Soliton Solutions and Bilinear Residue Formula for the Super Kadomtsev-Petviashvili Hierarchy

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1. This note is devoted to the study of a wave superfield and a soliton solution of the super Kadomtsev-Petviashvili (SKP) hierarchy. The SKP hierarchy was introduced by Manin-Radul [4] and extensively studied by the present authors [8.9.12] and Mulase [5]. Especially, in our previous paper [9], we revealed that the SKP hierarchy is equivalently transformed to the super Grassmann equation that connects a point in USGM (the universal super Grassmann manifold) with an initial data of a solution. In that argument, the Birkhoff (the Riemann-Hilbert) decomposition in the group of super micro-differetial operators plays a key role. However this operator formalism is inconvenient for treating geometrical solutions such as super soliton solutions, super quasi-periodic solutions which may be defined on superconformal curves. Therefore, as in the case of the ordinary soliton theory, we require a wave superfield.

Section 2 is a quick review of a wave function, the bilinear residue formula and the direct construction of soliton solutions to the KP hierarchy. In section 3 we introduce a wave superfield, the bilinear residue formula characterizing it, and construct directly super soliton solutions to the SKP hierarchy.

Analytical setting of the theory of the SKP hierarchy is completed by this note together with the previous paper [9] and the forthcoming paper [11]. We expect that quasi-periodic solutions can be constructed in our framework and that the theory of the super-moduli of superconformal curves will be clarified from the viewpoint of the SKP hierarchy (cf. [10]).

Details of this note will appear in [11].

2. Let us first introduce a wave operator:

$$W = W(x,t,\frac{\partial}{\partial x}) = \sum_{j=0}^{\infty} w_j(x,t)(\frac{\partial}{\partial x})^{-j} , \tag{2.1}$$

which is a monic microdifferential operator of 0-th order (i.e., $w_0 = 1$), and $t = (t_1, t_2, \cdots)$ denote infinite number of time variables. The KP hierarchy is linearized by the Sato equation [6]:

$$\frac{\partial W}{\partial t_n} = B_n W - W (\frac{\partial}{\partial x})^n , \tag{2.2}$$

where $B_n = (W(\frac{\partial}{\partial x})^n W^{-1})_+$ (the symbol "+" stands for taking the differential operator part). The compatibility conditions for (2.2) give rise to the Zakharov-Shabat representation of the KP hierarchy:

$$[\frac{\partial}{\partial t_m} - B_m, \frac{\partial}{\partial t_n} - B_n] = 0 .$$

A wave function and its dual version are introduced by

$$w(x,t,\lambda) = W(x,t,\frac{\partial}{\partial x})(\exp(x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n))$$

$$= (\sum_{j=0}^{\infty} w_j(x,t)\lambda^{-j})\exp(x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n) \ , \qquad (2.3)$$

$$w^*(x,t,\lambda) = (W^*(x,t,\frac{\partial}{\partial x}))^{-1}(\exp(-x\lambda - \sum_{n=1}^{\infty} t_n\lambda^n)) \ , \qquad (2.4)$$

where $W^* = \sum_{j=0}^{\infty} (-\frac{\partial}{\partial x})^{-j} w_j(x,t)$ is the formal adjoint operator of W. They obey the following linear equations:

$$\frac{\partial}{\partial t_n} w(x,t,\lambda) = B_n w(x,t,\lambda), \quad \frac{\partial}{\partial t_n} w^*(x,t,\lambda) = - B_n{}^* w^*(x,t,\lambda) \ .$$

Now let us assume that the coefficients $w_j(x,t)$ belong to C[[x,t]], the algebra of formal power series in (x,t). (Formal power series solutions to the KP hierarchy correspond to points in the largest cell UGM^{ϕ} of the universal Grassmann manifold UGM.) Then a wave function is completely characterized by the bilinear residue formula.

Theorem 1 [2]. Formal functions of the form (2.3), (2.4) are a wave function and a dual wave function for the KP hierarchy if and only if they satisfy

$$\mathrm{Res}_{\lambda=\infty} (w(x,t,\lambda)w^*(x',t',\lambda)) = 0 \ , \qquad (2.5)$$

for any x,x',t,t'.

Soliton solutions to the KP hierarchy are constructed by Date's direct method: Let α_j, β_j, c_j $(1 \le j \le N)$ be generic constants and consider the conditions on a wave function:

$$w(x,t,\lambda) = \left(\sum_{j=0}^{N} w_j(x,t)\lambda^{-j}\right)\exp\left(x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n\right) ,$$

$$w(x,t,\alpha_j) = c_j w(x,t,\beta_j) \quad (1 \le j \le N) \quad . \tag{2.6}$$

The condition (2.6) yields certain linear algebraic equations for the coefficients w_j $(1 \le j \le N)$. One can solve them by an elementary algebraic procedure and get an N-soliton solution.

3. Let $\mathcal{A}$ be a supercommutative algebra over $\mathbf{C}$, and (x,θ) be $(1|1)$-dimensional superspace variables, and $t = (t_1, t_2, \cdots)$ supertime variables (t_{2k} is even, t_{2k-1} is odd). A supercommutative algebra of superfields is, by definition,

$$\mathcal{S} = \mathbf{C}[[x,\theta,t]] \otimes \mathcal{A} \ .$$

Superdifferential operator $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ and super vector fields $D_{2\ell} = \frac{\partial}{\partial t_{2\ell}}$, $D_{2\ell-1} = \frac{\partial}{\partial t_{2\ell-1}} + \sum_{k=1}^{\infty} t_{2k-1}\frac{\partial}{\partial t_{2\ell+2k-2}}$ act on $\mathcal{S}$. The algebra $\mathcal{D}$ of superdifferential operators is defined by $\mathcal{D} = \mathcal{S}[D]$. Adding the formal inverse element $D^{-1} = \theta + \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\right)^{-1}$ to $\mathcal{D}$, one obtains the algebra of super microdifferential operators. Precisely,

$$\mathcal{E} = \mathbf{C}[[x,\theta,t]]((D^{-1})) \otimes \mathcal{A} \supset \mathcal{D} \ .$$

The algebras $\mathcal{S}$ and $\mathcal{E}$ are endowed with natural $\mathbf{Z}_2$-gradation [9].

Let W be a super wave operator:

$$W = W(x,\theta,t,D) = \sum_{j=0}^{\infty} w_j(x,\theta,t)D^{-j} \in \mathcal{E}_{\underline{0}} \ , \quad w_0 = 1, \ w_j \in \mathcal{S}_{\underline{j}} \ .$$

The SKP hierarchy is linearized by the Sato equations:

$$D_n(W) = \varepsilon_n(B_n W - WD^n), \quad n = 1,2,\cdots, \tag{3.1}$$

where $B_n = (WD^nW^{-1})_+$, and $\varepsilon_n = (-)^{n(n+1)/2}$. Let

$$H = H(x,\theta,t,\lambda,\xi) = x\lambda + \sum_{\ell=1}^{\infty} (-)^{\ell} t_{2\ell}\lambda^{\ell} + (\xi+h(t,\lambda))(\theta+h(t,\lambda)),$$

where (λ,ξ) are $(1|1)$-dimensional parameters, $h(t,\lambda) = \sum_{\ell=1}^{\infty}(-)^{\ell}t_{2\ell-1}\lambda^{\ell}$. It is easy to see that

$$(D^2-\lambda)(e^H) = 0, \quad D_n(e^H) = \varepsilon_n D^n(e^H). \tag{3.2}$$

Set

$$w(x,\theta,t,\lambda,\xi) = W(x,\theta,t,D)(\exp H(x,\theta,t,\lambda,\xi)), \tag{3.3}$$

$$w^*(x,\theta,t,\lambda,\xi) = W^*(x,\theta,t,D)^{-1}(\exp -H(x,\theta,t,\lambda,\xi)), \tag{3.4}$$

where $W^*(x,\theta,t,D) = \sum_{j=0}^{\infty}(-)^j\varepsilon_j D^{-j}w_j(x,\theta,t)$ is the formal adjoint operator of W [1]. They are called a wave superfield, a dual wave superfield of the SKP hierarchy, respectively. From (3.1), (3.2), they satisfy

$$D_n w(x,\theta,t,\lambda,\xi) = \varepsilon_n B_n w(x,\theta,t,\lambda,\xi) ,$$

$$D_n w^*(x,\theta,t,\lambda,\xi) = -\varepsilon_n B_n{}^* w^*(x,\theta,t,\lambda,\xi).$$

Now we need a superanalogue of Cauchy's residue formula: Let (λ,ξ) be coordinates on the $(1|1)$-dimensional super complex plane, $\Delta(d\lambda/d\xi)$ be the volume form on it, which is odd quantity, $f(\lambda,\xi)$ holomorphic superfield in (λ,ξ). Then the residue formula is given by [3]

$$\mathrm{Res}_{\lambda=\alpha}\, \Delta(d\lambda/d\xi)\, \frac{\xi-\eta}{(\lambda-\alpha-\xi\eta)^{n+1}} f(\lambda,\xi) = \frac{1}{n!}\,(D_{\lambda,\xi}^{2n} f)(\alpha,\eta) ,$$

$$\mathrm{Res}_{\lambda=\alpha}\, \Delta(d\lambda/d\xi)\, \frac{f(\lambda,\xi)}{(\lambda-\alpha-\xi\eta)^{n+1}} = \frac{1}{n!}\,(D_{\lambda,\xi}^{2n+1} f)(\alpha,\eta) ,$$

where $D_{\lambda,\xi} = \frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\lambda}$, and α is an even constant, η an odd constant.

Using the *duality of super Fourier transform* (see our forthcoming paper [11]), we can show the following.

Proposition 2. Let $P,Q \in \mathcal{E}$ be even operators. Then $PQ \in \mathcal{D}$ if and only if

$$\mathrm{Res}_{\lambda=\infty}\,(\Delta(d\lambda/d\xi)P(e^{\lambda x+\xi\theta})Q^{*}(e^{-\lambda x'-\xi\theta'})) = 0 \ ,$$

for any x,x',θ,θ'.

From this proposition we obtain the main theorem of this note.

Theorem 3 (*Bilinear residue formula*). Even superfields of the form (3.3), (3.4) are a wave superfield and its dual of the SKP hierarchy if and only if they enjoy

$$\mathrm{Res}_{\lambda=\infty}\,(\Delta(d\lambda/d\xi)w(x,\theta,t,\lambda,\xi)w^{*}(x',\theta',t',\lambda,\xi)) = 0, \qquad (3.5)$$

for any x,x',θ,θ',t,t'.

We proceed to the construction of soliton solutions. Let α_ν,β_ν, c_ν be even generic elements in $\mathcal{A}$ and η_ν,ω_ν be odd ones $(-2N\le\nu\le-1)$. Consider the following conditions on a wave superfield:

$$w(x,\theta,t,\lambda,\xi) = (\sum_{j=0}^{2N} w_j(x,\theta,\lambda)D^{-j})(\exp H(x,\theta,\lambda,t)),$$

where
$$D^{-2j}(\exp H) = \lambda^{-j}(\exp H),$$

$$D^{-2j+1}(\exp H) = \lambda^{-j}(\lambda\theta-\xi-h(t,\lambda))(\exp H),$$

and
$$w(x,\theta,t,\alpha_\nu,\eta_\nu) = c_\nu w(x,\theta,t,\beta_\nu,\omega_\nu) \quad \text{for even } \nu,$$

$$((\hat{D}^{-1})^{*}w)(x,\theta,t,\alpha_\nu,\eta_\nu) = c_\nu((\hat{D}^{-1})^{*}w)(x,\theta,t,\beta_\nu,\omega_\nu) \quad \text{for odd } \nu. \qquad (3.6)$$

The operator $\hat{D}$ is, by definition,

$$\hat{D} = \xi + \lambda\frac{\partial}{\partial\xi}\ , \quad \hat{D}^{-1} = \frac{\partial}{\partial\xi} + \lambda^{-1}\xi ,$$

and $\quad (\hat{D}^{-1})^{*}$ = the formal adjoint operator of $\hat{D}^{-1}$

$$= -\frac{\partial}{\partial\xi} + \lambda^{-1}\xi\ .$$

We remark that

$$((\hat{D}^{-1})^{*}w)(x,\theta,t,\alpha,\eta)$$

$$= \mathrm{Res}_{\lambda=\alpha}(\Delta(d\lambda/d\xi)W(x,\theta,t,D)D^{-1}(\exp H)\frac{\xi-\eta}{\lambda-\alpha-\xi\eta}\ ).$$

The conditions (3.6) read as the following linear equation:

$$(w_1,\cdots,w_{2N})\begin{pmatrix}(\varphi_{j,-2\nu})_{\substack{1\le j\le 2N\\ 1\le\nu\le N}} & \bigg| & (\varphi_{j+1,-2\nu+1})_{\substack{1\le j\le 2N\\ 1\le\nu\le N}}\end{pmatrix}$$

$$= -\ ((\varphi_{0,-2\nu})_{1\le\nu\le N}\ |\ (\varphi_{1,-2\nu+1})_{1\le\nu\le N}\ )\ , \tag{3.7}$$

where

$$\varphi_{j,\nu} = (D^{-j}e^{H})(\alpha_\nu,\eta_\nu) - c_\nu(D^{-j}e^{H})(\beta_\nu,\omega_\nu)\ .$$

Solving this equation, one gets an *N-soliton solution* to the SKP hierarchy. We can rewrite (3.7) into the following super Grassmann equation:

$${}^{t}\vec{w}\ \Phi\ \Xi = 0\ ,$$

where $\vec{w} = (w_{-j})_{j\in Z}$ $\quad (w_{-j} = 0$ for $j\ge 1)$,

$$\Phi = \exp\ (x\Lambda^2 + \theta\Lambda + \sum_{n=1}^{\infty} t_n\Gamma^n)\ , \quad \Lambda = (\delta_{\mu+1,\nu}),\ \Gamma = ((-)^{\nu}\delta_{\mu+1,\nu}),$$

$$\Xi = (\Xi_{\mu,\nu})_{\mu\in Z,\nu\in N^c} \quad \text{with}$$

$$\Xi_{\mu,\nu} = \begin{cases} \alpha_\nu^{\ \mu/2} - c_\nu\beta_\nu^{\ \mu/2} & (\mu:\text{even}) \\ -\eta_\nu\alpha_\nu^{\ (\mu-1)/2} + c_\nu\omega_\nu\beta_\nu^{\ (\mu-1)/2} & (\mu:\text{odd}) \end{cases} \quad \text{for } -2N\le\nu\le -1,\ \nu:\text{even}$$

$$\Xi_{\mu,\nu}=\begin{cases} -\eta_\nu \alpha_\nu{}^{\mu/2\ -1} + c_\nu \omega_\nu \beta_\nu{}^{\mu/2\ -1} & (\mu:\text{even}) \\ \alpha_\nu{}^{(\mu-1)/2} - c_\nu \beta_\nu{}^{(\mu-1)/2} & (\mu:\text{odd}) \end{cases} \quad \text{for } -2N\leq\nu\leq-1,\ \nu:\text{odd}$$

$$\Xi_{\mu,\nu}=\delta_{\mu\nu} \quad \text{for } -\infty<\nu<-2N\ .$$

The point of USGM represented by the superframe Ξ corresponds to the initial data of the N-soliton solution.

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