: The entangling region is a strip. The two length scales associated to this geometry is the width L of the strip and a long-distance cutoff $\widetilde{L}$.

shaped region in flat space (see Figure 3.3). For a strip in $d \geq 3$ dimensions, the entanglement entropy contains only the leading area-law divergence and a universal piece [38, 49]:

$$S_{\text{ent}} = \frac{\text{Area}}{\epsilon^{d-2}} + S_{\text{finite}}\,, \tag{3.13}$$

The strip is particularly convenient, because there are no sub-leading power-law divergences beyond the leading area-law term and S_{finite} is independent of the UV-cutoff (so no $\log \epsilon$ dependence).[1] Let L be the width of the strip and let $\widetilde{L}$ be an IR length scale associated to the transverse directions. For conformally invariant fixed points, one finds $S_{\text{finite}} \propto c_d \left(\widetilde{L}/L\right)^{d-2}$, where c_d are the known A-type central charges when d is even. The strip geometry is special, because the non-universal power-law divergent piece is independent of the width L of the strip. This means that the universal piece can be extracted rather easily by taking the derivative with respect to L,

$$S_{\text{finite}} \propto L\,\partial_L S_{\text{ent}}\,. \tag{3.14}$$

The right-hand side we shall call the renormalized entanglement entropy of the strip (in analogy to the renormalized entanglement entropy of the sphere [53]). In the case where a theory flows between two conformally invariant fixed points, it was suggested in [49] that the renormalized entanglement entropy would be a good candidate c-function:

$$c_d(L) = \beta_d \left(\frac{L}{\widetilde{L}}\right)^{d-2} L\,\partial_L S_{\text{ent}}\,. \tag{3.15}$$

The prefactor β_d is a dimensionless constant that depends on the number of dimensions as

$$\beta_d = \frac{1}{\sqrt{\pi}\,2^d\,\Gamma\left(\frac{d}{2}\right)} \left(\frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\Gamma\left(\frac{d}{2(d-1)}\right)}\right)^{d-1}. \tag{3.16}$$

The function c_d was constructed in such a way that it reduces to the known A-type central charges at conformally invariant fixed points.

In our situation, one of the fixed points we are interested in is not conformally invariant, so it is not a priori clear whether it makes sense to interpret (3.15) as a c-function. However, we will compute (3.15) holographically, in which case one finds that the computations of the entanglement entropy

[1] For a more generic entangling geometry that is not flat, one finds curvature-dependent power-law divergences that are sub-leading compared to the leading area-law term. Such terms typically do depend on L and even though its dependence can be scaled away by rescaling the UV-cutoff ϵ, it is far simpler to consider a strip for the purpose of finding a quantity that behaves as a c-function.

done either in AdS or in Lifshitz spacetime go through in precisely the same manner. The monotonicity of (3.15) for non-Lorentz invariant situations was recently discussed in [54]. One can easily check that our setup meets the requirements of [54]. In particular, the null energy condition is satisfied, $u_\mu u_\nu T^{\mu\nu} = \frac{m^2}{2}(u^\mu A_\mu)^2 \geq 0$, where u^μ is a future-directed null vector. Furthermore, the Ryu–Takayanagi formula holds in the massive vector bulk model, so our computation will be very similar to the known AdS/CFT computations in Einstein gravity.[2]

We will use the function (3.15) to see how the effective number of degrees of freedom decrease along the RG flow. Before we do so, however, we will first derive a simple formula for $c_d(L)$ using holography.

3.1.2 A simple formula for the entanglement c-function

Generically it is rather difficult to compute the entanglement entropy away from a scale-invariant fixed point. This why we will use holography. The holographic formula for the entanglement entropy associated to some subregion A was proposed by Ryu and Takayanagi [52]. It is given by the area (in Planck units) of a minimal surface in the bulk that is suspended from the boundary ∂A of the subregion A.[3] So, the entanglement entropy is given by the on-shell value of the Nambu–Goto type action

$$S_{\text{ent}} = \frac{1}{4G_d}\int d^{d-1}x\,\sqrt{\gamma}\,, \tag{3.17}$$

where $\gamma_{ab} = g_{\mu\nu}(x)\,\partial_a x^\mu\,\partial_b x^\nu$ is the induced metric on the hypersurface and G_d is Newton's constant in d dimensions. Consider the situation in which the metric at constant time is given by

$$ds^2\Big|_{t=\text{const}} = f(r)\,d\vec{x}^{\,2} + dr^2\,. \tag{3.18}$$

Note that this includes both AdS as well as Lifshitz spacetime. Focusing on the case where the entangling subregion is a strip yields

$$S_{\text{ent}} = \int_0^L dx\,\mathcal{L} \qquad \mathcal{L}(r,\dot r) = \frac{\widetilde{L}^{d-2}}{4G_d}\sqrt{f(r)\big(f(r)+\dot r^2\big)} \tag{3.19}$$

where $\dot r = dr/dx$. We have chosen our coordinates such that the strip lies perpendicular to the coordinate x in such a way that it covers the interval

[2] One can see that the Wald charge (or improvements thereof) reduces to the area formula in the massive vector model.

[3] This formula is incomplete e.g. when higher derivatives are taken into account.

$-L < x < L$, so x has been rescaled by a factor of $1/2$ compared to the standard one. Furthermore, we have used the symmetry $x \to -x$, such that the integral runs from $0 < x < L$ rather than $-L < x < L$. These two redefinitions generate two factors of 2, which mutually cancel.

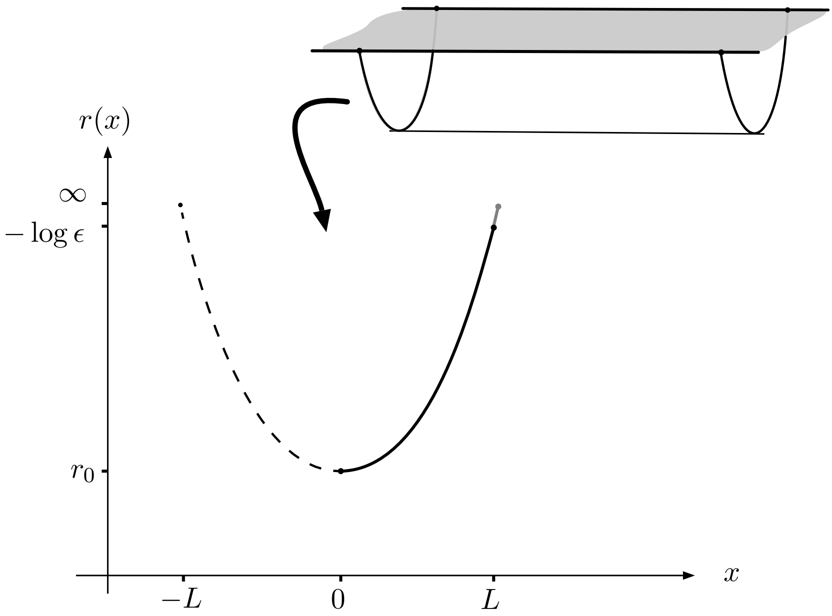


Figure 3.4: The minimal surface at fixed y. We have chosen our coordinates such that x runs from $-L$ to L. We use the symmetry $x \to -x$ to reduce the problem such that the coordinate x runs from 0 to L instead.

One can associate a Hamiltonian to x-evolution,

$$H(r,p) \;=\; p\,\dot{r} - \mathcal{L}(r,\dot{r})\Big|_{\dot{r}(p,r)}\,, \tag{3.20}$$

which is conserved, such that $H(r,p) = E$. The on-shell action is a function of the boundary data $r(0) = r_0$ and $r(L) = -\log\epsilon$. The integration constant r_0 is related to the constant of motion E by imposing that r_0 is the turning point of the minimal surface in the bulk, see Figure 3.4. Thus,

$$0 \;=\; \dot{r}(0) \;=\; \frac{\partial H}{\partial p}\big(r_0, E\big)\,. \tag{3.21}$$

In our case this yields $E = \frac{\widetilde{L}}{2G_d} f(r_0)$. So it seems that given ϵ and E, we get a value of the on-shell action. However, the physical input that we give the system is L rather than E, so we need to express E in terms of L. In summary, the boundary conditions are set by the two integration constants L and ϵ via:

$$f(r_0) = E(L)\,, \qquad\qquad r(L) = \log\frac{1}{\epsilon} \tag{3.22}$$

Because the Hamiltonian is a constant of motion, we can write the on-shell action as the Legendre transform of the so-called characteristic function $W(\epsilon, E)$,[4]

$$S_{\text{ent}}(\epsilon, L) \;=\; -E\,L + W(\epsilon, E)\,, \qquad\qquad L = \frac{\partial W}{\partial E}\,. \tag{3.23}$$

Now, we assume that the characteristic function is separable, by which we mean that it splits up into two pieces:

$$W(\epsilon, E) \;=\; W_\epsilon(\epsilon) + W_E(E)\,. \tag{3.24}$$

The first piece W_ϵ contains the area-law divergence, while the second one W_E contains information about the finite piece of the entanglement entropy (3.13). The above separation is justified if there is a clean separation between the UV and IR, which is the case when the entangling surface is a strip (3.13). The reason why we want the separation (3.24) is that $L = \frac{\partial W}{\partial E} = W_E'(E)$ depends only on E this way. The renormalized entanglement entropy (3.15) then becomes simply

$$\boxed{c_d(L) \;=\; -\beta_d\,\frac{L^{d-1}}{\widetilde{L}^{d-2}}\;E(L)} \tag{3.25}$$

where $E(L)$ is obtained by inverting the relation[5]

$$L(E) \;=\; \int_{r_0(E)}^{\infty}\frac{dr}{\dot r(r,E)} \;=\; \int_{r_0(E)}^{\infty} dr\,\left(\frac{\partial H}{\partial p}\right)^{-1}\Bigg|_{p=p(E,r)}\,. \tag{3.26}$$

We have thus reduced the problem of finding the renormalized entanglement entropy of a strip to inverting $L(E)$ to $E(L)$.

A relation between bulk and boundary length scales. It is known that the physical scale, i.e. the scale at which one probes the theory, is related to a radial scale in the bulk as $\mu \sim \mathrm{e}^{r/\ell}$. Although this relation between bulk and boundary scales formally true, it is not always easy to make this more precise. The holographic version of the renormalized entanglement entropy is a nice quantity to consider, because it gives an *explicit* relation between a boundary scale $\mu = 1/L$ and a bulk scale r_0 (or E).

[4]See e.g. Chapter 10 of [55].

[5]This integral would not converge if we were not allowed to separate the characteristic function as in (3.24). In other words, we would need to introduce the cut-off ϵ. The integral would then run up to $r = 1/\epsilon$ instead of all the way to $r \to \infty$, which would introduce a dependence of E on ϵ.

3.1.3 Lifshitz-to-AdS RG Flow

Let us put formula (3.25) to good use. We restrict ourselves to $d = 3$ boundary dimensions henceforth. Besides the Lifshitz geometry,

$$\begin{aligned} ds^2 &= dr^2 - e^{2zr/\ell_{\rm Lif}}\, dt^2 + e^{2r/\ell_{\rm Lif}}\, d\vec{x}^{\,2}\,, \\ A &= \sqrt{\frac{2(z-1)}{z}}\, e^{zr}\, dt\,, \end{aligned} \tag{3.27}$$

the massive vector theory also has an AdS_4 solution:

$$\begin{aligned} ds^2 &= dr^2 + e^{2r/\ell_{\rm AdS}} \left(-dt^2 + d\vec{x}^{\,2}\right)\,, \\ A &= 0\,, \end{aligned} \tag{3.28}$$

Both these backgrounds have the same equal-t geometry (3.18), where the function $f(r) = e^{2r/\ell}$; the curvature length scale is either $\ell = \ell_{\rm AdS}$ or $\ell = \ell_{\rm Lif}$. Moreover, the ratio of the length scales fixed in the massive vector model, which is $\ell_{\rm AdS}/\ell_{\rm Lif} = \sqrt{3/5}$.

All we need to do in order to compute the entanglement c-function is plug in $f(r) = e^{2r/\ell}$ and then run the machinery we just developed. First of all, we have

$$\dot{r}(r, E) = e^r \sqrt{e^{4(r-r_0)/\ell} - 1}\,, \tag{3.29}$$

where r_0 is given in terms of E via $E = \frac{\widetilde{L}}{4G}\, e^{2r_0/\ell}$. Then, we find

$$L(E) = \sqrt{\frac{\ell^2}{2G}\,\frac{\widetilde{L}}{\beta_3 E}}\,, \tag{3.30}$$

which can easily be inverted to $E(L)$. The central charges for the AdS and Lifshitz fixed points are thus given by

$$c_{\rm AdS} = \frac{\ell_{\rm AdS}^2}{2G}\,, \qquad c_{\rm Lif} = \frac{\ell_{\rm Lif}^2}{2G}\,. \tag{3.31}$$

As a first consistency check, we see that $c_{\rm AdS}/c_{\rm Lif} = 3/5 < 1$, which gives credence to the statement that the flow must be from a Lifshitz-type fixed point in the UV to a conformally invariant fixed point in the IR. In the previous chapter we choose the normalization of the Lifshitz central charges (2.11) such that $c_{\rm Lif} = C_1$, cf. (2.102).

Using formula (3.25) it is actually quite easy to evaluate the renormalized entanglement entropy on a numerical background, for which we take the

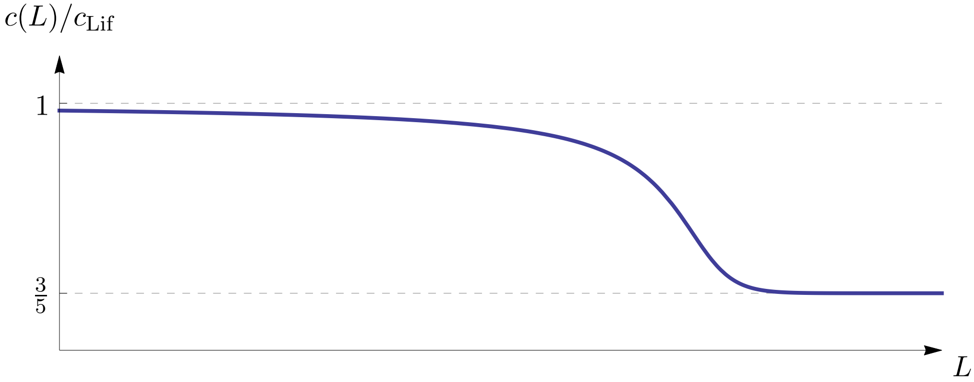


Figure 3.5: The renormalized entanglement entropy is evaluated on the numerical background. Along the horizontal axis we have the width of the strip L. We divided by the Lifshitz value $c_{\mathrm{Lif}} = \frac{\ell_{\mathrm{Lif}}^2}{2G}$.

interpolating solution computed in Appendix VII. The result of this is shown in Figure 3.5. We see that at small L (probing short distances), the c-function tends to the value associated to the Lifshitz fixed point. Then, when L becomes large enough (probing long distances), the minimal surface dips into the bulk deep enough to become sensitive to the AdS_4 part of the geometry.

3.2 Holographic Renormalization

The previous methods for renormalizing this system involved the use of counterterms that depend explicitly on the radial cut-off. The explicit r dependence was more involved than what one encounters in the presence of Weyl-type anomalies. In this section we show that the $z = 2$ massive-vector model can be renormalized by using only covariant counterterms. Moreover, we find a closed-form expression for the renormalized on-shell action.

Boundary conditions. We would like to impose the boundary condition $\delta_r \approx \delta_D$, where henceforth '$\approx$' means equality up to $1/r$ corrections. The implicit radial derivative is simply:

$$\delta_r \;=\; \int d^d x \left(\partial_r N \, \frac{\delta}{\delta N} + \partial_r h_{ij} \, \frac{\delta}{\delta h_{ij}} + \partial_r \psi \, \frac{\delta}{\delta \psi} \right) . \qquad (3.32)$$

In the previous chapter, we used the following representation of the dilatation operator:

$$\delta_D \;=\; \int d^d x \left(2N \frac{\delta}{\delta N} + 2h_{ij}\,\frac{\delta}{\delta h_{ij}} - \lambda\psi\,\frac{\delta}{\delta\psi} \right) \tag{3.33}$$

where λ was given in (2.69):

$$\lambda \;=\; \frac{1}{2}\left(z + d_s - \sqrt{(z+d_s)^2 + 8(z-1)(z-d_s)} \right) . \tag{3.34}$$

Now, notice that $\lambda = 0$ when $z = d_s = 2$. On the other hand, we do know that the leading behavior of ψ is not a constant, but it depends on r. Namely, for the asymptotic solution (see Appendix VIII), we know that $\psi \approx -\frac{1}{r}$. This means that we need to improve δ_D in order for it to asymptote to δ_r at large r. The main thing that changes due to the logarithmic behavior is that the radial derivative acts non-linearly on ψ:

$$\partial_r \psi \;\approx\; \psi^2 . \tag{3.35}$$

So, for $z = 2$, we must define δ_D differently:

$$\delta_D \;=\; \int d^d x \left(2N \frac{\delta}{\delta N} + 2h_{ij}\,\frac{\delta}{\delta h_{ij}} + \psi^2\,\frac{\delta}{\delta\psi} \right) \tag{3.36}$$

The asymptotic boundary condition $\delta_r \approx \delta_D$ is then imposed in much the same way as in §2.3.1. The result is:

$$\mathcal{L} \;=\; -6 - 2\,\psi - 2\,\psi^2 - 2\,\psi^3 + O(\psi^4) . \tag{3.37}$$

These coefficients agree with those found in (2.67).

The Hamiltonian. Translational invariance reduces the on-shell action to:

$$S \;=\; \int d^d x \sqrt{g}\,\mathcal{L}(\psi) , \tag{3.38}$$

The momenta are therefore given by

$$\begin{aligned} 2\pi^{AB} \;&=\; \left(-\mathcal{L}(\psi) + (\alpha+\psi)\,\mathcal{L}'(\psi) \right) \delta_0^A \delta_0^B + \mathcal{L}(\psi)\,\delta_{IJ}\,\delta_A^I \delta_B^J , \\ E^A \;&=\; \mathcal{L}'(\psi)\,\delta_0^A . \end{aligned} \tag{3.39}$$

The extrinsic curvature is given by $K_{AB} = G_{ABCD}\,\pi^{CD}$, cf. (2.63), so

$$K_{AB} \;=\; \frac{\mathcal{L} + (d_s - 1)(\alpha+\psi)\mathcal{L}'}{2d_s}\,\delta_A^0 \delta_B^0 + \frac{-\mathcal{L} + (\alpha+\psi)\mathcal{L}'}{2d_s}\,\delta_{IJ}\,\delta_A^I \delta_B^J \tag{3.40}$$

The Hamiltonian constraint is given by

$$\mathcal{H} \;=\; K_{AB}\,\pi^{AB} + \mathcal{V} , \tag{3.41}$$

where the non-derivative potential with $z = d_s = 2$ is

$$\mathcal{V} \;=\; 2\left(6 + 2\,\psi + \psi^2\right) . \tag{3.42}$$

In this section, we solve the Hamiltonian constraint $\mathcal{H} = 0$ directly. We get the Hamiltonian constraint by plugging in (3.39) and (3.40) into (3.41), which gives:

$$\mathcal{H} \;=\; -\frac{3}{8}\,\mathcal{L}^2 + \frac{\psi+1}{4}\,\mathcal{L}\mathcal{L}' + \frac{(\psi-1)(\psi+3)}{8}\,\mathcal{L}'^2 + 2\left(6 + 2\,\psi + \psi^2\right) \tag{3.43}$$

Solving the HJ equation. Let us denote the finite part of $\mathcal{L}(\psi)$ by $\mathcal{W}(\psi)$, such that the renormalized generation functional is

$$W \;=\; \lim_{r\to\infty} \int d^3x\sqrt{g}\,\mathcal{W}(\psi) . \tag{3.44}$$

Let us also define the counterterm Lagrangian through $\mathcal{W} = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$, where

$$\mathcal{L}_{\text{c.t.}} \;=\; 6 + 2\,\psi + 2\,\psi^2 + 2\,\psi^3 + O(\psi^4) \tag{3.45}$$

was fixed by the boundary conditions, cf. (3.37). By definition we have $\frac{dW}{dr} \approx 0$, such that

$$\frac{d\mathcal{W}}{dr} \;\approx\; -4\,\mathcal{W} , \qquad\qquad \frac{d\mathcal{L}_{\text{c.t.}}}{dr} \;\approx\; 0 . \tag{3.46}$$

This means that $\mathcal{W}$ is suppressed by a power of e^{-4r} compared to $\mathcal{L}_{\text{c.t.}}$, which allows us to solve the HJ equation perturbatively in powers of $\mathcal{W}$. Solving the leading-order HJ equation just gives the counterterms to higher order in powers of ψ, e.g.

$$\mathcal{L}_{\text{c.t.}} \;=\; 6 + 2\,\psi + 2\,\psi^2 + 2\,\psi^3 + \frac{5}{2}\,\psi^4 + \frac{5}{2}\,\psi^5 - \frac{17}{8}\,\psi^6 + O(\psi^7) . \tag{3.47}$$

The next-to-leading order gives us the following linear ODE for $\mathcal{W}(\psi)$:

$$\begin{aligned}\frac{\mathcal{W}'}{\mathcal{W}} \;&=\; \frac{3\mathcal{L}_{\text{c.t.}} - (\psi+1)\mathcal{L}'_{\text{c.t.}}}{(\psi+1)\mathcal{L}_{\text{c.t.}} + (\psi-1)(\psi+3)\mathcal{L}'_{\text{c.t.}}} \\ &=\; -\frac{4}{\psi^2} + \frac{10}{\psi} + O(\psi^0) .\end{aligned} \tag{3.48}$$

This is easily integrated:

$$\boxed{\mathcal{W} \;=\; w\,\mathrm{e}^{4/\psi}\,\psi^{10}\,\left(1 + O(\psi)\right) ,} \tag{3.49}$$

The integration constant w cannot we determined by the asymptotic boundary conditions; it is related to the geometry's renormalizable mode. From the asymptotic solution (see Appendix VIII) we know that ψ approaches 0 from below, i.e. $\psi \approx -\frac{1}{r}$. Superficially, it looks as though $\mathcal{W} \approx 0$ due to the ψ^{10} factor. We will see, however, that this is not true when we take the $O(r^{-2})$ correction in ψ into account. Before we look at any specific solution, though, we first discuss Weyl invariance.

Lifshitz-type Weyl invariance

Now, let us see whether W is Weyl invariant. As before, we write

$$W = \lim_{r\to\infty} \int d^3x \, N\sqrt{h} \, \mathcal{W}(\psi) \,. \tag{3.50}$$

The Weyl variation is computed using δ_D defined in (3.36), such that:

$$\delta_D W = \lim_{r\to\infty} \int d^3x \, N\sqrt{h} \, \big(4\mathcal{W} + \psi^2 \, \mathcal{W}'(\psi)\big) \tag{3.51}$$

Now, from (3.48), we see that $\psi^2 \, \mathcal{W}'(\psi) \approx -4\mathcal{W}$. It thus follows that the renormalized generating functional remains Weyl invariant:

$$\boxed{\delta_D W = 0} \tag{3.52}$$

We thus showed to that Lifshitz-type Weyl invariance remains unbroken for any translationally invariant configuration that satisfies the asymptotic boundary condition $\delta_r \approx \delta_D$.

Evaluate W on an asymptotic solution

In order to check whether this result makes sense, we consider the asymptotic solution found in [43], see Appendix VIII. One interesting fact about this solution is that there is a dynamically generated scale Λ. The ψ degree of freedom and $\sqrt{g}$ are given by

$$\begin{aligned} \psi &= -\frac{1}{r} - \frac{5\ln r + 2\ln\Lambda + 3 - \lambda}{2r^2} + O(r^{-3}) \,, \\ \sqrt{g} &= \hat{N}\hat{h}^2 \, \mathrm{e}^{4r} \left(1 + \frac{1}{r} + O(r^{-2})\right) \,. \end{aligned} \tag{3.53}$$

To see how the integration constants $\hat{N}$, $\hat{h}$, Λ and λ appear in the asymptotic solution, we refer to Appendix VIII. We thus see that the renormalized generating functional (3.49) is finite as required:

$$\lim_{r\to\infty} \sqrt{g}\,\mathcal{W} \;=\; \hat{N}\hat{h}^2\,\Lambda^4 e^{6-2\lambda}\,w \tag{3.54}$$

As the last step in our comparison to the asymptotic solution, let us express w in terms of the mass $\mathcal{M}$, which is the integration constant associated with the normalizable mode in the asymptotic solution. For convenience, we first define

$$\varepsilon(\psi) \;\equiv\; \mathrm{e}^{4/\psi}\,\psi^{10}\,, \tag{3.55}$$

such that:

$$w \;\approx\; \frac{\partial \mathcal{W}}{\partial \varepsilon} \;=\; \frac{\partial \psi}{\partial \varepsilon}\frac{\partial \mathcal{W}}{\partial \psi} \;=\; \frac{\mathcal{W}'(\psi)}{\varepsilon'(\psi)} \tag{3.56}$$

Next, we use the relation $\partial_r A_a = E_a = -\frac{A_a}{\sqrt{-A^2}}\,\mathcal{L}'(\psi)$ as well as $\mathcal{L} = \mathcal{W} - \mathcal{L}_{\text{c.t.}}$ to express w in terms of A_a and g_{ab} directly:

$$w \;=\; \lim_{r\to\infty}\,\frac{1}{\varepsilon'(\psi)}\left(\frac{A^a\partial_r A_a}{\sqrt{-A^2}} + \mathcal{L}_{\text{c.t.}}(\psi)\right)\Bigg|_{\psi=-1+\sqrt{-A^2}} \tag{3.57}$$

When evaluated on the asymptotic solution, this gives $w = e^{2\lambda-6}\mathcal{M}$, such that

$$\boxed{\lim_{r\to\infty} \sqrt{g}\,\mathcal{W} \;=\; \hat{N}\hat{h}^2\,\Lambda^4\,\mathcal{M}} \tag{3.58}$$

Conclusion

We have seen that the leading logarithmic modes that are present in the solution of the field equations in the massive-vector model can be interpreted as a marginally relevant operator in the $z = 2$ Lifshitz field theory. The theory flows to a conformal fixed point in the IR due to this marginally relevant operator. We derived a nice formula for the holographic entanglement c-function and we saw how it decreased along the RG flow. We found a way to renormalize the on-shell action without introducing explicitly r- dependent counterterms. This did require that we have an infinite number of counterterms, although we showed that one can obtain the renormalized on-shell action without knowing all counterterms explicitly. The fact that we need an infinite number of counterterms despite the fact that we are not dealing with an irrelevant operator seems to be an artifact of the non-analyticity of the renormalized on-shell action around the Lifshitz background value $A^2 = -1$.

One often encounters an ambiguity due to the freedom of being able to add finite local counterterms to the renormalized on-shell action. This ambiguity is absent in our case, because there are no finite local covariant counterterms when one assumes translational invariance in the boundary directions. This can also be seen in the free 3D Lifshitz scalar, $\dot{\phi}^2 + (\nabla^2\phi)^2$, where there are no finite counterterms at the non-derivative level.

The closed-form expression we found for the renormalized on-shell action is a non-analytic function of $\psi = -1 + \sqrt{-A^2}$, which reduces to a very simple expression once it is evaluated on the asymptotic solution. This may well be related to the onset of a $z = 2$ to $z < 2$ phase transition; we discuss this in the final conclusion. We investigated the possibility that Lifshitz scaling symmetry is broken by the appearance of the dynamically generated scale Λ, cf. Appendix VIII. Contrary to previous claims, including ones made by the author, we have shown once and for all that Lifshitz scaling is preserved for all translationally invariant geometries that are asymptotically locally Lifshitz in the sense that $\partial_r \approx \delta_D$. This includes the asymptotic solution of [43] presented in Appendix VIII.

Conclusion and Outlook

We have seen that it is possible to reproduce some general features of Lifshitz-type quantum field theories through holography. In order to remove UV/large-volume divergences we computed the necessary holographic counterterms. The asymptotic boundary conditions were imposed by taking serious the observation that dilatations are asymptotically generated by a radial shift. This way of specifying the boundary conditions was phrased as $\delta_r \approx \delta_D$, which admits renormalization of the on-shell action in the presence of arbitrary sources. In other words, it is a convenient way to impose the most lenient boundary conditions required to deem the on-shell action finite. It is easy to show that $\delta_r \approx \delta_D$ is equivalent to asymptotically locally AdS boundary conditions when the dynamical exponent is $z = 1$. This way of imposing the boundary conditions was especially useful for the case $z = d_s = 2$, which would otherwise be tricky due to the fact that the leading radial behavior of the fields is not purely power-law.

We investigated anomalous breaking of Lifshitz-type Weyl symmetry in two different models, one defined using a standard field theory quantization of an explicit classical action (2.20), the other defined using a holographic model. A precise definition of Lifshitz holography is still lacking, and a microscopic definition of the strongly coupled field theory remains unknown. It is therefore a priori not very meaningful to compare the two anomalies. Nevertheless, we found that the anomalies are quite similar. Namely, in both cases there are two possible central charges of which one vanishes, and as a consequence the two anomalies are directly proportional to each other. The ratio of the two anomalies is $2\ell^2/G$, with ℓ the curvature radius of the Lifshitz spacetime and G the 4D Newton constant.

It would be interesting to evaluate this quantity in explicit string theory embeddings of Lifshitz spacetimes to see how it scales with the various integer fluxes, as this will provide some measure of the effective number of degrees of freedom of the dual field theory. We know that in the relativistic case the central charges control the universal terms in the entanglement entropy in $d = 2, 4$. We studied this in a $z = 2$ holographic model. The computation of holographic entanglement entropy in a Lifshitz background is identical to those done in AdS/CFT, because the AdS and Lifshitz geometries look the same at constant time. In the process, we derived a nice new formula for the holographic entanglement c-function, where the entangling surface is a strip.

It is quite mysterious that the conformal anomaly only involves time derivatives, it is even more mysterious that there exists a conformal anomaly at all. According to [47], the dynamical critical exponent is in general renormalized, though there were some hints that $z = 2$ might be protected.[1] This would pose a problem, because the Lifshitz anomaly can no longer be written down as soon as the dynamical exponent is not precisely equal to the number of spatial dimensions. So either there is some unknown mechanism that protects the value of $z = 2$, or the conformal anomaly can be removed in the full quantum theory. In the latter case, one would be in the peculiar situation that one would need to include counterterms that diverge in the classical limit. Further work will be required to clarify this issue.

It is also of interest to explore other systems with anisotropic scale invariance to examine whether the conformal anomaly is still of the same form. For instance, if one allows that time reversal symmetry be broken, it is logically possible to have contributions with an odd number of time derivatives to the conformal anomaly. It is in principle straightforward to extend the analysis in Appendix IV to determine whether there are non-trivial terms of this type.

Knowledge of the general form of the Lifshitz anomaly allowed us to compute the Casimir energy and stress tensor in any $z = d_s$ Lifshitz theory, which is quite difficult to compute in general. This result is universal in the sense that it only depends on the central charges and the background geometry. This could be considered as a first step towards making contact with experiments. Although more work is required to make the relation to experiments more precise, one could imagine that, for instance, the central charge C_2 that is associated with spatial fluctuations of the background geometry might somehow be related to finite-size effects.

We have seen that the seemingly bad logarithmic modes that are naturally present solutions to the Einstein–Proca equations can be interpreted as originating from the presence of a marginally relevant operator in the dual $z = 2$

[1] It should be noted that the model that was used in [47] was not the massive vector model but the two-form/three-form model from [16]. Since the massive-vector model can be obtained from the two-form/three-form one by integrating out one of the forms, it is not unlikely that one would reach the same conclusions in the massive vector model. In particular it was found that the dynamical exponent is generically corrected due to the inclusion of higher-derivative (α'-like) corrections to the action. There were two special values of the dynamical exponent, however, that did not receive corrections at first order: $z = 1$ and $z = 2$.

Lifshitz field theory. Furthermore, we showed that this marginally relevant operator is responsible for an RG flow towards a conformally invariant fixed point in the IR. We saw that this observation is corroborated by the fact that the aforementioned entanglement c-function decreased monotonically along the holographic RG flow.

We found a way to properly renormalize the on-shell $z = 2$ massive-vector action without the need for introducing explicit dependence on the radial cutoff. This did require that we have an infinite number of counterterms, although we showed that one can obtain the renormalized on-shell action without knowing all counterterms explicitly. The fact that we need an infinite number of counterterms despite the fact that we are not dealing with an irrelevant operator seems to be an artifact of the non-analyticity of the renormalized on-shell action around the Lifshitz background value $A^2 = -1$. If we view ψ as an order parameter that is zero when the dynamical scaling is precisely $z = 2$ and non-zero when $z < 2$, we could interpret the non-analyticity of the free energy[2] as being related to a $z = 2$ to $z < 2$ phase transition. From the asymptotic solution we know that the dynamically generated scale Λ is the appropriate source for the marginally relevant operator rather than ψ itself, which gives $\frac{\partial \mathcal{F}}{\partial \Lambda} = -\frac{\psi^2}{\Lambda}\frac{\partial \mathcal{F}}{\partial \psi} = \frac{4\mathcal{F}}{\Lambda}$ (up to terms that vanish when the cut-off is removed). We thus see that repeated Λ-derivatives are all finite, which means that this phase transition is somewhat akin to a BKT transition. It would be interesting to understand this non-analyticity and its possible relation to a BKT-type phase transition better.

We find that the Lifshitz symmetry remains unbroken at the translationally-invariant level, as is expected from the generic form of the Lifshitz anomaly (2.11). Contrary to previous claims, we have seen that Lifshitz scaling is preserved for all translationally invariant geometries that are asymptotically locally Lifshitz in the sense that $\delta_r \approx \delta_D$, which includes the asymptotic solution from [43].

A possible extension of the work done in Chapter 3 might be to study the thermodynamic properties of these asymptotically Lifshitz geometries. In particular, it would be interesting to find an AdS-to-Lifshitz crossover in the free energy as a function of temperature. The temperature-scaling of the free energy depends on the dynamical exponent z, $\mathcal{F} \sim T^{1+d_s/z}$, so one may see a transition $\mathcal{F} \sim T^3$ to $\mathcal{F} \sim T^2$ as T is increased. Another extension of this work would be to let go of translation invariance, though at this stage this

[2] The free energy $\mathcal{F}$ is given by the on-shell Lagrangian in Euclidean signature, i.e. $\mathcal{F} = L_{\mathrm{E}}$. The on-shell Euclidean action is given by the integral of the Euclidean Lagrangian over imaginary time: $S_{\mathrm{E}} = \int d\tau\, L_{\mathrm{E}}$.

seems to complicate matters quite severely.

In AdS there is a natural ground state, which is usually taken to be pure AdS in global coordinates. For spacetimes with Lifshitz asymptotics this ground-state geometry is yet unknown. We know that the ground state in a 3D Lifshitz theory is invariant under spatial conformal transformations, cf. page 38. This suggests that the ground-state geometry would have to be a line times 3D hyperbolic space, $\mathbb{R} \times \mathcal{H}_3$. The ground state could be an interpolating solution between a $z = 0$ solution and $z = 2$ one. It is actually quite easy to construct such an interpolating solution numerically by using the techniques of [56]. This work is still ongoing. Besides finding the proper ground-state geometry, the main objective of this work is to figure out whether holography correctly captures the universal behavior of entanglement entropy as computed in [38, 39].

It would also be very interesting to figure out if there is indeed a relation between the central charge C_1 and the two-dimensional central c, as discussed in Section 2.5. Though, it should be noted that this possibility is not a very serious one at this stage.

Appendix

I Einstein–Proca Hamiltonian

In this section, we derive the Hamiltonian associated to the Einstein–Proca theory:

$$S = \int_M d^{d+1}x\sqrt{g}\left(-\tilde{R}+2\Lambda+\frac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab}+\frac{m^2}{2}\,\tilde{A}_a\tilde{A}^b\right)-\oint_{\partial M} d\sigma_a\Theta^a\,. \tag{A.1}$$

We use Wald's abstract index notation throughout this section and Θ^a is the Gibbons–Hawking term. Let us write the full geometry as a radial foliation consisting of a stack of equal-r slices Σ_r.[1] We split up the metric in terms of the outward-directed unit normal n^a, where

$$n_a \equiv N\,\nabla_a r\,. \tag{A.2}$$

The normalization constant is known as the lapse function. We then define the induced metric, or first fundamental form, as:

$$h_{ab} \equiv g_{ab}-\epsilon\, n_a n_b \tag{A.3}$$

Here, $\epsilon \equiv n^a n_a = \pm 1$ is the signature of the foliation. In this case we have a $\epsilon = 1$, but it is convenient to leave it arbitrary, so that we can use the same expressions for a time foliation as well. The induced metric $h^a{}_b$ can be used as a tangent projector, which pulls back tensorial quantities onto the hypersurface Σ_r. Similarly, $\epsilon\, n^a n_b$ can be seen as the normal projector. This allows us to decompose the vector in similar fashion:

$$\tilde{A}_a = \Phi\, n_a + A_a \tag{A.4}$$

where $\Phi \equiv \epsilon\, n^a \tilde{A}_a$ and $A_a \equiv h_a{}^b\,\tilde{A}_b$.

The canonical Lagrangian L, defined through $S = \int dr\, L$, is a function of the (induced) fields and their canonical dual velocities. In order to define the

[1] There might be global obstructions to such a foliation, but for our purposes it is enough to define the foliation only near the conformal boundary.

latter, we introduce the flow vector r^a, which is defined implicitly through $r^a n_a = \epsilon N$. The tangential component of r^a is known as the shift function, $N^a \equiv h^a{}_b\, r^b$. Thus, the radial flow vector field is decomposed as

$$r^a \;=\; N n^a + N^a \tag{A.5}$$

The canonical velocities are then defined via Lie transport along the radial flow r^a:

$$\dot{h}_{ab} \;\equiv\; \mathscr{L}_r h_{ab}\,, \qquad \dot{A}_a \;\equiv\; \mathscr{L}_r A_a\,. \tag{A.6}$$

A natural quantity to consider is the extrinsic curvature, or second fundamental form, as the Lie derivative of h_{ab} along the unit normal:[2]

$$K_{ab} \;\equiv\; \frac{1}{2}\mathscr{L}_n h_{ab} \;=\; h_a{}^c\,\nabla_c n_b \tag{A.7}$$

Let us also introduce the covariant derivative D_a, compatible with h_{ab}. This covariant derivative can be related to ∇_a as follows. Consider some arbitrary tangent vector $v^a = h^a{}_b\, v^b$. The two covariant derivatives are then related via:[3]

$$D_a v^b \;=\; h_a{}^c\,\nabla_c v^b \tag{A.8}$$

The metric-velocity and extrinsic curvature are related via:[4]

$$\begin{aligned}\dot{h}_{ab} \;&=\; \mathscr{L}_{(Nn)} h_{ab} + \mathscr{L}_N h_{ab}\\ &=\; N\mathscr{L}_n h_{ab} + \mathscr{L}_N h_{ab}\\ &=\; 2N\,K_{ab} + D_a N_b + D_b N_a\end{aligned} \tag{A.9}$$

Let us do the same for the vector velocity. We define $K_a \equiv \mathscr{L}_n A_a$, such that:

$$\begin{aligned}\dot{A}_a \;&=\; N\mathscr{L}_n A_a + \mathscr{L}_N A_a\\ &=\; N K_a - N^b F_{ab} + \nabla_a\left(N^b A_b\right)\end{aligned} \tag{A.10}$$

We are now ready to write the Lagrangian directly in terms of the canonical quantities.

[2]The property $h_a{}^c\,\nabla_c n_b = h_b{}^c\,\nabla_c n_a$ follows from hypersurface orthogonality; it can be shown straightforwardly by using the definition of the unit normal (A.2).

[3]This also works for higher-rank ternsors. Let $T_{abc\cdots}$ be a purely transversal tensor, i.e. $T_{abc\cdots} = h_a^{a'} h_b^{b'} h_c^{c'} \cdots T_{a'b'c'\ldots}$, then $D_a T_{bc\cdots} = h_a^{a'}\,\nabla_{a'} T_{bc\cdots}$.

[4]The property $\mathscr{L}_{(Nn)} h_{ab} = N\mathscr{L}_n h_{ab}$ follows directly from $n^a h_{ab} = 0$.

Gauss–Codazzi decomposition

Let us start with the metric. We shall use the Gauss–Codazzi decomposition. The Riemann tensor can be decomposed as follows:

$$\begin{aligned} h_a^e h_b^f h_c^g h_d^h \, \tilde{R}_{efgh} &= R_{abcd} + \epsilon \left(K_{ad} K_{bc} - K_{ac} K_{bd} \right) \\ n^e h_a^f h_b^g h_c^h \, \tilde{R}_{efgh} &= D_b K_{ac} - D_c K_{ab} \\ n^e h_a^f n^g h_b^h \, \tilde{R}_{efgh} &= -\mathscr{L}_n K_{ab} + K_{ac} K_b{}^c + D_b a_a - \epsilon \, a_a a_b \end{aligned} \quad \text{(A.11)}$$

where $a_a \equiv n^b \nabla_b n_a$ is the acceleration of observers that are at rest in the slices Σ_r. It then follows that the Ricci tensor and Ricci scalar can be decomposed as

$$\begin{aligned} h_a^c h_b^d \, \tilde{R}_{cd} &= R_{ab} + 2\epsilon \, K_{ac} K_b{}^c - \epsilon \, K_{ab} K - \epsilon \, \mathscr{L}_n K_{ab} + \epsilon \, D_b a_a - a_a a_b \\ n^c h_a^d \, \tilde{R}_{cd} &= D_a K - D^b K_{ab} \\ n^c n^d \, \tilde{R}_{cd} &= K^2 - K_{ab} K^{ab} + \nabla_a (a^a - n^a K) \\ \tilde{R} &= R + \epsilon \left(K^2 - K_{ab} K^{ab} \right) + 2\epsilon \, \nabla_a (a^a - n^a K) \end{aligned} \quad \text{(A.12)}$$

We only used the Leibniz rule a couple of times. We are now ready to start using the ADM variables again.

Now, let us decompose the Maxwell term. The field strength is decomposed as

$$\begin{aligned} h_a^c h_b^d \, \tilde{F}_{cd} &= F_{ab} \\ n^c h_a^d \, \tilde{F}_{cd} &= K_a - \epsilon D_a \Phi + a_a \Phi \end{aligned} \quad \text{(A.13)}$$

where we used $\nabla_a n_b = K_{ab} + \epsilon \, n_a a_b$.

$$\begin{aligned} \tilde{F}_{ab} \tilde{F}^{ab} &= (h^{ab} + \epsilon \, n^a n^b)(h^{cd} + \epsilon \, n^c n^d) \, \tilde{F}_{ac} \tilde{F}_{bd} \\ &= \left(h^{ab} h^{cd} + \epsilon \, h^{ab} \, n^c n^d + \epsilon \, h^{cd} \, n^a n^b + n^a n^b n^c n^d \right) \, \tilde{F}_{ac} \tilde{F}_{bd} \\ &= F_{ab} F^{ab} + \left(\epsilon \, h^{ab} \, n^c n^d + \epsilon \, h^{cd} \, n^a n^b \right) \, \tilde{F}_{ac} \tilde{F}_{bd} \\ &= F_{ab} F^{ab} + 2\epsilon \, h^{ab} \big(n^c h_a^d \, \tilde{F}_{cd} \big) \big(n^e h_b^f \, \tilde{F}_{ef} \big) \\ &= F_{ab} F^{ab} + 2\epsilon \left(K_a - \epsilon D_a \Phi + a_a \Phi \right) \left(K^a - \epsilon D^a \Phi + a^a \Phi \right) \end{aligned} \quad \text{(A.14)}$$

Now, before we write down the canonical Lagrangian L, let us focus our attention on the Einstein term:

$$
\begin{aligned}
-\int_M d^dx\,\sqrt{g}\;\tilde{R} - \int_M d\sigma_a \Theta^a \;=\; & -\int_M d^dx\, N\sqrt{h}\,\Big(R + \epsilon\,\Big(K^2 - K_{ab}K^{ab}\Big)\Big) \\
& - 2\epsilon \oint_{\partial M} d\sigma_a \left(a^a - K n^a + \frac{1}{2\epsilon}\,\Theta^a\right)
\end{aligned}
\tag{A.15}
$$

Let us have a look at the surface term. The Gibbons–Hawking term is $\Theta^a = 2\epsilon\,\tilde{n}^a\,\nabla_b \tilde{n}^b$, where $\tilde{n}^a$ is the outward-directed unit normal on ∂M. For simplicity, we consider the case where $\partial M = \Sigma_\infty$, with $\Sigma_\infty = \lim_{r\to\infty}\Sigma_r$, such that $\tilde{n}^a = n^a$. In that case, the surface term vanishes (using $n_a a^a = 0$).

The canonical Lagrangian is thus given by

$$
\begin{aligned}
L \;=\; \int d^dx\, N\sqrt{h}\,\Big\{ & -R + 2\Lambda + \epsilon\,\Big(K_{ab}K^{ab} - K^2\Big) + \frac{1}{4}F_{ab}F^{ab} \\
& + \frac{\epsilon}{2}\,\left(K_a - \epsilon D_a\Phi + a_a\Phi\right)\left(K^a - \epsilon D^a\Phi + a^a\Phi\right) \\
& + \frac{m^2}{2}\,\left(\epsilon\,\Phi^2 + A_a A^a\right)\Big\}
\end{aligned}
\tag{A.16}
$$

Canonical momenta

The momenta can be computed by taking the derivative of the Lagrangian with respect to the velocities:

$$
\begin{aligned}
\pi^{ab} &= \frac{1}{\sqrt{h}}\frac{\partial L}{\partial \dot{h}_{ab}} = \frac{1}{2N\sqrt{h}}\frac{\partial L}{\partial K_{ab}} = \epsilon\left(K^{ab} - K\,h^{ab}\right) \\
E^a &= \frac{1}{\sqrt{h}}\frac{\partial L}{\partial \dot{A}_a} = \frac{1}{N\sqrt{h}}\frac{\partial L}{\partial K_a} = \epsilon\left(K^a - \epsilon D^a\Phi + a^a\Phi\right)
\end{aligned}
\tag{A.17}
$$

This is easily inverted:

$$
K_{ab} = \epsilon\, G_{abcd}\,\pi^{cd}\,, \qquad K_a = \epsilon\left(E_a + D_a\Phi - \epsilon\, a_a\Phi\right) \tag{A.18}
$$

which also gives:

$$
\begin{aligned}
\dot{h}_{ab} &= 2\epsilon N\, G_{abcd}\,\pi^{cd} + 2D_{(a}N_{b)}\,, \\
\dot{A}_a &= \epsilon\left(E_a + D_a\Phi - \epsilon\, a_a\Phi\right) - N^b F_{ab} + \nabla_a\left(N^b A_b\right)
\end{aligned}
\tag{A.19}
$$

The Hamiltonian is then obtained as the Legendre transform that replaces the velocities for momenta:

$$
H = \int d^dx\,\sqrt{h}\,\left(\dot{h}_{ab}\,\pi^{ab} + \dot{A}_a E^a\right) - L \tag{A.20}
$$

For our purposes, it is convenient to express the Lagrangian in the following way:

$$L = \int d^dx \sqrt{h} \left(\frac{1}{2} \dot{h}_{ab} \pi^{ab} + \frac{1}{2} \dot{A}_a E^a - \mathcal{V} \right) , \tag{A.21}$$

which implicitly defines $\mathcal{V}$ to be

$$\begin{aligned} \mathcal{V} &= N \left(R - 2\Lambda - \frac{1}{4} F_{ab} F^{ab} - \frac{m^2}{2} \left(\epsilon \, \Phi^2 + A_a A^a \right) \right) \\ &\quad + N^a \left(E^b F_{ba} + A_a D_b E^b \right) \end{aligned} \tag{A.22}$$

The Hamiltonian thus becomes:

$$H = \int d^dx \sqrt{h} \left(\frac{1}{2} \dot{h}_{ab} \pi^{ab} + \frac{1}{2} \dot{A}_a E^a \right) + \mathcal{V} \tag{A.23}$$

The non-dynamical fields in this theory are N, N^a and Φ. We can vary the action (A.16) with respect to Φ to obtain the Proca constraint:[5]

$$\Phi = -\frac{1}{m^2} D_a E^a . \tag{A.24}$$

This fixes Φ. The most convenient gauge choice for lapse and shift functions is $N = 1$ and $N^a = 0$. The Hamiltonian then reduces to

$$H = \int d^dx \sqrt{h} \left(K_{ab} \pi^{ab} + \frac{1}{2} E_a E^a \right) + \mathcal{V} \tag{A.25}$$

with

$$\mathcal{V} = R - 2\Lambda - \frac{1}{4} F_{ab} F^{ab} + \frac{m^2}{2} \left(\epsilon \, \Phi^2 - A_a A^a \right) \tag{A.26}$$

where Φ is understood to be fixed by (A.24).

II Hamilton–Jacobi Formalism

The on-shell action is a function of the endpoints t_i and t_f and q_i and q_f, which specify the boundary conditions at the endpoints: $q(t_i) = q_i$ and $q(t_f) = q_f$. Let us focus our attention on the final endpoint only and we

[5] A useful relation is $D_a \Phi - \epsilon \, a_a \Phi = N^{-1} D_a(N\Phi)$.

will drop the 'f' subscript, so $S = S(q(t), t)$. The on-shell action can then be written as

$$S(q(t), t) \;=\; \int^{t} dt' \, (p\, \dot{q} - H) \;, \tag{A.27}$$

where p is the canonical momentum conjugate to q. Let us first vary the action with respect to q, keeping the endpoint fixed:

$$\begin{aligned} \delta S \;&=\; \int^{t} dt' \left(\dot{q}\, \delta p + p\, \delta \dot{q} - \delta H \right) \\ &=\; \int^{t} dt' \left(\dot{q}\, \delta p - \dot{p}\, \delta q - \delta H \right) + p\, \delta q \\ &= p\, \delta q \, . \end{aligned} \tag{A.28}$$

such that at the endpoint t, we have

$$p(t) \;=\; \frac{\partial S}{\partial q}\bigg|_{t'=t} \tag{A.29}$$

Now, let us vary the endpoint t, keeping the $q(t)$ fixed. We then find

$$\frac{\partial S}{\partial t} \;=\; \frac{dS}{dt} - \frac{\partial S}{\partial q}\, \dot{q} \;=\; L - p\, \dot{q} \;=\; -H \tag{A.30}$$

The Hamilton–Jacobi equation can then be summarized as

$$\frac{\partial S}{\partial t} + H \;=\; 0 \, , \qquad\qquad p \;=\; \frac{\partial S}{\partial q} \, . \tag{A.31}$$

The HJ equation has a natural extension to theories of gravity. In the context of this work, we replace time t by our radial coordinate r:

$$\frac{\partial S}{\partial r} + H \;=\; 0 \, , \qquad\qquad \pi^{ab} \;=\; \frac{1}{\sqrt{g}}\, \frac{\partial S}{\partial g_{ab}} \, , \tag{A.32}$$

where H is now the radial ADM Hamiltonian that we computed in the previous section. Because of general covariance, we know that the term $\frac{\partial S}{\partial t}$ vanishes by itself, so the HJ equaition reduces to solving the constraint $H = 0$.

III Higher Derivatives in AdS/CFT

To illustrate the power of the hybrid method presented in Section 1.2, we compute higher-derivative corrections to the $d = 4$ Weyl anomaly in order to lift the $a = c$ degeneracy. Let us consider the following bulk Euclidean action that includes a higher-derivative correction in the form of the ghost-free Gauss–Bonnet (GB) term:

$$S \;=\; -\int dr\, d^d x \,\sqrt{g}\,\Big\{\tilde{R} - 2\Lambda + \varepsilon\,\big(\tilde{R},\tilde{R}\big)_{\text{GB}}\Big\} + S_{\text{GH}}\,. \tag{A.33}$$

where $\tilde{R}$ is the Ricci scalar in $d+1$ dimensions; we gave it a twiddle to distinguish it from the d-dimensional one, R. We introduced the Gauss–Bonnet bracket:

$$\big(\tilde{R},\tilde{R}\big)_{\text{GB}} \;\equiv\; \tilde{R}_{abcd}\tilde{R}^{abcd} - 3\tilde{R}_{ab}\tilde{R}^{ab} + \tilde{R}^2\,, \tag{A.34}$$

which is symmetric and bilinear. The Gibbons–Hawking term is given by

$$S_{\text{GH}} \;=\; -2\int d^d x\,\sqrt{g}\,\left(K + \frac{\varepsilon}{d-3}\,g_{ab}\,\pi^{ab}_{\text{GB}}\right) \tag{A.35}$$

where π^{ab}_{GB} has the somewhat lengthy expression:

$$\begin{aligned}
\pi^{ab}_{\text{GB}} \;&\equiv\; \frac{1}{2}\frac{\partial \mathcal{L}_{\text{GB}}}{\partial K_{ab}} \\
&=\; 2RK^{ab} + 4R^{ab}K - 8R^{c(a}K^{b)}_{c} - 4R^{acbd}K_{cd} - 2g^{ab}\left(RK - 2R^{cd}K_{cd}\right) \\
&\quad - 2K^{ab}K^2 + 4K^{ac}K^{b}{}_{c}K + 2K^{ab}K_{cd}K^{cd} - 4K^{ac}K^{bd}K_{cd} \\
&\quad + \frac{2}{3}g^{ab}\left(K^3 - 3KK_{cd}K^{cd} + 2K^{a}{}_{b}K^{b}{}_{c}K^{c}{}_{a}\right)
\end{aligned} \tag{A.36}$$

The cosmological constant is related to the AdS curvature length scale ℓ as $-2\Lambda = d(d-1)/\ell^2$. The Gauss–Bonnet theory admits asymptotically AdS solutions with a corrected value for AdS length $\ell \to \ell_{\text{GB}}$, i.e. with a shifted cosmological constant:

$$-2\Lambda \;=\; \frac{d(d-1)}{\ell^2} \;=\; \frac{d(d-1)}{\ell^2_{\text{GB}}}\left(1 - \varepsilon\,\frac{(d-2)(d-3)}{\ell^2_{\text{GB}}}\right) \tag{A.37}$$

Previously, we chose to work in units such that $\ell = 1$. From now on, we will now set $\ell_{\text{GB}} = 1$ instead. The reason why we make this choice is especially nice when we extend the useful relation (1.56) to incorporate ε corrections below. So, the cosmological constant is effectively corrected:

$$-2\Lambda \;=\; (d)_2 - \varepsilon(d)_4 \tag{A.38}$$

where $(d)_n = d(d-1)\cdots(d-n+1)$ is the Pochhammer symbol.

The Hamiltonian constraint

Just like before, we write the Hamiltonian constraint as

$$\begin{aligned}\mathcal{H} &= K_{AB}\,\pi^{AB} + \mathcal{V}\,,\\ \mathcal{V} &= R - 2\Lambda + \varepsilon\,\big(R,R\big)_{\rm GB} + \frac{\varepsilon}{3}\,\big(\mathcal{K},\mathcal{K}\big)_{\rm GB}\,.\end{aligned} \tag{A.39}$$

where $\mathcal{K}_{abcd} \equiv K_{ad}K_{bc} - K_{ac}K_{bd}$. Notice that there no cross-term $(R,\mathcal{K})_{\rm GB}$ in $\mathcal{V}$. The momentum is related to the extrinsic curvature via:

$$\pi^{ab} = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial K_{ab}} = K^{ab} - Kg^{ab} + \varepsilon\,\pi^{ab}_{\rm GB}(K)\,. \tag{A.40}$$

where $\pi^{ab}_{\rm GB}$ was given in (A.36). Henceforth, we work perturbatively in ε, which allows us to invert the above relation:[1]

$$K_{AB} = G_{ABCD}\big(\pi^{CD} - \varepsilon\,\pi^{CD}_{\rm GB}(\pi)\big) + O(\varepsilon^2)\,. \tag{A.41}$$

Here, $\pi^{AB}_{\rm GB}$ is expressed in terms of π^{AB} to leading order in ε:

$$\pi^{AB}_{\rm GB}(\pi) \equiv \pi^{AB}_{\rm GB}(K)\,\Big|_{K_{ab}=G_{abcd}\pi^{cd}+O(\varepsilon)} \tag{A.42}$$

The way we solve the constraint $\mathcal{H}=0$ is by noticing that the contribution to $\mathcal{V}$ proportional to ε can be computed directly from the solution that we found in the previous section, where we had $\varepsilon=0$. Just like before, we solve the Hamiltonian constraint recursively, using the dilation-weight expansion.

Now, let us have a look at the useful relation (1.56). Let us expand the on-shell action to first order in ε, such that $S = S_0 + S_\varepsilon + O(\varepsilon^2)$, which gives a similar expansion for the momentum and extrinsic curvature, e.g. $\pi^{AB} = \pi_0^{AB} + \pi_\varepsilon^{AB}$. The reason why we chose $\ell_{\rm GB}=1$ is to make sure that the useful relation (1.56) continues to hold. It holds because $K^{(0)}_{AB}$ does not receive an $O(\varepsilon)$ correction, i.e. $K^{(0)}_{\varepsilon\;AB}=0$, so

$$\begin{aligned}2K^{(0)}_{AB}\,\pi^{(n)AB} &= 2K^{(0)}_{0\;AB}\,\pi_0^{(n)AB} + 2K^{(0)}_{0\;AB}\,\pi_\varepsilon^{(n)AB} + 2K^{(0)}_{\varepsilon\;AB}\,\pi_0^{(n)AB} + O(\varepsilon^2)\\ &= (d-n)\,\mathcal{L}^{(n)} + O(\varepsilon^2)\,. \qquad \checkmark\end{aligned} \tag{A.43}$$

Note that $\mathcal{L}^{(n)}$ is the full counterterm, which includes the ε correction. One thing that we need to be careful of is that $K^{(i)}_{AB}\,\pi^{(j)AB}$ is no longer symmetric under $i \leftrightarrow j$ due to (A.41), i.e. there is an ε correction:

$$K^{(n)}_{AB}\,\pi^{(0)AB} = K^{(0)}_{AB}\,\pi^{(n)AB} + \varepsilon\left(K^{(n)}_{AB}\,\pi^{(0)AB}_{\rm GB} - K^{(0)}_{AB}\,\pi^{(n)AB}_{\rm GB}\right) \tag{A.44}$$

This correction is determined in terms of the $\varepsilon=0$ quantities.

[1] $G_{ABCD} = \eta_{A(C}\,\eta_{D)A} - \frac{1}{d-1}\,\eta_{AB}\,\eta_{CD}$ is the DeWitt metric.

Solving the Hamiltonian constraint

We start at the lowest order.

$$0 \;=\; \mathcal{H}^{(0)} \;=\; K^{(0)}_{AB}\,\pi^{(0)AB} + \mathcal{V}^{(0)} \;=\; \frac{d}{2}\,\mathcal{L}^{(0)} + \mathcal{V}^{(0)} \tag{A.45}$$

We thus find:

$$\mathcal{L}^{(0)} \;=\; -2(d-1) + \frac{4\varepsilon}{3}\,(d-1)_3 + O(\varepsilon^2) \tag{A.46}$$

Next, we consider scaling weight $n = 2$:

$$\begin{aligned} 0 \;=\; \mathcal{H}^{(2)} \;&=\; K^{(0)}_{AB}\,\pi^{(2)AB} + K^{(2)}_{AB}\,\pi^{(0)AB} + \mathcal{V}^{(2)} \\ &=\; (d-2)\,\mathcal{L}^{(2)} + \mathcal{V}^{(2)} + \varepsilon\left(K^{(2)}_{AB}\,\pi^{(0)AB}_{\text{GB}} - K^{(0)}_{AB}\,\pi^{(2)AB}_{\text{GB}}\right) \end{aligned} \tag{A.47}$$

This gives the $n = 2$ counterterm:

$$\mathcal{L}^{(2)} \;=\; -\left(\frac{1}{d-2} + 2\varepsilon\,(d-3) + O(\varepsilon^2)\right) R \tag{A.48}$$

Finally, at $n = 4$, we have:

$$\begin{aligned} \mathcal{H}^{(4)} \;&=\; K^{(0)}_{AB}\,\pi^{(4)AB} + K^{(4)}_{AB}\,\pi^{(0)AB} + K^{(2)}_{AB}\,\pi^{(2)AB} + \mathcal{V}^{(4)} \\ &=\; (d-4)\,\mathcal{L}^{(4)} + K^{(2)}_{AB}\,\pi^{(2)AB} + \mathcal{V}^{(4)} + \varepsilon\left(K^{(4)}_{AB}\,\pi^{(0)AB}_{\text{GB}} - K^{(0)}_{AB}\,\pi^{(4)AB}_{\text{GB}}\right)\,, \end{aligned} \tag{A.49}$$

which yields:

$$\mathcal{L}^{(4)} \;=\; c_1\,R^2 + c_2\,R_{ab}R^{ab} + c_3\,R_{abcd}R^{abcd} \tag{A.50}$$

with

$$\begin{aligned} c_1 \;&=\; \frac{d}{4(d-4)(d-2)^2(d-1)} - \varepsilon\,\frac{3d^2-9d+4}{2(d-4)(d-2)(d-1)} + O(\varepsilon^2) \\ c_2 \;&=\; -\frac{1}{(d-4)(d-2)^2} + \varepsilon\,\frac{6d-14}{(d-4)(d-2)} + O(\varepsilon^2) \\ c_3 \;&=\; -\frac{\epsilon}{d-4} + O(\varepsilon^2) \end{aligned} \tag{A.51}$$

The curvature length can be restored by dimensional analysis, which comes down to replacing $\mathcal{L}^{(n)} \to \ell^{n-1}_{\text{GB}}\,\mathcal{L}^{(n)}$ and $\varepsilon \to \varepsilon/\ell^2_{\text{GB}}$. The result of [57] is reproduced by expressing ℓ_{GB} in terms of the non-corrected AdS length ℓ via (A.37). On page 31 we discuss how the inclusion of the Gauss–Bonnet term lifts the '$a = c$' degeneracy in the four-dimensional Weyl anomaly.

IV Wess–Zumino Condition and the Lifshitz Anomaly

Classification of possible terms in the anomaly

In this appendix we explore to what extent it is possible to remove total derivatives from the anomaly. This is achieved by adding appropriate scale invariant counterterms to the action that are not invariant under *local* scale transformations. Clearly, we can discuss the two-derivative and the four-derivative terms separately. Let us start with the former; there are only three possible scale-invariant terms that we can construct with two time derivatives:

$$K_{ij}K^{ij}\,, \qquad K^2\,, \qquad h^{ij}\partial_n\dot{K}_{ij}\,. \tag{A.52}$$

where introduced the 'normal' derivative $\partial_n \equiv N^{-1}\partial_t$. For instance, the extrinsic curvature is simply $K_{ij} = \frac{1}{2}\partial_n h_{ij}$.[1] It is straightforward to see that the two combinations

$$h^{ij}\,\partial_n K_{ij}\,, \qquad K_{ij}K^{ij} - \frac{1}{2}K^2\,, \tag{A.53}$$

are invariant under *local* scale transformations (up to total derivatives). These two terms are related by partial integration, and we now show that it is indeed possible to "partially integrate" inside the anomaly by adding an appropriate counterterm to the action. The most general form of the anomaly at the two derivative level is:

$$\delta W = \int dt\, d^2x\, N\sqrt{h}\left(a\, h^{ij}\,\partial_n K_{ij} + b\left(K_{ij}K^{ij} - \tfrac{1}{2}K^2\right)\right)\delta\omega\,, \tag{A.54}$$

where a and b are arbitrary numbers. The presence of the factor $\delta\omega$ prevents us from doing partial integration directly. What we can do, however, is add the following counterterm to the action:

$$W \;\to\; W' \;\equiv\; W + 4c\int dt\, d^2x\, N\sqrt{h}\;K^2\,. \tag{A.55}$$

It is then easy to check that

$$\delta W' = \int dt\, d^2x\, N\sqrt{h}\left((a-c)\,h^{ij}\,\partial_n K_{ij} + (b+2c)\left(K_{ij}K^{ij} - \tfrac{1}{2}K^2\right)\right)\delta\omega\,. \tag{A.56}$$

[1] We use foliation-preserving diffeomorphism invariance to kill the shift, $N_i = 0$.

Therefore we can pick $c = a$ and get rid of the first term, which is tantamount to integrating by parts, or discarding total derivatives in the anomaly.

Let us now consider the four derivative level. In this case we are interested in terms of the form $\nabla_i J^i$ in the anomaly. We ask ourselves to what extent it is possible to remove them by adding local counterterms G to the action. Both the total derivatives and the local counterterms must be scale invariant, therefore there is only a finite number of them. Let us choose a basis:

$$\begin{aligned} &J^i_a \qquad a = 1, \dots, A\,, \\ &G_b \qquad b = 1, \dots, B\,. \end{aligned} \tag{A.57}$$

The Weyl variation of a linear combination $q_b G_b$ can be written (after partial integration) as:

$$q_b\, \delta G_b = M_{ab}\, q_b \left(\frac{\nabla_i J^i_a}{N}\, \delta\omega \right)\,, \tag{A.58}$$

where the sum over repeated indices a and b is implied. If the variation of the effective action reads:

$$\delta W = \int \sqrt{h}\, \omega \left(\mathcal{A} + c_a \nabla_i J^i_a \right)\,, \tag{A.59}$$

we can get rid of the total derivatives if we can solve the system of linear equations:

$$\mathcal{M}_{ab}\, q_b = c_a\,. \tag{A.60}$$

If we are to remove all the possible total derivatives that can appear, the number of rows A of the matrix $\mathcal{M}_{ab}$ must be less than or equal to the number of columns B, and the rank of the matrix should be maximal. It is easy to check that there are 6 possible functionally independent scale invariant currents J^i, and we choose the following basis:

$$\begin{aligned} &J^i_1 = N\partial^i R && J^i_2 = (\partial^i N) R \\ &J^i_3 = (\partial^i N)(\tfrac{1}{N}\partial_j N)(\tfrac{1}{N}\partial^j N) && J^i_4 = (\partial^i N)(\tfrac{1}{N}\Delta N) \\ &J^i_5 = (\partial^j N)(\tfrac{1}{N}\nabla_j \partial^i N) && J^i_6 = \partial^i \Delta N \end{aligned} \tag{A.61}$$

Analogously, there are 12 functionally independent scale invariant counterterms, and we choose the basis:

$$
\begin{aligned}
G_1 &= R^2 & G_2 &= \Delta R \\
G_3 &= (\tfrac{1}{N}\Delta N)R & G_4 &= (\tfrac{1}{N}\partial_i N)(\tfrac{1}{N}\partial^i N)R \\
G_5 &= \big((\tfrac{1}{N}\partial_i N)(\tfrac{1}{N}\partial^i N)\big)^2 & G_6 &= (\tfrac{1}{N}\partial_i N)(\tfrac{1}{N}\partial^i N)(\tfrac{1}{N}\Delta N) \\
G_7 &= (\tfrac{1}{N}\Delta N)^2 & G_8 &= (\tfrac{1}{N}\partial^i N)(\tfrac{1}{N}\partial^j N)(\tfrac{1}{N}\nabla_i\partial_j N) \\
G_9 &= (\tfrac{1}{N}\partial^i N)\tfrac{1}{N}\partial_i\Delta N & G_{10} &= \tfrac{1}{N}\nabla_i\partial_j N\tfrac{1}{N}\nabla^i\partial^j N \\
G_{11} &= \tfrac{1}{N}\Delta^2 N & G_{12} &= \tfrac{1}{N}\partial^i N\partial_i R
\end{aligned}
\tag{A.62}
$$

While we have many more possible counterterms than currents, it is important to stress that not all the counterterms are independent, since we can always partially integrate inside the action. This means that some linear combinations of counterterms will have the same Weyl transformation. Furthermore, there can be Weyl invariant combinations of counterterms that do not help in removing total derivatives from the anomaly.

By taking the Weyl variation of the 12 terms G_b, it is straightforward to compute the matrix $\mathcal{M}_{ab}$, which is given by:

$$
\mathcal{M}_{ab} = \begin{pmatrix}
-4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
-4 & -2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 2 & -8 & -6 & 0 & -5 & -6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & -8 & 2 & 4 & -2 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 4 & 0 & -2 & 4 & -4 & 0 & 0 \\
0 & -2 & -2 & 0 & 0 & 0 & 4 & 0 & -4 & 4 & 0 & 2
\end{pmatrix}
\tag{A.63}
$$

It is easily checked that $\mathcal{M}_{ab}$ does *not* have maximal rank (which would be 6), but it has rank 5. In fact, $\mathcal{M}_{ab}$ has a 7 dimensional space of null vectors, which is spanned by the 6 total derivatives $\nabla_i J^i$ and a Weyl invariant term:

$$
\delta\int\sqrt{h}\,\nabla_i J^i_a = 0, \qquad \delta\int N\sqrt{h}\left(R + \frac{\Delta N}{N} - \frac{\partial_i N\partial^i N}{N^2}\right)^2 = 0. \tag{A.64}
$$

Since the rank of $\mathcal{M}_{ab}$ is 5, the Weyl variation of the most general counterterm spans a 5 dimensional subspace of the 6 dimensional space generated by $c_a\,\nabla_i J^i_a$. That means that we can find an orthonormal basis (with respect to the usual Euclidean scalar product δ_{ab}) for the currents where 5 are trivial (i.e. removable by counterterms) and 1 is non-trivial. In other words, we look for 5 vectors e_a such that $e_a = \mathcal{M}_{ab}q_b$ admits a solution. If we now take u_a to be the null vector of the transpose of $\mathcal{M}_{ab}$, it is obviously orthogonal to all the e_a since $u_a e_a = e_a M_{ab} q_b = 0$. We define the non-trivial current $\mathcal{J}^i$ to be:

$$\mathcal{J}^i = u_a J_a^i = J_1^i - J_2^i + J_4^i + J_5^i + 2J_6^i. \tag{A.65}$$

However, we will presently show that this current does *not* obey the Wess–Zumino consistency condition, therefore it cannot appear in the anomaly.

Wess–Zumino consistency condition and $\mathcal{J}^i$

The goal of this section is to figure out whether all possible terms that we found above satisfy the Wess–Zumino consistency conditions. To this end, we shall compute the quantities

$$\Omega_a \equiv \delta_1 \int d^2x \sqrt{h}\, \omega_2 \nabla_i J_a^i - \delta_2 \int d^2x \sqrt{h}\, \omega_1 \nabla_i J_a^i \tag{A.66}$$

$$= \int d^2x\, \delta_2\big(\sqrt{h}\, J_a^i\big)\, \partial_i \omega_1 - \int d^2x\, \delta_1\big(\sqrt{h}\, J_a^i\big)\, \partial_i \omega_2 \tag{A.67}$$

for each $a = 1, .., 6$. The main idea of this analysis is to find all possible linear combinations of the Ω's such that

$$c_a \Omega_a = 0 \tag{A.68}$$

If the vector space spanned by the vectors $\{c_a\}$ is six dimensional, all J_a^i's are Wess–Zumino-consistent. If, on the other hand, this vector space is five-dimensional then we must conclude that one of the J_a^i's is inconsistent. Since we already know that five currents can be generated by varying appropriate local scale invariant terms, these are manifestly consistent. Therefore the inconsistent current, if present, must be the non-trivial current of equation (A.65).

The way we shall carry out this computation is by first computing the first term in (A.67). The second term in (A.67) is then obtained from the first one by replacing the derivatives that act on ω_1 for derivatives that act on ω_2 by means of partial integration.

We shall start with Ω_1. The first term in (A.67) is[2]

$$\delta_2\big(\sqrt{h}\, J_1^i\big)\, \partial_i \omega_1 = \sqrt{h}\left(-\partial^i \omega_2\, NR - \partial^i \Delta \omega_2\, N\right)\, \partial_i \omega_1 \tag{A.69}$$

[2]For notational clarity, notice that the variation differs by a factor of two compared to before. For instance, $h_{ij} \to e^{\omega} h_{ij}$ rather than $h_{ij} \to e^{2\omega} h_{ij}$.

The second term is then

$$\begin{aligned}
\delta_1\big(\sqrt{h}\,J_1^i\big)\,\partial_i\omega_2 &= \sqrt{h}\left(-\partial^i\omega_1\,NR - \partial^i\Delta\omega_1\,N\right)\,\partial_i\omega_2 && \text{(A.70)}\\
&= \sqrt{h}\left(-\partial^i\omega_2\,NR - \nabla^i\nabla^j\big(\partial_j\omega_2\,N\big)\right)\,\partial_i\omega_1 && \text{(A.71)}\\
&= \sqrt{h}\Big(-\partial^i\omega_2\,NR - \partial^i\Delta\omega_2\,N && \text{(A.72)}\\
&\qquad - \Delta\omega_2\,\partial^i N - \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\Big)\,\partial_i\omega_1 && \text{(A.73)}
\end{aligned}$$

so that

$$\Omega_1 = \int d^2x\sqrt{h}\,\left(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\right)\,\partial_i\omega_1 \qquad \text{(A.74)}$$

Similarly, from J_2^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_2^i\big)\,\partial_i\omega_1 &= \sqrt{h}\left(\partial^i\omega_2\,NR - \Delta\omega_2\,\partial^i N\right)\,\partial_i\omega_1 && \text{(A.75)}\\
\delta_1\big(\sqrt{h}\,J_2^i\big)\,\partial_i\omega_2 &= \sqrt{h}\left(\partial^i\omega_1\,NR - \Delta\omega_1\,\partial^i N\right)\,\partial_i\omega_2 \\
&= \sqrt{h}\left(\partial^i\omega_2\,NR + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\right)\,\partial_i\omega_1 && \text{(A.76)}\\
\Omega_2 &= -\int d^2x\sqrt{h}\,\left(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\right)\,\partial_i\omega_1 && \text{(A.77)}
\end{aligned}$$

From J_3^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_3^i\big)\,\partial_i\omega_1 &= \sqrt{h}\left(\partial^i\omega_2\,\partial_j N\,\partial^j N + 2\partial_j\omega_2\,\partial^i N\,\partial^j N\right)\,\partial_i\omega_1 && \text{(A.78)}\\
\delta_1\big(\sqrt{h}\,J_3^i\big)\,\partial_i\omega_2 &= \sqrt{h}\left(\partial^i\omega_1\,\partial_j N\,\partial^j N + 2\partial_j\omega_1\,\partial^i N\,\partial^j N\right)\,\partial_i\omega_2 \\
&= \sqrt{h}\left(\partial^i\omega_2\,\partial_j N\,\partial^j N + 2\partial_j\omega_2\,\partial^j N\,\partial^i N\right)\,\partial_i\omega_1 && \text{(A.79)}\\
\Omega_3 &= 0 && \text{(A.80)}
\end{aligned}$$

From J_4^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_4^i\big)\,\partial_i\omega_1 &= \sqrt{h}\left(\partial^i\omega_2\,\Delta N + 2\partial_j\omega_2\,\tfrac{1}{N}\partial^i N\partial^j N + \Delta\omega_2\,\partial^i N\right)\,\partial_i\omega_1 && \text{(A.81)}\\
\delta_1\big(\sqrt{h}\,J_4^i\big)\,\partial_i\omega_2 &= \sqrt{h}\left(\partial^i\omega_1\,\Delta N + 2\partial_j\omega_1\,\tfrac{1}{N}\partial^i N\partial^j N + \Delta\omega_1\,\partial^i N\right)\,\partial_i\omega_2 \\
&= \sqrt{h}\left(\partial^i\omega_2\,\Delta N + 2\partial_j\omega_2\,\tfrac{1}{N}\partial^j N\partial^i N - \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\right)\,\partial_i\omega_1 && \text{(A.82)}\\
\Omega_4 &= \int d^2x\sqrt{h}\,\left(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\right)\,\partial_i\omega_1 && \text{(A.83)}
\end{aligned}$$

From J_5^i:

$$\delta_2\big(\sqrt{h}\,J_5^i\big)\,\partial_i\omega_1 = \sqrt{h}\,\big(\partial_j\omega_2\,\nabla^i\partial^j N + \partial^i\omega_2\,\tfrac{1}{N}\partial^j N\partial_j N + \nabla_j\partial^i\omega_2\,\partial^j N\big)\;\partial_i\omega_1 \tag{A.84}$$

$$\begin{aligned}\delta_1\big(\sqrt{h}\,J_5^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\big(\partial_j\omega_1\,\nabla^i\partial^j N + \partial^i\omega_1\,\tfrac{1}{N}\partial^j N\partial_j N + \nabla_j\partial^i\omega_1\,\partial^j N\big)\;\partial_i\omega_2 \\ &= \sqrt{h}\,\Big(\partial_j\omega_2\,\nabla^i\partial^j N + \partial^i\omega_2\,\tfrac{1}{N}\partial^j N\partial_j N - \nabla_j\big(\partial^{(i}\omega_1\,\partial^{j)}N\big)\Big)\;\partial_i\omega_1\end{aligned} \tag{A.85}$$

$$\Omega_5 = \int d^2x\sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 \tag{A.86}$$

From J_6^i:

$$\begin{aligned}\delta_2\big(\sqrt{h}\,J_6^i\big)\,\partial_i\omega_1 &= \sqrt{h}\,\partial^i\big(2\partial_j\omega_2\,\partial^j N + \Delta\omega_2\,N\big)\;\partial_i\omega_1 \\ &= \sqrt{h}\;\big(\partial^i\Delta\omega_2\,N + \Delta\omega_2\,\partial^i N + 2\partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1\end{aligned} \tag{A.87}$$

$$\begin{aligned}\delta_1\big(\sqrt{h}\,J_6^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\partial^i\big(2\partial_j\omega_1\,\partial^j N + \Delta\omega_1\,N\big)\;\partial_i\omega_2 \\ &= \sqrt{h}\,\big(\partial^i\Delta\omega_2\,N - \Delta\omega_2\,\partial^i N\big)\;\partial_i\omega_1\end{aligned} \tag{A.88}$$

$$\Omega_6 = 2\int d^2x\sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 \tag{A.89}$$

We thus find that each Ω_a is a multiple of

$$\int d^2x\sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1, \tag{A.90}$$

which means that there is one linear combination that does *not* satisfy the Wess–Zumino consistency conditions. In other words, all but one of the six J_a^i's can be made consistent. Since we have already found that five of the six J_a^i's can be canceled by variations of local terms, the one that cannot be canceled (which we called $\mathcal{J}^i$) must be inconsistent. We can make this more precise by noticing that the consistency equation

$$c_1 - c_2 + c_4 + c_5 + 2c_6 = 0 \tag{A.91}$$

describes a five-dimensional hypersurface of consistent linear combinations $c_a J_a^i$. The set of all such c_a-vectors can be defined as those that are orthogonal to the *inconsistent* vector, v_a say, such that $c_a v_a = 0$. The inconsistent vector is

$$\vec{v} = \begin{pmatrix}1 & -1 & 0 & 1 & 1 & 2\end{pmatrix} \tag{A.92}$$

As a consistency check on our computations, notice that this is precisely the five-dimensional hypersurface that we mentioned above, which may be defined as all vectors that are orthogonal to u_a (as defined in (A.65)). Namely, the vector u_a is the same as the inconsistent vector, i.e. $u_a = v_a$. The fact that $\mathcal{J}^i$ does not satisfy the Wess–Zumino condition means that it cannot appear as the variation of either local or non-local terms. The fact that there are precisely five total-derivative terms in the anomaly, all of which can be canceled by variations of local terms.

V Hamilton–Jacobi Renormalization of Lifshitz Spacetime

The goal of this section is to compute the (divergent piece of the) on-shell value of the above action using the Hamilton–Jacobi method (cf. page 22) as it appeared in [1, 2]. The HJ equation is a differential equation for the on-shell action, i.e. solving the HJ equation will give us the on-shell action S. We write the HJ equation $H = 0$ as

$$\{S, S\} + \mathcal{V} \; = \; 0 \tag{A.93}$$

where

$$\mathcal{V} \; = \; R - 2\Lambda - \frac{1}{4}F_{ab}F^{ab} - \frac{m^2}{2}\, A_a A^a \tag{A.94}$$

the brackets are given by the following expression. Let F and G be two arbitrary phase-space functionals, then the brackets are defined as

$$\{F, G\} \; \equiv \; \frac{1}{(\sqrt{g})^2}\Bigg[\left(g_{ac}g_{bd} - \frac{1}{d-1}g_{ab}g_{cd}\right)\frac{\delta F}{\delta g_{ab}}\frac{\delta G}{\delta g_{cd}} + \frac{1}{2}g_{ab}\frac{\delta F}{\delta A_a}\frac{\delta G}{\delta A_b} + \frac{1}{2m^2}D_a\frac{\delta F}{\delta A_a}\,D_b\frac{\delta G}{\delta A_b}\Bigg] \tag{A.95}$$

where g_{ab} and A_a are the induced fields pulled back onto the radial cut-off slice. The brackets (A.95) were introduced in [27]; see also [1]. It should be noted that these brackets are only introduced as a short-hand notation for the 'kinetic' part of the Hamiltonian contraint; they are *not* Poisson brackets (or any other type of special brackets).

Let us define the renormalized on-shell action as $W = S + S_{\text{c.t.}}$. Using the split $S = W - S_{\text{c.t.}}$, we may write the HJ equation as

$$0 \; = \; \{S_{\text{c.t.}}, S_{\text{c.t.}}\} - 2\{S_{\text{c.t.}}, W\} + \{W, W\} + \mathcal{V}\,. \tag{A.96}$$

In this approach, we use an Ansatz for the counterterm action. Let us expand the counterterm Lagrangian in terms of the number of derivatives, $\mathcal{L}_{\text{c.t.}} = \mathcal{L}^{(0)}_{\text{c.t.}} + \mathcal{L}^{(2)}_{\text{c.t.}} + \mathcal{L}^{(4)}_{\text{c.t.}} + \dots$. The Ansatz is chosen to be the most general local covariant counterterm action one can write down, for instance, at the constant level we take:

$$\mathcal{L}^{(0)}_{\text{c.t.}} \;=\; U(\psi)\,, \tag{A.97}$$

where $\psi = -1 + \sqrt{-A^2}$ is the deviation of the massive vector away from the time-like unit normal ($\psi = 0$ on the pure Lifshitz background). The derivative counterterms are suppressed by powers of e^{-r}, because each pair of derivatives comes with one factor of an inverse metric. This is why the boundary conditions will fix the leading terms of the non-derivative counterterm action $U(\psi)$.

Boundary condtions. The way we impose the boundary conditions that allow for arbitrary sources is by only fixing the asymptotic radial scaling: $\partial_r g_{tt} \approx 2z\, g_{tt}$, $\partial_r g_{ij} \approx 2g_{ij}$ and $\partial_r A_t \approx z\, A_t$. These can be translated to the leading behavior of the on-shell action by using the Hamiltonian equations of the type $\dot{q} = \partial H/\partial p$:

$$\begin{aligned}
\partial_r g_{ab} \;&=\; \frac{1}{\sqrt{g}}\,\frac{\delta H}{\delta \pi^{ab}} \;=\; 2\pi_{ab} - \frac{2}{d-1}\,\pi\, g_{ab} \\
&=\; \frac{U + (d_s - 1)(\alpha+\psi)U'}{d_s}\,\delta^t_a\delta^t_b \;+\; \frac{-U + (\alpha+\psi)U'}{d_s}\,\delta_{ij}\,\delta^i_a\delta^j_b \\
\partial_r A_a \;&=\; \frac{1}{\sqrt{g}}\,\frac{\delta H}{\delta E^a} \;=\; E_a \;=\; U'\,\delta^t_a\,.
\end{aligned} \tag{A.98}$$

The boundary conditions in terms of the radial scaling thus fix $U(0) = 6$ and $U'(0) = 2$. The 'potential' $\mathcal{V}$ involves only integer powers of ψ,

$$\mathcal{V} \;=\; 12 + 4\,\psi + 2\,\psi^2 + (\text{derivative terms})\,, \tag{A.99}$$

so it seems reasonable to assume that $U(\psi)$ can be expanded as

$$U(\psi) \;=\; \sum_{n\geq 0} u_n \psi^n\,, \tag{A.100}$$

where $u_0 = U(0)$ and $u_1 = U'(0)$ were just fixed by the boundary conditions. When we start solving the Hamiltonian constraint order by order, one find a discrete ambiguity for constant u_2. We must pick the value that corresponds to the non-normalizable mode, which gives $u_2 = 2$.[1] All higher order coefficients $u_{n\geq 3}$ are unambiguously determined in terms of $u_{0,1,2}$.

[1] If both modes were normalizable, we would have been able to pick either one.

The first thing we see is that $S_{\text{c.t.}} \sim \mathrm{e}^{4r}$, while the renormalized on-shell action $W \sim 1$ by definition.[2] We then see that we can work perturbatively in W. Solving leading order HJ equation will yield the counterterm action, while the solution to first order will give the renormalized on-shell action itself. Let us start by solving the leading-order HJ equation.

V.1 Two-derivative counterterms and anomaly

Throughout this section, we use the index (n) to denote the number of derivatives. The Hamiltonian constraint at the level of two spacetime derivatives is given by

$$\mathcal{H}^{(2)} \;=\; \{S_{\text{c.t.}}, S_{\text{c.t.}}\}^{(2)} + R - \frac{1}{4}F_{ab}F^{ab} \qquad \text{(A.101)}$$

At the level of two spacetime derivatives, the most general Ansatz is schematically given by

$$\mathcal{L}^{(2)}_{\text{c.t.}} \;\sim\; R\,,\; gg\,DADA\,,\; gAA\,DADA\,,\; AAAA\,DADA\,. \qquad \text{(A.102)}$$

When we take all contractions into account, we end up with 9 distinct terms. Each term, labeled by $i = 1, \ldots, 9$, has a coefficient $\mathcal{C}_i(\psi)$ that is a general function of ψ. Equation (A.101) can be solved straightforwardly by expanding the coefficient-functions around the Lifshitz background,

$$\mathcal{C}_i(\alpha) \;=\; \sum_{n\geq 0} c_{i(n)}(\alpha - \alpha_0)^n. \qquad \text{(A.103)}$$

The coefficients $c_{i(n)}$ are then found by solving the equation $\mathcal{H}^{(2)} = 0$ recursively order by order.

For the case of $z = 2$ in $d = 3$ boundary dimensions one finds a break-down of the recursive 'descent' equations, i.e. $\mathcal{H}^{(2)} \neq 0$ for any choice of coefficient functions $\mathcal{C}_i$. Such a finite remainder is directly related to the holographic Lifshitz anomaly, see Section 1.1.2 as well as (2.80):

$$\mathcal{A} \;=\; -\mathcal{H}^{(2)} \;=\; D_aA_b\,D^bA^a - \frac{1}{2}(D_aA^a)^2 + \text{four-derivative terms}\,, \qquad \text{(A.104)}$$

which is obviously affected by the ambiguity of adding finite local counterterms to the action. Using the time-like ADM decomposition from (2.81), we

[2] If W is not renormalizable, it would be reflected in a breakdown of the recursion relation at a level that is still power-law divergent.

get

$$\mathcal{A} \;=\; \frac{\ell^2}{16\pi G}\left(\hat{K}_{ij}\hat{K}^{ij}-\frac{1}{2}\hat{K}^2\right)+\text{four-derivative terms}\,. \tag{A.105}$$

We reinstated the four-dimensional Newton's constant G and the curvature length scale ℓ. Comparing with the generic Lifshitz anomaly (2.11) gives us the first central charge:

$$\boxed{C_1 \;=\; \frac{\ell^2}{2G}} \tag{A.106}$$

V.2 Four-derivative anomaly

One may repeat the above steps at the level of four derivatives.[3] The four-derivative Ansatz is

$$S^{(4)}_{\text{loc}} \;=\; \int d^dx\sqrt{g}\left(\mathcal{G}_1\,R^2+\mathcal{G}_2\,R_{ab}R^{ab}+\mathcal{G}_3\,R_{abcd}R^{abcd}+\mathcal{G}_4\,\Box R\right)+\ldots\,, \tag{A.107}$$

where the $\mathcal{G}_i$ are arbitrary functions of ψ and the ellipses denote terms that involve the Proca field A_a. All the terms that appear at this level are finite for our choice of boundary conditions, which means that they can only contribute with trivial total derivatives to the anomaly. In this case, we find the remainder

$$\mathcal{H}^{(4)} \;=\; \frac{1}{8}R^2-\frac{1}{4}R_{ab}R^{ab}+\ldots \tag{A.108}$$

where the ellipses denote once again terms that involve A_a. Writing this in terms of the two-dimensional Ricci tensor $\hat{R}$ gives:

$$\mathcal{H}^{(4)} \;=\; \frac{1}{4}\,\hat{R}_{ij}\hat{R}^{ij}-\frac{1}{8}\,\hat{R}^2+\ldots, \tag{A.109}$$

where these $\hat{R}$ and $\hat{R}_{ij}$ are the *two*-dimensional Ricci scalar and tensor and we have not written down terms that involve derivatives acting on N. We can use the off-shell identity that relates the Ricci tensor to the Ricci scalar, $\hat{R}_{ij}=\frac{1}{2}\,\hat{R}\,h_{ij}$, which is specific to two dimensions. When we plug this into (A.109), we find that the Ricci-squared terms cancel and we find that $\mathcal{H}^{(4)} =$

[3] One does not expect to find anomalous contributions that contain three derivatives (one time and two spatial), since terms that involve an odd number of time-derivatives are not invariant under time-reversal.

0. Now, we can use the identification $\mathcal{A}_{\text{4-deriv.}} = -\mathcal{H}^{(4)}$ to conclude that the second central charge vanishes:

$$C_2 = 0, \tag{A.110}$$

which interestingly seems to agree with the field theory computation.

Notice that while we were able to extract the coefficient C_2, we have not performed a complete analysis of the counterterms at the four-derivative level, which would be rather involved. Nevertheless, the complete answer has been computed using the results of [31] in [42] (cf. Chapter 2), which is in perfect agreement with our result $C_2 = 0$. In conclusion, the holographic anomaly is given by

$$\mathcal{A} \;=\; \frac{\ell^2}{16\pi G}\left(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2\right). \tag{A.111}$$

VI Computing the $z = 2$ Casimir energy and stress tensor

In this section we show the explicit computations that lead to the final expressions for the expectation values of the energy density (2.128) and the spatial stress tensor (**??**).

Working out $\delta(N\sqrt{h}\,G^{ijkl}K_{ij}K_{kl})/\delta N$.

This first term is relatively easy to compute. We get

$$\begin{aligned}\frac{\delta}{\delta N}\int dt\, d^n x\, N\sqrt{h}\, G^{ijkl}K_{ij}K_{kl} \;&=\; \frac{\delta}{\delta N}\int dt\, d^n x\, \frac{\sqrt{h}}{4N}\, G^{ijkl}\dot{h}_{ij}\dot{h}_{kl} \\ &=\; -\frac{\sqrt{h}}{4N^2}\, G^{ijkl}\dot{h}_{ij}\dot{h}_{kl}\,. \end{aligned} \tag{A.112}$$

We can then use the conformally flat background, i.e. $N = \mathrm{e}^{z\sigma} = \mathrm{e}^{2\sigma}$ and $h_{ij} = \mathrm{e}^{2\sigma}\,\delta_{ij}$. When we use the fact that $n = 2 + \varepsilon$, we find

$$\begin{aligned}\frac{\delta}{\delta N}\int dt\, d^n x\, N\sqrt{h}\, G^{ijkl}K_{ij}K_{kl} \;&=\; \frac{1}{2}(n-2)n\,\dot{\sigma}^2\,\mathrm{e}^{(n-4)\sigma} \\ &=\; \varepsilon\,\dot{\sigma}^2\,\mathrm{e}^{-2\sigma} \;+O(\varepsilon^2)\end{aligned} \tag{A.113}$$

Working out $\delta(N\sqrt{h}\,\widetilde{R}^2)/\delta N$.

Before we start, let us define the short-hand notation:

$$\widetilde{R}_{ij} \;\equiv\; R_{ij} + \frac{1}{N}\nabla_{(i}\nabla_{j)}N - \frac{1}{N^2}\nabla_{(i}N\,\nabla_{j)}N\,. \tag{A.114}$$

We want to compute the variation:

$$\frac{\delta}{\delta N}\int dt\,d^n x\,N\sqrt{h}\,\widetilde{R}^2 \;=\; \sqrt{h}\,\widetilde{R}^2 + 2\int dt\,d^n x\,N\sqrt{h}\,\widetilde{R}\,\frac{\delta\widetilde{R}}{\delta N} \tag{A.115}$$

Let us work out the variational derivative the second term.

$$\frac{\delta\widetilde{R}(x)}{\delta N(y)} \;=\; \left(-\frac{1}{N^2}\Delta N + \frac{2}{N^3}\nabla_i N\,\nabla^i N + \frac{1}{N}\Delta - \frac{2}{N^2}\nabla^i N\,\nabla_i\right)\delta(x-y) \tag{A.116}$$

such that

$$\begin{aligned} 2\int dt\,d^n x\,N\sqrt{h}\,\widetilde{R}\,\frac{\delta\widetilde{R}}{\delta N} \;&=\; 2\sqrt{h}\,\widetilde{R}\left(-\frac{1}{N}\Delta N + \frac{2}{N^2}\nabla_i N\,\nabla^i N\right) \\ &\quad + 2\sqrt{h}\left(\Delta\widetilde{R} + 2\nabla_i\left(\widetilde{R}\,\frac{1}{N}\nabla^i N\right)\right) \\ &=\; 2\sqrt{h}\,\frac{1}{N}\Delta\big(N\widetilde{R}\big) \end{aligned} \tag{A.117}$$

Now let us return to the thing we wanted to compute in the first place, namely (A.115). Again, we use the conformally flat background $N = \mathrm{e}^{2\sigma}$ and $h_{ij} = \mathrm{e}^{2\sigma}\,\delta_{ij}$. First, we notice that $\widetilde{R}$ is of order $O(\varepsilon)$:

$$\begin{aligned} \widetilde{R} \;&=\; (n-2)\Big((3-n)\sigma_k\sigma_k - 2\sigma_{kk}\Big)\,\mathrm{e}^{-2\sigma} \\ &=\; \varepsilon\Big(\sigma_k\sigma_k - 2\sigma_{kk}\Big)\mathrm{e}^{-2\sigma} \;+ O(\varepsilon^2)\,, \end{aligned} \tag{A.118}$$

where we use the abbreviated notation $\sigma_i \equiv \partial_i\sigma$, $\sigma_{ij} \equiv \partial_i\partial_j\sigma$, etc. This means that the first term in (A.115) will be of order $O(\varepsilon^2)$ and hence will not contribute. So, the only contribution we find comes from the second term.

$$\frac{\delta}{\delta N}\int dt\,d^n x\,N\sqrt{h}\,\widetilde{R}^2 \;=\; 2\sqrt{h}\,\frac{1}{N}\Delta\big(N\widetilde{R}\big) \tag{A.119}$$

$$\begin{aligned} &=\; 2\varepsilon\partial_k\partial_k\Big(\sigma_l\sigma_l - 2\sigma_{ll}\Big)\,\mathrm{e}^{-2\sigma} \;+ O(\varepsilon^2) \\ &=\; -4\varepsilon\Big(\sigma_{kkll} - \sigma_k\sigma_{kll} - \sigma_{kl}\sigma_{kl}\Big)\,\mathrm{e}^{-2\sigma} \;+ O(\varepsilon^2) \end{aligned} \tag{A.120}$$

Working out $\delta(N\sqrt{h}\, G^{klmn}K_{kl}K_{mn})/\delta h_{ij}$.

Remember that $G_{ijkl} \equiv h_{i(k}h_{l)j} - \frac{1}{2}h_{ij}h_{kl}$. Let us look at

$$\begin{aligned}
&\frac{\delta}{\delta h_{ij}} \int dt\, d^n x\, N\sqrt{h}\, G^{klmn}K_{kl}K_{mn} \\
&\quad = \frac{\delta}{\delta h_{ij}} \int dt\, d^n x\, \sqrt{h}\, \frac{1}{4N}\, G^{klmn}\dot{h}_{kl}\dot{h}_{mn} \\
&\quad = \int dt\, d^n x\, \frac{1}{4N} \left(\frac{\delta(\sqrt{h}\, G^{klmn})}{\delta h_{ij}} \dot{h}_{kl}\dot{h}_{mn} + 2\sqrt{h}\, G^{klmn}\dot{h}_{kl}\frac{\delta \dot{h}_{mn}}{\delta h_{ij}} \right) \\
&\quad = \frac{1}{4N} \left(\frac{\partial(\sqrt{h}\, G^{klmn})}{\partial h_{ij}} \dot{h}_{kl}\dot{h}_{mn} - 2\partial_t \left(\sqrt{h}\, G^{ijkl}\dot{h}_{kl}\right) + 2\frac{\dot{N}}{N}\sqrt{h}\, G^{ijkl}\dot{h}_{kl} \right).
\end{aligned} \tag{A.121}$$

We used partial integration in going from the second to the third line. Let us work the first term first. We can introduce the rank-6 tensor Ω^{ijklmn} as defined in below in section VI.1, so that we find the first term in (A.121):

$$\frac{\partial(\sqrt{h}\, G^{klmn})}{\partial h_{ij}} \dot{h}_{kl}\dot{h}_{mn} = -\sqrt{h}\, \Omega^{ijklmn}\, \dot{h}_{kl}\dot{h}_{mn}\,, \tag{A.122}$$

We used the property that Ω^{ijklmn} is symmetric under permutations of the pairs of indices ij, kl and mn. We specifically used $\Omega^{klmnij} = \Omega^{ijklmn}$. The second term in (A.121) is given by

$$\begin{aligned}
-2\partial_t \left(\sqrt{h}\, G^{klij}\dot{h}_{kl}\right) &= -2\sqrt{h}\, G^{ijkl}\ddot{h}_{kl} - 2\frac{\partial(\sqrt{h}\, G^{ijkl})}{\partial h_{mn}}\dot{h}_{kl}\dot{h}_{mn} \\
&= -2\sqrt{h}\, G^{ijkl}\ddot{h}_{kl} + 2\sqrt{h}\, \Omega^{ijklmn}\, \dot{h}_{kl}\dot{h}_{mn}\,.
\end{aligned} \tag{A.123}$$

The third and last term in (A.121) does not need rewriting at this point.

We can now return to the expression we were initially interested in.

$$\begin{aligned}
&\frac{\delta}{\delta h_{ij}} \int dt\, d^n x\, N\sqrt{h}\, G^{klmn}K_{kl}K_{mn} \\
&\qquad = \frac{\sqrt{h}}{4N} \left(\Omega^{ijklmn}\dot{h}_{kl}\dot{h}_{mn} - 2G^{ijkl}\ddot{h}_{kl} + \frac{2\dot{N}}{N}\, G^{ijkl}\dot{h}_{kl} \right)
\end{aligned} \tag{A.124}$$

At this stage we are ready to start using $N = \mathrm{e}^{2\sigma}$ and $h_{ij} = \mathrm{e}^{2\sigma}\delta_{ij}$, which means in particular that $\dot{h}_{ij} = 2\dot{\sigma}\, h_{ij}$ and $\ddot{h}_{ij} = 2\ddot{\sigma}\, h_{ij} + 4\dot{\sigma}^2\, h_{ij}$. We can then use the identities (A.153), such that

$$\begin{aligned}
\frac{\delta}{\delta h_{ij}} \int dt\, d^n x\, N\sqrt{h}\, G^{klmn} K_{kl} K_{mn} &= \frac{1}{4}(n-2)\left(2\ddot{\sigma}+(n-4)\dot{\sigma}^2\right)\, \mathrm{e}^{(n-4)\sigma}\, \delta^{ij} \\
&= \frac{1}{2}\varepsilon\left(\ddot{\sigma}-\dot{\sigma}^2\right)\, \mathrm{e}^{-2\sigma}\, \delta^{ij} \; + O(\varepsilon^2)\,.
\end{aligned} \tag{A.125}$$

Working out $\delta(N\sqrt{h}\,\widetilde{R}^2)/\delta h_{ij}$.

Remember the short-hand notation that we introduced above:

$$\widetilde{R}_{ij} \equiv R_{ij} + \frac{1}{N}\nabla_{(i}\nabla_{j)}N - \frac{1}{N^2}\nabla_{(i}N\,\nabla_{j)}N\,. \tag{A.126}$$

As we will use it later, let us first compute the variation $\delta\widetilde{R}(x)/\delta h_{ij}(y)$:

$$\begin{aligned}
\frac{\delta\widetilde{R}(x)}{\delta h_{ij}(y)} &= \frac{\delta h^{kl}(x)}{\delta h_{ij}(y)}\widetilde{R}_{kl}(x) + h^{kl}(x)\frac{\delta\widetilde{R}_{kl}(x)}{\delta h_{ij}(y)} \\
&= \left(-\widetilde{R}^{ij} + \nabla^{(i}\nabla^{j)} - h^{ij}\Delta - \frac{1}{N}\partial^{(i}N\,\nabla^{j)} + \frac{1}{2N}h^{ij}\,\partial^k N\,\nabla_k\right)\delta(x-y)
\end{aligned} \tag{A.127}$$

where we used

$$\begin{aligned}
\frac{\delta h_{kl}(x)}{\delta h_{ij}(y)} &= h^i_{(k}h^j_{l)}\,\delta(x-y)\,, && \text{(A.128)} \\
\frac{\delta h^{kl}(x)}{\delta h_{ij}(y)} &= -h^{i(k}h^{l)j}\,\delta(x-y)\,, && \text{(A.129)} \\
h^{kl}\delta R_{kl} &= \left(\nabla^{(k}\nabla^{l)} - h^{kl}\Delta\right)\delta h_{kl}\,, && \text{(A.130)}
\end{aligned}$$

as well as

$$h^{kl}\frac{\delta}{\delta h_{ij}(y)}\left(\frac{1}{N}\nabla_k\partial_l N\right)(x) = \frac{1}{N}\left(-\partial^{(i}N\,\nabla^{j)} + \frac{1}{2}h^{ij}\,\partial^k N\,\nabla_k\right)\delta(x-y)\,. \tag{A.131}$$

The thing we are interested in is

$$
\begin{aligned}
\frac{1}{\sqrt{h}}\frac{\delta}{\delta h_{ij}}\int dt\, d^n x\, N\sqrt{h}\,\widetilde{R}^2 &= \frac{1}{\sqrt{h}}\int dt\, d^n x\, N\left(\frac{\delta\sqrt{h}}{\delta h_{ij}}\widetilde{R}^2 + 2\sqrt{h}\,\widetilde{R}\frac{\delta\widetilde{R}}{\delta h_{ij}}\right) \\
&= \frac{1}{2}N\,\widetilde{R}^2\,h^{ij} - 2N\widetilde{R}\,\widetilde{R}_{ij} \\
&\quad + 2\left(\nabla^{(i}\nabla^{j)} - h^{ij}\Delta\right)(N\widetilde{R}) \\
&\quad + 2\nabla^{(i}\left(\widetilde{R}\,\partial^{j)}N\right) - h^{ij}\,\nabla_k\left(\widetilde{R}\,\partial^k N\right). \qquad \text{(A.132)}
\end{aligned}
$$

We have color-coded the different contributions for easy visual distinction. Using partial integration, we made sure no derivatives act on the delta function $\delta(x-y)$. Let us now take $N = \mathrm{e}^{2\sigma}$ and $h_{ij} = \mathrm{e}^{2\sigma}\delta_{ij}$. It then follows that (we use the abbreviated notation $\sigma_i \equiv \partial_i\sigma$, $\sigma_{ij} \equiv \partial_i\partial_j\sigma$, etc.)

$$
\begin{aligned}
R_{ij} &= -\delta_{ij}\sigma_{kk} + (n-2)\left(\sigma_i\sigma_j - \sigma_{ij} - \delta_{ij}\,\sigma_k\sigma_k\right), && \text{(A.133)} \\
\frac{1}{N}\nabla_{(i}\nabla_{j)}N &= 2\sigma_{ij} + 2\delta_{ij}\,\sigma_k\sigma_k\,, && \text{(A.134)} \\
-\frac{1}{N^2}\nabla_i N\,\nabla_j N &= -4\,\sigma_i\sigma_j\,, && \text{(A.135)}
\end{aligned}
$$

where repeated lower indices are summed over implicitly. From this, it follows that:

$$
\widetilde{R}_{ij} = (n-6)\sigma_i\sigma_j - (n-4)\sigma_{ij} - \delta_{ij}\Big((n-4)\sigma_k\sigma_k + \sigma_{kk}\Big) \qquad \text{(A.136)}
$$

which implies in particular that (remember, $n = 2+\varepsilon$)

$$
\begin{aligned}
N\widetilde{R} &= -(n-2)(n-3)\sigma_k\sigma_k - 2(n-2)\sigma_{kk} \\
&= \varepsilon\left(\sigma_k\sigma_k - 2\sigma_{kk}\right) + O(\varepsilon^2)\,. \qquad \text{(A.137)}
\end{aligned}
$$

Notice that the first term in (A.132), $\frac{1}{2}N\widetilde{R}^2 h^{ij}$, will not contribute, because it is of order ε^2. In the second term of (A.132) we use that

$$
\widetilde{R}_{ij} = -2\big(2\sigma_i\sigma_j - \sigma_{ij}\big) + \delta_{ij}\left(2\sigma_k\sigma_k - \sigma_{kk}\right) + O(\varepsilon) \qquad \text{(A.138)}
$$

and since $\widetilde{R}$ is already of order ε, only the $O(1)$ piece of $\widetilde{R}_{ij}$ will contribute. In other words,

$$
-2N\widetilde{R}\,\widetilde{R}_{ij} = -2\varepsilon\left(\sigma_k\sigma_k - 2\sigma_{kk}\right)\left(-4\sigma_i\sigma_j + 2\sigma_{ij} + \delta_{ij}\left(2\sigma_l\sigma_l - \sigma_{ll}\right)\right) + O(\varepsilon^2)\,, \qquad \text{(A.139)}
$$

bearing in mind that $\widetilde{R}^{ij} = h^{ik}h^{jl}\widetilde{R}_{kl}$. Before working out the third term in (A.132), notice that for any scalar function Φ we have[1]

$$\begin{aligned}
\nabla_{(i}\nabla_{j)}\Phi &= \partial_i\partial_j\Phi - 2\sigma_{(i}\,\partial_{j)}\Phi + \delta_{ij}\,\sigma_k\,\partial_k\Phi \\
h_{ij}\nabla_k\nabla^k\Phi &= \delta_{ij}\Big(\Delta\Phi + (n-2)\sigma_k\,\partial_k\Phi\Big) \\
\Big(\nabla_{(i}\nabla_{j)} - h_{ij}\nabla_k\nabla^k\Big)\Phi &= \partial_i\partial_j\Phi - 2\sigma_{(i}\,\partial_{j)}\Phi - \delta_{ij}\big(\partial_k\partial_k\Phi + (n-3)\sigma_k\,\partial_k\Phi\big)
\end{aligned} \tag{A.140}$$

In order to work out the third term in (A.132), we replace Φ by $N\widetilde{R}$ as given by (A.137), which yields

$$\begin{aligned}
&2\Big(\nabla_{(i}\nabla_{j)} - h_{ij}\nabla_k\nabla^k\Big)(N\widetilde{R}) = \\
&\quad = 2\varepsilon\Big(\partial_i\partial_j - 2\sigma_{(i}\partial_{j)} - \delta_{ij}\big(\partial_k\partial_k - \sigma_k\partial_k\big)\Big)\,(\sigma_l\sigma_l - 2\sigma_{ll})\; + O(\varepsilon^2)
\end{aligned} \tag{A.141}$$

Finally, we work out the last two terms in (A.132):

$$\begin{aligned}
2\nabla_{(i}\left(\widetilde{R}\,\partial_{j)}N\right) &= 4\partial_{(i}\left(N\widetilde{R}\,\sigma_{j)}\right) - 4(\sigma_i\delta_{jk} + \sigma_j\delta_{ik} - \sigma_k\delta_{ij})\left(N\widetilde{R}\,\sigma_k\right) \\
&= 4\partial_{(i}\left(N\widetilde{R}\,\sigma_{j)}\right) - 4(2\sigma_i\sigma_j - \delta_{ij}\,\sigma_k\sigma_k)(N\widetilde{R}) \qquad \text{(A.142)} \\
&= 4\varepsilon\partial_{(i}\left[\sigma_{j)}\,(\sigma_k\sigma_k - 2\sigma_{kk})\right] - 4\varepsilon(2\sigma_i\sigma_j - \delta_{ij}\,\sigma_k\sigma_k)\,(\sigma_l\sigma_l - 2\sigma_{ll})
\end{aligned}$$

and

$$\begin{aligned}
-h_{ij}\nabla_k\left(\widetilde{R}\,\partial^k N\right) &= -2\delta_{ij}\,\partial_k\left(N\widetilde{R}\,\sigma_k\right) - 2(n-2)\delta_{ij}\,\sigma_k\sigma_k\,(N\widetilde{R}) \\
&= -2\varepsilon\,\delta_{ij}\,\partial_k\,(\sigma_k\sigma_l\sigma_l - 2\sigma_k\sigma_{ll})\; + O(\varepsilon^2)
\end{aligned} \tag{A.143}$$

When we put all these pieces together, we finally get

$$\begin{aligned}
\frac{1}{\varepsilon}\frac{h_{ik}h_{jl}}{\sqrt{h}}\frac{\delta}{\delta h_{kl}}\int dt\,d^2x\,N\sqrt{h}\,\widetilde{R}^2 &= -4\,(\sigma_{ijkk} - \sigma_k\sigma_{ijk} - \sigma_{ik}\sigma_{jk}) \\
&\quad + 4\delta_{ij}\,(\sigma_{kkll} - \sigma_k\sigma_{kll} - \sigma_{kl}\sigma_{kl})\; + O(\varepsilon)
\end{aligned}$$

VI.1 General Variations

Variation of the Riemann and Ricci tensor

We adopt the usual convention, in which the Riemann tensor is given by

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} + \Gamma^a_{ce}\Gamma^e_{bd} - (c \leftrightarrow d) \tag{A.144}$$

[1] Use the fact that $\Gamma^k_{ij} = \sigma_i\,\delta_{jk} + \sigma_j\,\delta_{ik} - \sigma_k\,\delta_{ij}$.

and its variation is given by

$$
\begin{aligned}
\delta R^a{}_{bcd} &= \partial_c\,\delta\Gamma^a_{bd} + \delta\Gamma^a_{ce}\,\Gamma^e_{bd} + \Gamma^a_{ce}\,\delta\Gamma^e_{bd} \;-\partial_d\,\delta\Gamma^a_{bc} - \delta\Gamma^a_{de}\,\Gamma^e_{bc} - \Gamma^a_{de}\,\delta\Gamma^e_{bc} \\
&= \partial_c\,\delta\Gamma^a_{bd} + \Gamma^a_{ce}\,\delta\Gamma^e_{bd} - \Gamma^e_{bc}\,\delta\Gamma^a_{de} - \Gamma^e_{cd}\,\delta\Gamma^a_{be} \\
&\qquad - \partial_d\,\delta\Gamma^a_{bc} - \Gamma^a_{de}\,\delta\Gamma^e_{bc} + \Gamma^e_{bd}\,\delta\Gamma^a_{ce} + \Gamma^e_{cd}\,\delta\Gamma^a_{be} \\
&= \nabla_c\,\delta\Gamma^a_{bd} - \nabla_d\,\delta\Gamma^a_{bc}
\end{aligned}
$$

In the second line we reshuffled the terms somewhat and we added and subtracted the term $\Gamma^e_{cd}\,\delta\Gamma^a_{be}$, which makes it obvious that we are dealing with covariant derivatives. The variation of the Christoffel symbols is

$$
\begin{aligned}
\delta\Gamma^a_{bd} &= \frac{1}{2}g^{ae}\Big(\partial_b\delta g_{ed} + \partial_d\delta g_{be} - \partial_e\delta g_{bd}\Big) + \frac{1}{2}\delta g^{ae}\Big(\partial_b g_{ed} + \partial_d g_{be} - \partial_e g_{bd}\Big) \\
&= \frac{1}{2}g^{ae}\Big(\partial_b\delta g_{ed} + \partial_d\delta g_{be} - \partial_e\delta g_{bd}\Big) - \frac{1}{2}g^{a(k}g^{l)e}\delta g_{kl}\Big(\partial_b g_{ed} + \partial_d g_{be} - \partial_e g_{bd}\Big) \\
&= \frac{1}{2}g^{ae}\Big(\partial_b\delta g_{ed} + \partial_d\delta g_{be} - \partial_e\delta g_{bd}\Big) - g^{a(k}\Gamma^{l)}_{bd}\,\delta g_{kl} \\
&= \frac{1}{2}g^{ae}\Big(\partial_b\delta g_{ed} + \partial_d\delta g_{be} - \partial_e\delta g_{bd} - 2\Gamma^k_{bd}\,\delta g_{ek}\Big) \\
&= \frac{1}{2}g^{ae}\Big(\nabla_b\delta g_{ed} + \nabla_d\delta g_{be} - \nabla_e\delta g_{bd}\Big)
\end{aligned}
$$

In the last step we used $\nabla_a\delta g_{bc} = \partial_a\delta g_{bc} - \Gamma^k_{ab}\delta g_{kc} - \Gamma^k_{ac}\delta g_{bk}$. This yields the following expression for the variation of the Riemann tensor.

$$
\begin{aligned}
\delta R^a{}_{bcd} \;=\; \frac{1}{2}g^{ae}\Big(&\nabla_c\nabla_b\,\delta g_{de} + \nabla_c\nabla_d\,\delta g_{be} - \nabla_c\nabla_e\delta g_{bd} \\
&- \nabla_d\nabla_b\,\delta g_{ce} - \nabla_d\nabla_c\,\delta g_{be} + \nabla_d\nabla_e\delta g_{bc}\Big)
\end{aligned}
\tag{A.145}
$$

Taking the trace over the a and c indices gives the variation of the Ricci tensor,

$$
\delta R_{bd} \;=\; \frac{1}{2}\Big(\nabla^a\nabla_b\,\delta g_{ad} + \nabla^a\nabla_d\,\delta g_{ab} - \Box\,\delta g_{bd} - \nabla_{(b}\nabla_{d)}\,\delta g\Big) \tag{A.146}
$$

By δg we mean $g^{ab}\,\delta g_{ab}$; we used the fact that the two covariant derivatives acting on a scalar is equal to its symmetric combination, e.g. $\nabla_a\nabla_b\,\delta g = \nabla_{(a}\nabla_{b)}\,\delta g$. A further contraction with the metric gives part of the variation of the Ricci scalar,

$$
g^{ab}\,\delta R_{ab} \;=\; \Big(\nabla^{(a}\nabla^{b)} - g^{ab}\,\Box\Big)\delta g_{ab} \tag{A.147}
$$

One minor thing we used here is that $\nabla^a\nabla^b\,\delta g_{ab} = \nabla^{(a}\nabla^{b)}\,\delta g_{ab}$, since δg_{ab} is symmetric in its indices.

DeWitt metric and more

A useful tensor that one can introduce is the DeWitt metric. It can be defined as

$$G^{abcd} \equiv -\frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, g^{ab})}{\partial g_{cd}} = g^{a(c} g^{d)b} - \frac{1}{2} g^{ab} g^{cd} \tag{A.148}$$

The DeWitt metric has the same symmetry properties as the Riemann tensor. In a similar way, we define the rank-6 tensor

$$\begin{aligned} \Omega^{abcdmn} &\equiv -\frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, G^{abcd})}{\partial g_{mn}} = -\frac{1}{2} G^{abcd} g^{mn} - \frac{\partial G^{abcd}}{\partial g_{ef}} \\ &= g^{m(a} g^{b)(c} g^{d)n} + g^{n(a} g^{b)(c} g^{d)m} + \frac{1}{4} g^{ab} g^{cd} g^{mn} \\ &\quad - \frac{1}{2} g^{c(m} g^{n)d} g^{ab} - \frac{1}{2} g^{m(a} g^{b)n} g^{cd} - \frac{1}{2} g^{a(c} g^{d)b} g^{mn} , \end{aligned} \tag{A.149}$$

where we used

$$-\frac{1}{2} g^{mn} G^{abcd} = \frac{1}{4} g^{ab} g^{cd} g^{mn} - \frac{1}{2} g^{a(c} g^{d)b} g^{mn} , \tag{A.150}$$

$$-\frac{\partial G^{abcd}}{\partial g_{mn}} = g^{m(a} g^{b)(c} g^{d)n} + g^{n(a} g^{b)(c} g^{d)m} - \frac{1}{2} g^{c(m} g^{n)d} g^{ab} - \frac{1}{2} g^{a(m} g^{n)b} g^{cd} , \tag{A.151}$$

The symmetry structure of G^{abcd} and Ω^{abcdmn} can be seen most easily in terms of the Young tableaux:[2]

$$G^{abcd} \sim \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} , \qquad \Omega^{abcdmn} \sim \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline m & n \\ \hline \end{array} . \tag{A.152}$$

The way to read the Young tableau for Ω^{abcdmn} is that e.g. the anti-symmetric combination $[ab]$ vanishes as well does the symmetric combination (acm). We also notice that Ω^{abcdmn} is symmetric under permutations of the pairs of indices ab, cd, and mn. Some useful identities that we use in the main text are:

$$g_{mn} \Omega^{abcdmn} = -\frac{1}{2}(n-4) G^{abcd} , \quad g_{cd} G^{abcd} = -\frac{1}{2}(n-2) g^{ab} . \tag{A.153}$$

[2]See e.g. [60], p. 35.

VII Numerical Lifshitz-to-AdS Flow

In this section we set up the numerical solution that interpolates between AdS in the interior and Lifshitz in the asymptotic region, see e.g. [61] for previous work on flows that involve a Lifshitz scaling region in the massive-vector model. We use the Ansatz consistent with translational invariance and we focus on scalar modes only. The Ansatz is:

$$ds^2 \;=\; -f(r)\,dt^2 + g(r)\left(dx^2+dy^2\right) + dr^2\,, \qquad A \;=\; h(r)\,dt\,. \tag{A.154}$$

In our numerical set-up, we shoot from the AdS solution outward. The AdS background is (absorbing ℓ factors into t, x, y)

$$f(r) \;=\; \mathrm{e}^{2r/\ell}\,, \qquad g(r) \;=\; \mathrm{e}^{2r/\ell}\,, \qquad h(r) \;=\; 0\,. \tag{A.155}$$

We work in coordinates such that the Lifshitz curvature scale is set to one, which fixes the AdS scale to $\ell=\sqrt{3/5}$. In order to ensure that the solution flows to Lifshitz quickly enough we turn on the source for the irrelevant operator discussed in the main text. The linearized mode that plays the role of this source is

$$\delta h(r) \;=\; \varepsilon\,\mathrm{e}^{\nu r/\ell}\,, \qquad \nu \;=\; \frac{1}{2}\left(-1+\sqrt{1+16\ell^2}\right)\,. \tag{A.156}$$

The small parameter ε sets the radial scale at which the irrelevant mode picks up speed. To be more precise, the crossover point is at $r\sim r_*$, where

$$r_* = \frac{\ell}{\nu}\,\log\left(1/\varepsilon\right)\,. \tag{A.157}$$

We will let the numerical integration run from $r=-40$ to $r=140$, and we set the crossover radius to zero, so $\varepsilon=1$. The result of this calculation is plotted in terms of $\alpha = A_a A^a$ in Figure A.1.

VIII Asymptotic $z = 2$ Lifshitz Solution

The asymptotic solution was found in [43]. Because the solution itself does not look too pretty we have kept it out of the main text. The non-zero metric and vector components are

$$\begin{aligned} g_{tt} \;=\; & -\hat{N}^2\,\frac{\Lambda^4\,\mathrm{e}^{4\rho}}{\rho^4}\left(1+\frac{10\log\rho+10-2\lambda}{\rho}+\ldots\right) \\ & + \frac{\mathcal{M}}{4\rho^2}\left(1+\frac{5\log\rho+\frac{41}{6}-2\lambda-\tilde{\lambda}}{\rho}+\ldots\right) \end{aligned} \tag{A.158}$$

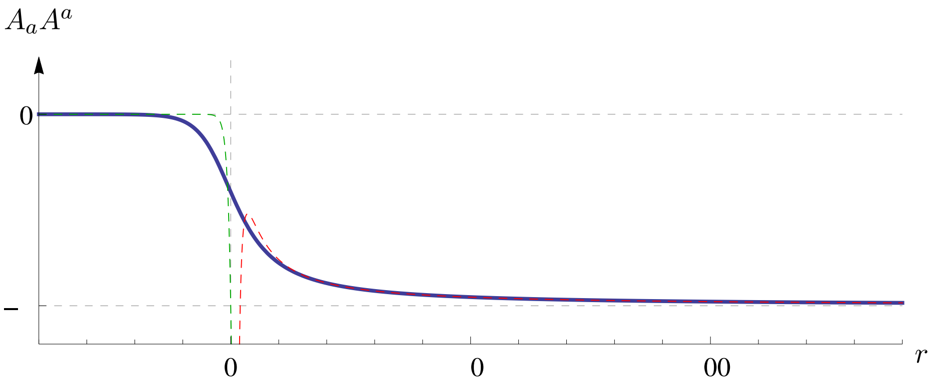


Figure A.1: The vector-squared $A_a A^a$ is evaluated on the numerical solution. On the left (IR) we have AdS, $\alpha = 0$, and the right (UV) we have log-Lifshitz, $\alpha = -1$. The dashed curves are the approximate analytic solutions $\alpha = \mathrm{e}^{\Delta(r-r_*)}$ with $\Delta = 2(\nu-1)/\ell$ (left, green) and $\alpha = -1 + \frac{2}{r-r_*}$ (right, red).

$$g_{ii} = \hat{h}\,\Lambda^2\,\rho^2\,\mathrm{e}^{2\rho}\left(1 + \frac{5\log\rho + 4 - \lambda}{\rho} + \ldots\right) + \frac{\mathcal{M}\,\rho^4\,\mathrm{e}^{-2\rho}}{8}\left(-1 + \frac{10\log\rho + \frac{13}{6} - \lambda + \tilde{\lambda}}{\rho} + \ldots\right) \tag{A.159}$$

$$A_t = \hat{N}\,\frac{\Lambda^2\,\mathrm{e}^{2\rho}}{\rho^2}\left(1 + \frac{5\log\rho + 4 - \lambda}{\rho} + \ldots\right) + \frac{5\mathcal{M}\,\mathrm{e}^{-2\rho}}{8}\left(1 - \frac{\tilde{\lambda} + \lambda - \frac{35}{6}}{\rho} + \ldots\right) \tag{A.160}$$

The ellipses denote terms that are sub-leading in $(\log\rho)/\rho$. The radial coordinate we use here is related to the one in [43] as $\rho = -\log(\Lambda r_{[43]})$. In terms of the radial coordinate r that we use throughout the rest of this note, we have $\rho = r - \log\Lambda$. One can see that the pure Lifshitz geometry is obtained by $\Lambda \to 0$ keeping r fixed (and rescaling t and $\vec{x}$ accordingly). The integration constants $(N, h, \lambda, \mathcal{M}, \tilde{\lambda})$ are related to the ones in [43] as

$$N = \sqrt{f_0}\,, \qquad h = p_0\,, \qquad \lambda = \lambda\,, \qquad \mathcal{M} = \frac{4\beta}{3\sqrt{2}}\,, \qquad \tilde{\lambda} = \frac{\alpha}{\beta}\,, \tag{A.161}$$

where α is an integration constant that appears in [43], it is *not* A^2. A useful contraction that we use in the main text is:

$$\begin{aligned}\alpha \;\equiv\; A^a A_a \;&=\; -1 + \frac{5\log\rho + 2 - \lambda}{\rho^2} + \frac{2}{\rho} + \dots \\ &\quad + \frac{3\mathcal{M}\,\rho^2\,\mathrm{e}^{4\rho}}{2}\left(-1 + \frac{5\log(\rho) + 1 + \tilde{\lambda}}{\rho}\right)\end{aligned} \tag{A.162}$$

A Lifshitz scaling transformation acts on the integration constants in the following way:

$$\left(\Lambda, \lambda, \mathcal{M}, \tilde{\lambda}\right) \;\to\; \left(\mathrm{e}^{\lambda'/2}\Lambda, \lambda + \lambda', \mathrm{e}^{-2\lambda'}\mathcal{M}, \tilde{\lambda} - \lambda'\right) \tag{A.163}$$

So a Lifshitz rescaling can be seen as a redefinition of the scale Λ. In the gauge chosen both here as well as in [43], there is one spurious integration constant, which must be removed from the solution. This spurious integration constant should be removed by fixing a dimensionless combination of integration constants, which is a combination that is invariant under (A.163). This is necessary so as not to break the Lifshitz symmetry explicitly. The Hamilton–Jacobi formalism will tell us uniquely which dimensionless combination we must fix. In [43] the extra integration constant is removed in a way that does not preserve (A.163), thereby breaking Lifshitz symmetry *explicitly*.

We will now show how the spurious integration constant is fixed in the HJ formalism. Consider the canonical momenta

$$\pi^{ab} \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta I_{\text{on-shell}}}{\delta g_{ab}} \qquad\qquad E^a \;=\; \frac{1}{\sqrt{g}}\,\frac{\delta I_{\text{on-shell}}}{\delta A_a} \tag{A.164}$$

which split up into $\pi^{ab} = \pi^{ab}_U + \pi^{ab}_W$ and $E^a = E^a_U + E^a_W$, where

$$\pi^{ab}_W \;=\; \frac{1}{2} g^{ab}\,W(\alpha) - A^a A^b\,W'(\alpha) \qquad\qquad E^a_W \;=\; 2A^a\,W'(\alpha) \tag{A.165}$$

and similarly for $W \to U$. It is useful to define the tensor

$$T^{ab} \;\equiv\; 2\pi^{ab} + E^{(a}A^{b)} \;=\; g^{ab}\,(U + W) \tag{A.166}$$

From these expressions we can isolate $W(\alpha)$ and $W'(\alpha)$ by respectively taking the trace of T^{ab} and by contracting E^a with $A_a/2\alpha$:

$$W(\alpha) \;=\; \frac{1}{3}\left(2g^{ab}\,\partial_r g_{ab} - A^a\,\partial_r A_a\right) - U(\alpha) \tag{A.167}$$

$$W'(\alpha) \;=\; -\frac{1}{2\alpha}\,A^a\,\partial_r A_a - U'(\alpha) \tag{A.168}$$

where we used the canonical relations $\pi_{ab} = -K_{ab} + g_{ab} K$ (with $K_{ab} = \frac{1}{2}\partial_r g_{ab}$) and $E_a = \partial_r A_a$. On the other hand, from the leading-order Hamilton–Jacobi equation for W one finds

$$\sqrt{g}\, W(\alpha) \;=\; \frac{(\alpha+1)^2}{2(5\alpha+9)}\sqrt{g}\, W'(\alpha) + \ldots \tag{A.169}$$

where the ellipses denote $\alpha + 1 \sim 1/\rho$ corrections. Thus, if one computes the renormalized on-shell action on the asymptotic solution using (A.167) and (A.168) one should get the same answer on both sides of the equation. On the left-hand side we get (up to $1/r$ corrections)[1]

$$\sqrt{g}\, W(\alpha) \;=\; -\frac{1}{3} N h\, \mathcal{M}\, (\lambda + \tilde{\lambda} - \frac{17}{6}) \tag{A.170}$$

while on the right-hand side we get

$$\frac{(\alpha+1)^2}{2(5\alpha+9)}\sqrt{g}\, W'(\alpha) \;=\; -N h\, \mathcal{M} \tag{A.171}$$

Comparing these two expressions gives

$$\lambda + \tilde{\lambda} \;=\; -\frac{1}{6}\,, \tag{A.172}$$

Notice that this does not affect the scaling transformation (A.163), because the combination $\lambda + \tilde{\lambda}$ is invariant. Finally, we should mention that one can expand in r rather than $\rho = r - \log \Lambda$. This comes down to performing a rescaling (A.163) with $\lambda' = -2 \log \Lambda$. This is the convention we use in the main text.

[1]Subtracting U in (A.167) ensures that all power-law ($\sim \mathrm{e}^{4\rho}$), logarithmic ($\sim \rho^{\#}$), and double-logarithmic ($\sim \log \rho$) divergences cancel. This can be checked explicitly up to arbitrarily high order in the asymptotic expansion.

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Samenvatting

In de theoretische fysica houdt men zich bezig met de theoretische beschrijving van *fysische systemen.* Het concept van een fysisch systeem is vrij breed en de precieze betekenis hangt af van de vragen die men stelt. Je zou je bijvoorbeeld kunnen afvragen hoe een bepaald stukje metaal elektrische stroom geleidt. Het systeem waar we het dan over hebben is het stukje metaal in een bepaalde meetopstelling in een laboratorium. Een andere vraag die je zou kunnen stellen is: hoe snel dijt het heelal uit? In dat geval is het hele heelal het fysische systeem.

Een theoretische beschrijving van een fysisch systeem noemt men een *theorie.* De term theorie heeft in de natuurkunde een iets preciezere betekenis dan in het dagelijks gebruik. Een natuurkundige theorie is een wiskundig model die men in staat stelt voorspellingen te doen over de uitkomst van experimentele metingen. Het meest bekende voorbeeld van een theorie zijn de wetten van Newton: 1) traagheid van massa, 2) kracht = massa×versnelling ($F = m \cdot a$) en 3) actie = −reactie. Deze theorie stelt ons in staat om bijvoorbeeld de beweging van een biljartbal te voorspellen.

Naast dat een natuurkundige theorie het voorspellen van de uitkomst van een metingen mogelijk maakt, geeft het veelal ook inzicht in het mechanisme dat verantwoordelijk is voor zo'n uitkomst. Dit laatste maakt de theoretische fysica zo interessant. Toch is het goed om wel op te passen voor de valkuil waarin een *waardoor*-vraag wordt verward met een *waarom*-vraag; natuurkundige theorieën zijn enkel in staat *waardoor*-vragen te beantwoorden.

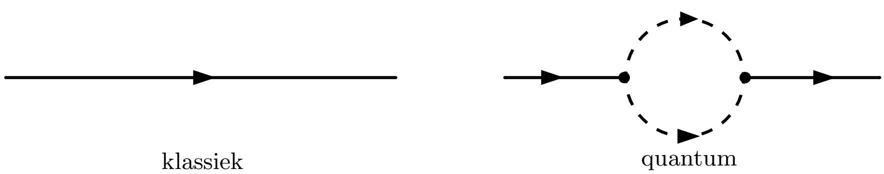
In sommige gevallen bestaan er meerdere theoretische beschrijving van een enkel fysisch systeem. Omdat de natuurkundige inhoud van de theorieën uiteindelijk hetzelfde moet zijn, zijn deze theorieën equivalent. Men spreek dan van een *dualiteit*; de theorieën zijn *duaal* aan elkaar. In dit proefschrift staat een bepaalde dualiteit centraal die afkomstig is uit de *snarentheorie.* Voordat we deze dualiteit bespreken is het handig om een klein beetje achtergrond van de snarentheorie te geven.

Snarentheorie

Snarentheorie is de uitkomst van een lange zoektocht naar een allesomvattende fundamentele theorie. Hoewel de snarentheorie nog lang niet compleet is, zijn er goede aanwijzingen dat men op het goede spoor zit. Bovendien is de snarentheorie op dit moment de enige serieuze kandidaat voor een allesomvattende theorie.

De belangrijkste taak van een allesomvattende theorie is dat het alle bestaande fundamentele theoriëen samenvoegt in één overkoepelende theorie. De fundamentele natuurkrachten zijn de zwaartekracht, de elektro-magnetische kracht en de zwakke en sterke kernkrachten. De elektro-magnetische kracht en de zwakke en sterke kernkrachten zijn al samengevoegd in het zogenaamde *standaard model van de deeltjesfysica*. Het standaard model beschrijft de interactie tussen subatomaire deeltjes met ongekende precisie. Vanwege de minuscule afmeting van zulke deeltjes is het nodig om *quantum-mechanica* te gebruiken in de beschrijving van het standaard model. De enige kracht die niet door het standaard model wordt beschreven is de zwaartekracht. Voor de zwaartekracht hebben we een andere theorie, namelijk de *algemene relativiteitstheorie*. In tegenstelling tot het quantum-mechanische standaard model, waar men hele kleine objecten beschrijft, is de algemene relativiteitstheorie juist van toepassing op hele grote objecten. Het is gebleken dat het ontzettend moeilijk is om de algemene relativiteitstheorie en de quantum-mechanica samen te voegen in één theorie.

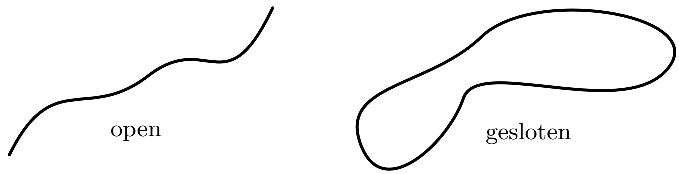
Een quantum-mechanische theorie is van toepassing als we te maken hebben met zeer kleine objecten. Heisenberg's onzekerheidsrelatie staat toe dat energiebehoud voor een zeer korte tijd wordt geschonden. Dit geeft de mogelijkheid tot het ontstaan van een deeltje/anti-deeltje paar op voorwaarde dat deze binnen zeer korte tijd weer op elkaar botsen en daarmee annihileren. Dit heeft grote gevolgen voor de beweging van subatomaire deeltjes. Het onderscheid tussen de paden die een klassiek en een quantum deeltje volgen is schematisch afgebeeld in Afbeelding 1. Hier is te zien dat een quantum deeltje kan opsplitsen in twee virtuele deeltjes die na zeer korte tijd weer



Afbeelding 1: Het pad van een klassiek en een quantum-mechanisch punt-deeltje. De stippellijn geeft het pad van een virtueel deeltje aan. De tijd loopt van links naar rechts.

samenkomen als het originele deeltje. De twee virtuele deeltjes vormen een gesloten lus die bekend staat als een *quantum-lus*. We zien straks dat het pad van een quantum-mechanisch snaartje een soortgelijke lus bevat.

Het belangrijkste inzicht van de snarentheorie is dat elementaire deeltjes kunnen worden geïnterpreteerd als verschillende trillingspatronen van hetzelfde snaartje. Zo worden alle deeltjes voortgebracht door een enkel elementair snaartje. Men maakt het onderscheid tussen open en gesloten snaren, zie Afbeelding 2.



Afbeelding 2: Open en gesloten snaren.

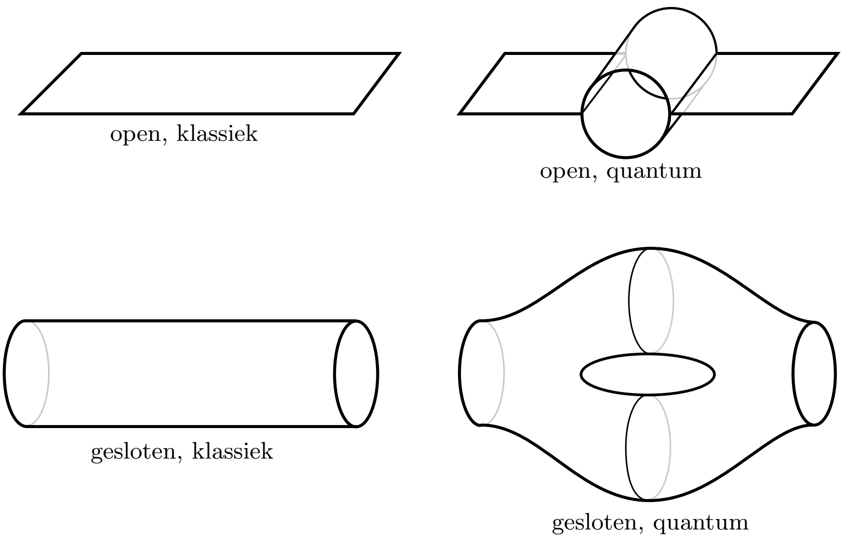
Holografische Dualiteit

Laten we gaan kijken naar de dualiteit waar dit proefschrift op gebaseerd is: *holografische dualiteit* of simpelweg *holografie*. Een holografische dualiteit is een bijzondere equivalentie tussen een zwaartekrachttheorie en een quantum-mechanische veldentheorie. Quantum-veldentheorie begrijpen we conceptueel veel beter dan quantum-zwaartekracht. Holografie geeft ons dus een handzame definitie voor quantum-zwaartekracht. Andersom kan holografie ook gebruikt worden om ingewikkelde problemen in quantum-veldentheorie op te lossen in termen van een simpel zwaartekrachtmodel.

Om een idee te krijgen wat de onderliggende reden is dat er zoiets kan bestaan als een equivalentie tussen een zwaartekrachttheorie en een quantum-mechanische veldentheorie hebben we twee ingrediënten nodig. Het eerste ingrediënt is dat men heeft bevonden dat de gesloten snaren verantwoordelijk zijn voor de zwaartekracht, terwijl de open snaren gerelateerd zijn aan deeltjes die enigszins lijken op de elementaire deeltjes uit het standaard model van de deeltjesfysica.[2] In het kort hebben we dus de volgende relaties:

$$\begin{aligned}\text{open snaren} \quad &\sim \quad \text{quantum-veldentheorie}\\ \text{gesloten snaren} \quad &\sim \quad \text{zwaartekracht}\end{aligned}$$

Voor het tweede ingrediënt bekijken wat voor pad een snaar ‘bewandelt’ in de tijd, zie Afbeelding 3. Het pad van een klassieke open snaar vormt een plat oppervlak en het pad een klassieke gesloten snaar vormt een buis-vormig oppervlak. Zoals we eerder zagen hebben we in de quantum-mechanica te maken met virtuele deeltjes. Op dezelfde manier kunnen we ook kijken naar het pad van een quantum-mechanisch snaartje. Zo’n snaartje kan opsplitsen (net als een quantum punt-deeltje) in twee virtuele snaartjes die na zeer korte tijd weer samenkomen.

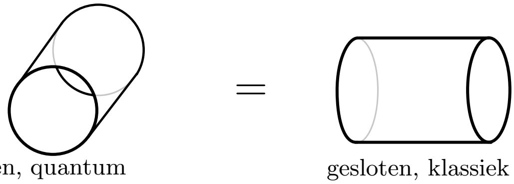


Afbeelding 3: Snaartjes die door de tijd bewegen volgen een twee-dimensionaal traject. De tijd loopt van links naar rechts.

Laten we nu even onze aandacht richten op de *quantum-lus* in het pad van het quantum-mechanische open snaartje. We zien dat deze buisvormig is. We zagen eerder al dat het klassieke pad van een gesloten snaartje ook buisvormig is. In Afbeelding 4 zetten we de twee paden naast elkaar.

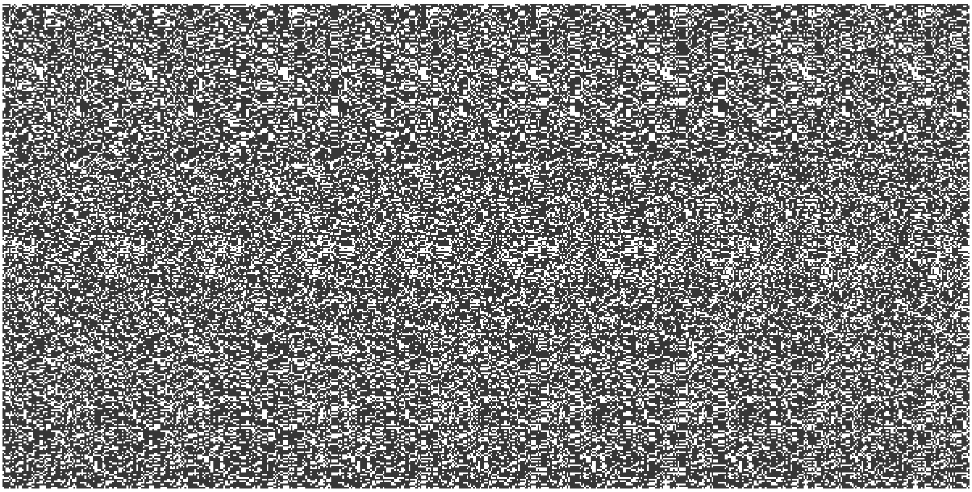
We zagen zojuist dat een quantum-mechanische open snaar kan worden geïnterpreteerd als een klassieke gesloten snaar. Uiteindelijk (na enig rekenwerk) volgt hieruit dat bepaalde quantum-veldentheorieëen (open snaren) kunnen worden vertaald naar een klassieke zwaartekrachttheorie (gesloten snaren).

[2] Het is helaas niet mogelijk om in een paar zinnen (of pagina’s) te beredeneren waarom dit het geval is. (Mijn excuses voor het weglaten van deze logische stap!)



Afbeelding 4: Het ‘traject’ van een quantum-mechanisch open snaartje is gelijk aan dat van een klassiek gesloten snaartje!

Waarom *holografie*? Een hologram is een drie-dimensionaal beeld dat wordt gegenereerd door een twee-dimensionaal vlak, zie bijvoorbeeld Afbeelding 5. De reden dat de equivalentie tussen deze quantum-veldentheoriëen en zwaartekrachttheoriëen ‘holografisch’ wordt genoemd heeft te maken met het feit dat de zwaartekrachttheorie is gedefiniëerd in één extra ruimtedimensie vergeleken met de quantum-veldentheorie.[3] Zwaartekracht kan dus worden gezien als een holografisch beeld van een lager-dimensionale quantum-veldentheorie.



Afbeelding 5: Een hologram. Als je lang genoeg naar dit plaatje staart kom er een drie-dimensionaal beeld tevoorschijn.

In dit proefschrift. De enige systemen waar tot nu toe een holografische dualiteit voor bestond zijn *relativistisch*, wat betekent dat de deeltjes in zo’n theorie heel snel bewegen. Er zijn echter veel systemen waarin de deeltjes niet heel snel bewegen. Zulke systemen worden dan *niet-relativistisch* genoemd.

[3]Zo kan bijvoorbeeld een drie-dimensionale zwaartekrachttheorie worden vertaald naar een twee-dimensionale quantum-veldentheorie.

In dit proefschrift is een versie van holografie ontwikkeld voor systemen die niet-relativistisch zijn.

Niet-Relativistische Systemen

Om een beetje een gevoel te krijgen wat het verschil is tussen relaticistische en niet-relativistische systemen. De kinetische energie $E_{\text{kin.}}$ van een relativistisch deeltje is evenredig met de kinetische impuls $p = mv$:[4]

$$E_{\text{kin.}} = pc\,. \qquad\qquad \text{(relativistisch / snel)}$$

De evenredigheidsconstante is de lichtsnelheid c. Aan de andere kant weten dat voor een niet-relativistisch deeltje de kinetische energie niet evenredig is met p, maar met het kwadraat van de impuls, p^2:

$$E_{\text{kin.}} = \frac{1}{2}mv^2 = \frac{p^2}{2m}\,. \qquad\qquad \text{(niet-relativistisch / langzaam)}$$

Hier is de evenredigheidsconstante $1/2m$. Deze laatste formule geldt voor deeltjes die met een relatief lage snelheid bewegen.

Zoals eerder vermeld, waren er tot nu toe alleen holografische dualiteiten bekend voor relativistische systemen, oftewel van het type $E = pc$. Het doel van dit promotieonderzoek is het opzetten van een versie van holografie voor niet-relativistische systemen, oftewel van het bekendere type $E = \frac{1}{2}mv^2$.

Een korte afleiding van de bovenstaande formules. Tot slot doen we voor de geïnteresseerde lezer een kleine berekening om de bovenstaande formules ($E_{\text{kin.}} = pc$ en $E_{\text{kin.}} = \frac{1}{2}mv^2$) af te leiden. De bekendste formule uit de relativiteitstheorie is $E = mc^2$, oftewel de energie E van een object is evenredig met zijn massa m (de evenredigheidsconstante is het kwadraat van de lichtsnelheid c). Om iets preciezer te zijn is het alleen de *rust*-energie die gelijk is aan mc^2. De totale energie E is de som van de rust-energie E_{rust} en de kinetische energie $E_{\text{kin.}}$, dus $E = E_{\text{rust}} + E_{\text{kin}}$, en de volledige formule luidt:

$$E = \gamma\, mc^2\,, \qquad\qquad \gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}\,.$$

De factor γ staat bekend als de Lorentz-factor en $p = mv$ is de kinetische impuls. Voor een relativistisch object is de kinetische impuls p veel groter dan

[4] m is de massa van het deeltje en v is zijn snelheid.

mc (de rustmassa keer de lichtsnelheid). Dit betekent $(p/mc)^2$ veel groter is dan 1, waardoor de Lorentz-factor reduceert tot:

$$\gamma \;=\; \sqrt{1+\Big(\frac{p}{mc}\Big)^2} \;\approx\; \sqrt{\Big(\frac{p}{mc}\Big)^2} \;=\; \frac{p}{mc}\,. \qquad \text{(relativistisch)}$$

Aan de andere kant, een niet-relativistisch object heeft een kinetische impuls p die juist veel kleiner is dan mc, dus $(p/mc)^2$ is dan veel kleiner dan 1. De Lorentz-factor reduceert in dat geval tot:

$$\gamma \;=\; \sqrt{1+\Big(\frac{p}{mc}\Big)^2} \;\approx\; 1+\frac{1}{2}\Big(\frac{p}{mc}\Big)^2. \qquad \text{(niet-relativistisch)}$$

Hier gebruikte we de standaard formule $\sqrt{1+x} \approx 1+\frac{1}{2}x$, die geldt als x een klein getal is.[5] Hieruit volgt dan dat de totale energie wordt gegeven door:

$$E \;=\; \gamma\, mc^2 \;\approx\; \begin{cases} pc & \text{(relativistisch)} \\ \\ mc^2+\frac{p^2}{2m} & \text{(niet-relativistisch)} \end{cases} \qquad \text{(A.173)}$$

In het niet-relativistische geval krijgen we het bekende resultaat, waar de rust-energie en kinetische energie worden gegeven door $E_{\text{rust}} = mc^2$ en $E_{\text{kin.}} = \frac{1}{2}mv^2$ (omdat $p = mv$). In het relativistische geval zien we dat de rust-energie helemaal geen bijdrage geeft aan de totale energie, zodat $E \approx E_{\text{kin.}} = pc$.

[5]Deze standaard formule is een voorbeeld van een *Taylor-benadering*.