Doctoral Thesis

Conformal aspects of quantum field theory and their applications to realistic physics

Esben Mølgaard

There is a single light of science,
and to brighten it anywhere is to brighten it everywhere.

- Isaac Asimov, 1920-1992

Conformal aspects of quantum field theory and their applications to realistic physics

Author	Esben Mølgaard
Advisor	Francesco Sannino
Co-Advisor	Claudio Pica

Dissertation for the degree of Doctor of Philosophy

CP^3-Origins

Centre for Cosmology and Particle Physics Phenomenology,
Department of Physics, Chemistry and Pharmacy

Submitted to the University of Southern Denmark
June 14, 2014

Acknowledgements

During my studies for the degree of *philosophiae doctor*, I have had the great pleasure of studying and researching in a stimulating and educational environment. The CP^3-Origins center is filled to the brim with incredibly talented, knowledgeable and helpful individuals to all of whom I owe great thanks. This particularly concerns my advisors Francesco Sannino and Claudio Pica, my collaborators Oleg Antipin, Stefano Di Chiara, Marc Gillioz, Jens Krog and Matin Mojaza, and my office mates Jakob Jark Jørgensen, Matin Mojaza, Ole Svendsen, Jussi Virkajärvi and Ulrik Søndergaard.

For six months, my work was relocated to the C.N. Yang Institute for Theoretical Physics at the University of Stony Brook where I had a pleasant and rewarding stay. I would like to thank my host, Robert Shrock, for his fruitful collaboration, and many interesting and illuminating conversations on physics and beyond. I also thank the staff, postdocs and students at the YITP who made me feel welcome so far from home.

Preface and publications

The focus of this dissertation is the study of four–dimensional, non-supersymmetric quantum field theory, a subject I have previously contributed to in the following scientific articles.

[1] O. Antipin, S. Di Chiara, M. Mojaza, E. Mølgaard, and F. Sannino, "A Perturbative Realization of Miransky Scaling", *Phys.Rev.*, vol. D86, p. 085 009, 2012. DOI: 10.1103/PhysRevD.86.085009. arXiv:1205.6157 [hep-ph].

[2] O. Antipin, M. Gillioz, E. Mølgaard, and F. Sannino, "The a theorem for Gauge-Yukawa theories beyond Banks-Zaks", *Phys.Rev.*, vol. D87, p. 125 017, 2013. DOI: 10.1103/PhysRevD.87.125017. arXiv:1303.1525 [hep-th].

[3] O. Antipin, J. Krog, E. Mølgaard, and F. Sannino, "Standard Model-like corrections to Dilatonic Dynamics", *JHEP*, vol. 1309, p. 122, 2013. DOI: 10.1007/JHEP09(2013)122. arXiv:1303.7213 [hep-ph].

[4] O. Antipin, M. Gillioz, J. Krog, E. Mølgaard, and F. Sannino, "Standard Model Vacuum Stability and Weyl Consistency Conditions", *JHEP*, vol. 1308, p. 034, 2013. DOI: 10.1007/JHEP08(2013)034. arXiv:1306.3234 [hep-ph].

[5] E. Mølgaard and R. Shrock, "Renormalization-Group Flows and Fixed Points in Yukawa Theories", *Phys.Rev.*, vol. D89, p. 105 007, 2014. DOI: 10.1103/PhysRevD.89.105007. arXiv:1403.3058 [hep-th].

[6] E. Mølgaard, "Decrypting gauge-Yukawa cookbooks", arXiv:1404.5550 [hep-th].

The main body of the text here is dedicated to reproducing the discussions and results of [2, 4–6], since these are the ones that contribute the most to our understanding of general quantum field theory.

Contents

English abstract

In this thesis, we will present the results of our studies into the nature of four–dimensional, non-supersymmetric quantum field theory, particularly that of renormalization group flows. This is of course a vast subject and we could not hope to cover it all. We begin our presentation by discussing two rather different models that we will return to several times during the course of the thesis. The first is the standard model of particle physics, where we put a great emphasis on the Higgs field and its particle, the second is an SU(N_c) toy model which has been found to have many interesting properties.

We then proceed to introduce the concept of renormalization, and dedicate an extended section to a new method that we have developed for the calculation of (especially) beta functions. We also take time to discuss fixed points in gauge theories, and how the presence or absence of these is determined by the parameters of the theory in question.

Next, we study the conjectured a theorem, i.e., the proposal that there exists a function a of the couplings in a four–dimensional quantum field theory which is monotonic along any renormalization group flow. We test the weak form of this conjecture, which states that a is larger at UV fixed points than at IR fixed points, and find that this holds in the toy model even when none of the fixed points is Gaussian.

From our investigations into the a theorem, we discovered that to preserve the symmetries of a gauge-Yukawa theory, it is necessary to run the gauge couplings with a beta function that is calculated to one higher loop order than the Yukawa beta functions, which must in turn be computed to one higher loop order than the quartic beta functions. We use this very important result to refine computations done previously by others regarding the stability of the standard model vacuum.

Finally, we consider the renormalization group flows of a model inspired by the standard model lepton sector when the beta functions are computed to different loop orders. We use this to give quantitative statements regarding the trustworthiness of perturbation theory.

Dansk sammenfatning

I denne afhandling præsenterer vi resultaterne af vore studier udi firedimensionel, ikke-supersymmetrisk kvantefeltteoris natur, især med hensyn til renormeringsgruppestrømme. Dette er naturligvis et omfattende emne, og vi har intet håb om at dække det hele her. Vi begynder vores præsentation med en diskussion af to ganske forskellige modeller som vi vil vende tilbage til flere gange i løbet af afhandlingen. Den første er partikelfysikkens standardmodel, hvor vi lægger stor vægt på Higgsfeltet og dets tilhørende partikel, den anden er en SU(N_c) *toy model* som har mange interessante egenskaber.

Vi introducerer derefter emnet renormering, og dedikerer et længere afsnit til nye metoder vi har udviklet med henblik på at udregne (især) betafunktioner. Vi afsætter også plads til at diskutere fikspunkter i gaugeteorier, og hvordan deres tilstedeværelse eller fravær bliver afgjort af teoriens parametre.

Det næste vi studerer er det foreslåede a-teorem, formodningen om at der eksisterer en funktion a af koblingerne i en firedimensionel kvantefeltteori der er monotonisk langs enhver renormeringsgruppestrøm. Vi tester den svage udgave af formodningen, der siger at a er større i UV-fikspunkter end i IR-fikspunkter, og vi finder at dette er sandt for den specifikke teori vi undersøger, selv når ingen af fikspunkterne er gaussiske.

Fra vores undersøgelse af a-teoremet finder vi at for at bevare symmetrierne i en gauge-Yukawa-teori er det nødvendigt at lade gaugekoblingen løbe med en betafunktion der er udregnet til én højere løkkeorden end Yukawa-betafunktionen, der igen skal udregnes til én højere løkkeorden end den kvartiske betafunktion. Vi bruger dette vigtige resultat til at forfine andres beregninger af vakuums stabilitet i standardmodellen.

Endeligt betragter vi renormeringsgruppestrømmene i en model der er inspireret af standardmodellens leptonsektor når betafunktionernerne er udregnet til forskellige løkkeordener. Vi bruger dette til at fremsætte kvantitative udsagn om perturbationsteoriens troværdighed.

Introduction

The border between mathematics and physics has always been blurry. Sir Isaac Newton famously made great advances in the understanding of calculus while working on classical mechanics [7] and optics [8]. In later centuries, Carl Friedrich Gauss made essential contributions to electrostatics while working in pure mathematics. More recently, Edward Witten was awarded the Fields Medal for, among other things, applying ideas from physics to mathematics [9, 10].

It is tempting, for a physicist, to see mathematics purely, or primarily, as a tool for understanding the phenomena that are observed in experiments, but history has shown us that the study of mathematics and physics have a greater synergy than that. It has even been suggested [11] that mathematics is a fundamental part of the world, rather than merely a language by which it can be described.

Despite the fruitful symbiosis between mathematics and physics, the increasing specialization of the fields that we have seen during, in particular, the twentieth century means that very few people are capable of bridging the gap between the subjects. However, this specialization has also opened up new areas which neither attempts to describe Nature, nor can be said to be the purview of mathematics. This disputed territory is exactly where the focus of this thesis will be.

The greatest triumph of particle physics in the previous hundred years is without a doubt the formulation and vindication of what we now term the standard model. The sheer magnitude of the theory's success was, of course, not immediately obvious, but it has served as a benchmark for the majority of new physics speculations for decades. However, there are several things that the standard model does not explain, and which makes finding a theory of Nature that goes beyond it desirable. We will not even try to make an exhaustive list, but a few of these are the hierarchy and cosmological constant problems, charge quantization, and the nature of dark matter.

Much theoretical effort has been put into developing beyond the standard model physics, but with the Large Hadron Collider (and the Super Proton Synchrotron, the TeVatron and the Large Electron-Positron Collider before it) having discovered no direct evidence for any such physics, we have not found extensions of the standard model to be a fruitful avenue of research.

Instead, we will take a step back and consider the framework the standard model is built within; that of non-supersymmetric renormalizable quantum field theory in four dimensions. Despite the obvious importance of understanding the framework that is our basis for our best predictions about Nature, there are still a large number of open questions about it. Of these we can only hope to answer a small fraction in a thesis such as this, but we will try to address the central issues concerning the energy dependence of the theory, or its *renormalization group flow*.

In particular, we will discuss and investigate the a theorem conjecture which in its strong form asserts that renormalization group flows are irreversible in coupling space, and in its weak form merely that there exists a function a which is larger at high energy fixed points than at low energy ones.

Following the insights gained in this analysis of pure field theory, we will detail their consequences for the proper running of coupling constants in a theory with many different interactions. Since the standard model is such a theory, we find that our study of field theory has implications for Nature, and we will show how they impact the standard model predictions for the stability of the vacuum.

Last, we will perform a detailed study of how different loop orders impact the renormalization group flows in a model inspired by the standard model lepton sector.

CHAPTER 1

GAUGE-YUKAWA THEORY

Quantum field theory is a very general framework that can accommodate a great variety of different particles and interactions, subject to an even greater variety of symmetries and constraints. Quite naturally, this means that quantum field theory is almost never studied in its full generality.

For physical applications, it is almost always assumed that the theory under consideration is invariant under translation and Lorentz transformations (together forming the Poincaré group). In this thesis we will consider breaking these symmetries as a mathematical trick in Section 3.2, but this should not be taken literally. Some work has been done to study Lorentz symmetry breaking as an approximation to quantum gravity (see e.g. [12] and references therein), but we will not consider that in this work.

Poincaré symmetry is generally considered to be implicit when working with quantum field theory, and the first really interesting class of symmetries we will consider is therefore gauge symmetries. Gauge symmetry is an invariance under a *local* Lie group transformation. This symmetry was present, but not considered at the time, in both Maxwell's theory of electromagnetism, and Hilbert's formulation of General relativity [13]. The thought of the day was that this was a curious accidental symmetry, useful only for simplifying calculations. However, if one takes the opposite view and considers gauge symmetry to be a fundamental property of Nature, it becomes a very powerful principle. Imposing invariance under U(1) gauge transformations on a quantum theory of electrons automatically implies the existence of massless vector particles that can be identified with photons, and extends the theory to the full quantum electrodynamics [14]. It should be pointed out that even in a gauge invariant theory, there are some quantities that depend on the choice of gauge. However, since the gauge transformations do not correspond to any physical change, observables must be explicitly gauge invariant.

One very common class of quantum field theories are the supersymmetric ones [15, 16], where the Poincaré symmetry is extended to also include anti-commuting generators, and each bosonic degree of freedom is matched by a corresponding fermionic one. These theories have many desirable qualities and are particularly well-behaved. For decades, they have also been considered as very likely candidates for beyond the standard model physics phenomenology [17, 18]. However, despite several generations of particle accelerators and experimental physicists looking for them, no definite signs of supersymmetry have been seen in Nature.

The ideas of renormalization, in particular the Wilsonian picture (see Section 2.1) make it common to deal only with interactions where the coupling constant has a non-negative mass dimension. In four dimensions, this limits us to gauge interactions, fermion and scalar masses, scalar cubic and quartic couplings, and fermion-fermion-scalar Yukawa interactions. This thesis will precisely deal with non-supersymmetric examples of this class of theories. Often, we will limit ourselves further by only considering clasically conformal theories, where the only the interactions with dimensionless coupling constants are considered, that is the gauge, Yukawa and quartic interactions. Some of our discussion will hold in general, but a very large part of it will deal with specific models, which we will introduce here before going into their more intricate details in later chapters.

We will primarily consider two models; the standard model of particle physics and a toy model first introduced in [19] with fermions transforming under the fundamental and adjoint representations of an $SU(N_c)$ gauge symmetry, and a large scalar sector that is a singlet under the gauge symmetry. In Chapter 5, we further study two variations of a different model inspired by the standard model lepton sector.

1.1 The standard model

The standard model of particle physics is perhaps the most frustratingly successful theory in the history of physics. Since its introduction in the 1960's [20–26], it has only undergone very few significant revisions. The near non-observation of flavor-changing neutral currents demanded a mechanism for the suppression of these, which was provided in the

GIM mechanism [27], and suitably extended upon the observation of CP-violation in the quark sector [28, 29]. The discovery of neutrino masses through neutrino oscillation is likewise in contradiction with the standard model as originally proposed. However, since it is a relatively simple matter to introduce a neutrino mass in the standard model through the addition of right-handed sterile neutrinos, this is not considered a problem for the theory. The reason right handed neutrinos are not generally considered to be part of the standard model is that it is not clear from experiments how they would enter.

At high energies, the standard model describes massless fields interacting with each other via strong and electroweak gauge interactions and Yukawa interactions. At low energy, the electroweak symmetry is spontaneously broken via the Higgs mechanism[1], which has the interesting property that it gives masses to the fundamental scalars, fermions and the three weak gauge bosons, leaving only the gluons and photon massless. In this thesis, we will mostly be interested in the standard model in the high energy regime.

Mathematically, we describe the standard model in terms of its Lagrangian and the transformation properties of its fundamental fields (see Table 1.1)

$$\mathcal{L}_{sm} = \mathcal{L}_{kin} + \mathcal{L}_{mass} + \mathcal{L}_{yuk} + \mathcal{L}_{quart} \tag{1.1}$$
$$\mathcal{L}_{mass} = -\mu^2 H^\dagger H \tag{1.2}$$
$$\mathcal{L}_{yuk} = -Y^E \bar{L} H E - Y^D \bar{Q} H D - Y^U \bar{Q} \tilde{H} U + h.c. \tag{1.3}$$
$$\mathcal{L}_{quart} = -\hat{\lambda}(H^\dagger H)^2, \tag{1.4}$$

where $\mathcal{L}_{kin}$ contains the canonically normalized kinetic terms, and Y^E, Y^D and Y^U are the Yukawa matrices of the electron, down, and up-type fields respectively. In flavor space, these are 3×3 matrices. Without loss of generality, it is possible to choose a basis where Y^E and Y^U are diagonal, and $Y^D = V\tilde{Y}^D$, where $\tilde{Y}^D$ is diagonal and V is the unitary CKM-matrix.

We will discuss some particulars of the standard model in Chapter 4, in particular how the Higgs mechanism gives mass to the gauge bosons and the eponymous Higgs boson. We do this as a preliminary exercise before studying the stability of its vacuum at very high scales.

[1]Many people have contributed to the development of this mechanism, and it is something of a historical accident that Peter Higgs gets the vast majority of the credit. Of course, the more fitting name of Anderson-Brout-Englert-Guralnik-Hagen-Kibble-Higgs-Weinberg-'t Hooft mechanism [25, 30–34] is rather cumbersome, and would quite likely still leave some invaluable contributors feeling slighted.

Fields	$[SU(3)_c]$	$[SU(2)_W]$	$[U(1)_Y]$	Chirality
L	1	$\square$	$-\frac{1}{2}$	L
E	1	1	-1	R
Q	$\square$	$\square$	$\frac{1}{6}$	L
D	$\square$	1	$-\frac{1}{3}$	R
U	$\square$	1	$\frac{2}{3}$	R
H	1	$\square$	$\frac{1}{2}$	

Table 1.1: Transformation properties of the standard model fields under the three constituent gauge groups. As mentioned in the text, it would be a simple matter to add a right-handed neutrino field transforming in the (1,1,0) representation of the standard model gauge group. However, since these have not been directly observed, we shall omit them. For the non-Abelian groups, we are using the Young tableau notation where $\square$ refers to the fundamental representation of the group, 1 to the singlet representation and $\overline{\square}$ would refer to the conjugate representation. For the U(1) group, we are using the charge convention that $Q = T_3 + Y$, where Q is electric charge, T_3 is the z component of the weak isospin, and Y is the weak hypercharge.

1.1.1 The role of the Higgs

Without a doubt, the most famous particle of recent years has been the Higgs particle of the standard model, so it behooves us to spend time describing its properties and importance. While its importance has been blown somewhat out of proportions by the mainstream media, there is no denying that the Higgs particle does an impressive job in the standard model, fulfilling multiple roles in a way the other particles do not.

Masses

The first and most celebrated role of the Higgs is to bridge the gap between the high energy regime where Nature exhibits the $SU(2)_W \otimes U(1)_Y$ electroweak gauge symmetry, and the low energy one where only the $U(1)_{em}$ of electromagnetism remains and where the W and Z bosons are massive. Massive gauge bosons are in contradiction with the principle of gauge invariance, but is accommodated by the Higgs mechanism, where the scalar field acquires a non-zero vacuum expectation value (vev). This leads to an effective theory where the gauge symmetry is broken and the gauge bosons are massive.

To properly appreciate the second role of the Higgs, giving masses to the standard model fermions, we will start by asking a question that to the untrained eye seems the province of fruitless philosophizing; "what would happen if left and right were switched around?" Or in a more formal language: "what would be the result of a parity transformation of the entire universe?"

In classical physics, and every day experience, the answer is simple; nothing at all. This holds true equally well in the quantum mechanics of the early 20th century, and even in quantum electrodynamics (QED), and quantum chromodynamics (QCD). However, it is not true for all physical processes. An example is the Z^0-decay polarization asymmetry,

$$A_{LR}^{f} = \frac{\Gamma(Z^0 \to f_L \bar{f}_R) - \Gamma(Z^0 \to f_R \bar{f}_L)}{\Gamma(Z^0 \to f_L \bar{f}_R) + \Gamma(Z^0 \to f_R \bar{f}_L)} \tag{1.5}$$

which is a non-zero number that changes sign under a parity transformation. The reason for this asymmetry is that the weak force (unlike the electromagnetic and strong forces) distinguishes between particles of left- and right-handed chirality.

This chiral nature of the weak force puts very stringent bounds on what kinds of mass terms the particles of the standard model can form. In particular, invariance under the weak gauge symmetry implies that none of the standard model fermions can have a fundamental mass. However, they *can* form Yukawa interactions with the Higgs field, and once the Higgs field acquires its vev, these interaction terms give effective masses to the standard model fermions. If we were to include a right-handed neutrino field N, this would be able to form a Majorana mass term, $\frac{1}{2}m_N NN$ in addition to a Yukawa coupling with the Higgs and Lepton fields, $Y^N \bar{L}\tilde{H}N$.

It is, however, very important to stress that this is not actually the source of most mass in the universe. The masses of the up and down quarks are (according to the Particle Data Group) 2.3 MeV and 4.8 MeV respectively [35], giving a naïve estimate for the proton mass of 9.4 MeV, two orders of magnitude less than its physical mass of 938.3 MeV [35]. The remainder of the mass comes from non-perturbative QCD effects. The inner workings of the proton and other hadrons is an immensely fascinating and important subject that we sadly do not have space to cover in this work.

Unitarity

Although the ability to provide masses for the standard model particles is the Higgs boson's most celebrated role, it is by no means its only one.

It has even been argued that it is not its most important one [36]. We will not here make any such arguments, but merely point out that the Higgs has at least one other very important role to play in the standard model.

Quantum field theory, and indeed all quantum theory, is by its nature probabilistic. When two particles interact, it is impossible to say with certainty what the result of the interaction will be, it is merely possible to calculate the probabilities of each possible outcome. One particular interaction where the Higgs boson plays a crucial role is the longitudinal scattering of W bosons (see Figure 1.1).

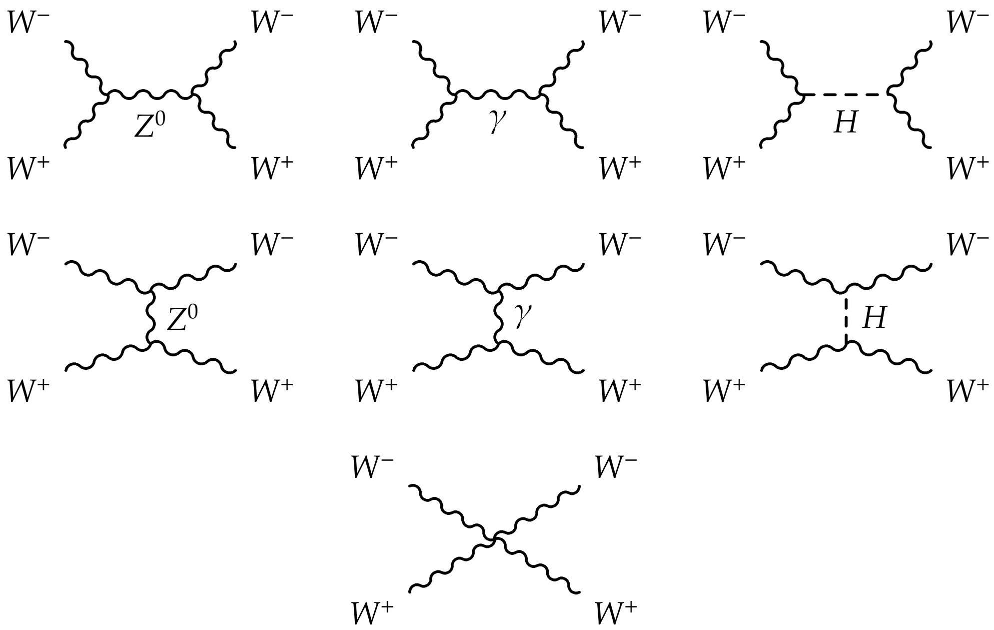


Figure 1.1: Diagrams contributing to the longitudinal scattering of W bosons.

Let us, for a moment, assume that Nature does not contain a Higgs boson, or that its coupling to the W bosons is not what the standard model predicts. Then we would find that the perturbative prediction of the probability of the longitudinal scattering of W bosons occurring would diverge as the center of mass energy in the interaction increased [37, 38]. However, it can be shown that non-perturbative effects can still render the total probability finite [39]. If there is a standard model Higgs, its contribution to the amplitude exactly cancels the divergence and renders the the perturbative prediction finite [37];

$$T(W_L^+W_L^- \to W_L^+W_L^-) = -\sqrt{2}G_F M_H^2 \left(\frac{s}{s-M_H^2} + \frac{t}{t-M_H^2} \right) . \tag{1.6}$$

Here G_F is the Fermi constant, s and t (and u which does not enter here) are the Mandelstam variables, and M_H the mass of the Higgs boson. This amplitude (and the others like it, see [37] for details) is well behaved for a Higgs boson mass

$$M_H \leq M_c = \sqrt{8\pi\sqrt{2}/(3G_F)} \approx 1\,\text{TeV} . \tag{1.7}$$

Renormalizability

It is a well known and much lamented fact that a very large fraction of the integrals central to quantum field theory calculations are divergent. Fortunately, the method of renormalization (which we will expound greatly on in Section 2.1) has been developed to make these divergences tractable for a very large class of theories. In four dimensions, this includes all theories with gauge, Yukawa and scalar quartic self-couplings, as well as fermion and scalar mass terms and scalar trilinear couplings.

However, the fact that non-Abelian gauge fields, and in particular massive ones, are renormalizable was not known, and was in fact suspected not to be the case, before the seminal work of 't Hooft and Veltman [40–42]. In [41], it is shown that for massive gauge fields to be renormalizable, it is absolutely crucial that the masses are acquired via a Higgs mechanism, and are not intrinsic properties of the gauge fields.

1.2 SU(N_c) toy model

The goals of this thesis is to understand Nature in terms of the standard model, and to explore aspects of quantum field theory both as they relate to it, and on their own terms. To accomplish the second goal, it is useful to have a toy model that can be used as a playground for quantum field theory, both to investigate phenomena that do not occur in the standard model, and to elucidate various aspects of ones that do.

1.2.1 Previous work

The toy model we consider was first introduced in [19] to study the dilaton, the Goldstone boson associated with the breaking of scale invariance.

The motivation for this study was twofold; several theories of beyond the standard model (BSM) physics predict that the Higgs may in fact be such a dilaton, and secondly the spontaneous breaking of conformal symmetry in quantum field theory is an interesting phenomenon on its own account.

The analysis of the breaking of conformal symmetry in this theory was further expanded in [1], where it was discovered that the two–loop beta functions of the theory have a fixed point that can be perturbatively turned off at a non-zero value of the couplings, leading to the celebrated phenomenon of walking coupling constants, and Miransky scaling [43–45]. While these features were extremely appealing, we later discovered that the two–loop beta function analysis is invalid because it violates the Weyl consistency conditions [4, 46, 47] (see also Section 3.2).

In [3], we brought the toy model closer to the standard model by considering the impact on the dilaton of the theory coming from an electroweak sector and a top quark-like Yukawa interaction. We found that for a large range of the model's parameters, the salient features were preserved, adding credence to the idea that the Higgs boson could be a dilaton of some underlying theory.

A prominent feature of the model is that it possesses several fixed points, and for certain values of the parameters, these include non-Banks-Zaks type fixed points. Since we are interested in the nature of conformality in gauge-Yukawa theories, we studied the conjectured a theorem[2] within this theory [2]. We showed that the $\tilde{a}$ function of Jack and Osborn [46] does indeed satisfy the (weak version of the) a theorem in the present theory. We will return to this analysis in great detail in Chapter 3.

Lagrangian and fields

We now turn our attention to the particulars of the model under investigation. It has an SU(N_c) gauge symmetry under which there are fermions transforming in the fundamental, conjugate fundamental and adjoint representations, and it also features a global SU$(N_f)_L \times$ SU$(N_f)_R$ symmetry (see Table 1.2 for the details). The fundamental and conjugate fundamental Weyl fermions can be thought of as forming a Dirac vector fermion in analogy with QCD, but we find the present description more convenient. In the original analysis, only one adjoint fermion was considered, but we will also investigate cases where it is illuminating to regard ℓ, the number

[2]The c theorem is a fully proved theorem in two dimensional conformal field theory. The a theorem is a *conjectured* equivalent in four dimensions. We apologize for this awkward naming convention and will return to the subject in Section 3.1.

of these, to be different from 1. If $\ell > 1$, this will lead the introduction of to an additional global SU(ℓ) symmetry. The Lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda \not{D}\bar{\lambda} + i\bar{q}\not{D}q + i\bar{\tilde{q}}\not{D}\tilde{q} + \partial_\mu H^\dagger \partial^\mu H + (y_H \tilde{q} H q + h.c.) - u_1(\mathrm{Tr}[H^\dagger H])^2 - u_2\,\mathrm{Tr}[(H^\dagger H)^2]\,, \quad (1.8)$$

where λ is the adjoint fermion, q and $\tilde{q}$ are (anti)fundamental fermions and H is a singlet scalar field.

Fields	$[SU(N_c)]$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_V$	$U(1)_{AF}$	Chirality
λ	Adj	1	1	0	1	L
q	$\square$	$\overline{\square}$	1	$\frac{N_f-N_c}{N_c}$	$-\frac{N_c}{N_f}$	L
$\tilde{q}$	$\overline{\square}$	1	$\square$	$-\frac{N_f-N_c}{N_c}$	$-\frac{N_c}{N_f}$	L
H	1	$\square$	$\overline{\square}$	0	$\frac{2N_c}{N_f}$	
G_μ	Adj	1	1	0	0	

Table 1.2: The field content of the model and the related symmetries.

Superficially, this model bears very little relation to the standard model, however, it can be regarded in several fashions that make it relevant to realistic physics. The fundamental and anti-fundamental fermions coupled together by a scalar field bears a striking resemblance to the quark sector of the standard model, and the scalar quartic coupling is identical to that of the SU(N_f)$\otimes$SU(N_f) linear sigma model, which captures many features of mesonic physics. Another angle to bring the model into contact with Nature is to consider it an approximation of an effective Technicolor theory, with the scalar sector imitating the technimesons which give rise to the masses of the W and Z bosons, and possibly also dark matter (see e.g. [48]). The variation of the model without adjoint fermions is also being studied as an approximation of the bosonized version of the gauged Nambu–Jona-Lasinio model [49–51]. This version has also been explored in the context of asymptotic safety, and found to have this desirable property for all the coupling constants [52].

CHAPTER 2

The renormalization group

As mentioned in Section 1.1.1, divergent integrals are near-omnipresent in quantum field theory calculations. In this chapter, we will concern ourselves with ultraviolet divergences arising from integrations over the internal momenta in Feynman diagrams with loops. Taming these divergences will lead us to the renormalization group and a deeper understanding of quantum field theory.

2.1 Renormalization

Renormalization is the name given to a number of related methods for bringing ultraviolet divergences in quantum field theory under control. We will begin our study of the subject with a definition that appears overly naïve, but allows for a very useful classification of all possible quantum field theories.

2.1.1 Superficial divergence

The structure of the loop diagrams is that for each loop, there will be d integrations over momenta, and for each propagator, there will be one (for fermions) or two (for bosons) powers of momentum in the denominator of the integrand. This leads to the *superficial degree of divergence*

$$\begin{aligned} D &\equiv (\text{power of } k \text{ in numerator}) - (\text{power of } k \text{ in the denominator}) \quad (2.1)\\ &= dL - P_f - 2P_b, \quad (2.2) \end{aligned}$$

where d is the number of space-time dimensions, L is the number of loops, P_f the number of fermion propagators, and P_b the number of boson propagators. A priori, we would expect an integral to be polynomially divergent in the ultraviolet if $D > 0$, logarithmically divergent if $D = 0$, and convergent if $D < 0$. Things are, unsurprisingly, more complicated than that, but we will not go into a detailed discussion of it here.

In theories with only a few types of interactions, it is convenient to rewrite Equation 2.2 in terms of the number of external legs and the number of vertices. For QED, this can be done and the superficial degree of divergence is [14]

$$D = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-1}{2}N_e, \tag{2.3}$$

where V is the number of vertices, N_γ the number of external photon legs, and N_e the number of external electron legs. We immediately notice that in four dimensions, this has the very peculiar result that D is independent of V. This implies that D does not increase with loop order. Had this not been the case, QED in four dimensions would not have been a renormalizable theory. It is tempting at this stage to speculate on whether this is a deep truth, implying that four space-time dimensions is somehow privileged, or essential for life as we know it; or if it is a coincidence, and would not have seemed significant at all if mathematics and theoretical physics had developed along different lines. There may be a more down-to-earth answer to this question, however, as we will discuss in Section 2.1.3.

While the superficial degree of divergence is not the whole story, it does allow us to divide all possible theories into three distinct kinds based on whether D decreases or increases with the numbers of loops [14]

Super-renormalizable theory:	Only a finite number of Feynman diagrams superficially diverge.
Renormalizable theory:	Only a finite number of amplitudes superficially diverge; however, divergences occur at all orders in perturbation theory.
Non-renormalizable theory:	All amplitudes are divergent at a sufficiently high order in perturbation theory.

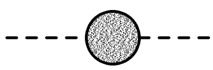
In renormalizable and super-renormalizable theories, the divergences can be cured as described in the next section, in non-renormalizable, they cannot. However, non-renormalizable theories can still be useful as effective theories up to a certain cutoff scale. Historically, many non-renormalizable theories have been exceedingly successful in describing Nature, perhaps most notably the Fermi theory of beta decay [53, 54].

2.1.2 Renormalized perturbation theory

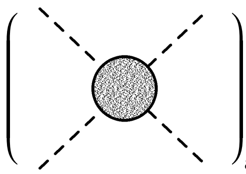
The way to deal with divergences in renormalizable and super-renormalizable theories without introducing a cutoff involves renaming the fields and couplings in the Lagrangian as the *bare* fields and couplings, and then decomposing these in terms of well-behaved *renormalized* fields, physical *renormalized* couplings, and *counterterms*. In the case of ϕ^4 theory, this can be done:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 && (2.4)\\
&= \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}Zm_0^2\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4 && (2.5)\\
&= \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\
&\quad + \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4\,. && (2.6)
\end{aligned}$$

The physical couplings and counterterms are fixed order by order in terms of specific Feynman diagrams evaluated at specific momentum scales,



$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2) \qquad (2.7)$$



amputated

$$= -i\lambda \qquad \text{at } s = 4m^2, t = u = 0\,. \qquad (2.8)$$

Once these *renormalization conditions* have been imposed, all other calculations within the theory can be performed and give finite predictions.

With the renormalization conditions in place, we find that when we calculate loop diagrams rather than simple tree-level ones, the amplitudes depend on the external momenta. However, since we identify the coupling constants with diagrams at specific external momenta, this *running* of the effective coupling is not visible at tree level. This means that depending on the energy at which a diagram is evaluated, it is possible that the next order in perturbation theory gives a very large contribution. Since the implicit assumption of perturbation theory is that each new order only gives a small correction to the previous, this behavior is undesirable.

In order to overcome this issue, we may adopt Wilson's approach to renormalization theory [55], where we integrate out the high momentum modes, and find that this yields perturbations of the bare coupling constants, and the introduction of all higher dimensional (non-renormalizable) operators. The introduction of these new operators seems worrying, but they turn out to be well under control [14], and in fact they can be exploited to understand the full non-perturbative behaviour of the theory using the *functional renormalization group* [56–61], but that is sadly beyond the scope of this thesis.

2.1.3 Renormalization group flows

To demonstrate Wilson's method, let us consider the generating functional for ϕ^4 theory with a cutoff Λ,

$$Z = \int [\mathcal{D}\varphi]_\Lambda \exp\left(-\int \mathrm{d}^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4\right]\right) \tag{2.9}$$

$$[\mathcal{D}\varphi]_\Lambda = \prod_{|k|<\Lambda} \mathrm{d}\varphi(k), \tag{2.10}$$

where k is the Euclidean momentum. We can now integrate out the high momentum modes with $b\Lambda < |k| < \Lambda$ for some $0 < b < 1$. We do this by redefining

$$\varphi(k) = \phi(k) + \hat{\phi}(k), \tag{2.11}$$

where

$$\phi(k) = \begin{cases} \varphi(k) & \text{for } k < b\Lambda \\ 0 & \text{for } b\Lambda \leq k < \Lambda \end{cases} \tag{2.12}$$

$$\hat{\phi}(k) = \begin{cases} 0 & \text{for } k < b\Lambda \\ \varphi(k) & \text{for } b\Lambda \leq k < \Lambda \end{cases} \tag{2.13}$$

With these definitions, we can rewrite the generating functional as

$$\begin{aligned} Z = \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi} \exp\Bigg(- \int \mathrm{d}^d x \Big[\frac{1}{2}(\partial_\mu \phi + \partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 (\phi + \hat{\phi})^2 \\ + \frac{\lambda}{4!}(\phi + \hat{\phi})^4 \Big] \Bigg) \end{aligned} \quad (2.14)$$

$$\begin{aligned} = \int \mathcal{D}\phi \mathrm{e}^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi} \exp\Bigg(- \int \mathrm{d}^d x \Big[\frac{1}{2}(\partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 \\ + \lambda \left(\frac{1}{6}\phi^3 \hat{\phi} + \frac{1}{4}\phi^2 \hat{\phi}^2 + \frac{1}{6}\phi \hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4 \right) \Big] \Bigg), \end{aligned} \quad (2.15)$$

where the quadratic terms $\phi\hat{\phi}$ and $\partial_\mu \phi \partial_\mu \hat{\phi}$ vanish since the Fourier modes of different wavenumbers are orthogonal.

The integral over $\hat{\phi}$ is in general cumbersome, but it can be performed, leading to an effective Lagrangian

$$\begin{aligned} \int \mathrm{d}^d x \mathcal{L}_{eff} = \int \mathrm{d}^d x \Big[\frac{1}{2}(1 + \Delta Z)(\partial_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 \\ + \frac{1}{4!}(\lambda + \Delta\lambda)\phi^4 + \Delta C (\partial_\mu \phi)^4 + \Delta D \phi^6 + \dots \Big] \end{aligned} \quad (2.16)$$

where the "..." covers higher order terms, and the corrections ΔZ, Δm^2, $\Delta\lambda$, ΔC, ΔD, etc. all arise from the integral over $\hat{\phi}$. These values will, of course, depend on our choice of b. We can incorporate this dependency into the effective Lagrangian by rescaling the momenta and distances,

$$k' = k/b \qquad x' = xb, \quad (2.17)$$

which yields

$$\begin{aligned} \int \mathrm{d}^d x \mathcal{L}_{eff} = \int \mathrm{d}^d x' b^{-d} \Big[\frac{1}{2}(1 + \Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 \\ + \frac{1}{4!}(\lambda + \Delta\lambda)\phi^4 + \Delta C b^4 (\partial'_\mu \phi)^4 + \Delta D \phi^6 + \dots \Big]. \end{aligned} \quad (2.18)$$

We can now absorb b into the field and coupling constants,

$$\phi' = [b^{2-d}(1+\Delta Z)]^{1/2}\phi \quad (2.19)$$
$$m'^2 = (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2} \quad (2.20)$$
$$\lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4} \quad (2.21)$$
$$C' = (C+\Delta C)(1+\Delta Z)^{-2}b^{d} \quad (2.22)$$
$$D = (D+\Delta D)(1+\Delta Z)^{-3}b^{-2d-6}, \quad (2.23)$$

and write the effective Lagrangian

$$\int \mathrm{d}^d x \mathcal{L}_{eff} = \int \mathrm{d}^d x' \Big[\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + C'(\partial'_\mu\phi')^4 + D'\phi'^6 + \ldots\Big], \quad (2.24)$$

with similar substitutions for the higher order terms included in the ellipsis.

The combined process of integrating out the high momentum modes and rescaling the distances and momenta has thus been rewritten as a transformation of the Lagrangian. This process can be performed successively, and if we choose each b to be infinitesimally close to 1, this is a continuous process. These continuously generated transformations are, for historical reasons, known as *the renormalization group*[1] (RG), and one can think of them generating a flow in the theory space of all possible Lagrangians with all possible operators composed of the constituent fields.

Let us take a moment to consider the simplest place in this theory space; where $m^2 = \lambda = C = D = \ldots = 0$. Here the theory is governed by the free-field Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2, \quad (2.25)$$

this is the *Gaussian fixed point*[2], and here the renormalization group transformation leaves the Lagrangian unchanged. If we make a small perturbation around this point, we may ignore $\Delta Z, \Delta m^2$, etc. in the transformations and Equations (2.20)-(2.23) get the very simple form

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = Cb^d, \quad D' = Db^{2d-6}, \quad \ldots \quad (2.26)$$

[1]Since the process of integrating out the degrees of freedom is not invertible, it is not formally a group.

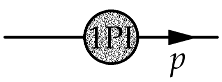
[2]It is called a Gaussian fixed point because at this point, the probability distribution for the field is Gaussian, making the theory solvable.

Since $b < 1$, it is very easy to see that the operators that are multiplied by positive powers of b will become increasingly less important as we perform the successive transformations, and vice versa for those with negative powers of b. For this reason, we name the former *irrelevant* operators and the latter *relevant*. Those that are multiplied by b^0 are termed *marginal*, and to determine their behavior under the renormalization group, we must consider loop corrections, which is exactly the focus of the next section.

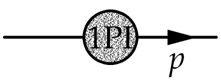
Before we consider the running of marginal operators, let us take a moment to address the more philosophical question posed in Section 2.1.1, regarding the fact that QED is a renormalizable theory only in four space-time dimensions. In the viewpoint we adopted there, it was considered essential for a theory to be fundamental, that it had a well-defineds limit as $\Lambda \to \infty$. However, in this Wilsonian setup, we consider the presence of a cutoff, where the theory stops being sensible, to be a fundamental property of any theory, and we have indicated that any theory defined at high energy can be expressed in terms of relevant and marginal operators at low energy. Thus, it is entirely possible that the fundamental theory behind QED is not at all renormalizable, or well behaved, but due to the renormalization group flow from the high to the low scale, the *effective theory* that we experience is *guaranteed* to be a renormalizable one.

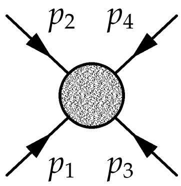
2.1.4 The beta function

While the Wilsonian picture is extremely useful in giving us a better understanding of the nature of renormalization, and of quantum field theories in general, it can be rather cumbersome to work with. It is therefore desirable to combine the ideas contained within with those of renormalized perturbation theory described above. We do this by imposing modified renormalization conditions, not at the scale of the physical mass, but rather at an arbitrary *renormalization scale* μ



$= 0 \qquad \text{at } p^2 = -\mu^2;$ (2.27)

$\dfrac{\mathrm{d}}{\mathrm{d}p^2}\Bigg($  $\Bigg) = 0 \qquad \text{at } p^2 = -\mu^2;$ (2.28)



$$= -i\lambda \quad \text{at } (p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -\mu^2 \,. \tag{2.29}$$

Note here that the first two renormalization conditions are for one particle irreducible (1PI) diagrams only.

Since the renormalization scale is arbitrary, it cannot enter into any physical quantities. This implies that the only impact it can have on a Green's function is a rescaling of the field. Under a change of renormalization scale, the coupling constant and field will naturally also change

$$\mu \to \mu + \delta\mu \tag{2.30}$$
$$\lambda \to \lambda + \delta\lambda \tag{2.31}$$
$$\phi \to (1+\delta\eta)\phi, \tag{2.32}$$

leading to a change in the n-point Green's function

$$G^{(n)}(x_1,\ldots,x_n) = \langle\Omega|T\phi_1(x_1)\cdots\phi_n(x_n)|\Omega\rangle_{\text{connected}} \tag{2.33}$$
$$\to (1+n\delta\eta)G^{(n)}(x_1,\ldots,x_n). \tag{2.34}$$

We may consider $G^{(n)}$ to be a function of μ and λ, in which case we can express this as

$$\mathrm{d}G^{(n)} = \frac{\partial G^{(n)}}{\partial\mu}\delta\mu + \frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda = n\delta\eta G^{(n)}, \tag{2.35}$$

and define

$$\beta \equiv \frac{\mu}{\delta\mu}\delta\lambda \qquad\qquad \gamma \equiv -\frac{\mu}{\delta\mu}\delta\eta. \tag{2.36}$$

Using these definitions, we can write down the Callan-Symanzik equation [62, 63]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right]G^{(n)}(\{x_i\};M,\lambda) = 0, \tag{2.37}$$

where we have made explicit the fact that β and γ do not depend on n or the x_i.

The *anomalous dimension* γ is of great importance in characterizing conformal field theories, and serves an essential role in the class of BSM theories known as Walking Technicolor, however we will not address it further in this thesis. For some of our work which does involve the anomalous dimension in a crucial way, see [64].

Instead, we will turn our full attention to the beta function. From its definition (2.36), we see that it describes the change in λ needed to preserve the value of the Greens function as μ changes. Since the Green's function is independent of the renormalization scale up to a rescaling of the fields, this implies that it must depend only on the bare coupling λ_0 and the cutoff of the theory Λ. Furthermore in a renormalizable theory, the cutoff can be taken to infinity, and its dependence removed entirely. We may thus write

$$\beta(\lambda) = \mu \left.\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda\right|_{\lambda_0} . \tag{2.38}$$

This equation tells us exactly how the beta function governs the running of the coupling constant as a function of the renormalization scale, and we may use it to make predictions about a theory at wildly different scales. In Section 4.4 of this thesis, we will use it to make predictions about the standard model at energies up to the Planck scale, 16 orders of magnitude above the reach of current particle physics experiments.

Because we often consider renormalization scales of such different magnitudes, it is convenient to define the *renormalization group time* $t = \ln(\mu/\mu_0)$ where μ_0 is some arbitrary reference scale. In terms of the renormalization group time, the beta function can be expressed as

$$\beta = \frac{\mathrm{d}\lambda}{\mathrm{d}t} , \tag{2.39}$$

where we see explicitly that the choice of reference scale makes no difference.

2.2 Calculating beta functions

The values of the beta functions and anomalous dimensions of a gauge-Yukawa theory can be computed in terms of Feynman diagrams. This yields a perturbative expansion as a polynomial in the relevant coupling constants where each loop order contributes a higher order term and a loop factor of $1/(4\pi)^2$. It can be illuminating to express the beta function as

$$\beta_i = \sum_{\ell=1}^{\infty} \frac{b_i^{(\ell)}}{(4\pi)^{2\ell}} , \tag{2.40}$$

where the index i distinguishes between the different couplings of the gauge-Yukawa theory. At each new loop order, the gauge and Yukawa couplings contribute with twice the power of the quartic coupling, this mismatch makes it convenient to rescale the coupling constants, and we can also take the opportunity to absorb the factors of $1/(4\pi)$. For general gauge, Yukawa and quartic couplings g, y and λ, the rescaling would be

$$a_g = \frac{g^2}{(4\pi)^2} \qquad a_y = \frac{y^2}{(4\pi)^2} \qquad a_\lambda = \frac{\lambda}{(4\pi)^2} \,. \tag{2.41}$$

In terms of these, the beta functions can be expanded as

$$\beta_{a_i} = \sum_{\ell=1}^{\infty} b_{a_i}^{(\ell)} \,. \tag{2.42}$$

When it is relevant to specify to which loop order a beta function is calculated, we will write $\beta_{a_i, n\ell}$ where n is the number of loops in question.

For the anomalous dimension of the scalar and fermion wave functions, the actual calculation is done by extracting the coefficient of the divergent part of the scalar and fermion propagator respectively [65]. The beta functions are extracted in a similar fashion, but from the vector propagator, Yukawa interaction and quartic self-interactions respectively. In the latter two cases, there are also contributions from the lower loop level fermion and scalar wave function anomalous dimensions that must be taken into account [65–67].

With this well-defined procedure in mind, it is a straightforward, if computationally involved, matter to calculate the quantities of interest in any well-defined gauge-Yukawa theory. However, it is also possible to do these calculations in a completely general setting without specifying the theory. This yields general formulas where one simply needs to insert the details of one's favorite theory and they then produce expressions for beta functions, anomalous dimensions, and more. The computation of such general formulas has been ongoing for a long time, and is still proceeding (see for example [65–70]).

Before proceeding further, it behooves us to remark on the gauge and scheme dependence of beta functions, anomalous dimensions and related quantities. The renormalization scheme refers to the way in which counterterms are introduced to cancel the divergences of quantum field theory, as detailed in the previous section (see for instance Equations (2.27)-(2.29)). The formulas we have used are all computed in the $\overline{\text{MS}}$ scheme, but the new notation introduced here is independent of the choice of scheme. Since none of the quantities under consideration are physical observables, we cannot be sure of their gauge invariance, and indeed we find [65] that the anomalous dimensions of the scalar and fermion fields are explicitly dependent on a gauge parameter. The beta functions on the other hand are gauge independent in the $\overline{\text{MS}}$ scheme, but not necessarily in others.

2.2.1 General Lagrangian

The general Lagrangian that is assumed inmost general analyses [65–70] has the form

$$\mathcal{L} = \mathcal{L}_{kin} - \frac{1}{2}\left(y_{JK;A}\Psi_J\Psi_K\Phi_A + h.c.\right) - \frac{1}{4!}\lambda_{ABCD}\Phi_A\Phi_B\Phi_C\Phi_D, \tag{2.43}$$

where Ψ_J is an all-encompassing fermion field transforming under some (in general) reducible representation of the gauge group. Without loss of generality, we may assume that it has a definite chirality, and that all of its component fields are Weyl spinors. Similarly, Φ_A is an all-encompassing scalar field transforming under a reducible representation of the gauge group. Likewise without loss of generality, we can take these to be real. Here and in the following, we are summing over repeated indices.

Scalar or fermion mass terms, and scalar cubic terms can be included by introducing a non-propagating dummy real scalar field $\Phi_{\hat{D}}$ which is not summed over [71]. The relevant operators can then be expressed as

$$\begin{aligned}\mathcal{L}_1 = -\frac{1}{2}\left(y_{JK;\hat{D}}\Psi_J\Psi_K\Phi_{\hat{D}} + h.c.\right) - \frac{1}{4!}\lambda_{AB\hat{D}\hat{D}}\Phi_A\Phi_B\Phi_{\hat{D}}\Phi_{\hat{D}} \\ - \frac{1}{4!}\lambda_{ABC\hat{D}}\Phi_A\Phi_B\Phi_C\Phi_{\hat{D}},\end{aligned} \tag{2.44}$$

where $y_{JK;\hat{D}}\Phi_{\hat{D}} = (m_f)_{JK}$, $\lambda_{AB\hat{D}\hat{D}}\Phi_{\hat{D}}\Phi_{\hat{D}} = 2m^2_{AB}$ and $\lambda_{ABC\hat{D}}\Phi_{\hat{D}} = h_{ABC}$. The beta functions for the relevant operators can then be obtained from those of the corresponding marginal operators.

Any gauge-Yukawa theory can, in principle be put on the form (2.43). How to actually accomplish this in a general way is not at all obvious, and it is the focus of the present discussion.

To accomplish this task, we introduce an object which organizes the specific fields (such as $\bar{L}$, E and H in the standard model lepton sector, see Section 1.1 for details) within the abstract fields Ψ_J and Φ_A, and keeps track of overall indices as well as specific field indices. We call this the *structure delta*

$$\Delta^S_{J;\{j\}} \tag{2.45}$$

where S takes values from the names of the fields in the theory ($\bar{L}$, E and H); J is the overall fermion index from (2.43); and $\{j\}$ covers the gauge and flavor indices of the field S.

This new symbol obeys the following summation rule

$$\Delta^{S}_{J;\{j\}}\Delta^{S'}_{J;\{j'\}} = \delta^{SS'} \prod_{\{j\},\{j'\}} \delta_{jj'}, \tag{2.46}$$

and has the fundamental property that $\Psi_J \Delta^{S}_{J;\{j\}} = S_{\{j\}}$ (or $\Phi_A \Delta^{S}_{A;\{a\}} = S_{\{a\}}$ if S is a scalar field). In the specific case of the standard model lepton sector, this is realized in the following manner

$$\Psi_J \Delta_J^{\bar{L};g_2,f_L} = \bar{L}^{g_2,f_L}, \quad \Psi_J \Delta^{E}_{J;f_E} = E_{f_E} \quad \text{and} \quad \Phi_A \Delta^{H}_{A;g_2,c} = H_{g_2,c}. \tag{2.47}$$

2.2.2 Yukawa interactions

We first illustrate how to express general Yukawa interactions by considering the leptonic part of the standard model Yukawa interaction (see Equation (1.3)) with all flavor and gauge indices written explicitly. It is

$$\mathcal{L}_{Yuk,Lep} = (Y^E)^{f_E}_{f_L} \bar{L}^{g_2,f_L} H_{g_2} E_{f_E} + h.c.\,, \tag{2.48}$$

where g_2 is the SU(2) gauge index, and f_L and f_E are the flavor indices of the lepton doublet and electron-like singlet respectively. There is a subtlety here because the standard model Higgs is a complex scalar, and thus have twice as many degrees of freedom as its gauge index would suggest. We take this into account by adding a complex index c and introducing the symbol

$$o^c = \begin{cases} 1 & \text{for } c = 1 \\ i & \text{for } c = 2 \end{cases} \tag{2.49}$$

Real scalars are also normalized differently from complex ones, so we have

$$\mathcal{L}_{Yuk,Lep} = \frac{1}{\sqrt{2}} (Y^E)^{f_E}_{f_L} o^c \bar{L}^{g_2,f_L} H_{g_2,c} E_{f_E} + h.c.\,. \tag{2.50}$$

Using the structure delta (2.45), we can now write the lepton Yukawa Lagrangian (2.50) as

$$\mathcal{L}_{Yuk,Lep} = \frac{1}{2\sqrt{2}}(Y^E)^{f_E}_{f_L} o^c \left(\Psi_J \Delta_J^{\bar{L};g_2,f_L} \Psi_K \Delta^E_{K;f_E} + \Psi_K \Delta_K^{\bar{L};g_2,f_L} \Psi_J \Delta^E_{J;f_E}\right) \Phi_A \Delta^H_{A;g_2,c} + h.c. \tag{2.51}$$

$$= \frac{1}{2\sqrt{2}}(Y^E)^{f_E}_{f_L} o^c \left(\Delta_J^{\bar{L};g_2,f_L} \Delta^E_{K;f_E} + \Delta_K^{\bar{L};g_2,f_L} \Delta^E_{J;f_E}\right) \Delta^H_{A;g_2,c} \Psi_J \Psi_K \Phi_A + h.c.\,, \tag{2.52}$$

and simply read off

$$y^{(Lep)}_{JK;A} = \frac{1}{\sqrt{2}}(Y^E)^{f_E}_{f_L} o^c \left(\Delta_J^{\bar{L};g_2,f_L} \Delta^E_{K;f_E} + \Delta_K^{\bar{L};g_2,f_L} \Delta^E_{J;f_E}\right) \Delta^H_{A;g_2,c}. \tag{2.53}$$

An equivalent procedure can be used to find $y^{(Up)}_{JK;A}$ and $y^{(Down)}_{JK;A}$, and then the Yukawa coupling for the entire standard model is just the sum,

$$y_{JK;A} = y^{(Lep)}_{JK;A} + y^{(Up)}_{JK;A} + y^{(Down)}_{JK;A}\,. \tag{2.54}$$

2.2.3 Quartic interactions

We can find the quartic coupling λ_{ABCD} in an analogous way. We start out by taking the quartic part of the standard model Lagrangian (1.4),

$$\mathcal{L}_{quart} = \hat{\lambda}\left(H^\dagger H\right)^2. \tag{2.55}$$

H can be written as a complex vector,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{1,1} + iH_{1,2} \\ H_{2,1} + iH_{2,2} \end{pmatrix}, \tag{2.56}$$

thus

$$H^\dagger H = \frac{1}{2}(H_{1,1} - iH_{1,2})(H_{1,1} + iH_{1,2}) + \frac{1}{2}(H_{2,1} - iH_{2,2})(H_{2,1} + iH_{2,2}) \tag{2.57}$$

$$= \frac{1}{2}(H_{1,1}^2 + H_{1,2}^2 + H_{2,1}^2 + H_{2,2}^2), \tag{2.58}$$

which we can write as

$$H^\dagger H = \frac{1}{2} H_{g_2,c} H_{g_2,c}\,, \tag{2.59}$$

where a sum over g_2 and c is again assumed. Thus, in terms of the structure deltas we have

$$\mathcal{L}_{quart,H} = \frac{\hat{\lambda}}{4} H_{g_2,c} H_{g_2,c} H_{g_2',c'} H_{g_2',c'} \tag{2.60}$$

$$= \frac{\hat{\lambda}}{4}\frac{1}{4!} \sum_{perms} \Delta^H_{A;g_2,c} \Delta^H_{B;g_2,c} \Delta^H_{C;g_2',c'} \Delta^H_{D;g_2',c'} \Phi_A \Phi_B \Phi_C \Phi_D, \tag{2.61}$$

where $\sum_{perms}$ is a sum over all possible permutation of A, B, C, D, and the factor of $\frac{1}{4!}$ enters to compensate for the number of perturbations.

We can now read off

$$\lambda_{ABCD} = \frac{\hat{\lambda}}{4} \sum_{perms} \Delta^H_{A;g_2,c} \Delta^H_{B;g_2,c} \Delta^H_{C;g_2',c'} \Delta^H_{D;g_2',c'} \tag{2.62}$$

2.2.4 Generators

Following the notation of [69], we refer to the generator of the (reducible) scalar representation as S^α_{AB}, where A and B are general scalar indices, and α is the group index running from 1 to $d(G)$. In the case of a semi-simple group, this is generalized to $S^{t;\alpha}_{AB}$ where t labels the simple subgroup. Equivalently, the generator of the spinor representation is given as R^α_{JK}, and $R^{t;\alpha}_{JK}$ if the group is semi-simple.

The normalization of S and R is such that

$$\mathrm{Tr}(S^\alpha S^\beta) = \delta^{\alpha\beta} T(S), \qquad S^\alpha_{AC} S^\alpha_{CB} = C_2(S)_{AB}, \tag{2.63}$$

$$\mathrm{Tr}(R^\alpha R^\beta) = \delta^{\alpha\beta} T(R), \qquad R^\alpha_{JL} R^\alpha_{LK} = C_2(R)_{JK}, \tag{2.64}$$

$$f^{\alpha\gamma\delta} f^{\beta\gamma\delta} = \delta^{\alpha\beta} C_2(G), \qquad \delta^{\alpha\alpha} = d(G), \tag{2.65}$$

where $T(\cdot)$ is the Dynkin index of the representation, $C_2(\cdot)$ is the quadratic Casimir of the representation, with $C_2(G)$ in particular being the quadratic Casimir of the adjoint, and $d(G)$ is the dimension of the group. As is well known, the quadratic Casimir of an irreducible representation is just a number times the relevant identity matrix. In a reducible representation, this is slightly more complicated, here the quadratic Casimir is a block-diagonal matrix with the quadratic Casimir of an irreducible component in each block.

We can find S^α_{AB} by summing over the generators and structure deltas of each scalar species. For U(1) this is particularly simple as the generator is just the charge of the field. Since we are decomposing the complex scalars into their real components, each set of indices corresponds to a single real scalar field. Furthermore, since the generators must be Hermitian, this implies that they must be imaginary and anti-symmetric in A and B. The expression for the U(1) generator is thus

$$S^{\alpha}_{AB} = \sum_{S} -iQ_S \epsilon^{c}{}_{c'} \delta^{f_S}_{f'_S} \Delta^{S}_{A;c,f_S} \Delta^{S;c',f'_S}_{B}, \tag{2.66}$$

where Q_S is the charge of the scalar species S. The sign is conventional, and has been chosen such that the charges of the standard model fields are as listed in Table 1.1.

For a non-Abelian group, things are more complicated as the fields now transform in non-trivial representations of the group. Since each scalar field under consideration is still real, we must again ensure that each term in the generator is imaginary and anti-symmetric under an exchange of *all* of the indices associated with the field.

We first observe that since the generators of the irreducible representations, $T^{\alpha;a}{}_{b}$, only carry gauge indices, the flavor indices are contracted through a delta function. Secondly, the generators are either real and symmetric, or imaginary and anti-symmetric. To ensure that the final expression is imaginary and anti-symmetric, we must multiply by $i\epsilon^{c}{}_{c'}$ in the former case, in complete analogy with the U(1) case above, but in the latter we must instead multiply by $\delta^{c}_{c'}$. We show that the following construction is imaginary and anti-symmetric in either case:

$$T^{\alpha;a}{}_{b}(\delta^{c}_{c'} + i\epsilon^{c}{}_{c'}) - T^{\alpha;a}_{b}(\delta^{c}_{c'} - i\epsilon^{c}{}_{c'}) \tag{2.67}$$

$$= \quad T^{\alpha;a}{}_{b}\delta^{c}_{c'} + iT^{\alpha;a}{}_{b}\epsilon^{c}{}_{c'} - T^{\alpha;a}_{b}\delta^{c}_{c'} + iT^{\alpha;a}_{b}\epsilon^{c}{}_{c'} \tag{2.68}$$

$$\underset{\text{symmetric}}{=} \quad 2iT^{\alpha;a}{}_{b}\epsilon^{c}{}_{c'} \tag{2.69}$$

$$\underset{\text{antisymmetric}}{=} \quad 2T^{\alpha;a}{}_{b}\delta^{c}_{c'}\ , \tag{2.70}$$

where $T^{\alpha;a}_{b}$ is the transpose of $T^{\alpha;a}{}_{b}$, and the former equality holds if $T^{\alpha;a}_{b} = T^{\alpha;a}{}_{b}$ and the latter if $T^{\alpha;a}_{b} = -T^{\alpha;a}{}_{b}$.

The final expression for the generator of the scalar representation is thus:

$$S^{\alpha}_{AB} = \sum_{S} -\frac{1}{2}\left(T^{\alpha;a}_{S}{}_{b}(\delta^{c}_{c'} + i\epsilon^{c}{}_{c'}) - T^{\alpha}_{S;b}{}^{a}(\delta^{c}_{c'} - i\epsilon^{c}{}_{c'})\right)\delta^{f_S}_{f'_S}\Delta^{S}_{A;c,f_S,a}\Delta^{S;c',f'_S,b}_{B} \tag{2.71}$$

where $T^{\alpha;a}_{S}{}_{b}$ is the generator of the representation under which the scalar species S transform, a and b are the gauge indices of the representation, α is the gauge index of the group, and f_S, f'_S are the flavor indices of the scalar species S. The sign is again conventional and a change of sign here will correspond to a change in the sign of the generators of the representation.

The fermion case is equivalent to the scalar case, but simpler since we do not need to keep track of the complex index c. In the Abelian case, we have

$$R^{\alpha}_{JK} = \sum_{S} Q_S \delta^{f_S}_{f'_S} \Delta^{S}_{J;f_S} \Delta^{S;f'_S}_{K}. \tag{2.72}$$

and in the non-Abelian

$$R^{\alpha}_{JK} = \sum_{S} T^{\alpha;k}_{S\ \ j} \delta^{f_S}_{f'_S} \Delta^{S}_{J;f_S,j} \Delta^{S;f'_S,b}_{K} \tag{2.73}$$

The expressions for the generators can be generalized to a semi-simple group by including an index t for the subgroup in question and a product $\prod_{t' \neq t} \delta^{a'_t}_{b'_t}$ over the other subgroups.

2.3 Fixed points in gauge-Yukawa theory

In Section 2.1.2, we discussed that in the vicinity of the Gaussian fixed point, the running of the couplings is particularly simple, and that exactly at the fixed point itself, there is no running. It is interesting to ask if there are other such fixed points around which the running is simple.

Later, in Section 2.1.4, we saw that the beta function is what governs the flow the coupling constants at different renormalization scales. We may well imagine that for a sufficiently complicated theory, other fixed points will appear, and this is indeed what we find. It is perhaps more surprising that we do not even need a particularly complicated theory in order to find a non-trivial fixed point in four dimensions. Simply consider an SU(N) gauge theory with N_f flavors of fermions, and we find a rich structure with fixed points for many possible choices of N, N_f and the fermion representation (see [72, 73] for details).

To make things explicit, let us consider the two–loop beta function for just such a theory with N_f Dirac fermions transforming in the fundamental representation SU(N). It is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11N}{3} - \frac{2N_f}{3}\right) - \frac{g^5}{(4\pi)^4}\left(\frac{34N^2}{3} + \frac{N_f}{N} - \frac{13}{3}NN_f\right). \tag{2.74}$$

Setting this to zero, we can easily find the non-trivial Banks-Zaks fixed point [72],

$$g^* = 4\pi\sqrt{\frac{2N_f - 11N}{34N^2 + \left(\frac{3}{N} - 13N\right)N_f}} \tag{2.75}$$

$$= \frac{4\pi}{\sqrt{N}}\sqrt{\frac{11 - 2x}{13x - 34 - \frac{3}{N^2}}}\,, \tag{2.76}$$

where in the second equality we have introduced $x = N_f/N$. It is easy to see that the $3/N^2$ term can largely be ignored, and we can safely do our analysis considering only the remaining terms. Here, we see that for $\frac{34}{13} \lesssim x < \frac{11}{2}$, a fixed point exists. The lower limit is dictated by the positivity of the denominator, and the upper by the positivity of the numerator. The fixed point value is zero for $x = \frac{11}{2}$, and diverges as it approaches its lower limit.

At such a fixed point, the theory does not run, and it is therefore invariant under dilations (scale transformations). This is particularly interesting since one of the most studied subgenres of quantum field theory is conformal field theories. In addition to Poincaré invariance, these are also invariant uner dilatations and special conformal transformations (which are an entirely separate group of transformations, that, like Poincaré and scale transformations, preserve the angles between vectors). Conformal field theories are surprisingly constrained and it is possible to make far stronger statements about them than one can about general field theories. In particular, the two- and three-point functions are completely determined up to a constant [74], and the four- and higher n-point functions can be expressed solely in terms of the lower-point functions through the operator product expansion.

While conformal field theories are interesting, the connection to general quantum field theories at fixed points is not immediately clear, as these are not, a priori, invariant under the special conformal transformations. However, it is known from classical field theory that dilation-invariant, four-dimensional unitary and renormalizable theories are also conformally invariant [75, 76]. Much recent work (see Section 3.1) has discussed whether this result also holds for quantum theories [77, 78], and there seems to be strong indications that it does.

Chapter 3

The a theorem

The so-called a theorem is perhaps the most misleadingly named concept in physics. It is not a theorem, it is not clear to a non-expert what a is, and it does not actually deal with a. Before going into too much detail, let us introduce some notation. We will in this chapter concern ourselves with classically conformal field theories, that is theories given by Lagrangians of the form

$$\mathcal{L} = \mathcal{L}_{kin} + g_i \mathcal{O}^i \,, \tag{3.1}$$

where $\mathcal{L}_{kin}$ contains the kinetic terms of the fields, g_i are all dimensionless coupling constants of the theory and $\mathcal{O}^i$ are marginal operators. The beta functions of the theory are given by

$$\beta_i = \frac{\mathrm{d}g_i}{\mathrm{d}t} \,, \tag{3.2}$$

where $t = \log(\mu/\mu_0)$ is the *renormalization group time* at the scale μ in relation to an arbitrary reference scale μ_0.

The name a theorem comes from a desire to generalize the more aptly named c theorem, proven by Zamolodchikov [79] for two dimensional field theories. It states that

1. There exists a function $c(g) \geq 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} c = \beta_i(g) \frac{\partial}{\partial g_i} c(g) \leq 0 \,, \tag{3.3}$$

where the equality only holds at fixed points where $\beta_i(g^*) = 0$.

2. At the fixed points, the two dimensional field theory has an infinite dimensional conformal symmetry whose generators form a Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12}(n^3 - n)\delta_{n+m,0}\,, \tag{3.4}$$

   where the central charge $\tilde{c}$ is in general different for each fixed point, $\tilde{c} = \tilde{c}(g^*)$.

3. The value of c at each fixed point is the same as the corresponding central charge, $c(g^*) = \tilde{c}(g^*)$.

This theorem clearly makes very strong statements about the nature of two dimensional field theories, but alas the proof relies crucially on the special properties of field theories in two dimensions.

3.1 In four dimensions

The conformal group in two space-time dimensions is very different from the conformal group in higher dimensions. In particular, only in two dimensions do the generators of the conformal symmetry form an infinite dimensional Virasoro algebra, in higher dimensions they instead form the Poincaré, dilatation and *special conformal* algebras. This means that the properties 2 and 3 of Zamolodchikov's c theorem cannot possibly be satisfied in higher dimensions. Despite this, much work has been done to find a different function which satisfies the fist property in four space-time dimensions [46, 80, 81] (see also [82] and references therein).

The function a, proposed by Cardy [80], is identified with the integral of the trace of the energy-momentum tensor over the four-sphere and the coefficient of the Euler density in the conformal anomaly on a curved space-time manifold [82]. However, it can be shown [81] that a (or β_b in the notation of [81]) is explicitly not monotonous along renormalization group flows, but that

$$\tilde{a} = a + w^i \beta_i\,, \tag{3.5}$$

where w^i is a function of the couplings of the theory (and differs from Osborn's notation [81] by a factor of $\frac{1}{8}$), *is* monotonous to leading order in perturbation theory. It also has the convenient property that it equals a at fixed points where the beta functions vanish.

More recent work [83, 84] has helped rekindle interest in this problem by linking the scattering amplitude of a new diatonic field to the $\tilde{a}$ function. In particular, the analyticity of this scattering amplitude supports the conjecture that $\tilde{a}$ is monotonic along renormalization group flows.

While the above mentioned work has mainly concerned itself with generalizing property 1 of the c theorem, progress has also been made in making generalizations of the second property. As pointed out in Section 2.3, a quantum field theory at a fixed point is automatically dilatation invariant, and it has long been known that in classical field theory, this would imply that it is also fully conformally invariant [75, 76]. Recent work [77, 78, 85] indicates that this may also hold true for a unitary *quantum* field theory in four space-time dimensions. If true, this is perhaps as close to a generalization of the second property of the c theorem as we are likely to get. Since there is no object in the four dimensional conformal algebra that corresponds to the central charge, the third property is unlikely to be generalized.

A completely different candidate function for fulfilling the first property of the c theorem in four dimensions has been proposed in [86, 87]. It is based on the number of perturbative degrees of freedom in the theory, and we will not discuss it here.

3.2 The Weyl consistency conditions

As mentioned above, the a function can be considered either in the context of the trace of the energy momentum tensor of the theory, or in that of the Weyl anomaly. Here, we will consider both approaches, starting with the former and returning to the latter once we have established the a function.

In four dimensions and for a general quantum field theory, the vacuum expectation value of the trace of the energy–momentum tensor for a locally flat metric $\gamma_{\mu\nu}$ reads

$$\left\langle T^{\mu}_{\mu} \right\rangle = c\, W^2(\gamma_{\mu\nu}) - a\, E_4(\gamma_{\mu\nu}) + \dots\ , \tag{3.6}$$

where a and c are real coefficients, $E_4(\gamma_{\mu\nu})$ the Euler density and $W(\gamma_{\mu\nu})$ the Weyl tensor. The dots represent contributions coming from operators that can be constructed out of the fields defining the theory. Their contribution is proportional to the beta functions of their couplings. E_4 and W^2 are defined in terms of the curvature tensors,

$$W^2 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2 \tag{3.7}$$

$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \tag{3.8}$$

The coefficient a is the one used in Cardy's conjecture [80], and for a free field theory it is [88]

$$a_{\text{free}} = \frac{1}{90(8\pi)^2}\left(n_s + \frac{11}{2}n_f + 62n_v\right) , \tag{3.9}$$

where n_s, n_f and n_v are respectively the number of real scalars, Weyl fermions and gauge bosons.

The change of a along the RG flow is directly related to the underlying dynamics of the theory via the beta functions. To study this, we follow the second approach mentioned at the beginning of this section and consider the Weyl anomaly.

Keeping track of the classical conformal symmetry after the theory has been renormalized is not straightforward. A convenient way to do so is to promote the couplings to functions of space-time, i.e. $g_i = g_i(x)$, and to work in an arbitrary curved background $\gamma_{\mu\nu}$. Under these assumptions, a conformal transformation of the space-time metric $\gamma_{\mu\nu} \to e^{2\sigma(x)}\gamma_{\mu\nu}$ is partially compensated by a change in the renormalized coupling as $g_i(\mu) \to g_i(e^{-\sigma(x)}\mu)$, up to a number of terms that vanish in the limit of flat space-time and constant couplings. This can be explicitly encoded in the infinitesimal variation of the generating functional $W = \log\left[\int \mathcal{D}\Phi\, e^{i\int \mathrm{d}^4x\mathcal{L}}\right]$, parametrized as

$$\Delta_\sigma W \equiv \int \mathrm{d}^4x\, \sigma(x)\left(2\gamma_{\mu\nu}\frac{\delta W}{\delta\gamma_{\mu\nu}} - \beta_i\frac{\delta W}{\delta g_i}\right) \tag{3.10}$$

$$= \sigma\left(aE_4(\gamma) + \chi^{ij}\partial_\mu g_i\partial_\nu g_j G^{\mu\nu}\right) + \partial_\mu\sigma w^i\,\partial_\nu g_i G^{\mu\nu} + \ldots \tag{3.11}$$

where a and w^i are functions of the renormalized couplings that we introduced in Equation (3.5), χ^{ij} is a new function of the couplings which we will consider in detail later and $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}R$ is the Einstein tensor. The right-hand side of the equation contains all possible dimension-four Lorentz scalars constructed out of the metric and derivatives of the couplings, $\partial_\mu g_i$, and only the three terms relevant to our discussion have been shown here. The functions a, χ^{ij} and w^i are completely determined by the theory and can be explicitly computed in a perturbative expansion in the couplings g_i. The essence of the Weyl consistency conditions is that these functions are not independent of each other. In particular, the Weyl anomaly expressed by Equation (3.10) is of abelian nature, and therefore must satisfy

$$\Delta_\sigma \Delta_\tau W = \Delta_\tau \Delta_\sigma W. \tag{3.12}$$

This equation gives a number of relations between the terms to the right-hand side of Equation (3.11), among which the one we will make the most use of is[1] [47]

$$\partial^i \tilde{a} = -\chi^{ij}\beta_j + (\partial^i w^j - \partial^j w^i)\beta_j \,, \tag{3.13}$$

where $\tilde{a}$ is defined in Equation (3.5) and $\partial^i = \frac{\partial}{\partial g_i}$. From this equation it follows that

$$\frac{d}{d\mu}\tilde{a} = -\chi^{ij}\beta_i\beta_j \,, \tag{3.14}$$

so that $\tilde{a}$ is monotonically decreasing along the RG flow, provided that χ is a positive definite matrix. χ is indeed positive definite at lowest order in perturbation theory [46], however not necessarily non-perturbatively (see e.g. Reference [2]). Establishing the positivity of χ beyond perturbation theory would immediately prove that the first property of the c theorem could be generalized to the a theorem [80] and the irreversibility of the RG flow in four dimensions[2]. We stress that equation (3.13) relies neither on the space dependence of the couplings nor the space-time metric. Henceforth we will work in ordinary Minkowski background.

For a generic gauge-Yukawa theory, the function w^i turns out to be an exact one-form at the lowest orders in perturbation theory [46], so that the terms in Equation (3.13) involving derivatives of w^i cancel out, and we will use in the following the simplified consistency condition

$$\frac{\partial \tilde{a}}{\partial g_i} = -\beta^i \,, \qquad \beta^i \equiv \chi^{ij}\beta_j \,. \tag{3.15}$$

χ^{ij} can be seen as a metric in the space of couplings, used in this case to raise and lower the latin coupling space indices. We note that if χ^{ij} is invertible, all beta functions can be derived from the same quantity $\tilde{a}$,

[1] It is worth mentioning that the Weyl consistency conditions used above assume that the trace of the energy–momentum tensor vanishes when all the beta functions are zero simultaneously. Exceptions are known to exist [89] and in this case one modifies the consistency conditions [77, 78] in order to build $\tilde{a}$.

[2] Using analyticity arguments, it was shown recently that the function $\tilde{a}$ in the ultraviolet (UV) is bigger than in the infrared (IR) [83, 84]. However, this method does not permit to draw any conclusions on the behaviour of $\tilde{a}$ along the RG flow.

which has profound implications. The flow generated by the modified beta functions β^i is a gradient flow, implying in particular

$$\frac{\partial \beta^j}{\partial g_i} = \frac{\partial \beta^i}{\partial g_j}\,, \tag{3.16}$$

which gives relations between the beta functions of different couplings. These consistency conditions can be used as a check of a known computation, but could, in principle, also be used to determine some unknown coefficients at a higher loop order in perturbation theory. Sadly, this is impractical as it requires calcluations of χ^{ij} and w^i that are in general at least as complicated as the ones that would be required to find the beta functions directly.

3.3 The *a* theorem in gauge–Yukawa theories

We consider a Lagrangian of the basic form (2.43), but restrict ourselves to a simple gauge group, and scalars that are singlets under it. To exemplify our results, we will consider gauge theories for which the Yukawa and quartic interactions only depend on a single parameter in the following manner

$$y_{JK;A} \equiv y\, T_{JK;A}\,, \qquad \lambda_{ABCD} \equiv \lambda\, T_{ABCD}\,, \tag{3.17}$$

where the T's are specified by the specific theory in question. There are therefore three couplings in our setup: gauge g, Yukawa y and quartic λ. In analogy with Equation (2.41), we define

$$\alpha_g = \frac{g^2}{(4\pi)^2}, \quad \alpha_y = \frac{y^2}{(4\pi)^2}, \quad \alpha_\lambda = \frac{\lambda}{(4\pi)^2}. \tag{3.18}$$

The generic structure of the associated beta functions reads[3]

$$\beta_{\alpha_g} = -2\alpha_g^2\left[b_0 + b_1\alpha_g + b_y\alpha_y + b_2\alpha_g^2 + b_3\alpha_g\alpha_y + b_4\alpha_y^2\right]\,, \tag{3.19}$$

$$\beta_{\alpha_y} = 2\alpha_y\left[c_1\alpha_y + c_2\alpha_g + c_3\alpha_g\alpha_y + c_4\alpha_g^2 + c_5\alpha_y^2 + c_6\alpha_y\alpha_\lambda + c_7\alpha_\lambda^2\right]\,, \tag{3.20}$$

$$\beta_{\alpha_\lambda} = d_1\alpha_\lambda^2 + d_2\alpha_\lambda\alpha_y + d_3\alpha_y^2\,. \tag{3.21}$$

The expansion to three loops in the gauge coupling, two loops in the Yukawa and one in the quartic coupling leads to a consistent expression

[3] The factors of 2 in the definition of the gauge and Yukawa beta functions follow from the definitions (3.18). One has for example $\beta_{\alpha_g}/\alpha_g = 2\beta_g/g$.

for $\tilde{a}$ to order α^3. If the scalars were charged under the gauge group, terms proportional to $\alpha_g \alpha_\lambda$ would also appear.

Having established the generic form of the beta functions, we move to determining the metric χ and one–form w. They can be found by examining the relevant Feynman diagrams which enter the computation of the trace anomaly, as shown in Appendix A. We find

$$\chi = \begin{pmatrix} \frac{\chi^{gg}}{\alpha_g^2}\left(1 + A\alpha_g + B_1\alpha_g^2 + B_2\alpha_g\alpha_y\right) & B_0 & 0 \\ B_0 & \frac{\chi^{yy}}{\alpha_y}\left(1 + B_3\alpha_y + B_4\alpha_g\right) & 0 \\ 0 & 0 & \chi^{\lambda\lambda} \end{pmatrix} . \tag{3.22}$$

The coefficient χ^{gg} enters at the one–loop order, A and χ^{yy} at two loops, while $\chi^{\lambda\lambda}$ and the B_i's appear only at three loops. Similarly, the one–form w takes the form

$$\begin{aligned} w^g &= \frac{1}{\alpha_g}\left(D_0 + D_1\alpha_g + C_1\alpha_g^2 + C_2\alpha_g\alpha_y\right) , \\ w^y &= D_2 + C_3\alpha_y + C_4\alpha_g , \\ w^\lambda &= D_3\alpha_\lambda . \end{aligned} \tag{3.23}$$

The general structure of χ confirms that it is sufficient for all our purposes to consider the Yukawa beta function (3.20) to two–loop order and the quartic one (3.21) to one–loop only.

The leading coefficients χ^{gg}, χ^{yy} and $\chi^{\lambda\lambda}$ are [46]

$$\chi^{gg} = \frac{d(G)}{128\pi^2}, \tag{3.24a}$$

$$\chi^{yy} = \frac{1}{128\pi^2}\left(\frac{1}{3}T_{JK;A}T^*_{JK;A}\right), \tag{3.24b}$$

$$\chi^{\lambda\lambda} = \frac{1}{128\pi^2}\left(\frac{1}{72}T_{ABCD}T_{ABCD}\right) , \tag{3.24c}$$

where we used the T's defined in Equation (3.17) and $d(G)$ denotes the dimension of the adjoint representation G of the underlying gauge group, i.e. the number of gluons. A is given by [46]

$$A = 17C_2(G) - \frac{10}{3}N_R\, T(R), \tag{3.25}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint, N_R the number of Weyl fermions in the representation R and $T(R)$ is the trace normalization satisfying $T(R)\delta^{\alpha\beta} = \mathrm{Tr}(R^\alpha R^\beta)$, R^α being the generators of the fermions under the gauge group (see also Section 2.2.4). With these coefficients, the metric χ is positive definite near the origin of the coupling constant space. It is however clear that in the absence of a theorem, the positivity of χ away from the origin is not guaranteed. The remaining coefficients of χ and w are yet to be determined, but, as we shall show, they are not needed to determine $\tilde{a}$ at the fixed points to the order investigated here.

3.3.1 Power of the consistency relations and the $\tilde{a}$ function

The system of first order differential equations in Equation (3.13) allows us to derive the following conditions relating the different coefficients of the beta functions as well as χ and w,

$$\chi^{gg} b_y = -\chi^{yy} c_2 \,, \tag{3.26a}$$

$$\chi^{yy} c_6 = \chi^{\lambda\lambda} d_3 \,, \tag{3.26b}$$

$$4\chi^{yy} c_7 = \chi^{\lambda\lambda} d_2 \,, \tag{3.26c}$$

$$2\chi^{gg} b_4 + \chi^{yy} \left(B_4 c_1 + B_3 c_2 + c_3\right) = 2\left(B_0 - C_2 + C_4\right) c_1 \,, \tag{3.26d}$$

$$\begin{aligned} \chi^{gg} \left(B_2 b_0 + A b_y + b_3\right) + 2\chi^{yy} \left(B_4 c_2 + c_4\right) = &\left(B_0 - C_2 + C_4\right) c_2 \\ &+ 2\left(B_0 + C_2 - C_4\right) b_0 \,. \end{aligned} \tag{3.26e}$$

In the first three equations, we note that c_2, d_2 and d_3 are one–loop coefficients, and b_y, c_6 and c_7 are two–loop coefficients, meaning that these consistency conditions link together beta function terms of different loop orders. They can thus be used to either test or determine some of the higher order coefficients of the beta functions since we know the metric factors. The remaining equations can be used in a similar fashion. However, given that the B_i coefficients have not been explicitly computed we use the knowledge of the beta functions to deduce, for example, B_3 and B_4 assuming that c_2 does not vanish.

For $\tilde{a}$ to cubic order in the couplings, and using the consistency relations above, we have

$$\tilde{a} = a_{\text{free}} + \tilde{a}^{(1)} + \tilde{a}^{(2)} + \tilde{a}^{(3)} + \ldots \,, \tag{3.27}$$

where a_{free} is the free–field theory value (3.9), and the one–, two– and three–loop coefficients are

$$
\begin{aligned}
\tilde{a}^{(1)} &= -2\chi^{gg} b_0 \alpha_g \,, && (3.28)\\
\tilde{a}^{(2)} &= -\chi^{gg}\left(b_1 + A b_0\right)\alpha_g^2 - 2\chi^{gg} b_y \alpha_g \alpha_y + \chi^{yy} c_1 \alpha_y^2 \,, && (3.29)\\
\tilde{a}^{(3)} &= -\chi^{gg}\Big[\frac{2}{3}\left(b_2 + A b_1\right)\alpha_g^3 + \left(b_3 + A b_y\right)\alpha_g^2\alpha_y + 2 b_4 \alpha_g \alpha_y^2 \\
&\quad + \frac{1}{3}\frac{c_1}{c_2}\left(4 b_4 - \frac{c_1}{c_2}\left(b_3 + A b_y\right)\right)\alpha_y^3\Big] + \chi^{yy}\Big[\frac{2}{3}\left(c_5 - \frac{c_1}{c_2}c_3 + \left(\frac{c_1}{c_2}\right)^2 c_4\right)\alpha_y^3 \\
&\quad + c_6 \alpha_y^2 \alpha_\lambda + 2 c_7 \alpha_y \alpha_\lambda^2\Big] + \frac{1}{3}\chi^{\lambda\lambda} a_1 \alpha_\lambda^3 + \frac{\beta_{\alpha_g}}{\alpha_g^2} f\left(\alpha_i^3\right) + \frac{\beta_{\alpha_y}^2}{\alpha_y}\frac{B_0 - C_2 + C_4}{4 c_2} \,, && (3.30)
\end{aligned}
$$

where we defined

$$
f\left(\alpha_i^3\right) = \chi^{gg}\left(\frac{B_1}{3}\alpha_g^3 + \frac{B_2}{2}\alpha_g^2\alpha_y - \frac{B_2}{6}\left(\frac{c_1}{c_2}\right)^2\alpha_y^3\right) + \frac{B_0 + C_2 - C_4}{3}\left(\frac{c_1}{c_2}\right)^2\alpha_y^3 \,. \quad (3.31)
$$

Remarkably, the unknown coefficients B_i and C_i appear only in the last two terms of $\tilde{a}^{(3)}$, where they are multiplied by beta functions and hence vanish at fixed points[4]. This was also observed to occur in supersymmetric theories [90].

It is instructive to calculate a to the second order using (3.5) and recalling that the leading coefficients entering the one–form w are $D_0 = \chi^{gg}$, $D_1 = \frac{1}{2}A\chi^{gg}$ and $D_2 = \frac{1}{2}\chi^{yy}$ [46]. We find the simple expression

$$
a = a_{\text{free}} + \chi^{gg}\left(b_1\alpha_g^2 + b_y\alpha_g\alpha_y\right) + \mathcal{O}\left(\alpha_i^3\right) \,, \quad (3.32)
$$

where, remarkably, the term linear in α_g, the one quadratic in α_y as well as the term linear in A canceled out. Because the signs of b_1 and b_y depend on the gauge theory, a is not generally a monotonically decreasing function along the perturbative RG flow.

3.3.2 $\tilde{a}$ at fixed points

We can now move on to the study of the fixed points and determine the variation of $\tilde{a}$ between two of them. A convenient way to search for the zeros of the system of beta functions (3.19)-(3.21) is to first solve

[4]We have taken the liberty of adding higher order terms in order to rewrite the coefficients as beta functions. These terms are irrelevant when computing $\tilde{a}$ between perturbative fixed points.

analytically for $\beta_{\alpha_\lambda} = 0$ which permits us to relate α_λ to α_y, then set β_{α_y} to zero, further relating α_y to α_g, so that finally we can search for the zeros of the following effective beta function in α_g

$$\beta_{\alpha_g}^{\text{eff}} = -2\alpha_g^2 \left[b_0 + b_1^{\text{eff}} \alpha_g + b_2^{\text{eff}} \alpha_g^2 \right], \tag{3.33}$$

where[5]

$$b_1^{\text{eff}} = b_1 - \frac{c_2}{c_1} b_y \,, \tag{3.34}$$

$$b_2^{\text{eff}} = b_2 - \frac{c_2}{c_1} b_3 + \left(\frac{c_2}{c_1}\right)^2 b_4 - \frac{b_y}{c_1} \left[c_4 - \frac{c_2}{c_1} c_3 + \left(\frac{c_2}{c_1}\right)^2 c_5^{\text{eff}} \right] , \tag{3.35}$$

with

$$c_5^{\text{eff}} = c_5 - \frac{d_3}{d_1} c_7 - \frac{d_2}{2d_1} \left(c_6 - \frac{d_2}{d_1} c_7 \right) \left(1 \pm \sqrt{1 - \frac{4 d_1 d_3}{d_2^2}} \right) . \tag{3.36}$$

At this order in perturbation theory, there can be at most two perturbative fixed points for each sign in c_5^{eff}, if both b_0 and b_1^{eff} are tuned to be small. An example of this is provided in the following section. Using Equation (3.34) and (3.35), the difference in the function $\tilde{a}$ — or equivalently a — between the UV and IR fixed points can then be written as

$$\Delta \tilde{a}_{\text{perturbative}} \equiv (\tilde{a}^{UV} - \tilde{a}^{IR})_{\text{perturbative}} \tag{3.37}$$

$$\begin{aligned} = &- 2\chi^{gg} \left[b_0 \left(\alpha_g^{UV} - \alpha_g^{IR} \right) + \frac{1}{2} \left(b_1^{\text{eff}} + A b_0 \right) \left((\alpha_g^{UV})^2 - (\alpha_g^{IR})^2 \right) \right. \\ &\left. + \frac{1}{3} \left(b_2^{\text{eff}} + A b_1^{\text{eff}} + B b_0 \right) \left((\alpha_g^{UV})^3 - (\alpha_g^{IR})^3 \right) \right] , \end{aligned} \tag{3.38}$$

where α_g^{UV} and α_g^{IR} denote the values of the gauge coupling at the UV and IR fixed point respectively, and we defined

$$B \equiv B_1 - \frac{c_2}{c_1} \left(B_2 + \frac{B_0}{\chi^{gg}} \right) . \tag{3.39}$$

The expression (3.38) reduces to the case of a gauge theory without Yukawa interactions by replacing b_1^{eff} and b_2^{eff} with b_1 and b_2.

Inspecting the effective beta function, there is a perturbative fixed point for small b_0, which reads

$$\alpha_g^{BZ} = -\frac{b_0}{b_1^{\text{eff}}} + O\left(b_0^2\right) . \tag{3.40}$$

[5] Note that there was a typo in the original paper [2] where the coefficient of b_4 in b_2^{eff} was written as $(c_1/c_2)^2$ rather than $(c_2/c_1)^2$.

For positive b_0 and negative b_1^{eff}, this is the generalized Banks–Zaks (BZ) IR fixed point (see also Section 2.3). The situation in which the BZ fixed point is of UV nature is equally possible. This occurs by reversing the signs of both b_0 and b_1^{eff}. The trivial fixed point at the origin will in the first case be an UV fixed point and in the second an IR one. The finite change in $\tilde{a}$ between the UV and IR fixed points can be computed either way, and we obtain

$$\Delta\tilde{a}^{BZ} = \mp\chi^{gg}\frac{b_0^2}{b_1^{\text{eff}}} \,. \tag{3.41}$$

Here, the sign reflects the sign of b_1^{eff} (which in turn reflects the sign of b_0), such that $\Delta\tilde{a}$ is positive for any physical fixed point. However, $\Delta\tilde{a}$ can formally become negative when the value of the coupling α_g at the fixed point is on the unphysical negative axis.

To three–loop order in the effective beta function, one can have the two following physical zeros

$$\alpha_g^{BZ} = -\frac{b_1^{\text{eff}}}{2b_2^{\text{eff}}}\left(1-\sqrt{1-\frac{4b_0b_2^{\text{eff}}}{(b_1^{\text{eff}})^2}}\right), \qquad \alpha_g^{\cancel{BZ}} = -\frac{b_1^{\text{eff}}}{2b_2^{\text{eff}}}\left(1+\sqrt{1-\frac{4b_0b_2^{\text{eff}}}{(b_1^{\text{eff}})^2}}\right) \tag{3.42}$$

For small values of b_0, the solution with negative sign corresponds to the usual BZ fixed point, with the following corrections

$$\alpha_g^{BZ} = -\frac{b_0}{b_1^{\text{eff}}}\left(1+\frac{b_0b_2^{\text{eff}}}{(b_1^{\text{eff}})^2}+\mathcal{O}\left(b_0^2\right)\right) \,. \tag{3.43}$$

This expression holds provided $b_0/(b_1^{\text{eff}})^2$ is small. We shall see below that there are cases where this is not true. Using (3.43), we can compute the three–loop corrections to the variation of $\tilde{a}$,

$$\Delta\tilde{a}^{BZ} = \mp\chi^{gg}\frac{b_0^2}{b_1^{\text{eff}}}\left(1-(Ab_1^{\text{eff}}-2b_2^{\text{eff}})\frac{b_0}{3(b_1^{\text{eff}})^2}\right) \,. \tag{3.44}$$

We now turn our attention to the second zero, $\alpha_g^{\cancel{BZ}}$. The first observation is that for a generic value of b_1^{eff}, this fixed point occurs at a non–perturbative value of the coupling. This is what happens in general for gauge theories with fermionic matter in a given irreducible representation of the gauge group [91]. However, for gauge theories with Yukawa interactions and/or multiple matter representations, the possibility that both b_0 and b_1^{eff} are small exists. An explicit example is provided below. Furthermore, when

$b_2^{\rm eff} = (b_1^{\rm eff})^2/(4b_0)$, the two fixed points merge. This phenomenon can happen within the purview of perturbation theory. At the merger, one has $\alpha_{\rm merger} = -2b_0/b_1^{\rm eff}$, which, when plugged into Equation (3.38), gives

$$\Delta\tilde{a}^{BZ}\Big|_{\rm merger} = \mp\chi^{gg}\frac{4}{3}\frac{b_0^2}{b_1^{\rm eff}} . \tag{3.45}$$

The virtues of this expression is studied in more detail elsewhere [92].

Having in our hands the explicit tools, we can explore the a theorem for gauge theories with interesting fixed point structures.

3.4 A concrete example

We consider the $SU(N_c)$ gauge theory introduced in Section 1.2. For convenience, we repeat the Lagrangian of the theory,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda\not{D}\bar{\lambda} + i\bar{q}\not{D}q + i\bar{\tilde{q}}\not{D}\tilde{q} + \partial_\mu H^\dagger\partial^\mu H \\ + (y_H\tilde{q}Hq + h.c.) - u_1(\mathrm{Tr}[H^\dagger H])^2 - u_2\,\mathrm{Tr}[(H^\dagger H)^2] , \tag{3.46}$$

Throughout this section, we will work with rescaled couplings which enable a finite Veneziano limit of the theory with ℓ, the number of adjoint fermions, fixed. That is, we let $N_c, N_f \to \infty$ while keeping $x \equiv N_f/N_c$ fixed. The appropriately rescaled couplings are

$$a_g = \frac{g^2N_c}{(4\pi)^2} , \quad a_H = \frac{y_H^2N_c}{(4\pi)^2} , \; z_1 = \frac{u_1N_f^2}{(4\pi)^2} , \; z_2 = \frac{u_2N_f}{(4\pi)^2} . \tag{3.47}$$

This model was introduced in [19] to investigate near–conformal dynamics, at the one–loop level, and its impact on the spectrum of the theory with special attention payed to the properties of the dilaton. The model was further investigated at the two–loop level in [1]. To compute $\tilde{a}$, following the previous section, we need to determine the three–loop contribution to the gauge beta function.

Using [6, 65–67, 69, 93, 94] and taking the Veneziano limit, we find

$$\beta_{a_g} = -\frac{2}{3}a_g^2\Bigg[11 - 2\ell - 2x + (34 - 16\ell - 13x)\,a_g + 3x^2 a_H + \frac{81x^2}{4}a_g a_H \\ -\frac{3x^2(7+6x)}{4}a_H^2 + \frac{2857 + 112x^2 - x(1709 - 257\ell) - 1976\ell + 145\ell^2}{18}a_g^2\Bigg]\,, \tag{3.48}$$

$$\beta_{a_H} = a_H\Bigg[2(x+1)a_H - 6a_g + (8x+5)a_g a_H + \frac{20(x+\ell) - 203}{6}a_g^2 \\ - 8xz_2 a_H - \frac{x(x+12)}{2}a_H^2 + 4z_2^2\Bigg]\,, \tag{3.49}$$

$$\beta_{z_2} = 2\left(2z_2 a_H + 4z_2^2 - x a_H^2\right)\,. \tag{3.50}$$

Here one can see that the double trace coupling z_1 does not participate in the running of the remaining couplings. In addition, using (3.24) and (3.25), the metric coefficients for this theory can be found:

$$\chi^{gg} = \frac{N_c^2}{2^7\pi^2}\,,\quad \chi^{yy} = \frac{N_f^2}{3\cdot 2^7\pi^2}\,,\quad \chi^{z_2z_2} = \frac{N_f^2}{3\cdot 2^6\pi^2}\,,\quad A = 17 - \frac{10}{3}(x+\ell)\,. \tag{3.51}$$

One can check that the expressions above satisfy the consistency relations given in (3.26a)-(3.26c), and therefore it constitutes an independent check of the correctness of the beta functions. We now turn to the fixed point analysis of the model which will reveal an interesting perturbative structure.

3.4.1 Leading order analysis: Banks–Zaks fixed point

In order to see a fixed point of the Banks–Zaks type (see Section 2.3 and [72]), the one–loop coefficient of the gauge beta function has to be small and the signs of b_0 and b_1^{eff} have to be opposite. Therefore, our first task is to find a region in the parameter space of the model where the physical BZ fixed point exists. We use (3.34)

$$b_0 = \frac{1}{3}\big(11 - 2(\ell + x)\big)\,,\qquad b_1^{\text{eff}} = \frac{1}{3}\left(34 - 16\ell - 13x + \frac{9x^2}{(x+1)}\right). \tag{3.52}$$

From the asymptotic freedom (AF) boundary condition $b_0 = 0$, we obtain that $x = (11 - 2\ell)/2$. Substituting this value of x into b_1^{eff}, we have

$$b_{1AF}^{\text{eff}} = -\frac{25}{2} - \ell - \frac{3(11-2\ell)^2}{4\ell - 26}\,, \tag{3.53}$$

where the last term comes from the Yukawa interactions. We immediately notice that $b^{\rm eff}_{1AF}$ can be made to vanish, which happens when $\ell^* \approx 0.37$. Below, we will consider the cases $\ell = 1$ for which $b^{\rm eff}_{1AF}$ is negative and $\ell = 0$ for which it is positive. In the first case we have a standard IR BZ fixed point, and in the second, we obtain a new UV BZ fixed point. It is worth noticing that in the absence of Yukawa interactions, $b^{\rm eff}_{1AF}$ is always negative, and therefore the physical BZ fixed point can only be the standard IR fixed point.

$\ell = 1$ case

In this case, there exists a perturbative IR fixed point regardless of whether we consider the presence of Yukawa interactions. In Figure 3.1.a, we show the leading order result for the change in $\tilde{a}$ between the Gaussian (trivial) fixed point and the BZ IR one at leading order, both in the presence (blue line) and absence (red line) of Yukawa interactions. Both curves cross zero at $x^* = 9/2$, when asymptotic freedom is lost. For $x > 9/2$, where $b_0 < 0$, there is an unphysical BZ UV fixed point with negative α_g^* yielding a negative $\Delta\tilde{a}$. The Yukawa interactions in (3.53) imply $|b_1^{\rm eff}| < |b_1|$, which leads to a larger $\Delta\tilde{a}$ in the case of the gauge theory with scalars.

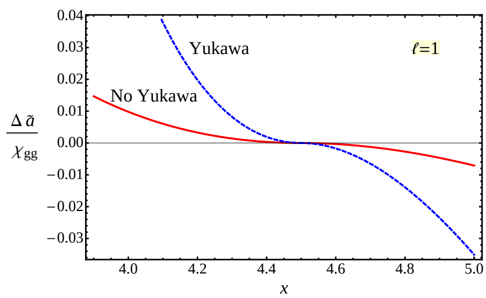


Figure 3.1.a $\Delta\tilde{a}$ between the Gaussian and BZ fixed points normalised to χ^{gg} at leading order for the $\ell = 1$ case. The solid red (dashed blue) line corresponds to the model without (with) Yukawa interactions. In both cases the physical BZ fixed point is an IR one.

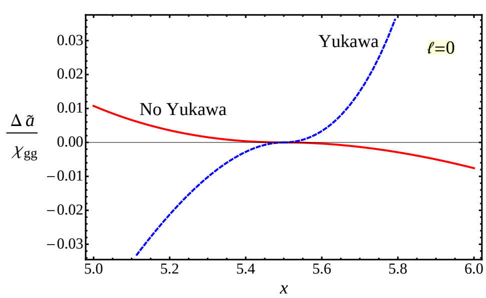


Figure 3.1.b $\Delta\tilde{a}$ between the Gaussian and BZ fixed points normalised to χ^{gg} at leading order for the $\ell = 0$ case. In the absence (presence) of Yukawa interactions, the physical BZ fixed pont is an IR (UV) one. The color code is the same as on the left panel.

$\ell = 0$ case

We now turn to the $\ell = 0$ case where, rather than having a BZ IR fixed point, the theory develops a UV fixed point when asymptotic freedom is lost, i.e. when $b_0 < 0$. Of course, this is possible only because of the presence of the Yukawa interactions. In Figure 3.1.b, the leading order result for the change in $\tilde{a}$ between the Gaussian and BZ fixed points without Yukawa interactions is shown in red, and the one with Yukawa interactions in blue. Both curves cross zero for $x^* = 11/2$ when asymptotic freedom is lost.

3.4.2 Next–to–leading order analysis: Fixed point merger

At the next perturbative order, we deal with the full system of Equations (3.48)–(3.50) and from now on, we concentrate only on the physical fixed points.

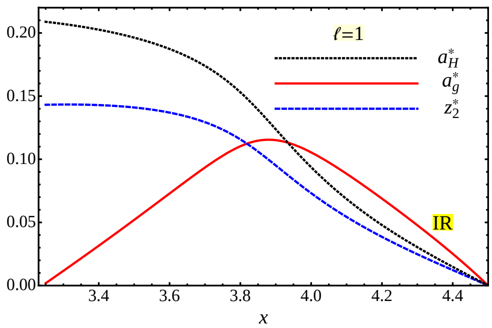


Figure 3.2.a The next–to–leading order physical fixed point structure for the $\ell = 1$ case with Yukawa and quartic interactions.

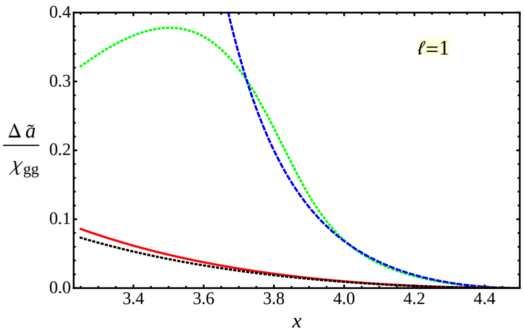


Figure 3.2.b $\Delta\tilde{a}$ normalised to χ^{gg} for the $\ell = 1$ case. The red and dashed blue lines are leading order results from Figure 3.1.a while the dotted black and green lines are the next–to–leading order corrections.

$\ell = 1$ case

We start again with the $\ell = 1$ theory and in Figure 3.2.a, we display the fixed point structure for the model with Yukawa and quartic interactions. We notice that at $x \approx 3.25$, the fixed point value of the gauge coupling vanishes. However, this happens in the region beyond applicability of perturbation theory since the two remaining coupling constants are large. In Figure 3.2.b, we plot the change in the $\tilde{a}$–function for the next–to–leading order BZ IR fixed point and compare it with the corresponding leading order results from Figure 3.1.a. As a general feature, we notice that the next–to–leading order corrections reduce the value of $\Delta\tilde{a}$ in the perturbative regime. It is clear from the plots that for the theory with Yukawa interactions, perturbation theory breaks down earlier when

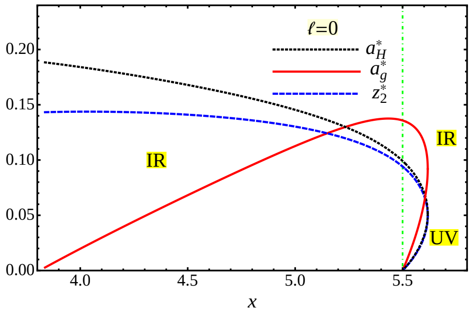


Figure 3.3.a The next–to–leading order physical fixed point structure for the $\ell = 0$ case *with* Yukawa and quartic interactions. The vertical dash–dotted green line represents the point where asymptotic freedom is lost.

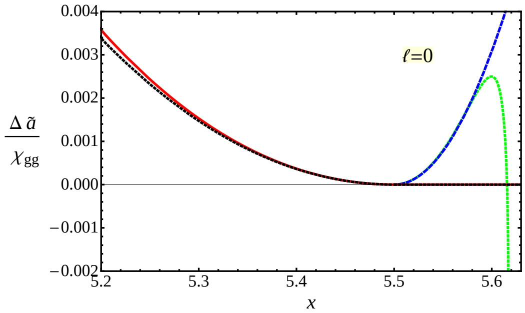


Figure 3.3.b $\Delta\tilde{a}$ normalised to χ^{gg} for the $\ell = 0$ case. The red and dashed blue lines are leading order results from Figure 3.1.b while the dotted black and green lines are next–to–leading order corrections.

moving away from the critical value $x^* = 9/2$, than in the theory without Yukawa interactions.

$\ell = 0$ case

We now turn to the $\ell = 0$ theory where, as discussed above, there is no BZ IR fixed point. However, in the asymptotically free regime there is a new physical IR fixed point emerging at the next–to–leading order. This non–BZ IR fixed point is present also when asymptotic freedom is lost, i.e. for $x > 11/2$. In this region there is also a BZ UV fixed point that was discussed in Section 3.4.1. The complete fixed point structure is shown in Figure 3.3.a. The UV and IR fixed points merge around $x \approx 5.6$. In Figure 3.3.b, we plot the change in the $\tilde{a}$–function at next–to–leading order together with the corresponding leading order results from Figure 3.1.b. We notice that $\Delta\tilde{a}$ becomes negative just before the merger which is incompatible with the *a* theorem. We interpret this effect as the breakdown of the perturbative expansion, rather than a counterexample to the conjecture, since the fixed point values of the couplings at the merger are quite large, as can be seen from Figure 3.3.a. See also discussion below.

So far, all our calculations of $\Delta\tilde{a}$ were for the flow connecting the trivial fixed point at the origin of the coupling constant space with the BZ one. However, it is relevant also to determine $\Delta\tilde{a}$ for the branch connecting the two non–trivial fixed points. In the theory with $\ell = 0$ and $x > 11/2$ this is the RG flow between the BZ UV fixed point and the non–BZ IR one.

We display the change in the $\tilde{a}$–function in Figure 3.4.a. Of course, at the merger $\Delta\tilde{a}$ vanishes.

$\ell = 0.35$ case: The perturbative merger

If we regard both ℓ and x as continuous parameters, it is formally possible to study the merging phenomenon within the perturbative regime. This happens around $\ell^* \approx 0.37$. Therefore we provide an example with $\ell = 0.35$. The change in the $\tilde{a}$–function for the two RG flows are shown in Figure 3.4.b. Since for this value of ℓ, perturbation theory holds, we observe a positive and well behaved $\Delta\tilde{a}$ all the way to the merger. This supports our interpretation from above that the negative value of $\Delta\tilde{a}$ was due to a breakdown of perturbation theory.

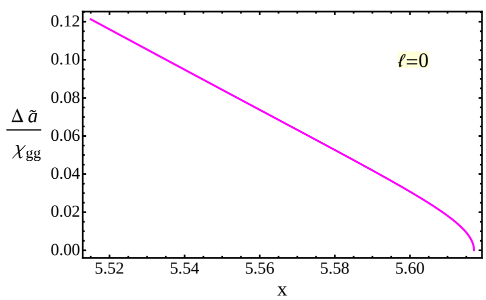


Figure 3.4.a $\Delta\tilde{a}$ normalised to χ^{gg} for the RG flow between the BZ UV fixed point and the non–BZ IR fixed point.

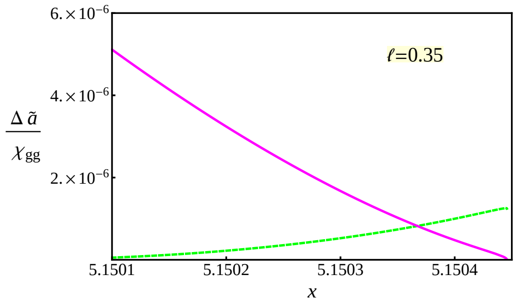


Figure 3.4.b $\Delta\tilde{a}$ normalised to χ^{gg} for the $\ell = 0.35$ case. The magenta curve corresponds to the flow between the BZ UV and the non–BZ IR fixed points. The green curve is the result between the Gaussian IR fixed point and the UV BZ one.

CHAPTER 4

VACUUM STABILITY IN THE STANDARD MODEL

One of the most celebrated features of the standard model Higgs field is that its potential does not have its global minimum at the origin (see also Section 1.1). To understand the Higgs potential in detail, it is instructive to consider the simplest realization of the Higgs mechanism; the abelian Higgs model.

4.1 The Higgs mechanism

We consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |D_\mu\phi|^2 - V(\phi)\,, \tag{4.1}$$

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \tag{4.2}$$

where $D_\mu = \partial_\mu + ieA_\mu$, $\mu^2 > 0$ and $\lambda > 0$. This Lagrangian is invariant under the combined U(1) gauge transformation

$$\phi(x) \to \mathrm{e}^{i\alpha(x)}, \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)\,. \tag{4.3}$$

With both μ^2 and λ positive, it is easy to see that the potential is at its minimum when

$$\phi = \phi_0 = \sqrt{\frac{\mu^2}{\lambda}}\,, \tag{4.4}$$

or any other value related to this by a U(1) transformation. It is therefore convenient to expand the complex field $\phi(x)$ in terms of two real fields,

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\Big(\phi_1(x) + i\phi_2(x)\Big). \tag{4.5}$$

Using this expansion, the potential becomes

$$V(\phi_1, \phi_2) = -\frac{\mu^4}{\lambda} + \frac{1}{2}2\mu^2\phi_1^2 + \mathcal{O}(\phi_i^3), \tag{4.6}$$

yielding a theory where ϕ_1 is the massive Higgs particle, with a mass of $\sqrt{2}\mu$, and ϕ_2 is a massless Goldstone boson coming from the breaking of the U(1) symmetry (4.3).

We can further apply this expansion to the kinetic energy term and find

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}e\phi_0 A_\mu\partial^\mu\phi_2 + e^2\phi_0^2 A_\mu A^\mu + \dots \tag{4.7}$$

where the ellipsis contains all terms of third order or higher in the fields A_μ, ϕ_1 and ϕ_2.

The most interesting term here in the context of the standard model is of course the gauge boson mass term $e^2\phi_0^2 A_\mu A^\mu$ the generalization of which is what gives the experimental mass of the W and Z bosons. There are several other subtleties involved in this, concerning among other things the disappearance of the Goldstone boson and the appearance of a longitudinal mode for the massive vector bosons, but we will not go into further detail here.

4.1.1 The Higgs potential

What we will concern ourselves with is the Higgs potential (4.2). As we saw, for low values of the field, or low energies, this can best be expressed by expanding around its minimum. The location of the minimum can be measured experimentally, and is identified with the Higgs field vacuum expectation value $v \approx 246\,\text{GeV}$. In Figure 4.1, we show a sketch of this potential for relatively modest values of the Higgs field.

As we saw above, the masses of the Higgs particle and the gauge bosons come from the expansion around the global minimum of the potential. It is meaningful to make this expansion because objects in Nature will naturally tend towards the minimum of their respective potential functions. The question we will be investigating in this section is

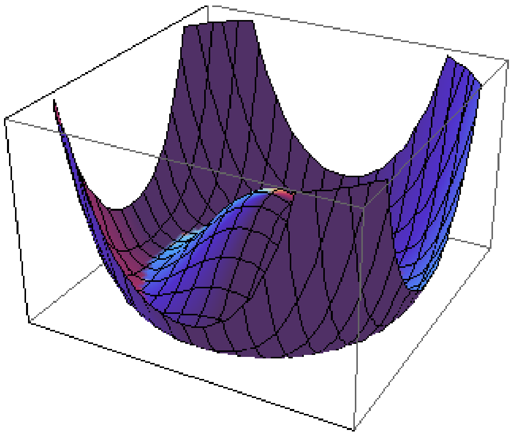


Figure 4.1: Sketch of the standard model Higgs potential for values of the Higgs field not much larger than twice the vacuum expectation value. We notice the so-called "Mexican hat" or "wine bottle" shape characterized by a raised center around a circular valley, and rising against at the edges.

whether the minimum we have chosen to expand around is in fact the true minimum of the standard model, and what the consequences are if it is not.

Since the standard model is a quantum theory, two crucial things can happen to disrupt the situation we have described in the preceding section. First, as established in Section 2.1, the coupling constants of a quantum field theory are not actually constants, but rather functions of the energy at which they are probed. In this case it is possible that if we run the couplings of the theory towards higher energies, the sign of the quartic coupling λ may change. If this happens, the Higgs potential either develops a new minimum or becomes unbounded from below, and the minimum around which we expand is not a true global minimum of the theory. Classically, this would not pose a problem, but in a quantum theory, particles may tunnel through a potential barrier and into the true minimum, without having sufficient energy to traverse the barrier classically.

If the Higgs coupling never changes sign, or does so above the cut off of the standard model (which is commonly taken to be the Planck scale $M_P = 1.22 \times 10^{19}$ GeV), the theory is stable and there is no problem. If the time it takes to tunnel through the barrier is larger than the lifetime of the universe so far, there is no contradiction with our observation of the universe's existence (and we expect that there will not be a problem with its future existence either), and we say that the theory is metastable. If, however, the standard model Higgs potential is unbounded from below or has a very deep second minimum, and the tunneling time is significantly lower than the lifetime of the universe, the theory predicts that the universe is unstable and would, in all probability, already have decayed to its true vacuum. Since this has clearly not happened, we may use the continued existence of the universe as proof that the theory is false.

In this chapter, we will investigate whether or not this actually does happen in the standard model, and if so what the consequences are. However, we will first consider the Weyl consistency conditions introduced in Sections 3.2 and 3.3.1, and the implications they have on the running of the standard model coupling constants.

4.2 The Weyl consistency conditions in the standard model

The central set of conditions we arrived at in Section 3.2, and which we will refer to as the Weyl consistency conditions, is, to leading order,

$$\frac{\partial \beta^j}{\partial g_i} = \frac{\partial \beta^i}{\partial g_j}, \qquad \beta^i \equiv \chi^{ij}\beta_j\,, \tag{4.8}$$

which relates the beta functions of different couplings to each other through the coupling space metric χ.

We now specialize these conditions to the important case of the standard model of particle interactions. The couplings we consider are the gauge couplings, the top-Yukawa and the quartic interaction of the Higgs field. Due to the nature of the perturbative corrections, it is convenient to redefine the coupling set $\{g_i\}$ as $\{\alpha_1, \alpha_2, \alpha_3, \alpha_t, \alpha_\lambda\}$, where

$$\alpha_1 = \frac{g_1^2}{(4\pi)^2}, \quad \alpha_2 = \frac{g_2^2}{(4\pi)^2}, \quad \alpha_3 = \frac{g_3^2}{(4\pi)^2}, \quad \alpha_t = \frac{y_t^2}{(4\pi)^2}, \quad \alpha_\lambda = \frac{\lambda}{(4\pi)^2}\,. \tag{4.9}$$

Here, g_1, g_2, g_3 are the $U(1)_Y$, $SU(2)_W$ and $SU(3)_c$ gauge couplings respectively. Similarly, we denote by $\beta_1, \beta_2, \beta_3, \beta_t$ and β_λ their respective beta functions, defined[1] as $\beta_i \equiv \mu^2 \frac{d\alpha_i}{d\mu^2}$. At leading order in the couplings, the matrix χ is diagonal, and reads [46]

$$\chi = \mathrm{diag}\left(\frac{1}{\alpha_1^2}, \frac{3}{\alpha_2^2}, \frac{8}{\alpha_3^2}, \frac{2}{\alpha_t}, 4\right)\,. \tag{4.10}$$

[1]Note that this convention is different from the one we introduced in Section 2.1.4. It is related to that one through $\mu^2 \frac{d\alpha_i}{d\mu^2} = \mu^2 \frac{d\mu}{d\mu^2}\frac{d\alpha_i}{d\mu} = \frac{1}{2}\mu\frac{d\alpha_i}{d\mu}$.

We find that after using the coupling space metric to raise the coupling space index, β^g features two fewer orders of α_g than the original β_g; the Yukawa β^t is related to β_t with one less power of α_t, while β^λ carries the same powers in α_λ as β_λ. We note that this would exactly cancel out the universal prefactors of α_g^2 and α_y that were present in Equations (3.19)-(3.21), the generic beta functions of a gauge-Yukawa theory.

The condition (4.8) therefore plays an important role, since it relates coefficients of different beta functions at different loop orders. Explicitly, the lowest order consistency conditions that we obtain are

$$2\frac{\partial}{\partial\alpha_t}\beta_\lambda = \frac{\partial}{\partial\alpha_\lambda}\left(\frac{\beta_t}{\alpha_t}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11a}$$

$$4\frac{\partial}{\partial\alpha_1}\beta_\lambda = \frac{\partial}{\partial\alpha_\lambda}\left(\frac{\beta_1}{\alpha_1^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11b}$$

$$\frac{4}{3}\frac{\partial}{\partial\alpha_2}\beta_\lambda = \frac{\partial}{\partial\alpha_\lambda}\left(\frac{\beta_2}{\alpha_2^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11c}$$

$$2\frac{\partial}{\partial\alpha_1}\left(\frac{\beta_t}{\alpha_t}\right) = \frac{\partial}{\partial\alpha_t}\left(\frac{\beta_1}{\alpha_1^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11d}$$

$$\frac{2}{3}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta_t}{\alpha_t}\right) = \frac{\partial}{\partial\alpha_t}\left(\frac{\beta_2}{\alpha_2^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11e}$$

$$\frac{1}{4}\frac{\partial}{\partial\alpha_3}\left(\frac{\beta_t}{\alpha_t}\right) = \frac{\partial}{\partial\alpha_t}\left(\frac{\beta_3}{\alpha_3^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11f}$$

$$\frac{1}{3}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta_1}{\alpha_1^2}\right) = \frac{\partial}{\partial\alpha_1}\left(\frac{\beta_2}{\alpha_2^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11g}$$

$$\frac{1}{8}\frac{\partial}{\partial\alpha_3}\left(\frac{\beta_1}{\alpha_1^2}\right) = \frac{\partial}{\partial\alpha_1}\left(\frac{\beta_3}{\alpha_3^2}\right) + \mathcal{O}\left(\alpha_i^2\right) , \tag{4.11h}$$

$$\frac{3}{8}\frac{\partial}{\partial\alpha_3}\left(\frac{\beta_2}{\alpha_2^2}\right) = \frac{\partial}{\partial\alpha_2}\left(\frac{\beta_3}{\alpha_3^2}\right) + \mathcal{O}\left(\alpha_i^2\right) . \tag{4.11i}$$

We can now proceed to test these relations for the standard model beta functions. We take them from Reference [95–97], and make the slight modification that we do not use the $SU(5)$ normalisation for the hypercharge:

$$\begin{aligned}
\beta_1 = 2\alpha_1^2\Bigg\{ & \frac{1}{12} + \frac{10n_G}{9} + \left(\frac{1}{4} + \frac{95n_G}{54}\right)\alpha_1 + \underbrace{\left(\frac{3}{4} + \frac{n_G}{2}\right)\alpha_2}_{Eq.\ (4.11g)} + \underbrace{\frac{22n_G}{9}\alpha_3}_{Eq.\ (4.11h)} \\
& + \left(\frac{163}{1152} - \frac{145n_G}{81} - \frac{5225n_G^2}{1458}\right)\alpha_1^2 + \left(\frac{87}{64} - \frac{7n_G}{72}\right)\alpha_1\alpha_2 \\
& - \frac{137n_G}{162}\alpha_1\alpha_3 + \left(\frac{3401}{384} + \frac{83n_G}{36} - \frac{11n_G^2}{18}\right)\alpha_2^2 \\
& + \left(\frac{1375n_G}{54} - \frac{242n_G^2}{81}\right)\alpha_3^2 - \frac{n_G}{6}\alpha_2\alpha_3 + \underbrace{\alpha_\lambda\left(\frac{3}{4}\alpha_1 + \frac{3}{4}\alpha_2 - \frac{3}{2}\alpha_\lambda\right)}_{Eq.\ (4.11b)} \\
& + \alpha_t\Bigg[\underbrace{-\frac{17}{12}}_{Eq.\ (4.11d)} - \frac{785}{64}\alpha_2 - \frac{29}{6}\alpha_3 + \left(\frac{113}{32} + \frac{101n_t}{16}\right)\alpha_t\Bigg]\Bigg\},
\end{aligned} \tag{4.12a}$$

$$\begin{aligned}
\beta_2 = 2\alpha_2^2\Bigg\{ & -\frac{43}{12} + \frac{2n_G}{3} + \underbrace{\left(\frac{1}{4} + \frac{n_G}{6}\right)\alpha_1}_{Eq.\ (4.11g)} + \left(-\frac{259}{12} + \frac{49n_G}{6}\right)\alpha_2 \\
& + \underbrace{2n_G\alpha_3}_{Eq.\ (4.11i)} + \left(\frac{163}{1152} - \frac{35n_G}{54} - \frac{55n_G^2}{162}\right)\alpha_1^2 + \left(\frac{187}{64} + \frac{13n_G}{24}\right)\alpha_1\alpha_2 \\
& - \frac{n_G}{18}\alpha_1\alpha_3 + \left(-\frac{667111}{3456} + \frac{3206n_G}{27} - \frac{415n_G^2}{54}\right)\alpha_2^2 \\
& + \frac{13n_G}{2}\alpha_2\alpha_3 + \left(\frac{125n_G}{6} - \frac{22n_G^2}{9}\right)\alpha_3^2 + \underbrace{\alpha_\lambda\left(\frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2 - \frac{3}{2}\alpha_\lambda\right)}_{Eq.\ (4.11c)} \\
& + \alpha_t\Bigg[\underbrace{-\frac{3}{4}}_{Eq.\ (4.11e)} - \frac{593}{192}\alpha_1 - \frac{729}{64}\alpha_2 - \frac{7}{2}\alpha_3 + \left(\frac{57}{32} + \frac{45n_t}{16}\right)\alpha_t\Bigg]\Bigg\},
\end{aligned} \tag{4.12b}$$

$$\begin{aligned}\beta_3 = 2\alpha_3^2 \Bigg\{ &-\frac{11}{2} + \frac{2n_G}{3} + \underbrace{\frac{11n_G}{36}\alpha_1}_{Eq.\ (4.11h)} + \underbrace{\frac{3n_G}{4}\alpha_2}_{Eq.\ (4.11i)} + \left(-51 + \frac{38n_G}{3}\right)\alpha_3 \\ &+ \left(-\frac{65n_G}{432} - \frac{605n_G^2}{972}\right)\alpha_1^2 - \frac{n_G}{48}\alpha_1\alpha_2 + \frac{77n_G}{54}\alpha_1\alpha_3 \\ &+ \left(\frac{241n_G}{48} - \frac{11n_G^2}{12}\right)\alpha_2^2 + \frac{7n_G}{2}\alpha_2\alpha_3 \\ &+ \left(-\frac{2857}{4} + \frac{5033n_G}{18} - \frac{325n_G^2}{27}\right)\alpha_3^2 \\ &+ \alpha_t\Big[\underbrace{-1}_{Eq.\ (4.11f)} -\frac{101}{48}\alpha_1 - \frac{93}{16}\alpha_2 - 20\alpha_3 + \left(\frac{9}{4} + \frac{21n_t}{4}\right)\alpha_t\Big]\Bigg\},\end{aligned} \tag{4.12c}$$

$$\begin{aligned}\beta_t = 2\alpha_t \Bigg\{ &\frac{9}{4}\alpha_t - \underbrace{4\alpha_3}_{Eq.\ (4.11f)} - \underbrace{\frac{17}{24}\alpha_1}_{Eq.\ (4.11d)} - \underbrace{\frac{9}{8}\alpha_2}_{Eq.\ (4.11e)} + \underbrace{3\alpha_\lambda^2 - 6\alpha_t\alpha_\lambda}_{Eq.\ (4.11a)} - 6\alpha_t^2 \\ &+ 18\alpha_3\alpha_t + \alpha_3^2\left(-\frac{202}{3} + \frac{40n_G}{9}\right) + \alpha_t\left(\frac{131}{32}\alpha_1 + \frac{225}{32}\alpha_2\right) \\ &+ \frac{1187}{432}\alpha_1^2 - \frac{3}{8}\alpha_1\alpha_2 + \frac{19}{18}\alpha_1\alpha_3 - \frac{23}{8}\alpha_2^2 + \frac{9}{2}\alpha_3\alpha_2\Bigg\},\end{aligned} \tag{4.12d}$$

$$\begin{aligned}\beta_\lambda = &\underbrace{\frac{9}{16}\alpha_2^2 - \frac{9}{2}\alpha_\lambda\alpha_2}_{Eq.\ (4.11c)} + \underbrace{\frac{3}{16}\alpha_1^2 - \frac{3}{2}\alpha_\lambda\alpha_1}_{Eq.\ (4.11b)} + \underbrace{\frac{3}{8}\alpha_1\alpha_2}_{Eqs.\ (4.11b\ \&\ 4.11c)} + 12\alpha_\lambda^2 \\ &+ \underbrace{6\alpha_\lambda\alpha_t - 3\alpha_t^2}_{Eq.\ (4.11a)}\ .\end{aligned} \tag{4.12e}$$

Here n_G is the number of generations, which we set to 3 in actual calculations, and n_t is the number of top quarks, i.e. 1. These parameters are kept general here to enable more detailed comparison with other calculations. Note that although we considered the gauge beta functions to three loops, we show only the two–loop top-Yukawa and the one–loop Higgs quartic beta functions. This, as we will demonstrate momentarily, leads to a Weyl consistent expansion in the couplings up to $O(\alpha_i^3)$.

To help the reader immediately identify the terms in the beta functions that must satisfy the Weyl consistency conditions given in Equation (4.11), we have color-coded the contributions. Furthermore, beneath each relevant term we have noted the equation number of the Weyl consistency condition it refers to. Note that the term $\frac{3}{8}\alpha_1\alpha_2$ in β_λ enters into both Equation (4.11b) and Equation (4.11c).

This illustrates that the one–loop coefficients of the quartic β_λ-function are related to the two–loop coefficient of the Yukawa β_t-function, and to the three–loop beta functions of the electroweak gauge couplings. Restricting the computation to these orders, namely adopting a 3-2-1 loop counting in the gauge, Yukawa and quartic beta functions, corresponds to a truncation of the function $\tilde{a}$ at order α_i^3. For illustration, we show the terms in the function $\tilde{a}$ which contribute to the one–loop quartic β_λ-function:

$$-\tilde{a} = \ldots + \underbrace{\frac{9}{4}\alpha_2^2\alpha_\lambda - 9\alpha_\lambda^2\alpha_2}_{Eq.\ (4.11c)} + \underbrace{\frac{3}{4}\alpha_1^2\alpha_\lambda - 3\alpha_\lambda^2\alpha_1}_{Eq.\ (4.11b)} + \underbrace{\frac{3}{2}\alpha_1\alpha_2\alpha_\lambda}_{Eqs.\ (4.11b\,\&\,4.11c)} + 16\alpha_\lambda^3 + \underbrace{12\alpha_\lambda^2\alpha_t - 12\alpha_t^2\alpha_\lambda}_{Eq.\ (4.11a)} + \ldots \tag{4.13}$$

4.3 A consistent perturbative expansion

When considering terms in the beta functions of higher order than the ones present in Equation (4.12), one implicitly includes terms of order α_i^4 or higher in $\tilde{a}$. For instance, let us study a typical two–loop term in the quartic beta function,

$$\beta_\lambda = \ldots + \frac{45}{4}\alpha_2\alpha_t\alpha_\lambda + \ldots . \tag{4.14}$$

It involves a term of the form $\alpha_2\alpha_t\alpha_\lambda^2$ in $\tilde{a}$[2], whose presence demands a term of order $\alpha_2\alpha_t\alpha_\lambda^2$ in β_t, which only appears at the three–loop level, and another of order $\alpha_2^2\alpha_t\alpha_\lambda^2$ in β_2, which is a four-loop term.[3] When truncating all beta functions to three loops, the absence of these terms explicitly violates the Weyl consistency conditions.

The central point presented in [4] and this chapter is that for any analysis requiring the running of multiple couplings, a consistent perturbative

[2]Note that NLO corrections in χ and w mean that several terms in $\tilde{a}$ will contribute to each beta function.

[3]It is important to note, however, that one cannot simply infer the form of these terms directly from Equation (4.14), since the metric χ^{ij} contains corrections of higher order in α_i, not shown in Equation (4.10). Some of these corrections have been computed in Reference [46].

expansion must be adopted in the function $\tilde{a}$, from which the counting of couplings in the various beta functions should then be derived. Truncating $\tilde{a}$ to order α_i^3 corresponds to the 3-2-1 counting mentioned above. Similarly, truncations at order α_i^4 or α_i^5 in $\tilde{a}$ yield respectively the 4-3-2 or 5-4-3 Weyl-consistent countings. If, for instance, the three–loop terms are included for the quartic beta function [98], then the 5-4-3 counting should be adopted. This requires an additional theoretical effort to compute the gauge and Yukawa beta functions to the corresponding order. The key point for any renormalization group analysis, as shown above, is that the beta functions are linked through $\tilde{a}$. This implies that any perturbative truncation made at the level of $\tilde{a}$ will be consistent. Conversely if the truncation is made at the level of the beta functions, unphysical features may well appear.

In [1], we performed an analysis of the toy model described in Section 1.2 to two loops in each of the beta functions, and did indeed find such unphysical features.

4.4 Vacuum stability analysis

The analysis of the vacuum stability requires knowledge of the effective potential of the model at hand. The standard model effective potential is known up to two loops [99], and its explicit form is given in the appendix of Reference [98, 99]. For large field values $\phi \gg v = 246$ GeV, the potential is very well approximated by its RG-improved tree-level expression,

$$V_{\rm eff}^{tree} = \frac{\lambda(\mu)}{4}\phi^4 \,, \tag{4.15}$$

where μ is on the order of ϕ. Therefore if one is simply interested in the condition of absolute stability of the potential, it is possible to study the RG evolution of λ and determine the largest scale $\Lambda < M_P$, with M_P the Planck scale, above which the coupling becomes negative.

We now compare the RG evolution of the standard model Higgs quartic coupling within the 3-2-1 Weyl consistent counting to the 3-3-3 counting.[4] The RG evolution of the standard model Higgs self interaction coupling in both counting schemes is shown in Figure 4.2.a, where we used the PDG value for the top mass $M_t = 173.5 \pm 1.4$ GeV [35] and the CMS measurement of the Higgs mass, $M_H = 125.7 \pm 0.6$ GeV [103]. We observe that in both counting schemes λ crosses zero at the scale $\Lambda \approx 10^{10}$

[4] For the 3-3-3 counting scheme we use the state-of-the-art three–loop standard model beta functions Refs.[95–97, 100–102].

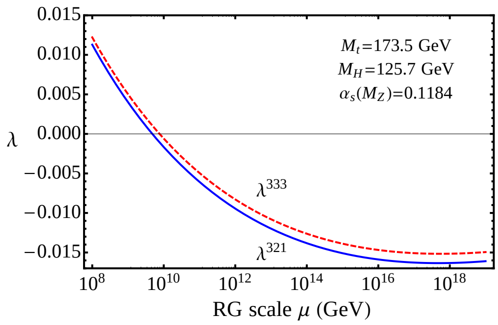


Figure 4.2.a

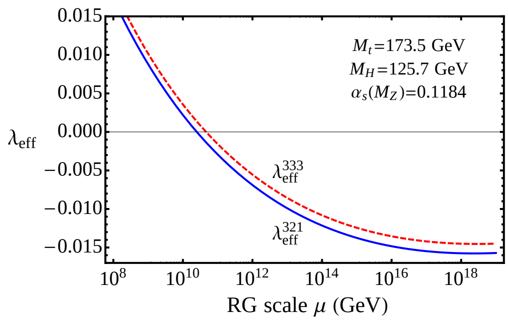


Figure 4.2.b

Figure 4.2: The RG evolution of the standard model Higgs quartic coupling (a) and effective coupling (b). In (a), λ_{333} (λ_{321}) shows the evolution of λ according to the 3-3-3 (3-2-1) scheme, and in (b) $\lambda^{333}_{\rm eff}$ ($\lambda^{321}_{\rm eff}$) shows the evolution of $\lambda_{\rm eff}$ according to the 3-3-3 (3-2-1) scheme.

GeV, although the crossing happens at a slightly lower scale in the 3-2-1 counting.

However, an accurate determination of the scale Λ has to take into account the full structure of the Higgs potential. As was shown in [104, 105], one can always define an effective coupling $\lambda_{\rm eff}$ such that for $\phi \gg v$ the effective potential assumes the form

$$V_{\rm eff} = \frac{\lambda_{\rm eff}(\mu)}{4}\phi^4 \, . \tag{4.16}$$

The explicit expression for $\lambda_{\rm eff}$, up to two–loop order, can be found in [98, 99]. Within the 3-2-1 counting scheme, we only consider $\lambda_{\rm eff}$ to one–loop order, to be consistent with the one–loop running of the quartic coupling. On the other hand in the 3-3-3 scheme we keep the full two–loop expression. The direct comparison between the running of the effective quartic couplings in the two schemes is shown in Figure 4.2.b. We note a very similar pattern to the one for λ given in Figure 4.2.a. The difference is that the scale where $\lambda_{\rm eff}$ crosses zero is roughly one order of magnitude larger, $\Lambda \approx 10^{11}$GeV.

We have also studied the possibility that the standard model is in a metastable vacuum that may in principle decay at a later time. However, if the time it takes for the vacuum to decay is longer than the lifetime of the universe, this is not of immediate concern. To illustrate the situation we have plotted the stability of the standard model as a function of the top and Higgs masses (see Figure 4.3). The criterion for stability is that the quartic coupling is positive at least all the way to the Planck scale. On the other hand, for metastability we must require that the probability (with certain standard approximations, see [106] for details) of the false vacuum decaying within the lifetime of the universe is less than one. This can be expressed as

$$\lambda(\phi) > -\frac{8\pi^2/3}{4\log[\phi T_U e^{\gamma_E}/2]}, \tag{4.17}$$

where T_U is the age of the universe and γ_E is the Euler-Mascheroni constant.

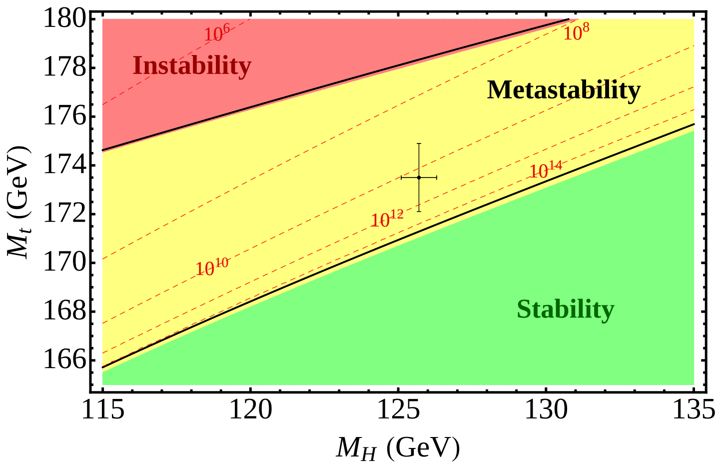


Figure 4.3: Standard model stability analysis based on the effective standard model Higgs quartic coupling. The red region indicates instability, the yellow metastability and the green absolute stability following the 3-2-1 counting. For comparison, the black lines indicate the bounds from the 3-3-3 counting. The point with error bars shows the experimental values of the top [35] and Higgs [103] masses. The red dashed lines show the value in GeV at which $\lambda_{\rm eff}^{321}$ crosses zero.

In addition to the vacuum stability analysis, we consider the case where the electroweak vacuum is the true ground state, but an unstable minimum exists at higher values of the Higgs field. The condition for such a second vacuum close to the point when $\lambda_{\rm eff}$ vanishes is the simultaneous vanishing of $\beta_{\rm eff} = d\lambda_{\rm eff}/d\ln\phi$ on the new minimum. Typically these two conditions are met by lowering the value of the top mass. To verify this possibility we show in the left and right panels of Figs. 4.4 the evolution of the quartic couplings, as done in Figs. 4.2.a and 4.2.b, but adopting a

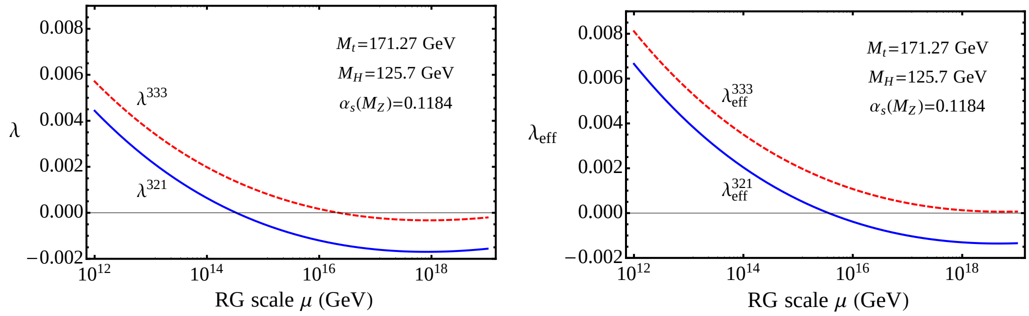


Figure 4.4: RG evolution of the (effective) standard model Higgs quartic coupling. The mass of the top is tuned such that for $\lambda_{\text{eff}}^{333}$ the potential develops a minimum at high energy, which is degenerate with the electroweak one.

lower value of the top mass, i.e. $M_t = 171.27$ GeV. It is clear from the picture, that for this value of the top mass and within the 3-3-3 counting scheme, the conditions for the existence of a second vacuum, degenerate in energy with the electroweak one, are met. Indeed, in the right panel of Figure 4.4 we observe that $\lambda_{\text{eff}}^{333}$ crosses zero at $\Lambda \approx 10^{19}$ GeV with a near zero slope, i.e. $\beta_{\text{eff}} \approx 0$. However, within the 3-2-1 counting scheme, the situation differs as $\lambda_{\text{eff}}^{321}$ crosses zero about three orders of magnitude earlier, with non-vanishing β_{eff}, for the same value of the top mass. We have to substantially lower the top mass to circa $M_t \approx 171.05$ GeV in this Weyl consistent scheme to accommodate the emergence of a degenerate minimum, giving a deviation of the order 2σ from the central value of the top mass.

CHAPTER 5

A STUDY OF RENORMALIZATION GROUP FLOWS

In this chapter, we will investigate the renormalization group (RG) flows of two closely related Yukawa theories at various loop orders in the beta functions. The purpose of this is to explore the limits of perturbation theory, and to illustrate just how desireable a well motivated counting scheme such as the Weyl consistent one presented in Section 3.3.1, and further explored in Chapter 4, is. Our investigations will closely follow the discussion in [5].

5.1 Renormalization group flow analysis

The dependence of the coupling constants in a quantum field theory on the Euclidean momentum scale μ, at which they are measured is of fundamental importance. As established prior in this thesis (see Section 2.1.4), this behavior is described by the beta functions for the couplings [62, 63, 107–110]. In a theory with two or more couplings, a change in μ thus induces a renormalization group flow in the space of couplings. The RG flow typically involves some infrared (IR) or ultraviolet (UV) fixed points, and one can characterize these as being attractive or repulsive along certain directions in the space of couplings. If the couplings are sufficiently small, then the respective beta functions can be reliably calculated perturbatively. As one or more of these couplings increases in magnitude, higher-loop contributions to the various beta functions become important, motivating calculations of these beta functions to higher loop order to obtain reliable results for RG flows and fixed points. If one or more couplings become too large, then it may not be possible to describe the RG flows, or indeed the properties of the theory, using perturbative calculations.

A general criterion for the reliability of a perturbative calculation is that if one calculates some quantity to a given loop order, then there should not be a large fractional change in this quantity if one computes it to one higher order in the loop expansion. Thus, in a situation where a putative fixed point occurs at moderately strong coupling, it is important to study how the value of the coupling(s) at this fixed point change(s) if one calculates the beta function(s) to higher loop order. For example, an asymptotically free non-Abelian gauge theory with sufficiently many fermions in a given representation has an IR fixed point (IRFP) [111, 112]. If the number of fermions is only slightly less than the maximum allowed by the constraint of asymptotic freedom, this IRFP occurs at weak coupling [72]. As the number of fermions is decreased, the IRFP moves to stronger coupling, and studies of the effect of higher-loop terms in the beta function of the gauge coupling have been carried out in this case [91, 113–118]. One may also investigate a possible ultraviolet fixed point (UVFP) in an infrared-free theory such as U(1) gauge theory with higher-loop calculations (e.g., [119–121] and references therein).

It is also of considerable interest to investigate renormalization group flows in the more complicated case of quantum field theories that depend on more than one interaction coupling. There have been many studies of such flows for theories and ranges of momentum scale μ where the couplings are reasonably weak, so that perturbative calculations are reasonably accurate. This is the case for computations of RG flows of the $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$ gauge couplings in the standard model or the minimal supersymmetric standard model from a reference scale of, say, 1 TeV, up to higher scales such as 10^{16} GeV. There has also been interest in calculating the RG flow of the elements of Yukawa matrices in the standard model and minimal supersymmetric standard model, and the quartic Higgs coupling λ_{SM} in the standard model, from the 1 TeV scale to higher scales. Again, these RG flows can be reasonably well described by perturbative calculations, although with the measured value of the Higgs-like boson observed by the LHC, $m_H \simeq 126$ GeV (whence in the standard model, $\lambda_{SM}(\mu) \simeq 0.13$ at $\mu = m_H$), in the absence of new physics effects at intermediate scales, it follows that $\lambda_{SM}(\mu)$ would decrease through zero at a high scale $\mu \sim 10^{10\pm1}$ GeV, implying that the standard model, by itself, would be metastable above this scale [4, 93, 95, 96, 98, 101, 122–135]. See also the analysis in the preceding chapter, in particular Figure 4.2.

Here, we will study renormalization group flows in Yukawa theories and assess the reliability of perturbative calculations of these flows for a substantial range of Yukawa and quartic scalar couplings. The method that we use for this purpose is to compare the properties of flows that we obtain with the beta functions of these couplings calculated to different orders in the loop expansion. To better focus on the essential features in as simple a framework as possible, we study scalar-fermion models without any gauge fields. We construct these models so that the global symmetries forbid any Dirac or Majorana fermion mass terms, and we also consider the limit where scalar masses are negligibly small relative to the scales of interest, μ. These models depend on two dimensionless couplings, a quartic self-coupling λ for the scalar field and a Yukawa coupling y. The beta functions for these couplings comprise a set of coupled first-order ordinary differential equations describing how the couplings vary as functions of μ. Integrating this set of differential equations, we determine their renormalization group flows as functions of μ. To do this, we choose an initial scale, μ_0, where the magnitudes of the couplings are sufficiently small that perturbative calculations may be reliable, and then perform the integration. Our method is to compare RG flows calculated using different loop orders for the two beta functions. We recall the basic fact that in these theories, the quartic scalar self-coupling λ must be positive for the boundedness of the energy and equivalently the stability of the theory. As will be evident in our results, RG flows may take a theory with positive λ to one with negative λ. In this case, a comment is necessary. Strictly speaking, for a sufficiently small range of negative λ the theory may still be metastable, with a sufficiently long tunneling time that our perturbative calculations may be physically meaningful. However, for negative values of λ of sufficiently large magnitude, the theory is simply unstable, and the perturbative analysis is not applicable or meaningful. In most of our analytic discussions, therefore, we will implicitly take λ to be positive.

We remark on some earlier related work on Yukawa models. As is well known, Yukawa proposed such models [136] as an approach to understanding the binding of nucleons in nuclei, and pion exchange between nucleons does, indeed, play an important role in this binding. Of course, the physics here involves the exchange of a light approximate Nambu-Goldstone boson between two baryons, with the baryons being much heavier than the exchanged π meson, as indicated by the ratio of masses $m_\pi/m_N \approx 0.15$. This is quite different from our models, for which, by construction, a global chiral symmetry forbids any fermion mass and the scalar mass is taken to be negligibly small relative to the interval of Euclidean momentum scales μ for which we integrate the beta functions to calculate the RG flows. Some early studies of perturbative RG equations for standard model Yukawa couplings included Refs. [68, 137–140]. It was recognized early on that the one–loop beta function for a scalar theory without fermions is positive, therefore such a theory is, perturbatively, IR-free; that is, as $\mu \to 0$, $\lambda(\mu) \to 0$. However, it was also recognized that if one adds fermions to this scalar theory to get a full scalar-fermion Yukawa theory, then the fermions contribute a negative term proportional to y^4 in the beta function $\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$, and hence, for sufficiently large y, this can reverse the sign of the full one–loop term in this beta function and hence possibly render the scalar coupling in the Yukawa theory nontrivial [137–140]. This motivated fully nonperturbative investigations, and these were carried out using lattice studies with dynamical fermions [141–152] (some recent work includes [153–155]). One may obtain a Yukawa theory starting from a full gauge-fermion-Higgs theory by turning off the gauge couplings. In this framework, a natural approach is to start with a chiral gauge theory (exemplified by the standard model), which forbids bare fermion masses in the Lagrangian. However, owing to fermion doubling on the lattice, it has been challenging to implement chiral gauge theories in lattice calculations. We believe, therefore, that there is continuing interest in pursuing analyses of renormalization group evolution of continuum Yukawa theories using perturbatively calculated beta functions. Indeed, simple scalar-fermion models have been of recent interest in studies of quasi-scale invariant behavior (e.g., [1, 19, 156]; see also [2, 4, 135, 157]).

5.1.1 Example models

In this chapter, we will regard two closely related models. Both are ungauged Yukawa theories featuring two fermion fields and a scalar field. The first will be invariant under a global $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ symmetry, and the second under a global $\mathrm{SU}(N) \otimes \mathrm{SU}(N_f) \otimes \mathrm{U}(1)$ symmetry. In the former, we will use the Lagrangian level coupling constants y and λ, and later absorb the loop factors into them (see Equation (2.41)). However, for the latter model, it will be convenient to absorb a factor of N into each of the rescaled couplings so that we may take the limit of infinite N and N_f, and we will refer to these as $\bar{a}_y$ and $\bar{a}_\lambda$ (see Equation (5.34)).

5.2 Beta Function details

As discussed above, the beta functions in a simple Yukawa theory form a set of two coupled differential equations. We integrate these for each of the two models that we study below to calculate the resultant RG flows. A point in the multidimensional space of couplings where all of the beta functions vanish simultaneously is, formally, a renormalization group fixed point. In general, RG flows may include the presence of one or more ultraviolet fixed point(s) if the beta functions vanish as $\mu \to \infty$ and/or infrared fixed point(s), where the beta functions vanish as $\mu \to 0$. In general, a fixed point may be stable along some directions and unstable along others. If the particle content of the theory does not change along the RG flow from the reference scale μ_0 to the fixed point, then it is an exact UV or IR fixed point. In the vicinity of a (formal) fixed point, the RG flows are slow, so that the theories exhibit approximate scale-invariance.

For our comparative study, we will perform the integrations to compute the RG flows with the beta functions β_{a_y} and β_{a_λ} calculated to various different loop orders. Here, we will use the notation introduced in Section 2.2. For the SU(2)$\otimes$U(1) model, the calculation to n and k loops, using the $\beta_{a_y,n\ell}$ and $\beta_{a_\lambda,k\ell}$ beta functions respectively is denoted (n,k). The specific cases for which we perform the integrations are

- (1,1), i.e., $\beta_{a_y,1\ell}$ and $\beta_{a_\lambda,1\ell}$
- (1,2), i.e., $\beta_{a_y,1\ell}$ and $\beta_{a_\lambda,2\ell}$
- (2,1), i.e., $\beta_{a_y,2\ell}$ and $\beta_{a_\lambda,1\ell}$
- (2,2), i.e., $\beta_{a_y,2\ell}$ and $\beta_{a_\lambda,2\ell}$

We will use the same notation to describe the four equivalent cases for the SU(N) $\otimes$ SU(N_f) $\otimes$ U(1) model, so that in this context, the case (1,1) refers to an RG calculation using $\beta_{\bar{a}_y,1\ell}$ and $\beta_{\bar{a}_\lambda,1\ell}$ and so forth for the other cases. Some remarks are in order here. For a perturbative calculation of quantities in a theory with multiple couplings, the naïve procedure would be to calculate to similar orders in the various couplings if they are equally large and significant for the physics, and to calculate to higher order in a coupling that is larger. Thus, for example, in a standard model process, one may only need to calculate to lowest order in electroweak couplings, but to higher order in the QCD coupling.

However, from a pure field theory point of view, the correct way to run multiple couplings simoultanenously is to use the beta functions$\beta_{a_g,(n+2)\ell}$, $\beta_{a_y,(n+1)\ell}$, and $\beta_{a_\lambda,n\ell}$, where g denotes a gauge coupling and $a_g \equiv g^2/(4\pi)^2 =$

$\alpha/(4\pi)$. This follows from the Weyl consistency conditions [47] and was described in detail in Sections 3.2 and 3.3.1 (see also Refs. [4, 135, 158]). In the present analysis, this means that if we desired to make accurate predictions about the behavior of the theory, we would use the (2,1) counting scheme. However, since the purpose of the present chapter is to study the consequences of including different loop orders, we will not stress this issue further.

In this type of study there are several obvious caveats. First, clearly, as couplings increase in strength, perturbative calculations become progressively less reliable. This is, indeed, a motivation for our present work - to assess quantitatively where this reduction in reliability occurs in the case of scalar-fermion models depending on two coupling constants. Second, higher-loop terms in beta functions of multi-coupling theories are generically scheme-dependent, and the positions of fixed points are hence also scheme-dependent. Indeed, scheme dependence is also present in higher-loop calculations in quantum chromodynamics (QCD). As in common practice in QCD, we use results computed with the $\overline{\text{MS}}$ scheme [159, 160]. One can assess the effect of scheme dependence of RG flows and fixed points by comparing these in different schemes [91, 113–118]. However, many scheme transformations that are acceptable in the vicinity of a fixed point at zero coupling (e.g., a UVFP in an asymptotically free gauge theory, or an IRFP in an infrared-free theory) are not acceptable at a fixed point that occurs at a moderately strong coupling, because they produce various unphysical pathologies [120, 121, 161–163]. A third caveat, related to the first, is that if one or more of the couplings is (are) sufficiently large, the Yukawa and/or quartic scalar self-interaction may lead to nonperturbative phenomena such the formation of a fermion condensate, a vacuum expectation value (vev) for the scalar field, and/or fermion-fermion bound states[1] (see, e.g., [2, 157]). In the case where the coefficient of the quadratic term in the scalar potential V is zero, there is the related possibility of a nonperturbative generation of a nonpolynomial term in V, whose minimum could lead to a nonzero vev for the scalar field [166, 167]. Early studies of the stability of a theory in the presence of this phenomenon and associated related bounds on fermion and Higgs masses include [137–140, 168–171].

[1]There have been many studies of nonrelativistic bound states due to Yukawa interactions, but these are not directly relevant to our work, since our models are constructed to be invariant under global chiral symmetries and hence to avoid any bare fermion mass terms, so that the fermions are ultrarelativistic. Some explorations of possible relativistic fermion-fermion bound states resulting from a strong Yukawa interaction include [164, 165].

If fermion condensation occurs at some scale μ_c in the vicinity of a formal IR fixed point, then the originally massless fermions gain dynamical masses, spontaneously breaking the approximate scale invariance in the theory near to an apparent RG fixed point. In the low-energy effective field theory applicable for scales $\mu < \mu_c$, one integrates these fermions out, thereby obtaining different beta functions. Thus, in this case, the formal fixed point would only be approximate rather than exact, since after the fermion condensation, the beta functions and flows would be different. This spontaneous symmetry breaking of the approximate scale invariance generically leads to the appearance of a corresponding Nambu-Goldstone boson, the dilaton. This dilaton is not massless, since the beta functions in the vicinity of the fixed point were small, but not precisely zero.

If $\mu_\phi^2 < 0$, so that there is a vev for the scalar field, then the Yukawa coupling leads to a mass for the fermion field(s) of the form $m_f \propto yv$. However, since the vev $v = (-\mu_\phi^2/\lambda)^{1/2}$ and since we assume that $|\mu_\phi|$ is much smaller than the reference scales μ over which we integrate the renormalization group equations, it follows that for moderate values of the ratio y^2/λ, the resultant fermion masses $m_f \propto y(-\mu_\phi^2/\lambda)^{1/2}$ are negligible relative to the interval of μ that we study.

5.3 SU(2) ⊗ U(1) Model

5.3.1 Field Content and Symmetry Group

The first model that we study is motivated by the leptonic sector of the standard model, with the gauge interactions turned off. It includes a fermion ψ_L^a which is a doublet under SU(2) with weak hypercharge Y_ψ and a χ_R, which is a singlet under SU(2) with weak hypercharge Y_χ, together with the usual scalar field ϕ^a transforming as a doublet under SU(2) with weak hypercharge Y_ϕ. Here, $a = 1, 2$ is an SU(2) group index which will often be suppressed in the notation. We assume that these hypercharges are nonzero and that $Y_\psi \neq Y_\chi$. Since we have set the gauge couplings to zero, the SU(2) ⊗ U(1) is a global symmetry group. As in the standard model, we set

$$Y_\phi = Y_\psi - Y_\chi \tag{5.1}$$

to ensure that the Yukawa interaction term is invariant under the global symmetry. The Lagrangian for this model is

$$\mathcal{L} = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\chi}_R i\partial\!\!\!/ \chi_R - [y\bar{\psi}_L\chi_R\phi + h.c.] \\ + \partial_\mu\phi^\dagger \partial^\mu\phi - \mu_\phi^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \,. \quad (5.2)$$

Without loss of generality, we can make $y(\mu_0)$ real and positive at a given value μ_0 (by changing the phase of ψ_L or χ_R or ϕ). We assume that this is done. We allow μ_ϕ^2 of either sign, but assume that $|\mu_\phi^2|$ is negligibly small compared with the range of μ^2 of interest for our study of RG flows[2] (see also the end of Section 5.2). The global SU(2) symmetry forbids the Majorana bilinear $\psi_L^{a\,T}C\psi_L^b$ and the Dirac bilinear $\bar{\psi}_{a,L}\chi_R$ from occurring in $\mathcal{L}$. Since Y_χ is taken to be nonzero, the U(1) symmetry forbids the Majorana bilinear $\chi_R^TC\chi_R$ (as well as $\psi_L^{a\,T}C\psi_L^b$ and $\bar{\psi}_{a,L}\chi_R$ bilinears). Thus, the condition that $\mathcal{L}$ be invariant under this global symmetry group implies that the fermions are massless.

5.3.2 Beta Functions

The one–loop and two–loop coefficients in the beta functions β_y and β_λ can be extracted, with the requisite changes to match our normalizations, from previous calculations (which were done in the $\overline{\text{MS}}$ scheme) [66–68, 71, 93, 95, 96, 99, 101, 172–175] (see also Section 2.2). They are

$$b_y^{(1)} = \frac{5}{2}y^3 \quad (5.3)$$

$$b_y^{(2)} = 3y(-y^4 - 4y^2\lambda + 2\lambda^2) \quad (5.4)$$

$$b_\lambda^{(1)} = 2(12\lambda^2 + 2y^2\lambda - y^4) \quad (5.5)$$

$$b_\lambda^{(2)} = -312\lambda^3 - 48y^2\lambda^2 - y^4\lambda + 10y^6 \,. \quad (5.6)$$

[2] The purpose of this assumption of negligibly small m_ϕ compared with the range of μ of interest for our RG flows is to ensure that the ϕ field is dynamical; if m_ϕ were $\gg \mu$ for the values of μ of interest, then we could integrate it out, obtaining a low-energy effective field theory consisting of just the fermions ψ and χ with a resultant four-fermion operator $\propto (1/m_\phi^2)\sum_a[\bar{\psi}_{a,L}\chi_R][\bar{\chi}_R\psi_L^a] + h.c.$.

In terms of the variables a_y and a_λ used for the figures,

$$b^{(1)}_{a_y} = 5a_y^2 \tag{5.7}$$

$$b^{(2)}_{a_y} = 6a_y(-a_y^2 - 4a_y a_\lambda + 2a_\lambda^2) \tag{5.8}$$

$$b^{(1)}_{a_\lambda} = 2(12a_\lambda^2 + 2a_y a_\lambda - a_y^2) \tag{5.9}$$

$$b^{(2)}_{a_\lambda} = -312a_\lambda^3 - 48a_y a_\lambda^2 - a_y^2 a_\lambda + 10a_y^3 \ . \tag{5.10}$$

We comment on some properties of β_y or equivalently, β_{a_y}. We recall that at the initial point μ_0 where we start our integrations of the renormalization group equations, we have, with no loss of generality, rendered y real and positive. A first comment is that because β_y has an overall factor of y, and β_{a_y} has an overall factor of a_y, it follows that the flow in y can never take y through zero to negative values of y, and the flow in a_y can never take a_y through zero to negative values of a_y.

The fact that $b^{(1)}_{a_y} > 0$ means that for sufficiently small a_y and a_λ, $\beta_{a_y} > 0$, i.e., as μ decreases from the UV to the IR, the Yukawa coupling y decreases as well. At the two–loop level,

$$b^{(2)}_{a_y} > 0 \quad \text{if} \quad a_\lambda > (1 + \sqrt{3/2}\,)a_y = 2.2247a_y \ , \tag{5.11}$$

to the given floating-point accuracy. If these conditions are satisfied, then the two–loop coefficient contributes to β_{a_y} with the same sign as the one–loop coefficient and increases the rate of change of a_y as a function of μ. If, on the other hand $a_\lambda < (1 + \sqrt{3/2}\,)a_y$, then $b^{(2)}_{a_y} < 0$, so $b^{(2)}_{a_y}$ contributes to β_{a_y} with a sign opposite to that of $b^{(1)}_{a_y}$. In this case, it is possible for β_{a_y} to vanish at the two–loop level. The condition for this to happen is that either $a_y = 0$ for some μ or (suppressing the argument, μ) that

$$5a_y + 6(-a_y^2 - 4a_y a_\lambda + 2a_\lambda^2) = 0 \ . \tag{5.12}$$

Solving this equation for a_y yields the physical solution

$$a_y = \frac{5}{12} - 2a_\lambda + \frac{1}{12}\sqrt{864a_\lambda^2 - 240a_\lambda + 25} \ . \tag{5.13}$$

(The polynomial in the square root is positive-definite.) Equivalently, solving Equation (5.12) for a_λ yields

$$a_\lambda = a_y + \frac{1}{6}\sqrt{3a_y(18a_y - 5)} \ , \tag{5.14}$$

which is physical if $a_y \geq 5/18$, i.e., $y \geq (4\pi/3)\sqrt{5/2} = 6.623$. Evidently, this zero of $\beta_{a_y,2\ell}$ is only possible for such large values of y that one must anticipate significant corrections from higher-loop terms in β_{a_y}. In passing, we note that the other solution of Equation (5.12) for λ with a minus sign in front of the square root is unphysical, since it can lead to a negative λ. (As noted before, we do not attempt to consider a metastable situation with a negative λ of small magnitude.) Also, the other solution of Equation (5.12) for a_y with a minus sign in front of the square root in Equation (5.13) is unphysical because it can lead to a value of $a_y < 5/18$. Setting $a_y = 5/18$ in Equation (5.14) yields $a_\lambda = a_y = 5/18$, and similarly, setting $a_\lambda = 5/18$ in Equation (5.13) yields $a_y = a_\lambda = 5/18$.

We next remark on some properties of β_{a_λ}. We find that

$$b^{(1)}_{a_\lambda} = 0 \quad \text{if} \quad a_\lambda = \frac{(\sqrt{13}-1)}{12}\, a_y = 0.21713 a_y \tag{5.15}$$

and

$$b^{(1)}_{a_\lambda} > 0 \quad \text{if} \quad a_\lambda > \frac{(\sqrt{13}-1)}{12}\, a_y \,, \tag{5.16}$$

or equivalently, $a_y < (1+\sqrt{13}\,)a_\lambda = 4.60555 a_\lambda$. The condition that $b^{(2)}_{a_\lambda} = 0$ is a cubic equation in a_λ and separately a cubic equation in a_y. We find that if $a_y = (1+\sqrt{13}\,)a_\lambda$, such that $b^{(1)}_\lambda = 0$, then

$$b^{(2)}_{a_\lambda} = \frac{2(13+55\sqrt{13})}{(4\pi)^6}\,\lambda^3 = (1.073\times 10^{-4})\lambda^3 \,, \tag{5.17}$$

which is clearly positive.

In the special case where $a_y = 0$, we find that if we consider $\beta_{a_\lambda,2\ell}$, a non-trivial fixed point appears at

$$a^*_\lambda = \frac{1}{13} = 0.076923 \,. \tag{5.18}$$

This fixed point is repulsive in the a_y-direction, since for lower values of a_λ (while keeping $a_y = 0$), $b^{(1)}_{a_\lambda}$ drives the flow down, and for higher, $b^{(2)}_{a_\lambda}$ drives it up.

We next give some illustrative numerical evaluations. Let us consider that the theory is such that at some reference scale μ_0, $y(\mu_0)$ and $\lambda(\mu_0)$ have the values $y(\mu_0) = 1$ and $\lambda(\mu_0) = 1$. If one were to consider turning on gauge fields (and adding quarks so that this theory is free of gauge anomalies), then these would be rather large physical values of these couplings. For reference, considering only the third generation in the standard model and using the relation for a fermion mass in terms of the Yukawa coupling and the Higgs vacuum expectation value, $\langle\phi\rangle_0$, namely

$$y_f\langle\phi\rangle_0 = y_f\frac{v}{\sqrt{2}} = m_f \ , \tag{5.19}$$

where $v = 246$ GeV, one has the rough values $y_\tau \simeq 1\times 10^{-2}$, $y_b \simeq 2\times 10^{-2}$, and $y_t \simeq 1$. Further, using the relation for the Higgs boson mass m_H in the standard model, namely,

$$m_H = (2\lambda)^{1/2}v \tag{5.20}$$

one has $\lambda(\mu) = 0.13$ at $\mu = m_H = 126$ GeV, as noted above. So the illustrative reference values $y(\mu_0) = \lambda(\mu_0) = 1$ that we have taken may be considered to be reasonably large. Nevertheless, the variables that enter in the beta functions are then rather small because they involve a factor of $1/(4\pi^2)$; $a_y(\mu) = \lambda(\mu) = 1/(4\pi)^2 = 0.6333\times 10^{-2}$. In the beta function β_{a_y}, the one–loop term $b^{(1)}_{a_y} = 2.005\times 10^{-4}$, and the two–loop term term $b^{(2)}_{a_y} = -0.4571\times 10^{-5}$, so that the ratio of the two–loop to one–loop terms is

$$y = \lambda = 1 \ \Rightarrow\ \frac{b^{(2)}_{a_y}}{b^{(1)}_{a_y}} = -0.02280 \ . \tag{5.21}$$

In the beta function β_{a_λ}, the one–loop term $b^{(1)}_{a_\lambda} = 1.043\times 10^{-3}$ and the two–loop $b^{(2)}_{a_\lambda} = -0.89135\times 10^{-4}$, such that

$$y = \lambda = 1 \ \Rightarrow\ \frac{b^{(2)}_{a_\lambda}}{b^{(1)}_{a_\lambda}} = -0.0855 \ . \tag{5.22}$$

We also note the values of the one–loop and two–loop beta functions for a_y and a_λ:

$$y = \lambda = 1 \ \Rightarrow\ \frac{\beta_{a_y,1\ell}}{\beta_{a_\lambda,1\ell}} = \frac{b^{(1)}_{a_y}}{b^{(1)}_{a_\lambda}} = 0.1923 \tag{5.23}$$

and

$$y = \lambda = 1 \ \Rightarrow\ \frac{\beta_{a_y,2\ell}}{\beta_{a_\lambda,2\ell}} = \frac{b^{(1)}_{a_y} + b^{(2)}_{a_y}}{b^{(1)}_{a_\lambda} + b^{(2)}_{a_\lambda}} = 0.2055 \tag{5.24}$$

Thus, for this illustrative case with $y(\mu_0) = \lambda(\mu_0) = 1$, the two–loop term in β_{a_y} makes only a small contribution relative to the one–loop term, so that the perturbative expansion for β_{a_y} is reasonably reliable to this two–loop order, and similarly for β_{a_λ}.

5.3.3 RG Flows

To study the RG flows in this model, we begin by finding the fixed points, that is the solutions to the simultaneous conditions $\beta_{a_y,n\ell} = 0$, $\beta_{a_\lambda,k\ell} = 0$ for the values of loop orders (n,k) that we consider. We first note that the IR-free (trivial) fixed point

$$a_y^* = 0, \quad a_\lambda^* = 0\,, \tag{5.25}$$

is a solution to the beta functions for any of our (n,k) cases. Beyond this IR-free fixed point, we find that the choice of loop order (n,k) in the beta functions is quite important for the appearance and location of fixed points. From Equations (5.7)-(5.10), we calculate the fixed point to be as follows:

$$\text{case } (1,1) \;\Rightarrow\; \text{no nonzero fixed points.} \tag{5.26}$$

$$\text{case } (1,2) \;\Rightarrow\; a_y^* = 0, \quad a_\lambda^* = \frac{1}{13} = 0.07692. \tag{5.27}$$

$$\begin{aligned} \text{case } (2,1) \Rightarrow a_y^* &= \frac{5}{318}(13\sqrt{13} - 17) = 0.4697, \\ a_\lambda^* &= \frac{5}{638}(31 - 5\sqrt{13}) = 0.1020\,. \end{aligned} \tag{5.28}$$

$$\text{case } (2,2) \;\Rightarrow\; \text{two fixed points :}$$

$$a_y^* = 0, \quad a_\lambda^* = \frac{1}{13} = 0.07692 \;\text{ and} \tag{5.29}$$

$$a_y^* = 0.4104, \quad a_\lambda^* = 0.1247\,. \tag{5.30}$$

The presence of a fixed point for such a low value of a_λ as $1/13$ means that only a very small region of coupling space is independent of the choice of (n,k). In Figure 5.1, we see that the flows change character based on (n,k) when both a_y and a_λ are larger than approximately 0.04. In this and the other figures, our convention is to start the analysis at a high value of μ in the UV, integrate the renormalization group equations for a_y

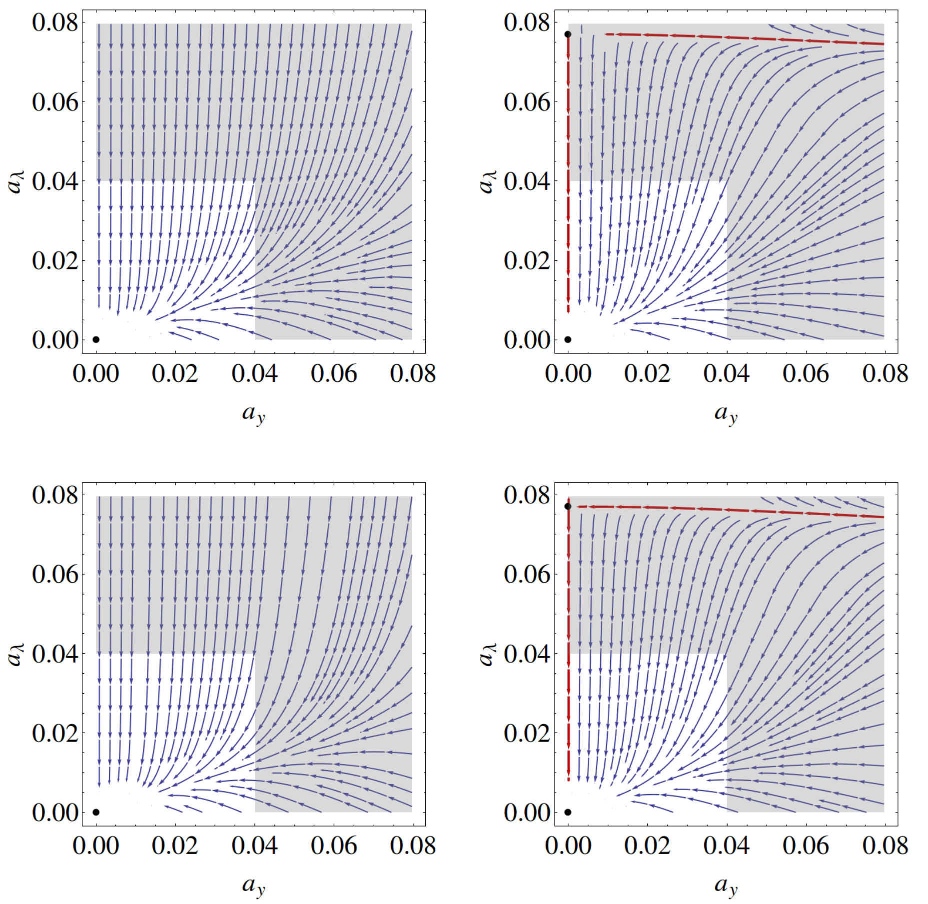


Figure 5.1: The renormalization group flows for the SU(2) ⊗ U(1) model with $0 < a_y < 1/(4\pi)$ and $0 < a_\lambda < 1/(4\pi)$. In this and the other figures, the arrows for the flows point in the direction from the UV to the IR. The white square region is where $0 < a_y < 0.04$ and $0 < a_\lambda < 0.04$, and the gray region occupies the rest of the plot. The figures correspond to the following different choices of loop order in the beta functions: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). The red flows for the cases (1,2) and (2,2) originate along the eigendirections of the fixed points.

and a_λ, and follow the flow from the UV to the IR, and this is indicated by the direction of the arrows. Note, in particular, that the plots where the two–loop term $b^{(2)}_{a_\lambda}$ is included in β_{a_λ} have, in the upper-right hand area, concave flows towards the trivial fixed point, whereas the ones where it is not have convex flows towards the same in this region.

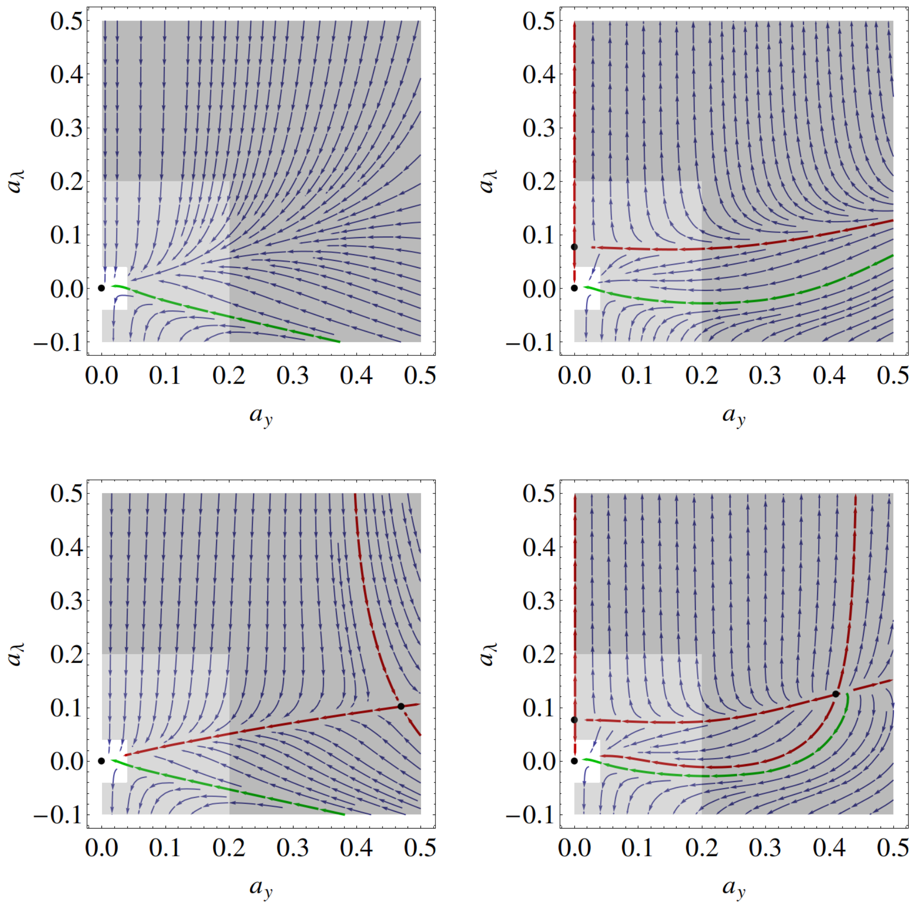


Figure 5.2: The renormalization group flows for the $SU(2) \otimes U(1)$ model with $0 < a_y < 0.5$ and $-0.1 < a_\lambda < 0.5$. The white region is where $0 < a_y < 0.04$ and $0 < |a_\lambda| < 0.04$; the light gray region is where $0.04 < a_y < 0.2$ and $0.04 < |a_\lambda| < 0.2$; and the dark gray region occupies the rest of the figure. The figures correspond to the following different choices of loop order in the beta functions: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). The green flows are the stable manifolds in coupling constant space which bound the basins of attraction of the fixed point at the origin. The red flows in (1,2), (2,1) and (2,2) originate along the eigendirections of the fixed points.

If we let a_y and a_λ increase beyond $1/(4\pi)$, changes appear quite rapidly (see Figure 5.2), which means that one cannot trust the perturbative analysis to these orders in this region of couplings. With this caveat in mind, we shall proceed to describe the RG flows. The first striking difference is that if the two–loop term $\beta^{(2)}_{a_\lambda}$ in the beta function β_{a_λ} is included, then the flow ending in the partially attractive fixed point at $a^*_y = 0, a^*_\lambda = 1/13$ is a separatrix which divides a region where the flows end in the trivial fixed point at the origin, from one where they increase to large values of a_λ. The plots in this and the other figures were generated using the `Mathematica` StreamPlot routine. (Because the integration routine can lose some numerical accuracy when the beta functions approach zero near fixed points, it does not show arrows and associated RG flows very close to these fixed points.)

The second is that including the two–loop term in the Yukawa beta function produces a fixed point where neither of the couplings is zero. However, the impact that this has on the flow is very different in the (2,1) and (2,2) cases. In the (2,1) case, the fixed point is partially attractive, and the flow that reaches it from above forms a separatrix, separating a region where the flows end at the origin from a region where they move toward larger values of a_y in the IR. In the (2,2) case, the fixed point is totally repulsive, and the dominant term in the beta functions is the a_λ^3 term in equation (5.10). This term drives every flow, above the one originating in the eigendirection of positive a_y from the fixed point (marked in red on Figure 5.2), towards larger a_λ in the IR, which in turn means that the dominant term in $\beta_{a_y,2\ell}$ will eventually be the $a_y a_\lambda^2$ term, which drives $a_y \to 0$ in the IR.

For the (2,2) flows that originate at the totally repulsive fixed point and go in the direction of negative a_λ, there is a delicate balance between terms driving them towards the origin and terms driving them towards highly negative a_λ in the IR. This balance is manifested in the stable manifold (marked in green on Figure 5.2) which separates the regions of convergence to the origin and flow to (unphysical) negative values.

5.4 SU(N) $\otimes$ SU(N_f) $\otimes$ U(1) Model

5.4.1 *Field Content, Symmetry Group, and LNN Limit*

In this section, we study a model that is a two-fold generalization of the model in the previous section. First, we construct the model so that it is invariant under a global symmetry group

$$G = \mathrm{SU}(N) \otimes \mathrm{SU}(N_f) \otimes \mathrm{U}(1)\,, \tag{5.31}$$

rather than the SU(2) $\otimes$ U(1) group of the previous model. We include an N_f-fold replication of the left-handed and right-handed fermions. The fermion content consists of (i) $\psi^a_{j,L}$, transforming as a $(\Box,\Box)$ representation of $SU(N)\otimes SU(N_f)$, where a is an $SU(N)$ group index taking on the values $a = 1,...,N$, and j is a flavor index, taking on the values $j = 1,...,N_f$; and (ii) $\chi_{j,R}$, with $j = 1,...,N_f$, transforming as a $(1,\Box)$ representation of $SU(N)\otimes SU(N_f)$. The model also has a scalar field ϕ^a transforming as a $(\Box,1)$ representation of $SU(N)\otimes SU(N_f)$. The hypercharges are again taken to be nonzero and to satisfy the conditions that $Y_\psi \neq Y_\chi$ and Equation (5.1). The transformations of $\psi^a_{j,L}$ and $\chi_{j,R}$ under $SU(N_f)$ are

$$\psi^a_{j,L} \to \sum_{k=1}^{N_f} U_{jk}\psi^a_{k,L}\,, \quad \chi_{j,R} \to \sum_{k=1}^{N_f} U_{jk}\chi_{k,R} \tag{5.32}$$

where $U \in SU(N_f)$.

The Lagrangian of this model is

$$\begin{aligned}\mathcal{L} = \sum_{j=1}^{N_f}\left[\bar\psi_{j,L} i\partial\!\!\!/\psi_{j,L} + \bar\chi_{j,R} i\partial\!\!\!/\chi_{j,R}\right] - y\sum_{j=1}^{N_f}[\bar\psi_{j,L}\chi_{j,R}\phi + h.c.] \\ + \partial_\mu\phi^\dagger\,\partial^\mu\phi - \mu_\phi^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,,\end{aligned} \tag{5.33}$$

where we have suppressed $SU(N)$ indices in the notation. The $SU(N)\otimes U(1)$ symmetry forbids the fermion bilinears $\psi^{a\,T}_{j,L}C\psi^b_{k,L}$, $\chi^T_{j,R}C\chi_{k,R}$, and $\bar\psi_{a,j,L}\chi_{k,R}$, so the fermions are massless. Our requirement of $SU(N_f)$ invariance restricts the Yukawa coupling to the form given in Equation (5.33). As before, we allow either sign of μ^2_ϕ and impose the condition that $|\mu_\phi|$ be negligibly small relative to the range of μ over which we calculate the RG flows (see also the end of Section 5.2).

One of the motivations for this generalization is that it enables us to take the combined limit

$$\begin{aligned}&N\to\infty\,, \quad N_f\to\infty \;\text{ with } r\equiv\frac{N_f}{N} \text{ fixed, and}\\ &y\to 0\,, \quad \lambda\to 0 \quad \text{with}\\ &\bar a_y = \frac{yN}{(4\pi)^2}\,, \quad \bar a_\lambda = \frac{\lambda N}{(4\pi)^2} \quad \text{finite.}\end{aligned} \tag{5.34}$$

We will use the symbol $\lim_{LNN}$ for this limit[3], where "LNN" stands for "large N and N_f."

[3]Note that this limit is equivalent with what we referred to as the Veneziano limit in Section 3.4.

5.4.2 Beta Functions

To simplify the analysis, we take the LNN limit (5.34). In this limit, from [66, 67, 71, 99, 172–175] (see also [93, 95, 96, 101]) we find

$$b^{(1)}_{\bar{a}_y} = (1+2r)\bar{a}_y^2 \tag{5.35}$$

$$b^{(2)}_{\bar{a}_y} = -3r\bar{a}_y^3 \tag{5.36}$$

$$b^{(1)}_{\bar{a}_\lambda} = 2(2\bar{a}_\lambda^2 + 2r\bar{a}_y\bar{a}_\lambda - r\bar{a}_y^2) \tag{5.37}$$

and

$$b^{(2)}_{\bar{a}_\lambda} = r\bar{a}_y(-8\bar{a}_\lambda^2 - 3\bar{a}_y\bar{a}_\lambda + 2\bar{a}_y^2)\ . \tag{5.38}$$

We remark on some general properties of these terms. First, because $\beta_{\bar{a}_y}$ has an overall factor of $\bar{a}_y$, it follows that the flow in $\bar{a}_y$ can never take $\bar{a}_y$ through zero to negative values of $\bar{a}_y$. For $y \neq 0$, the one–loop term in $\beta_{\bar{a}_y}$, namely $b^{(1)}_{\bar{a}_y}$, is positive-definite and independent of $\bar{a}_\lambda$. Hence, provided that the initial values of y and λ at the starting point of the integration are such that one can apply these perturbative calculations, $\bar{a}_y$ decreases toward zero as μ decreases from the UV to the IR. Since for $y \neq 0$, the two–loop term, $b^{(2)}_{\bar{a}_y}$, is negative, it follows that the full two–loop beta function, $\beta_{\bar{a}_y,2\ell} = \bar{a}_y^2[(1+2r) - 3r\bar{a}_y]$ has a zero, which occurs at

$$\bar{a}_y^* = \frac{1+2r}{3r}\ , \tag{5.39}$$

independently of $\bar{a}_\lambda$. For weaker Yukawa couplings, i.e., $\bar{a}_y < \bar{a}_y^*$, $\beta_{\bar{a}_y,2\ell} > 0$, so the UV to IR flow is to still weaker Yukawa couplings, while for $\bar{a}_y > \bar{a}_y^*$, $\beta_{\bar{a}_y,2\ell} < 0$, so that the direction of the UV to IR flow is to larger $\bar{a}_y$. Note that as r decreases toward 0, $\bar{a}_y^*$ gets sufficiently large that we cannot trust the perturbative calculations, so this discussion is restricted to moderate values of r. These results are shown in Figure 5.3. For the range of r shown in Figure 5.3, $\bar{a}_y^* \sim 1$. As is evident from Equation (5.39), as $r \to \infty$, $\bar{a}_y$ approaches the limit 2/3 from above.

We next discuss the one–loop and two–loop terms in $\beta_{\bar{a}_\lambda}$. The analysis here is more complicated than that for $\beta_{\bar{a}_y}$, because whereas the one–loop and two–loop terms in $\beta_{\bar{a}_y}$ depended only on $\bar{a}_y$, the one–loop and two–loop terms in $\beta_{\bar{a}_\lambda}$ depend on both $\bar{a}_\lambda$ and $\bar{a}_y$. We find that the one–loop term $b^{(1)}_{\bar{a}_\lambda}$ is positive (negative) if $\bar{a}_\lambda$ is larger (smaller) than the value

$$\bar{a}_\lambda = \frac{1}{2}\Big[-r + \sqrt{r(r+2)}\,\Big]\bar{a}_y \tag{5.40}$$

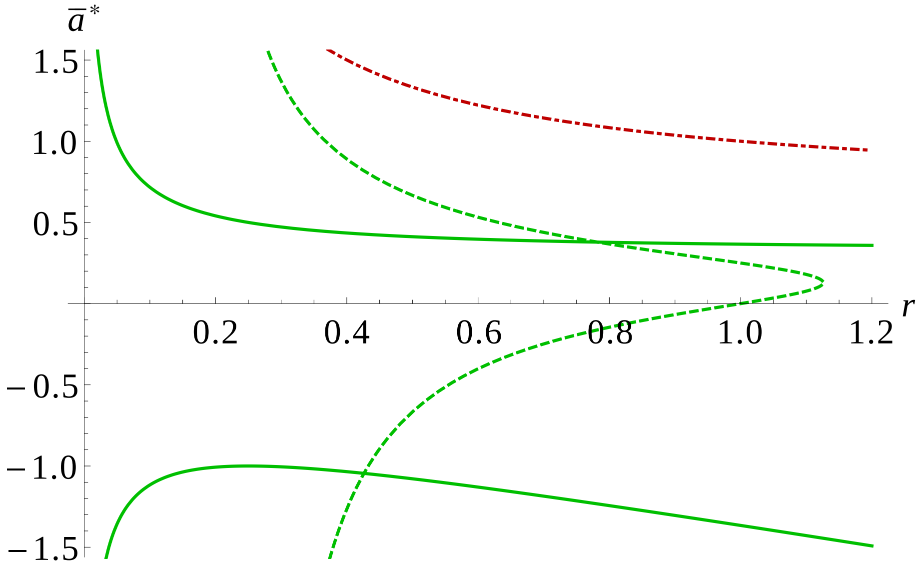


Figure 5.3: The fixed point values of (i) $\bar{a}_y$, denoted as $\bar{a}_y^*$ and shown as the red, dot-dashed curve, and (ii) $\bar{a}_\lambda$, denoted as $\bar{a}_\lambda^*$ and shown as the green solid curve for the case (2,1) and green dashed curve for the (2,2) case, plotted as functions of $r = N_f/N$ (with the LNN limit understood). The curve for $\bar{a}_y^*$ is the same for the (2,1) and (2,2) cases, since, as discussed in the text, $\beta_{\bar{a}_y}$ is independent of $\bar{a}_\lambda$ to two–loop order. The curves with $\bar{a}_\lambda$ negative are only formal, since the theory is unstable for $\bar{a}_\lambda < 0$, i.e., $\lambda < 0$.

and zero if the equality in Equation (5.40) holds. The condition in Equation (5.40) is equivalent to $\bar{a}_y = [1 + \sqrt{1 + (2/r)}\,]\bar{a}_\lambda$. The solution for $\bar{a}_\lambda$ in Equation (5.40) is one of the two solutions of the quadratic equation $b^{(1)}_{\bar{a}_\lambda} = 0$; the solution with the minus sign in front of the square root is unphysical because it leads to a negative λ, and similarly in the equivalent solution for $\bar{a}_y$, the other root with the minus sign in front of the square root is unphysical. The fact that $b^{(1)}_{\bar{a}_\lambda} > 0$ for $\bar{a}_\lambda$ larger than the value on the right-hand side of Equation (5.40) means that if the initial value of $\bar{a}_\lambda$ satisfies this condition, then along the RG flow from the UV to the IR, $\bar{a}_\lambda$ decreases, and similarly, if the initial value of $\bar{a}_\lambda$ is smaller than the value on the right-hand side of Equation (5.40), then $\bar{a}_\lambda$ increases along the RG flow from the UV to IR.

We come next to the two–loop term in $\beta_{\bar{a}_\lambda}$, namely $b^{(2)}_{\bar{a}_\lambda}$. Because this factorizes into a linear times a quadratic factor in the LNN limit that we consider here, it is somewhat simpler to analyze than $b^{(2)}_{a_\lambda}$ for the

SU(2) ⊗ U(1) model. We find that $b^{(2)}_{\bar{a}_\lambda}$ is negative (positive) if $\bar{a}_\lambda$ is larger (smaller) than the value

$$\bar{a}_\lambda = \frac{1}{16}(-3 + \sqrt{73}\,)\,\bar{a}_y = 0.34650\bar{a}_y\ . \tag{5.41}$$

(The solution of the quadratic with the opposite sign in front of the square root is unphysical, since it renders λ negative.) The two–loop term $b^{(2)}_{\bar{a}_\lambda}$ vanishes if $\bar{a}_y = 0$ or if the condition in Equation (5.41) is satisfied. Thus, for large $\bar{a}_\lambda$ relative to $\bar{a}_y$, at least to the extent that our perturbative calculations still apply, we thus find that the one–loop and two–loop terms in the $\beta_{\bar{a}_\lambda,2\ell}$ have the opposite signs; $b^{(1)}_{\bar{a}_\lambda} > 0$, while $b^{(2)}_{\bar{a}_\lambda} < 0$. Similarly, for sufficiently small $\bar{a}_\lambda$ relative to $\bar{a}_y$, these terms again have opposite signs; $b^{(1)}_{\bar{a}_\lambda} < 0$, while $b^{(2)}_{\bar{a}_\lambda} > 0$. It is thus plausible that the full two–loop $\beta_{\bar{a}_\lambda,2\ell}$ would have a zero, where these terms cancel each other.

In Figure 5.3 we show our solutions for the value of the fixed point in the variable $\bar{a}_\lambda$ as a function of r. (Here and elsewhere, it is implicitly understood that the LNN limit has been taken.) The value of r determines the value of the fixed point in $\bar{a}_y$, the existence or non-existence of a fixed point in $\bar{a}_\lambda$, and, in the former case, its value. The solutions that yield a fixed point $\bar{a}^*_\lambda$ at negative values are only formal, since the theory is unstable for $\bar{a}_\lambda < 0$, i.e., $\lambda < 0$. If $\bar{a}_\lambda$ is negative but $|\bar{a}_\lambda|$ is sufficiently small, the theory may be metastable, but considerations of metastability and estimates of tunneling times are beyond the scope of our present analysis (see instead Section 4.4 for that analysis within the context of the standard model). Thus, as regards $\bar{a}_\lambda$, there is only a single physical fixed point, $\bar{a}^*_\lambda$, and the calculation for the (2,1) case yields a value of $\bar{a}^*_\lambda \simeq 0.5$ in the range of r shown, for which perturbation theory may be reliable down to $r \simeq 0.2$. As $r \to \infty$, this curve for $\bar{a}^*_\lambda$ approaches the limit 1/3. For the (2,2) case, if $r < 1$, there is also only one physical (positive) fixed point, $\bar{a}^*_\lambda$, but its value grows more rapidly as r decreases, so we anticipate significant corrections to the two–loop perturbative result already for r decreasing below $r \simeq 0.4$. In the narrow interval of r between $r = 1$ and the value

$$r^{(2,2)}_{merger} = \frac{31 + 12\sqrt{3}}{46} = 1.12575\ , \tag{5.42}$$

there are two physical fixed points for $\bar{a}_\lambda$. We shall refer to these as the upper and lower fixed points. As r increases through the value $r^{(2,2)}_{merger}$, the upper and lower fixed points in $\bar{a}_\lambda$ merge and disappear. This is exactly the point where the solution to the equation $\beta_{\bar{a}_y,2\ell} = \beta_{\bar{a}_\lambda,2\ell} = 0$ becomes complex.

5.4.3 RG Flows

Here we present the results of our integration of the beta functions calculated to various loop orders. In Figure 5.4, we plot the RG flows for $r = 0.5$ and

$$\bar{a}_y < \frac{1}{4\pi}\,, \quad \bar{a}_\lambda < \frac{1}{4\pi}\,, \qquad i.e., \qquad \frac{y^2N}{4\pi} < 1\,, \quad \frac{\lambda N}{4\pi} < 1\,. \tag{5.43}$$

We find that for this value of r and range of $\bar{a}_y$ and $\bar{a}_\lambda$, the theory has only the IR fixed point at the IR-free point

$$(\bar{a}_y^*, \bar{a}_\lambda^*) = (0, 0)\,. \tag{5.44}$$

This can be understood as a result of the fact that the one–loop expression for β_{a_y}, namely, $\beta_{a_y,1\ell}$, is positive and independent of a_λ, so as μ decreases from the UV to the IR, $\bar{a}_y$ always decreases. Although the one–loop result for $\beta_{\bar{a}_\lambda}$, namely $\beta_{\bar{a}_\lambda,1\ell}$, could initially be negative if the initial value of $\bar{a}_y$ is such that $\bar{a}_y > (1+\sqrt{13}\,)\bar{a}_\lambda$, as discussed above, $\beta_{\bar{a}_\lambda,1\ell}$ will eventually pass through zero and become positive as $\bar{a}_y$ decreases through this zero, and as the flow continues toward the IR thereafter, $\beta_{\bar{a}_\lambda,1\ell}$ will remain positive. This causes $\bar{a}_\lambda$ to vanish in the IR.

These results also provide an answer to a question that we posed at the beginning, namely how robust the perturbative calculation of the RG flows are to the inclusion of higher-loop terms in the beta function. For this range (5.43) of $\bar{a}_y$ and $\bar{a}_\lambda$, all four cases (1,1), (1,2), (2,1), and (2,2) yield qualitatively similar flows. This serves as a strong indication that for this range (5.43), our perturbative calculations are reliable.

Next, we increase r from 0.5 to 1.1. The results are shown in Figure 5.5. We reach the same qualitative conclusions for this case $r = 1.1$ as for $r = 0.5$.

We then study a larger range of $\bar{a}_y$ and $\bar{a}_\lambda$, namely $0 < \bar{a}_y < 1.5$ and $0 < \bar{a}_\lambda < 1.5$. We show the RG flows for $r = 0.5$ and $r = 1.1$ in Figs. 5.6 and 5.7.

For reference, in these plots we distinguish three regions: (i) a white square region where $0 < \bar{a}_y < 1/(4\pi)$ and $0 < \bar{a}_\lambda < 1/(4\pi)$; (ii) a light gray region where $1/(4\pi) < \bar{a}_y < 1$ and $1/(4\pi) < \bar{a}_\lambda < 1$ $\big(1/(4\pi) < \bar{a}_y < 0.75$ and $1/(4\pi) < \bar{a}_\lambda < 0.75$ in Figure 5.7$\big)$; and (iii) a dark gray region where $1 < \bar{a}_y < 1.5$ and $1 < \bar{a}_\lambda < 1.5$ $\big(0.75 < \bar{a}_y < 1.5$ and $0.75 < \bar{a}_\lambda < 1.5$ in Figure 5.7$\big)$. In the case where $r = 0.5$ (Figure 5.6), the four light gray regions are still quite similar, but now the inclusion of the two–loop term

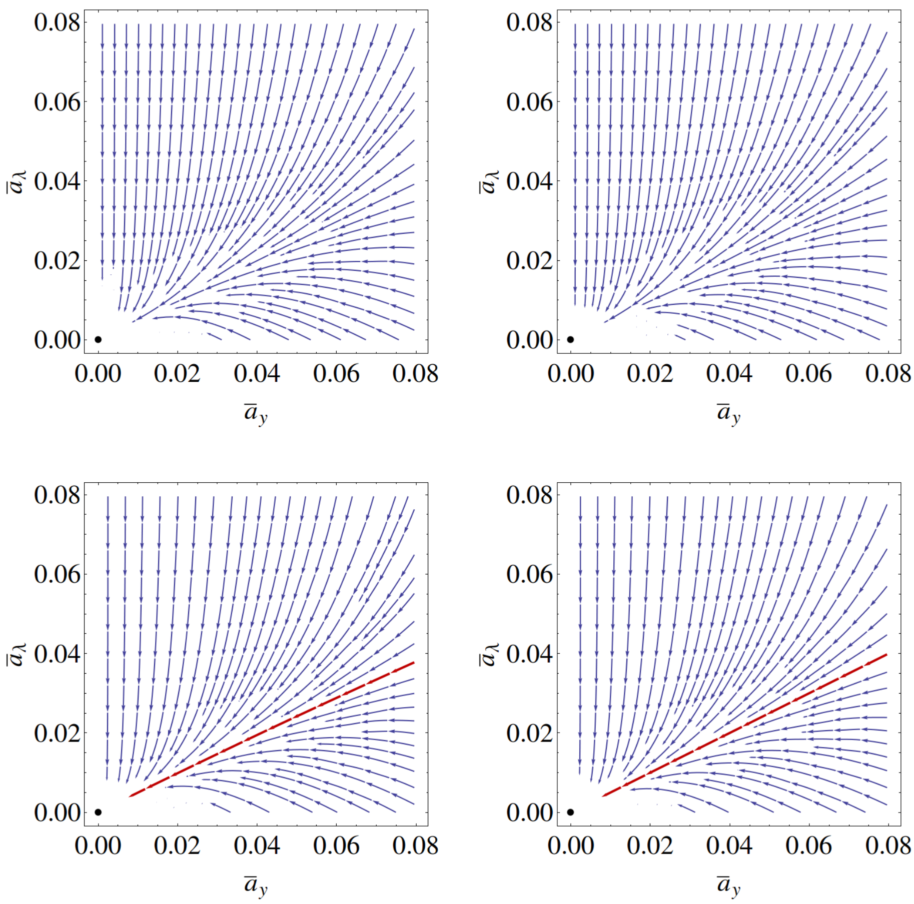


Figure 5.4: The renormalization group flows for $r = 0.5$ with $0 < \bar{a}_y < 1/(4\pi)$ and $0 < \bar{a}_\lambda < 1/(4\pi)$. The figures correspond to the following choices of inclusion of different-loop terms in the beta functions: upper left: (1,1); upper right: (1,2); lower left: (2,1); lower right: (2,2). The red flows in the (2,1) and (2,2) cases originate along the eigendirection of the upper fixed point (see Figure 5.3).

in $\beta_{\bar{a}_\lambda}$ has a significant effect. In the left-hand plots where this term is not included, we note that the flows that reach the fixed points seem to be attracted to a central flow, which, in the (2,1) (lower left) plot is identified with the one flowing in the eigendirection from the upper fixed point. In the right-hand plots that include the two–loop term in $\beta_{\bar{a}_\lambda}$, this behavior is reversed for relatively large values of $\bar{a}_y$, and they are instead repulsed by this line. In (1,1) and (2,1) cases, the RG flows in the light gray region where $\bar{a}_y < 1$ and $\bar{a}_\lambda < 1$, look similar to the flows in the white square

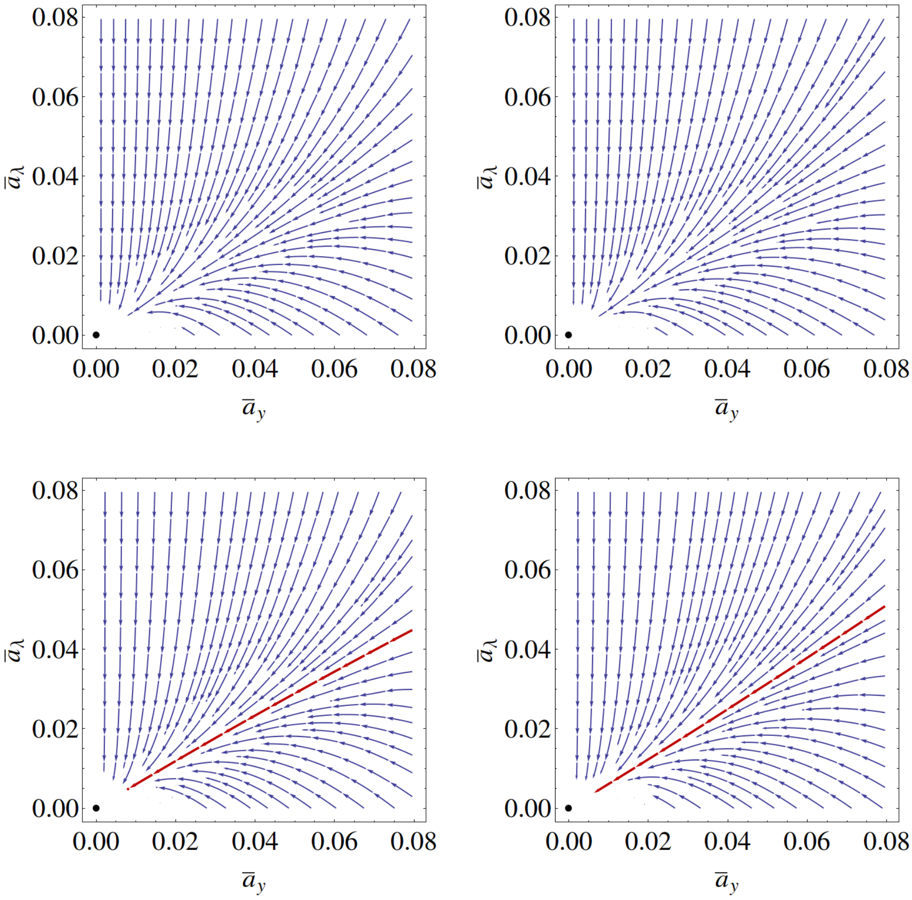


Figure 5.5: The renormalization group flows for $r = 1.1$ with $0 < \bar{a}_y < 1/(4\pi)$ and $0 < \bar{a}_\lambda < 1/(4\pi)$. The figures correspond to the following choice of inclusion of different loop-order terms in the beta functions: upper left: (1,1); upper right: (1,2); lower left: (2,1); lower right: (2,2). The red flows in the (2,1) and (2,2) cases originate along the eigendirection of the upper fixed point (see Figure 5.3).

region where $\bar{a}_y < 1/(4\pi)$ and $\bar{a}_\lambda < 1/(4\pi)$.

The largest changes in the flows occur in the dark gray area where $\bar{a}_y$ and $\bar{a}_\lambda$ are largest. When considering this region, it is important to recall that this is where we expect perturbation theory to break down, partly because higher-order terms in the beta functions are of comparable size compared with lower-order terms, and partly because completely nonperturbative effects such as fermion condensates can appear for such strong values of the couplings. However, continuing in the context of

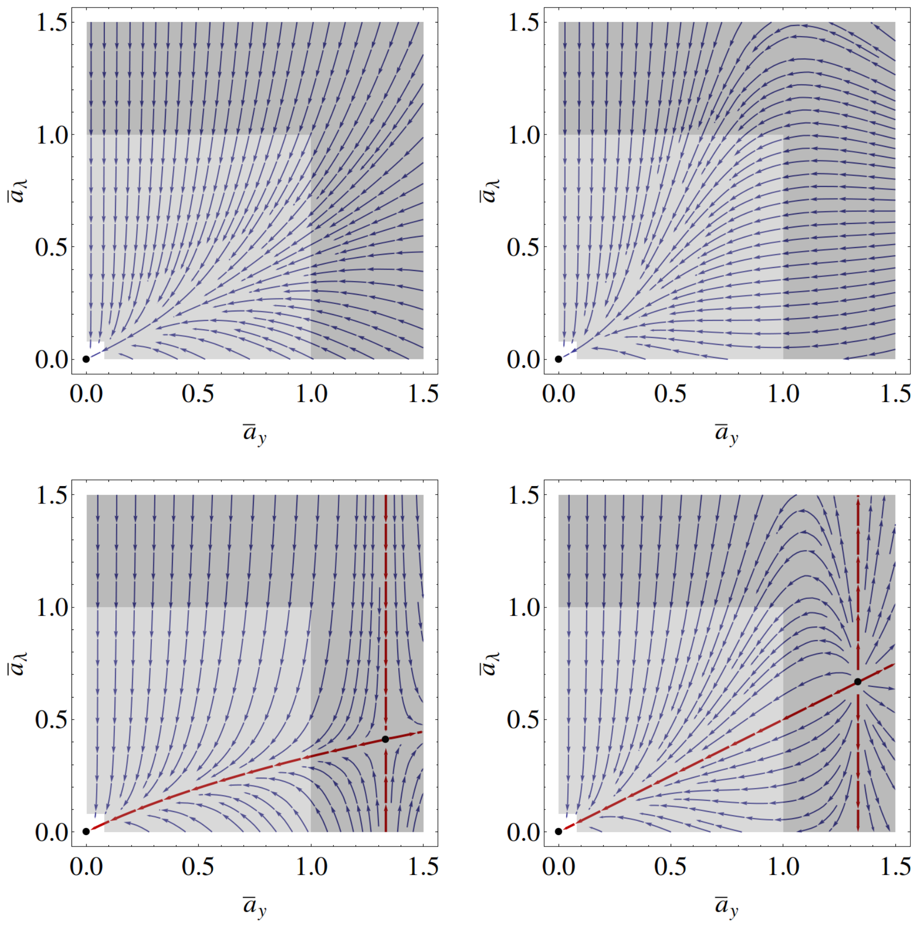


Figure 5.6: The renormalization group flows for $r = 0.5$ with $0 < \bar{a}_y < 1.5$ and $0 < \bar{a}_\lambda < 1.5$. The white square region is where $0 < \bar{a}_y < 1/(4\pi)$ and $0 < \bar{a}_\lambda < 1/(4\pi)$; the light gray region is where $1/(4\pi) < \bar{a}_y < 1$ and $1/(4\pi) < \bar{a}_\lambda < 1$; and the dark gray region occupies the rest of the plot. The figures correspond to the following choices of inclusion of different loop-order terms in the beta functions: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). The red flows in (2,1) and (2,2) originate along the eigendirections of the fixed points.

the perturbative analysis, we see that fixed points appear in the (2,1) and (2,2) plots, and correspondingly the flows are changed by their presence.

The inclusion of the two–loop term in $\beta_{\bar{a}_\lambda}$ fundamentally changes the nature of the fixed points. In the (2,1) plot, we see that the non-trivial fixed point is attractive along the vertical direction, and repulsive along the approximately horizontal direction, but the fixed point in the (2,2) case

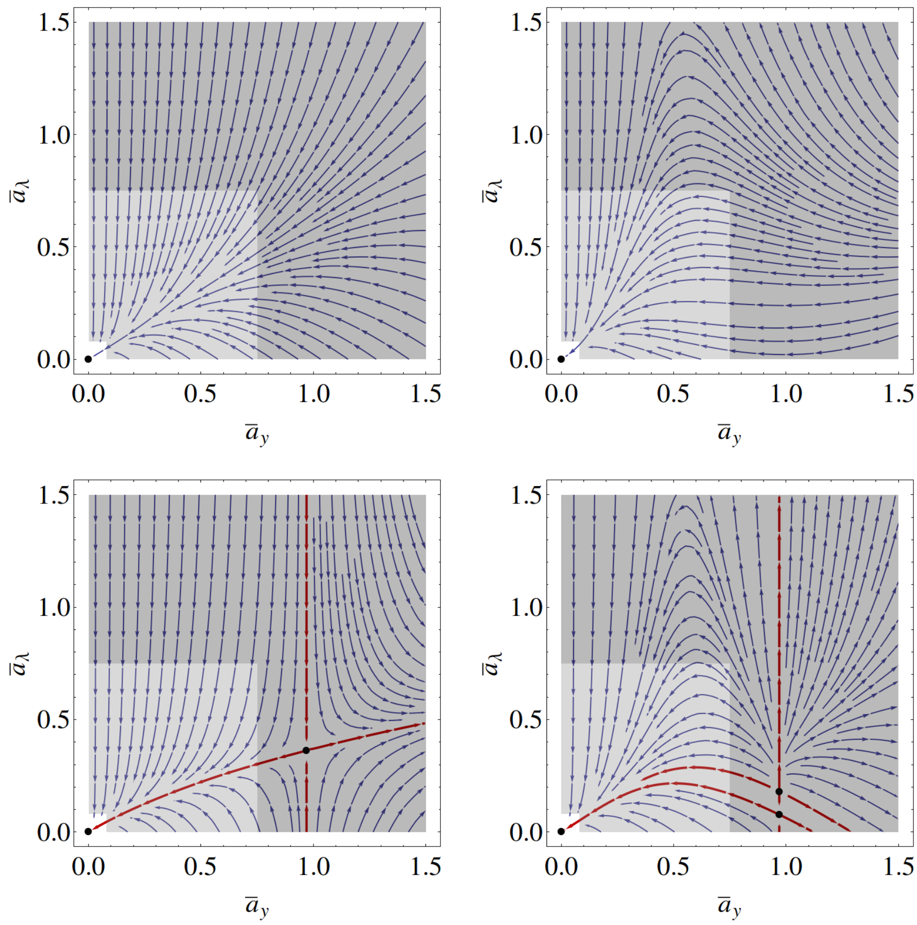


Figure 5.7: The renormalization group flows for $r = 1.1$ with $0 < \bar{a}_y < 1.5$ and $0 < \bar{a}_\lambda < 1.5$. The white square region is where $0 < \bar{a}_y < 1/(4\pi)$ and $0 < \bar{a}_\lambda < 1/(4\pi)$; the light gray region is were $1/(4\pi) < \bar{a}_y < 0.75$ and $1/(4\pi) < \bar{a}_\lambda < 0.75$; and the dark gray occupies the rest of the plot. The figures correspond to the following choices of inclusion of different loop-order terms in the beta functions: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). The red flows in (2,1) and (2,2) originate along the eigendirections of the fixed points.

occurs at a roughly similar position, it is now repulsive in all directions.

In Figure 5.7, we note that (1,1), (1,2), and (2,1) plots are similar to those in Figure 5.6, except that the fixed point in the (2,1) plot now occurs at a value of $\bar{a}_y < 1$. However, in the (2,2) plot, the flows are very different. Most dramatically, the lower fixed point (see Figure 5.3) has become positive, and is very close to merging with the upper one.

Thus, our comparative calculations of RG flows for these (1,1), (1,2), (2,1), and (2,2) cases in this model show that a perturbative calculation of the RG flows and fixed points is reasonably reliable for the region $0 < \bar{a}_y \lesssim 1/(4\pi)$ and $0 < \bar{a}_\lambda \lesssim 1/(4\pi)$ but is unreliable when these variables increase to sizes of order 1 or greater.

Conclusion

Quantum field theory is a very large subject which can be realized in a myriad of different ways. For the purpose of describing Nature, all experimental evidence points to the standard model of particle physics, a non-supersymmetric four dimensional gauge-Yukawa theory. In this thesis, we have elucidated several aspects of this subject in a general manner and applied some of them to the standard model itself, leading to modified predictions for its very high energy behavior.

The early part of our work was devoted to introducing not only the framework of gauge-Yukawa theory, but also the two example models that are used in the majority of this thesis. These are the standard model, where we put a special emphasis on several aspects of the Higgs field and its associated particle. With the basic setup established, we proceeded to describe the concept of renormalization in quantum field theory, with an emphasis the importance of renormalization group flows and the beta function.

This allowed us to discuss a new method for making calculations in any particular gauge-Yukawa theory. Much previous work has been done to establish an expansive literature of calculations done in completely general gauge-Yukawa theories, in particular the beta functions have been computed up to three loops in the gauge coupling, and two loops in the Yukawa and quartic couplings. Our work provides a straightforward procedure for applying these formulas to any particular theory. This should be especially helpful for beyond the standard model theories where a complicated new physics sector is often present.

We then proceeded to discuss one of the deepest issues of current studies of quantum field theory. The conjectured *a* theorem in its strong form stipulates that there should exist a function which is monotonic along renormalization group flows. If true, this would immediately ensure that the renormalization group flows are also irreversible and would put strong constraints on what is possible to achieve within the framework of quantum field theory. In its weak form, it merely states that there exists a function which is always larger at a high energy fixed point than at a low energy one. We showed that the weak form holds true in a specific theory, even in a case where none of the fixed points under consideration were Gaussian in nature.

However, the most important lesson that we drew from our study of the a theorem came from its relation to the Weyl consistency conditions. These imply a set of highly non-trivial relations between the beta functions of a theory with multiple couplings, such as the standard model which features three gauge couplings, a top quark Yukawa coupling (and several other subleading ones), and a quartic scalar self-interaction. In order to satisfy these conditions at lowest order, the beta functions of such a theory must be run at three loops in the gauge coupling, two in the Yukawa and one in the quartic.

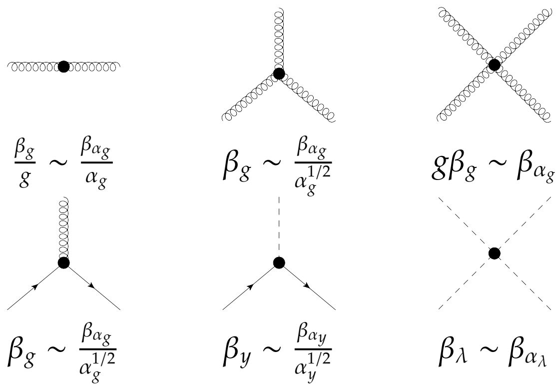
Using these conditions, we went on to explore the stability of the standard model vacuum, and found that the Weyl consistency conditions imply that the standard model is less stable than the naïve running of three loops in each beta function would have led us to believe.

Partly inspired by the discovery of the implications of the Weyl consistency conditions, we moved on to perform a study of the renormalization group flow within a simple ungauged Yukawa theory. Our procedure was to study how the renormalization group flowed as we changed the number of loops to which the beta function under consideration is calculated. We found that this can have very significant implications, and thus that having definite conditions that dictate the orders we should consider is very helpful.

Appendix A

Trace anomaly diagrams

The structure of the coupling space metric (3.22) and one–form (3.23) entering in the trace anomaly can be determined by looking at vacuum polarisation diagrams containing in addition to the usual vertices of the quantum field theory the following counterterms:



The terms entering the metric (3.22) are the ones proportional to two powers of the beta functions while the ones with one power of β_i fix the one form w (3.23). All the vacuum polarisation diagrams up to three–loop order as well as the form of their contribution are shown in Table A.1. Note that more diagrams would be present if the scalar field were charged under the gauge group.

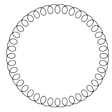
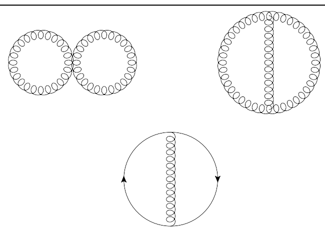
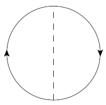
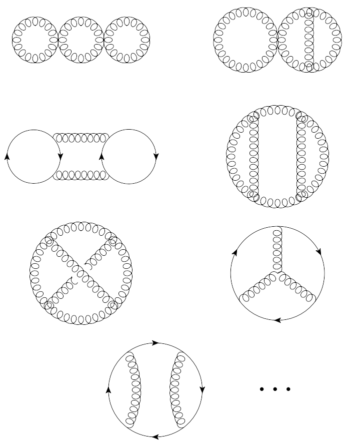
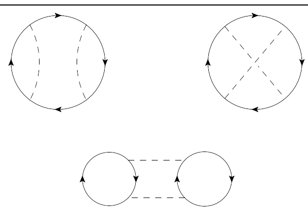
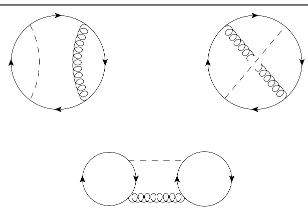
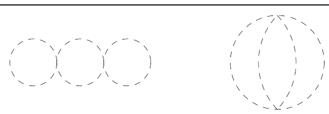
Diagrams	Contributions to χ	Contributions to w
	$\frac{\beta_{\alpha_g}^2}{\alpha_g^2}$	$\frac{\beta_{\alpha_g}}{\alpha_g}$
	$\frac{\beta_{\alpha_g}^2}{\alpha_g}$	β_{α_g}
	$\frac{\beta_{\alpha_y}^2}{\alpha_y}$	β_{α_y}
 ...	$\beta_{\alpha_g}^2$	$\alpha_g\beta_{\alpha_g}$
	$\beta_{\alpha_y}^2$	$\alpha_y\beta_{\alpha_y}$
	$\frac{\beta_{\alpha_g}^2}{\alpha_g}\alpha_y$, $\frac{\beta_{\alpha_y}^2}{\alpha_y}\alpha_g$, $\beta_{\alpha_g}\beta_{\alpha_y}$	$\alpha_y\beta_{\alpha_g}$, $\alpha_g\beta_{\alpha_y}$
	$\beta_{\alpha_\lambda}^2$	$\alpha_\lambda\beta_{\alpha_\lambda}$

Table A.1: One, two and three–loop vacuum polarisation diagrams entering the computation of the metric (3.22) and the one–form (3.23).

Bibliography

[1] O. Antipin, S. Di Chiara, M. Mojaza, E. Mølgaard, and F. Sannino, "A Perturbative Realization of Miransky Scaling", *Phys.Rev.*, vol. D86, p. 085 009, 2012. doi: 10.1103/PhysRevD.86.085009. arXiv:1205.6157 [hep-ph].

[2] O. Antipin, M. Gillioz, E. Mølgaard, and F. Sannino, "The *a* theorem for Gauge-Yukawa theories beyond Banks-Zaks", *Phys.Rev.*, vol. D87, p. 125 017, 2013. doi: 10.1103/PhysRevD.87.125017. arXiv:1303.1525 [hep-th].

[3] O. Antipin, J. Krog, E. Mølgaard, and F. Sannino, "Standard Model-like corrections to Dilatonic Dynamics", *JHEP*, vol. 1309, p. 122, 2013. doi: 10.1007/JHEP09(2013)122. arXiv:1303.7213 [hep-ph].

[4] O. Antipin, M. Gillioz, J. Krog, E. Mølgaard, and F. Sannino, "Standard Model Vacuum Stability and Weyl Consistency Conditions", *JHEP*, vol. 1308, p. 034, 2013. doi: 10.1007/JHEP08(2013)034. arXiv:1306.3234 [hep-ph].

[5] E. Mølgaard and R. Shrock, "Renormalization-Group Flows and Fixed Points in Yukawa Theories", *Phys.Rev.*, vol. D89, p. 105 007, 2014. doi: 10.1103/PhysRevD.89.105007. arXiv:1403.3058 [hep-th].

[6] E. Mølgaard, "Decrypting gauge-Yukawa cookbooks", arXiv:1404.5550 [hep-th].

[7] I. Newton, *Philosophiae naturalis principia mathematica*. J. Societatis Regiae ac Typis J. Streater, 1687. [Online]. Available: http://books.google.com/books?id=-dVKAQAAIAAJ.

[8] I. Newton and W. Innys, *Opticks: Or, A Treatise of the Reflections, Refractions, Inflections and Colours of Light*. William Innys at the West-End of St. Paul's., 1730. [Online]. Available: http://books.google.com/books?id=GnAFAAAAQAAJ.

[9] E. Witten, "Topological Quantum Field Theory", *Commun.Math.Phys.*, vol. 117, p. 353, 1988. DOI: 10.1007/BF01223371.

[10] ——, "Topological Sigma Models", *Commun.Math.Phys.*, vol. 118, p. 411, 1988. DOI: 10.1007/BF01466725.

[11] M. Tegmark, "Is the theory of everything merely the ultimate ensemble theory?", *Annals Phys.*, vol. 270, pp. 1–51, 1998. DOI: 10.1006/aphy.1998.5855. arXiv:gr-qc/9704009 [gr-qc].

[12] S. Liberati, "Tests of Lorentz invariance: a 2013 update", *Class.Quant.Grav.*, vol. 30, p. 133 001, 2013. DOI: 10.1088/0264-9381/30/13/133001. arXiv:1304.5795 [gr-qc].

[13] A. Pickering, *Constructing Quarks: A Sociological History of Particle Physics*. University of Chicago Press, 1984.

[14] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory (Frontiers in Physics)*. Westview Press, 1995, ISBN: 0201503972. [Online]. Available: http://physics.weber.edu/schroeder/qftbook.html.

[15] J. Wess and B. Zumino, "Supergauge Transformations in Four-Dimensions", *Nucl.Phys.*, vol. B70, pp. 39–50, 1974. DOI: 10.1016/0550-3213(74)90355-1.

[16] ——, "A Lagrangian Model Invariant Under Supergauge Transformations", *Phys.Lett.*, vol. B49, p. 52, 1974. DOI: 10.1016/0370-2693(74)90578-4.

[17] S. Dimopoulos and H. Georgi, "Softly Broken Supersymmetry and SU(5)", *Nucl.Phys.*, vol. B193, p. 150, 1981. DOI: 10.1016/0550-3213(81)90522-8.

[18] G. Degrassi, S. Heinemeyer, W. Hollik, P. Slavich, and G. Weiglein, "Towards high precision predictions for the MSSM Higgs sector", *Eur.Phys.J.*, vol. C28, pp. 133–143, 2003. DOI: 10.1140/epjc/s2003-01152-2. arXiv:hep-ph/0212020 [hep-ph].

[19] O. Antipin, M. Mojaza, and F. Sannino, "Light Dilaton at Fixed Points and Ultra Light Scale Super Yang Mills", *Phys.Lett.*, vol. B712, pp. 119–125, 2012. DOI: 10.1016/j.physletb.2012.04.050. arXiv:1107.2932 [hep-ph].

[20] M. Gell-Mann, "Symmetries of baryons and mesons", *Phys.Rev.*, vol. 125, pp. 1067–1084, 1962. DOI: 10.1103/PhysRev.125.1067.

[21] ——, "A Schematic Model of Baryons and Mesons", *Phys.Lett.*, vol. 8, pp. 214–215, 1964. DOI: 10.1016/S0031-9163(64)92001-3.

[22] G. Zweig, "An SU(3) model for strong interaction symmetry and its breaking. Version 1", 1964.

[23] ——, "An SU(3) model for strong interaction symmetry and its breaking. Version 2", 1964.

[24] S. Glashow, "Partial Symmetries of Weak Interactions", *Nucl.Phys.*, vol. 22, pp. 579–588, 1961. DOI: 10.1016/0029-5582(61)90469-2.

[25] S. Weinberg, "A Model of Leptons", *Phys.Rev.Lett.*, vol. 19, pp. 1264–1266, 1967. DOI: 10.1103/PhysRevLett.19.1264.

[26] A. Salam, "Weak and Electromagnetic Interactions", *Conf.Proc.*, vol. C680519, pp. 367–377, 1968.

[27] S. Glashow, J. Iliopoulos, and L. Maiani, "Weak Interactions with Lepton-Hadron Symmetry", *Phys.Rev.*, vol. D2, pp. 1285–1292, 1970. DOI: 10.1103/PhysRevD.2.1285.

[28] N. Cabibbo, "Unitary Symmetry and Leptonic Decays", *Phys.Rev.Lett.*, vol. 10, pp. 531–533, 1963. DOI: 10.1103/PhysRevLett.10.531.

[29] M. Kobayashi and T. Maskawa, "CP Violation in the Renormalizable Theory of Weak Interaction", *Prog.Theor.Phys.*, vol. 49, pp. 652–657, 1973. DOI: 10.1143/PTP.49.652.

[30] P. W. Anderson, "Plasmons, gauge invariance, and mass", *Phys. Rev.*, vol. 130, pp. 439–442, 1 1963. DOI: 10.1103/PhysRev.130.439. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRev.130.439.

[31] F. Englert and R. Brout, "Broken Symmetry and the Mass of Gauge Vector Mesons", *Phys.Rev.Lett.*, vol. 13, pp. 321–323, 1964. DOI: 10.1103/PhysRevLett.13.321.

[32] G. Guralnik, C. Hagen, and T. Kibble, "Global Conservation Laws and Massless Particles", *Phys.Rev.Lett.*, vol. 13, pp. 585–587, 1964. DOI: 10.1103/PhysRevLett.13.585.

[33] P. W. Higgs, "Broken Symmetries and the Masses of Gauge Bosons", *Phys.Rev.Lett.*, vol. 13, pp. 508–509, 1964. DOI: 10.1103/PhysRevLett.13.508.

[34] G. 't Hooft, "Renormalizable Lagrangians for Massive Yang-Mills Fields", *Nucl.Phys.*, vol. B35, pp. 167–188, 1971. DOI: 10.1016/0550-3213(71)90139-8.

[35] J. Beringer *et al.*, "Review of Particle Physics (RPP)", *Phys.Rev.*, vol. D86, p. 010 001, 2012. DOI: 10.1103/PhysRevD.86.010001.

[36] M. Veltman, "The higgs particle", Della Pietra Public Lecture, Simons Center for Geometry and Physics, 2014, [Online]. Available: http://scgp.stonybrook.edu/archives/9540.

[37] B. W. Lee, C. Quigg, and H. Thacker, "Weak Interactions at Very High-Energies: The Role of the Higgs Boson Mass", *Phys.Rev.*, vol. D16, p. 1519, 1977. DOI: 10.1103/PhysRevD.16.1519.

[38] J. Chang, K. Cheung, C.-T. Lu, and T.-C. Yuan, "WW scattering in the era of post-Higgs-boson discovery", *Phys.Rev.*, vol. D87, no. 9, p. 093 005, 2013. DOI: 10.1103/PhysRevD.87.093005. arXiv:1303.6335 [hep-ph].

[39] C. Quigg and R. Shrock, "Gedanken Worlds without Higgs: QCD-Induced Electroweak Symmetry Breaking", *Phys.Rev.*, vol. D79, p. 096 002, 2009. DOI: 10.1103/PhysRevD.79.096002. arXiv:0901.3958 [hep-ph].

[40] G. 't Hooft, "Renormalization of Massless Yang-Mills Fields", *Nucl.Phys.*, vol. B33, pp. 173–199, 1971. DOI: 10.1016/0550-3213(71)90395-6.

[41] ——, "Renormalizable Lagrangians for Massive Yang-Mills Fields", *Nucl.Phys.*, vol. B35, pp. 167–188, 1971. DOI: 10.1016/0550-3213(71)90139-8.

[42] G. 't Hooft and M. Veltman, "Regularization and Renormalization of Gauge Fields", *Nucl.Phys.*, vol. B44, pp. 189–213, 1972. DOI: 10.1016/0550-3213(72)90279-9.

[43] V. Miransky, "Dynamics of Spontaneous Chiral Symmetry Breaking and Continuum Limit in Quantum Electrodynamics", *Nuovo Cim.*, vol. A90, pp. 149–170, 1985. DOI: 10.1007/BF02724229.

[44] V. Miransky and K. Yamawaki, "On Gauge Theories with Additional Four Fermion Interaction", *Mod.Phys.Lett.*, vol. A4, pp. 129–135, 1989. DOI: 10.1142/S0217732389000186.

[45] V. Miransky and K. Yamawaki, "Conformal phase transition in gauge theories", *Phys.Rev.*, vol. D55, pp. 5051–5066, 1997. DOI: 10.1103/PhysRevD.56.3768, 10.1103/PhysRevD.55.5051. arXiv:hep-th/9611142 [hep-th].

[46] I. Jack and H. Osborn, "Analogs for the *c* Theorem for Four-dimensional Renormalizable Field Theories", *Nucl.Phys.*, vol. B343, pp. 647–688, 1990. DOI: 10.1016/0550-3213(90)90584-Z.

[47] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories", *Nucl.Phys.*, vol. B363, pp. 486–526, 1991. DOI: 10.1016/0550-3213(91)80030-P.

[48] R. Foadi, M. T. Frandsen, and F. Sannino, "Technicolor Dark Matter", *Phys.Rev.*, vol. D80, p. 037 702, 2009. DOI: 10.1103/PhysRevD.80.037702. arXiv:0812.3406 [hep-ph].

[49] Y. Nambu and G. Jona-Lasinio, "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. 1.", *Phys.Rev.*, vol. 122, pp. 345–358, 1961. DOI: 10.1103/PhysRev.122.345.

[50] ——, "Dynamical Model Of Elementary Particles Based On An Analogy With Superconductivity. II", *Phys.Rev.*, vol. 124, pp. 246–254, 1961. DOI: 10.1103/PhysRev.124.246.

[51] T. Karavirta, J. Krog, M. Mojaza, and F. Sannino, In preparation, 2014.

[52] D. F. Litim and F. Sannino, "Asymptotic safety guaranteed", 2014. arXiv:1406.2337 [hep-th].

[53] E. Fermi, "Trends to a Theory of beta Radiation. (In Italian)", *Nuovo Cim.*, vol. 11, pp. 1–19, 1934. DOI: 10.1007/BF02959820.

[54] ——, "An attempt of a theory of beta radiation. 1.", *Z.Phys.*, vol. 88, pp. 161–177, 1934. DOI: 10.1007/BF01351864.

[55] K. Wilson and J. B. Kogut, "The Renormalization group and the epsilon expansion", *Phys.Rept.*, vol. 12, pp. 75–200, 1974. DOI: 10.1016/0370-1573(74)90023-4.

[56] C. Wetterich, "Exact evolution equation for the effective potential", *Phys.Lett.*, vol. B301, pp. 90–94, 1993. DOI: 10.1016/0370-2693(93)90726-X.

[57] J. Berges, N. Tetradis, and C. Wetterich, "Nonperturbative renormalization flow in quantum field theory and statistical physics", *Phys.Rept.*, vol. 363, pp. 223–386, 2002. DOI: 10.1016/S0370-1573(01)00098-9. arXiv:hep-ph/0005122 [hep-ph].

[58] D. F. Litim, "Optimization of the exact renormalization group", *Phys.Lett.*, vol. B486, pp. 92–99, 2000. DOI: 10.1016/S0370-2693(00)00748-6. arXiv:hep-th/0005245 [hep-th].

[59] ——, "Optimized renormalization group flows", *Phys.Rev.*, vol. D64, p. 105 007, 2001. DOI: 10.1103/PhysRevD.64.105007. arXiv:hep-th/0103195 [hep-th].

[60] A. Pelissetto and E. Vicari, "Critical phenomena and renormalization group theory", *Phys.Rept.*, vol. 368, pp. 549–727, 2002. DOI: 10.1016/S0370-1573(02)00219-3. arXiv:cond-mat/0012164 [cond-mat].

[61] J. M. Pawlowski, "Aspects of the functional renormalisation group", *Annals Phys.*, vol. 322, pp. 2831–2915, 2007. DOI: 10.1016/j.aop.2007.01.007. arXiv:hep-th/0512261 [hep-th].

[62] J. Callan Curtis G., "Broken scale invariance in scalar field theory", *Phys.Rev.*, vol. D2, pp. 1541–1547, 1970. DOI: 10.1103/PhysRevD.2.1541.

[63] K. Symanzik, "Small distance behavior in field theory and power counting", *Commun.Math.Phys.*, vol. 18, pp. 227–246, 1970. DOI: 10.1007/BF01649434.

[64] O. Antipin, E. Mølgaard, and F. Sannino, In preparation, 2014.

[65] M. E. Machacek and M. T. Vaughn, "Two–Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization", *Nucl.Phys.*, vol. B222, p. 83, 1. DOI: 10.1016/0550-3213(83)90610-7.

[66] ——, "Two–Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings", *Nucl.Phys.*, vol. B236, p. 221, 1984. DOI: 10.1016/0550-3213(84)90533-9.

[67] ——, "Two–Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings", *Nucl.Phys.*, vol. B249, p. 70, 1985. DOI: 10.1016/0550-3213(85)90040-9.

[68] T. Cheng, E. Eichten, and L.-F. Li, "Higgs Phenomena in Asymptotically Free Gauge Theories", *Phys.Rev.*, vol. D9, p. 2259, 1974. DOI: 10.1103/PhysRevD.9.2259.

[69] A. Pickering, J. Gracey, and D. Jones, "Three loop gauge beta function for the most general single gauge coupling theory", *Phys.Lett.*, vol. B510, pp. 347–354, 2001. DOI: 10.1016/S0370-2693(01)00624-4. arXiv:hep-ph/0104247 [hep-ph].

[70] L. Mihaila, "Three-loop gauge beta function in non-simple gauge groups", *PoS*, vol. RADCOR2013, p. 060, 2014.

[71] M.-x. Luo, H.-w. Wang, and Y. Xiao, “Two–loop renormalization group equations in general gauge field theories”, *Phys.Rev.*, vol. D67, p. 065 019, 2003. DOI: 10.1103/PhysRevD.67.065019. arXiv:hep-ph/0211440 [hep-ph].

[72] T. Banks and A. Zaks, “On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions”, *Nucl.Phys.*, vol. B196, p. 189, 1982. DOI: 10.1016/0550-3213(82)90035-9.

[73] D. D. Dietrich and F. Sannino, “Conformal window of SU(N) gauge theories with fermions in higher dimensional representations”, *Phys.Rev.*, vol. D75, p. 085 018, 2007. DOI: 10.1103/PhysRevD.75.085018. arXiv:hep-ph/0611341 [hep-ph].

[74] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, ser. Graduate texts in contemporary physics. New York, NY: Springer, 1997.

[75] J. Callan Curtis G., S. R. Coleman, and R. Jackiw, “A New improved energy - momentum tensor”, *Annals Phys.*, vol. 59, pp. 42–73, 1970. DOI: 10.1016/0003-4916(70)90394-5.

[76] S. R. Coleman and R. Jackiw, “Why dilatation generators do not generate dilatations?”, *Annals Phys.*, vol. 67, pp. 552–598, 1971. DOI: 10.1016/0003-4916(71)90153-9.

[77] J.-F. Fortin, B. Grinstein, and A. Stergiou, “Limit Cycles and Conformal Invariance”, *JHEP*, vol. 1301, p. 184, 2013. DOI: 10.1007/JHEP01(2013)184. arXiv:1208.3674 [hep-th].

[78] M. A. Luty, J. Polchinski, and R. Rattazzi, “The a-theorem and the Asymptotics of 4D Quantum Field Theory”, *JHEP*, vol. 1301, p. 152, 2013. DOI: 10.1007/JHEP01(2013)152. arXiv:1204.5221 [hep-th].

[79] A. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory”, *JETP Lett.*, vol. 43, pp. 730–732, 1986.

[80] J. L. Cardy, “Is There a c Theorem in Four-Dimensions?”, *Phys.Lett.*, vol. B215, pp. 749–752, 1988. DOI: 10.1016/0370-2693(88)90054-8.

[81] H. Osborn, “Derivation of a Four-dimensional c Theorem”, *Phys.Lett.*, vol. B222, p. 97, 1989. DOI: 10.1016/0370-2693(89)90729-6.

[82] D. Kutasov, “New results on the ’a theorem’ in four-dimensional supersymmetric field theory”, 2003. arXiv:hep-th/0312098 [hep-th].

[83] Z. Komargodski, "The Constraints of Conformal Symmetry on RG Flows", *JHEP*, vol. 1207, p. 069, 2012. DOI: 10.1007/JHEP07(2012)069. arXiv:1112.4538 [hep-th].

[84] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions", *JHEP*, vol. 1112, p. 099, 2011. DOI: 10.1007/JHEP12(2011)099. arXiv:1107.3987 [hep-th].

[85] A. Dymarsky, Z. Komargodski, A. Schwimmer, and S. Theisen, "On Scale and Conformal Invariance in Four Dimensions", 2013. arXiv:1309.2921 [hep-th].

[86] T. Appelquist, A. G. Cohen, and M. Schmaltz, "A New constraint on strongly coupled gauge theories", *Phys.Rev.*, vol. D60, p. 045 003, 1999. DOI: 10.1103/PhysRevD.60.045003. arXiv:hep-th/9901109 [hep-th].

[87] T. Appelquist, A. G. Cohen, M. Schmaltz, and R. Shrock, "New constraints on chiral gauge theories", *Phys.Lett.*, vol. B459, pp. 235–241, 1999. DOI: 10.1016/S0370-2693(99)00616-4. arXiv:hep-th/9904172 [hep-th].

[88] M. Duff, "Observations on Conformal Anomalies", *Nucl.Phys.*, vol. B125, p. 334, 1977. DOI: 10.1016/0550-3213(77)90410-2.

[89] J.-F. Fortin, B. Grinstein, and A. Stergiou, "Limit Cycles in Four Dimensions", *JHEP*, vol. 1212, p. 112, 2012. DOI: 10.1007/JHEP12(2012)112. arXiv:1206.2921 [hep-th].

[90] D. Freedman and H. Osborn, "Constructing a c function for SUSY gauge theories", *Phys.Lett.*, vol. B432, pp. 353–360, 1998. DOI: 10.1016/S0370-2693(98)00649-2. arXiv:hep-th/9804101 [hep-th].

[91] C. Pica and F. Sannino, "Ultraviolet and infrared zeros of gauge theories at the four–loop order and beyond", *Phys.Rev.*, vol. D83, p. 035 013, 2011. DOI: 10.1103/PhysRevD.83.035013. arXiv:1011.5917 [hep-ph].

[92] O. Antipin, M. Gillioz, and F. Sannino, "a New Conformal Window Bound from the a theorem", 2013. arXiv:1303.1547 [hep-ph].

[93] L. N. Mihaila, J. Salomon, and M. Steinhauser, "Renormalization constants and beta functions for the gauge couplings of the Standard Model to three-loop order", *Phys.Rev.*, vol. D86, p. 096 008, 2012. DOI: 10.1103/PhysRevD.86.096008. arXiv:1208.3357 [hep-ph].

[94] R. V. Harlander, L. Mihaila, and M. Steinhauser, "The SUSY-QCD beta function to three loops", *Eur.Phys.J.*, vol. C63, pp. 383–390, 2009. DOI: 10.1140/epjc/s10052-009-1109-9. arXiv:0905.4807 [hep-ph].

[95] M. Holthausen, K. S. Lim, and M. Lindner, "Planck scale Boundary Conditions and the Higgs Mass", *JHEP*, vol. 1202, p. 037, 2012. DOI: 10.1007/JHEP02(2012)037. arXiv:1112.2415 [hep-ph].

[96] K. Chetyrkin and M. Zoller, "Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model", *JHEP*, vol. 1206, p. 033, 2012. DOI: 10.1007/JHEP06(2012)033. arXiv:1205.2892 [hep-ph].

[97] L. N. Mihaila, J. Salomon, and M. Steinhauser, "Gauge Coupling Beta Functions in the Standard Model to Three Loops", *Phys.Rev.Lett.*, vol. 108, p. 151 602, 2012. DOI: 10.1103/PhysRevLett.108.151602. arXiv:1201.5868 [hep-ph].

[98] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, *et al.*, "Higgs mass and vacuum stability in the Standard Model at NNLO", *JHEP*, vol. 1208, p. 098, 2012. DOI: 10.1007/JHEP08(2012)098. arXiv:1205.6497 [hep-ph].

[99] C. Ford, I. Jack, and D. Jones, "The Standard model effective potential at two loops", *Nucl.Phys.*, vol. B387, pp. 373–390, 1992. DOI: 10.1016/0550-3213(92)90165-8. arXiv:hep-ph/0111190 [hep-ph].

[100] A. Bednyakov, A. Pikelner, and V. Velizhanin, "Yukawa coupling beta-functions in the Standard Model at three loops", *Phys.Lett.*, vol. B722, pp. 336–340, 2013. DOI: 10.1016/j.physletb.2013.04.038. arXiv:1212.6829 [hep-ph].

[101] K. Chetyrkin and M. Zoller, "β-function for the Higgs self-interaction in the Standard Model at three-loop level", *JHEP*, vol. 1304, p. 091, 2013. DOI: 10.1007/JHEP04(2013)091. arXiv:1303.2890 [hep-ph].

[102] A. Bednyakov, A. Pikelner, and V. Velizhanin, "Higgs self-coupling beta-function in the Standard Model at three loops", 2013. arXiv:1303.4364 [hep-ph].

[103] "Combination of standard model Higgs boson searches and measurements of the properties of the new boson with a mass near 125 GeV", CERN, Geneva, Tech. Rep. CMS-PAS-HIG-13-005, 2013.

[104] J. Casas, J. Espinosa, and M. Quiros, "Improved Higgs mass stability bound in the standard model and implications for supersymmetry", *Phys.Lett.*, vol. B342, pp. 171–179, 1995. DOI: 10.1016/0370-2693(94)01404-Z. arXiv:hep-ph/9409458 [hep-ph].

[105] J. Casas, J. Espinosa, M. Quiros, and A. Riotto, "The Lightest Higgs boson mass in the minimal supersymmetric standard model", *Nucl.Phys.*, vol. B436, pp. 3–29, 1995. DOI: 10.1016/0550-3213(94)00508-C. arXiv:hep-ph/9407389 [hep-ph].

[106] G. Isidori, G. Ridolfi, and A. Strumia, "On the metastability of the standard model vacuum", *Nucl.Phys.*, vol. B609, pp. 387–409, 2001. DOI: 10.1016/S0550-3213(01)00302-9. arXiv:hep-ph/0104016 [hep-ph].

[107] E. Stueckelberg and A. Petermann, "Normalization of constants in the quanta theory", *Helv.Phys.Acta*, vol. 26, pp. 499–520, 1953.

[108] M. Gell-Mann and F. Low, "Quantum electrodynamics at small distances", *Phys.Rev.*, vol. 95, pp. 1300–1312, 1954. DOI: 10.1103/PhysRev.95.1300.

[109] N. Bogolyubov and D. Shirkov, "Prilozhenie Renormalizatsionnoi Gruppy K Uluchsheniyu Formul Teorii Vozmushchenii", *Doklady Akademii Nauk SSSR*, vol. 103, no. 3, 391–394, 1955, ISSN: 0002-3264.

[110] K. G. Wilson, "The Renormalization Group and Strong Interactions", *Phys.Rev.*, vol. D3, p. 1818, 1971. DOI: 10.1103/PhysRevD.3.1818.

[111] W. E. Caswell, "Asymptotic Behavior of Nonabelian Gauge Theories to Two–Loop Order", *Phys.Rev.Lett.*, vol. 33, p. 244, 1974. DOI: 10.1103/PhysRevLett.33.244.

[112] D. Jones, "Two-loop diagrams in Yang-Mills theory", *Nucl.Phys.*, vol. B75, p. 531, 1974. DOI: 10.1016/0550-3213(74)90093-5.

[113] E. Gardi and M. Karliner, "Relations between observables and the infrared fixed point in QCD", *Nucl.Phys.*, vol. B529, pp. 383–423, 1998. DOI: 10.1016/S0550-3213(98)00392-7. arXiv:hep-ph/9802218 [hep-ph].

[114] E. Gardi, G. Grunberg, and M. Karliner, "Can the QCD running coupling have a causal analyticity structure?", *JHEP*, vol. 9807, p. 007, 1998. DOI: 10.1088/1126-6708/1998/07/007. arXiv:hep-ph/9806462 [hep-ph].

[115] T. A. Ryttov and R. Shrock, "Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions", *Phys.Rev.*, vol. D83, p. 056 011, 2011. DOI: 10.1103/PhysRevD.83.056011. arXiv:1011.4542 [hep-ph].

[116] ——, "Comparison of Some Exact and Perturbative Results for a Supersymmetric SU(N_c) Gauge Theory", *Phys.Rev.*, vol. D85, p. 076 009, 2012. DOI: 10.1103/PhysRevD.85.076009. arXiv:1202.1297 [hep-ph].

[117] R. Shrock, "Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories", *Phys.Rev.*, vol. D87, p. 105 005, 2013. DOI: 10.1103/PhysRevD.87.105005. arXiv:1301.3209 [hep-th].

[118] ——, "Higher-loop calculations of the ultraviolet to infrared evolution of a vectorial gauge theory in the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed", *Phys.Rev.*, vol. D87, p. 116 007, 2013. DOI: 10.1103/PhysRevD.87.116007. arXiv:1302.5434 [hep-th].

[119] B. Holdom, "Large N flavor beta-functions: a recap", *Phys.Lett.*, vol. B694, pp. 74–79, 2010. DOI: 10.1016/j.physletb.2010.09.037. arXiv:1006.2119 [hep-ph].

[120] R. Shrock, "Study of Scheme Transformations to Remove Higher-Loop Terms in the β Function of a Gauge Theory", *Phys.Rev.*, vol. D88, p. 036 003, 2013. DOI: 10.1103/PhysRevD.88.036003. arXiv:1305.6524 [hep-ph].

[121] ——, "Study of Possible Ultraviolet Zero of the Beta Function in Gauge Theories with Many Fermions", *Phys.Rev.*, vol. D89, p. 045 019, 2014. DOI: 10.1103/PhysRevD.89.045019. arXiv:1311.5268 [hep-th].

[122] J. Ellis, J. Espinosa, G. Giudice, A. Hoecker, and A. Riotto, "The Probable Fate of the Standard Model", *Phys.Lett.*, vol. B679, pp. 369–375, 2009. DOI: 10.1016/j.physletb.2009.07.054. arXiv:0906.0954 [hep-ph].

[123] M. Shaposhnikov and C. Wetterich, "Asymptotic safety of gravity and the Higgs boson mass", *Phys.Lett.*, vol. B683, pp. 196–200, 2010. DOI: 10.1016/j.physletb.2009.12.022. arXiv:0912.0208 [hep-th].

[124] F. Bezrukov and M. Shaposhnikov, “Standard Model Higgs boson mass from inflation: Two loop analysis”, *JHEP*, vol. 0907, p. 089, 2009. DOI: 10.1088/1126-6708/2009/07/089. arXiv:0904.1537 [hep-ph].

[125] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto, *et al.*, “Higgs mass implications on the stability of the electroweak vacuum”, *Phys.Lett.*, vol. B709, pp. 222–228, 2012. DOI: 10.1016/j.physletb.2012.02.013. arXiv:1112.3022 [hep-ph].

[126] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, “Higgs Boson Mass and New Physics”, *JHEP*, vol. 1210, p. 140, 2012. DOI: 10.1007/JHEP10(2012)140. arXiv:1205.2893 [hep-ph].

[127] L. A. Anchordoqui, I. Antoniadis, H. Goldberg, X. Huang, D. Lust, *et al.*, “Vacuum Stability of Standard Model^{++}”, *JHEP*, vol. 1302, p. 074, 2013. DOI: 10.1007/JHEP02(2013)074. arXiv:1208.2821 [hep-ph].

[128] I. Masina, “Higgs boson and top quark masses as tests of electroweak vacuum stability”, *Phys.Rev.*, vol. D87, no. 5, p. 053 001, 2013. DOI: 10.1103/PhysRevD.87.053001. arXiv:1209.0393 [hep-ph].

[129] F. Jegerlehner, “The hierarchy problem of the electroweak Standard Model revisited”, 2013. arXiv:1305.6652 [hep-ph].

[130] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, *et al.*, “Investigating the near-criticality of the Higgs boson”, *JHEP*, vol. 1312, p. 089, 2013. DOI: 10.1007/JHEP12(2013)089. arXiv:1307.3536.

[131] V. Branchina and E. Messina, “Stability, Higgs Boson Mass and New Physics”, *Phys.Rev.Lett.*, vol. 111, p. 241 801, 2013. DOI: 10.1103/PhysRevLett.111.241801. arXiv:1307.5193 [hep-ph].

[132] H. Gies, C. Gneiting, and R. Sondenheimer, “Higgs Mass Bounds from Renormalization Flow for a simple Yukawa model”, *Phys.Rev.*, vol. D89, p. 045 012, 2014. DOI: 10.1103/PhysRevD.89.045012. arXiv:1308.5075 [hep-ph].

[133] E. Eichten, “The higgs boson and naturalness”, Origin of Mass 2013 Conference, 2013, [Online]. Available: http://cp3-origins.dk/events/meetings/mass2013/mass2013talks/mass2013-conf-talks.

[134] C. T. Hill, "Is the Higgs Boson Associated with Coleman-Weinberg Dynamical Symmetry Breaking?", *Phys.Rev.*, vol. D89, p. 073 003, 2014. DOI: 10.1103/PhysRevD.89.073003. arXiv:1401.4185 [hep-ph].

[135] O. Antipin, M. Mojaza, and F. Sannino, "Conformal Extensions of the Standard Model with Veltman Conditions", *Phys.Rev.*, vol. D89, p. 085 015, 2014. DOI: 10.1103/PhysRevD.89.085015. arXiv:1310.0957 [hep-ph].

[136] H. Yukawa, "On the interaction of elementary particles", *Proc. Phys. Math. Soc. Jap.*, vol. 17, pp. 48–57, 1935.

[137] L. Maiani, G. Parisi, and R. Petronzio, "Bounds on the Number and Masses of Quarks and Leptons", *Nucl.Phys.*, vol. B136, p. 115, 1978. DOI: 10.1016/0550-3213(78)90018-4.

[138] R. A. Flores and M. Sher, "Upper Limits to Fermion Masses in the Glashow-Weinberg-Salam Model", *Phys.Rev.*, vol. D27, p. 1679, 1983. DOI: 10.1103/PhysRevD.27.1679.

[139] M. Beg, C. Panagiotakopoulos, and A. Sirlin, "Mass of the Higgs Boson in the Canonical Realization of the Weinberg-Salam Theory", *Phys.Rev.Lett.*, vol. 52, p. 883, 1984. DOI: 10.1103/PhysRevLett.52.883.

[140] M. Lindner, "Implications of Triviality for the Standard Model", *Z.Phys.*, vol. C31, p. 295, 1986. DOI: 10.1007/BF01479540.

[141] I.-H. Lee and R. E. Shrock, "Chiral Symmetry Breaking Phase Transition in Lattice Gauge Higgs Theories With Fermions", *Phys.Rev.Lett.*, vol. 59, p. 14, 1987. DOI: 10.1103/PhysRevLett.59.14.

[142] ——, "Study of the Chiral Transition in a U(1) Gauge Higgs Fermion Theory With Yukawa Couplings", *Phys.Lett.*, vol. B199, p. 541, 1987. DOI: 10.1016/0370-2693(87)91624-8.

[143] J. Shigemitsu, "Yukawa Coupling Generated Fermion Masses on the Lattice", *Phys.Lett.*, vol. B189, p. 164, 1987. DOI: 10.1016/0370-2693(87)91290-1.

[144] I.-H. Lee and R. E. Shrock, "The Chiral Transition in an SU(2) Gauge Higgs Fermion Theory With Yukawa Couplings", *Nucl.Phys.*, vol. B305, p. 305, 1988. DOI: 10.1016/0550-3213(88)90298-2.

[145] J. Shigemitsu, "Renormalized Yukawa Couplings From the Lattice", *Phys.Lett.*, vol. B226, p. 364, 1989. DOI: 10.1016/0370-2693(89)91211-2.

[146] A. Hasenfratz and T. Neuhaus, "Nonperturbative Study of the Strongly Coupled Scalar Fermion Model", *Phys.Lett.*, vol. B220, p. 435, 1989. DOI: 10.1016/0370-2693(89)90899-X.

[147] I.-H. Lee, J. Shigemitsu, and R. E. Shrock, "Lattice Study of a Yukawa Theory With a Real Scalar Field", *Nucl.Phys.*, vol. B330, p. 225, 1990. DOI: 10.1016/0550-3213(90)90309-2.

[148] ——, "Study of Different Lattice Formulations of a Yukawa Model With a Real Scalar Field", *Nucl.Phys.*, vol. B334, p. 265, 1990. DOI: 10.1016/0550-3213(90)90664-Y.

[149] A. Hasenfratz, W.-q. Liu, and T. Neuhaus, "Phase Structure and Critical Points in a Scalar Fermion Model", *Phys.Lett.*, vol. B236, p. 339, 1990. DOI: 10.1016/0370-2693(90)90994-H.

[150] L. Lin, I. Montvay, H. Wittig, and G. Munster, "A U(1)-L x U(1)-R symmetric Yukawa model in the phase with spontaneously broken symmetry", *Nucl.Phys.*, vol. B355, pp. 511–548, 1991. DOI: 10.1016/0550-3213(91)90124-G.

[151] J. Shigemitsu, "Higgs-Yukawa chiral models", *Nucl.Phys.Proc.Suppl.*, vol. 20, pp. 515–527, 1991. DOI: 10.1016/0920-5632(91)90968-K.

[152] R. E. Shrock, "Lattice yukawa models", in *Quantum Fields on the Computer*, ch. 4, pp. 150–210. DOI: 10.1142/9789814439732_0004. [Online]. Available: http://www.worldscientific.com/doi/abs/10.1142/9789814439732_0004.

[153] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, "New Higgs physics from the lattice", *PoS*, vol. LAT2007, p. 056, 2007. arXiv:0710.3151 [hep-lat].

[154] P. Gerhold and K. Jansen, "Lower Higgs boson mass bounds from a chirally invariant lattice Higgs-Yukawa model with overlap fermions", *JHEP*, vol. 0907, p. 025, 2009. DOI: 10.1088/1126-6708/2009/07/025. arXiv:0902.4135 [hep-lat].

[155] ——, "Upper Higgs boson mass bounds from a chirally invariant lattice Higgs-Yukawa model", *JHEP*, vol. 1004, p. 094, 2010. DOI: 10.1007/s13130-010-0464-1. arXiv:1002.4336 [hep-lat].

[156] B. Grinstein and P. Uttayarat, "A Very Light Dilaton", *JHEP*, vol. 1107, p. 038, 2011. DOI: 10.1007/JHEP07(2011)038. arXiv:1105.2370 [hep-ph].

[157] O. Antipin, M. Mojaza, and F. Sannino, "Jumping in the Light Higgs Conformal Window", 2012. arXiv:1208.0987 [hep-ph].

[158] B. Grinstein, A. Stergiou, and D. Stone, "Consequences of Weyl Consistency Conditions", *JHEP*, vol. 1311, p. 195, 2013. DOI: 10.1007/JHEP11(2013)195. arXiv:1308.1096 [hep-th].

[159] G. 't Hooft, "Dimensional regularization and the renormalization group", *Nucl.Phys.*, vol. B61, pp. 455–468, 1973. DOI: 10.1016/0550-3213(73)90376-3.

[160] W. A. Bardeen, A. Buras, D. Duke, and T. Muta, "Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories", *Phys.Rev.*, vol. D18, p. 3998, 1978. DOI: 10.1103/PhysRevD.18.3998.

[161] T. A. Ryttov and R. Shrock, "Scheme Transformations in the Vicinity of an Infrared Fixed Point", *Phys.Rev.*, vol. D86, p. 065 032, 2012. DOI: 10.1103/PhysRevD.86.065032. arXiv:1206.2366 [hep-ph].

[162] ——, "An Analysis of Scheme Transformations in the Vicinity of an Infrared Fixed Point", *Phys.Rev.*, vol. D86, p. 085 005, 2012. DOI: 10.1103/PhysRevD.86.085005. arXiv:1206.6895 [hep-th].

[163] T. A. Ryttov, "Higher Loop Corrections to the Infrared Evolution of Fermionic Gauge Theories in the RI' Scheme", *Phys.Rev.*, vol. D89, p. 016 013, 2014. DOI: 10.1103/PhysRevD.89.016013. arXiv:1309.3867 [hep-ph].

[164] S. D. Glazek, A. Harindranath, S. Pinsky, J. Shigemitsu, and K. Wilson, "On the relativistic bound state problem in the light front Yukawa model", *Phys.Rev.*, vol. D47, pp. 1599–1619, 1993. DOI: 10.1103/PhysRevD.47.1599.

[165] M. Mangin-Brinet, J. Carbonell, and V. Karmanov, "Relativistic bound states in Yukawa model", *Phys.Rev.*, vol. D64, p. 125 005, 2001. DOI: 10.1103/PhysRevD.64.125005. arXiv:hep-th/0107235 [hep-th].

[166] S. R. Coleman and E. J. Weinberg, "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking", *Phys.Rev.*, vol. D7, pp. 1888–1910, 1973. DOI: 10.1103/PhysRevD.7.1888.

[167] E. Gildener and S. Weinberg, "Symmetry Breaking and Scalar Bosons", *Phys.Rev.*, vol. D13, p. 3333, 1976. DOI: 10.1103/PhysRevD.13.3333.

[168] P. Q. Hung, "Vacuum Instability and New Constraints on Fermion Masses", *Phys.Rev.Lett.*, vol. 42, p. 873, 1979. DOI: 10.1103/PhysRevLett.42.873.

[169] A. D. Linde, "Vacuum Instability, Cosmology and Constraints on Particle Masses in the Weinberg-Salam Model", *Phys.Lett.*, vol. B92, p. 119, 1980. DOI: 10.1016/0370-2693(80)90318-4.

[170] H. D. Politzer and S. Wolfram, "Bounds on Particle Masses in the Weinberg-Salam Model", *Phys.Lett.*, vol. B82, pp. 242–246, 1979. DOI: 10.1016/0370-2693(79)90746-9.

[171] M. Sher, "Electroweak Higgs Potentials and Vacuum Stability", *Phys.Rept.*, vol. 179, pp. 273–418, 1989. DOI: 10.1016/0370-1573(89)90061-6.

[172] M. Fischler and J. Oliensis, "Two–Loop Corrections to the Evolution of the Higgs-Yukawa Coupling Constant", *Phys.Lett.*, vol. B119, p. 385, 1982. DOI: 10.1016/0370-2693(82)90695-5.

[173] J. Oliensis and M. Fischler, "Two–Loop Calculations of M(b)/M(tau) and Heavy Fermion Masses in the SU(5) Model", *Phys.Rev.*, vol. D28, p. 194, 1983. DOI: 10.1103/PhysRevD.28.194.

[174] I. Jack and H. Osborn, "General Background Field Calculations With Fermion Fields", *Nucl.Phys.*, vol. B249, p. 472, 1985. DOI: 10.1016/0550-3213(85)90088-4.

[175] V. D. Barger, M. Berger, and P. Ohmann, "Supersymmetric grand unified theories: Two–loop evolution of gauge and Yukawa couplings", *Phys.Rev.*, vol. D47, pp. 1093–1113, 1993. DOI: 10.1103/PhysRevD.47.1093. arXiv:hep-ph/9209232 [hep-ph].