

On Ashtekar's Formulation of General Relativity

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Abstract. We review Ashtekar's variables in physics and in mathematics notation. The first one uses local indices, the second one is given by global objects within fibre bundles.

1. Introduction

The coalescence of gravity and quantum theory has remained one of the greatest challenges of modern physics. Over almost one century there have been many attempts to solve that issue. One of the most promising ones was initiated a quarter of a century ago, when Abhay Ashtekar presented his “new variables” [1] in 1986. These new variables ultimately turned gravity into an $SU(2)$ gauge field theory with constraints and opened the door to a Dirac-like canonical quantization of gravity. This theory is called loop quantum gravity today. The original success of Ashtekar's variables consisted in a remarkable simplification of the constraints in comparison to the standard Wheeler-DeWitt approach, as they removed non-polynomialities that spoiled quantization. Nevertheless, loop quantum gravity is by far not a theory of full quantum gravity yet; in particular, many dynamical issues have remained unsolved so far.

Although Ashtekar's variables form the heart of loop quantum gravity and although the early success of loop quantum gravity is heavily based on its mathematical rigor, a geometric description of Ashtekar's variables themselves in more mathematical terms has been missing for a long time. Only in recent years, there have been some efforts [2, 3, 4, 5] to describe these variables in the language of differential geometry using fibre bundles. In this short paper, we are going to summarize these developments together with a review of the usual formulation of Ashtekar's variables in physics.

We start in Section 2 with the very basic structures in classical canonical gravity. After fixing some notation in Section 3, we will review the standard index definitions of Ashtekar's variables in Section 4. We will introduce the notion of dreibeinen which lead to the first half of Ashtekar's variables, and then give the notion of spin connection and extrinsic curvature that lead to the Ashtekar connection, i.e., the second half of Ashtekar's variables. This section is closed by listing the constraints of Ashtekar gravity. Then, in Section 5, we will represent Ashtekar's variables within the language of fibre bundles. First, we are doing this for the tangent bundle, where the expressions are particularly simple. This comprises frames (they correspond to dreibeinen, see Subsection 5.1), some vector field operation (generalizing the vector product on \mathbb{R}^3) and the Weingarten mapping (encoding the extrinsic curvature, for both see Subsection 5.2). This will give us a covariant derivative in the tangent bundle. Using associate bundle techniques and a spin structure, we may deduce from this the $SU(2)$ Ashtekar connection living in a certain principal fibre bundle, the so-called spin bundle (Subsection 5.4). After a naive derivation

of the constraints and the application of Ashtekar's variables to Friedmann-Robertson-Walker spacetimes, we conclude by regaining the index notations in Section 6.

2. Canonical Gravity

Since its very beginning, general relativity has been described by a metric g of Lorentzian signature on some spacetime manifold, this means a 4-dimensional, connected, time-oriented and oriented Lorentz manifold. Its behaviour is governed by two (up to certain technicalities, e.g., along boundaries) equivalent formulations. First, g is required to solve Einstein's equations¹

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor w.r.t. g , and R the corresponding Ricci scalar. Or, second, g is a stationary point of the Einstein-Hilbert action

$$\int_M \sqrt{\det g} R, \quad (2)$$

corrected with appropriate boundary terms. In order to rewrite general relativity as a canonical theory, we need some splitting of M into $\mathbb{R} \times \Sigma$, where the \mathbb{R} part is supposed to carry the time variable and Σ to denote some embedded spacelike hypersurface. For this to be possible, one usually assumes that the spacetime is globally hyperbolic as it is the case iff (M, g) is isometric² to $(\mathbb{R} \times \Sigma, -f d\tau^2 + g_\tau)$ for some $f : \mathbb{R} \rightarrow \mathbb{R}$ and a family g_τ of Riemannian metrics on Σ . [6, 7] It is somewhat surprising that this equivalence, although conjectured for a long time, has been established many years after the invention of Ashtekar's variables that had already heavily used such a splitting. Given such a splitting, g can be uniquely reconstructed from the reduction q of g to Σ and the second fundamental form (extrinsic curvature) K describing the shape of Σ as a submanifold of M . However, not every choice of a Riemannian metric q and of a symmetric bilinear form K originates in some spacetime metric g . Indeed, certain constraints are to be fulfilled. If they are met, classical results [8, 9, 10] guarantee the (at least local) solvability of the corresponding initial-value problem. The metric q on Σ and some expression that is (up to some weight) linear in K , form a set of canonically conjugate variables.

Based on this formulation, there have been several attempts to quantize gravity. As already mentioned, one of the main obstacles has been the complicated structure of constraints. The Ashtekar variables have provided an option to overcome at least some of these problems. For the moment, we would only like to mention that the Ashtekar variables consist of an appropriately densitized dreibein field together with an $SU(2)$ connection. Below we are now going to describe them a bit more in detail both from the physical and from the mathematical point of view, the analysis remaining classical all over the paper.

3. Notation

Throughout the whole paper, let Σ be some oriented three-dimensional manifold, and let M be some manifold diffeomorphic to $\mathbb{R} \times \Sigma$. We will identify Σ , if necessary, with $\{0\} \times \Sigma$. Next, let q be some Riemannian metric on Σ . We write $\langle X, Y \rangle := q(X, Y)$ for all vector fields X, Y on Σ . Moreover, we let g be a metric on M inducing q on Σ . Recall, that all objects we are dealing with are in the smooth category. When working with indices, we denote spatial indices by a, b, c, \dots and "internal" $SO(3)$ or $SU(2)$ indices by i, j, k, \dots (for expansions w.r.t. some Lie algebra basis) or I, J, K, \dots (if they are matrix indices). Lifting as well as lowering of indices is

¹ Here, we have ignored any matter or cosmological terms as we will do throughout the paper.

² Unless otherwise noted, all functions, metrics, isometries etc. are assumed to be smooth.

made by means of the metric q_{ab} in the spatial, and by the Kronecker delta δ_{ij} in the internal case, respectively.

To end up with the preliminaries, we would like to mention some deviation from a nowadays predominant convention. In contrast to the original version, Thiemann [11] defines the Ashtekar field as the weight-one densitized dreibein E divided by the Barbero-Immirzi parameter β . The deeper reason for this is that E/β is the canonically conjugate variable to the Ashtekar connection. However, as we are less interested in the dynamics here, we give the old version without dividing by β .

4. From indices ...

Common to any definition of Ashtekar's variables is that first the phase space of classical general relativity gets extended. But, how this is done has changed a little bit over the time. Originally [1, 12], Ashtekar started his construction by introducing the so-called soldering form σ and its canonically conjugate momentum C , a density of weight 1. The soldering form plays a similar rôle for $SU(2)$ as the dreibein field does for $SO(3)$. As the soldering form is determined only up to internal $SU(2)$ transformations, C is constrained by $C^{[ab]} = 0$. Now, given a phase space point (σ, C) , Ashtekar defined two connections \mathcal{D}^\pm by

$$\mathcal{D}_a^\pm a_{bI} := \nabla_a a_{bI} \pm i \frac{G}{\sqrt{2 \det q}} \left[C_{aI}{}^J - \frac{1}{2} \sigma_{aI}{}^J \sigma^c{}_{KL} C_c{}^{KL} \right] a_{bJ}. \quad (3)$$

Here, G is the gravitational constant. Note that the bracket term plays the same rôle w.r.t. C , as the extrinsic curvature does for the canonically conjugate momentum to q . Now, the two connections correspond to connection one-forms $(A^\pm)_{aI}^J$ via

$$\mathcal{D}_a^\pm a_I = \partial_a a_I + G A_{aI}{}^J a_J. \quad (4)$$

One can prove that A^\pm and $\sqrt{\det q} \sigma$ are (up to some constant $\pm i/\sqrt{2}$) canonically conjugate variables. If then F^\pm is the curvature two-form corresponding to A^\pm , one gets the constraints

$$\text{tr} \sigma^a F_{ab}^\pm = 0 \quad \text{and} \quad \text{tr} \sigma^a \sigma^b F_{ab}^\pm = 0. \quad (5)$$

Moreover, $C^{[ab]} = 0$ can be rewritten as $\mathcal{D}_a^\pm \sigma^a{}_{IJ} = 0$. It was mostly the polynomial form of these constraints (they are even Yang-Mills like) that triggered the success of Ashtekar's variables. This was also despite the fact that one lets oneself in for complex structures due to the i in the definition of the connection. Indeed, in order to return to the real structure, certain reality conditions had to be imposed.

The situation got somewhat changed in the mid-90s. Barbero [13] and Immirzi [14] observed that the constant i in the definition above can be replaced by any non-zero complex number β , later called Barbero-Immirzi parameter. In particular, for real β one gets rid of the reality conditions, hence one is dealing with $SU(2)$ instead of $SL_C(2)$ connections. This ultimately paved the road for loop quantum gravity, although its foundations (in particular, the Ashtekar-Lewandowski measure [15] needing compact gauge groups) had been laid a few years earlier. However, this switch came with a drawback: In the scalar (Hamilton) constraint, a non-polynomial term prefactored by $1 + \beta^2$, popped up. Hence, the constraints are of simple polynomial form iff $\beta = \pm i$. But, this puzzle was solved as early as Thiemann [16] observed that the new term can be written by means of certain Poisson brackets.

About that time, the derivation of Ashtekar's variables changed a bit (without modifying the final point). A standard reference for this is Thiemann's book [11] that we will follow here closely (up to the overall factor β^{-1} for the Ashtekar field mentioned in Section 3). We will start with the definition of the Ashtekar fields based on dreibeinen and then define the Ashtekar connections by means of spin connection and extrinsic curvature.

4.1. Dreibeine

Consider a local³ (oriented) orthonormal basis (e_1, e_2, e_3) of the tangent space $T\Sigma$. Using some local coordinate system x^a , we may expand each basis vector into $e_i = e_i^a \partial_a$ defining the dreibein⁴ e_i^a . At each single point in Σ , we may view e_i^a as a non-singular matrix. Its inverse is denoted by e_a^i and called co-dreibein. The metric q on Σ can be reconstructed via $q_{ab} = \delta_{ij} e_a^i e_b^j$. From this, we may define the first part E_i^a of the Ashtekar variables: $E_i^a := \sqrt{\det q} e_i^a$ is the densitized dreibein of weight 1. Both e_i^a and e_a^i can be used to replace spatial by internal indices and vice versa. One immediately checks that multiplying e_i^a with any $\mathfrak{so}(3)$ matrix keeps q_{ab} invariant. Note that, using the isomorphism between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$, one can regard e_a^i also as a local $\mathrm{SU}(2)$ -valued one-form.

4.2. Ashtekar Connection

The second type of the Ashtekar variables is an $\mathrm{SU}(2)$ connection A_a^i on Σ that is built up from the spin connection and the extrinsic curvature. The spin connection, on the other hand, comes from the Levi-Civita connection associated to our given Riemannian metric q on Σ . Indeed, one first extends the spatial covariant derivative ∇ from standard tensors on Σ to tensors on Σ carrying also internal $\mathrm{SO}(3)$ indices. So, e.g., we have

$$\nabla_a a_i := \partial_a a_i + \Gamma_{ai}^j a_j \quad (6)$$

and extend this to arbitrary tensors by linearity, Leibniz rule and contractivity. Imposing metricity $\nabla_a e_b^i = 0$, the Γ_{ai}^j terms are given by

$$\Gamma_{ai}^j = e_i^b [\partial_a e_b^j - \Gamma_{ab}^c e_c^j] \quad (7)$$

with Γ_{ab}^c being the Christoffel symbols for q . One checks that Γ_a takes values in $\mathrm{SO}(3)$. Let now $\{M_1, M_2, M_3\}$ be the basis of $\mathfrak{so}(3)$ with $(M_i)_{jk} = \varepsilon_{ikj}$. We use them to expand Γ_a into $\Gamma_a^i M_i$, giving the spin connection Γ_a^i . We may interpret the internal $\mathrm{SO}(3)$ indices also as $\mathrm{SU}(2)$ indices w.r.t. the adjoint representation as this is isomorphic to the fundamental representation at Lie algebra level.

Next, let us introduce the extrinsic curvature term. For this, we now need that Σ is considered as a hypersurface of M , whose normal w.r.t. g will be denoted by n . The extrinsic curvature is then just half the Lie derivative of g w.r.t. n . More explicitly, it is given by $K_{ab} := g(\partial_a, \nabla_b n)$. Using the dreibein as above to replace the spatial index b by the internal index i , we get K_a^i . Altogether, we define the Ashtekar $\mathrm{SU}(2)$ connection by

$$A_a^i := \Gamma_a^i + \beta K_a^i. \quad (8)$$

The corresponding covariant derivative will be denoted by ∇^A .

4.3. Constraints

Directly from metricity of the dreibein, we get the Gauß constraint

$$G_i = \nabla_a^A E_i^a \equiv \partial_a E_i^a + \varepsilon_{ij}^k A_a^j E_k^a = 0 \quad (9)$$

which shows exactly the structure of a gauge field theory constraint. Indeed, this constraint is the justification for considering Ashtekar gravity as an $\mathrm{SU}(2)$ gauge theory. Additional constraints are the (spatial) diffeomorphism constraint (also called vector constraint)

$$V_a = F_{ab}^i E_i^b = 0 \quad (10)$$

³ As Σ is orientable, it has a spin structure and is parallelizable. [17] Therefore we may assume w.l.o.g. that this basis is even a global basis.

⁴ also called triad

and the Hamilton constraint (also called scalar constraint)

$$H = \left[F_{ab}^i - (1 + \beta^2) \varepsilon^i{}_{lm} K_a^l K_b^m \right] \frac{\varepsilon_i{}^{jk} E_j^a E_k^b}{\sqrt{\det q}} = 0 \quad (11)$$

(both provided the Gauß constraint is met).

5. ... to Differential Geometry ...

Now, we are going to overcome the index notation and discuss the mathematical structures behind the Ashtekar variables. Dreibeine will correspond to (oriented) orthonormal frames. They might be discussed in any dimension, whereas the Ashtekar connection will heavily rely on the fact that the manifold Σ is three-dimensional. The deeper reason for this is the fact that the Ashtekar connection is constructed by means of the vector product, which is defined only for dimension 3. For simplicity, we will therefore restrict ourselves to dimension 3 throughout.

The presentation in this section follows⁵ closely that in [5], partially as an excerpt.

5.1. Orthonormal Frames

As already mentioned, dreibeine correspond to certain frames in the tangent bundle $T\Sigma$. To be specific, a frame at $x \in \Sigma$ is a vector space isomorphism

$$e : \mathbb{R}^3 \longrightarrow T_x \Sigma. \quad (12)$$

Given a frame e , we define the corresponding Ashtekar field E to be the densitized frame field

$$E := \frac{1}{\det e} e \quad (13)$$

of weight 1. Here, the determinant is taken w.r.t. some (local) basis of $T_x \Sigma$ and the canonical basis of \mathbb{R}^n . The frame can be reconstructed from the Ashtekar field via

$$e = \frac{1}{\sqrt{\det E}} E. \quad (14)$$

We call a frame orthonormal w.r.t. the metric q on Σ iff it is an isometry, with \mathbb{R}^3 given the Euclidean metric. We call it oriented iff it preserves the orientation. Whereas each frame e via

$$\tilde{q}(X, Y) := \langle e^{-1}(X), e^{-1}(Y) \rangle_{\text{Eucl}} \quad \text{for } X, Y \in T_x \Sigma \quad (15)$$

yields the unique metric \tilde{q} on Σ , such that e is orthonormal w.r.t. \tilde{q} , there exist several frames that are orthonormal w.r.t. a given metric. More precisely, two frames e and e' are isometries for the same q iff $e' = e \circ L_g$ for some $g \in \text{O}(3)$. Here, L_g denotes the left translation by g .

Completely analogously to the definition of the tangent bundle as an appropriately topologized union of the tangent vectors all over the points of Σ , one defines the frame bundle $\text{Gl}(\Sigma)$ as such a collection of frames. The orthonormal frames form a subbundle $\text{O}_q(\Sigma)$ of $\text{Gl}(\Sigma)$; it can be obtained via reducing the structure group from $\text{Gl}(3)$ to $\text{O}(3)$. Further reduction to $\text{SO}(3)$ provides us with the bundle $\text{O}_q^+(\Sigma)$ of oriented orthonormal frames.

⁵ This does not refer to the discussion of constraints.

5.2. Ashtekar Connection

Let us now turn the Ashtekar connection into an intrinsic object of the tangent bundle.

For any tangent vectors X, Y on Σ and any oriented q -orthonormal frame e , we define

$$X \bullet Y := e[e^{-1}(X) \times e^{-1}(Y)], \quad (16)$$

with \times being the standard vector product on \mathbb{R}^3 . As the vector product is invariant w.r.t. $SO(3)$, one immediately checks that \bullet depends on the metric q only rather than on the oriented q -orthonormal frame e itself. The definition can be extended immediately to a smooth operation on vector fields. This can also be seen from the relation $X \bullet Y = *(X \wedge Y)$, where \wedge is the standard wedge product and $*$ the Hodge operator w.r.t. q . The product \bullet inherits all relevant properties of the vector product, in particular,

$$X \bullet Y = -Y \bullet X \quad (17)$$

$$\langle X \bullet Y, Z \rangle = \langle X, Y \bullet Z \rangle \quad (18)$$

$$X \bullet (Y \bullet Z) = \langle X, Z \rangle Y - \langle X, Y \rangle Z \quad (19)$$

and the Jacobi identity

$$X \bullet (Y \bullet Z) + Y \bullet (Z \bullet X) + Z \bullet (X \bullet Y) = 0. \quad (20)$$

The second ingredient for the Ashtekar connection is the so-called Weingarten mapping. It corresponds to the second fundamental form which is usually called extrinsic curvature in physics. The Weingarten mapping encodes the respective shape of Σ within M ; indeed, we will now need to assume that Σ is an embedded submanifold of a four-dimensional Lorentzian manifold (M, g) and that q is the reduction of g to Σ . The Weingarten mapping is defined by

$$\begin{aligned} W : T\Sigma &\longrightarrow T\Sigma. \\ X &\longmapsto {}^4\nabla_X n \end{aligned} \quad (21)$$

Here, n is the normal to Σ within (M, g) , and ${}^4\nabla$ is the Levi-Civita connection for g on M . As ${}^4\nabla$ is metric and torsion-free, the Weingarten mapping is well defined and symmetric. Moreover, it is $C^\infty(\Sigma)$ -linear. The corresponding second fundamental form K is the quadratic form $K(X, Y) = \langle W(X), Y \rangle$ defined by the Weingarten mapping. Of course, W can uniquely be re-obtained from K , as q is non-degenerate.

Mixing all these ingredients appropriately, we get the Ashtekar connection. For this, fix some non-zero complex number β . This number is the Barbero-Immirzi parameter introduced above. Recall that the imaginary part of β may be non-zero. So, for non-real β , we will tacitly assume in the following that all structures (connections etc.) will be complexified.

Now, the Ashtekar connection w.r.t. β is defined by

$$\nabla_X^A Y := \nabla_X Y + \beta W(X) \bullet Y. \quad (22)$$

One can easily show that ∇^A is metric and obeys the Leibniz rule

$$\nabla_X^A (Y \bullet Z) = \nabla_X^A Y \bullet Z + Y \bullet \nabla_X^A Z. \quad (23)$$

Its torsion is given by

$$T^A(X, Y) = \beta [W(X) \bullet Y - W(Y) \bullet X] \quad (24)$$

and its curvature by

$$\begin{aligned} R^A(X, Y)Z &= R(X, Y)Z + \beta [(\nabla_X W)Y - (\nabla_Y W)X] \bullet Z \\ &\quad + \beta^2 [W(X) \bullet W(Y)] \bullet Z. \end{aligned} \quad (25)$$

5.3. Reconstruction

Note that both the metric q and the second fundamental form K can be reconstructed from E and ∇^A . In fact, first reconstruct q from E as in Section 5.1. Then, let ∇ be the Levi-Civita connection to this metric q . Using $\beta \neq 0$, we now find $W(X) \bullet Y$ for all vector fields $X, Y \in T\Sigma$. Next, with the standard orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of \mathbb{R}^3 , we get

$$W(X) = \frac{1}{2} \sum_{i=1}^3 e(\mathbf{e}_i) \bullet (W(X) \bullet e(\mathbf{e}_i)). \quad (26)$$

Finally, the second fundamental form is easily derived from W as mentioned above.

5.4. Spin Bundle

In order to identify Ashtekar's variables as entities of an $SU(2)$ gauge field theory, we will now go from the tangent bundle to principal fibre bundles. For this, we will first transfer the structures to the oriented orthonormal frame bundle. This bundle, however, is an $SO(3)$ bundle, whence we will have to introduce a spin structure to go to the $SU(2)$ spin bundle. Of course, the tangent bundle is associated to both principal fibre bundles. Note that in the standard index notation of Subsections 4.1 and 4.2 it is barely impossible to distinguish between the $SU(2)$ and the $SO(3)$ case; in fact, the internal indices are Lie algebra indices, but the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic. Within principal fibre bundles, however, the structure *group* is encoded.

To implement this for the $SO(3)$ case, consider the canonical action ρ of $SO(3)$ on \mathbb{R}^3 . Given any metric q on Σ , the usual isomorphism

$$\begin{aligned} O_q^+(\Sigma) \times_{(SO(3), \rho)} \mathbb{R}^3 &\cong T\Sigma \\ [e, \mathbf{r}] &\longmapsto e(\mathbf{r}) \end{aligned} \quad (27)$$

identifies the tangent bundle as associated to the bundle of oriented orthonormal frames. Moreover, it induces an isomorphism between the space of metric covariant derivatives on $T\Sigma$ and the space of metric connection 1-forms in the oriented orthonormal frame bundle. In order to apply this identification to the Ashtekar connection, take some q -orthonormal local frame e on $U \subseteq \Sigma$ and define the local connection 1-form A by

$$A := \langle \nabla^A e, e \rangle : TU \longrightarrow \mathfrak{gl}(3). \quad (28)$$

More explicitly, $A(X)$ is uniquely given by

$$\langle A(X)\mathbf{r}, \mathbf{\eta} \rangle_{\text{Eucl}} = \langle \nabla_X^A [e(\mathbf{r})], e(\mathbf{\eta}) \rangle \quad (29)$$

for all vector fields X and $\mathbf{r}, \mathbf{\eta} \in \mathbb{R}^3$, viewing both sides as functions on U . One immediately checks that, as e is q -orthonormal, A maps to $\mathfrak{so}(3)$. Now, the Ashtekar connection ω_A in the oriented orthonormal frame bundle is simply the connection that is patched together from the local connections, i.e., mapped to A by the different pull-backs $f^*\omega_A$ with f being a mapping from some U to $O_q^+(\Sigma)$.

To finally establish the Ashtekar connection as an $SU(2)$ connection, first recall [18] that a spin structure on (Σ, q) is a pair $(S(\Sigma), \Lambda)$ consisting of an $SU(2)$ principal fibre bundle $\tilde{\pi} : S(\Sigma) \longrightarrow \Sigma$ and a double covering $\Lambda : S(\Sigma) \longrightarrow O_q^+(\Sigma, q)$, such that the following diagram commutes:

$$\begin{array}{ccc} S(\Sigma) \times SU(2) & \longrightarrow & S(\Sigma) \\ \text{both } 2:1 \downarrow \Lambda \times \lambda & & \downarrow \Lambda \\ O_q^+(\Sigma) \times SO(3) & \longrightarrow & O_q^+(\Sigma) \end{array} \quad \begin{array}{c} \nearrow \tilde{\pi} \\ \searrow \pi \end{array} \quad \Sigma.$$

In the diagram, the structure groups act horizontally, and λ denotes the double covering from $SU(2)$ to $SO(3)$. Using the isomorphism $\lambda_* : \mathfrak{su}(2) \longrightarrow \mathfrak{so}(3)$, we may lift any connection ω in $O_q^+(\Sigma)$ to a spin bundle connection $\tilde{\omega}$ via

$$\lambda_* \circ \tilde{\omega} := \Lambda^* \omega. \quad (30)$$

This way, the Levi-Civita connection is lifted to the spin connection $\tilde{\omega}_{LC}$, as well as the $SO(3)$ Ashtekar connection ω_A is lifted to the celebrated $SU(2)$ Ashtekar connection $\tilde{\omega}_A$. Note that the tangent bundle is associated also to the spin bundle via

$$T\Sigma \cong S(\Sigma) \times_{(SU(2), \rho \circ \lambda)} \mathbb{R}^3. \quad (31)$$

5.5. Constraints

So far, we have not been able to really derive the constraints within a purely differential-geometric framework. For this, one needs to carefully implement techniques from symplectic reduction and canonical transformations. For this short article, we refrained from doing this. Instead, we simply translate the index expressions for the constraints (see Subsection 4.3). Therefore, please be aware that the expressions might get modified later, if the “genuine” derivation has been done.

Denote the image of the standard basis of \mathbb{R}^3 under the frame e by (e_1, e_2, e_3) . Then we have (up to constant prefactors)

$$G \sim \sum_i \nabla_{e_i}^A e_i \quad (32)$$

$$V \sim \sum_{i,j} R^A(e_i, e_j)[e_i \bullet e_j] \quad (33)$$

$$H \sim R^A + (1 + \beta^2)[(\text{tr } W)^2 - \text{tr}(W^2)]. \quad (34)$$

5.6. Friedmann-Robertson-Walker Spacetimes

In highly symmetric cosmological models, the Ashtekar connection takes a particularly simple form. Here, we will review the results for the Friedmann-Robertson-Walker spacetime, where Σ is assumed to have constant sectional curvature κ . The Weingarten mapping is simply the Hubble “constant” h times the identity. From this, we get for the Ashtekar connection

$$\nabla_X^A Y = \nabla_X Y + \beta h X \bullet Y, \quad (35)$$

its torsion and curvature

$$T^A(X, Y) = 2\beta h X \bullet Y \quad (36)$$

$$R^A(X, Y)Z = [(\beta h)^2 - \kappa](X \bullet Y) \bullet Z, \quad (37)$$

as well as the constraints

$$G = 0 \quad (38)$$

$$D = 0 \quad (39)$$

$$H \sim 6(\kappa + h^2). \quad (40)$$

6. ... and Back to Indices

We have already mentioned that Σ is parallelizable, whence it has a global oriented orthonormal basis for $T\Sigma$. Therefore the internal indices can be assumed to be global indices. This need not be true for the spatial ones. In fact, it might happen that the manifold Σ cannot be covered by a single chart. Therefore, the expressions derived using indices in Section 4 will, in general, be of local nature only. To rederive them from our index-free quantities, let $\chi : U \rightarrow \mathbb{R}^3$ be a chart for some open $U \subseteq \Sigma$; this defines a local basis $\{\partial_1, \partial_2, \partial_3\}$ for the tangent space. Moreover, let $\{M_1, M_2, M_3\}$ be a basis of $\mathfrak{so}(3)$ with $[M_i, M_j] = \varepsilon_{ij}^{k} M_k$, e.g., we may take the choice $(M_i)_{jk} = \varepsilon_{ijk}$ from Subsection 4.2.

Taking some orthonormal frame e , we get the dreibein from $e(\mathfrak{e}_i) =: e_i^a \partial_a$ by means of the canonical basis (\mathfrak{e}_i) of \mathbb{R}^3 . The transition to the Ashtekar field E_i^a is now obvious. On the other hand, one can calculate [4] that

$$e^* \omega_A(\partial_a) = A_a^i M_i \quad (41)$$

reproduces the Ashtekar connection A_a^i . In the same way, the extrinsic curvature K_a^i can be obtained from

$$\widetilde{K}(X) = \langle W(\pi_* X), e \bullet e \rangle \quad (42)$$

with π being the bundle projection, as well as the spin connection Γ_a^i can be obtained from the bundle version of the Levi-Civita connection. Finally, the $\mathrm{SO}(3)$ indices can be reduced to the $\mathrm{SU}(2)$ indices by simply using the isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$.

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References

- [1] Ashtekar A 1986 New Variables for classical and quantum gravity *Phys. Rev. Lett.* **57** 2244–2247
- [2] Funkner S 2006 Die Geometrie der Ashtekarvariablen (Diplomarbeit) Universität Leipzig
- [3] Graupner S 2010 Ashtekar-Variablen: Bündeltheorie und Schwarze Löcher (Diplomarbeit) Universität Leipzig
- [4] Levermann Ph 2009 Geometrie des Ashtekar-Zusammenhangs (Diplomarbeit) Universität Hamburg
- [5] Fleischhack Ch and Levermann Ph 2011 Ashtekar Variables: Structures in Bundles *Preprint* 1112.1262 [math-ph]
- [6] Bernal A N and Sánchez M 2003 On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem *Commun. Math. Phys.* **243** 461–470 (*Preprint* gr-qc/0306108)
- [7] Bernal A N and Sánchez M 2006 Further Results on the smoothability of Cauchy hypersurfaces and Cauchy time functions *Lett. Math. Phys.* **77** 183–197 (*Preprint* gr-qc/0512095)
- [8] Fourès-Bruhat Y 1952 Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires *Acta Math.* **88** 141–225
- [9] Straumann N 2004 *General Relativity: With Applications to Astrophysics* (Springer)
- [10] Ringström H 2009 *The Cauchy Problem in General Relativity* (European Mathematical Society)
- [11] Thiemann Th 2007 *Modern Canonical Quantum General Relativity* (Cambridge University Press)
- [12] Ashtekar A 1987 New Hamiltonian formulation of general relativity *Phys. Rev.* **D36** 1587–1602
- [13] Barbero F 1995 Real Ashtekar Variables for Lorentzian Signature Space-Times *Phys. Rev.* **D51** 5507–5510 (*Preprint* gr-qc/9410014)
- [14] Immirzi G 1997 Real and complex connections for canonical gravity *Class. Quant. Grav.* **14** L177–L181 (*Preprint* gr-qc/9612030)
- [15] Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy C^* algebras *Knots and Quantum Gravity (1993, Riverside)* ed J C Baez (Oxford University Press) (*Preprint* gr-qc/9311010)
- [16] Marolf D, Mourão J M and Thiemann Th 1997 The status of diffeomorphism superselection in Euclidean (2+1) gravity *J. Math. Phys.* **38** 4730–4740 (*Preprint* gr-qc/9701068)

- [17] Kirby R C 1989 *The Topology of 4-Manifolds* (Springer)
- [18] Friedrich Th 1997 *Dirac-Operatoren in der Riemannschen Geometrie* (Vieweg)