Joint Institute for Nuclear Research
Bogoliubov Laboratory of Theoretical Physics

SQS'05

Proceedings of International Workshop

Supersymmetries
and
Quantum Symmetries

Joint Institute for Nuclear Research
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SUPERSYMMETRIES AND QUANTUM SYMMETRIES

(SQS'05)

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The Proceedings include talks given at the International Workshop “Supersymmetries and Quantum Symmetries” (SQS’05) hold on 27 − 31 July 2005 in Dubna, Russia. The SQS workshops were initiated by Victor I. Ogievetsky and, since 1993, have being organized by JINR in Dubna. Since 1997 they are biennial and are intended to commemorate the contribution to theoretical physics of V. I. Ogievetsky, who passed away in 1996.

В сборник вошли доклады, представленные на международном рабочем совещании “Суперсимметрии и квантовые симметрии” (SQS’05), которое проходило с 27 по 31 июля 2005 г. в Дубне, Россия. Проведение рабочих совещаний по суперсимметрии было инициировано В. И. Огиевецким в 1993 г. в ОИЯИ. С 1997 г. эти совещания проходят каждые два года в память о выдающемся вкладе в теоретическую физику В. И. Огиевецкого, ушедшего из жизни в 1996 г.

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Foreword

The International Workshop *"Supersymmetries and Quantum Symmetries 2005"* (SQS'05) was held at the Bogoliubov Laboratory of Theoretical Physics from 27 to 31 July 2005. It continued a series of biennial SQS workshops initiated by Professor V. I. Ogievetsky (1928 – 1996). The previous SQS event was held in July 2003 also in Dubna.

The program of SQS'05, like in the previous years, covered several "hot" directions of modern theoretical physics. This time the basic subjects were string theory, quantum and geometric aspects of supersymmetric theories, the theory of higher spins, supersymmetric integrable models, quantum groups and noncommutative geometry, as well as the Standard Model and its supersymmetric extensions. The sessions included both the plenary talks presented by the world-recognized experts and shorter original reports on quite fresh results. Special attention was paid to such extremely hot topics as the theory of higher spins and its relationships with branes, fermions on superbranes with fluxes, the string theory-inspired approach to the problem of dark matter, matrix models, as well as the supergravity-inspired two-dimensional cosmological models, conformal field theories in higher dimension, string theory-motivated non-anticommutative deformations of supersymmetric theories, twistor and harmonic methods in gauge theories and strings, noncommutative geometry and noncommutative cousins of integrable and quantum-mechanics systems. A separate session was reserved for new developments in the quantum inverse scattering method and quantum groups.

Like the previous SQS workshop, SQS'05 featured the extraordinary activity of the talented young researchers, both from the West and East. The workshop was a natural continuation of the traditional Dubna Advanced Summer School on Modern Mathematical Physics this time basically devoted to supersymmetry and string theory. Many senior speakers and young participants of SQS'05 participated in this preceding event too.

The workshop was organized and financially supported by the Bogoliubov Laboratory of Theoretical Physics, JINR (Dubna). We should like to acknowledge the support from RFBR (grant 05-02-26060-г), as well as from the Heisenberg–Landau, Bogoliubov–Infeld and Votruba–Blokhintsev Programs.

Evgeny A. Ivanov, Boris M. Zupnik,

Editors

Contents

1 STRINGS, BRANES AND HIGHER SPINS

NonBPS-brane Decay as a Model for Cosmological Dark Energy

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Abstract

There are many different phenomenological models describing the cosmological dark energy and accelerating Universe by choosing adjustable functions. We consider a scenario based on the fundamental superstring theory. In this scenario the Universe is considered as a nonBPS 3-brane related with NSR string embedded in the 10-dimensional space time. Its dynamics is derived from the NSR string field theory, it has a tension and it is unstable due to a present of tachyon leaving in GSO$-$ sector. In flat case there is an exact compensation of the D3brane tension and the vacuum energy of the tachyon field. We explain a small value of the present day energy density (the cosmological constant) by a small deviation in a non-flat case from this exact compensation in the flat case. Studying the evolution of the string tachyon in the Friedmann metric we also advocate that the equation of state parameter of tachyon field becomes less then -1.

1 Introduction

Two observational projects evaluated the distance versus redshift relation for high values of the redshift with the supernova type Ia as standard candle arrived at the conclusion that the Universe is presently accelerating[1, 2].

The observations suggest that the bulk of energy density in the Universe is gravitationally repulsive and appears like an unknown form of energy (dark energy) with negative pressure. It is believed that 2/3 of the total density of the universe is in a form of dark energy.

Measurements of the cosmic microwave background [3] and the galaxy power spectrum [4] also indicate the existence of the dark energy.

There exist many different phenomenological models of dark energy. It is convenient to describe them by using the equation of state parameter $w = p/\rho$, where p is a pressure and ρ is the energy density. The analysis of the current observation data shows that w lies in the range

$$-1.61 < w < -0.78 \tag{1.1}$$

at 95% confidence level [5, 3, 4]. According the last ref. in [2] with a more restricted sample of 176 SN type Ia,

$$w = -1.02^{+0.13}_{-0.19}.$$

The precise value of the parameter w is one of the most important task in observational cosmology today. Note that, in spite of the fact that the evaluation of w from the observational data depends on the background model, on the sample of data and on the way the analysis is performed, a possibility of $w < -1$ is not excluded. In [6] it has been proposed a direct search strategy for $w < -1$.

From the theoretical point of view the specified domain of w (1.1) covers three essentially different cases: $w > -1$, $w = -1$ and $w < -1$.

- The first case is achieved in cosmological models with a scalar matter field and roughly speaking such types of models do not have theoretical problems except for a question of an origin of this scalar field. This scalar field should be extra light and hence it does not belong to the Standard Model set of fields.
- The second case is $w = -1$. This possibility is realized by means of the cosmological constant. This is a simplest candidate for dark energy. It is acceptable from a general point of view except for a problem of an order of the magnitude of the cosmological constant. It should be 10^{120} times less the natural theoretical prediction.
- The third case is $w < -1$. It is achieved in cosmological models with a scalar field with a "wrong" kinetic term (phantom scalar field). In this case all natural energy conditions are violated and there are problems of stability at classical and quantum levels. Thus, phantom becomes a great challenge for the theory while its support according to the supernovae data is strong.

Let us note in all tree cases there is a problem of a small value of the present day energy density.

One of possible ways to get a zero energy provides a supersymmetry, and a small deviation from zero energy could give a smooth supersymmetry breaking.

Here we propose to use another mechanism of compensation [7]. This compensation mechanism is related with Sen's conjectures. According to the Sen conjectures in the perturbative string vacuum there are unstable branes and a tension of branes is equal to the energy of non-perturbative tachyon vacuum (for review see [9],[10]).

We assume that this equilibrium of energies taking place in the flat space and is broken in non-flat cases. A disbalance of the brane tension and the energy of non-perturbative vacuum in a non-flat case can be found from a requirement of existence of a rolling solution starting in the perturbation vacuum and ending in the non-perturbative vacuum. In this talk we present results of numerical study of such solutions. In [8] has been consider a stringy model admitted such a solution and the energy density in this model is of order $M_s^2/g_o^2M_{Pl}^2$, where M_s^2 is a string mass scale, M_{Pl}^2 is a Plank mass scale and g_o is an open string coupling constant.

We also show that for nonlocal tachyon the condition $w < -1$ is realized without problems with unstability.

2 Non-BPS tachyon in Friedmann space-time

We consider a non-BPS tachyon leaving on 3-brane and interacting with gravity with the following action

$$S = \frac{M_p^2}{2}\int \sqrt{-g}d^4x\, R + S_{tach}\ . \tag{2.2}$$

where

$$S_{tach} = \int \sqrt{-g}d^4x\left(-\frac{q^2}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\phi^2 - \frac{1}{4}\Phi^4\right), \tag{2.3}$$

$\Phi = \exp(\frac{1}{2}\Box_g)\phi$, $\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, $q^2 = const < 1$. Here we assume that all constants are absorbed into M_p^2. The action (2.3) generalizes the non-BPS tachyon action obtained from low level truncated SFT to the case of a non-flat metric [11].

On space homogeneous configurations in the Friedmann metric

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{2.4}$$

the action (2.3) takes the form

$$S_{tach}[\phi] = \int \sqrt{-g}dt\left[\frac{1}{2}\phi^2(t) + \frac{q^2}{2}\dot{\phi}(t)^2 - \frac{1}{4}\Phi^4(t)\right], \tag{2.5}$$

where $\Phi = \exp(\frac{1}{2}\mathcal{D})\phi$, $\mathcal{D} = -\partial_t^2 - 3H(t)\partial_t$ and $H(t) = \dot{a}/a$, $\dot{a} = \partial_t a$. The Einstein equations have the form

$$3H^2 = \frac{1}{M_p^2}\ \rho \tag{2.6}$$

$$H^2 + 2\ddot{a}/a = -\frac{1}{M_p^2}\ p \tag{2.7}$$

with the energy and pressure densities are given by [11]

$$\rho = \frac{q^2}{2}(e^{-\frac{1}{2}\mathcal{D}}\dot{\Phi})^2 - \frac{1}{2}(e^{-\frac{1}{2}\mathcal{D}}\Phi)^2 + \frac{1}{4}\Phi^4 + \mathcal{E}_1 + \mathcal{E}_2] \tag{2.8}$$

$$p = \frac{q^2}{2}(e^{-\frac{1}{2}\mathcal{D}}\dot{\Phi})^2 + \frac{1}{2}(e^{-\frac{1}{2}\mathcal{D}}\Phi)^2 - \frac{1}{4}\Phi^4 - \mathcal{E}_1 + \mathcal{E}_2. \tag{2.9}$$

where

$$\mathcal{E}_1 = -\frac{1}{2}\int_0^1 d\rho(e^{\frac{1}{2}\tau\mathcal{D}}\Phi^3)\mathcal{D}e^{-\frac{1}{2}\tau\mathcal{D}}\Phi \tag{2.10}$$

$$\mathcal{E}_2 = -\frac{1}{2}\int_0^1 d\tau(\partial_t e^{\frac{1}{2}\tau\mathcal{D}}\Phi^3)\partial_t e^{-\frac{1}{2}\tau\mathcal{D}}\Phi \tag{2.11}$$

Equation of motion for the scalar field is

$$\left(q^2\mathcal{D} + 1\right)e^{-\mathcal{D}}\Phi = \Phi^3. \tag{2.12}$$

3 Rolling solution in flat space-time

Taking $H = 0$ in (2.12) we get the following equation in the flat space

$$\left(-q^2\partial_t^2 + 1\right) e^{\partial^2}\Phi = \Phi^3. \tag{3.13}$$

This equation contains infinite number of time derivatives, and actually can be written in the integral form. It has been shown numerically that for q^2 small enough there is a solution that interpolates between non trivial vacua $\Phi(\pm\infty) = \pm 1$ and $\Phi(0) = 0$ [12]. One can get an approximation to this solution expanding the exponent in (3.13) in powers of derivatives and keeping only the second derivatives,

$$\left((1-q^2)\partial_t^2 + 1\right)\Phi(t) = \Phi^3(t). \tag{3.14}$$

This equation describes a particle moving in the potential $V = \frac{(\Phi^2-1)^2}{4(q^2-1)} + const.$ For $q^2 < 1$ the factor $q^2 - 1$ flips the potential.

Equation (3.14) for $q^2 < 1$ has the kink solution Φ_{kink}. Kink interpolates between two vacua during infinitely long time and it is represented in Fig.1a by a thin line.

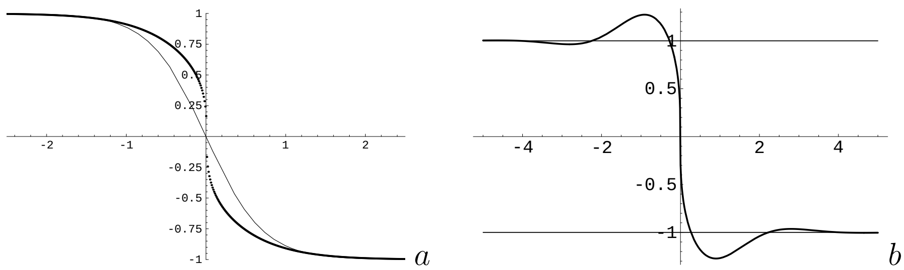

Figure 1: a) kink $\Phi_{kink}(t)$ (thin line) and $\Phi_0(t)$ (think line); b) $\Phi(t)$ for $q^2 = 0.96$;

Equation (3.13) for $q = 0$ (the p-adic string equation of motion for $p = 3$) also has a interpolating solution [13, 14, 12]. We denote it $\Phi_0(t)$ and plot it in Fig. 1a by think line. Note that the function $\Phi_0(t)$ is monotonic. From Fig.1a we see that Φ_{kink} and Φ_0 have different profiles, but this difference is not too big for large times. There is an essential difference at small time. Φ_{kink} has the finite first derivative at $t = 0$, meanwhile the first derivative of $\Phi_0(t)$ becomes infinite at $t = 0$. Note, that the derivative of the initial scalar field ϕ related with Φ via $\phi = e^{-\frac{1}{2}\mathcal{D}}\Phi$ is finite at $t = 0$. Therefore, higher derivatives in (3.13) change the profile of $\Phi_{kink}(t)$ only at small time and do not change the asymptotic behavior at large time.

Note, that small q^2 also does not change too much a profile of a solution to (3.13) interpolating between two vacua. This solution is plotted in Fig.1b. The profile of this solution is not a monotonic function. It can be presented as $\Phi(t) = \Phi_0(t) + \phi(t)$, where $\phi(t)$ describes oscillations around Φ_0 with decreasing amplitude. These oscillations are presented in Fig.1b.

4 Approximate solution of system of equations for Non-BPS tachyon in Friedmann space-time

Motivated by the flat case we make in (2.12) an approximation

$$\exp(\partial_t^2 + 3H(t)\partial_t)\Phi \approx (\ 1 + \partial_t^2 + 3H(t)\partial_t\ )\ \Phi \tag{4.15}$$

and keep only terms linear on $(1+\partial_t^2+3H(t)\partial_t\ )$. It is evident that this equation can be obtained from the action

$$S'_{scalar} = \int \sqrt{-g}d^4x \left(\frac{1-q^2}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right)\ . \tag{4.16}$$

We see that for $q^2 < 1$ we get the ghost sign in front of the kinetic terms. Assuming that $q^2 < 1$ we take for simplicity in the following formula $q^2 = 0$ (such q^2 can be achieved just by rescaling of time). The corresponding Einstein equations have the form (2.8), (2.9) with

$$\rho = -\frac{1}{2}\dot{\Phi}^2 + V(\Phi), \tag{4.17}$$

$$p = -\frac{1}{2}\dot{\Phi}^2 - V(\Phi) \tag{4.18}$$

and the equation for Φ field read

$$\ddot{\Phi} + 3H\dot{\Phi} = V'_\Phi \tag{4.19}$$

The equation state parameter w

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\Phi}^2 + V(\Phi)}{\frac{1}{2}\dot{\Phi}^2 - V(\Phi)} \tag{4.20}$$

is always less then -1, since w can be represented also as

$$w = -\frac{3H^2 + 2\dot{H}}{3H^2} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} \tag{4.21}$$

and from the equation of motions follows

$$\dot{H} = \frac{1}{2M_p^2}\dot{\Phi}^2, \tag{4.22}$$

i.e. $\dot{H}$ is positive.

5 Numerical solutions

Let us examine numerically solution of the system of equations (4.17) and (4.19) for the potential

$$V(\Phi) = \frac{1}{4}\left(\Phi^2 - 1\right)^2 \tag{5.23}$$

There are two independent initial conditions for $\Phi(0)$ and $\dot{\Phi}(0)$. If the initial position $\Phi(0)$ is on the the top of the hill (for the flip potential, Fig.1.b), $\Phi(0) = -1$, and the initial velocity is very small $\dot{\Phi}(0) \simeq 0$ (this corresponds to $H(0) \simeq 0$) then after some time Φ reaches the largest position and goes back to the bottom, and then performs few oscillations and stops at the bottom. The final value of H is $1/2\sqrt{3}$. The evolutions of the scalar field and log-derivative of the scale factor are represented in Fig.2.a and Fig.2.b.

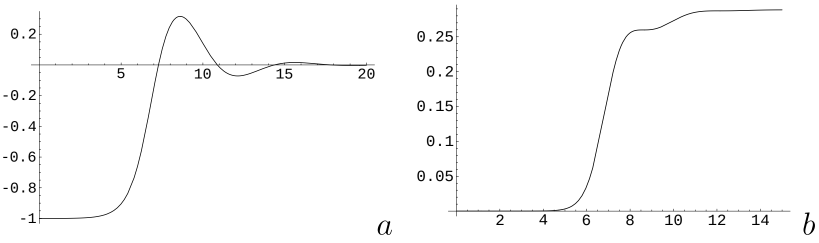

Figure 2: a) Plot of $\Phi = \Phi(t)$ with $\Phi(0) = -1$ and $\dot{\Phi} \simeq 0$; b) plot of $H = H(t)$

The evolution of the state equation parameter w is plotted in Fig.3a,b. It starts from -1, becomes a very big negative number when the field passes the bottom of the flip potential Fig.3a and goes with small fluctuations to -1 at large times. Fig.3.b shows that these fluctuations do not exceed -1.

To reach the top of the hill $\Phi = 1$ one has to increase the velocity, but since there is a restriction on the initial velocity $\dot{\Phi}(0)^2 \leq 2V(0)$, (the initial energy should be positive), one has to add a positive constant V_0 to the potential to be able to increase the initial velocity.

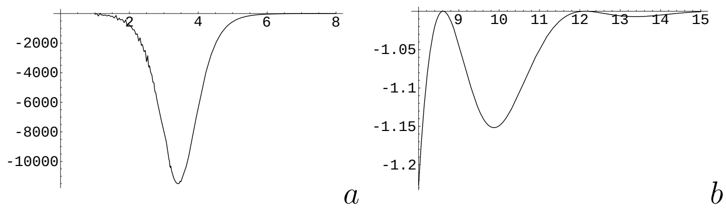

Figure 3: Plot of $w = w(t)$ for a) $0 < t < 8$ and b) for $8 < t < 15$

For large M_p and a suitable V_0 there is a solution that starts from the top of one hill with a non-zero velocity and reach the top of other hill during an infinite time, Fig.4. In this case during the initial stages of evolution the field is near the top of the hill, $\Phi = -1$ and the acceleration is small. At later times the field begins to evolve more rapidly towards the local minimum of the flip potential and the equation state parameter w becomes rather big. Finally, in very late time the field comes closed to the top of other hill, $\Phi = 1$

$$\Phi = 1 - Ae^{-\alpha t}, \tag{5.24}$$

where A is an arbitrary constant, $\alpha = (\frac{\sqrt{3V_0}}{M_p} + \sqrt{\frac{3V_0}{M_p^2} + 8})/2$ and a period of $w \simeq -1$ begins. This period is infinitely long because the flip potential has the maximum at $\Phi = 1$.

6 Exact Solution for Stringy Phantom Model

In [8] we have considered the phantom model with the potential

$$V(\phi) = \frac{1}{2}(1 - \phi^2)^2 + \frac{1}{12m_p^2}\phi^2(3 - \phi^2)^2. \tag{6.25}$$

and we have found that

$$\phi(t) = \tanh(t - t_0). \tag{6.26}$$

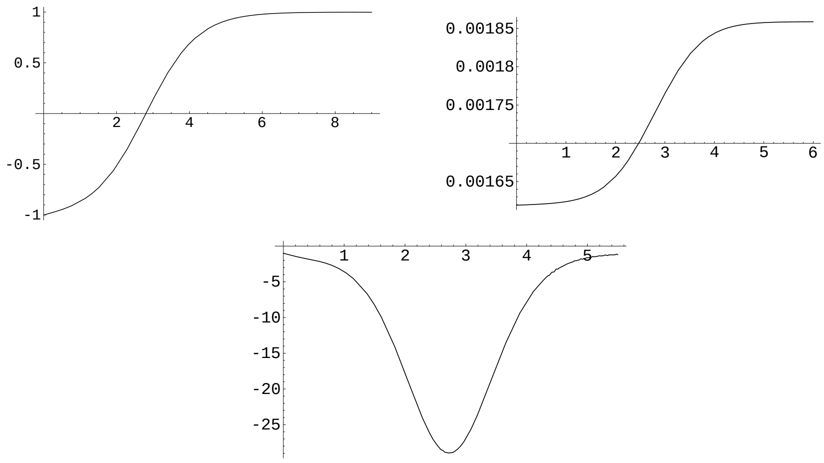

Figure 4: Plots of $\phi(t)$, $H(t)$ and $w(t)$ for $\phi(0) = -1, \dot{\phi}(0) = 0.1, V_0 = 0.02$

is a solution of the corresponding Friedman equations. The Hubble parameter for this case have the form

$$H = \frac{1}{2m_p^2}\phi\left(1 - \frac{1}{3}\phi^2\right) = \frac{1}{2m_p^2}\tanh(t)\left(1 - \frac{1}{3}\tanh(t)^2\right) \tag{6.27}$$

and goes asymptotically to $1/(3m_p^2)$ when t tends to infinity. The scale factor also can be written explicitly

$$a(\phi) = a_0 \frac{e^{\phi^2/(12m_p^2)}}{(1-\phi^2)^{1/(6m_p^2)}}, \tag{6.28}$$

The corresponding plots are drawn in Fig. 5 (Here we assume $m_p = 1$). We see that

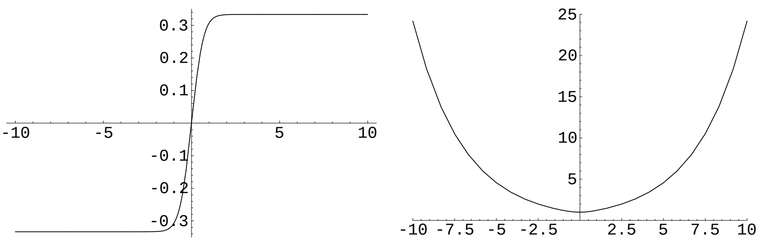

Figure 5: $H(t)$ (left) and $a(t)$ (right), $a_0 = 1$

behavior of H presented on Fig.4 for $t > 2.5$ and $H(t)$, Fig.5, $t > 0$ are very similar. More examples on can find in [15]

7 Conclusion

We have studied the evolution of open GSO $-$ NSR string tachyon in the Friedmann space-time. The corresponding solution in the flat space-time is known as a rolling tachyon and it describes the decay of the space filling D3 brane corresponding to the unstable perturbative vacuum to the local stable vacuum. We have performed calculations under the following approximations and assumptions:

- the level truncation and an approximation of a slow varying axillary field;
- a direct generalization of the tachyon nonlocal action to the Friedmann space-time;
- an effective local action approximation.

We have found that in the effective field theory approximation the equation of state parameter $w < -1$, i.e. one has a phantom Universe, but there is no problem with quantum instability. We have found that to reach the nonperturbative vacuum one has to add to the action a brane tension that larger that is required by the Sen hypothesis. This large brane tension can be interpreted as an effect of the closed string excitations.

Acknowledgements
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The Dirac Operator on Branes with Fluxes and Super–Potential Generation

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Abstract

We review the derivation of the Dirac equations for the worldvolume fermions on M–theory branes and the Type IIB D3–brane interacting with bulk supergravity fluxes and analyze conditions under which brane instantons wrapping a compactifying submanifold may generate a superpotential.

1 Introduction

Branes and fluxes of tensor gauge fields play an important role in compactifications of string/M-theory that may lead to a realistic model of fundamental interactions and cosmology. In particular, they may generate a potential for scalar fields in the effective 4D theory which may single out a proper physical vacuum. This is the so called moduli stabilization problem.

The scalar potential can be generated basically by two mechanisms i) perturbatively, by gauge field fluxes and ii) non-perturbatively, by gaugino condensation or by brane instantons. In search for the realistic theory one should take into consideration all possible mechanisms. In this contribution we shall discuss the possibility of generating the superpotential by non–perturbative brane instanton effects.

The study of non–perturbative corrections to the effective field theory due to brane instantons was put forward in [1, 2, 3]. In particular, Witten showed that in M–theory compactified on a complex Calabi–Yau 4–fold X_8 which preserves $N=2$ supersymmetry in an effective three–dimensional space–time M_3, under certain conditions an N=2 superpotential in M_3 can be generated by instanton effects produced by Euclidean M5–branes whose six–dimensional worldvolume wraps a holomorphic submanifold D_6 (a complex divisor) of X_8. Via M–theory/F–theory duality this is related to a type IIB string theory compactified to four dimensions with $N=1$ supersymmetry where an N=1 superpotential can be generated by instanton effects produced by D3-branes wrapping a complex submanifold D_4 of the compact space X_6.

For the M5–brane wrapped on D_6 to produce a non–zero contribution to the $N=2, D=3$ superpotential it should have at least two fermionic zero modes. To see roughly how this happens, consider a coupling of fermions to scalar fields in an effective $N=2, D=3$ supergravity with matter which is determined by the structure of the superpotential,

$$L_{int}^{N=2,D=3}(\varphi,\chi,\psi) = e^{K(\varphi)/2}\, W(\varphi)\, \bar{\psi}_m \gamma^{mn} \bar{\psi}_n - e^{K(\varphi)/2}\, D_{\mathcal{A}} D_{\mathcal{B}} W\, \chi^{\mathcal{A}} \chi^{\mathcal{B}} + \cdots\,, \tag{1}$$

where $K(\varphi)$ is a Kahler potential, $W(\varphi)$ is (the leading term of) the superpotential of chiral supermultiplets $\varphi^{\mathcal{A}}$, $\chi^{\mathcal{A}}$ in the effective $N=2, D=3$ theory and ψ_m are $N=2, D=3$ gravitini ($m=0,1,2$). To determine whether W receives corrections due to M5–brane wrapped over a 6–dimensional divisor D_6, we can consider e.g. the VEV of the fermion bilinear $\chi^{\mathcal{A}}\chi^{\mathcal{B}}$ in this instanton background. This is schematically given by the expression

$$\begin{aligned}
\langle \chi^{\mathcal{A}} \chi^{\mathcal{B}} \rangle &= \int \mathcal{D}\varphi\, \mathcal{D}\chi\, \mathcal{D}\psi\, e^{-I_{bulk}} \chi^{\mathcal{A}} \chi^{\mathcal{B}} \int \mathcal{D}x\, \mathcal{D}\theta\, e^{-\int_{D_6}(\sqrt{g}+iC_6(x)+\theta \not{D}\, \theta + \Psi\, \mathcal{V}\theta + \cdots)} \\
&= \int \mathcal{D}\varphi\, \mathcal{D}\chi\, \mathcal{D}\psi\, e^{-I_{bulk}} \chi^{\mathcal{A}} \chi^{\mathcal{B}}\, e^{-V_{D_6} - i\int C_6} \int \mathcal{D}x\, \mathcal{D}\theta\, e^{-\int (\theta \not{D}\, \theta + \Psi\, \mathcal{V}\theta + \cdots)},
\end{aligned} \tag{2}$$

where the second functional integral is taken over 5 transverse bosonic physical modes $x(\xi)$ and 16 fermionic physical modes $\theta(\xi)$ of the M5–brane.

$$S_{M5} = \int_{D_6} (\sqrt{g} + iC_6(x) + \theta \not{D}\, \theta + \Psi\, \mathcal{V}\theta + \cdots) \tag{3}$$

is the M5–brane worldvolume action [4] (where we neglected the contribution of the worldvolume chiral 2–form field) which describes the coupling of the M5–brane to the $D=11$ metric g_{MN}, the gravitino field Ψ_M ($M=0,1,\cdots,10$) and the dual potential 6–form C_6. Both $\chi^{\mathcal{A}}$ and ψ_m arise in the dimensional reduction of Ψ_M. V_{D_6} and $\int C_6$ are the effective 3d scalar field moduli associated, respectively, with the volume of the D_6 manifold and with the tensor field C_6. Finally, $\not{D}$ is a Dirac operator on the M5 brane worldvolume and $\mathcal{V}\theta$ is a vertex operator with $\mathcal{V}$ given by $\mathcal{V} = \Gamma_a\,(1+\Gamma_{M_5})$ ($a=1,\cdots,5,6$).

Now, in perturbation theory, the first term that can contribute to the fermion bilinear VEV is

$$\langle \chi^{\mathcal{A}} \chi^{\mathcal{B}} \rangle =$$

$$\int \mathcal{D}\varphi\, \mathcal{D}\chi\, \mathcal{D}\psi\, e^{-I_{bulk}} \chi^{\mathcal{A}} \chi^{\mathcal{B}}\, e^{-V_{D_6} - i\int C_6} \int \mathcal{D}x\, \mathcal{D}\theta\, (\int_{D_6} \chi^{\mathcal{A}}\, \mathcal{V}_{\mathcal{A}}\theta)(\int_{D_6} \chi^{\mathcal{B}}\, \mathcal{V}_{\mathcal{B}}\theta)\, e^{-\int \theta \not{D}\, \theta} + \cdots$$

If the Dirac operator $\not{D}$ on D_6 has exactly two zero modes (this is the minimal number of zero modes that arises when the M5–brane breaks half the supersymmetry), then this expression has the right number of θ insertions for the $\mathcal{D}\theta$ integration not to vanish. One can thus argue that if the brane instanton breaks half of $N = 2, D = 3$ supersymmetry and has only the two zero modes due to this breaking, then such an instanton can generate a superpotential. If the Dirac operator has additional (accidental) zero modes the situation becomes much more complicated. The above expression will then vanish, but higher order fermionic terms such as $\theta^3\Psi$, θ^4 *etc.*, whose exact structure has not been derived yet, could give rise to nonvanishing contributions. In such a situation one may at least try to formulate some general requirements under which the superpotential can be generated.

In the case of M theory compactified down to $N = 2, D = 3$, in the absence of the background and worldvolume fluxes, Witten argues that for an M5–brane generated superpotential to be possible, the submanifold D_6 should have certain topological properties, namely D_6 must have arithmetic genus $\chi_D = 1$, where

$$\chi_D = \sum_{n=0}^{3} (-1)^n h_n \,, \tag{4}$$

and h_n is the dimension of the space of $H^{0,n}(D_6)$. The index characterizes the anomaly of the $U(1)$ symmetry of the M5–brane action corresponding to rotations in the normal bundle structure group of the brane. In the dimensional reduced theory, this $U(1)$ symmetry descends to an R-symmetry, the R-charge of the chiral superspace measure $d^2\theta$ being -1, the R-charge of the factor $e^{-V_{D_6}-i\int C_6}$ in the superpotential W given by χ_D. The assumption that the complete theory is $U(1)$ invariant hence requires that for $\chi_D \neq 1$, W vanishes.

Corresponding topological restrictions also hold for the D3 brane wrapped over the divisor D_4 in type IIB String Theory (see [2] for details).

Since, fluxes should certainly be involved into the compactification process, branes will interact with them and then the question arises whether the interactions of M5, D3 and other branes with background and worldvolume fluxes may change the geometrical conditions under which non–perturbative superpotentials are generated?

To answer this question, one should get an explicit form of the Dirac operator for fermions on the branes coupled to fluxes and study its zero modes. To this end one should know either the explicit form of the brane action in the quadratic approximation in the worldvolume fermions θ, or the brane fermion equations linear in θ. Quadratic actions for fermions on branes coupled to a generic supergravity background with fluxes were derived in [5] for the M2–brane, in [6] for a D3–brane, in [7] for the Dp–branes, and in [8] for the M5–brane.

Using these results, the zero modes of the Dirac operators of brane fermions interacting with fluxes have been analyzed in [9, 10] for M5–brane instantons in M–theory on $M_3 \times X_8$, and in [11, 12, 13] for D3–brane instantons in type IIB String Theory on $M_4 \times X_6$. It has been shown that in some cases in which without fluxes brane instantons could not contribute to the superpotential, the interaction of brane fermions with fluxes can result in non–perturbative corrections to the superpotential. In this contribution we shall consider an example when this takes place. The back reaction of brane instanton corrections on the compactification setup has been discussed in [14], and in a more general context in [15].

2 Dirac Lagrangian for D3 brane fermions in type IIB supergravity

In the quadratic approximation for worldvolume fermions the Dirac Lagrangian on a D3 brane interacting with bosonic part of type IIB supergravity has the following form [7],

$$L_f^{D3} = \frac{1}{2} e^{-\Phi} \sqrt{-\det g}\, \bar{\Theta}(1 - \Gamma_{D3})[\Gamma^\alpha \delta\Psi_\alpha - \delta\lambda]\Theta(\xi)\,, \tag{5}$$

where $g_{\alpha\beta}$ is the induced worldvolume metric, $\Theta^I = (\theta^1, \theta^2)$ are 10D Majorana-Weyl spinors of the same chirality, $\Gamma_\alpha = \partial_\alpha x^M(\xi) E_M^A(x)\Gamma_A$ and $\Psi_\alpha = \partial_\alpha x^M \Psi_M$ are pullbacks of 10D gamma–matrices and of the gravitino on the D3 brane worldvolume parametrized by the coordinates ξ^α, and Φ is the 10D dilaton.

$$\delta\Psi_\alpha\Theta \equiv \partial_\alpha x^M(\delta_\Theta \Psi_M) = \mathcal{D}_\alpha \Theta\,, \quad \delta\lambda\Theta \equiv \delta_\Theta \lambda\,, \tag{6}$$

stand for local supersymmetry transformations of the 10D gravitino and gaugino pulled back on the D3 brane worldvolume, with the supersymmetry parameter being replaced by the fields $\Theta^I(\xi)$. The appearance of these terms in the brane action reflects the well known fact that fermionic fields on the branes are Volkov–Akulov goldstinos of spontaneously broken bulk supersymmetries. For the explicit form of (6) see [12]. To simplify things, in the Lagrangian (5) we have put to zero the worldvolume Dirac–Born–Infeld field strength $\mathcal{F}_2 = dA - B_2|_{D3} = 0$.

In the absence of the DBI field the matrix $\frac{1}{2}(1 - \Gamma_{D3})$ in (5), which is the D3 brane kappa–symmetry projector, has the following form

$$1 - \Gamma_{D3} = 1 - \frac{i}{4!}\sigma^2 \varepsilon^{abcd}\Gamma_{abcd} \equiv 1 - \sigma^2\gamma^5, \tag{7}$$

where σ^2 is the Pauli matrix acting on the Type IIB index I and γ^5 stands for the antisymmetrized product of four gamma–matrices along the D3 brane worldvolume (the indices a, b, c, d are worldvolume tangent space indices). The kappa–symmetry transformations of Θ^I are

$$\delta_\kappa \Theta = (1 - \sigma^2\gamma^5)\kappa(\xi)\,, \quad \delta_\kappa \bar{\Theta} = \bar{\kappa}(\xi)(1 + \sigma^2\gamma^5)\,. \tag{8}$$

They allow one to eliminate half of Θ^I. A possible gauge choice, which because of the form of (8) is consistent with any background, is to impose the condition

$$(1 - \sigma^2\gamma^5)\Theta = 0 \quad \Rightarrow \quad \theta^2 = i\gamma^5\theta^1\,. \tag{9}$$

Upon fixing the kappa–symmetry gauge (9) and making a Wick rotation we arrive at the following Dirac Lagrangian on D3

$$L_f^{D3} = \sqrt{+\det g}\,\theta^1(2e^{-\phi}\Gamma^a\nabla_a + \frac{1}{4}\tilde{G}_{abi}\Gamma^{abi} - \frac{1}{12}\tilde{G}_{ijk}\Gamma^{ijk} + \frac{i}{2}\,\nabla_a\tilde{\tau}\,\Gamma^a\,)\theta^1\,, \tag{10}$$

where $\tilde{G}_3 \equiv e^{-\Phi}H_3 + i(F_3 - C_0 H_3)\,\gamma_5$, is the combination of the NS flux $H_3 = dB_2$ and the RR flux $F_3 = dC_2$, and $\tilde{\tau} = C_{(0)}\,\gamma_5 + ie^{-\Phi}$ is the type IIB axion–dilaton. $\nabla = d + \omega + A$ is the worldvolume covariant derivative which contains a worldvolume spin connection ω and the normal bundle gauge connection A.

From the form of (10) we conclude that, at least in the absence of the worldvolume flux, the dynamical D3 brane fermions do not couple to the self–dual RR flux F_5, and that the fermions couple only to those 3–form fluxes H_3 and F_3 which have all three legs or only one leg in the directions orthogonal to the D3 brane worldvolume. These normal directions are indicated by the indices $i, j, k = 1, 2, \cdots, 6$, while $a, b = 1, 2, 3, 4$ stand for the tangent space directions along the brane.

Apart from the condition that the DBI field is zero and, hence, the worldvolume pullback of B_2 is the pure gauge, i.e. $H_3|_{D3} = 0$, the Lagrangian (10) describes the coupling of the D3 brane fermions to a *generic* type IIB supergravity bosonic background. Since we are interested in brane instanton effects in type IIB String Theory compactified on $M_4 \times X_6$ with the D3-brane wrapped over a $D_4 \subset X_6$, we should further specify the setup:

- we assume that flux vacuum expectation values do not break Lorentz invariance of M_4 space-time, thus the fluxes with all indices orthogonal to the D3–brane are zero $\tilde{G}_{ijk} = 0$, and $\tilde{G}_{abi}$ has only X_6 indices, i.e. $a, b \subset D_4$ and $i = 1, 2$;
- the axion $C_0(x)$ and the dilaton $\Phi(x)$ are assumed to be constant;
- the compactification is assumed to preserve $N = 1$ supersymmetry in the effective 4D theory. This imposes certain restrictions on the flux G_3 in X_6 [16] to be specified below.

With the above assumptions the Dirac equations which one gets from the Dirac Lagrangian (10) have a rather simple form

$$(\Gamma^a \nabla_a + \frac{1}{8}\tilde{G}_{abi}\Gamma^{abi})\theta^1 = 0\,. \tag{11}$$

3 The analysis of the Dirac equations

To study the solutions of eq. (11) on the complex manifold $D_4 \subset X_6$ it is useful to switch to complex notation. We replace the ‘real’ indices $a, b = 1, 2, 3, 4$ of D_4 with the ‘complex’ index $a = 1, 2$ and its conjugate $\bar{a} = \bar{1}, \bar{2}$ and the index $i = 1, 2$ of the directions of X_6 normal to D_4 with $z, \bar{z}$. In this notation the Hermitian conjugate gamma matrices are

$$\Gamma^a,\ \Gamma^i \ \Rightarrow\ (\Gamma^a, \Gamma^z), \quad (\Gamma^{\bar{a}}, \Gamma^{\bar{z}})\,. \tag{12}$$

We can now establish a one–to–one correspondence between the components of the spinor and anti–holomorphic forms on X_6 and on $D_4 \subset X_6$ as follows. We define a Clifford vacuum $|\Omega\rangle$ as a spinor ‘state’ annihilated by half of the gamma matrices (12)

$$\Gamma^a|\Omega\rangle = 0\,, \qquad \Gamma^z|\Omega\rangle = 0\,. \tag{13}$$

Then the components of a generic spinor $\eta = (\eta_+, \eta_-)$ on X_6 are generated by acting on the Clifford vacuum with the rising operators $\Gamma^{\bar{a}}$ and $\Gamma^{\bar{z}}$:

$$\eta_+ = \phi\,|\Omega\rangle + \phi_{\bar{a}}\,\Gamma^{\bar{a}}|\Omega\rangle \ + \phi_{\bar{a}\bar{b}}\,\Gamma^{\bar{a}\bar{b}}|\Omega\rangle\,, \tag{14}$$

$$\eta_- = \phi_{\bar{z}}\,\Gamma^{\bar{z}}|\Omega\rangle \ + \phi_{\bar{a}\bar{z}}\,\Gamma^{\bar{a}\bar{z}}|\Omega\rangle \ + \phi_{\bar{z}\bar{a}\bar{b}}\,\Gamma^{\bar{z}\bar{a}\bar{b}}|\Omega\rangle\,, \tag{15}$$

such that

$$\eta_\pm = \pm \frac{1}{2}(\Gamma^z\Gamma^{\bar z} - \Gamma^{\bar z}\Gamma^z)\,\eta_\pm := \pm\,\Gamma^{z\bar z}\,\eta_\pm \tag{16}$$

are eigenstates of the generator of the $U(1)$ rotations in the D_4 normal bundle.

In the case under consideration, the eight fields $\phi(\xi)$, $\phi_{\bar z}(\xi)$, $\phi_{\bar a}(\xi)$, $\phi_{\bar a\bar b}(\xi)$, $\phi_{\bar a\bar z}(\xi)$, $\phi_{\bar a\bar b\bar z}(\xi)$ on the D3 brane worldvolume $D_4 \subset X_6$ are components of $(0,n)$–forms ($n = 0,1,2,3$) in X_6. Instead of splitting the X_6 spinor into the parts of the positive and negative $U(1)$ charge, we can split it into the two spinors of positive and negative X_6 chirality

$$\eta = (\eta_6^+\,, \eta_6^-)\,, \quad \eta_6^\pm = \pm\Gamma^{1\bar 1 2\bar 2 z\bar z}\,\eta_6^\pm\,, \tag{17}$$

$$\eta_6^+ = \phi\,|\Omega\rangle + \phi_{\bar a\bar z}\,\Gamma^{\bar a\bar z}|\Omega\rangle\; + \phi_{\bar a\bar b}\,\Gamma^{\bar a\bar b}|\Omega\rangle\,, \tag{18}$$

$$\eta_6^- = \phi_{\bar z}\,\Gamma^{\bar z}|\Omega\rangle\; + \phi_{\bar a}\,\Gamma^{\bar a}|\Omega\rangle\; + \phi_{\bar z\bar a\bar b}\,\Gamma^{\bar z\bar a\bar b}|\Omega\rangle\,. \tag{19}$$

Then, the 10D positive chirality spinor of the D3 brane Dirac Lagrangian (10) on $M_4\times X_6$ can be represented as

$$\theta^1 = \theta\otimes\eta_6^+ \quad\oplus\quad \bar\theta\otimes\eta_6^- \tag{20}$$

where θ and $\bar\theta$ are, respectively chiral and anti-chiral (two–component) spinors on M_4.

Note that in the example of the compactification on $X_6 = K3\times T^2/Z_2$ with the D3 brane wrapping $K3$, which we consider below, the covariantly constant spinor on $K3$ has a definite (say positive) chirality. In this case the components $\phi_{\bar a}$ and $\phi_{\bar a\bar z}$ of (14), (15), (18) and (19) vanish, and the spinors (14), (15) of the positive and negative $U(1)$ charge coincide with the spinors (18) and (19) of positive and negative X_6 chirality, respectively.

Let us now substitute (20) into the Dirac equations (11) and rewrite them as equations for the $(0,n)$–forms ϕ. For the flux compactification to preserve $N=1$ supersymmetry, the flux $G_3 = F_3 - iH_3$ in X_6 should be a primitive (2,1)–form [16]. This implies that in (11) the non–zero components of G_3 are $G_{ab\bar z}$ and $G_{za\bar b}$ which in addition satisfy the primitivity condition $G_{za\bar b}g^{a\bar b} = 0$, where $g^{a\bar b}$ is a Kahler metric on D_4. Taking this into account we reduce the Dirac equations (11) to the following equations on the (0,2)–forms

$$\partial_{\bar a}\phi + 4g^{\bar bc}\nabla_c\phi_{\bar b\bar a} = 0\,, \quad g^{\bar ba}\nabla_a\phi_{\bar b} = 0\,, \quad \nabla_{[\bar a}\phi_{\bar b]} = 0\,, \quad \nabla_{[\bar a}\phi_{\bar b]\bar z} = 0\,,$$
$$\nabla_{\bar a}\phi_{\bar z} + 4g^{\bar bc}\nabla_c\phi_{\bar b\bar a\bar z} - 2i\bar G_{\bar a\bar z b}\phi^b = 0\,, \quad g^{\bar ab}\nabla_b\phi_{\bar a\bar z} + 4iG_{ab\bar z}\phi^{ab} = 0\,, \quad \phi^{ab} = g^{a\bar a}g^{b\bar b}\,\phi_{\bar a\bar b}\,, \tag{21}$$

where we have skipped the M_4 part of the 10D spinor field (20).

The analysis of the equations (21) shows [12] that the $(0,n)$–forms ϕ are harmonic, i.e. closed $\bar\partial\phi = 0$ and co–closed $\partial^{\bar a}\,\phi_{\bar a\,\cdots} = 0$. Thus the problem of counting the zero modes of the Dirac operator reduces to the counting of the dimension $h_{(0,n)}$ of the space of the harmonic $(0,n)$–forms on D_4. To get this number we should specify X_6 and its submanifold D_4.

4 The example of the orientifold $X_6 = K3\times T^2/Z_2$

Consider the type IIB string compactified on $X_6 = K3\times T^2/Z_2$. It is of interest to note that $K3$ is a Ricci–flat compact space with a self–dual Riemann tensor and so is regarded as a gravitational instanton in the theory of quantum gravity. The D3 brane wraps $K3$ and is at a fixed point of the orientifold T^2/Z_2. The fixed point is invariant under the reflection of the T^2 coordinates $z \to -z, \;\; \bar z \to -\bar z$. The reflection acts on the 10D spinors as $\Theta^I \to \Gamma^{z\bar z}(\sigma^2\Theta)^I$, therefore at the fixed point Θ must satisfy the orientifold projection

$$(1-\sigma^2\Gamma^{z\bar z})\Theta = 0 \qquad \Rightarrow \qquad \theta^2 = i\Gamma^{z\bar z}\theta^1\,. \tag{22}$$

The orientifolding projection commutes with the kappa–symmetry condition (9), which means that the both conditions are compatible and result in the additional constraint on the independent spinor θ^1 (20) to be chiral on X_6. Then η_6^- defined in (19) is zero and θ^1 is also chiral on M_4 [1].

Let us analyze the number of the solutions of the equations (21). First, consider the case of the zero flux, the effective 4D theory being $N=2$ supersymmetric. On $K3$ there are no non–trivial harmonic one–forms, $h_{(0,1)}=0$, hence $\phi_{\bar a} = \phi_{\bar a \bar z} = 0$. As we have found, the combination of the orientifolding projection with κ–symmetry gauge fixing eliminates η_6^- (19). Hence, we must put to zero also $\phi_{\bar z}$ and $\phi_{\bar a\bar b\bar z}$. Thus, if the flux is zero, we are left with the harmonic zero–form ϕ and the two–form $\phi_{\bar a\bar b}$. The dimension of the corresponding spaces of these forms on $K3$ is $h_{0,0}=h_{0,2}=1$. Therefore the number of the zero modes of the Dirac operator is $4 = 2(h_{0,0}+h_{0,2})$, where the factor 2 takes into account the 2 components of the chiral spinor (20) on M_4. Following our general arguments in the Introduction (about at least two zero modes) one might assume that in this case the superpotential can be generated. However, the index which characterizes the $U(1)$–anomaly produced by the zero mode integration measure is given by $\chi_{D3} = h_{0,0}+h_{0,2} = 2$. Generically, we define the index $\chi_{D3} = \frac{1}{2}(N_+ - N_-)$ as the difference between the number of the D3 zero modes with the positive (14) and negative (15) $U(1)$–charge [12]. This index is analogous and actually is related via F–theory uplifting to the M–theory index (4) [2]. The index should be equal to one in order to generate a superpotential and, therefore, we conclude that in this case the superpotential cannot be generated.

Let us now switch on the flux which breaks $N=2$ supersymmetry in M_4 down to $N=1$. In $X_6 = K3\times T^2/Z^2$ [17] it has the form $G_3 = \Omega\wedge d\bar z$, where Ω is a harmonic (2,0)–form on $K3$. Since on $K3$ $h_{(2,0)} = h_{(0,2)} = 1$, the anti–holomorphic form $\phi_{\bar a\bar b}$ can only be proportional to $\bar\Omega$, i.e. $\phi_{\bar a\bar b} = c\bar\Omega_{\bar a\bar b}$. Taking into account that on $K3$ $\phi_{\bar a\bar z}=0$, the non–zero flux equation in (21) takes the form

$$4iG_{ab\bar z}\phi^{ab} = 4ic\;\Omega_{ab}\bar\Omega^{ab} = 0 \quad\Rightarrow\quad c=0 \quad\Rightarrow\quad \phi_{\bar a\bar b} = 0\,. \tag{23}$$

We are left with a single non–zero ϕ, which implies that the Dirac operator has two zero modes with the positive $U(1)$ charge and the index $\chi_{D3} = h_{0,0} = 1$. We therefore conclude that in this case , with fluxes, a superpotential can be generated.

[1] Note that the κ–symmetry gauge fixing condition $\theta^2=0$, which can also be written as $(1-\sigma^3)\,\Theta = 0$, *anti*–commutes with (22) and hence is incompatible with orientifolding. To see this one should (before imposing any gauge fixing condition) to analyze how θ^1 and θ^2, which are *subject* to the orientifolding projection (22), transform under kappa–symmetry (8) whose parameter is also subject to orientifolding. If we do this we shall find that $\delta\theta^1 = P\kappa^1$, $\delta\theta^2 = P\kappa^2$, where $P = (1+\gamma^5\Gamma^{z\bar z})$ is the X_6 chirality projector. Because of this additional projector only half of the components of θ^1 or of θ^2 can be eliminated, but not all of them. Another possibility of seeing this is just to note that by orientifolding θ^1 and θ^2 are related $\theta^1 = i\Gamma^{z\bar z}\theta^2$, so if we put $\theta^2=0$, also θ^1 would be zero and we would not have any physical fermions on the brane.

5 Comment on anti–brane instantons

The discussion above concerned the corrections to the superpotential of the holomorphic type
$e^{-V_{D3}-i\int C_4}\langle\psi\psi\rangle$. Since the full effective Lagrangian is real there should also be anti–holomorphic corrections of the type $e^{-V_{D3}+i\int C_4}\langle\bar\psi\bar\psi\rangle$, where C_4 enters with the opposite sign. The object which has the opposite C_4 charge is the anti–D3 brane and as such it also has the opposite sign in the kappa–symmetry projector in comparison with that of the D3–brane (7). The analysis of the zero modes of the anti–D3 brane Dirac operator can be carried out in the way discussed above. As a result one can find that the anti–D3 brane wrapping $K3$ and interacting with the primitive flux G_3 will have two fermionic zero modes corresponding to the form $\phi_{\bar a\bar b\bar z}$ and to anti–chiral fermions $\bar\theta$ on M_4, see (17)-(19). The $U(1)$ charge of these modes is negative and the corresponding index is $\chi_{\bar D3} = -h_{(0,2)} = -1$ indicating that the anti–D3 brane can generate the complex conjugate part of the superpotential in accordance with (anti)instantons in Yang–Mills theories. From the point of view of our conventions, the author of [13] studied non–perturbative corrections due to anti–brane (anti)instantons.

6 Concluding remarks

We have demonstrated that the potential of scalar fields in effective N=1, D=4 theory obtained by compactifications of string theory can receive non-perturbative corrections due to brane instantons wrapping compact submanifolds and interacting with non-zero fluxes of the compactification background. Other examples have been considered in [9]–[13]. Brane instanton corrections to the superpotential can stabilize scalar moduli and hence should be taken into account when carrying out the analyses in search for phenomenologically relevant models of particle interactions and cosmology derived from string theory.

It would be interesting to look for more examples of brane instantons generating scalar field potentials and to analyze whether also the *worldvolume* fluxes on D-branes and the M5-brane can non-trivially contribute into the moduli stabilization and try to understand (at least in some cases) an explicit structure of the pre-exponential factor of the instanton corrections to the superpotential.

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Ten-Dimensional Supergravity Revisited

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Abstract

We show that the exisiting supergravity theories in ten dimensions can be extended with extra gauge fields whose rank is equal to the spacetime dimension. These gauge fields have vanishing field strength but nevertheless play an important role in the coupling of supergravity to spacetime filling branes.

We discuss the role of these gauge fields in the construction of string theories with sixteen supercharges and mention their relation with a conjectured hyperbolic symmetry underlying string theory and M theory. We conjecture the existence of a solitonic supersymmetric and kappa-symmetric NS 9-brane in both Type IIA and Type IIB string theory.

Introduction

The Type II supergravity theories in ten dimensions form a starting point from which all lower dimensional maximal supergravities can be derived. The Type IIB [1] and IIA theory [2], with two supercharges of equal (opposite) chirality were both constructed around 1984. The Type IIA theory follows by dimensional reduction from $D = 11$ supergravity. It was extended in 1986 to include a massive parameter [3]. The IIB theory does not appear to have a higher dimensional origin. The bosonic fields of the two theories are

$$\text{IIA}: \qquad g_{\mu\nu},\ \phi,\ B_{(2)},\ C_{(1)},\ C_{(3)}\,, \tag{1}$$

$$\text{IIB}: \qquad g_{\mu\nu},\ \phi,\ B_{(2)},\ C_{(0)},\ C_{(2)},\ C_{(4)}\,. \tag{2}$$

The subscripts (n) indicate the rank of an antisymmetric tensor gauge field, or n-form field. The IIB 4-form satisfies a self-duality relation, which prevents the construction of a covariant action.

A natural extension in both theories is the addition of duals of the n-form fields. In this way one can associate to every n-form field an $8-n$-form field ($n \geq 0$). The $n+1$-form curvatures are then related by a duality relation to the corresponding $9-n$-form curvatures. These forms therefore do not introduce new degrees of freedom, but instead provide a alternative way to view the role of these propagating fields. This is particularly profitable in the coupling of these fields to extended objects or branes. A p-brane, with p spatial extensions, couples in a natural way to a $p+1$-form field. The dual forms are therefore useful in studying the properties of p-branes with $p \geq 4$ (for $p = 3$ the brane couples to $C_{(4)}$, which is its own dual). The introduction of dual forms for the RR potentials $C_{(n)}$ has led to a completely "democratic" formulation of IIA and IIB supergravity, where all RR forms appear simultaneously [4].

It is also possible to introduce n-form fields with rank $n \geq 9$. These do not carry propagating degrees of freedom, and are therefore not dual to the physical supergravity fields. Nevertheless, they also have interesting applications. In [5] the $C_{(9)}$ field in the massive IIA theory played an essential role in understanding the 8-brane domain wall. The dual of the curvature $G_{(10)}$ plays the role of a cosmological constant.

Ten-form fields couple to space-time filling branes. These are related to truncations of the IIB theory to $N = 1$ supersymmetric theories. A 9-brane charge is by itself inconsistent. This can be resolved by adding opposite charge on an orientifold plane, which triggers the truncation to a Type I string theory. The introduction of 10-form fields in IIB supergravity and the corresponding truncation to $N = 1$ were considered in [7]. There two 10-forms were obtained. One is an RR form $C_{(10)}$, which will reappear in our present work. The other, called $B_{(10)}$, has the wrong tension to be understood as the S-dual of $C_{(10)}$, so that these two fields could not arise from an $SU(1,1)$ [8] (see Section 2) doublet. This problem will be resolved in this talk.

Supersymmetry, $SU(1,1)$ and form-fields

Since 9- and 10-forms do not carry physical degrees of freedom their number is not a priori limited. Of course they must be consistent with supersymmetry, and this turns out to lead to restrictions. The purpose of our work [6] is precisely to establish how many of these forms are possible in IIB supergravity, and to classify them in the correct $SU(1,1)$ representations. A similar investigation of IIA is presently under way [9], see below.

The starting point of our analysis is IIB supergravity without dual potentials. The theory exhibits an explicit $SU(1,1)$ symmetry, which acts on the two bosonic fields. The scalars parametrize an $SU(1,1)/U(1)$ coset. The scalars and fermions in the theory each has a charge associated with the local $U(1)$ symmetry, the gauge fields have zero charge. Under $SU(1,1)$ the fields $B_{(2)}$ and $C_{(2)}$ form a doublet $A^\alpha_{(2)}$ (satisfying $A^1_{(2)} = (A^2_{(2)})^*$, while $C_{(4)}$ (also written as $A_{(4)}$) corresponds to a singlet. The scalars are conveniently written as a matrix U:

$$U = \begin{pmatrix} V^1_- & V^1_+ \\ V^2_- & V^2_+ \end{pmatrix} . \tag{3}$$

Here $V\pm^\alpha$, with charge ± 1 and with $\alpha = 1, 2$, form doublets of $SU(1,1)$. They are constrained by the relation

$$V^\alpha_- V^\beta_+ - V^\alpha_+ V_-\beta = \epsilon^{a\beta} . \tag{4}$$

The supersymmetry transformations are, to terms bilinear in fermions:

$$\begin{aligned}
\delta e_\mu{}^a &= i\bar\epsilon\gamma^a\psi_\mu + i\bar\epsilon_C\gamma^a\psi_{\mu C} \quad ,\\
\delta\psi_\mu &= D_\mu\epsilon + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon + \frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon_C \quad ,\\
\delta A^\alpha_{\mu\nu} &= V^\alpha_-\ \bar\epsilon\gamma_{\mu\nu}\lambda + V^\alpha_+\ \bar\epsilon_C\gamma_{\mu\nu}\lambda_C + 4iV^\alpha_-\ \bar\epsilon_C\gamma_{[\mu}\psi_{\nu]} + 4iV^\alpha_+\ \bar\epsilon\gamma_{[\mu}\psi_{\nu]C} \quad ,
\end{aligned}$$

$$
\begin{aligned}
&\delta A_{\mu\nu\rho\sigma} = \bar\epsilon\gamma_{[\mu\nu\rho}\psi_{\sigma]} - \bar\epsilon_C\gamma_{[\mu\nu\rho}\psi_{\sigma]C} - \frac{3i}{8}\epsilon_{\alpha\beta}A^\alpha_{[\mu\nu}\delta A^\beta_{\rho\sigma]} \quad , \\
&\delta\lambda = iP_\mu\gamma^\mu\epsilon_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon \quad , \\
&\delta V^\alpha_+ = V^\alpha_-\ \bar\epsilon_C\lambda \quad , \\
&\delta V^\alpha_- = V^\alpha_+\ \bar\epsilon\lambda_C \quad .
\end{aligned} \tag{5}
$$

Here we have introduced

$$
\begin{aligned}
P_\mu &= -\epsilon_{\alpha\beta}V^\alpha_+\partial_\mu V^\beta_+ \, , \\
Q_\mu &= -i\epsilon_{\alpha\beta}V^\alpha_-\partial_\mu V^\beta_+ \, , \\
G_{\mu\nu\rho} &= -\epsilon_{\alpha\beta}V^\alpha_+F^\beta_{\mu\nu\rho} \, , \\
F^\alpha_{\mu\nu\rho} &= 3\partial_{[\mu}A^\alpha_{\nu\rho]} \, , \\
F_{\mu\nu\rho\sigma\tau} &= 5\partial_{[\mu}A_{\nu\rho\sigma\tau]} + \frac{5i}{8}\epsilon_{\alpha\beta}A^\alpha_{[\mu\nu}F^\beta_{\rho\sigma\tau]} \, .
\end{aligned} \tag{6}
$$

Q_μ is the $U(1)$ gauge field which is implicitly present in the covariatizations in (5). An important property of these transformations is that the commutator of two supersymmetry transformations on the bosonic gauge fields closes on translations and gauge transformations.

The way to obtain extensions of the supergravity multiplet above is to use this property of closure: we assume an initial form of the supersymmetry transformation of a proposed field, including free parameters, and determine these parameters by requiring closure. Since no new degrees of freedom can be introduced, closure will also require a relation between the additional and original fields. For the 6-form and 8-form fields this leads to a unique extension. For the 10-forms no relation with fields of the original IIB multiplet exists, so that the number of 10-forms is not determined a priori.

We find that the following fields can be introduced in IIB supergravity:

6-forms There is a doublet of 6-forms $A^\alpha_{(6)}$, with $A^1_{(6)} = (A^2_{(6)})^*$, satisfying the duality relation

$$
F^\alpha_{(7)\mu_1\ldots\mu_7} = -\frac{i}{3!}\epsilon_{\mu_1\ldots\mu_7\mu\nu\rho}S^{\alpha\beta}\epsilon_{\beta\gamma}F^{\gamma;\mu\nu\rho}_{(3)} \, , \tag{7}
$$

where

$$
S^{\alpha\beta} = V^\alpha_-V^\beta_+ + V^\alpha_+V^\beta_- \, . \tag{8}
$$

8-forms There is a triplet of 8-forms $A^{\alpha\beta}_{(8)}$, symmetric in α,β, satisfying a reality condition

$$
(A^{11}_{(8)})^* = A^{22}_{(8)} \, , \qquad (A^{12}_{(8)})^* = A^{12}_{(8)} \, , \tag{9}
$$

and a duality relation

$$
F^{\alpha\beta}_{(9)\,\mu_1\ldots\mu_9} = i\epsilon_{\mu_1\ldots\mu_9}{}^\sigma\{V^\alpha_+V^\beta_+P^*_\sigma - V^\alpha_-V^\beta_-P_\sigma\} \, . \tag{10}
$$

However, the three 8-forms are related to each other through a condition on the field-strengths,

$$
\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}V^\alpha_+V^\beta_-F^{\gamma\delta}_{(9)} = 0 \, . \tag{11}
$$

This implies that in the 8-form sector there are only two degrees of freedom, the 'duals' of the dilaton ϕ and the axion $C_{(0)}$. Note that the three potentials are not related by a local condition. The existence of a triplet of 8-forms, and the relation between these forms, was also discussed in [10, 11].

10-forms There are 10-forms in two $SU(1,1)$ representations: a doublet $A^{\alpha}_{(10)}$, with the usual reality condition, and a quadruplet $A^{\alpha\beta\gamma}_{(10)}$, symmetric in α, β, γ, satisfying

$$(A^{111})^* = A^{222}\,, \qquad (A^{112})^* = A^{122}\,. \tag{12}$$

There are no conditions relating the 10-forms to other fields. However, the doublet and the quadruplet differ in an important respect. The doublet does not transform under n-form gauge transformations with $n < 9$, while the fields of the quadruplet transform under all lower rank gauge transformations.

Truncation to $N = 1$

We can truncate our results for $N = 2$ supergravity to find the $N = 1$ algebra [6]. Since in $D = 10$ the $N = 1$ supergravity is unique, there is only one independent truncation, all others being related by field redefinitions. In spite of this, since there are two inequivalent $N = 1$ string theories, it is instructive to truncate the $N = 2$ theory in two different ways leading to the low energy limits of $D = 10$ heterotic and type I string theory. Hence we perform the "heterotic" and the "type I" truncations [7].
The heterotic truncation can be derived from the IIB algebra by setting

$$\epsilon = \epsilon_C. \tag{13}$$

This projects out the following fields from the IIB spectrum:

$$C_{(0)}, C_{(2)}, C_{(4)}, C_{(6)}, C_{(8)}, B_{(8)}, C_{(10)}, E_{(10)}, \mathcal{E}_{(10)}. \tag{14}$$

Further, $B_{(10)}$ and $\mathcal{D}_{(10)}$ turn out to be dependent fields of the form $e^{x\phi}\epsilon_{(10)}$ (where $\epsilon_{(10)}$ is the volume form) for some x in the truncated theory. Therefore, the field contents of the heterotic truncation of the $D = 10$, IIB supergravity is given by

$$\phi, B_{(2)}, B_{(6)}, D_{(8)}, D_{(10)}. \tag{15}$$

The supersymmetry algebra which is realised on these fields can easily be obtained by setting all truncated and dependent fields to zero in the IIB algebra as presented in [6].
The type I truncation can be derived from the IIB algebra by setting

$$\epsilon = i\epsilon_C. \tag{16}$$

This projects out the following fields from the IIB spectrum:

$$C_{(0)}, B_{(2)}, C_{(4)}, B_{(6)}, C_{(8)}, B_{(8)}, D_{(10)}, B_{(10)}, \mathcal{D}_{(10)}. \tag{17}$$

Similarly to the heterotic case, we find that $C_{(10)}$ and $\mathcal{E}_{(10)}$ turn out to be dependent fields. Therefore the field contents of the type I truncation is given by

$$\phi, C_{(2)}, C_{(6)}, D_{(8)}, E_{(10)}. \tag{18}$$

Coupling to branes

We now wish to investigate to which kind of branes the different n-form potentials couple. In particular we would like to know the tension of the corrersponding branes. These tensions can be determined from the supersymmetry rules as follows. To be concrete let us consider the 8-form potentials, related by the condition (11). After gauge-fixing the generic supersymmetry rule of the 3 different potentials is as follows (in string frame):

$$\delta A_{(8)} \sim f(\tau,\bar{\tau})\bar{\epsilon}\gamma(a\psi_\mu + b\gamma_\mu\lambda) \quad + \quad \cdots \quad , \tag{19}$$

where a, b are constants, the dots stand for other terms and the scalars have been expressed in terms of a complex scalar τ. The function $f(\tau,\bar{\tau})$ can be expressed in terms of the dilaton ϕ and axion $C_{(0)}$ via the relation $\tau = C_{(0)} + ie^{-\phi}$. For our present purposes it is sufficient to consider the axion-independent part of the tension, for the full result, see [9].

The 8-form potentials may occur as Wess-Zumino terms in a supersymmetric action as follows:

$$\mathcal{L}_{\text{brane}} \sim \underbrace{e^{-\alpha\phi}}_{\text{brane tension at } C_{(0)}=0} \sqrt{-g} \quad + \quad A_{(8)} \quad + \quad \cdots \tag{20}$$

Before fixing kappa-symmetry the first, Nambu-Goto, term and the second, Wess-Zumino, term are separately supersymmetric. After gauge-fixing kappa-symmetry the (linear) supersymmetry variations of the two terms should cancel. At $C_{(0)} = 0$ this is only possible if the function $f(\tau,\bar{\tau})$ is proportional to the brane tension $e^{-\alpha\phi}$. To achieve this one must consider particular combinations of the 8-form potentials. This enables us to read off the brane tensions from the supersymmetry rules. This indeed works for two of the three 8-form potentials, leading to the combinations $C_{(8)}$ and $B_{(8)}$ in Table 1. They couple to the D7-brane and the S-dual $\widetilde{\text{D7}}$-brane (with exotic brane tension g_s^{-3}), respectively. However, for the third 8-form potential $D_{(8)}$ we find that the tension vanishes for zero axion. Therefore, $D_{(8)}$ does not couple to an independent (solitonic) 7-brane. Nevertheless, the axion-dependent factor in front of the Nambu-Goto term plays a role in deriving the tension formula of the other nonlinear doublet of 7-branes, see [9]. Note that, after a type I/heterotic truncation, $D_{(8)}$ can be identified as the dual of the dilaton[1] and the other two potentials are truncated away. The constraint (11) vanishes after truncation.

A similar analysis can be performed for the 10-form potentials. The result is summarized in Table 2. Note that, like $D_{(8)}$, the 10-form potentials $D_{(10)}$ and $E_{(10)}$ cannot be associated in this way with an independent supersymmetric 9-brane. For more details, see [9].

Very extended symmetry groups

Collecting all n-form potentials of the IIB theory we find that they transform nonlinearly under the bosonic gauge transformations in the following generic form:

[1]Since $D_{(8)}$ does not transform to a gravitino under supersymmetry, no supersymmetric 7-brane can be associated with this dual dilaton.

potential	associated brane	tension	truncation
$C_{(8)}$	D7	g_s^{-1}	—
$D_{(8)}$	—	—	type I/heterotic
$B_{(8)}$	$\widetilde{\mathrm{D7}}$	g_s^{-3}	—

Table 1: The triplet of 8-form potentials and the corresponding nonlinear doublet of 7-branes. The last column indicates whether a potential survives a type I and/or heterotic truncation.

potential	associated brane	tension	truncation
$C_{(10)}$	D9	g_s^{-1}	type I
$D_{(10)}$	—	—	heterotic
$E_{(10)}$	—	—	type I
$B_{(10)}$	exotic	g_s^{-4}	heterotic
$\mathcal{D}_{(10)}$	solitonic	g_s^{-2}	heterotic
$\mathcal{E}_{(10)}$	exotic	g_s^{-3}	type I

Table 2: The quadruplet and doublet of 10-form potentials and the corresponding nonlinear and linear doublets of 9-branes. The last column indicates whether a potential survives a type I or heterotic truncation.

$$\delta A = d\Lambda + F \wedge \Lambda\,, \qquad\qquad F = dA + A \wedge F\,. \tag{21}$$

Since the gauge transformation rules only contain gauge-invariant curvatures, the bosonic gauge algebra is Abelian. Surprisingly, it turns out that the bosonic gauge transformations can all be rewritten in terms of

$$\Lambda_{(2n)} \equiv d\Lambda_{(2n-1)}\,. \tag{22}$$

After an appropriate (field-dependent) redefinition of the gauge fields and the parameters, all transformation rules become linear but the resulting bosonic gauge algebra is non-Abelian. A similar analysis has been performed in [12]. Schematically, we thus obtain the following non-trivial commutators (the numbers indicate the $SU(1,1)$ representations of the potentials/gauge transformations):

$$[\mathbf{2},\mathbf{2}] = 4\,, \qquad [\mathbf{2},4] = 6\,, \qquad [\mathbf{2},6] = 8\,,\cdots \tag{23}$$

We thus see an interesting structure arising: all gauge fields can be obtained by applying a number of times the basic **2** gauge transformation. This number is the so-called level of the gauge field. A similar structure arises in the IIA case where the basic building blocks are the RR 1-form **1** and the NS 2-form **2**:

$$[\mathbf{1},\mathbf{1}] = 0\,, \qquad [\mathbf{1},\mathbf{2}] = 3\,, \qquad [\mathbf{1},3] = 0\,, \qquad [\mathbf{2},3] = 5\,, \qquad [\mathbf{1},5] = 6\,,\cdots \tag{24}$$

A similar symmetry structure has been found both in the IIA [13] and in the IIB [14] case.

The above is very reminiscent to recent work on a hyperbolic E_{11}-symmetry that might underly string and/or M-theory, see [6] for a list of references. In particular, in [15], representations of the E_{11} algebra are worked out for different embeddings of a bosonic $GL(10)$ subalgebra[2]. This leads to the Dynkin diagrams of Figure 1 [15]. In these diagrams the horizontal line represents the $GL(10)$ subalgebra whereas the empty dots are related to our basic building blocks in the following way (the numbers 9,10,11 refer to the numbers in the Figure):

$$\mathrm{IIA} \quad : \qquad 10 \leftrightarrow \mathbf{1}\,, \qquad 11 \leftrightarrow \mathbf{2} \tag{25}$$

$$\mathrm{IIB} \quad : \qquad 9, 10 \leftrightarrow \mathbf{2}\,. \tag{26}$$

It would be interesting to pursue this relationship further.

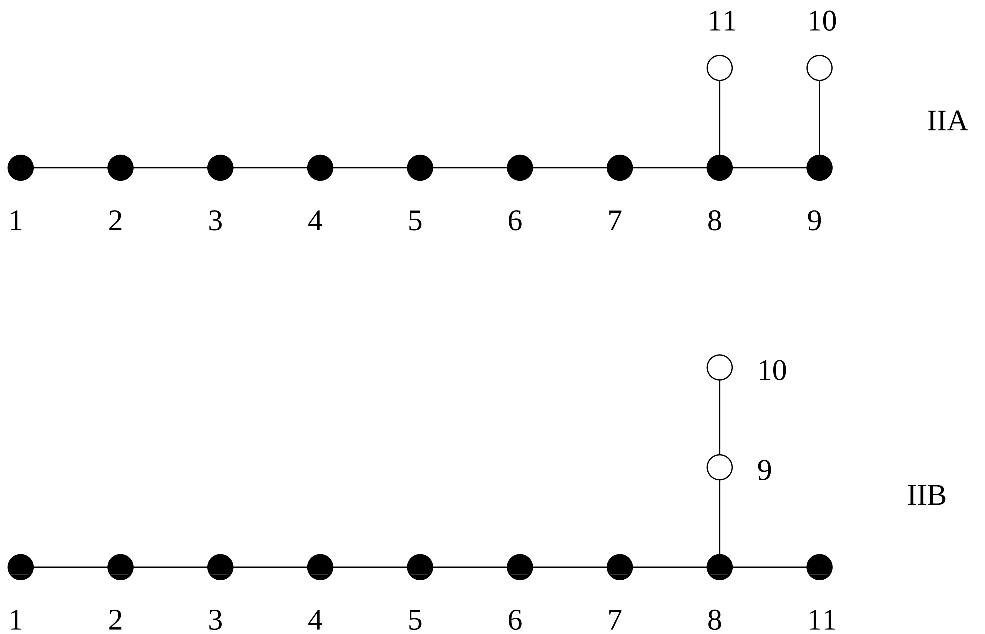

Figure 1: The Dynkin diagrams leading to IIA and IIB representations

Strings with 16 supercharges

It is well-known that the IIB ten-form potential $C_{(10)}$ and the kappa-symmetric D9-brane it couples to play an important role in the construction of the Type I SO(32) superstring. It has been suggested that, similarly, the S-dual IIB ten-form potential $B_{(10)}$ and the S-dual NS 9-brane play a role in the construction of the S-dual Heterotic SO(32) superstring theory [17, 18]. However, the 9-brane the potential $B_{(10)}$ couples to has an exotic g_s^{-4} tension which does not seem to occur in string loop calculations. On the other hand, there is a supersymmetric *solitonic* NS 9-brane. This is the one that couples to $\mathcal{D}_{(10)}$.

[2]The relation between our result and the E_{11} predictions have been recently analysed in [16]

At the same time, all we know is that the Type I and Heterotic SO(32) theories are S-dual to each other but this does not necessarily imply that the S-dual D9-branes underly the Heterotic SO(32) theory. Since solitonic objects, like the solitonic NS 5-brane, do naturally occur in string theory, it is interesting to investigate the possibility that it is the supersymmetric solitonic 9-brane that underlies the Heterotic SO(32) string theory. Solitonic 9-branes have recently been mentioned in the context of open heterotic strings [19]. It is an interesting challenge to write down a kappa-symmetric action for the solitonic NS 9-brane. We conjecture that such an action exists. It remains to be seen what exactly the corresponding $\mathbb{Z}_2$ truncation and solitonic orientifold is.

In an upcoming paper [9] we will show that IIA supergravity allows two 10-form potentials, only one of which couples to a supersymmetric solitonic NS 9-brane. Applying T-duality we get a picture where for every string theory with 16 supercharges there is a 9-form or 10-form potential and a corresponding 8-brane or 9-brane that underlies its construction. The situation is summarized in the Table below. It will be interesting to see whether this conjectured relation between string theories with 16 supercharges and D-branes and solitonic NS 9-branes can be supported by further circumstantial evidence.

IIA/IIB	string theory	potential	tension
IIA	Type I′ SO(16) $\times$ SO(16)	$C_{(9)}$	g_s^{-1}
IIA	Heterotic $E_8 \times E_8$	$B_{(10)}$	g_s^{-2}
IIB	Type I SO(32)	$C_{(10)}$	g_s^{-1}
IIB	Heterotic SO(32)	$D_{(10)}$	g_s^{-2}

Table 3: Each of the 4 string theories with 16 supercharges correspond with a D-brane or a solitonic NS 9-brane.

We finally note that *no* higher-form potentials can be added to D=11 supergravity. In particular, none of the two 10-form potentials of the IIA theory can be uplifted to D=11 in a Lorentz-covariant way. The extended gauge algebra in the presence of a dual 6-form potential is given by:

$$[3,3] = 6 \qquad\qquad [3,6] = 0\,. \tag{27}$$

To obtain a higher-form potential with $[3,6] \neq 0$ and to relate it to an underlying E_{11} algebra it seems that one needs to consider a dual version of the graviton, see e.g. the tables in [15]. It is of interest to check whether the D=11 supersymmetry algebra can be realized in the presence of such a dualized graviton field.

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Conifold Geometries, Matrix Models and Quantum Solutions

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Abstract

Open topological B-models describing D-branes on 2-cycles of local Calabi–Yau geometries with conical singularities are reviewed. The paper expands in particular on two aspects: the gauge fixing problem in the reduction to two dimensions and the quantum matrix model solutions.

1 Introduction

Singular Calabi–Yau manifolds represent one of the most interesting developments in string compactifications. For instance, the presence of a conifold point in a Calabi-Yau opens new prospects: in conjunction with fluxes and branes it may allow for warped compactifications, which in turn may create the conditions for moduli stabilization and for large hierarchies of physical scales. On the other hand singular Calabi-Yau compactifications with conical singularities seem to realize favorable conditions for low energy theory models with realistic cosmological features. A conifold singularity can be smoothed in two different ways, by means of a 2-sphere (resolution) or a 3–sphere (deformation). This leads, from a physical point of view, to a geometric transition that establishes a duality relation between theories defined by the two nonsingular geometries (gauge–gravity or open–closed string duality),[1, 2]. In summary, conifold singularities are at the crossroads of many interesting recent developments in string theory. It is therefore important to study theories defined on conifolds, i.e. on singular non compact Calabi-Yau threefolds, as well–defined and (partially) calculable models to approximate more realistic situations.

In [3], building on previous literature, we started to elaborate on an idea that is receiving increasing attention: how data describing the geometry of a local Calabi–Yau can be encoded, via a topological field theory, in a (multi–)matrix model and how they can be efficiently calculated. The framework we considered was IIB string theory with spacetime filling D5–branes wrapped around resolving two–dimensional cycles. This geometry defines a 4D gauge theory, [4, 5, 6]. On the other hand one can consider the open topological B model describing strings on the conifold. The latter has been shown long ago by Witten to be represented by a six-dimensional holomorphic Chern–Simons theory, [8]. When reduced to a two-dimensional cycle this theory can be shown to reduce to a matrix model. In particular, if one wishes to represent a wide class of deformations of the complex structure satisfying the Calabi–Yau condition, one must resort to very general multi–matrix models, [7]. In [3] we concentrated on the topological string theory part of the story, [9, 11, 10, 12], in particular on the formal aspects of the reduction from the six-dimensional holomorphic Chern–Simons theory to a two–dimensional field theory and on the analysis of the matrix model potentials originated from the Calabi-Yau complex structure deformations. Finally we concentrated on the subclass of matrix models represented by two–matrix models with bilinear coupling. In this case the functional integral can be explicitly calculated with the method of orthogonal polynomials. Using old results we showed how one can find explicit solutions by solving the *quantum equations of motion* and utilizing the recursiveness granted by integrability. All the data turn out to be encoded in a Riemann surface, which we called *quantum Riemann surface* in order to distinguish it from the Riemann surface of the standard saddle point approach.

In this paper we would like to review and expand on some topics that were only partially developed in [3]. In particular, in section 2, after a concise review of the reduction to from CS theory to matrix models, we explain in detail the gauge fixing problem in this process. Subsequently we return to the problem of solving two–matrix models with bilinear couplings by means of the orthogonal polynomials method via the solution of the quantum equation of motion. Our main purpose in section 3 is to clarify the similarities and differences with other methods, in particular with the semiclassical saddle point method. We discuss in particular the result of [3] that the quantum equations of motion admit in general more vacua than the saddle point method. We interpret these additional solutions as 'quantum' cycles that have no classical analog.

2 Reduction to the brane

In this section we summarize the reduction of the topological open string field theory (B model) to a holomorphic 2-cycle in a local Calabi-Yau threefold [3]. Let us consider a holomorphic $\mathbb{C}^2-$fibre bundle $X \to \Sigma$, where Σ is a Riemann surface. The space X is obtained as a deformation of the complex structure of the total space of a rank-2 vector bundle V on Σ. Given an atlas $\{U_\alpha\}$ on Σ, the transition functions for X can be written

$$\begin{aligned} z_{(\alpha)} &= f_{(\alpha\beta)}(z_{(\beta)}) \\ \omega^i_{(\alpha)} &= M^i_{j(\alpha\beta)}\left(z_{(\beta)}\right)\left[\omega^j_{(\beta)} + \Psi^j_{(\alpha\beta)}\left(z_{(\beta)}, \omega_{(\beta)}\right)\right] \ , \quad i,j = 1,2 \end{aligned}$$

where $f_{(\alpha\beta)}$ are the transition functions on the base, $M^i_{j(\alpha\beta)}$ the transition functions of the vector bundle V and $\Psi^j_{(\alpha\beta)}$ are the deformation terms, holomorphic on the intersections $(U_\alpha \cap U_\beta) \times \mathbb{C}^2$.

The Calabi-Yau condition on the space X, i.e. the existence of a nowhere vanishing holomorphic top-form $\Omega = \mathrm{d}z \wedge \mathrm{d}w^1 \wedge \mathrm{d}w^2$, puts conditions on the vector bundle and on the deformation terms. The determinant of the vector bundle has to be equal to the canonical line bundle on Σ and for the transition functions this means $\det M_{(\alpha\beta)} \times f'_{(\alpha\beta)} = 1$. For the deformation terms we have $\det(1 + \partial\Psi) = 1$, where $(1+\partial\Psi)^i_j = \delta^i_j + \partial_j \Psi^i$. The solution of this condition can be given in terms of a set of potential functions $X_{(\alpha\beta)}$, the *geometric potential*, which generates the deformation via

$$\epsilon_{ij} w^i_{(\alpha\beta)} \mathrm{d}w^j_{(\alpha\beta)} = \epsilon_{ij}\omega^i_{(\alpha)}\mathrm{d}\omega^j_{(\alpha)} - \mathrm{d}X_{(\alpha\beta)},$$

where we define the *singular coordinates* $w^i_{(\alpha\beta)} = \omega^i_{(\alpha)} + \Psi^i_{(\alpha\beta)}(z^{(\beta)}, \omega_{(\beta)})$.

The topological open B-model on X can be obtained from open string field theory and reduces [8] to the holomorphic Chern-Simons (hCS) theory on X for a (0,1)-form connection on a $U(N)$ bundle E, where N is the number of space-filling B-branes. We will restrict ourselves to the case in which E is trivial. The action of hCS is

$$S(\mathcal{A}) = \frac{1}{g_s}\int_X \mathcal{L}, \qquad \mathcal{L} = \Omega \wedge Tr\left(\frac{1}{2}\mathcal{A}\wedge\bar{\partial}\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \tag{2.1}$$

where $\mathcal{A} \in T^{(0,1)}(X)$. The dynamics of B-branes on a 2-cycle $\Sigma \subset X$ can be described by reducing the open string field theory from the space X to the B-brane world-volume Σ.

To obtain the reduced action for the *linear geometry* ($\Psi_i \equiv 0$), first we split the form $\mathcal{A}$ into horizontal and vertical components using a reference connection Γ on the vector bundle, then we impose the independence of the fields on the vertical directions and finally we "integrate along the fiber" using a bilinear structure K on the bundle. If the connection Γ is the generalized Chern connection for the bilinear structure, then the result is independent of the particular (Γ, K) chosen.

Let us define $\mathcal{A}_{\bar{z}} = A_{\bar{z}} - A_{\bar{k}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}\bar{w}^{\bar{j}}$ and $\mathcal{A}_{\bar{i}} = A_{\bar{i}}$, where Γ is a reference connection and impose that the components $(A_{\bar{z}}, A_{\bar{i}})$ are independent on the coordinates along $\mathbb{C}^2$, obtaining for the Lagrangian

$$L = \frac{1}{2}\Omega\wedge\mathrm{Tr}\left\{A_{\bar{i}}D_{\bar{z}}A_{\bar{j}} + A_{\bar{i}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}A_{\bar{k}}\right\}\mathrm{d}w^{\bar{i}}\wedge\mathrm{d}\bar{z}\wedge\mathrm{d}w^{\bar{j}} \tag{2.2}$$

where $D_{\bar{z}}$ is the covariant derivative w.r.t. the gauge structure.

Now let us consider a bilinear structure K in V, i.e. a local section $K \in \Gamma(V\otimes\bar{V})$, the components $K^{i\bar{j}}$ being an invertible complex matrix at any point. The "integration along the fiber" is realized contracting the hCS (3,3)-form Lagrangian by the two bi-vector fields $k = \frac{1}{2}\epsilon_{ij}K^{i\bar{l}}K^{j\bar{k}}\frac{\partial}{\partial\bar{w}^l}\frac{\partial}{\partial\bar{w}^k}$ and $\rho = \frac{1}{2}\epsilon^{ij}\frac{\partial}{\partial w^i}\frac{\partial}{\partial w^j}$

$$\mathcal{L}_{red} = i_{\rho\wedge k}L = \frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}(\det K)\epsilon^{\bar{i}\bar{j}}\mathrm{Tr}\left[A_{\bar{i}}D_{\bar{z}}A_{\bar{j}} + A_{\bar{i}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}A_{\bar{k}}\right]. \tag{2.3}$$

Defining the field components $\varphi^i = i_{V^i}A \in V$, where $V^i = K^{i\bar{j}}\frac{\partial}{\partial\bar{w}^j}$, one gets

$$\mathcal{L}_{red} = \frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}\left[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j + (\det K)\varphi^m\varphi^n\epsilon^{\bar{i}\bar{j}}\left(K_{m\bar{i}}\partial_{\bar{z}}K_{n\bar{j}} + K_{m\bar{i}}K_{n\bar{k}}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}\right)\right] \tag{2.4}$$

where $K_{\bar{i}j}$ are the components of the inverse bilinear structure, that is $K_{\bar{i}j}K^{j\bar{l}} = \delta^{\bar{l}}_{\bar{i}}$. In order to have a result which is independent of the trivialization, just set the reference connection to be the generalized Chern connection of the bilinear structure K, that is $\Gamma^{\bar{k}}_{\bar{z}\bar{j}} = K_{\bar{j}l}\partial_{\bar{z}}K^{l\bar{k}}$. The action for the reduced theory is given by

$$S_{red} = \frac{1}{g_s}\int_\Sigma \mathcal{L}_{red} = \frac{1}{2g_s}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}\left[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j\right].$$

In the *rational case* $\Sigma \simeq \mathbb{P}^1$ with non vanishing deformation terms Ψ_i, X is a deformation of a vector bundle $V \simeq \mathcal{O}_{\mathbb{P}^1}(n)\oplus\mathcal{O}_{\mathbb{P}^1}(-n-2)$ for some n.

Let us start with the reduction in the *Abelian* $U(1)$ *case*. In this case the cubic term in the hCS Lagrangian is absent and the reduction is almost straightforward. In the singular coordinates (φ^1, φ^2) we obtain that

$$\mathcal{L}_{red} = \frac{1}{2}\epsilon_{ij}\varphi^i\partial_{\bar{z}}\varphi^j dz\mathrm{d}\bar{z} \tag{2.5}$$

in both charts of the standard atlas $\{U_N, U_S\}$ of $\mathbb{P}^1$. The potential term X gives the deformation of the action due to the deformation of the complex structure. Passing to the non singular coordinates (ϕ_1, ϕ_2), one obtains

$$S_{red} = \frac{1}{2\,g_s}\left[\int_{\mathbb{P}^1} \mathrm{d}z\mathrm{d}\bar{z}\epsilon_{ij}\phi^i\partial_{\bar{z}}\phi^j + \oint \frac{\mathrm{d}z}{2\pi i}X(z,\phi)\right] \tag{2.6}$$

where $\oint$ is a contour integral along the equator. Therefore, the reduced theory gives a b–c (β–γ) system on the two hemispheres with a junction interaction along the equator.

The *non–Abelian case* is a bit more complicated than the Abelian one because of the tensoring with the (trivial) gauge bundle. This promotes the vector bundle sections to matrices and therefore it is not immediate how to unambiguously define the potential function X in the general case. The easiest way to avoid matrix ordering prescriptions is to restrict to the case in which $X(z,\omega)$ does not depend, say, on ω^2. Defining $B := \omega^1\Psi^2 + X$, one obtains $\Psi^1 = 0$, $\Psi^2 = \partial_{\omega^1}B$ and the reduced action

$$S \equiv S_{red} = \frac{1}{g_s}\left[\int_{\mathbb{P}^1} -\mathrm{Tr}(\phi^2 D_{\bar{z}}\phi^1)\mathrm{d}z\mathrm{d}\bar{z} + \oint \mathrm{Tr}B(z,\phi^1)\mathrm{d}z\right] \tag{2.7}$$

2.1 Gauge fixing

In order to calculate its partition function, let us now discuss the gauge fixing of the theory. The following discussion is a refinement of the derivation given in [4]. Our starting action is (2.7) and we follow the standard BRST quantization.

The BRST invariance in the minimal sector is

$$sA_{\bar{z}} = -(Dc)_{\bar{z}}, \quad s\phi^i = [c,\phi^i], \quad sc = \frac{1}{2}[c,c]$$

while we add a further non minimal one to implement the gauge fixing with

$$s\bar{c} = b, \quad sb = 0.$$

The gauge fixed action is obtained by adding to S a gauge fixing term

$$S_{gf} = S + s\Psi, \quad \text{where} \quad \Psi = \frac{1}{g_s}\int_{P^1} \mathrm{Tr}\bar{c}\partial_z A_{\bar{z}}$$

which implements a holomorphic version of the Lorentz gauge. Actually we have

$$s\Psi = \frac{1}{g_s}\int_{P^1} \mathrm{Tr}\left[b\partial_z A_{\bar{z}} - \partial_z\bar{c}(Dc)_{\bar{z}}\right].$$

Our partition function is then the functional integral

$$Z_B = \int \mathcal{D}\left[\phi^i, A_{\bar{z}}, c, \bar{c}, b\right] e^{-S_{gf}}$$

The calculation can proceed as follows. Let us first integrate along the gauge connection $A_{\bar{z}}$ which enters linearly the gauge fixed action and find

$$Z_B = \int \mathcal{D}\left[\phi^i, c, \bar{c}, b\right] e^{-\frac{1}{g_s}\left[-\int_{P^1}\mathrm{Tr}\phi^2\partial_{\bar{z}}\phi^1 - \partial_z\bar{c}\partial_{\bar{z}}c + \oint \mathrm{Tr}B(z,\phi^1)\right]}\delta\left\{\partial_z b + [\partial_z\bar{c}, c] + [\phi^1,\phi^2]\right\}$$

Now we integrate along the field b. By solving the constraint we obtain

$$Z_B = \int \mathcal{D}\left[\phi^i, c, \bar{c}\right] e^{-\frac{1}{g_s}\left[-\int_{P^1} \mathrm{Tr}\phi^2 \partial_{\bar{z}}\phi^1 - \partial_z \bar{c}\partial_{\bar{z}} c + \oint \mathrm{Tr}B(z,\phi^1)\right]} \frac{1}{\det{}'\partial_z}$$

where $\det{}'$ is the relevant functional determinant with the exclusion of the zero modes. Then we integrate along the $(c, \bar{c})$ ghosts and get

$$Z_B = \int \mathcal{D}\left[\phi^i\right] e^{-\frac{1}{g_s}\left[-\int_{P^1} \mathrm{Tr}\phi^2 \partial_{\bar{z}}\phi^1 + \oint \mathrm{Tr}B(z,\phi^1)\right]} \frac{\det{}'\partial_z \partial_{\bar{z}}}{\det{}'\partial_z}$$

Finally, since the geometric potential B does not depend on ϕ^2, we can also integrate along this variable and obtain

$$Z_B = \int \mathcal{D}\left[\phi^1\right] e^{-\frac{1}{g_s}\left[\oint \mathrm{Tr}B(z,\phi^1)\right]} \delta(\partial_{\bar{z}}\phi^1) \frac{\det{}'\partial_z \partial_{\bar{z}}}{\det{}'\partial_z}$$

The delta function constrains the field ϕ^1 to span the $\partial_{\bar{z}}$-zero modes and once it is solved it produces a further $\left(\det{}'\partial_{\bar{z}}\right)^{-1}$ multiplicative term that cancel the other determinants. Therefore, all in all we get

$$Z_B = \int_{\mathrm{Ker}\partial_{\bar{z}}} d\phi^1 e^{-\frac{1}{g_s}\left[\oint \mathrm{Tr}B(z,\phi^1)\right]}.$$

Lastly we can expand $\phi^1 = \sum_{i=0}^n X_i \xi_i$ along the basis $\xi_i(z) \sim z^i$ of $\mathrm{Ker}\partial_{\bar{z}}$ with $N \times N$ matrix coefficients X_i. Finally we find the multi-matrix integral

$$Z_B = \int \prod_{i=0}^n dX_i e^{-\frac{1}{g_s}\mathcal{W}(X_0,\ldots,X_n)} \tag{2.8}$$

where we defined

$$\mathcal{W}\left(X_0, \ldots, X_n\right) = \oint \mathrm{Tr}B(z, \sum_i X_i z^i) \tag{2.9}$$

This is the result of our gauge fixing procedure which covers the details needed to complete the derivation given in [3] and confirms the conjecture in [7].

3 Solving two–matrix models.

The second part of this paper is devoted to solving some of the matrix models introduced above, eq.(2.8), the two–matrix models with bilinear coupling. The main point we insist on is that for these models there is the possibility to solve the quantum problem exactly. To this purpose the method of orthogonal polynomials turns out to be particularly fit. This method allows one to explicitly perform the path integration, so that one is left with quantum equations of motion and the flow equations of an integrable linear systems. Our approach for solving two–matrix models consists in solving the quantum equations of motion and, then, using the recursiveness intrinsic to integrability (the flow equations), in finding explicit expressions for the correlators.

The model of two Hermitian $N \times N$ matrices M_1 and M_2 with bilinear coupling is defined by the partition function

$$Z_N(t,c) = \int dM_1 dM_2 e^{trW}, \qquad W = V_1 + V_2 + cM_1M_2 \tag{3.1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1,2. \tag{3.2}$$

where p_α are finite numbers. We denote by $\mathcal{M}_{p_1,p_2}$ the corresponding two–matrix model. With reference to eq.(2.9), this model descends from the geometric potential B defined by

$$B(z,\omega) = \frac{1}{z}\left[V_1(\omega) + V_2\left(\frac{\omega}{z}\right)\right] + \frac{c}{2z^2}\omega^2 \tag{3.3}$$

We are interested in computing correlation functions (CF's) of the operators $\tau_k = trM_1^k$ and $\sigma_k = trM_2^k, \qquad \forall k$. For this reason we complete the above model by replacing (3.2) with the more general potentials $V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r$, $\alpha = 1,2$, where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \le p_\alpha$. The CF's are defined by

$$< \tau_{r_1} \dots \tau_{r_n} \sigma_{s_1} \dots \sigma_{s_m} >= \frac{\partial^{n+m}}{\partial t_{1,r_1} \dots \partial t_{1,r_n} \partial t_{2,s_1} \dots \partial t_{2,s_m}} \ln Z_N(t,g) \tag{3.4}$$

where, in the RHS, all the $t_{\alpha,r}$ are set equal to $\bar{t}_{\alpha,r}$ for $r \le p_\alpha$ and the remaining ones are set to zero.

By introducing monic orthogonal polynomials it is possible to explicitly perform (3.1) and obtain

$$Z_N(t,c) = \text{const } N! \prod_{i=0}^{N-1} h_i \tag{3.5}$$

where h_i are the normalization factors of the orthogonal polynomials.

The crucial ingredient to solve these models are the *quantum equations of motion.* They are written as

$$P^\circ(1) + V_1'(Q(1)) + cQ(2) = 0, \qquad cQ(1) + V_2'(Q(2)) + \widetilde{\mathcal{P}}^\circ(2) = 0, \tag{3.6}$$

$Q(1), Q(2), P^\circ(1), P^\circ(2)$ represent the multiplication by λ_1, λ_2 and the derivative by the same parameters, respectively, in the basis of orthogonal polynomials. $Q(1), Q(2)$ are infinite Jacobi matrices:

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad \widetilde{\mathcal{Q}}(2) = I_+ + \sum_i \sum_{l=0}^{m_2} b_l(i) E_{i,i-l} \tag{3.7}$$

where $I_+ = \sum_{i=0} E_{i,i+1}$ and $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$, and $m_1 = p_2 - 1$, $m_2 = p_1 - 1$. Eqs.(3.6) can be considered the quantum analog of the classical equations of motion.

Once the quantum equations of motion have been solved, i.e. once we know $Q(1)$ and $Q(2)$ explicitly, we use the reconstruction formula for the partition function

$$\frac{\partial}{\partial t_{\alpha,r}} \ln Z_N(t,c) = \sum_{i=0}^{N-1} \Big(Q^r(\alpha)\Big)_{ii}, \qquad \alpha = 1,2 \tag{3.8}$$

Since the correlators are nothing but derivatives of Z_N with respect to $t_{\alpha,r}$, and the derivatives of the Q matrices with respect to $t_{\alpha,r}$ are known via the flow equations of the Toda lattice hierarchy

$$\frac{\partial}{\partial t_{\alpha,k}} Q(1) = [Q(1), \quad Q^k(\alpha)_-], \qquad \frac{\partial}{\partial t_{\alpha,k}} Q(2) = [Q^k(\alpha)_+, \quad Q(2)], \tag{3.9}$$

we can explicitly compute all the correlators.

For instance

$$\frac{\partial^2}{\partial t_{1,1}^2} \ln Z_N(t,c) = a_1(N), \qquad \frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}} \ln Z_N(t,c) = R(N) \tag{3.10}$$

where $R(i+1) \equiv h_{i+1}/h_i$.

Let us turn now to two explicit examples. By solving the quantum equations of motion we determine the 'lattice fields' $a_i(n), b_i(n)$ and $R(n)$. Once these are known we can compute all the correlation functions starting from (3.8) by repeated use of the Toda lattice hierarchy flows. In the sequel we will concentrate on solving the equations of motion, since the calculus of correlators is of algorithmic nature and, therefore, not particularly interesting; in any case, it has already been illustrated in a number of examples, [15, 14, 3]. The equations of motion are definitely more interesting, because some aspects of them have not been stressed enough or ignored in the existing literature.

3.1 The cubic model

The full $\mathcal{M}_{3,2}$ model has been discussed at length in [15] and, in particular, in [3]. Here we would like to consider its decoupling limit $c = 0$ and single out the cubic potential part, which amounts to considering the cubic one–matrix model $\mathcal{M}_3$. In the genus 0 limit this model is described by the discrete algebraic equations

$$a_0^3 + \frac{t_2}{t_3} a_0^2 + \frac{2}{9}\left(\frac{t_2}{t_3}\right)^2 a_0 - \frac{n}{3t_3} = 0 \tag{3.11}$$

$$a_1 = -\frac{1}{2} a_0^2 - \frac{1}{3}\frac{t_2}{t_3} a_0 \tag{3.12}$$

where, for simplicity and without loss of generality, we have set $t_1 = 0$. One can extract from these equations a_0 and a_1 and calculate all the correlators with the algorithm described above. Here we are not interested in this, but rather in analyzing eq.(3.11) and its solutions.

In the large N limit we shift to $x = \frac{n}{N}$ and, in order to make contact with section 4 of [16] for a comparison, we simplify a bit further our notation setting $t_2 = -\frac{N}{2}$ and $t_3 = -Ng$, where g is the cubic coupling constant there. Moreover we denote $z = 3ga_0$. Then eq.(3.11) becomes

$$18g^2 x + z(1+z)(1+2z) = 0 \tag{3.13}$$

This can be solved exactly for z and gives the three solutions

$$z_1 = -\frac{1}{2} + \frac{1}{2I(x)} + \frac{I(x)}{6} \tag{3.14}$$

$$z_2 = -\frac{1}{2} + \frac{1+i\sqrt{3}}{4I(x)} + \frac{1-i\sqrt{3}I(x)}{12} \tag{3.15}$$

$$z_3 = -\frac{1}{2} + \frac{1-i\sqrt{3}}{4I(x)} + \frac{1+i\sqrt{3}I(x)}{12} \tag{3.16}$$

where

$$I(x) = 3^{1/3}\left(-324g^2x + \sqrt{3}\sqrt{-1+34992g^4x^2}\right)^{1/3} \tag{3.17}$$

From these we can extract three solutions for a_0 and, consequently, for a_1. For small x the three solutions can be expanded as follows

$$z_1 = -18g^2x - 972g^4x^2 - 93312g^6x^3 - 11022480g^8x^4 + O(x^5) \tag{3.18}$$

$$z_2 = -1 - 18g^2x + 972g^4x^2 - 93312g^6x^3 + 11022480g^8x^4 + O(x^5) \tag{3.19}$$

$$z_3 = -\frac{1}{2} + 36g^2x + 186624g^6x^3 + O(x^5). \tag{3.20}$$

The best way to analyze these solutions is to notice that they represent a plane curve in the complex z, x plane. It is a genus 0 Riemann surface with punctures at $x = 0$ and $x = \infty$, made of three sheets joined through cuts running from $z = -1/(\sqrt{3}108g^2)$ to $z = 1/(108\sqrt{3}g^2)$. The solutions (3.18,3.19,3.20) correspond to the values z takes near $x = 0$, away from the cuts. In order to pass from one solution to another we have to cross the cuts. We call the Riemann surface so constructed the *quantum Riemann surface associated to the model*. This Riemann surface picture is the clue to understanding the solutions with multiple brane configurations, as was explained in [3].

Let us analyze the meaning of these three solutions. To this end it is useful to make a comparison with [16]: we see that the first solution corresponds to the unique solution found there, which corresponds to the minimum of the classical potential (see below). In fact the correspondence with [16] can be made more precise: one can easily verify that eqs.(46) there are nothing but eqs.(3.11,3.12), provided we make the identifications: $a + b = a_0$ and $(b - a)^2 = 4a_1$ and the rescalings $a_0 \to \sqrt{x}a_0, a_1 \to xa_1$ and $g \to g/\sqrt{x}$. In [16] the interval $(2a, 2b)$ represents a cut in the eigenvalue λ plane. In [16] we can therefore introduce an auxiliary Riemann surface. The latter is not to be confused with the quantum Riemann surface defined above, although the two are related.

Let us now discuss the correspondence between our three solutions and the classical extrema of the potential. The classical potential for the continuous eigenvalue function $\lambda(x)$ (which is $\lambda_n/\sqrt{N}$ in the large N limit), is $V_{cl} = \frac{1}{2}\lambda^2 + g\lambda^3$. It has extrema at $\lambda = 0$ and $\lambda = -1/3g$. To find the classical limit in our quantum approach instead, we rescale t_k, $k = 2, 3$ as: $t_k \to t_k/\hbar$, and take $\hbar \to 0$. This amounts to dropping the last term in eq.(3.11). The extrema are three, $z = 0, -1, -1/2$, which corresponds to $a_0 = 0, -1/3g, -1/6g$, not two as in the classical case. $z = 0$ corresponds to the minimum of the potential, $z = -1$ corresponds to the maximum, while $z = -1/2$ to the flex. The latter solution does not have a classical analog. This result is somewhat puzzling, but we should remember that the saddle point method is semiclassical: one cannot exclude that the quantum problem admits solution without classical analog. This is precisely what happens in the present case. One can phrase it also by saying that, in general, the large N limit and the $\hbar \to 0$ limit do not commute.

The next question is: what is the meaning of the third solution, $z = -1/2$? Let us recall what the other two solutions at $z = 0$ and $z = -1$ mean. On the basis of the discussion in section 2, we know that they represent two Riemann spheres located at the minimum and the maximum of the potential. They replace the continuous family of $\mathbb{P}^1$ which characterizes the conifold geometry before the deformation W is introduced. This is the interpretation on the basis of classical geometry. What we learn now is that solving the quantum problem we obtain a third solution, which we can interpret as a *quantum* $\mathbb{P}^1$, located at the flex of the potential. This is a pure quantum geometry effect.

Before we end this section we would like to make a few remarks. First we notice that the classical extrema are characterized by the fact that $a_1 = 0$, while the pure quantum solution corresponds to a non-vanishing a_1. Moreover, after setting $a_1 = 0$ we get for a_0 an equation that coincides with the classical eigenvalue equation. From this simple example we learn three important pieces of informations.

- The number of solutions of the quantum problem (i.e. the number of solutions to eq.(3.13)) is in general larger than the number of the extrema of the classical potential.
- The field a_0 can be regarded as the quantum version of the classical eigenvalue function.
- The classical extrema are obtained by neglecting the n term in eq.(3.11) and setting $a_1 = 0$.

These conclusion are valid in general, except for the fact that, the condition $a_1 = 0$ in the last remark must be replaced by the fields $a_1, a_2, ..., b_1, b_2, ...$, being set to zero in the general case.

3.2 The $\mathcal{M}_{3,3}$ model

We study the model in the case $t_1 = s_1 = 0$ and limit ourselves to writing down the genus 0 quantum equations of motion:

$$
\begin{aligned}
&3t_3 c a_0^2 + 2t_2 c a_0 - 36 s_3 t_3 b_0 R + c^2 b_0 - 12 s_2 t_3 R = 0\\
&3s_3 c b_0^2 + 2s_2 c b_0 - 36 s_3 t_3 a_0 R - 12 s_3 t_2 R + a_0 c^2 = 0\\
&nc + Rc^2 - 18 s_3 t_3 R^2 - 36 s_3 t_3 a_0 b_0 R - 12 s_2 t_3 a_0 R - 12 t_2 s_3 b_0 R - 4 s_2 t_2 R = 0 \qquad (3.21)\\
&a_1 = -\frac{6s_3}{c} b_0 R - \frac{2s_2}{c} R, \qquad a_2 = -\frac{3s_3}{c} R^2\\
&b_1 = -\frac{6t_3}{c} a_0 R - \frac{2t_2}{c} R, \qquad b_2 = -\frac{3t_3}{c} R^2
\end{aligned}
$$

As in the previous subsection we see that we are indeed interested in finding all the solutions that have an analytic expansion in $x = n/N$ around $x = 0$. In order to compute all possible solutions of this type, that is the quantum vacua, we therefore drop the first term in the lhs of (3.21) and solve the resulting system. The third equation, in particular, admits the solution $R = 0$.

$$R = 0 \tag{3.22}$$
$$3t_3a_0^2 + 2t_2a_0 + cb_0 = 0 \tag{3.23}$$
$$3s_3b_0^2 + 2s_2b_0 + ca_0 = 0 \tag{3.24}$$

which give rise to four (in general) distinct solutions. The alternative $R \neq 0$ leads to

$$\begin{aligned} &\frac{9}{2}c\frac{s_3t_3}{s_2}a_0^2 + 3(\frac{s_3t_2}{s_2}c + 8s_2t_3)a_0 + 108\, s_3t_3a_0b_0 + 54\frac{s_362t_2}{s_2}b_0^2 + 162\frac{s_3^2t_3}{s_2}a_0b_0^2 \\ &\qquad = 3\frac{s_3}{s_2}(c^2 - 16s_2t_2) + c^2 - 4s_2t_2, \end{aligned} \tag{3.25}$$
$$\begin{aligned} &\frac{9}{2}c\frac{s_3t_3}{t_2}b_0^2 + 3(\frac{t_3s_2}{t_2}c + 8s_3t_2)a_0 + 108\, s_3t_3a_0b_0 + 54\frac{s_2t_3^2}{t_2}a_0^2 + 162\frac{s_3t_3^2}{t_2}a_0^2b_0 \\ &\qquad = 3\frac{t_3}{t_2}(c^2 - 16s_2t_2) + c^2 - 4s_2t_2, \end{aligned} \tag{3.26}$$

This leads to an algebraic equation of order 10 for a_0, for instance. Therefore, generically, we have 10 (possibly complex) solutions for a_0, each of which gives rise to two different values for b_0. Altogether we are going to find 24 different quantum vacua. Once again it is interesting to compare these solutions with the classical ones. To this end in the above equations we set $a_2 = b_2 = a_1 = b_1 = 0$ as well as $R = 0$, from which we get eqs.(3.23,3.24).

From the first we can get $b_0 = -\frac{1}{c}(3t_3a_0^2 + 2t_2a_0)$, whence we get either $a_0 = 0$ or the cubic equation

$$27s_3t_3^2a_0^3 + 36t_2s_3t_3a_0^2 + (12s_3t_2^2 - 6cs_2t_3)a_0 + c(c^2 - 4s_2t_2) = 0$$

Therefore in general we have four classical extrema. In ref.[3] it is shown how to find an explicit series expansion in x about each of these solutions.

Before we end this section it is interesting to discuss the geometric meaning of the first three equations in (3.21). We can think of the third equation as a definition of the complex $x = n/N$ plane. The two remaining ones are quadratic equations in a_0, b_0, R. Introducing homogeneous coordinates they can be seen to represent two hypersurfaces in $\mathbb{P}^3$. The intersection is a genus 1 Riemann surface.

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BRST Approach to Higher Spin Field Theories

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Abstract

We develop the BRST approach to Lagrangian formulation for massive bosonic and massless fermionic higher spin fields on a flat space-time of arbitrary dimension. General procedure of gauge invariant Lagrangian construction describing the dynamics of the fields with any spin is given. No off-shell constraints on the fields (like tracelessness) and the gauge parameters are imposed. The procedure is based on construction of new representations for the closed algebras generated by the constraints defining irreducible representations of the Poincare group. We also construct Lagrangians describing propagation of all massive bosonic fields and massless fermionic fields simultaneously.

1 Introduction

Construction of higher spin field theory is one of the fundamental problems of high energy theoretical physics. At present, there exist the various approaches to this problem (see e.g. [1] for reviews). This paper is a brief review of recent development of BRST approach to free higher spin field theory. It is based on two papers [2, 3] devoted to Lagrangian construction of free fermionic massless higher spin fields and Lagrangian construction of free bosonic massive higher spin fields respectively. The main motivation for using BRST approach is to try to construct the theory of interacting higher spin fields analogously to string field theory. The first natural step in constructing a higher spin interacting model is formulation of the corresponding free theory.

The paper is organized as follows. In sections 2 and 3 we discuss operator algebras generated by primary constraints which define irreducible representations of the Pioncare group both in the massless fermionic and massive bosonic cases respectively. The sructure of the algebras proved to be the same and the method of Lagrangian construction is explained in section 4 on the base of a toy model. Then in sections 5 and 6 we applay this method for Lagrangian construction both for massless fermionic and massive bosonic fields respectively. Section 7 summarizes the obtained results.

2 Massless fermionic theory. Algebra of the constraints

It is well known that the totally symmetrical tensor-spinor field $\Psi_{\mu_1\cdots\mu_n}$ (the Dirac index is suppressed), describing the irreducible spin $s = n + 1/2$ representation must satisfy the following constraints (see e.g. [4])

$$\gamma^\nu \partial_\nu \Phi_{\mu_1\cdots\mu_n} = 0, \qquad \gamma^\mu \Phi_{\mu\mu_2\cdots\mu_n} = 0. \tag{1}$$

Here γ^μ are the Dirac matrices $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = (+, -, \ldots, -)$.

In order to describe all higher tensor-spinor fields together it is convenient to introduce Fock space generated by creation and annihilation operators a^+_μ, a_μ with vector Lorentz index $\mu = 0, 1, 2, \ldots, D-1$ satisfying the commutation relations

$$[a_\mu, a^+_\nu] = -\eta_{\mu\nu}. \tag{2}$$

These operators act on states in the Fock space

$$|\Phi\rangle \;\; = \;\; \sum_{n=0}^{\infty} \Phi_{\mu_1\cdots\mu_n}(x) a^{+\mu_1} \cdots a^{+\mu_n} |0\rangle \tag{3}$$

which describe all half-integer spins simultaneously if the following constraints are taken into account

$$T_0|\Phi\rangle \;\; = \;\; 0, \qquad T_1|\Phi\rangle = 0, \tag{4}$$

where

$$T_0 = \gamma^\mu p_\mu, \qquad T_1 = \gamma^\mu a_\mu, \tag{5}$$

with $p_\mu = -i\frac{\partial}{\partial x^\mu}$. If constraints (4) are fulfilled for the general state (3) then constraints (1) are fulfilled for each component $\Phi_{\mu_1\cdots\mu_n}(x)$ in (3) and hence the relations (4) describe all free higher spin fermionic fields together. The constraints T_0, T_1 are primary constraints. They generate all the constraints on the space of ket-vectors (3). Thus we get three more constraints

$$L_0|\Phi\rangle = 0, \qquad L_1|\Phi\rangle = 0, \qquad L_2|\Phi\rangle = 0, \tag{6}$$

where

$$L_0 = -p^2, \qquad L_1 = a^\mu p_\mu, \qquad L_2 = \tfrac{1}{2}\, a_\mu a^\mu. \tag{7}$$

Our purpose is to construct Lagrangian for the massless fermionic higher spin fields on the base of BRST approach, therefore we must construct Hermitian BRST operator. In the case under consideration the constraints T_0, L_0 are Hermitian, $T_0^+ = T_0$, $L_0^+ = L_0$, however the constraints T_1, L_1, L_2 are not Hermitian. Therefore we extend the set of the constraints adding three new operators

$$T_1^+ = \gamma^\mu a^+_\mu, \qquad L_1^+ = a^{+\mu} p_\mu, \qquad L_2^+ = \tfrac{1}{2}\, a^+_\mu a^{+\mu} \tag{8}$$

to the initial constraints (5) and (7). As a result, the set of operators T_0, T_1, T_1^+, L_0, L_1, L_2, L_1^+, L_2^+ is invariant under Hermitian conjugation. Taking hermitian conjugation of (4) and (6) we see that the operators (8) together with T_0 and L_0 are constraints on the space of ket-vectors

$$\langle\Phi|T_0 = \langle\Phi|T_1^+ = \langle\Phi|L_0 = \langle\Phi|L_1^+ = \langle\Phi|L_2^+ = 0. \tag{9}$$

Algebra of operators (5), (7), (8) is open in terms of commutators of these operators. We will suggest the following procedure of consideration. We want to use the BRST construction in the simplest (minimal) form coresponding to closed algebras. To get such an algebra we add to the above set of operators, all operators generated by the commutators of (5), (7), (8). Doing such a way we obtain one new operator

$$G_0 \quad = \quad -a_\mu^+ a^\mu + \tfrac{D}{2}, \tag{10}$$

which arises from the commutators

$$-\tfrac{1}{2}[T_1, T_1^+] = [L_2, L_2^+] = G_0, \tag{11}$$

and which is not a constraint neither in the space of ket-vectors nor in the space of bra-vectors. The resulting operators algebra may be found in [2].

Let us summarize what we have at the moment. The structure of the operator algebra in the fermionic case is as follows. First we have hermitian operators T_0, L_0, G_0. Two of them T_0 and L_0 are constraints both in the space of ket-vectors and in the space of bra-vectors, another G_0 is not a constrint neither in the space of ket-vectors nor in the space of bra-vectors. Then we have pairs of mutually conjugated operators (T_1, T_1^+), (L_1, L_1^+), (L_2, L_2^+). One representative from the pairs is constraint in the space of ket-vectors another representative is a constraint on the space of bra-vectors. The problem is to find BRST operator which reproduce equations of motion (1) up to gauge transformations.

Let us turn to the massive bosonic case.

3 Massive bosonic theory. Algebra of the constraints

It is well known that the totally symmetric tensor field $\Phi_{\mu_1\cdots\mu_s}$, describing the irreducible spin-s massive representation of the Poincare group must satisfy the following constraints (see e.g. [4])

$$(\partial^2 + m^2)\Phi_{\mu_1\cdots\mu_s} = 0, \qquad \partial^{\mu_1}\Phi_{\mu_1\mu_2\cdots\mu_s} = 0, \qquad \eta^{\mu_1\mu_2}\Phi_{\mu_1\cdots\mu_s} = 0. \tag{12}$$

Analogously to the fermionic case, in order to describe all higher integer spin fields simultaneously we introduce Fock space generated by creation and annihilation operators a_μ^+, a_μ satisfying the commutation relations (2) and define the operators

$$L_0 = -p^2 + m^2, \qquad L_1 = a^\mu p_\mu, \qquad L_2 = \tfrac{1}{2}a^\mu a_\mu, \tag{13}$$

where $p_\mu = -i\frac{\partial}{\partial x^\mu}$. These operators act on states in the Fock space

$$|\Phi\rangle \quad = \quad \sum_{s=0}^{\infty} \Phi_{\mu_1\cdots\mu_s}(x) a^{\mu_1+}\cdots a^{\mu_s+}|0\rangle \tag{14}$$

which describe all integer spin fields simultaneously if the following constraints on the states take place

$$L_0|\Phi\rangle = 0, \qquad L_1|\Phi\rangle = 0, \qquad L_2|\Phi\rangle = 0. \tag{15}$$

If constraints (15) are fulfilled for the general state (14) then constraints (12) are fulfilled for each component $\Phi_{\mu_1\cdots\mu_s}(x)$ in (14) and hence the relations (15) describe all free massive higher spin bosonic fields simultaneously.

Constraints (13) are all constraints in the space of ket-vectors (14). Again, as in the fermionic case, in order to be possible to construct hermitian BRST operator we must add to the constraints (13) their hermitian conjugated operators. Since $L_0^+ = L_0$ we add two operators

$$L_1^+ = a^{+\mu}p_\mu, \qquad L_2^+ = \tfrac{1}{2}\, a_\mu^+ a^{+\mu} \tag{16}$$

to the initial constraints (13). As a result, the set of operators L_0, L_1, L_2, L_1^+, L_2^+ is invariant under Hermitian conjugation. Taking hermitian conjugation of (15) we see that the operators (16) together with L_0 are constraints in the space of bra-vectors

$$\langle\Phi|L_0 = \langle\Phi|L_1^+ = \langle\Phi|L_2^+ = 0. \tag{17}$$

Algebra of the constraints (13), (16) is not closed and in order to construct BRST operator we must include in the algebra all the operators generated by (13), (16). Thus we have to include in the algebra two more hermitian operator

$$m^2 \qquad \text{and} \qquad G_0 = -a_\mu^+ a^\mu + \tfrac{D}{2}. \tag{18}$$

which are obtained from the commutators

$$[L_1, L_1^+] = L_0 - m^2, \qquad [L_2, L_2^+] = G_0, \tag{19}$$

and which are not not constraints neither in the space of ket-vectors nor in the space of bra-vectors. The resulting operator algebra can be found in [3].

Let us summarize the structure of the operator algebra in the bosonic case. It is the same as in the fermionic case. First we have hermitian operators T_0, L_0, m^2, G_0. Two of them T_0 and L_0 are constraints both in the space of ket-vectors and in the space of bra-vectors, another m^2 and G_0 are not constraints neither in the space of ket-vectors nor in the space of bra-vectors. Then we have pairs of mutually conjugated operators (L_1, L_1^+), (L_2, L_2^+). One representative from the pairs is constraint in the space of ket-vectors another representative is a constraint on the space of bra-vectors.

In order to understand better the method used for construction of BRST operator leading to the proper equations of motion (1), (15) it is useful to consider a toy model.

4 A simlified model

Let us consider a model where the 'physical' states are defined by the equations

$$L_0|\Phi\rangle = 0, \qquad L_1|\Phi\rangle = 0, \tag{20}$$

with some operators L_0 and L_1. Let us also suppose that some scalar product $\langle\Phi_1|\Phi_2\rangle$ is defined for the states $|\Phi\rangle$ and let L_0 be a Hermitian operator $(L_0)^+ = L_0$ and let L_1 be non-Hermitian $(L_1)^+ = L_1^+$ with respect to this scalar product. In this section we show how to construct Lagrangian which will reproduce (20) as equations of motion up to gauge transformations.

In order to get the Lagrangian within BRST approach we should begin with the Hermitian BRST operator. However, the standard prescription does not allow to construct such a Hermitian operator on the base of operators L_0 and L_1 if L_1 is non-Hermitian. We assume to define the nilpotent Hermitian operator in the case under consideration as follows.

Let us consider the algebra generated by the operators L_0, L_1, L_1^+ and let this algebra takes the form

$$[L_0, L_1] = [L_0, L_1^+] = 0, \tag{21}$$

$$[L_1, L_1^+] = L_0 + C, \qquad C = const \neq 0. \tag{22}$$

It is known (see e.g. [5]) that in the case $C = 0$ if we construct Hermitian BRST operators as if all the operators L_0, L_1, L_1^+ were the first class constraints then this BRST operator will reproduce the proper equations of motion (20) up to gauge transformations.

Now let us consider the case $C \neq 0$. In this case the central charge C plays the role analogous to m^2 and G_0 in the algebras of the two previous sections. If we construct BRST operator as if the operators L_0, L_1, L_1^+, C are the first class constraints we get a solution $|\Phi\rangle = 0$ [3] what contradicts to (20). This happens because we treat the operator C as a constraint.

But the case $C = 0$ may serve as a hint about solution to our problem. Namely, we construct new representation of the algebra (21), (22) with operator $C_{new} = 0$ in this representation.

Thus the solution is as follows. We enlarge the representation space of the operator algebra (21), (22) by introducing the additional (new) creation and annihilation operators and construct a new representation of the algebra bringing into it an arbitrary parameter h. The basic idea is to construct such a representation where the new operator C_{new} has the form $C_{new} = C + h$. Since parameter h is arbitrary and C is a central charge, we can choose $h = -C$ and the operator C_{new} will be zero in the new representation. After this we proceed as if operators L_{0new}, L_{1new}, L_{1new}^+ are the first class constraints.

For example, we can construct new representation of the operator algebra (21), (22) as follows

$$L_{0new} = L_0, \qquad\qquad C_{new} = C + h, \tag{23}$$

$$L_{1new} = L_1 + hb, \qquad\qquad L_{1new}^+ = L_1^+ + b^+. \tag{24}$$

Here we have introduced the new bosonic creation and annihilation operators b^+, b with the standard commutation relations $[b, b^+] = 1$.

In principle, we could set $h = -C$ and get $C_{new} = 0$, but there is one more equivalent scheme. Namley we still consider C_{new} as nonzero operator including the arbitrary parameter h, but demand for state vectors and gauge parameters to be independent on ghost η_C as before. It can be shown [3] that these conditions reproduce that h should be equal to $-C$.

Now if we introduce the BRST construction taking the operators in new representation as if they were the first class constarints

$$Q_h = \eta_0 L_0 + \eta_C C_{new} + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ - \eta_1^+ \eta_1 (\mathcal{P}_0 + \mathcal{P}_C), \qquad Q_h^2 = 0. \quad (25)$$

we shall get [3] that equation $Q_h|\Psi\rangle = 0$, where

$$|\Psi\rangle = \sum_{k=0}^{\infty} \sum_{k_i=0}^{1} (\eta_0)^{k_1} (\eta_1^+)^{k_2} (\mathcal{P}_1^+)^{k_3} (b^+)^k |\Phi_{kk_1k_2k_3}\rangle. \quad (26)$$

reproduces (20) up to gauge transformations.

Let us pay attention that operators L_{1new} and L_{1new}^+ are not mutually conjugate in the new representation if we use the usual rules for Hermitian conjugation of the additional creation and annihilation operators $(b)^+ = b^+$, $(b^+)^+ = b$. To consider the operators L_{1new}, L_{1new}^+ as conjugate to each other we change a definition of scalar product for the state vectors (26) $\langle\Psi_1|\Psi_2\rangle_{new} = \langle\Psi_1|K_h|\Psi_2\rangle$, with

$$K_h = \sum_{n=0}^{\infty} |n\rangle \frac{h^n}{n!} \langle n|, \qquad |n\rangle = (b^+)^n |0\rangle. \quad (27)$$

Now the new operators L_{1new}, L_{1new}^+ are mutually conjugate and the operator Q_h is Hermitian relatively the new scalar product (4) since the following relations take place

$$K_h L_{1new}^+ = (L_{1new})^+ K_h, \quad K_h L_{1new} = (L_{1new}^+)^+ K_h, \quad Q_h^+ K_h = K_h Q_h. \quad (28)$$

Finally we note that the proper equations of motion may be derived using the following Lagrangian

$$\mathcal{L} = \int d\eta_0 \langle\Psi| K_{-C} \Delta Q_{-C} |\Psi\rangle \quad (29)$$

where subscripts $-C$ means that we substitute $-C$ instead of h. Here the integral is taken over Grassmann odd variable η_0.

5 Lagrangians for massless fermionic fields

5.1 New representation

Let us first construct new representation for the operator algebra. Ones find

$$L_{2new}^+ = \tfrac{1}{2} a_\mu a^\mu + b^+, \qquad L_{2new} = \tfrac{1}{2} a_\mu^+ a^{+\mu} + (b^+ b + d^+ d + h) b, \quad (30)$$

$$T_{1new}^+ = \gamma^\mu a_\mu + 2b^+ d + d^+, \qquad T_{1new} = \gamma^\mu a_\mu^+ - 2(b^+ b + h) d - d^+ b, \quad (31)$$

$$G_{0new} = -a_\mu^+ a^\mu + \tfrac{D}{2} + 2b^+ b + d^+ d + h, \quad (32)$$

with the other operators being unchanged. Here b^+, b are bosonic creation and annihilation operators and d^+, d are fermionic ones with the standard commutation relations

$[b, b^+] = 1$, $\{d, d^+\} = 1$. Then we introduce the scalar product in the Fock space so that $\langle\Phi_1|\Phi_2\rangle_{new} = \langle\Phi_1|K|\Phi_2\rangle$, with operator K

$$K = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(|n\rangle\langle n|\, C(n,h) - 2d^+|n\rangle\langle n|d\, C(n+1,h) \Big), \qquad |n\rangle = (b^+)^n|0\rangle, \quad (33)$$

$$C(n,h) = h(h+1)(h+2)\ldots(h+n-1), \qquad C(0,h) = 1. \quad (34)$$

Now we construct BRST operator as if all the operators were the first class constraints

$$\begin{aligned} \tilde{Q} = {} & q_0 T_0 + q_1^+ T_{1new} + q_1 T_{1new}^+ + \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_{2new} + \eta_2 L_{2new}^+ \\ & + \eta_G G_{0new} + i(\eta_1^+ q_1 - \eta_1 q_1^+)p_0 - i(\eta_G q_1 + \eta_2 q_1^+)p_1^+ + i(\eta_G q_1^+ + \eta_2^+ q_1)p_1 \\ & + (q_0^2 - \eta_1^+\eta_1)\mathcal{P}_0 + (2q_1 q_1^+ - \eta_2^+\eta_2)\mathcal{P}_G + (\eta_G\eta_1^+ + \eta_2^+\eta_1 - 2q_0 q_1^+)\mathcal{P}_1 \\ & + (\eta_1\eta_G + \eta_1^+\eta_2 - 2q_0 q_1)\mathcal{P}_1^+ + 2(\eta_G\eta_2^+ - q_1^{+2})\mathcal{P}_2 + 2(\eta_2\eta_G - q_1^2)\mathcal{P}_2^+. \end{aligned} \quad (35)$$

Let us notice that the BRST operator (35) is selfconjugate in the following sense $\tilde{Q}^+ K = K\tilde{Q}$, with operator K (33).

5.2 Lagrangians for the free fermionic fields of single spin

It can be shown [2] that from equation $\tilde{Q}|\Psi\rangle = 0$ using gauge transformations we can remove dependence of the fields and the gauge parameters on the ghost fields η_0, $\mathcal{P}_0$, q_0, p_0 and obtain equations of motion for field with given spin $s = n + 1/2$

$$\Delta Q_\pi|\chi_0^0\rangle_n + \frac{1}{2}\{\tilde{T}_0, \eta_1^+\eta_1\}|\chi_0^1\rangle_n = 0, \qquad \tilde{T}_0|\chi_0^0\rangle_n + \Delta Q_\pi|\chi_0^1\rangle_n = 0. \quad (36)$$

Here $|\chi_0^0\rangle_n$ and $|\chi_0^1\rangle_n$ are states with ghost numbers 0 and -1 respectively and subscript n indicates that the corresponding field obeying the condition

$$\pi|\chi\rangle_n = (n + (D-4)/2)|\chi\rangle_n, \quad (37)$$

with

$$\pi = G_0 + 2b^+b + d^+d - iq_1p_1^+ + iq_1^+p_1 + \eta_1^+\mathcal{P}_1 - \eta_1\mathcal{P}_1^+ + 2\eta_2^+\mathcal{P}_2 - 2\eta_2\mathcal{P}_2^+. \quad (38)$$

Next $\tilde{T}_0 = T_0 - 2q_1^+\mathcal{P}_1 - 2q_1\mathcal{P}_1^+$, $\{A, B\} = AB + BA$ and Q_π is the part of $\tilde{Q}$ (35) which independent of η_G, $\mathcal{P}_G$, η_0, $\mathcal{P}_0$, q_0, p_0 with substitution $h \to -\pi$ [2].

These field equations (36) can be deduced from the following Lagrangian

$$\begin{aligned} \mathcal{L}_n = {} & {}_n\langle\chi_0^0|K_\pi\tilde{T}_0|\chi_0^0\rangle_n + \frac{1}{2}\, {}_n\langle\chi_0^1|K_\pi\{\tilde{T}_0, \eta_1^+\eta_1\}|\chi_0^1\rangle_n \\ & + {}_n\langle\chi_0^0|K_\pi\Delta Q_\pi|\chi_0^1\rangle_n + {}_n\langle\chi_0^1|K_\pi\Delta Q_\pi|\chi_0^0\rangle_n, \end{aligned} \quad (39)$$

where the standard scalar product for the creation and annihilation operators is assumed and the operator K_π is the operator K (33) where the following substitution is done $h \to -\pi$ [2].

The equations of motion (36) and the Lagrangian (39) are invariant under the gauge transformations

$$\delta|\chi_0^0\rangle_n = \Delta Q_\pi|\Lambda_0^0\rangle_n + \frac{1}{2}\{\tilde{T}_0, \eta_1^+\eta_1\}|\Lambda_0^1\rangle_n, \qquad \delta|\chi_0^1\rangle_n = \tilde{T}_0|\Lambda_0^0\rangle_n + \Delta Q_\pi|\Lambda_0^1\rangle_n, \quad (40)$$

which are reducible

$$\delta|\Lambda^{(i)0}{}_0\rangle_n = \Delta Q_\pi|\Lambda^{(i+1)0}{}_0\rangle_n + \frac{1}{2}\{\tilde{T}_0, \eta_1^+\eta_1\}|\Lambda^{(i+1)1}{}_0\rangle_n, \qquad |\Lambda^{(0)0}{}_0\rangle_n = |\Lambda^0_0\rangle_n, \tag{41}$$

$$\delta|\Lambda^{(i)1}{}_0\rangle_n = \tilde{T}_0|\Lambda^{(i+1)0}{}_0\rangle_n + \Delta Q_\pi|\Lambda^{(i+1)1}{}_0\rangle_n, \qquad |\Lambda^{(0)1}{}_0\rangle_n = |\Lambda^1_0\rangle_n, \tag{42}$$

with finite number of reducibility stages $i_{max} = n - 1$ for spin $s = n + 1/2$. It can be shown [2] that the Lagrangian (39) can be transformed to the Fang-Fronsdal Lagrangian [6] in four dimensions after eliminating the auxiliary fields.

5.3 Lagrangian for all half-integer spin fields

Now we turn to construction of Lagrangian describing propagation of all half-integer spin fields simultaneously. It can be show [2] that it looks like

$$\begin{aligned} \mathcal{L} &= \langle\chi^0_0|K_\pi\tilde{T}_0|\chi^0_0\rangle + \frac{1}{2}\langle\chi^1_0|K_\pi\{\tilde{T}_0, \eta_1^+\eta_1\}|\chi^1_0\rangle \\ &\quad + \langle\chi^0_0|K_\pi\Delta Q_\pi|\chi^1_0\rangle + \langle\chi^1_0|K_\pi\Delta Q_\pi|\chi^0_0\rangle, \end{aligned} \tag{43}$$

where $|\chi^0_0\rangle$ and $|\chi^1_0\rangle$ are states with ghost numbers 0 and -1 respectively. Then we have the following gauge transformations for the fields

$$\delta|\chi^0_0\rangle = \Delta Q_\pi|\Lambda^0_0\rangle + \frac{1}{2}\{\tilde{T}_0, \eta_1^+\eta_1\}|\Lambda^1_0\rangle, \qquad \delta|\chi^1_0\rangle = \tilde{T}_0|\Lambda^0_0\rangle + \Delta Q_\pi|\Lambda^1_0\rangle. \tag{44}$$

which are also reducible

$$\delta|\Lambda^{(i)0}{}_0\rangle = \Delta Q_\pi|\Lambda^{(i+1)0}{}_0\rangle + \frac{1}{2}\{\tilde{T}_0, \eta_1^+\eta_1\}|\Lambda^{(i+1)1}{}_0\rangle, \qquad |\Lambda^{(0)0}{}_0\rangle = |\Lambda^0_0\rangle, \tag{45}$$

$$\delta|\Lambda^{(i)1}{}_0\rangle = \tilde{T}_0|\Lambda^{(i+1)0}{}_0\rangle + \Delta Q_\pi|\Lambda^{(i+1)1}{}_0\rangle, \qquad |\Lambda^{(0)1}{}_0\rangle = |\Lambda^1_0\rangle. \tag{46}$$

Since the fields $|\chi^0_0\rangle$ and $|\chi^1_0\rangle$ contain infinite number of spins and since the order of reducibility grows with the spin value, then the order of reducibility of the gauge symmetry will be infinite.

6 Lagrangians for massive bosonic fields

6.1 New representation for the algebra

To construct new representation, we introduce two pairs of additional bosonic annihilation and creation operators b_1, b_1^+, b_2, b_2^+ with the standard commutation relations $[\,b_1, b_1^+] = [\,b_2, b_2^+] = 1$ and construct new representation as follows

$$m^2_{new} = 0, \qquad G_{0new} = -a^+_\mu a^\mu + \tfrac{D}{2} + b_1^+ b_1 + \tfrac{1}{2} + 2b_2^+ b_2 + h, \tag{47}$$

$$L^+_{1new} = a^\mu p_\mu + m b_1^+, \qquad L_{1new} = a^{+\mu} p_\mu + m b_1, \tag{48}$$

$$L^+_{2new} = \tfrac{1}{2}a^\mu a_\mu - \tfrac{1}{2}b_1^{+2} + b_2^+, \qquad L_{2new} = \tfrac{1}{2}a^{+\mu}a_\mu - \tfrac{1}{2}b_1^2 + (b_2^+ b_2 + h)b_2, \tag{49}$$

with the other operators being unchanged. Then we change the definition of scalar product of vectors in the new representation $\langle\Phi_1|\Phi_2\rangle_{new} = \langle\Phi_1|K|\Phi_2\rangle$, with operator K in the form

$$K = \sum_{n=0}^{\infty} |n\rangle\frac{C(n,h)}{n!}\langle n|, \qquad |n\rangle = (b_2^+)^n|0\rangle. \tag{50}$$

with $C(n,h)$ given in (34).

Next we introduce the operator $\tilde{Q}$ as if all the operators were the first class constarints

$$\tilde{Q} = \eta_0 L_0 + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+ L_{2new} + \eta_2 L_{2new}^+ + \eta_G G_{0new} - \eta_1^+\eta_1\mathcal{P}_0 - \eta_2^+\eta_2\mathcal{P}_G$$
$$+ (\eta_G\eta_1^+ + \eta_2^+\eta_1)\mathcal{P}_1 + (\eta_1\eta_G + \eta_1^+\eta_2)\mathcal{P}_1^+ + 2\eta_G\eta_2^+\mathcal{P}_2 + 2\eta_2\eta_G\mathcal{P}_2^+, \quad (51)$$

One can show that the operator (51) satisfy the relation $\tilde{Q}^+K = K\tilde{Q}$, which means that this operator is Hermitian relatively the new scalar product with operator K (50).

6.2 Lagrangians for the massive bosonic field with given spin

It can be shown [3] that we can construct Lagrangian for the field with given spin as

$$\mathcal{L}_n = \int d\eta_0 \; {}_n\langle\chi|K_\sigma Q_\sigma|\chi\rangle_n. \quad (52)$$

Here field $|\chi\rangle_n$ subject to the condition

$$\sigma|\chi\rangle_n = (n + (D-6)/2)|\chi\rangle_n. \quad (53)$$

with operator σ being

$$\sigma = G_0 + b_1^+b_1 + 2b_2^+b_2 + \eta_1^+\mathcal{P}_1 - \eta_1\mathcal{P}_1^+ + 2\eta_2^+\mathcal{P}_2 - 2\eta_2\mathcal{P}_2^+. \quad (54)$$

Next Q_σ is the part of operator $\tilde{Q}$ (51) independent of the ghost fields η_G, $\mathcal{P}_G$ with the substitution $h \to -\sigma$. Analogouly, operator K_σ is operator (50) where substitution $h \to -\sigma$ be done.

The gauge symmetry induced by nilpotency of the operator Q_σ will be reducible with the first stage of reducibility

$$\delta|\chi\rangle_n = Q_\sigma|\Lambda\rangle_n \quad gh(|\Lambda\rangle_n) = -1, \quad (55)$$
$$\delta|\Lambda\rangle_n = Q_\sigma|\Omega\rangle_n, \quad gh(|\Omega\rangle_n) = -2. \quad (56)$$

6.3 Unified description of all massive integer spin fields

It is evident, the fields with different spins $s = n$ may have different masses which we denote m_n. First of all we introduce the state vectors with definite spin and mass as follows

$$|\chi, m\rangle_{n,m_n} = |\chi\rangle_n \, \delta_{m,m_n}, \quad (57)$$

with $|\chi\rangle_n$ being defined in (53) and m in (57) is now a new variable of the states $|\chi, m\rangle_{n,m_n}$. Second, we introduce the mass operator M acting on the variable m so that the states $|\chi, m\rangle_{n,m_n}$ are eigenvectors of the operator M with the eigenvalues m_n

$$M|\chi, m\rangle_{n,m_n} = m_n|\chi, m\rangle_{n,m_n} = m|\chi, m\rangle_{n,m_n}. \quad (58)$$

Construction of the Lagrangian decribing unified dynamics of fields with all spins is realized in terms of a single state $|\chi\rangle$ containing the fields of all spins (57)

$$|\chi\rangle = \sum_{n=0}^{\infty} |\chi, m\rangle_{n,m_n}. \quad (59)$$

This Lagrangian describing a propagation of all integer spin fields with different masses simultaneously looks like [3]

$$\mathcal{L} \;=\; \int d\eta_0 \;\langle\chi|K_\sigma Q_{\sigma M}|\chi\rangle. \tag{60}$$

Let us turn to the gauge transformations. Analogously to (57) we introduce the gauge parameters for the fields with given spin and mass

$$|\Lambda,m\rangle_{n,m_n} = |\Lambda\rangle_n\,\delta_{m,m_n}, \qquad\qquad |\Omega,m\rangle_{n,m_n} = |\Omega\rangle_n\,\delta_{m,m_n} \tag{61}$$

and analogously to (59) we denote

$$|\Lambda\rangle = \sum_{n=0}^{\infty}|\Lambda,m\rangle_{n,m_n}, \qquad\qquad |\Omega\rangle = \sum_{n=0}^{\infty}|\Omega,m\rangle_{n,m_n}. \tag{62}$$

Summing up (55), (56) over all n we find gauge transformation for the field $|\chi\rangle$ (59) and transformation for the gauge parameter $|\Lambda\rangle$

$$\delta|\chi\rangle = Q_{\sigma M}|\Lambda\rangle, \qquad\qquad \delta|\Lambda\rangle = Q_{\sigma M}|\Omega\rangle. \tag{63}$$

7 Summary

We have developed the BRST approach to derivation of gauge invariant Lagrangians both for massless fermionic and massive bosonic higher spin fields. We investigated the (super)algebras generated by the constraints which are necessary to define these irreducible representations of the Poincare group and found that the algebras have an identical structure. In particular, the algebras contain operators which are not constraints neither in the space of bra-vectors nor in the space of ket-vectors. For the operators which are not constraints to be made harmless this method includes construction of a new representation of the algebra, after which the BRST operator can be obtained as if all the operators were the first class constraints.

The main obtained results are

- The Lagrangians for free arbitrary spin fields are constructed in terms of completely symmetric tensor(-spinor) fields (see eq. (39) for massless fermionic fields and eq. (52) for massive bosonic fields) in concise form. No off-shell constraints (including tracelessness) on the fields and the gauge parameters are used. All the equations which define an irreducible representation of the Poincare group (including tracelessness of the fields) are consequences of the Lagrangian equations of the motion and the gauge fixing.

- The models under consideration are reducible gauge theories. In the bosonic case the models have the first order of reducibility and in the fermionic case the order of reducibility grows with the value of spin.

- Lagrangian describing propagation of all massless fermionic fields simultaneously is constructed (43). Lagrangian describing propagation of all bosonic massive fields (with different massess) simultaneously is constructed (60).

There are several possibilities for extending our results. This approach can be applied to Lagrangian construction of fermionic massive fields and to Lagrangian construction of higher spin fields (both massive and massless) with mixed symmetry of Lorentz indeces (see [7] for corresponding bosonic massless case).

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Higher Spin Particles with Bosonic Counterpart of Supersymmetry

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Abstract

We propose the relativistic point particle models invariant under the bosonic counterpart of SUSY. The particles move along the world lines in four dimensional Minkowski space extended by N commuting Weyl spinors. The models provide after first quantization the non–Grassmann counterpart of chiral superfields, satisfying Klein–Gordon equation. Free higher spin fields obtained by expansions of such chiral superfields satisfy the $N = 2$ Bargman–Wigner equations in massive case and Fierz–Pauli equations in massless case.

1. Introduction. Higher spin fields (see e.g. [1]-[3]) were investigated recently mainly due to their relations to string theory. For the description of higher spin fields the usual space–time is often extended by additional coordinates, e.g. commuting tensorial coordinates and/or commuting spinorial variables [1]-[6] having twistorial origin [7]. Higher spin fields do appear as component fields in expansions of fields with respect to additional coordinate variables. It appears that the system of all higher spin fields possesses symmetry which is an extension of standard Poincare or conformal symmetries. In four dimensional space–time the system of massless higher spin fields has $Sp(8)$ symmetry or its supersymmetric extensions $OSp(N|8)$ ($N = 1, 2$) (see e.g. [8]).

In this report which is based on our paper [9] we propose new particle models invariant under bosonic counterpart of SUSY. The quantization of these particles produce infinite number of higher spin fields with all spins (helicities in massless case). The particle model with a trace of 'bosonic' SUSY has been considered in [10] for description of the relativistic particle with fixed spin (helicity). The realizations of 'even' superalgebra was used also in [11] for the description of spectrum of the critical open $N = 2$ string in 2+2 dimensions.

The plan of our report is the following. In Sect. 2 we define the model describing the particle trajectory in the Minkowski space extended by N Weyl commuting spinors. We

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determine the complete set of constraints and classify them. In Sect. 3 and 4 using Gupta–Bleuler method we perform the quantization of the models. The wave function describing first–quantized theory satisfies the Klein–Gordon equation and the bosonic counterpart of chirality condition. In expansion of wave function with respect to commuting spinors the component fields describe (anti)self-dual field strenghts and in massless case satisfy the Fierz-Pauli equations. It appears that in the case of bosonic counterpart of $N = 2$ SUSY one can obtain also the linear Bargmann–Wigner equations for $D = 4$ higher spin fields. In last section we shall summarize obtained results and present some unsolved questions related with our framework.

2. Action with bosonic SUSY and the constraints.

We describe the classical mechanics of higher spin particles by the following action

$$S = \int d\tau\, \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{2e}(\dot{\omega}_\mu \dot{\omega}^\mu + e^2 m^2) - im(a_{ij}\dot{\lambda}_i^\alpha \lambda_{\alpha j} - \bar{a}_{ij}\bar{\lambda}_{\dot{\alpha} j}\dot{\bar{\lambda}}_i^{\dot{\alpha}})\,. \tag{1}$$

The action (1) describes propagation of the particle in Minkowski space extended by commuting complex Weyl spinors coordinates $\lambda_i^\alpha(\tau)$, $\bar{\lambda}_i^{\dot{\alpha}} = (\overline{\lambda_i^\alpha})$. We shall consider $N = 2$ case ($i = 1, 2$) and $N = 1$ case (no internal subindices). The constant matrix a_{ij} is symmetric, $a_{ij} = a_{ji}$; if $a_{ij} = -a_{ji}$ the last terms in (1) are total derivatives because $a_{ij}\dot{\lambda}_i^\alpha \lambda_{\alpha j} = \frac{1}{2}(a_{ij}\lambda_i^\alpha \lambda_{\alpha j})^{\cdot}$. The variable e in Lagrangian (1) describes the einbein. Constant m is the mass of the particle.

The ω–form can be written in general case as follows

$$\dot{\omega}^\mu = \dot{x}^\mu - i\kappa_{ij}(\dot{\lambda}_i^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\lambda}_j^{\dot{\beta}} - \lambda_j^\alpha \sigma^\mu_{\alpha\dot{\beta}} \dot{\bar{\lambda}}_i^{\dot{\beta}}) \tag{2}$$

where constant matrix $\kappa_{ij} = \kappa_{ji}$ can be choose in the form $\kappa_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}$ with real κ by linear redefinitions of spinors λ_i^α in $N = 2$ internal space.

The action (1) is invariant under the following spinorial bosonic transformation

$$\delta x^\mu = i\kappa_{ij}(\lambda_i^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\varepsilon}_j^{\dot{\beta}} - \varepsilon_i^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\lambda}_j^{\dot{\beta}})\,, \qquad \delta\lambda_i^\alpha = \varepsilon_i^\alpha\,, \qquad \delta\bar{\lambda}_i^{\dot{\alpha}} = \bar{\varepsilon}_i^{\dot{\alpha}} \tag{3}$$

where ε_i^α is a constant commuting Weyl spinors. Conserved Noether spinorial charges corresponding to the transformations (3) are

$$R_{\alpha i} \equiv \pi_{\alpha i} - i\kappa_{ij} p_{\alpha\dot{\beta}} \bar{\lambda}_j^{\dot{\beta}} - im a_{ij}\lambda_{\alpha j}\,, \qquad \bar{R}_{\dot{\alpha} i} \equiv \bar{\pi}_{\dot{\alpha} i} + i\kappa_{ij}\lambda_j^\beta p_{\beta\dot{\alpha}} + im\bar{a}_{ij}\bar{\lambda}_{\dot{\alpha} j} \tag{4}$$

where p_μ, $\pi_{\alpha i}$, $\bar{\pi}_{\dot{\alpha} i}$ are the canonical momenta. Using the canonical Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu\,, \qquad \{\lambda_i^\alpha, \pi_{\beta j}\} = \delta^\alpha_\beta \delta_{ij}\,, \qquad \{\bar{\lambda}_i^{\dot{\alpha}}, \bar{\pi}_{\dot{\beta} j}\} = \delta^{\dot{\alpha}}_{\dot{\beta}}\delta_{ij} \tag{5}$$

we obtain the PB algebra

$$\{R_{\alpha i}, \bar{R}_{\dot{\beta} j}\} = -2i\kappa_{ij} p_{\alpha\dot{\beta}}\,, \qquad \{R_{\alpha i}, R_{\beta j}\} = 2im a_{ij}\epsilon_{\alpha\beta}\,, \qquad \{\bar{R}_{\dot{\alpha} i}, \bar{R}_{\dot{\beta} j}\} = -2im\bar{a}_{ij}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{6}$$

which is classical (Poisson bracket) realization of bosonic counterpart of $N = 2$ supersymmetry algebra with central charges $Z_{ij} = ma_{ij}$, $\bar{Z}_{ij} = m\bar{a}_{ij}$. Since the spinor variables are commuting and the Poisson brackets in (6) are even, the quantum realization of the algebra (6) is constructed in terms of the commutators

$$[R_{\alpha i}, \bar{R}_{\dot\beta j}] = 2\kappa_{ij} p_{\alpha\dot\beta}\,, \qquad [R_{\alpha i}, R_{\beta j}] = -2m a_{ij}\epsilon_{\alpha\beta}\,, \qquad [\bar{R}_{\dot\alpha i}, \bar{R}_{\dot\beta j}] = 2m\bar{a}_{ij}\epsilon_{\dot\alpha\dot\beta} \tag{7}$$

in contrast to the algebra of anticommutators in standard $N = 2$ supersymmetry.

The model (1) has the following nontrivial constraints (we omit the constraint which implies pure gauge character of the einbein e)

$$T \equiv p^2 - m^2 \approx 0\,, \tag{8}$$

$$D_\alpha \equiv \pi_{\alpha i} + i\kappa_{ij} p_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta}_j + i m a_{ij}\lambda_{\alpha j} \approx 0\,, \qquad \bar{D}_{\dot\alpha} \equiv \bar{\pi}_{\dot\alpha i} - i\kappa_{ij}\lambda^{\beta}_j p_{\beta\dot\alpha} - i m\bar{a}_{ij}\bar{\lambda}_{\dot\alpha j} \approx 0\,. \tag{9}$$

Nonvanishing Poisson brackets of the constraints (8)–(9) are

$$\{D_{\alpha i}, \bar{D}_{\dot\beta j}\} = 2i\kappa_{ij} p_{\alpha\dot\beta}\,, \qquad \{D_{\alpha i}, D_{\beta j}\} = -2im a_{ij}\epsilon_{\alpha\beta}\,, \qquad \{\bar{D}_{\dot\alpha i}, \bar{D}_{\dot\beta j}\} = 2im\bar{a}_{ij}\epsilon_{\dot\alpha\dot\beta}\,. \tag{10}$$

The constraint (8) $T \approx 0$ is a first class constraint. For classifying of the spinor constraints (9) we look for the determinant of the matrix

$$\mathcal{C} = \begin{pmatrix} \{D_{\alpha i}, D_{\beta j}\} & \{D_{\alpha i}, \bar{D}_{\dot\beta j}\} \\ \{\bar{D}_{\dot\alpha i}, D_{\beta j}\} & \{\bar{D}_{\dot\alpha i}, \bar{D}_{\dot\beta j}\} \end{pmatrix} = \begin{pmatrix} -2im a_{ij}\epsilon_{\alpha\beta} & 2i\kappa_{ij} p_{\alpha\dot\beta} \\ -2i\kappa_{ij} p_{\beta\dot\alpha} & 2im\bar{a}_{ij}\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}. \tag{11}$$

If matrix (a_{ij}) is diagonal it follows for $N = 1, 2$ that in massive case $\det\mathcal{C}$ is always nonzero, therefore all the constraints (9) are of second class.

In case of antidiagonal matrix (a_{ij}) the matrix (11) has vanishing determinant when $\kappa = -|a_{12}|^2 < 0$. Only in such a case the first class constraints are present in the model (1).[1] Thus in massive case if we wish to have spinorial first class constraints we should consider $N \geq 2$ bosonic supersymmetry.

If $N = 2$ we shall consider a simple choice $\kappa = -a_{12} = -1$. In such a case the formulation (1) has an attractive interpretation if we pass to the commuting four–component Dirac spinor $\psi_a = \begin{pmatrix} \lambda_{\alpha 1} \\ \bar{\lambda}^{\dot\alpha}_2 \end{pmatrix}$, $\bar{\psi}^a = (\psi^+\gamma_0)^a = (\lambda^\alpha_2, \bar{\lambda}_{\dot\alpha 1})$, where $a = 1, 2, 3, 4$ is Dirac index. The Lagrangian (1) takes the simple form

$$\mathcal{L} = -\tfrac{1}{2e}(\dot\omega_\mu\dot\omega^\mu + e^2m^2) - im(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi})\,, \tag{12}$$

$$\dot\omega^\mu = \dot{x}^\mu + i(\dot{\bar{\psi}}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\dot{\psi})\,. \tag{13}$$

In massless case ($m = 0$) the matrix (11) has vanishing determinant and even if $N = 1$ the half of the spinorial constraints are first class.

3. Gupta–Bleuler quantization of the model with $N = 1$ bosonic SUSY. We shall perform the quantization using Gupta–Bleuler technique what implies the split of the second class constraints into complex–conjugated pairs, with holomorphic and antiholomorphic parts forming separately the subalgebras of first class constraints.

[1] We note that in case of usual $N = 2$ massive superparticle [12] when spinor variables are Grassmannian and the matrix (a_{ij}) is skew–symmetric, the first class constraints are present if $\kappa = |a_{12}|^2 > 0$.

In massive $N=1$ case the algebra (10) of the constraints (9) does not satisfy the Gupta–Bleuler requirements. However, the redefined constraints

$$\mathcal{D}_\alpha = D_\alpha + \tfrac{b}{m} p_{\alpha\dot\beta} \bar D^{\dot\beta} \,, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \bar D_{\dot\alpha} + \tfrac{b}{m} D^\beta p_{\beta\dot\alpha} \tag{14}$$

have the following algebra (we take $a_{11}=1$ without the loss of generality and we obtain $b = 1 \pm \sqrt{2}$)

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \tfrac{2i}{m}\epsilon_{\alpha\beta} T \,, \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = -\tfrac{2i}{m}\epsilon_{\dot\alpha\dot\beta} T \,, \qquad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -8bi p_{\alpha\dot\beta} - \tfrac{2b^2 i}{m^2} p_{\alpha\dot\beta} T \tag{15}$$

i. e. are suitable for application of Gupta–Bleuler quantization method. The wave function which satisfies the Klein–Gordon constraint (8) and spinorial wave equations ($\bar{\mathcal{D}}_{\dot\alpha}\Psi = 0$ (chiral case) or $\mathcal{D}_\alpha \Psi = 0$ (antichiral case)) provide the bosonic (non–Grassmann) counterpart of $D=4$ $N=1$ chiral superfield. It is possible to introduce new spinorial variables λ'^α, $\bar\lambda'^{\dot\alpha}$, π'_α, $\bar\pi'_{\dot\alpha}$ via canonical transformation (see details in [9]) in which new constraints (14) have the form

$$\mathcal{D}_\alpha = \pi'_\alpha - 4bi p_{\alpha\dot\beta}\bar\lambda'^{\dot\beta} \approx 0 \,, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \bar\pi'_{\dot\alpha} + 4bi \lambda'^\beta p_{\beta\dot\alpha} \approx 0 \,. \tag{16}$$

Solving chirality condition we obtain that the expansion of the wave function with respect to new spinorial variables contains infinite number space–time fields $\psi_{\alpha_1\cdots\alpha_n}(x) = \psi_{(\alpha_1\cdots\alpha_n)}(x)$. They satisfy the Klein–Gordon equation ($\Box \equiv \partial_\mu\partial^\mu$)

$$(\Box + m^2)\psi_{\alpha_1\cdots\alpha_n}(x) = 0 \qquad (n = 1, 2, \ldots) \,. \tag{17}$$

In antichiral case do appear in the expansion of the wave function the infinite number of fields $\bar\psi_{(\dot\alpha_1\cdots\dot\alpha_n)}(x)$ with dotted Weyl indices.

In massless $N=1$ case the spinorial constraints

$$D_\alpha = \pi_\alpha + i p_{\alpha\dot\beta}\bar\lambda^{\dot\beta} \approx 0 \,, \qquad \bar D_{\dot\alpha} = \bar\pi_{\dot\alpha} - i\lambda^\beta p_{\beta\dot\alpha} \approx 0 \tag{18}$$

are the mixture of first and second class constraints. The spinorial bosonic first class constraints are obtained from (18) by the multiplication with $p_{\alpha\dot\beta}$:

$$F^{\dot\alpha} = p^{\dot\alpha\beta} D_\beta \approx 0 \,, \qquad \bar F^\alpha = \bar D_{\dot\beta} p^{\dot\beta\alpha} \approx 0 \,. \tag{19}$$

Unfortunately these constraints are reducible since

$$p_{\alpha\dot\beta} F^{\dot\beta} \approx 0 \qquad \bar F^\beta p_{\beta\dot\alpha} \approx 0 \,. \tag{20}$$

Irreducible separation of first and second class constraints is obtained by the projection of spinorial constraints (18) on spinors λ^α and $\bar\lambda_{\dot\alpha} p^{\dot\alpha\alpha}$. The constraints

$$G \equiv \lambda^\alpha D_\alpha \approx 0 \,, \qquad\qquad \bar G \equiv \bar D_{\dot\alpha}\bar\lambda^{\dot\alpha} \approx 0 \tag{21}$$

are second class whereas the constraints

$$F \equiv \bar\lambda_{\dot\alpha} p^{\dot\alpha\alpha} D_\alpha \approx 0 \,, \qquad\qquad \bar F \equiv \bar D_{\dot\alpha} p^{\dot\alpha\alpha}\lambda_\alpha \approx 0 \tag{22}$$

are first class.

Because the spinors λ^α and $\bar\lambda_{\dot\alpha} p^{\dot\alpha\alpha}$ are independent the pair of constraints $G \approx 0$ and $F \approx 0$ is equivalent to the constraints $D_\alpha \approx 0$. Similarly the constraints $\bar G \approx 0$ and $\bar F \approx 0$ are equivalent to the constraints $\bar D_{\dot\alpha} \approx 0$. Thus we have two sets of the wave equations: 'bosonic chiral' case

$$T\,\Psi = 0\,, \qquad F\,\Psi = 0\,, \qquad \bar D_{\dot\alpha}\,\Psi = 0 \tag{23}$$

or 'bosonic antichiral' one

$$T\,\Psi = 0\,, \qquad \bar F\,\Psi = 0\,, \qquad D_\alpha\,\Psi = 0\,. \tag{24}$$

In the representation

$$p_\mu = -i\partial_\mu\,, \qquad \pi_\alpha = -i\partial_\alpha\,, \qquad \bar\pi_{\dot\alpha} = -i\bar\partial_{\dot\alpha} \tag{25}$$

the equations in chiral case

$$\Box\,\Psi = 0\,, \qquad \bar D_{\dot\alpha}\,\Psi = (-i\bar\partial_{\dot\alpha} - \lambda^\beta \partial_{\beta\dot\alpha})\,\Psi = 0\,, \qquad -i\bar\lambda_{\dot\alpha}\partial^{\dot\alpha\alpha} D_\alpha\,\Psi = -\bar\lambda_{\dot\alpha}\partial^{\dot\alpha\alpha}\partial_\alpha\,\Psi = 0$$

give only the dependence of the wave function on left–chiral variables

$$z_L \equiv (x^\mu_L = x^\mu + i\lambda\sigma^\mu\bar\lambda,\ \lambda^\alpha)\,. \tag{26}$$

We obtain the expansion

$$\Psi(x_L,\lambda) = \sum_{n=0}^{\infty} \lambda^{\alpha_1}\ldots\lambda^{\alpha_n}\phi_{\alpha_1\ldots\alpha_n}(x_L)\,. \tag{27}$$

The component fields are completely symmetric in spinor indices, $\phi_{\alpha_1\ldots\alpha_n} = \phi_{(\alpha_1\ldots\alpha_n)}$ and satisfy Fierz–Pauli equations for component fields

$$\partial^{\dot\beta\beta}\,\phi_{\beta\alpha_2\ldots\alpha_n} = 0\,. \tag{28}$$

Scalar component field satisfies only the d'Alembert equation $\Box\,\phi = 0$. The fields $\phi_{\alpha_1\ldots\alpha_n}(x)$ in the expansion of the wave function (27) are self–dual field strenghts of massless particles with helicities $n/2$. In antichiral case we obtain analogously anti-self–dual field strenghts of massless particles.

4. Quantum states describing particles with $N = 2$ bosonic SUSY.

The constraints (8)–(9) at $\kappa = -a_{12} = -1$, written in Dirac notation, are the following

$$T \equiv p^2 - m^2 \approx 0\,, \qquad D^a \equiv \pi^a + i\bar\psi^b(\hat p - m)_b{}^a \approx 0\,, \qquad \bar D_a \equiv \bar\pi_a - i(\hat p - m)_a{}^b\psi_b \approx 0\,. \tag{29}$$

Here π^a and $\bar\pi_a$ are the conjugate momenta for ψ_a and $\bar\psi^a$; its Poisson brackets are

$$\{\psi_a, \pi^b\} = \delta^b_a \qquad \{\bar\psi^a, \bar\pi_b\} = \delta^a_b \tag{30}$$

where we use notation $\hat p \equiv \gamma^\mu p_\mu$.

From nonvanishing Poisson brackets of the constraints

$$\{\bar D_a, D^b\} = -2i(\hat p - m)_a{}^b \tag{31}$$

we obtain directly that half of the spinorial constraints (29) are first class constraints. The projectors $\mathcal{P}_\pm \equiv \frac{1}{2m}(m \pm \hat{p})$ define respectively the first class constraints

$$F^a = D^b(\hat{p}+m)_b{}^a\,, \qquad \bar{F}_a = (\hat{p}+m)_a{}^b\bar{D}_b \tag{32}$$

and the second class

$$G^a = D^b(\hat{p}-m)_b{}^a\,, \qquad \bar{G}_a = (\hat{p}-m)_a{}^b\bar{D}_b\,. \tag{33}$$

But due to the reducibility conditions

$$F^b(\hat{p}-m)_b{}^a \approx 0 \qquad (\hat{p}-m)_a{}^b\bar{F}_b \approx 0 \tag{34}$$

if $T = p^2 - m^2 \approx 0$ the eight constraints $(F^a, \bar{F}_a)$ has only four (real) independent constraints. Analogously, the constraints $(G^a, \bar{G}_a)$ contain also four (real) independent constraints.

In a way depending on the choice of second class constraints imposed on the wave function, we obtain (see details in [9]) that the wave function satisfies or the 'bosonic chiral' equations

$$T\,\Psi = 0\,, \qquad \bar{D}_a\,\Psi = 0\,, \qquad F^a\,\Psi = 0 \tag{35}$$

or the 'bosonic antichiral' ones

$$T\,\Psi = 0\,, \qquad D^a\,\Psi = 0\,, \qquad \bar{F}_a\,\Psi = 0\,. \tag{36}$$

In 'bosonic chiral' case the wave equations

$$(\Box^2 + m^2)\Psi = 0\,, \qquad -i[\frac{\partial}{\partial\bar{\psi}^a} - (i\hat{\partial}+m)_a{}^b\psi_b]\Psi = 0\,, \qquad i\frac{\partial}{\partial\psi_b}(i\hat{\partial}-m)_b{}^a\Psi = 0 \tag{37}$$

have the general solution

$$\Psi(x,\psi,\bar{\psi}) = e^{\bar{\psi}(i\hat{\partial}+m)\psi}\sum_{n=0}^{\infty}\psi_{a_1}\cdots\psi_{a_n}\phi^{a_1\cdots a_n}(x) \tag{38}$$

where the component fields $\phi^{a_1\cdots a_n}(x)$ are completely symmetric with respect to all Dirac indices, $\phi^{a_1\cdots a_n}(x) = \phi^{(a_1\cdots a_n)}(x)$, and satisfy the Dirac equations

$$(i\hat{\partial}-m)_{a_1}{}^b\phi^{a_1a_2\cdots a_n}(x) = 0\,. \tag{39}$$

From (39) follows the Klein–Gordon equation (37). Finally we have obtained the Bargmann–Wigner fields describing massive particles of spins $n/2$.

5. Conclusions. We have considered the models of the relativistic point particles propagating on fourdimensional Minkowski space extended by commuting Weyl spinors. The models are invariant under bosonic (non–Grassmann) counterpart of SUSY. The main results are the following:

- Higher spin fields emerge as the result of first quantization of the proposed models.
- In massless case one obtained infinite set of field strenghts with all helicities satisfying linear Fierz–Pauli equations.

- If we quantize the massive particle with $N = 1$ bosonic counterpart of SUSY we obtain the massive free fields with any spin which satisfy only Klein–Gordon equation. We stress however that the first order equations of motions are missing.

- For massive particle with $N = 2$ bosonic counterpart of SUSY we get after quantization the wave function described by Bargmann–Wigner equations.

Let us note that some questions still should be answered. For instance, we do not understand the relation of our formalism with the unfolded formulation of higher spin fields by Vasiliev (see e.g. [1]) and link with the formulation using tensorial coordinates (see e.g. [5]). Also in our approach appears nonstandard relation between spin and statistics: both integer and half–integer spin fields have the same bosonic statistic. Here one should add that the analogous situation with statistics appears also in higher spin fields theory formulated on twistor spaces [7], [13], [14].

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Notes on Harmonic Superspace and Pure Spinor String Theory

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Abstract

We elucidate some relations between Harmonic Superspace and Pure Spinor String Theory. The example of massless hypermultiplet for N=2 D=4 is considered and the action is derived from a String Field Theory action.

1 Introduction

Harmonic superspace is a very useful tool for study supersymmetric models with extended supersymmetry. See [1] for a complete review of the subject and two useful accounts of the subject can be found in [2]. Projective harmonic superspace has been introduced in [3] and the application to the AdS/CFT correspondence is studied in [4]. Recent developments of $N = 4$ harmonic superspace for SYM can be found in [5].

On string theory side, an important revolution was started with the construction of Pure Spinor String Theory by N. Berkovits in [6]. This new formalism is based on sigma model for the superspace coordinates with the additional of some bosonic fields (in the following denoted by λ^α). The latter are to be understood as ghost fields and they are needed in order to implement the BRST symmetry of the theory. They are constrained to satisfy a quadratic equation $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ which is known as *Pure Spinor Constraint*.

It has been noticed in [7] that by a suitable ansatz, the pure spinor constraint is solved and the solution parametrize the same cosets of the harmonic coordinates of the harmonic superspace. Therefore, using the idea of adding new ghost fields pursued in [8], we derived the N=3 harmonic superspace action and the equations of motion in [7].

In [7], we consider only the case of N=3 SYM, but the same technique can be applied to the D=4, N=2 hypermultiplet. In the present note, under a suggestion of E. Sokatchev and E. Ivanov, we wrote the complete derivation of the action for N=3 (sec. 3) and we describe the case on the hypermultiplet (sec. 4). This last part is original and it has been presented at the conference SQS'5 in Dubna in 2005. In sec. 2, we briefly review the coordinates of harmonic superspaces derived from string theory.

2 N=4,3,2 harmonic superspace from pure spinors

The notations are taken from [7] to which we refer for a complete discussion.

We substitute the decomposition $\lambda^{\hat\alpha} = (\lambda^\alpha_I, \bar\lambda^{\dot\alpha I})$ (where $\alpha, \dot\alpha = 1, 2$ and $I = 1, \dots 4$) into the pure spinor constraints $\lambda\gamma^m\lambda = 0$ obtaining the six plus four equations

$$\lambda^\alpha_I \epsilon_{\alpha\beta}\lambda^\beta_J + \frac{1}{2}\epsilon_{IJKL}\bar\lambda^{\dot\alpha K}\epsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta L} = 0\,, \qquad \lambda^\alpha_I\bar\lambda^{\dot\alpha I} = 0\,. \tag{1}$$

To solve these constraints we adopt the ansatz $\lambda^\alpha_I = \lambda^\alpha_a u^a_I\,, \bar\lambda^{\dot\alpha J} = \bar\lambda^{\dot\alpha}_a \bar v^{aJ}$, where $a = 1, 2$. The new variables u^a_I and $\bar v^{aJ}$ are complex and commuting. They carry $GL(2, \mathbf{C})$ and $SU(4)$ indices. The spinors $\lambda^\alpha_a, \bar\lambda^{\dot\alpha}_a$ are also complex and commuting, and carry a representation of $SL(2, \mathbf{C})$ and $GL(2, \mathbf{C})$. This decomposition is left invariant by the gauge transformations

$$u^a_I \to M^a{}_b u^b_I\,, \qquad \lambda^\alpha_a \to \lambda^\alpha_b (M^{-1})^b{}_a\,, \qquad \bar v^{aJ} \to \bar M^a{}_b \bar v^{bJ}\,, \qquad \bar\lambda^{\dot\alpha}_a \to \bar\lambda^{\dot\alpha}_b (\bar M^{-1})^b{}_a\,, \tag{2}$$

where M and $\bar M$ are independent $GL(2, \mathbf{C})$ matrices. The factorization plus the gauge invariance (2) yields 16 complex parameters. To reduce to the usual 11 independent complex parameters of pure spinors, we further impose the following two covariant constraints

$$u^a{}_I \bar v^{bI} = 0\,, \qquad \lambda^\alpha_a \epsilon_{\alpha\beta}\epsilon^{ab}\lambda^\beta_b + \bar\lambda^{\dot\alpha}_a\epsilon_{\dot\alpha\dot\beta}\epsilon^{ab}\bar\lambda^{\dot\beta}_b = 0\,. \tag{3}$$

The first constraint in (3) and the gauge transformations in (2) reduce the 16 complex components of u^a_I and $\bar v^{aI}$ to 8 real parameters. This is the same number as the number of independent parameters of the coset $\frac{U(4)}{U(2)\times U(2)} = \frac{SU(4)}{S(U(2)\times U(2))}$ used in [2] (see also [5] and [4]). The restriction of $U(2)\times U(2)$ to the subgroup $S(U(2)\times U(2))$ is due to second constraint of (3). The latter is preserved by the transformations M and $\bar M$ only after the identification $\det M = \det \bar M$.

Let us turn to N=3 harmonic superspace. If we decompose the λ^α_I's and the $\bar\lambda^{\dot\alpha I}$'s into N=3 vectors and N=3 scalars we have $\lambda^\alpha_I = (\lambda^\alpha_i, \psi^\alpha)$ and $\bar\lambda^{\dot\alpha I} = (\bar\lambda^{\dot\alpha i}, \bar\psi^{\dot\alpha})$. In that basis, the pure spinor constraints in (2.1) become

$$\lambda^\alpha_i\epsilon_{\alpha\beta}\lambda^\beta_j + \epsilon_{ijk}\bar\lambda^{\dot\alpha k}\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} = 0\,, \qquad \lambda^\alpha_i\epsilon_{\alpha\beta}\psi^\beta + \epsilon_{ijk}\bar\lambda^{\dot\alpha j}\epsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta k} = 0\,, \qquad \lambda^\alpha_i\bar\lambda^{\dot\alpha i} + \psi^\alpha\bar\psi^{\dot\alpha} = 0\,. \tag{4}$$

The reduction to the N=3 case is obtained by setting $\psi^\alpha = \bar\psi^{\dot\alpha} = 0$. Inserting this ansatz into the first two equations of (4), we obtain

$$\lambda^\alpha_i\epsilon_{\alpha\beta}\lambda^\beta_j = 0\,, \qquad \bar\lambda^{\dot\alpha j}\epsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta k} = 0\,, \tag{5}$$

which is equivalent to requiring that all determinants of order 2 of the matrices λ^α_i and $\bar\lambda^{\dot\alpha i}$ vanish. This means that the pure spinors can be factorized into $\lambda^\alpha_i = \lambda^\alpha u_i\,, \bar\lambda^{\dot\alpha i} = \bar\lambda^{\dot\alpha}\bar v^i$ and the equations (4) are solved by $\psi^\alpha = \bar\psi^{\dot\alpha} = 0\,, \qquad u_i\bar v^i = 0$.

So for the N=3 case no constraint is needed for λ^α and $\bar\lambda^{\dot\alpha}$. Notice that the two complex vectors u_i and $\bar v^i$ are defined up to a gauge transformation

$$u_i \to \rho u_i\,, \qquad \lambda^\alpha \to \rho^{-1}\lambda^\alpha\,, \qquad \bar v^i \to \sigma\bar v^i\,, \qquad \bar\lambda^{\dot\alpha} \to \sigma^{-1}\bar\lambda^{\dot\alpha} \tag{6}$$

where $\rho, \sigma \in \mathbf{C}$. The two real parameters $|\rho|$ and $|\sigma|$ are used to impose the normalizations $u_i\bar u^i = 1$ and $v_i\bar v^i = 1$. If one also gauges away the overall phases of u_i and $\bar v^i$, the space of harmonic coordinates u_i and $\bar{v}^i$ is parameterized by six real parameters. This coincides with the number of free parameters of the coset $SU(3)/U(1) \times U(1)$. Indeed, we can construct 3×3 matrices $(u_i^1, u_i^2, u_i^3) = (u_i^{(1,0)}, u_i^{(0,-1)}, u_i^{(-1,1)})$ as follows $u_i^1 \equiv u_i^{(1,0)} = u_i\,, \quad u_i^2 \equiv u_i^{(-1,1)} = \epsilon_{ijk}\bar{v}^j \bar{u}^k\,, \quad u_i^3 \equiv u_i^{(0,-1)} = v_i$ where $\bar{u}^i = (u_i)^*$ and $v_i = (\bar{v}^i)^*$. Fixing the phases of u_i^1 and u_i^3, the u_i^I form $SU(3)$ matrices which are coset representatives of $\frac{SU(3)}{U(1)\times U(1)}$. The $U(1) \times U(1)$ transformations generate the phases $\arg(\rho)$ and $\arg(\sigma)$. The notation $u_i^{(a,b)}$ indicates the $U(1) \times U(1)$ charges of the harmonic variables and they satisfy the hermiticity property $\overline{u_i^{(a,b)}} = u^{i(-a,-b)}$. We denote by u_I^i the inverse harmonics $u_I^i u_i^J = \delta_I{}^J\,, \quad u_i^I u_I^j = \delta_i{}^j\,, \quad \det u = \epsilon^{ijk} u_i^1 u_j^2 u_k^3 = 1$.

Finally, we consider a further reduction to N=2. We decompose the N=3 pure spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha} i}$ into a vector of N=2 and a singlet, $\lambda_i^\alpha = (\lambda_{\mathcal{I}}^\alpha, \lambda_3^\alpha)$ and $\bar{\lambda}^{\dot{\alpha} i} = (\bar{\lambda}^{\dot{\alpha}\mathcal{I}}, \bar{\lambda}^{\dot{\alpha}3})$ where $\mathcal{I} = 1, 2$. We set λ_3^α and $\bar{\lambda}_3^{\dot{\alpha}}$ to zero. The pure spinor equations (5) reduce then to

$$\lambda_{\mathcal{I}}^\alpha \epsilon_{\alpha\beta} \lambda_{\mathcal{J}}^\beta \epsilon^{\mathcal{I}\mathcal{J}} = 0\,, \qquad \bar{\lambda}^{\dot{\alpha}\mathcal{J}} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}\mathcal{K}} \epsilon_{\mathcal{J}\mathcal{K}} = 0\,, \qquad \lambda_{\mathcal{I}}^\alpha \bar{\lambda}^{\dot{\alpha}\mathcal{I}} = 0\,. \tag{7}$$

The first two equations imply that $\lambda_{\mathcal{I}}^\alpha$ and $\bar{\lambda}^{\dot{\alpha}\mathcal{I}}$ are factorized into $\lambda_{\mathcal{I}}^\alpha = \lambda^\alpha u_{\mathcal{I}}$ and $\bar{\lambda}^{\dot{\alpha}\mathcal{J}} = \bar{\lambda}^{\dot{\alpha}} \bar{v}^{\mathcal{J}}$ where $u_{\mathcal{I}} \bar{v}^{\mathcal{I}} = 0$. The vector $\bar{v}^{\mathcal{I}}$ is proportional to $\epsilon^{\mathcal{I}\mathcal{J}} u_{\mathcal{J}}$. Hence without loss of generality one may write

$$\lambda_{\mathcal{I}}^\alpha = \lambda^\alpha u_{\mathcal{I}}\,, \qquad \bar{\lambda}^{\dot{\alpha}\mathcal{J}} = \bar{\lambda}^{\dot{\alpha}} \epsilon^{\mathcal{I}\mathcal{J}} u_{\mathcal{I}}\,. \tag{8}$$

With this parametrization of the N=2 case there are neither constraints on the λ's nor on the u's.

The vector $u_{\mathcal{I}}$ yields the usual parametrization of N=2 harmonic superspace [1]. Namely, one introduces the $SU(2)$ matrix $(u_{\mathcal{I}}^+, u_{\mathcal{I}}^-)$ where $u_{\mathcal{I}}^+ = u_{\mathcal{I}}$ and $u_{\mathcal{I}}^- = (u^{+\mathcal{I}})^*$ with $u_{\mathcal{J}}^+ = \epsilon_{\mathcal{J}\mathcal{K}} u^{+\mathcal{K}}$. The coset $SU(2)/U(1)$ is obtained by dividing by the subgroup $U(1)$ which generates the phases $u_{\mathcal{I}}^\pm \to e^{\pm i\alpha} u_{\mathcal{I}}^\pm$. In fact, the decompositions are defined up to a rescaling of $\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}$ and of $u_{\mathcal{I}}$ given by $u_{\mathcal{I}} \to \rho u_{\mathcal{I}}$, for $\rho \neq 0$. This yields the compact space $\mathbf{CP}^1$.

3 N=3 Harmonic SYM from String Field Theory

The field equation for $D = 4, N = 3$ SYM-theory in ordinary (not harmonic) superspace are given by [10]

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = \epsilon_{\alpha\beta} \bar{W}^{ij}\,, \qquad \{\bar{\nabla}_{\dot{\alpha}i}, \bar{\nabla}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}} W_{ij}\,, \qquad \{\nabla_\alpha^i, \bar{\nabla}_{\dot{\beta}j}\} = \delta_j^i \nabla_{\alpha\dot{\beta}}\,. \tag{9}$$

The coordinates for this N=3 superspace, $(x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$, are obtained by imposing the constraint $\theta_4^\alpha = \bar{\theta}^{\dot{\alpha}4} = 0$. Since θ's transform into λ's under BRST transformations we also impose for consistency $\lambda_4^\alpha = \bar{\lambda}^{\dot{\alpha}4} = 0$.

Using the decomposition of the N=3 spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha}i}$ given above, and contracting the harmonic variables with the operators $d_{\hat{\alpha}}$ representing the covariant derivatives yields to the BRST charge

$$Q_G = \lambda^\alpha d_\alpha^1 + \bar{\lambda}^{\dot{\alpha}} \bar{d}_{3\dot{\alpha}}\,.$$

where $d_\alpha^1 = u_i d_\alpha^i = u_i^1 d_\alpha^i = u_i^{(1,0)} d_\alpha^i$ and $\bar{d}_{3\dot{\alpha}} = \bar{v}^i \bar{d}_{\dot{\alpha}i} = u_3^i \bar{d}_{\dot{\alpha}i} = u^{i(0,1)} \bar{d}_{\dot{\alpha}i}$. The operator d_α^1 corresponds to $\xi_i D_\alpha^i$ and $\bar{d}_{3\dot{\alpha}}$ to $\eta^i \bar{D}_{\dot{\alpha}i}$ in [1].

Due to the constraints on the u's, the operators d^1_α and $\bar{d}_{3\dot{\alpha}}$ satisfy the commutation relations

$$\{d^1_\alpha, d^1_\beta\} = 0\,, \qquad \{d^1_\alpha, \bar{d}_{3\dot{\beta}}\} = 0\,, \qquad \{\bar{d}_{3\dot{\alpha}}, \bar{d}_{3\dot{\beta}}\} = 0\,. \tag{10}$$

Hence Q_G (where G stands for Grassmann) is nilpotent for any λ^α and $\bar{\lambda}^{\dot{\alpha}}$.

The BRST operator Q_G implements naturally the G-analyticity on the space of superfields $\Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$. A superfield with ghost number zero is given by $\Phi(x, \theta, \bar{\theta}, u)$ and G-analyticity means $Q_G\Phi = 0$ which implies $D^1_\alpha \Phi = \bar{D}_{3\dot{\alpha}}\Phi = 0$ (since $\{d^1_\alpha,\, \Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)\}$ $= D^1_\alpha \Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$ and similarly for $\bar{d}_{3\dot{\alpha}}$). Such a superfield is called a G-analytic superfield. A generic superfield $\Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$ with ghost number one can be parametrized in terms of two u-dependent spinorial superfields $A_\alpha, \bar{A}_{\dot{\alpha}}$ as $\Phi^{(1)}(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u) = \lambda^\alpha A_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{A}_{\dot{\alpha}}$ and $\{Q_G, \Phi^{(1)}\} = 0$ implies the following constraints on these superfields

$$D^1_\alpha A_\beta + D^1_\beta A_\alpha = 0\,, \qquad \bar{D}_{3\dot{\alpha}} \bar{A}_{\dot{\beta}} + \bar{D}_{3\dot{\beta}} \bar{A}_{\dot{\alpha}} = 0\,, \qquad D^1_\alpha \bar{A}_{\dot{\beta}} + \bar{D}_{3\dot{\beta}} A_\alpha = 0\,. \tag{11}$$

Assuming that A_α and $A_{\dot{\alpha}}$ factorize in the same way as $D^1_\alpha = u_i D^i_\alpha$ and $\bar{D}_{3\dot{\alpha}} = \bar{v}^i \bar{D}_{\dot{\alpha} i}$, the equations (11) reproduce (9). Gauge transformations are generated by a ghost-number zero scalar superfield $\Omega^{(0)}$. To lowest order in $\Phi^{(1)}$ they read $\delta\Phi^{(1)} = \{Q_G, \Omega^{(0)}\}$ which yields $\delta A_\alpha = D_\alpha \Omega$ and $\delta A_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \Omega$. The Q_G-cohomology in the space of superfields with ghost number 1 is empty.

To determine on which harmonic variables superfields depend, we construct a second BRST operator Q_H which is constructed from the $SU(3)$ generators

$$d^a{}_b = u^a_i \partial_{u^b_i} - u^i_b \partial_{u^i_a} = u^a_i p^i_b - u^i_b p^a_i\,. \tag{12}$$

where p^i_b can be represented by $\partial / \partial u^b_i$ and similarly for p^b_i. These generators split into three raising operators $d^1_2 = d^{(2,-1)}, d^2_3 = d^{(-1,2)}, d^1_3 = d^{(1,1)}$, three lowering operators $d^2_1 = d^{(-2,1)}, d^3_2 = d^{(1,-2)}, d^3_1 = d^{(-1,-1)}$, and two Cartan generators d^1_1 and d^2_2. The raising operators operators commute with Q_G and form an algebra, in particular $[d^{(2,-1)}, d^{(-1,2)}] = d^{(1,1)}$. This suggests to construct a new nilpotent BRST operator Q_H

$$Q_H = \xi^3_1\, d^1_3 + \xi^2_1\, d^1_2 + \xi^3_2\, d^2_3 - \beta^1_3 \xi^2_1 \xi^3_2\,, \tag{13}$$

where we introduced new pairs of anticommuting (anti)ghosts (ξ^3_1, β^1_3), (ξ^2_1, β^1_2), (ξ^3_2, β^2_3) with canonical anticommutation relations. Since Q_H and Q_G anticommute their sum Q_{tot} is obviously nilpotent. The harmonic weights of the superfields follow from requiring that $\Phi^{(1)}$ has zero harmonic weight, just like the BRST charge Q_{tot}. Note that $\Phi^{(1)}$ depends only upon the variables $x, \theta, \bar{\theta}, \lambda, \bar{\lambda}$'s and u's and not upon the conjugated momenta as a consequence of quantum mechanical rules. This forbids ghost-number one combinations of the form $\beta\xi\xi, \beta\xi\lambda, \ldots$.

The equations of motion for N=3 SYM follow from the BRST-cohomology equations

$$\{Q_{tot}, \Phi^{(1)}\} + \frac{1}{2}\{\Phi^{(1)}, \Phi^{(1)}\} = 0\,. \tag{14}$$

To reduce this equation to the field equations of harmonic superspace, we use the fact that Q_G has no cohomology. We decomposed Φ into $\Phi_G + \Phi_H$. This implies that equation (16) is solved by a pure gauge superfield $\Phi^{(1)}_G = e^{-i\Delta}\Big(Q_G e^{i\Delta}\Big)$ where Δ is a ghost-number zero superfield known in the literature as the *bridge*. In the harmonic superspace framework, one usually employs the bridge superfield $\Delta(x,\theta,\bar{\theta},u)$ to bring the spinorial covariant derivatives to the 'pure gauge' form. Here the bridge is seen as a solution of (14). By making a finite gauge transformation which sets $\Phi^{(1)}_G = 0$, the gauge transformed $\Phi^{(1)}_H$ is given by

$$e^{-i\Delta}(\Phi^{(1)}_H + Q_H)e^{i\Delta} = \xi^3_1\, V^{(1,1)} + \xi^2_1\, V^{(2,-1)} + \xi^3_2\, V^{(-1,2)}\,. \tag{15}$$

And inserting this ansatz in (14) one finds the SYM equations of motion of N=3 harmonic superspace [1]. Those equations can be derived by the action

$$S_{N=3} = \int d\mu \Big(\Phi^{(1)}_H Q_H \Phi^{(1)}_H + \frac{2}{3}\Phi^{(1)}_H \star \Phi^{(1)}_H \star \Phi^{(1)}_H\Big) \tag{16}$$

where $\star$ denotes conventional matrix multiplication. The measure $d\mu$ has to be determined. This can be done by observing that $[Q_G, S_{N=3}] = 0\,, [Q_G, d\mu_H] = 0$ where $d\mu_H$ is the invariant measure in the space of the zero modes of $x^\mu, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha} i}, u^I_i$ and $\xi^3_1, \xi^2_1, \xi^3_2$. In addition, $S_{N=3}$ has zero ghost number, while $d\mu_H$ has ghost number three. Since we know that $d\mu_H \in H^3(Q_H)$. This implies that $d\mu_H = d\xi^3_1 d\xi^2_1 d\xi^3_2 d\mu'$ where the measure $d\mu' = d\mu'(x^\mu, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha} i}, u^I_i)$ has to be fixed by the G-analyticity $[Q_G, d\mu'] = 0$.

In order to obtain the action from the string field theory action, we have to integrate over the ghost fields $\xi^3_1, \xi^2_1, \xi^3_2$. Since they are anticommuting, the integration is a Berezin integral. This means that we have several contributions: one contribution is coming by taking two ghost fields from the expansion of $\Phi^{(1)}_H$ as in (15) and one from the BRST charge in the first term $S_{N=3}$. There is a contribution from the fourth term in (13) acting on one of the two $\Phi^{(1)}_H$ and by extracting one ghost from the other $\Phi^{(1)}_H$. Finally, there is a contribution from the interaction term. After this operation the resulting action is the same as given in [1].

4 N=2 Harmonic Hypermultiplet from String Theory

Let us know consider the case on N=2 harmonic superspace. We recall that the equations of motion for the N=2 hypermultiplet in d=4 are

$$D_{\alpha(\mathcal{I}}\varphi_{\mathcal{J})} = 0\,, \qquad \bar{D}_{\dot{\alpha}(\mathcal{I}}\varphi_{\mathcal{J})} = 0\,. \tag{17}$$

These constraints reduce the number of independent components and the resulting superfield describes an on-shell hypermultiplet. To prove this, one has to act with the superderivatives on the equations (17) and contracting the N=2 indices. As shown in [1] this system is studied more easily using the harmonic superspace. Here we show that the action for the hypermultiplet has a simple interpretation from string theory. Therefore, we first define the BRST charge, then the vertex operator and finally the action.

The BRST implementing the Grassmann analitycity is now given by

$$Q_G = \lambda^\alpha u^{\mathcal{I}} D_{\alpha\mathcal{I}} + \bar{\lambda}^{\dot{\alpha}} u^{\mathcal{I}} \epsilon_{\mathcal{I}\mathcal{J}} \bar{D}^{\mathcal{J}}_{\dot{\alpha}} \tag{18}$$

and it is nilpotent because we have solved the pure spinor constraint in sec. 2. However, in order to reproduce the on-shell hypermultiplet, we have to impose a new constraint. We recall that on the $SU(2)/U(1)$ space we can define the following differential operators

$$D = u^{\mathcal{I}} \epsilon_{\mathcal{I}\mathcal{J}} \partial_{\bar{u}_J}\,, \quad \bar{D} = \bar{u}_{\mathcal{I}} \epsilon^{\mathcal{I}\mathcal{J}} \partial_{u^J}\,, \quad D_0 = u^{\mathcal{I}} \partial_{u^J} - \bar{u}_{\mathcal{I}} \partial_{\bar{u}_J} \tag{19}$$

which satisfy the Lie algebra $[D, \bar{D}] = D_0, [D, D_0] = D$ and $[\bar{D}, D_0] = -\bar{D}$. The second BRST operator is defined by picking only the positive root of the Lie algebra D. This operator commutes with Q_G and the new BRST operator is obtained by introducing a new anticommuting ghost field c and by constructing the nilpotent charge

$$Q_H = cD \,. \tag{20}$$

The vertex operator is now identified with ghost number zero superfield. It needs an harmonic charge +1 (see [1] for a complete analysis of the cohomology of Q_H). Since the cohomology of Q_G is empty, we can consider only vertex operator $\Phi^{(0)}_H$ which are invariant under Q_G. Since there is only one anticommuting ghost field c we can write the string field theory action as follows

$$S_{N=2} = \int d\mu \Big(\bar{\Phi}^{(0)}_H Q_H \Phi^{(0)}_H + V(\Phi^{(0)}_H, \Phi^{(1)}_H) \Big) \,, \tag{21}$$

where the measure can be decomposed as $d\mu = d\mu' dc$ where $d\mu'$ is BRST invariant and it coincides with the harmonic superspace measure. The integral over c is Berezin integral and by integrating over it, the action (21) reproduces the action for the hypermultiplet given [1]. The field $\Phi^{(1)}_H$ is the dual to $\Phi^{(0)}_H$ since there is only one ghost c, namely $\Phi^{(1)}_H = c\,\Phi^{(0)}_H$.

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Superparticles with Constrained Generalized Supersymmetries

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Talk delivered by F. Toppan

Abstract

This talk is based on the paper appeared in JHEP 0505 (2005) 060 [1], where a classification of the constrained complex generalized supersymmetries is presented. The generalized superparticle models (i.e., whose target superspaces are generalized supersymmetries) are formulated in arbitrary space-times. The consistency conditions for the constrained generalized complex superparticles are derived.

1 Introduction

The elusive nature of the M-theory forces us to understand the role of the bosonic tensorial central charges appearing in the M-algebra and going beyond the Haag-Łopuszański-Sohnius scheme [2]. This is particularly true if we want to understand the dynamics of the non-minkowskian twelve-dimensional F-theory [3], based on the F-algebra presentation of the M-algebra, see e.g. [4], admitting only higher-rank bosonic tensors and no translations at all.

From this point of view, in order to understand this generalized dynamical setting, it is quite convenient to investigate at first the simplest classes of models that can be based on generalized supersymmetries. The generalized superparticles models fit nicely into this framework. It is worth recalling that the first theory of this kind was introduced by Rudychev-Sezgin [5] as a generalization of the Brink-Schwarz superparticle [6], in terms of a generalized supersymmetric target with extra, tensorial, bosonic coordinates. The [5] model was based on real spinors. Later, Bandos-Lukierski [7] analyzed a corresponding model for complex spinors. They surprisingly proved, see also [8], that the dynamical content of the four-dimensional superparticle model with six extra rank-two bosonic coordinates, describes a tower of higher helicity massless particles, making the physical implications of these theories, orginally regarded as toy-models, particularly deep.

In this talk we discuss several aspects of this class of models. We point out that they can be derived in a unified framework, dimensionally reduced models being obtained from the associated oxidized (read, maximal) form of the generalized supersymmetries. Inequivalent models are specified in terms of the different admissible choices for the spinorial metric. Complex generalized supesymmetries can, finally, be consistently constrained. In various cases, these admissible algebraic constraints lead to admissible constraints on the Equations Of Motion of their associated complex generalized superparticles.

2 Constrained complex generalized supersymmetries

A complex generalized supersymmetry algebra is expressed in terms of complex spinors Q_a and their complex conjugate ${Q^*}_{\dot{a}}$. The most general (with a saturated r.h.s.) algebra is in this case given by

$$\{Q_a, Q_b\} = \mathcal{P}_{ab} \qquad , \qquad \{{Q^*}_{\dot{a}}, {Q^*}_{\dot{b}}\} = {\mathcal{P}^*}_{\dot{a}\dot{b}}, \tag{2.1}$$

together with

$$\{Q_a, {Q^*}_{\dot{b}}\} \quad = \quad \mathcal{R}_{a\dot{b}}, \tag{2.2}$$

where the matrix $\mathcal{P}_{ab}$ (${\mathcal{P}^*}_{\dot{a}\dot{b}}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{R}_{a\dot{b}}$ is hermitian.

The maximal number of allowed components in the r.h.s. is given, for complex fundamental spinors with n complex components, by $n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ complex matrix $\mathcal{P}_{ab}$ plus n^2 (real) bosonic components entering the hermitian $n \times n$ complex matrix $\mathcal{R}_{a\dot{b}}$.

The saturated r.h.s. are given by the most general combination of rank-k antisymmetric tensors which are either symmetric in the $a \leftrightarrow b$ exchange (they are constructed with the help of the charge conjugation matrix C) or hermitian (these tensors are constructed with the matrix A used to define barred spinors).

The following division-algebra compatible constraints can be imposed on both $\mathcal{P}$ and $\mathcal{R}$. We obtain the table, whose entries specify the total number of bosonic components (in the real counting), while the columns represent the restrictions on $\mathcal{R}$ and the rows the restrictions on $\mathcal{P}$ (an imaginary condition on $\mathcal{P}$ is equivalent to the reality condition and therefore is not reported)

$\mathcal{P}\backslash\mathcal{R}$	1) *Full*	2) *Real*	3) *Imag.*	4) *Abs.*
a) *Full*	$2n^2+n$	$\frac{3}{2}(n^2+n)$	$\frac{1}{2}(3n^2+n)$	n^2+n
b) *Real*	$\frac{1}{2}(3n^2+n)$	n^2+n	n^2	$\frac{1}{2}(n^2+n)$
c) *Abs.*	n^2	$\frac{1}{2}(n^2+n)$	$\frac{1}{2}(n^2-n)$	0

(2.3)

Some comments are in order. The above list of constraints is not necessarily implemented for any given supersymmetric dynamical system. One should check, e.g., that the above restrictions are indeed compatible with the equations of motion. On a purely algebraic basis, however, they are admissible restrictions which require a careful investigation.

One can notice that certain numbers appear twice as entries in the above table. This is related with the fact that the same constrained superalgebra can admit a different, but equivalent, presentation. We refer to these equivalent presentations as "dual formulations" of the constrained supersymmetries. It is worth stressing that in application to dynamical systems, which need more data than just superalgebraic data, one should explicitly verify whether the above related constraints indeed lead to equivalent theories.

The inequivalent constrained generalized supersymmetries can be listed as follows

$$
\begin{array}{|c|c|llc|}
\hline
I & (a1) & 2n^2+n, & k=3, & l=1\\
\hline
II & (a2) & \frac{3}{2}(n^2+n), & k=3, & l=0\\
\hline
III & (a3\,\&\,b1) & \frac{1}{2}(3n^2+n), & k=2, & l=1\\
\hline
IV & (a4\,\&\,b2) & n^2+n, & k=2, & l=0\\
\hline
V & (b3\,\&\,c1) & n^2, & k=1, & l=1\\
\hline
VI & (b4\,\&\,c2) & \frac{1}{2}(n^2+n), & k=1, & l=0\\
\hline
VII & (c3) & \frac{1}{2}(n^2-n), & k=0, & l=1\\
\hline
\end{array}
\tag{2.4}
$$

The integral numbers k, l have the following meaning. For the given constrained supersymmetry the bosonic r.h.s. can be presented in the following form

$$
Z \;=\; kX + lY, \quad k=0,1,2,3, \quad l=0,1, \tag{2.5}
$$

where X and Y denote the bosonic sectors associated with the VI and respectively VII constrained supersymmetry.

In association with the maximal Clifford algebras in D-dimensional spacetimes (with no dependence on their signature), the X and Y bosonic sectors are given by the following set of rank-k antisymmetric tensors

$$
\begin{array}{|c|c|c|}
\hline
 & X & Y\\
\hline
D=3 & M_1 & M_0\\
\hline
D=5 & M_2 & M_0+M_1\\
\hline
D=7 & M_0+M_3 & M_1+M_2\\
\hline
D=9 & M_0+M_1+M_4 & M_2+M_3\\
\hline
D=11 & M_1+M_2+M_5 & M_0+M_3+M_4\\
\hline
D=13 & M_2+M_3+M_6 & M_0+M_1+M_4+M_5\\
\hline
\end{array}
\tag{2.6}
$$

Formula (2.5) specifies the admissible class of division-algebra related, constrained bosonic sectors.

3 Superparticles with tensorial central charges

The most general action S involving real spinors is constructed in terms of the real superspace coordinates X^{ab}, Θ^a conjugated to the superalgebra generators $\mathcal{Z}_{ab}$ and Q_a [5] (X^{ab} is symmetric in the $a \leftrightarrow b$ exchange). We have

$$
S \;=\; \frac{1}{2}\int d\tau tr\left[\mathcal{Z}\cdot\Pi - e(\mathcal{Z})^2\right], \tag{3.7}
$$

where

$$
\Pi^{ab} \;=\; dX^{ab} - \Theta^{(a}d\Theta^{b)}, \tag{3.8}
$$

while e^{ab} denotes the Lagrange multipliers whose (anti)symmetry property is the same as the one of the charge conjugation matrix C^{ab}, i.e.

$$e^T = \varepsilon e \qquad for \qquad C^T = \varepsilon C. \tag{3.9}$$

By construction

$$(\mathcal{Z})^2{}_{ab} \;\; = \;\; \mathcal{Z}_{ac}C^{cd}\mathcal{Z}_{db}, \tag{3.10}$$

namely the charge conjugation matrix is used as a metric to raise and lower spinorial indices.

The massless constraint

$$(\mathcal{Z})^2_{ab} \;\; = \;\; 0 \tag{3.11}$$

is obtained from the variation δe^{ab} of the Lagrange multipliers.

A symmetric charge conjugation matrix ($\varepsilon = 1$) allows us [5] to construct a massive model by simply performing a shift $\mathcal{Z} \to \mathcal{Z} + mC$ in the action (3.7).

In order to introduce the action for the superparticle with complex spinors we mimick, as much as possible, the real formulation. The bosonic matrix $\mathcal{Z}_{ab}$ is now replaced by the pair of matrices $\mathcal{P}_{ab}$ and $\mathcal{R}_{a\dot{b}}$ (respectively symmetric and hermitian) entering (2.1) and (2.2). They can be accommodated in a symmetric matrix $\mathbf{P}$ ($\mathbf{P}^T = \mathbf{P}$) as follows

$$\mathbf{P} \;\; = \;\; \left(\begin{array}{cc} \mathcal{P} & \mathcal{R} \\ \mathcal{R}^* & \mathcal{P}^* \end{array} \right). \tag{3.12}$$

The supercoordinates conjugated to $\mathcal{P}_{ab}$, $\mathcal{R}_{a\dot{b}}$, Q_a and $Q^*{}_{\dot{a}}$ are given by X^{ab}, $Y^{a\dot{b}}$, Θ^a and $\Theta^{*\dot{a}}$.

It is convenient to use the notation

$$\mathbf{\Pi} \;\; = \;\; \left(\begin{array}{cc} dX - \Theta d\Theta & dY - \Theta d\Theta^* \\ dY^* - \Theta^* d\Theta & dX^* - \Theta^* d\Theta^* \end{array} \right). \tag{3.13}$$

We will also need the matrix

$$\mathbf{P}^2 \;\; = \;\; \mathbf{P}\mathcal{C}\mathbf{P}, \tag{3.14}$$

whose indices are raised by the metric $\mathcal{C}$. There are three possible choices for $\mathcal{C}$, given by

i)

$$\mathcal{C} \;\; = \;\; \left(\begin{array}{cc} C & 0 \\ 0 & C \end{array} \right), \tag{3.15}$$

in this case $\mathcal{C}$ is (anti)symmetric in accordance with the sign of ϵ;

ii)

$$\mathcal{C} \;\; = \;\; \left(\begin{array}{cc} 0 & A \\ \xi A^* & 0 \end{array} \right), \tag{3.16}$$

where ξ is an arbitrary sign ($\xi = \pm 1$); in this case the (anti)symmetry property of $\mathcal{C}$ is specified by the sign of $\delta\xi$;

iii)

$$\mathcal{C} = \begin{pmatrix} C & A \\ \epsilon\delta A^* & C^* \end{pmatrix}, \tag{3.17}$$

the (anti)symmetry property of $\mathcal{C}$ is specified by the sign of ϵ. It should be noticed that in this last case an (anti)symmetric matrix $\mathbf{P}^2$ ($\mathbf{P}^2 = \mathbf{P}\mathcal{C}\mathbf{P}$) is only possible, for both non-vanishing $\mathcal{P}$, $\mathcal{R}$ entering $\mathbf{P}$, if the condition

$$\epsilon = \delta \tag{3.18}$$

is matched.

The (anti)-symmetry property of $\mathbf{P}^2$ coincides with the (anti)-symmetry property of $\mathcal{C}$.

The Lagrange multipliers enter a matrix

$$\mathbf{E} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \tag{3.19}$$

In general, for any $\mathbf{U}$ (for our purposes $\mathbf{U} \equiv \mathbf{P}^2$) s.t.

$$\mathbf{U} = \begin{pmatrix} U & V \\ \lambda\mu V^* & U^* \end{pmatrix} \tag{3.20}$$

with $U^T = \lambda U$, $V^\dagger = \mu V$ (therefore $\mathbf{U}^T = \lambda\mathbf{U}$), the reality of the term $tr(\mathbf{EU})$ requires

$$\begin{aligned} g &= \lambda\mu f^*, \\ h &= e^*. \end{aligned} \tag{3.21}$$

A reality (imaginary) condition imposed on either $\mathbf{U}$ or $\mathbf{V}$ implies a reality (imaginary) condition for the lagrange multipliers e and f respectively.

We are now in the position to write the action S for the superparticle with bosonic tensorial central charges and complex spinors as

$$S = \frac{1}{2}\int d\tau tr\left[\mathbf{P\Pi} - \mathbf{E}(\mathbf{P})^2\right]. \tag{3.22}$$

As in the real case, a massive model can be introduced in correspondence of a symmetric $\mathcal{C}$ through the shift $\mathbf{P} \rightarrow \mathbf{P} + m\mathcal{C}$ in the action (3.22).

4 Constrained complex superparticles with tensorial central charges

In the previous Section we formulate the complex generalized superparticle model. It is clear at this point that we can investigate whether its equations of motion are compatible with the constraints on complex generalized supersymmetries discussed in Section **2**. This investigation should be performed for each one of the three available choices for the spinorial metric $\mathcal{C}$. As a necessary condition for the consistency of the theory, the number of lagrange multipliers constraints should not exceed the number of bosonic degrees of freedom entering $\mathcal{P}$ and $\mathcal{R}$.

The complete list of results, which we cannot report here for lack of space, has been furnished in [1]. Here we limit ourselves to mention that the constraints II and III of (2.4) are never compatible with the equations of motion of the (constrained) generalized complex superparticles. The remaining constraints, on the other hand, can be imposed for suitable values of the ϵ, δ, ξ signs entering the construction of the model, as discussed in Section **3**. For "generic" values of the space-time we obtain the following table which reports the set of consistent constraints for the allowed choices of the metric $\mathcal{C}$

	i	ii	iii
I	yes	yes	yes
$IV\,(a4)$	yes	yes	no
$IV\,(b2)$	yes	yes	$yes^*\,(\epsilon=1)$
$V\,(b3)$	yes	yes	$yes^*\,(\epsilon=1)$
$V\,(c1)$	yes	yes	no
$VI\,(b4)$	$yes^*\,(\epsilon=-1)$	yes	no
$VI\,(c2)$	$yes^*\,(\epsilon=-1)$	yes	no
VII	$yes^*\,(\epsilon=-1)$	yes	no

(4.23)

The "$*$" denotes which choices are consistent only for a specific value of ϵ.

The above result is the starting point for investigating the consequences of the constrained generalized supersymmetries in a dynamical setting. The importance of (one class of) constrained generalized supersymmetries was noticed in [9]. It was proven that they are required in order to perform the functional quantization of any model constructed with the minkowskian M-algebra.

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The b-Field in Pure Spinor Quantization of Superstrings

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Abstract

In the framework of the pure spinor approach of superstring theories, we describe the Y-formalism and use it to compute the picture raised b-field. At the end we discuss briefly the new, non-minimal formalism of Berkovits and the related non-minimal b-field.

The new superstring formulation of Berkovits [1]-[5], based on pure spinors, has solved the old problem of quantization of superstrings with manifest super-Poincaré invariance. It can be considered at present as a complete and consistent formulation of superstring theories, alternative to the NSR and GS ones that shares the advantages of these two formulations without suffering from their disadvantages.

To be specific, let us consider the heterotic string. The pure spinor approach is based on the BRST charge

$$Q = \oint dz(\lambda^\alpha d_\alpha), \tag{1}$$

and the action

$$I = \int d^2z(\frac{1}{2}\partial X^a \bar{\partial} X_a + p_\alpha \bar{\partial}\theta^\alpha - \omega_\alpha \bar{\partial}\lambda^\alpha) + S_{left}, \tag{2}$$

where the ghost λ^α is a pure spinor satisfying an equation

$$(\lambda\Gamma^a\lambda) = 0. \tag{3}$$

Moreover, $\Pi^a = \partial X^a + ...$ and $d_\alpha = p_\alpha + ...$ are the supersymmetrized momenta of the superspace coordinates $Z^M = (X^m, \theta^\mu)$ and ω_α is the momentum of λ^α. Due to the pure spinor constraint, the action I is invariant under the local ω-symmetry

$$\delta\omega_\alpha = \epsilon_a(\Gamma^a\lambda)_\alpha. \tag{4}$$

Finally, S_{left} is the action for the heterotic fermions. (For type II superstrings, S_{left} is the (free) action of the left-handed pairs $(\hat{p}_\alpha, \hat{\theta}^\alpha)$ and $(\hat{\omega}_\alpha, \hat{\lambda}^\alpha)$, and one must add to Q the left-handed BRST charge $\hat{Q} = \oint \hat{\lambda}^\alpha \hat{d}_\alpha$.)

Taking into account the pure spinor constraint, the action I describes a critical string with vanishing central charge and the BRST charge Q is nilpotent. Moreover it has been proved [2]-[3] that the cohomology of Q reproduces the correct physical spectrum. The recipe to compute tree amplitudes [4] and higher-loop amplitudes [5] was proposed and all the checks done untill now give support to the full consistency of this formulation.

The statement that the pure spinor approach provides a super-Poincaré covariant quantization of superstring theories is correct but deserves a warning. The non-standard pure spinor constraint, which is assumed to hold in a strong sense [1] and implies that only 11 of the 16 components of λ are independent, gives rise to the following problems:

i) The $\omega - \lambda$ OPE cannot be a standard free OPE since $\omega_\alpha(y)(\lambda\Gamma^a\lambda)(z) \neq 0$.

ii) The ω-symmetry requires to be gauge fixed but the gauge fixing cannot be done in a covariant way. The only gauge invariant fields involving ω are the ghost current J, the Lorentz current N^{ab} and the stress-energy tensor $T^{\omega\lambda}$ for the (ω, λ) system. At the classical level they are respectively $J = (\omega\lambda)$, $N^{ab} = \frac{1}{2}(\omega\Gamma^{ab}\lambda)$ and $T^{\omega\lambda} = (\omega\partial\lambda)$. Notice that all of them have ghost number zero.

iii) In the pure spinor approach, the antighost b (ghost number -1), needed to compute higher-loop amplitudes, is a compound field which cannot be written in a Lorentz invariant way. Indeed ω is the only field with negative ghost number but it can arise only in gauge invariant compound fields with zero (or positive) ghost number.

From i), ii) and iii) a violation of (target space) Lorentz symmetry, at intermediate steps, seems to be unavoidable. Indeed in [1],[4] the pure spinor constraint is resolved, thereby breaking SO(10) to U(5), and a U(5) formalism is used to compute the OPE's between gauge invariant quantities. Here we would like to describe a different but related approach, the so called Y-formalism, that proved to be useful to compute OPE's and to deal with the b-field [9],[11].

Let us define the <u>non-covariant</u> spinor

$$Y_\alpha = \frac{v_\alpha}{(v\lambda)},$$

where v_α is a constant pure spinor, so that

$$(Y\lambda) = 1.$$

(and $(Y\Gamma^a Y) = 0$). Then consider the projector

$$K_\alpha{}^\beta = \frac{1}{2}(\Gamma^a\lambda)_\alpha(Y\Gamma_a)^\beta. \tag{5}$$

[1]For different strategies, see Refs. [6], [7], [8].

that projects a 5-D subspace of the 16-D spinorial space (since $TrK = 5$). One has

$$(\lambda\Gamma^a\lambda) = 0 \Longleftrightarrow \lambda^\alpha K_\alpha{}^\beta = 0,$$

(so that λ has 11 independent components and 5 components of ω are pure gauge) and

$$(1-K)_\alpha{}^\beta(\Gamma^a\lambda)_\beta = 0.$$

Using this formalism, the correct $\omega - \lambda$ OPE is

$$\omega_\alpha(y)\lambda^\beta(z) = \frac{(1-K(z))_\alpha{}^\beta}{(y-z)}. \tag{6}$$

(Indeed with this equation, we obtain the OPE $\omega(y)(\lambda\Gamma^a\lambda)(z) = 0$.)

Using these rules (as well as free field OPE's for X^m and (p,θ)) one can compute all OPE's for composite fields and in particular for the covariant and gauge invariant fields involving ω (when they are suitably defined). Indeed, if Y_α enters into the game, ∂Y_α has the same ghost number and conformal weight as ω, and as a result in the definitions of J, N^{ab} and $T^{\omega\lambda}$ terms like $(\partial Y, \lambda)$, $(\partial Y\Gamma^{ab}\lambda)$, $\partial(Y\Gamma^{ab}\lambda)$ etc. can arise. The coefficients of these Y-dependent terms are fixed by requiring that the algebra of OPE 's closes, i.e., that these spurious terms do not arise in the r.h.s. of OPE 's. With the choice

$$N^{ab} = \frac{1}{2}[(\omega\Gamma^{ab}\lambda) + \frac{1}{2}(\partial Y\Gamma^{ab}\lambda) - 2\partial(Y\Gamma^{ab}\lambda)], \tag{7}$$

$$J = (\omega\lambda) - \frac{7}{2}(\partial Y\lambda), \tag{8}$$

$$T^{\omega\lambda} = (\omega\partial\lambda) + \frac{3}{2}\partial(Y\partial\lambda), \tag{9}$$

one recovers [11] the correct OPE's with the right levels (-3 for N, -4 for J) and ghost anomaly 8, as first given by Berkovits in the U(5)-formalism. Notice that all the Y-dependent terms in N^{ab}, J and $T^{\omega\lambda}$ are BRST exact. In conclusion, J, N^{ab} and $T^{\omega\lambda}$, defined in eqs.(7)-(9) are primary and Lorentz covariant fields, and their OPE's are the right ones with correct central charges, levels and ghost-number anomaly.

Now let us come back to the b-field. b is a field with ghost number -1 and weight 2 which is essential to compute higher-loop amplitudes. It satisfies the important condition

$$\{Q, b\} = T, \tag{10}$$

where T is the stress-energy tensor. In the pure spinor approach the recipe to compute higher loops [5] is based on three ingredients:

i) A Lorentz invariant measure factor for pure spinor ghosts.

ii) BRST closed, picture changing operators (PCO) to absorb the zero modes of the bosonic ghosts, that is, Y_C for the 11 zero modes of λ and Z_B, Z_J for the $11g$ zero modes of ω at genus g.

iii) $3g-3$ insertions of the b-field folded into Beltrami parameters $\mu(z,\bar{z})$, i.e., $b[\mu] = \int d^2 z b(z)\mu(z)$ at genus $g>1$ (1 at genus 1 and 0 at tree level).

At a schematic level, the recipe for computing N-point amplitudes, at genus g ($g \geq 2$)(for tipe II closed superstrings), is

$$\mathcal{A} = \int d^{3g-3}\tau < |\prod_{i=1}^{3g-3} b[\mu_i] \prod_{j=1}^{10g} Z_{B_j}(z_j) \prod_{h=1}^{g} Z_J(z_h) \prod_{r=1}^{11} Y_{C_r}(z_r)|^2 \prod_{s=1}^{N} \int d^2 z_s U(z_s) >,$$

where τ are Teichmuller parameters, $\int U$ are integrated vertex operators and $< \quad >$ denotes the path integral measure (that we shall not discuss here). For $g=1$, one integrated vertex is replaced by one unintegrated vertex V and there is only one b-insertion. At $g=0$, three integrated vertices are replaced by unintegrated ones.

In standard string theories, b is the antighost of diffeomorphism. In pure spinor approach, in the absence of diff. ghosts, b is a compound field, which, as already noted, cannot be written as a Lorentz scalar. Using the Y-formalism, an expression for b that satisfies the fundamental condition (10), is [4]

$$b = \frac{1}{2}(Y\Pi^a\Gamma_a d) + (\tilde{\omega}\partial\theta) \equiv Y_\alpha G^\alpha, \tag{11}$$

where $\tilde{\omega}$ is the non-covariant but gauge-invariant ghost

$$\tilde{\omega}_\alpha = (1-K)_\alpha{}^\beta \omega_\beta, \tag{12}$$

and

$$G^\alpha = \frac{1}{2} : \Pi^a(\Gamma_a d)^\alpha : -\frac{1}{4}N_{ab}(\Gamma^{ab}\partial\theta)^\alpha - \frac{1}{4}J\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha, \tag{13}$$

the last term in the r.h.s. of (13) coming from normal ordering. Whereas G^α is Lorentz covariant, b, due to its dependence on Y_α, is not Lorentz invariant. However, it turns out that the Lorentz variation of b is BRST exact. In an attempt to understand the origin of the pure spinor approach [9] the b-field (11) has been interpreted as the twisted current of the second w.s. susy charge of an N=2 superembedding approach, the first twisted charge being the BRST charge of the pure spinor approach. Even if this analysis was done only at a classical level (and only for the heterotic string), it is suggestive of an N=2 topological origin of the pure spinor approach. The singularity of b at $(v\lambda)=0$ due to its dependence on Y_α is problematic in presence of the picture changing operators $Y_C = C_\alpha\theta^\alpha\delta(C_\beta\lambda^\beta)$ that cancel the zero modes of λ, C_α being a constant spinor. Therefore this b-field does not seem suitable to compute higher loops.

Since covariant and ω-invariant fields with ghost number -1, needed to get a b-field, do not exist, the idea of Berkovits [5] was to combine T with a picture raising operator Z_B with ghost number $+1$ and use as insertion, a picture raised, compound field b_B such that

$$\{Q, b_B\} = TZ_B. \tag{14}$$

Then, this b_B makes it possible to define a bilocal field $\tilde{b}_B(y,z)$ [5] such that

$$\{Q, \tilde{b}_B(y,z)\} = T(y)Z_B(z). \tag{15}$$

Then $3g-3$ $b[\mu]$ insertions (1 at $g=1$), together with $3g-3$ picture-raising operators Z_B (1 at $g=1$), are replaced by $3g-3$ (1 at $g=1$) insertions of the newly-introduced $\tilde{b}_B(y,z)$ folded into Beltrami parameters.

To explain this recipe we need more details about the picture raising operators $Z = (Z_B, Z_J)$ that absorb the zero modes of ω included in N^{ab} and J :

$$Z_B = \frac{1}{2}(\lambda\Gamma^{ab}d)B_{ab}\delta(N^{cd}B_{cd}),$$

$$Z_J = (\lambda^\alpha d_\alpha)\delta(J),$$

where B_{ab} is an antisymmetric constant tensor. Then in general

$$Z = \lambda^\alpha Z_\alpha,$$

and

$$\{Q, Z\} = 0.$$

It follows (by explicit computation or from general arguments plus pure spinor constraint) that:

$$\{Q, Z_\alpha\} = \lambda^\beta Z_{\beta\alpha},$$

$$\{Q, Z_{\beta\alpha}\} = \lambda^\gamma Z_{\gamma\beta\alpha},$$

$$\{Q, Z_{\gamma\beta\alpha}\} = \lambda^\delta Z_{\delta\gamma\beta\alpha} + \partial\lambda^\delta \Upsilon_{\delta\gamma\beta\alpha},$$

where $Z_{\beta\alpha}, Z_{\gamma\beta\alpha}, Z_{\delta\gamma\beta\alpha}$ and $\Upsilon_{\delta\gamma\beta\alpha}$ are Γ_5-traceless, i.e., they vanish when saturated with $(\Gamma_{a_1...a_5})^{\alpha_i\alpha_{i+1}}$ between two adjacent indices. Their expressions can be found in [5] or [11]. Moreover ∂Z_B and ∂Z_J are BRST exact.

As shown by Berkovits [5], starting from G^α there exist fields $H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\delta}$ (and $S^{\alpha\beta\gamma}$) defined modulo Γ_1-traceless terms (that is modulo fields $h_i^{\alpha_1..(\alpha_i,\alpha_{i+1})..\alpha_n}$ which vanish if saturated with $\Gamma^a_{\alpha_i\alpha_{i+1}}$), such that

$$\{Q, G^\alpha\} = \lambda^\alpha T, \tag{16}$$

$$\{Q, H^{\alpha\beta}\} = \lambda^\alpha G^\beta + ..., \tag{17}$$

$$\{Q, K^{\alpha\beta\gamma}\} = \lambda^\alpha H^{\beta\gamma} + ..., \tag{18}$$

$$\{Q, L^{\alpha\beta\gamma\delta}\} = \lambda^\alpha K^{\beta\gamma\delta} + ..., \tag{19}$$

where the dots denote Γ_1-traceless terms. Moreover, since we have $\lambda^\alpha L^{\beta\gamma\delta\epsilon} = 0 + ...$, an equation

$$L^{\alpha\beta\gamma\delta} = \lambda^\alpha S^{\beta\gamma\delta} + ...,$$

is obtained. Then the picture raised b-field that satisfies eq.(14) is

$$b_B = b_1 + b_2 + b_3 + b_4^{(a)} + b_4^{(b)}, \tag{20}$$

where

$$b_1 = G^\beta Z_\beta, \qquad b_2 = H^{\beta\gamma} Z_{\beta\gamma}, \qquad b_3 = -K^{\alpha\beta\gamma} Z_{\alpha\beta\gamma}.$$

$$b_4^{(a)} = -L^{\alpha\beta\gamma\delta} Z_{\alpha\beta\gamma\delta}, \qquad b_4^{(b)} = -S^{\alpha\beta\gamma}\partial\lambda^\delta \Upsilon_{\delta\alpha\beta\gamma}.$$

The expression of b_B is quite complicated and Berkovits in [5] presented only the expressions of G^α and $H^{\alpha\beta}$. The technical device of using the non-covariant Y_α as an intermediate step helps us to obtain the full expression of b_B with a reasonable effort [10], [11]. In order to compute $H^{\alpha\beta}, K^{\alpha\beta\gamma}, S^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$ one makes the ansatz such that these fields can be constructed using only the building blocks

$$\lambda^\alpha, \quad (\Gamma_{ab}\lambda)^\alpha, \quad (\Gamma_a\tilde{\omega})^\alpha, \quad (\Gamma_a d)^\alpha,$$

(as well as Π^a in $H^{\alpha\beta}$); then one writes their most general expressions in terms of these blocks and imposes the condition that in the superfields H and K any dependence on Y_α(which is implicit in $\tilde{\omega}$) should be absent; then one requires that these superfields satisfy the recursive equations (17) - (19). Consequently, we have found

$$H^{\alpha\beta} = -\frac{1}{16}(\Gamma^a d)^\alpha(\Gamma_a d)^\beta - \frac{1}{2}\lambda^\alpha\Pi^a(\Gamma_a\tilde{\omega})^\beta + \frac{1}{16}[\Pi^a(\Gamma_b\Gamma_a\lambda)^\alpha(\Gamma^b\tilde{\omega})^\beta - (\alpha \leftrightarrow \beta)] + ..., \quad (21)$$

$$K^{\alpha\beta\gamma} = \frac{1}{16}\lambda^\alpha(\Gamma^a\tilde{\omega})^\beta(\Gamma_a d)^\gamma + \frac{1}{32}[(\tilde{\omega}\Gamma^a)^\alpha\lambda^\beta(\Gamma_a d)^\gamma + (\alpha \leftrightarrow \gamma)]$$
$$+\frac{1}{96}[(\tilde{\omega}\Gamma^a)^\alpha(\Gamma_{ab}\lambda)^\beta(\Gamma_a d)^\gamma - (\alpha \leftrightarrow \gamma)] + ..., \quad (22)$$

$$S^{\alpha\beta\gamma} = -\frac{1}{32}(\tilde{\omega})^\alpha\lambda^\beta(\Gamma_a\tilde{\omega})^\gamma - \frac{1}{96}(\tilde{\omega}\Gamma^a)^\alpha(\Gamma_{ab}\lambda)^\beta(\Gamma^b\tilde{\omega})^\gamma + ..., \quad (23)$$

and

$$L^{\alpha\beta\gamma\delta} = \lambda^\alpha S^{\beta\gamma\delta} + ..., \quad (24)$$

where again the dots denote Γ_1-traceless terms.

All these expressions are invariant under ω-symmetry (since $\tilde{\omega}$ is invariant). Moreover H and K are Lorentz covariant (being independent of Y_α) and therefore they depend on ω only through J and N^{ab}. Indeed, modulo Γ_1-traceless terms, the previous expressions of H and K can be rewritten as

$$H^{\alpha\beta} = \frac{1}{16}(\Gamma_a)^{\alpha\beta}(N^{ab}\Pi_b - \frac{1}{2}J\Pi_a) + \frac{1}{384}(\Gamma_{abc})^{\alpha\beta}[(d\Gamma^{abc}d) + 24N^{ab}\Pi^c] + \frac{1}{8}(\Gamma_a)^{\alpha\beta}\partial\Pi^a, \quad (25)$$

which coincides with the result of Berkovits and

$$K^{\alpha\beta\gamma} = -\frac{1}{48}(\Gamma_a)^{\alpha\beta}(\Gamma_b d)^\gamma N^{ab} - \frac{1}{192}(\Gamma_{abc})^{\alpha\beta}(\Gamma^a d)^\gamma N^{bc}$$
$$+\frac{1}{192}(\Gamma_a)^{\gamma\beta}\left[(\Gamma_b d)^\alpha N^{ab} + \frac{3}{2}(\Gamma^a d)^\alpha J\right] + \frac{1}{192}(\Gamma_{abc})^{\gamma\beta}(\Gamma^a d)^\alpha N^{bc} - \frac{1}{32}\Gamma_a^{\beta\gamma}(\Gamma^a\partial d)^\alpha. \quad (26)$$

Again the last terms in the r.h.s. of eqs.(25) and (26) come from normal ordering.

$L^{\alpha\beta\gamma\delta}$ and $S^{\alpha\beta\gamma}$ have a residual dependence on Y. However, when $S^{\beta\gamma\delta}$ is saturated with $\partial\lambda^\epsilon\Upsilon_{\epsilon\beta\gamma\delta}$ to get $b_4^{(b)}$, this dependence on Y drops out so that

$$b_4^{(b)} = -B_{ab}\delta(B_{cd}N^{cd})[T^{\omega\lambda}N^{ab} + \frac{1}{4}J\partial N^{ab} - \frac{1}{4}N^{ab}\partial J - \frac{1}{2}N^a_c\partial N^{bc}]. \quad (27)$$

Furthermore, it turns out that all the Y-dependent terms of $L^{\alpha\beta\gamma\delta}$ (linear and quadratic in Y) are Γ_1-traceless and therefore vanish when saturated with $Z_{\alpha\beta\gamma\delta}$ so that also $b_4^{(a)}$

does not depend on Y_α.

It is interesting to notice the relation between the non-covariant b-field b, given in (11) and the picture raised b-field b_B. Since

$$\{Q, bZ\} = TZ = \{Q, b_B\},$$

the quantity $bZ - b_B$ is closed. In [10], it has been shown that this quantity is also BRST exact:

$$b_B(z) = b(z)Z(z) + \{Q, X(z)\}, \tag{28}$$

so that b_B and bZ are cohomologically equivalent. Then, we also have

$$\tilde{b}_B(y,z) = b(y)Z(z) + \{Q, \tilde{X}(y,z)\}.$$

This result is interesting since it can be used to show that the insertion of $b_B[\mu](z) \equiv \int \mu(y)\tilde{b}_B(y,z)$ does not depend on the point z of the insertion. Indeed, since $\partial Z(z)$ is BRST exact, let say, $\partial Z(z) = \{Q, R(z)\}$ and $\{Q, b(y)\} = T(y)$ one has

$$\partial b[\mu](z) = \int \mu(y)T(y)R(z) + \{Q, \cdot\}, \tag{29}$$

and, modulo an exact term, the r.h.s. is the total derivative w.r.t. a Teichmuller parameter τ and vanishes after integration over τ.

Let us conclude this report by describing briefly a very interesting, new proposal of Berkovits [12], the non-minimal pure spinor formalism, that in addition leads to the construction of a covariant b-field. The main idea behind this work was to add to the fields involved in the pure spinor formalism a BRST quartet of fields $\bar{\lambda}_\alpha, \bar{\omega}^\alpha, r_\alpha, s^\alpha$ such that their BRST variations are $\delta\bar{\lambda}_\alpha = r_\alpha$, $\delta s^\alpha = \bar{\omega}^\alpha$, $\delta\bar{\omega}^\alpha = 0$, $\delta r_\alpha = 0$. $\bar{\lambda}_\alpha$ is a bosonic pure spinor with ghost number -1, r_α is a fermionic field that satisfies the constraint $(\bar{\lambda}\Gamma^a r) = 0$ and $\bar{\omega}^\alpha$ and s^α are the conjugate momenta of $\bar{\lambda}_\alpha$ and r_α, respectively. The action is obtained by adding to the action I in eq.(2), $\tilde{I}$ given by the BRST variation of the "Gauge fermion" $F = -\int(s\bar{\partial}\bar{\lambda})$ so that

$$I_{nm} = I + \tilde{I} = \int d^2z(\frac{1}{2}\partial X^a\bar{\partial}X_a + p_\alpha\bar{\partial}\theta^\alpha - \omega_\alpha\bar{\partial}\lambda^\alpha + s^\alpha\bar{\partial}r_\alpha - \bar{\omega}^\alpha\bar{\partial}\bar{\lambda}_\alpha) + S_{left}. \tag{30}$$

This action is invariant under gauge symmetries involving $\bar{\omega}$ and s, similar to the ω-symmetry so that, due to the constraints and these symmetries, each of the fields of the quartet has 11 components. The new BRST charge is

$$Q_{nm} = \int dz(\lambda^\alpha d_\alpha + \bar{\omega}^\alpha r_\alpha), \tag{31}$$

and the new (non-covariant) b-field corresponding to eq.(11) is

$$\tilde{b} = Y_\alpha G^\alpha + s^\alpha\partial\bar{\lambda}_\alpha. \tag{32}$$

Of course the quartet does not contribute to the central charge and has trivial cohomology w.r.t. the (new) BRST charge. Now let us define

$$b_{nm} = \tilde{b} + [Q_{nm}, W], \tag{33}$$

where

$$W = Y_\alpha \frac{\bar\lambda_\beta}{(\bar\lambda\lambda)} H^{[\alpha\beta]} + Y_\alpha \frac{\bar\lambda_\beta}{(\bar\lambda\lambda)^2} r_\gamma K^{[\gamma\beta\alpha]} - Y_\alpha \frac{\bar\lambda_\beta}{(\bar\lambda\lambda)^3} r_\gamma r_\delta L^{[\delta\gamma\beta\alpha]}, \tag{34}$$

and $H^{[\alpha\beta]}, K^{[\alpha\beta\gamma]}, L^{[\alpha\beta\gamma\delta]}$ are the fields defined in eqs.(21)-(26), antisymmetrized, e.g., $H^{[\alpha\beta]} = H^{\alpha\beta} - H^{\beta\alpha}$ etc. Then

$$b_{nm} = s^\alpha \partial \bar\lambda_\alpha + \frac{\bar\lambda_\alpha G^\alpha}{(\bar\lambda\lambda)} + \frac{\bar\lambda_\alpha r_\beta H^{[\alpha\beta]}}{(\bar\lambda\lambda)^2} - \frac{\bar\lambda_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\bar\lambda\lambda)^3} - \frac{\bar\lambda_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\bar\lambda\lambda)^4}, \tag{35}$$

which is the new non-minimal, covariant b-field defined in eq.(3.11) of [12].

As shown in [12], this non-minimal formalism is nothing but a critical topological string, so topological methods can be applied to compute multiloop amplitudes where a suitable regularization factor replaces the picture-changing operators to deal with zero modes. The regulator proposed in [12] allows us to compute loop amplitudes up to $g = 2$.

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2 NONCOMMUTATIVE THEORIES AND QUANTUM GROUPS

Construction of the Deformed Instantons in $\mathcal{N} = 1/2$ Super Yang-Mills Theory

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Abstract

Extending the exterior algebra on superspace to non(anti)commutative superspace, we formulate a non(anti)commutative version of the super ADHM construction which gives deformed instantons in $\mathcal{N} = 1/2$ super Yang-Mills theory with U(n) gauge group.

1 Introduction

It has been found that supersymmetric gauge theory defined on a kind of deformed superspace, called non(anti)commutative superspace, arises in superstring theory as a low energy effective theory on D-branes with constant graviphoton field strength [1]-[3]. In non(anti)commutative space, anticommutators of Grassmann coordinates become nonvanishing. Such a deformation of (Euclidean) four dimensional $\mathcal{N} = 1$ super Yang-Mills (SYM) theory has been formulated by Seiberg [2], which is sometimes called $\mathcal{N} = 1/2$ SYM theory.

It was argued by Imaanpur [4] that that the anti-self-dual (ASD) instanton equations should be modified in the $\mathcal{N} = 1/2$ SYM theory with self-dual (SD) non(anti)-commutativity. Solutions to those equations (deformed ASD instantons) have been studied by many authors [4]-[6]. It is well known that in the ordinary theory the instanton configurations of the gauge field can be obtained by the ADHM construction [7]. The authors of ref. [6] have studied string amplitudes in the presence of D(-1)-D3 branes with the background R-R field strength and derived constraint equations for the string modes ending on D(-1)-branes, which are the ADHM constraints for the deformed ASD instantons. We show that we can obtain these constraints in the purely field theoretic context, formulating a non(anti)commutative version of a superfield extension of the ADHM construction initiated by Semikhatov and Volovich [8]. We follow the notation and conventions in refs. [9, 10].

2 $\mathcal{N} = 1/2$ SYM theory

We will briefly describe the non(anti)commutative deformation of $\mathcal{N} = 1$ superspace and $\mathcal{N} = 1/2$ SYM theory formulated in [2].

The non(anti)commutative deformation of $\mathcal{N} = 1$ superspace is given by introducing non(anti)commutativity of the product of $\mathcal{N} = 1$ superfields. This deformation is realized by the following star product:

$$f * g = f \exp(P) g, \quad P = -\frac{1}{2} \overleftarrow{Q_\alpha} C^{\alpha\beta} \overrightarrow{Q_\beta}, \tag{2.1}$$

where f and g are $\mathcal{N} = 1$ superfields and Q_α is the (chiral) supersymmetry generator. $C^{\alpha\beta}$ is the non-anticommutativity parameter and is symmetric: $C^{\alpha\beta} = C^{\beta\alpha}$. The above star product gives the following relations among the chiral coordinates $(y^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$: $\{\theta^\alpha, \theta^\beta\}_* = C^{\alpha\beta}$, $[y^\mu, \cdot]_* = 0$, $[\bar{\theta}^{\dot{\alpha}}, \cdot\}_* = 0$. Turning on such a deformation, the original action formulated in the $\mathcal{N} = 1$ superfield formalism is deformed by the star product. The deformed $\mathcal{N} = 1$ SYM theory has $\mathcal{N} = 1/2$ supersymmetry, so that they are called $\mathcal{N} = 1/2$ SYM theory.

The action of $\mathcal{N} = 1/2$ SYM theory is given by

$$S = \frac{1}{16Ng^2} \int d^4x \left(\int d^2\theta \mathrm{tr} W^\alpha * W_\alpha + \int d^2\bar{\theta} \mathrm{tr} \bar{W}_{\dot{\alpha}} * \bar{W}^{\dot{\alpha}} \right) \tag{2.2}$$

where

$$W_\alpha = -\frac{1}{4} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \left(e_*^{-V} * D_\alpha e_*^V \right), \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4} D^\alpha D_\alpha \left(e_*^V * \bar{D}_{\dot{\alpha}} e_*^{-V} \right), \tag{2.3}$$

and $e_*^V \equiv \sum_n \frac{1}{n!} \overbrace{V * \cdots * V}^{n}$. Here $V = V^a T^a$ with V^a the vector superfields and T^a the hermitian generators which are normalized as $\mathrm{tr}[T^a T^b] = N\delta^{ab}$. We may redefine the component fields of V in the WZ gauge such that the component gauge transformation becomes canonical (the same as the undeformed case). In [2], such a field redefinition is found and then the component action becomes

$$S = \frac{1}{4Ng^2} \mathrm{tr} \int d^4x \left[-\frac{1}{4} v^{\mu\nu} v_{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu \mathcal{D}_\mu \lambda + \frac{1}{2} D^2 - \frac{i}{2} C^{\mu\nu} v_{\mu\nu} \bar{\lambda}\bar{\lambda} + \frac{1}{8} |C|^2 (\bar{\lambda}\bar{\lambda})^2 \right], \tag{2.4}$$

where $C^{\mu\nu} \equiv C^{\alpha\beta} (\sigma^{\mu\nu})_\alpha{}^\gamma \varepsilon_{\beta\gamma}$ and $|C|^2 \equiv C^{\mu\nu} C_{\mu\nu}$.

From the component action, we can see that the equations for SD instantons are unchanged compared to the undeformed case. Therefore, the SD instanton solutions are not affected by the deformation. On the other hand, the equations for ASD instantons should be modified. The action can be rewritten as [4]

$$S = \frac{1}{4Ng^2} \mathrm{tr} \int d^4x \left[-\frac{1}{2} \left(v_{\mu\nu}^{\mathrm{SD}} + \frac{i}{2} C_{\mu\nu} \bar{\lambda}\bar{\lambda} \right)^2 - i\bar{\lambda}\bar{\sigma}^\mu \mathcal{D}_\mu \lambda + \frac{1}{2} D^2 + \frac{1}{4} v^{\mu\nu} \tilde{v}_{\mu\nu} \right], \tag{2.5}$$

where $\tilde{v}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}v_{\rho\sigma}$. From this expression, we can see that configurations which satisfies the equations of motion and is connected to the ASD instantons when turning off the deformation are the solutions to the following deformed ASD instanton equations [4]:

$$v^{\mathrm{SD}}_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar{\lambda}\bar{\lambda} = 0, \quad \lambda = 0, \quad \mathcal{D}_\mu\sigma^\mu\bar{\lambda} = 0, \quad D = 0. \tag{2.6}$$

3 Differential forms in the deformed superspace

We will take a geometrical approach to formulate the deformed super ADHM construction by generalizing the exterior algebra: we extend the star product between superfields to the one including differential forms in superspace. The principle of our construction of the deformed exterior algebra is that the operators Q_α appearing in the star product are identified with the generators of supertranslation. Thus, the star product of differential forms is defined according to the representations of supersymmetry they belong to. Since the one-form bases e^A are supertranslation invariant, the action of Q_α on e^A is naturally defined as $Q_\alpha(e^A) = 0$. Then for a 1-form $\omega = e^A\omega_A$, it holds that $Q_\alpha(\omega) = (-)^{|A|}e^A Q_\alpha(\omega_A)$. Using this action of Q_α, we define the deformed wedge product of a p-form ω_p and a q-form ω_q as

$$\omega_p \overset{*}{\wedge} \omega_q \equiv \omega_p \wedge \exp\left(-\frac{1}{2}\overleftarrow{Q_\alpha}C^{\alpha\beta}\overrightarrow{Q_\beta}\right)\omega_q, \tag{3.1}$$

where $\overleftarrow{Q}$ ($\overrightarrow{Q}$) acts on ω_p (ω_q) from the right (left) and the normal wedge product is taken for the resulting (transformed) differential forms. Note that $\omega\overleftarrow{Q_\alpha} = (-)^{|\omega|}Q_\alpha(\omega)$. Hereafter we will suppress the wedge symbols. In the e^A-basis, the product of the p- and q-form is simply given by the star product of the coefficients:

$$\omega_p * \omega_q = (-)^{(|A_1|+\cdots+|A_q|)(|B_1|+\cdots+|B_q|)}e^{A_1}\cdots e^{A_p}e^{B_1}\cdots e^{B_q}(\omega_{pA_p\dots A_1} * \omega_{qB_q\dots B_1}), \tag{3.2}$$

The exterior derivative d is defined as a map from a p-form to a $p+1$-form by using the basis e^A:

$$d\omega_p = e^{A_1}\cdots e^{A_p}e^B D_B\omega_{pA_p\dots A_1} + \sum_{r-1}^{p}(-1)^{|A_{r+1}|+\cdots+|A_p|}e^{A_1}\cdots de^{A_r}\cdots e^{A_p}\omega_{pA_p\dots A_1} \tag{3.3}$$

where de^A is the same as the undeformed one.

We see that the deformed exterior algebra defined above is consistent with the $\mathcal{N} = 1/2$ SYM theory described in the previous section, in the sense that the curvature 2-from superfield will correctly reproduce the field strength superfield W_α and $\bar{W}_{\dot\alpha}$ in (2.3) (after imposing appropriate constraints as in the undeformed case [11]) based on the deformed exterior algebra. Given a connection 1-form superfield ϕ, the curvature superfields F_{AB} are obtained as the coefficient functions of the two-form superfield F constructed in a standard way: $F = d\phi + \phi * \phi$. Therefore, we find the curvature superfields F_{AB} as

$$F_{AB} = D_A\phi_B - (-)^{|A||B|}D_B\phi_A - [\phi_A, \phi_B\}_* + T_{AB}{}^C\phi_C, \tag{3.4}$$

where $T_{AB}{}^C$ is the torsion defined by $de^C = \frac{1}{2}e^A e^B T_{BA}{}^C$ whose non-vanishing elements are $T_{\alpha\dot\beta}{}^\mu = T_{\dot\beta\alpha}{}^\mu = 2i\sigma^\mu_{\alpha\dot\beta}$. The proper constraints for the curvature superfields to give the $\mathcal{N} = 1/2$ SYM theory turn out to be

$$F_{\alpha\beta} = 0, \quad F_{\dot\alpha\dot\beta} = 0, \quad F_{\alpha\dot\beta} = 0, \tag{3.5}$$

where the curvature superfields are given by (3.4) (see [11] for the undeformed case). We refer these constraints as the Yang-Mills constraints. These constraints are solved in a parallel way to the undeformed case and the invariant action with respect to super- and gauge symmetry can be constructed which coincides with the action S given in (2.2). Therefore, imposing the Yang-Mills constraints (3.5), the $\mathcal{N} = 1/2$ SYM theory can be correctly reproduced in a geometrical way based on the deformed exterior algebra.

4 Review of the $\mathcal{N} = 1$ super ADHM construction

Before describing the deformed version, we briefly review the $\mathcal{N} = 1$ super ADHM construction which was initiated by Semikhatov and Volovich [8]. Here we follow ref. [9].

The U(n) (or SU(n)) k instanton configurations can be given by the ADHM construction [7]. Define $\Delta_\alpha(x)$ such as

$$\Delta_\alpha(x) = a_\alpha + x_{\alpha\dot{\alpha}} b^{\dot{\alpha}} \tag{4.6}$$

where a_α and $b^{\dot{\alpha}}$ are constant $k \times (n+2k)$ matrices and $x_{\alpha\dot{\alpha}} \equiv i x_\mu \sigma^\mu_{\alpha\dot{\alpha}}$. We assume that Δ_α has maximal rank everywhere except for a finite set of points. Its hermitian conjugate $\Delta^{\dagger\alpha} \equiv (\Delta_\alpha)^\dagger$ is given by $\Delta^{\dagger\alpha}(x) = a^{\dagger\alpha} + b^\dagger_{\dot{\beta}} x^{\dot{\beta}\alpha}$. Then the gauge field v_μ is given by $v_\mu = -2iv^\dagger \partial_\mu v$, where v is the set of the normalized zero modes of Δ_α: $\Delta_\alpha v = 0$, $v^\dagger v = \mathbf{1}_n$. For later use we define f as the inverse matrix of the quantity $f^{-1} \equiv \frac{1}{2}\Delta_\alpha \Delta^{\dagger\alpha}$.

In the superfield formalism, the ASD super instanton equations are equivalent to the following super ASD condition [8]:

$$F_{\mu\dot{\alpha}} = 0, \quad \star F_{\mu\nu} = -F_{\mu\nu}. \tag{4.7}$$

Note that the latter equation follows from the former as long as the two-form F satisfies the Bianchi identities and the (undeformed) Yang-Mills constraints. The super ADHM construction gives the solutions to (4.7) [9]. Define a superfield extension of $\Delta_\alpha(x)$:

$$\hat{\Delta}_\alpha = \Delta_\alpha(y) + \theta_\alpha \mathcal{M}, \tag{4.8}$$

where $\Delta_\alpha(y)$ is the zero dimensional Dirac operator in the ordinary ADHM construction with replacing x^μ by the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\mathcal{M}$ is a $k \times (n+2k)$ fermionic matrix which includes the fermionic moduli. We suppose that $\hat{\Delta}_\alpha$ has a maximal rank almost everywhere as in the ordinary ADHM construction. Its ‡-conjugate [9] $\hat{\Delta}^{\ddagger\alpha}$ is found to be $\hat{\Delta}^{\ddagger\alpha} = \Delta^{\dagger\alpha}(y) + \theta^\alpha \mathcal{M}^\dagger$. As $\hat{\Delta}_\alpha$ has n zero modes we collect them in a matrix superfield $\hat{v}_{[n+2k]\times[n]}$ and require that $\hat{v}$ satisfies the normalization condition: $\hat{\Delta}_\alpha \hat{v} = 0$, $\hat{v}^\ddagger \hat{v} = 1$. (Its ‡-conjugate $\hat{v}^\ddagger$ satisfies $\hat{v}^\ddagger \hat{\Delta}^{\ddagger\alpha} = 0$.) Then the connection one-form superfield ϕ is given by

$$\phi = -\hat{v}^\ddagger d\hat{v}. \tag{4.9}$$

where d is exterior derivative of superspace. The connection ϕ defines the curvature

$$F = d\phi + \phi\phi = \hat{v}^\ddagger d\hat{\Delta}^{\ddagger\alpha} \hat{K}_\alpha{}^\beta d\hat{\Delta}_\beta \hat{v}, \tag{4.10}$$

where $\hat{K}^{-1}{}_\alpha{}^\beta \equiv \hat{\Delta}_\alpha \hat{\Delta}^{\ddagger\beta}$ and $\hat{K}_\alpha{}^\beta$ is defined such that $\hat{K}^{-1}{}_\alpha{}^\beta \hat{K}_\beta{}^\gamma = \hat{K}_\alpha{}^\beta \hat{K}^{-1}{}_\beta{}^\gamma = \delta^\gamma_\alpha \mathbf{1}_k$. The curvature superfield $F_{\mu\nu}$ becomes ASD if $\hat{K}$ satisfies $\hat{\Delta}_\alpha \hat{\Delta}^{\ddagger\beta} \propto \delta^\beta_\alpha$ and thus

$$\hat{K}^{-1}{}_\alpha{}^\beta = \delta^\beta_\alpha \hat{f}^{-1} \tag{4.11}$$

where $\hat{f}^{-1} \equiv \frac{1}{2}\hat{\Delta}_\alpha\hat{\Delta}^{\ddagger\alpha}$ is a $k\times k$ matrix superfield. There exists $\hat{f}$ because we have assumed that $\hat{\Delta}_\alpha$ has maximal rank. The above condition (4.11) leads to both the bosonic and fermionic ADHM constraints. When eq. (4.11) holds, i.e., the parameters in $\hat{\Delta}_\alpha$ are satisfying both bosonic and fermionic ADHM constraints, we can check that the above F satisfies the Yang-Mills constraints and the super ASD condition.

To ensure the WZ gauge of the superfields obtained by the super ADHM construction, we impose on the zero mode $\hat{v}$ of $\hat{\Delta}_\alpha$ the conditions $\bar{D}_{\dot{\alpha}}\hat{v} = 0$ and $\hat{v}^\ddagger \frac{\partial}{\partial\theta^\alpha}\hat{v} = 0$. Then $\hat{v}$ is determined as $\hat{v} = v + \theta^\gamma(\Delta_\gamma^\dagger f\mathcal{M}v) + \theta\theta(\frac{1}{2}\mathcal{M}^\dagger f\mathcal{M}v)$, and the connection ϕ_μ in (4.9) correctly gives the super instanton configuration in the WZ gauge: Its lowest component is the instanton gauge field and the θ-component is the fermion zero mode.

5 Deformed super ADHM construction

The deformed super ASD condition turns out to be of the same form as the super ASD condition (4.7) but the product replaced with the star product (2.1):

$$F_{\mu\dot{\alpha}} = 0, \quad \star F_{\mu\nu} = -F_{\mu\nu}, \tag{5.12}$$

where the curvature superfields F_{AB} are given by eq.(3.4). We can prove the equivalence of the condition (5.12) and the deformed equations (2.6).

One would expect that solutions to eq. (5.12) can be constructed by the super ADHM construction, replacing each product with the star product (2.1). For the deformed super ASD instantons, ϕ_μ in the WZ gauge becomes $\phi_\mu = -\frac{i}{2}\left[v_\mu + i\theta\sigma_\mu\bar{\lambda}\right](y)$. This leads us to adopt $\hat{\Delta}_\alpha$ in our super ADHM construction with the same form as before:

$$\hat{\Delta}_\alpha = \Delta_\alpha(y) + \theta_\alpha\mathcal{M}. \tag{5.13}$$

Then, according to the $\ddagger$-conjugation rules, we have $\hat{\Delta}^{\ddagger\alpha} = \Delta^{\ddagger\alpha}(y) + \theta^\alpha\mathcal{M}^\ddagger$. We collect the n zero modes of $\hat{\Delta}$ into a matrix form $\hat{u}_{[n+2k]\times[n]}$ and require it to be normalized with respect to the star product: $\hat{\Delta}_\alpha * \hat{u} = 0$, $\hat{u}^\ddagger * \hat{u} = \mathbf{1}_n$. Define $k\times k$ matrices $\hat{K}_{*\alpha}{}^\beta$ $(\alpha,\beta = 1,2)$ as the "inverse" matrices of $\hat{K}_*^{-1}{}_\alpha{}^\beta \equiv \hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta}$ such that $\hat{K}_*^{-1}{}_\alpha{}^\beta * \hat{K}_{*\beta}{}^\gamma = \hat{K}_{*\alpha}{}^\beta * \hat{K}_*^{-1}{}_\beta{}^\gamma = \delta_\alpha^\gamma\mathbf{1}_k$. Then we have a relation $\hat{u} * \hat{u}^\ddagger = \mathbf{1}_{n+2k} - \hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * \hat{\Delta}_\beta$.

With the use of the zero modes $\hat{u}$ of $\hat{\Delta}_\alpha$, the connection ϕ is given by $\phi = -\hat{u}^\ddagger * d\hat{u}$, and the curvature two-form becomes

$$F = d\phi + \phi * \phi = \hat{u}^\ddagger * d\hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * d\hat{\Delta}_\beta * \hat{u} \tag{5.14}$$

which reads $F_{AB} = -\hat{u}^\ddagger * D_{[A}\hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * D_{B\}}\hat{\Delta}_\beta * \hat{u}$, especially $F_{\mu\nu} = \hat{u}^\ddagger * b_{\dot{\alpha}}^\dagger\bar{\sigma}_{[\mu}^{\dot{\alpha}\alpha}\hat{K}_{*\alpha}{}^\beta\sigma_{\nu]\beta\dot{\beta}}$ $b^{\dot{\beta}} * \hat{u}$. Thus $F_{\mu\nu}$ becomes ASD (see eq. (5.12)) if $\hat{K}_*$ commutes with the sigma matrices σ_μ:

$$\hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta} = \hat{K}_*^{-1}{}_\alpha{}^\beta \propto \delta_\alpha^\beta. \tag{5.15}$$

Then we immediately find that $F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0$ and $F_{\mu\dot{\alpha}} = 0$, because $\hat{\Delta}_\alpha$ is a chiral superfield. We can also check that $F_{\alpha\beta} = 0$ with the use of the constraint (5.15), the relations $D_\beta\hat{\Delta}_\alpha = \varepsilon_{\alpha\beta}(\mathcal{M} + 4\bar{\theta}_{\dot{\beta}}b^{\dot{\beta}})$ and $D_\beta\hat{\Delta}^{\ddagger\alpha} = \delta_\beta^\alpha(\mathcal{M}^\ddagger + 4b_{\dot{\beta}}^\dagger\bar{\theta}^{\dot{\beta}})$, and the fact that $F_{\alpha\beta}$ is symmetric with respect to α and β. Therefore, we have shown that the above described super ADHM construction gives curvature superfields that satisfy the Yang-Mills constraints (3.5) and the ASD conditions (5.12) if the condition (5.15) is imposed.

Since we can write $\hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta} = \hat{\Delta}_\alpha \hat{\Delta}^{\ddagger\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma}C^{\gamma\beta}\mathcal{M}\mathcal{M}^\ddagger$, the requirement (5.15) leads to the following deformed bosonic ADHM constraint $\Delta_\alpha \Delta^{\ddagger\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma}C^{\gamma\beta}\mathcal{M}\mathcal{M}^\ddagger \propto \delta_\alpha^\beta$ and the fermionic ADHM constraint $\Delta_\alpha \mathcal{M}^\ddagger + \mathcal{M}\Delta^\ddagger{}_\alpha = 0$. These constraints agree with those in [6] obtained by considering string amplitudes. We can rewrite the deformed bosonic ADHM constraints in another form as follows. Let us denote

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} J^\ddagger_{[k]\times[n]} & \bar{z}_2 \mathbf{1}_k + B_2{}^\ddagger_{[k]\times[k]} & \bar{z}_1 \mathbf{1}_k + B_1{}^\ddagger_{[k]\times[k]} \\ I_{[k]\times[n]} & -z_1 \mathbf{1}_k - B_{1[k]\times[k]} & z_2 \mathbf{1}_k + B_{2[k]\times[k]} \end{pmatrix}, \tag{5.16}$$

where $z_1 \equiv y_{2\dot{1}}$, $z_2 \equiv y_{2\dot{2}}$ and $I \equiv \omega_2$, $J^\ddagger \equiv \omega_1$, $B_1 \equiv a'_{2\dot{1}}$, $B_2 \equiv a'_{2\dot{2}}$. Then the bosonic ADHM constraints reads

$$II^\ddagger - J^\ddagger J + [B_1, B_1{}^\ddagger] + [B_2, B_2{}^\ddagger] - C^{12}\mathcal{M}\mathcal{M}^\ddagger = 0, \tag{5.17}$$

$$IJ + [B_2, B_1] - \frac{1}{2}C^{11}\mathcal{M}\mathcal{M}^\ddagger = 0. \tag{5.18}$$

We can give an expression in terms of the ADHM data Δ_α and $\mathcal{M}$, of the general solution in the WZ gauge obtained by our construction, and have shown in [10] that it gives the known U(2) one instanton solution. In summary, we have correctly deformed the super ADHM construction to give solutions to the deformed ASD instantons in $\mathcal{N} = 1/2$ SYM theory. We see that deformation terms emerge in the bosonic ADHM constraints (see also [6]), which are comparable with the U(1) terms due to space-space noncommutativity [12]. Our formulation reveals the geometrical meaning of those deformation terms as non(anti)commutativity of superspace. However, it needs a further study to clarify how those terms can be interpreted in the hyper-Kähler quotient construction [13].

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Chiral Effective Potential in Non-anticommutative Wess-Zumino Model

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Abstract

We study a structure of holomorphic quantum contributions to the effective action for $\mathcal{N} = \frac{1}{2}$ noncommutative Wess-Zumino model. Using the symbol operator techniques we present the one-loop chiral effective potential in a form of integral over proper time of the appropriate heat kernel. We obtain the exact integral representation of the one-loop effective potential. Also we study the derivative expansion of the effective potential.

The deformation of superspace and construction the Moyal superstar product based on nontrivial (super)Poisson manifolds has been attracted much attention. It was a significant work of Seiberg and Witten [1] where they studied a star product in a certain class of quantum field theories on noncommutative (NC) Minkowski space-times, where (bosonic) directions become noncommutative. This result generated a modern activity in studying quantum field theories in NC space.

It should be noted that there are possible several different types of coordinate deformations [2]. Recently it was shown that the low-limit of superstring theory in the self-dual graviphoton background field $F^{\alpha\beta}$ leads to a four-dimensional supersymmetric field theory formulated in the deformed $\mathcal{N} = 1$ superspace [3] with fermionic coordinate satisfying the relation

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = 2\alpha'^2 F^{\alpha\beta} = C^{\alpha\beta} \ . \tag{1}$$

The other commutation relations are determined by the consistency of the algebra. In particular, the ordinary space-time coordinates x^m can not commute. In contrast to the space-time coordinates, the chiral coordinates $y^m = x^m + i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$ can be taken commuting. Note that because the anticommutation relation of $\bar{\theta}$ remains undeformed, $\bar{\theta}$ is not the complex conjugate of θ, that is possible only in the Euclidean space. The product of functions of θ on the chiral superspace is Weyl ordered by using the star-product, which is the fermionic version of the Moyal product:

$$f(\theta) \star g(\theta) = f(\theta) \, \exp\left(-\frac{1}{2} \, C^{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial\theta^\alpha} \frac{\overrightarrow{\partial}}{\partial\theta^\beta} \right) \, g(\theta) \ . \tag{2}$$

The supercharges are defined as follows $Q_\alpha = i\frac{\partial}{\partial\theta^\alpha}$, $\bar{Q}_{\dot{\alpha}} = i(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \frac{\partial}{\partial y^{\alpha\dot{\alpha}}})$. The star-product (2) is invariant under the action of supercharges Q_α. However, because $\bar{Q}_{\dot{\alpha}}$

depends explicitly on θ, it is clear that the star-product is not invariant under $\bar{Q}$. Such a deformation saves the $\mathcal{N}=\frac{1}{2}$ supersymmetry and has interesting properties in the field theory viewpoint. Replacing all ordinary products with the above $\star$-product, one can proceed studying a supersymmetric field theory in this non(anti)commuting superspace taking into account that this deformed supersymmetry algebra admits well-defined representations. Namely, one can define chiral and vector superfields much similarly to the ordinary $\mathcal{N}=1$ supersymmetry [3].

It is very interesting to study how the deformation (1) modifies the quantum dynamics of supersymmetric field theories, paying particular attention to consequences of nonlocality in the superspace caused by Eq.(2). Though new kinds of (anti)chiral superfields in $\mathcal{N}=\frac{1}{2}$ supersymmetric theory violate the holomorphy, the anti-holomorphy still remains. For deformed WZ-model, this leads to the non-renormalization theorem of the anti-chiral superpotential and vanishing of the vacuum energy. Moreover, one can show that such deformed theories preserve Lorentz symmetry and have finite number of divergent structures in their effective actions and hence, they are in fact renormalizable. This is primarily because although the theory contains operators of dimension five and higher, they are not accompanied by their hermitian conjugates which would be required to generate divergent diagrams.

In this work we develop a general approach to constructing the one-loop effective potential in $\mathcal{N}=\frac{1}{2}$ WZ model. The approach is based on use of the symbol operator techniques and heat kernel method and allows to carry out a straightforward calculation of one-loop finite quantum corrections. As a result we find an exact form of one-loop effective potential for the considered model in terms of a proper-time integral. Also we construct a new scheme for approximate evaluation of the effective potential and give a complete solution of the problem settled up in [4].

On the $\mathcal{N}=\frac{1}{2}$ noncommutative superspace the WZ model is given by the standard classical action where the point products of superfields are replaced with the star product (2):

$$S=\int d^8z\,\bar{\Phi}\star\Phi+\int d^6z\,(\frac{m}{2}\Phi\star\Phi+\frac{g}{3!}\Phi\star\Phi\star\Phi)+\int d^6\,\bar{z}(\frac{\bar{m}}{2}\bar{\Phi}\star\bar{\Phi}+\frac{\bar{g}}{3!}\bar{\Phi}\star\bar{\Phi}\star\bar{\Phi})\ . \quad (3)$$

The (anti)chiral superfields $\bar{\Phi},\Phi$ are defined by the ordinary relation $\bar{D}_{\dot{\alpha}}\Phi(y,\theta,\bar{\theta})=0$, $D_\alpha\bar{\Phi}(y,\theta,\bar{\theta})=0$. As it has been demonstrated in Ref. [3], the $\star$-product of the chiral superfields is again a chiral superfield; likewise, the $\star$-product of the antichiral superfields is again an antichiral superfield.

The model is formulated in Euclidean space where the fields $\Phi,\bar{\Phi}$ are considered as independent. Using the property $\int\Phi\star\Phi=\int\Phi\cdot\Phi$, performing the expansion of the star-product (2) and turning down total superspace derivatives, the cubic interaction terms reduce to the usual WZ interactions complemented by the terms violating $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=\frac{1}{2}$ supersymmetry.

$$\begin{aligned}S=&\int d^8z\,\bar{\Phi}\Phi+\int d^6z\,(\tfrac{m}{2}\Phi\Phi+\tfrac{g}{3!}\Phi\Phi\Phi)+\int d^6\bar{z}\,(\tfrac{\bar{m}}{2}\bar{\Phi}\bar{\Phi}+\tfrac{\bar{g}}{3!}\bar{\Phi}\bar{\Phi}\bar{\Phi})+\\&+\int d^6z\,(\tfrac{h}{3!}\Phi(Q^2\Phi)^2+\tfrac{1}{2\lambda}\Phi(Q^2\Phi))\ ,\end{aligned} \quad (4)$$

where $h=-\frac{g}{4}|\det C|$. Last term containing the coupling λ is added to provide a multiplicative renormalization of the model (see e.g. [4]). As a result we see that the action (3) is rewritten in terms of standard $\mathcal{N}=1$ superspace, i.e. without star-product. Hence, one can consider the deformed WZ model as ordinary WZ model, where superfield multiplication is standard, with a new addition to the F-term. Thus we treat the theory as some special model formulated in terms of $\mathcal{N}=1$ superspace and this circumstance allows us to use all the standard tools and techniques of superspace quantum field theory.

The one-loop correction to the effective action is formally written in the form $\Gamma_{(1)} = \frac{i}{2} \ln \mathrm{Det}(\hat{H})$, where $\hat{H}$ is the differential operator acting on superfields being the second variational derivatives over quantum (super)fields of the classical action. In order to find this operator in the framework of the loop expansion one have to split all fields of the model on quantum and background parts. We use the standard quantum-background splitting $\Phi \to \Phi + \varphi, \quad \bar{\Phi} \to \bar{\Phi} + \bar{\varphi}$, where Φ and φ stand for background and quantum fields respectively. The quadratic over quantum (super)fields part of the classical action is written in the form $S_{(2)} = \frac{1}{2} \int d^8z \, (\bar{\varphi}, \varphi) \hat{H} \begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix}$, where we denote

$$\hat{H} = \begin{pmatrix} D^2 \bar{D}^2 & (\bar{m} + \bar{g}\bar{\Phi}) D^2 \\ (m + \Lambda) \bar{D}^2 & \bar{D}^2 D^2 \end{pmatrix} , \quad \Lambda = g\Phi + h(Q^2\Phi)Q^2 + \frac{1}{\lambda} Q^2 \, . \tag{5}$$

Further we use the convenient denotations $m + g\Phi = \mu, \quad \bar{m} + \bar{g}\bar{\Phi} = \bar{\mu}$ and consider the constant space-time background $\Phi = \mathrm{Const}, \bar{\Phi} = \mathrm{Const}$ which is sufficient for calculation of the chiral effective potential (e.g. see a discussion in Refs. [3]).

After a number of simplifications the one-loop contribution to the effective potential can be finally presented in the following form

$$\Gamma_{(1)} = \frac{i}{2} \mathrm{Tr} \left(\frac{D^2 \bar{D}^2}{\Box} \ln(\Box - \mu\bar{\mu} - \bar{\mu}\tilde{\Lambda}) \right) \, . \tag{6}$$

We pay attention on appearance of the chiral projector in this relation showing that the effective action is given by an integral over a chiral subspace. Further calculations will be fulfilled using the symbol-operator techniques [5] which we shortly describe.

The main idea is based on the supersymmetric generalization of the well known trace formula for the operator $\hat{A} = a(\hat{q}, \hat{p})$: $\mathrm{Tr}(\hat{A}) = \int d\mu(\gamma) A(\gamma)$, where $\hat{q}, \hat{p}$ are the operators of coordinate and momentum, $\gamma = (q, p)$ are the coordinates on the phase-space, $d\mu(\gamma)$ is a measure on the phase-space, $A(\gamma)$ is a symbol of the operator $\hat{A}$ and integration goes over the full phase-space. The symbol of the operator $\hat{A}$ is function on phase space.

We apply the symbol operator techniques to calculation of traces for the operators depending on $\mathcal{N}=1$ superspace coordinates $z^M = (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and corresponding derivatives. The phase superspace is parameterized by z^M, p_M where $p_M = (p_m, \psi_\alpha, \bar{\psi}_{\dot{\alpha}})$. According to the symbol-operator techniques if the operator $\hat{A}$ consists from a function of the set of basic operators $\hat{\gamma}$, i.e. $\hat{A} = A(\hat{\gamma})$, then its symbol can be defined as $A(\gamma) = A(\gamma^\hbar) \times 1$ where $\gamma^\hbar$ are the special representation for the basic operators $\hat{\gamma} = (\hat{p}, \hat{q})$. In turn, the special representation can be calculated from the phase-space coordinates γ of the basic operators with the help of a star-operator U by the rule $\gamma^\hbar = U^{-1} \gamma U$. The appropriate operator U for $\mathcal{N}=1$ superspace field theories is found in [5] and for the case under consideration, has the form $U = \mathrm{e}^{-\bar{\partial}\bar{D}} \mathrm{e}^{\partial p \cdot \bar{\theta}} \mathrm{e}^{-\partial D} \mathrm{e}^{-i\partial_p \partial_x}$, where $\partial^\alpha = \frac{\partial}{\partial \psi_\alpha}, \bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\psi}_{\dot{\alpha}}}$ and the derivatives $D, \bar{D}, \partial_z$ act on the left while the operators $\frac{\partial}{\partial \psi}, \frac{\partial}{\partial \bar{\psi}}, \partial_p$ act on the right. The corresponding special representation operator are as follows $\bar{D}^\hbar_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}$, $D^\hbar_\alpha = \psi_\alpha - \bar{\partial}^{\dot{\alpha}} p_{\alpha\dot{\alpha}}$, $Q^\hbar_\alpha = i(\psi_\alpha + p_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}})$. These operators obey the initial algebra but act on functions of phase superspace coordinates. For the background field one can get the representation $\Phi^\hbar(y,\theta) = \Phi(y+i\partial_p, \theta+\partial_\psi) = \Phi + \partial^\alpha(D_\alpha\Phi) - \partial^2(D^2\Phi) + \mathcal{O}(\partial_y)$, here the derivatives $\partial = \frac{\partial}{\partial\psi}$ act through on the right. The terms including the y-derivatives of Φ can be omitted because we use the approximation of background fields slowly varying in space-time.

Using the ζ-function representation for Eq. (6) we obtain proper-time integral representation of the logarithm. It is convenient also to introduce a dimension parameter L^2 to make proper-time dimensionless. With the mentioned notations the ζ-function representation became

$$\Gamma_{(1)} == \left(-\frac{d}{ds}\right)\bigg|_{s=0} \int_0^\infty \frac{dT}{\Gamma(s)}\, T^{s-1} \int d^6z\, \bar{D}^2 K(\frac{T}{L^2}) \ , \tag{7}$$

where the heat kernel in the above equation is defined as

$$K(T) = \frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{-p^2} \left(\frac{L^2}{\bar{\mu}}\right)^s \bar{\mu}\ \mathrm{e}^{-T(p^2+m)} h(T) \ . \tag{8}$$

In the last expression we introduce a denotation

$$h(T) = \mathrm{e}^{-T(MQ_\hbar^2 + g\Phi_\hbar(y,\theta))} \times 1 \ . \tag{9}$$

In general, exact calculation of the heat kernel is impossible. The model under consideration is quite remarkable since it provides the exact evaluation of the heat kernel. The reason is the fact that for this model the heat kernel calculation is reduced to finding an evolution operator for a harmonical oscillator with the Grassmannian coordinate ψ and momentum $\partial/\partial\psi$. In order to calculate (8), according to the symbol-operator techniques, we have to disentangle derivatives in the exponent of the heat kernel. To do that we transfer all derivatives ∂_ψ on the right and act on unit. It means that after such a transformation all terms with derivatives must be omitted and a final contribution is resulted only from recommutations of the differential operators to the last right position. The rest part is the symbol of the heat kernel.

The expression in the exponent (9) can be simplified by introducing a new denotation $\tilde{\partial}_\alpha = \partial_\alpha - \frac{D_\alpha\Phi}{D^2\Phi}$, $\tilde{Q}^\hbar_\alpha = (\psi + \sqrt{\bar{\mu}}p\bar{\theta})_\alpha$ then we can extract ψ- and ∂_ψ-independent part

$$h(T) = e^{-Tg\Phi - Tg\frac{(D\Phi)^2}{D^2\Phi}} \cdot k(T) \times 1 \ , \quad k(T) = \mathrm{e}^{TM\tilde{Q}_\hbar^2 + Tg(D^2\Phi)\tilde{\partial}^2} \ . \tag{10}$$

Straightforward calculations shows that $\tilde{Q}^\hbar_\beta$ and $\tilde{\partial}^\alpha$ can be considered as the Grassmann coordinates and momenta. Let's introduce operators: $e_1 = \tilde{Q}^2_\hbar,\ e_2 = \tilde{\partial}^2,\ e_3 = \tilde{Q}^\hbar_\alpha\tilde{\partial}^\alpha - 1$. It is easy to see that these operators satisfy the commutation relations for the generators of $su(2)$ algebra: $[e_1, e_2] = e_3, \quad [e_3, e_1] = 2e_1, \quad [e_3, e_2] = -2e_2$. Hence, the exponent in $k(T)$ (10) is nothing but a group element of $SU(2)$: $k(T) = \mathrm{e}^{TMe_1 + TgFe_2}$. Since our goal is to find a symbol of the heat kernel, we should move all derivatives in the exponent (10) to right hand side and act on unit what is equivalent to drop them. The generators containing the derivatives in the group element $k(T)$ are e_2, e_3. It is most convenient to rewrite the group element $k(T)$ in the Gaussian form $k(T) = \mathrm{e}^{TMe_1 + TFe_2} = \mathrm{e}^{A(T)e_1}\mathrm{e}^{B(T)e_3}\mathrm{e}^{C(T)e_2}$. The solution for functions $A(T), B(T), C(T)$ can be founded

$$A = \sqrt{\frac{M}{F}}\tanh(T\sqrt{MF}) \ , B = -\ln\cosh(T\sqrt{FM}) \ , C = \sqrt{\frac{F}{M}}\tanh(T\sqrt{FM}) \ . \tag{11}$$

That gives us an exact expression for the symbol (9)

$$k(T)\times 1 = Q^2\left(A(T)+(1-e^B)^2\frac{g^2}{F^2}(D\Phi)^2\right)\exp\left(-B(T)+C(T)\frac{g^2}{F^2}(D\Phi)^2\right)\ , \quad (12)$$

where $(D\Phi)^2=\frac{1}{2}D^\alpha\Phi D_\alpha\Phi$. Now using (10, 12) in (7, 8) we obtain the exact expression for the one-loop chiral potential (see details in Ref. [6])

$$\Gamma_{(1)} = -\left.\frac{d}{ds}\right|_{s=0}\int d^6z\int_0^\infty \frac{dT}{\Gamma(s)}T^{s-1}\frac{1}{2(4\pi T)^2}\left(\frac{L^2}{\bar\mu}\right)^s\bar\mu^2\mathrm{e}^{-T(m+g\Phi)}\cdot\tilde k(T)\ , \quad (13)$$

where

$$\tilde k(T)=\sqrt{\frac{M}{F}}\sinh(T\sqrt{FM})\left(1-g\frac{Tg\Phi}{1-Tg\Phi}\left(\frac{\tanh(T\sqrt{FM}/2)}{T\sqrt{FM}/2}-1\right)\right)\ . \quad (14)$$

The expressions (13, 14) determine the final exact solution for the one-loop chiral effective potential in $\mathcal{N}=\frac{1}{2}$ WZ model. The various approximate results can be obtained using the various expansions of (14). Also we point out that the integral (13) is divergent at the low limit. To get a finite effective potential we should, as usual, to subtract in the integrand of (14) a first term in expansion of the integrand in T.

The exact result for the one-loop chiral effective potential is presented by the expressions (13, 14) in the form of an integral over proper time T which can not be written in an explicit form in terms of elementary or known special functions. To obtain the various approximate results we have to construct the expansions of the heat kernel and calculate the integral over proper time in an explicit form. In the paper [6] we present an independent procedure for the heat kernel expansion which allows to obtain the chiral effective potential in a form of a power expansion of spinor derivatives of Φ.

Let's present the exponent (9) as a sum $h(T)=e^{T(H_0+V)}\times 1=\sum_{n=0}^\infty h_n(T)\times 1\ ,\quad h_0=\mathrm{e}^{TH_0}\ ,\quad H_0=-MQ_\hbar^2\ ,\quad V=-g\Phi_\hbar(y,\theta)$, where the general term of the sum is given by the T-ordered iterated integral

$$h_n(T)\times 1=\int_0^T dt_n\int_0^{t_n}dt_{n-1}\dots\int_0^{t_2}dt_1\ \mathrm{e}^{(T-t_n)H_0}V\mathrm{e}^{(t_n-t_{n-1})H_0}V\dots V\mathrm{e}^{(t_2-t_1)H_0}V\mathrm{e}^{t_1H_0}\ , \quad (15)$$

(see details in Ref. [7]). The integral (15) for every fixed n can be calculated [6] and it induces the expansion for the one-loop effective action: $\Gamma_{(1)}=\sum_{n=0}^\infty R_n$, where every expansion term has the following structure

$$R_n=\int d^4xd^2\theta\sum_{k=0}^n C_{k;n}(M,\bar\mu,m)\Phi^{n-k}(\frac{M}{m}D^2\Phi)^k\ ,\quad \Phi=\Phi(y,\theta) \quad (16)$$

here $C_{k;n}(M,\bar\mu,m)$ are some functions. In particular to get the divergences it is enough to consider only terms coming from R_0:

$$R_0^{\mathrm{div}}=\int d^4xd^2\theta\,\frac{m}{2(4\pi)^2}(\bar m^2+2\bar gQ^2\Phi)(\frac{1}{\lambda}+hQ^2\Phi)\ln(\frac{m\bar\mu}{L^2})\ , \quad (17)$$

and R_1:

$$R_1^{\mathrm{div}}=\int d^4xd^2\theta\frac{g}{2(4\pi)^2}(\bar m^2+2\bar gQ^2\Phi)(\frac{1}{\lambda}+hQ^2\Phi)\Phi\ln(\frac{m\bar\mu}{L^2})\ . \quad (18)$$

In the above expressions we used relations $(\bar{m}+\bar{g}\bar{A})^2=\bar{m}^2+2\bar{g}\bar{G}$ and $\bar{G}=Q^2\Phi$. The sum of R_0 and R_1 gives the result which is completely consistent with all earlier results on one-loop divergences in the model under consideration obtained by the other methods [8].

Let's find the explicit calculations of several finite contributions to effective potential with higher orders of external Grassmannian momenta. It should be especially noted that calculations of the finite contributions R_n to the chiral effective potential is closely related with so called Mellin-Barnes representation of hypergeometrical functions of several variables. It should be noted that all terms R_n for $n>1$ in (16) are finite. We will consecutively consider several first finite contributions to the chiral effective potential. In this section we find an explicit expression for R_2 term. After a number of transformation we calculated this contribution in the form

$$\begin{aligned} R_2 = \tfrac{g^2}{4(4\pi)^2}\cdot\int d^4x d^2\theta \tfrac{M}{m}\bar{\mu}^2\Phi(y,\theta)\,{}_2F_1(1,1;\tfrac{3}{2};\tfrac{M}{4m}D^2)\Phi(y,\theta) = \\ = \tfrac{g^2}{4(4\pi)^2}\int d^4x d^2\theta\,\tfrac{M}{m}\bar{\mu}^2\left(\Phi^2+\tfrac{M}{6m}\Phi D^2\Phi\right)\ , \end{aligned} \tag{19}$$

where we use the explicit expression for ${}_2F_1(1,1;\frac{3}{2};z)=\frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}}$. In the bosonic sector this contribution has a simple form $R_2^b=\frac{g^2}{4(4\pi)^2}\int d^4x\,\frac{M}{m}\bar{m}^2\left(2AF+\frac{M}{6m}F^2\right)\ ,\quad M=hF+\frac{1}{\lambda}$. The expression for R_3 contribution will have a following form

$$\begin{aligned} R_3 = \tfrac{g^3}{6(4\pi)^2}\int d^4x d^2\theta\,(\tfrac{M}{m^2})\bar{\mu}^2\times \\ \times\left(\tfrac{1}{2}\Phi^3+\tfrac{1}{4}\tfrac{M}{4m}\Phi^2D^2\Phi+\tfrac{1}{4\cdot5}\left(\tfrac{M}{4m}\right)^2\Phi D^2\Phi D^2\Phi+\tfrac{1}{5\cdot6\cdot7}\left(\tfrac{M}{4m}\right)^3(D^2\Phi)^3\right)\ . \end{aligned} \tag{20}$$

In the bosonic sector this result gives $R_3^b=\frac{g^3}{6(4\pi)^2}\int d^4x\frac{M\bar{m}^2}{m^2}(\frac{3}{2}A^2F+\frac{1}{2}\frac{M}{4m}AF^2+\frac{1}{4\cdot5}(\frac{M}{4m})^2F^3)$.

During the calculation for higher R_n contributions, we will obtain the expressions containing the hypergeometric function of several variables, which for $n>2$ are called generalized Lauricella hypergeometric functions. In principal, all R_n can be calculated [6]. Moreover we can sum up all part from R_n terms without derivatives. Really let's take into account only $k=0$ term in the representation (16) for all R_n, i.e. we sum contributions from all R_n which have no Grassmannian derivatives. The sum can be calculated [6], that gives

$$\Gamma^{(0)}_{(1)}=\frac{1}{2(4\pi)^2}\int d^4xd^2\theta\, m(\bar{m}+\bar{g}\bar{\Phi})^2(hQ^2\Phi+\frac{1}{\lambda})\left(-\frac{g\Phi}{m}+(1+\frac{g\Phi}{m})\ln(1+\frac{g\Phi}{m})\right)\ . \tag{21}$$

The expression (21) is the chiral effective potential in approximation when all terms containing the $D^2\Phi$ can be neglected, however all terms without these derivatives are exactly summed up. The corrections to this approximation obligatory contain the terms with Grassmann derivatives of the background field. This result can be used to finding the effective potential in the bosonic component sector and obtaining the classical potential.

Finally we can say that the general approach for finding the one-loop effective potential in $\mathcal{N}=\frac{1}{2}$ noncommutative Wess-Zumino model was founded. Using the symbol-operator techniques we obtained a general expression for the effective potential in terms of a superfield heat kernel. The exact form of the effective potential including the complete dependence on Φ and $D^2\Phi$ in term of a single proper time integral was obtained. To clarify the structure of the effective potential in more details we calculate divergent contributions to the one-loop effective potential as well as a few first finite contributions.

The expansion of the effective potential has an enough simple structure and allows to organize resummation of the above series and to get a series in derivatives $D^2\Phi$ with the coefficients depending on Φ. We have demonstrated how to obtain the first term in this new expansion containing no derivatives but including all powers of Φ.

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Generic Chiral Superfield Model on Non-anticommutative $\mathcal{N} = \frac{1}{2}$ Superspace

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Abstract

We consider the generic nonanticommutative model of chiral-antichiral superfields on $\mathcal{N} = \frac{1}{2}$ superspace. The model is formulated in terms of an arbitrary Kählerian potential, chiral and antichiral superpotentials and can include the nonanticommutative supersymmetric sigma-model as a partial case. We study a component structure of the model and derive the component Lagrangian in an explicit form with all auxiliary fields contributions. We show that the infinite series in the classical action for generic nonanticommutative model of chiral-antichiral superfields in $D = 4$ dimensions can be resumed in a compact expression which can be written as a deformation of standard Zumino's lagrangian and chiral superpotential. Problem of eliminating the auxiliary fields in the generic model is discussed and the first perturbative correction to the effective scalar potential is obtained.

Supersymmetric field theories on deformed superspaces with nonanticommuting coordinates possess the interesting properties in classical and quantum domains. Remarkable class of such theories based on special deformation of $\mathcal{N} = 1$ supersymmetry was proposed by Seiberg [1]. Seiberg's type of superspace deformation introduces the nonanticommutativity both even and odd coordinates but preserves anticommutativity in the chiral sector. As a result, the corresponding deformed superspace breaks the supersymmetry in the antichiral sector and therefore it is called $\mathcal{N} = \frac{1}{2}$ superspace. Formulation of analogous deformation in $\mathcal{N} = 2$, $D = 4$ superspace was given in [2]. Studying of various aspects of $\mathcal{N} = \frac{1}{2}$ supersymmetric theories has been carried out in a number of recent papers (see e.g. [3], [4], [5], [6] for $D = 4$ models and [7], [8] for $D = 2$ models).

To interpret the $\mathcal{N} = \frac{1}{2}$ supersymmetric theories as conventional field models and to clarify their dynamics it is necessary to rewrite such superfield theories in the component form. Finding the component structure of the nonanticommutative theories is a highly nontrivial technical problem because of the very complicated superspace structure and therefore it demands a special study. Component form of actions for nonanticommutative theories in addition to standard terms always will contain the terms dependent on the superspace deformation parameter. Since a half of supersymmetries is broken down a symmetry between chiral and antichiral superspace coordinates is absent and some component fields can enter in the action in very cumbersome combinations.

In the papers [1], the component structure of $D=4$, $\mathcal{N}=\frac{1}{2}$ supersymmetric models Yang-Mills theory and the Wess-Zumino was studied. For this case it was shown that the deformed theory is renormalizable [4], [5] in spite of the presence of higher dimensional terms in the Lagrangian.

In this paper we study the $D=4$ generic chiral superfield model in $\mathcal{N}=\frac{1}{2}$ superspace and derive its component structure. Explicit expressions for the Kählerian, chiral and antichiral superpotentials are not fixed. We show that the component action is represented as an infinite series in nonanticommutativity parameter with coefficients depending on the derivatives of above potentials. Despite this fact, it is possible to write down the action in a closed form via smoothing integrals of the Kählerian K and chiral W superpotential around the bosonic component of the chiral superfield Φ on a scale dependent on the deformation parameter and the auxiliary field $\sqrt{\det C}F$. This effect is in an agreement with an observations of Ref. [8] for $D=2$. For $\mathcal{N}=2$ sigma model nonanticommutativity induces simple deformations of the Zumino Lagrangian along with the holomorphic superpotential. This phenomena is interpreted as a fuzziness in the target space controlled by the vacuum expectation value of the auxiliary field.

We begin with consideration of the $\mathcal{N}=\frac{1}{2}$ deformed superspace. According to Seiberg, the coordinates of this superspace are defined such a way that Grassmannian coordinates are not complex conjugate to one another ($(\theta^\alpha)^* \neq \bar\theta^{\dot\alpha}$) and the anticommuting coordinates θ form a Clifford algebra

$$\{\hat\theta^\alpha, \hat\theta^\beta\} = C^{\alpha\beta} \; , \tag{1}$$

where $C^{\alpha\beta}=C^{\beta\alpha}$ is a symmetrical constant matrix. The other commutation relations are determined by the consistency of the algebra: $[x^m,\theta^\alpha]=iC^{\alpha\beta}\sigma^m_{\beta\dot\alpha}\bar\theta^{\dot\alpha}$, $\quad [x^m,x^n]=\bar\theta\bar\theta\, C^{mn}$, $\{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\}=0$, where $C^{mn}=C^{\alpha\beta}\varepsilon_{\beta\gamma}\sigma^{mn}{}_\alpha{}^\gamma$.

Because of the deformation (1) functions of θ must be ordered. The simplest possible ordering is the Weyl type. Reordering is implementing by a noncommutative $\star$-product which is defined as follows

$$\Phi\star\Psi = \Phi\, \mathrm{e}^{-C^{\alpha\beta}\overleftarrow{Q_\alpha}\overrightarrow{Q_\beta}}\,\Psi = \Phi\left(1 - C^{\alpha\beta}\overleftarrow{Q_\alpha}\overrightarrow{Q_\beta} + \lambda\overleftarrow{Q}^2\overrightarrow{Q}^2\right)\Psi \; , \tag{2}$$

where $\lambda=-\frac{1}{2}C^{\alpha\beta}C_{\alpha\beta}$. The star-product (2) is invariant under the action of Q_α but is not invariant under action of $\bar Q$. Described deformation preserves the half of $\mathcal{N}=1$ supersymmetry and has interesting properties in the field theory viewpoint. Replacing all ordinary products with the above $\star$-product, one can proceed studying a supersymmetric field theory in this nonanticommuting superspace taking into account that this deformed supersymmetry algebra admits well-defined chiral and antichiral representations determined by the standard relations $\bar D_{\dot\alpha}\Phi=0$, $D_\alpha\bar\Phi=0$. As it has been demonstrated in Ref. [1], the $\star$-product (2) of the chiral superfields is again a chiral superfield; likewise, the $\star$-product of the antichiral superfields is again an antichiral superfield. This observation allows to extend well-studied anticommutative theories on nonanticommutative versions by simple replacement the point product with the star product.

The action of the generic chiral superfield model on $\mathcal{N}=1/2$ superspace is

$$S_\star[\bar\Phi,\Phi] = \int d^4x d^4\theta\, K(\bar\Phi,\Phi)_\star + \int d^4x d^2\theta\, W(\Phi)_\star + \int d^4x d^2\bar\theta\, \bar W(\bar\Phi)_\star \; , \tag{3}$$

where $K(\bar\Phi,\Phi), W(\Phi), \bar W(\bar\Phi)$ are the arbitrary Kählerian potential, chiral and antichiral superpotentials respectively and the superfield multiplication is defined in terms of the star-product (2). Since the star-product (2) always begins with the point product, it is easy to understand that the action (3) can be written as a sum of the action for the general chiral superfield model on undeformed $\mathcal{N}=1$ superspace and some contributions higher dimensions resulting from deformation of the superspace. Further we will write the action (3) in component fields and study its structure.

We consider the chiral superpotential component form and write it as a Taylor series:

$$\int d^4xd^2\theta\, W(\Phi)_\star = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4xd^2\theta\, W_n(\phi)f_\star^n \ , \tag{4}$$

here W_n are the expansion coefficients taken at the point ϕ and the superfield f is defines as $f = \Phi(y,\theta) - \phi(y) = \theta\kappa - \theta^2 F$ and $f_\star^n = \underbrace{f \star f \star \cdots \star f}_{n}$. Our first aim is calculation of this star-product.

Consideration of several first orders and further induction leads to the following expression

$$f_\star^{2m} = -2m\theta^2\kappa^2(\lambda F^2)^{m-1} + (\lambda F^2)^m \ , \quad f_\star^{2m+1} = (\lambda F^2)^m f(\theta) + 2m\kappa^2(\lambda^m F^{2m-1}) \ . \tag{5}$$

Collecting from (5) terms with θ^2, which will survive after integration over chiral coordinates we obtain the component form of the chiral superpotential

$$\begin{aligned}\int d^6z W(\Phi)_\star \ &= \ \int d^4x \sum_{n=0}^{\infty}\Big(\frac{1}{(2n)!}W_{2n}(\phi)\cdot 2n\kappa^2(\lambda F^2)^{n-1} + \\ &\qquad \frac{1}{(2n+1)!}W_{2n+1}(\phi)\cdot\lambda^n F^{2n+1}\Big) \ .\end{aligned} \tag{6}$$

The antichiral superpotential expansion around the scalar field $\bar\phi$ is defined as a series

$$\int d^4xd^2\bar\theta\, \bar W_\star(\bar\Phi(\bar y,\bar\theta)) = \sum_{\bar n=0}^{\infty}\frac{1}{\bar n!}\int d^4xd^2\bar\theta\, \bar W_{\bar n}(\bar\phi)\bar f_\star^{\bar n} \ , \tag{7}$$

here $\bar f = \bar\Phi(\bar y,\bar\theta) - \bar\phi(y) = \bar\theta^{\dot\alpha}\bar\kappa_{\dot\alpha}(y) - \bar\theta^2\bar F(y) - i\theta^\alpha(\partial_{\alpha\dot\alpha}\bar\phi(y))\bar\theta^{\dot\alpha} + i\theta^\alpha\partial_{\alpha\dot\alpha}\bar\kappa^{\dot\alpha}(y)\bar\theta^2 + \theta^2\bar\theta^2\Box\bar\phi(y)$. Taking into account further integration over chiral coordinates $d^2\bar\theta$ we will consider only components proportional to $\bar\theta^2$. For example $\bar f_\star^2|_{\bar\theta^2} = -2\bar\kappa^2 + 2i\theta^\alpha(\partial_{\alpha\dot\alpha}\bar\phi)\bar\kappa^{\dot\alpha} + \theta^2\partial^{\alpha\dot\alpha}\bar\phi\partial_{\alpha\dot\alpha}\bar\phi + C^{\alpha\beta}\partial_\alpha^{\dot\alpha}\bar\phi\partial_{\beta\dot\alpha}\bar\phi$. The last term is equal to zero due to a property $\partial^{\dot\alpha}_{(\alpha}\bar\phi\partial_{\beta)\dot\alpha}\bar\phi = -\partial_{\dot\alpha(\alpha}\bar\phi\partial^{\dot\alpha}_{\beta)}\bar\phi \equiv 0$ and can be dropped. Therefore deformation doesn't affect on antichiral sector $\bar f \star \bar f = \bar f \cdot \bar f$ in accordance with Ref. [1]. All other orders is equal to zero, because each $\bar f$ contains $\bar\theta$ and $\bar f^2 \sim \bar\theta^2$, i.e. $\bar f_\star^n = 0, \ n > 2$. Finally write a component form for the antichiral superpotential

$$\int d^4xd^2\bar\theta\, \bar W_\star = \int d^4x\ \left(\bar W_{\bar 1}(\bar\phi)\bar F + \bar W_{\bar 2}(\bar\phi)\bar\kappa^2\right) \ , \tag{8}$$

where the expansion coefficients $\bar W_{\bar 1}, \bar W_{\bar 2}$ were defined in (7). As one can see the great difference between forms of chiral and antichiral superpotentials appears. Obviously the action doesn't have Hermiticity properties.

The most nontrivial calculation is related to the Kähler potential decomposition. We will suppose that its expansion is fully symmetrical in powers of $\bar f$ and f, i.e.

$$K(\Phi,\bar\Phi)_\star = \sum_{m=0}^{\infty}\frac{1}{m!}\left(f\frac{\partial}{\partial\Phi} + \bar f\frac{\partial}{\partial\bar\Phi}\right)_\star^m K(\Phi,\bar\Phi)|_{\Phi=\phi,\ \bar\Phi=\bar\phi} \ , \tag{9}$$

Such kind of ordering leads to the following expansion

$$K(\Phi,\bar{\Phi})_\star = \sum_n K_n f_\star^n + \sum_{\bar{n}} K_{\bar{n}} \bar{f}_\star^{\bar{n}} + \sum_{n,\bar{n}} K_{n\bar{n}} [f_\star^n \star \bar{f}_\star^{\bar{n}}] \ , \tag{10}$$

where $[\bar{f}^{\bar{n}} \star f^n]$ is a fully symmetrized star-product including all possible permutations. It is obviously that unmixed products like $f_\star^n$ for any n will not give contribution to the Kähler potential because they do not contain factor $\bar{\theta}^2$ we need for further integration over $\int d^2\bar{\theta}$. Unmixed star products $\bar{f}_\star^{\bar{n}}$ for $n=3$ and higher will vanish and hence, do not contribute to the action. Thus, we should study the star product $[\bar{f}^{\bar{n}} \star f^m]$ of arbitrary integer m with $\bar{n}=1,2$. Direct calculation gives us factors at the coefficients $K_{\bar{1}n}$:

$$\begin{aligned} \bar{f} \star f_\star^{2n}|_{\theta^2\bar{\theta}^2} &= 2n\kappa^2(\lambda F^2)^{n-1}\bar{F} + (\lambda F^2)^n \Box\bar{\phi} \ , \\ \bar{f} \star f_\star^{(2n+1)}|_{\theta^2\bar{\theta}^2} &= \lambda^n F^{2n+1}\bar{F} - i\kappa^\alpha \partial_{\alpha\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}} \lambda^n F^{2n} + 2n\kappa^2\lambda^n F^{2n-1}\Box\bar{\phi} \ . \end{aligned} \tag{11}$$

Next, we compute factors at the coefficients $K_{\bar{2}n}$ by the same way

$$\begin{aligned} f_\star^{2n} \star \bar{f}_\star^2|_{\theta^2\bar{\theta}^2} &= 2\kappa^2\bar{\kappa}^2 2n(\lambda F^2)^{n-1} + \lambda^n F^{2n}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi} \ , \\ f_\star^{2n+1} \star \bar{f}_\star^2|_{\theta^2\bar{\theta}^2} &= -(\lambda F^2)^n 2i\kappa^\alpha\bar{\kappa}^{\dot{\alpha}}(\partial_{\alpha\dot{\alpha}}\bar{\phi}) + 2\bar{\kappa}^2\lambda^n F^{2n+1} + 2n\kappa^2\lambda^n F^{2n-1}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi} \ . \end{aligned} \tag{12}$$

Using above expressions we write the full Lagrangian in component form for the $\mathcal{N}=\frac{1}{2}$ nonanticommutative generic chiral superfield model (3) as a infinite series expansion in the parameter deformation

$$\begin{aligned} \mathcal{L}_\star &= K(\Phi,\bar{\Phi})_\star|_{\theta^2\bar{\theta}^2} + W(\Phi)_\star|_{\theta^2} + \bar{W}(\bar{\Phi})_\star|_{\bar{\theta}^2} = \bar{W}_{\bar{1}}\bar{F} + \bar{W}_{\bar{2}}\bar{\kappa}^2 \\ &+ \sum_{n=0}^\infty \frac{\lambda^n F^{2n}}{(2n+1)!}\left(W_{2n+2}\kappa^2 + W_{2n+1}F\right) + \bar{F}\sum_{n=0}^\infty \frac{\lambda^n F^{2n}}{(2n+1)!}\left(K_{\bar{1}(2n+2)}\kappa^2 + K_{\bar{1}(2n+1)}F\right) \\ &+ \Box\bar{\phi}\sum_{n=0}^\infty \frac{\lambda^n F^{2n-1}}{(2n)!}\left(\frac{2n}{2n+1}K_{\bar{1}(2n+1)}\kappa^2 + K_{\bar{1}(2n)}F\right) + \sum_{n=0}^\infty \frac{\lambda^n F^{2n}}{(2n+1)!}K_{\bar{1}(2n+1)}(i\kappa^\alpha\partial_\alpha^{\dot{\alpha}}\bar{\kappa}_{\dot{\alpha}}) \\ &+ \frac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi}\sum_{n=0}^\infty \frac{\lambda^n F^{2n-1}}{(2n)!}\left(K_{\bar{2}(2n+1)}\frac{2n}{2n+1}\kappa^2 + K_{\bar{2}(2n)}F\right) + \sum_{n=0}^\infty \frac{\lambda^n F^{2n+1}}{(2n+1)!}K_{\bar{2}(2n+1)}\bar{\kappa}^2 \\ &+ \sum_{n=0}^\infty \left[\frac{1}{(2n)!}K_{\bar{2}(2n)}\left(2n\kappa^2\bar{\kappa}^2(\lambda F^2)^{n-1}\right) + \frac{1}{(2n+1)!}K_{\bar{2}(2n+1)}\left((\lambda F^2)^n i\kappa^\alpha(\partial_\alpha^{\dot{\alpha}}\bar{\phi})\bar{\kappa}_{\dot{\alpha}}\right)\right] \ , \end{aligned} \tag{13}$$

where all coefficients are calculated at the point ϕ. The Lagrangian (13) can be written as a sum $\mathcal{L}_\star = \mathcal{L} + \Delta\mathcal{L}(\lambda)$, here $\mathcal{L}$ is the component Lagrangian for the generic chiral superfield model in $\mathcal{N}=1$ superspace with the action (see e.g. Ref. [9])

$$S[\bar{\Phi},\Phi] = \int d^4x d^4\theta\, K(\bar{\Phi},\Phi) + \int d^4x d^2\theta\, W(\Phi) + \int d^4x d^2\bar{\theta}\, \bar{W}(\bar{\Phi}) \ . \tag{14}$$

In particular, being expanded around the bosonic fields $\phi, \bar{\phi}$, the component form for the Lagrangian (14) is written as

$$\mathcal{L} = -g\frac{1}{2}\partial^{\alpha\dot{\alpha}}\phi\partial_{\alpha\dot{\alpha}}\bar{\phi} + ig\kappa^\alpha\partial_\alpha^{\dot{\alpha}}\bar{\kappa}_{\dot{\alpha}} - K_{1\bar{2}}i\kappa^\alpha\bar{\kappa}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\bar{\phi} + gF\bar{F} \tag{15}$$

$$+K_{2\bar{1}}\kappa^2\bar{F}+K_{1\bar{2}}\bar{\kappa}^2F+W_1F+\bar{W}_{\bar{1}}\bar{F}+W_2\kappa^2+\bar{W}_{\bar{2}}\bar{\kappa}^2+K_{2\bar{2}}\kappa^2$$

where we introduced the Kählerian metrics $g=K_{1\bar{1}}(\bar{\phi},\phi)=\partial^2K(\bar{\phi},\phi)/\partial\phi\partial\bar{\phi}$. Such a form can be directly obtained from (13) as a coefficient at $n=0$.

The obtained representation for the action (13) is complicated and inaccessible even in the classical domain. Now we show that the infinite series (13) can be resummed in a compact expression similar to the standard Zumino's Lagrangian [10] with the deformed Kähler potential and the chiral superpotential plus a finite number of higher dimensional terms with field-dependent couplings. In the analogy with the trick used in the papers [8] we introduce "fuzzy fields" controlled by the auxiliary fields $\phi+\tau\sqrt{\lambda}F$ on interval $-1\leq\tau\leq 1$:

$$\mathcal{W}^{(0)}(\phi,F)=\frac{1}{2}\int_{-1}^{1}d\tau W(\phi+\tau\xi)\ ,\quad \xi=\sqrt{\lambda}F\ ,$$

$$\mathcal{K}^{(0)}(\phi,F,\bar{\phi})=\frac{1}{2}\int_{-1}^{1}d\tau K(\phi+\tau\xi,\bar{\phi})\ ,\quad \mathcal{K}^{(1)}(\phi,F,\bar{\phi})=\frac{1}{2}\int_{-1}^{1}d\tau\tau K(\phi+\tau\xi,\bar{\phi})\ ,$$

$$\mathcal{K}^{(-1)}(\phi,F,\bar{\phi})=\frac{1}{2}\int_{-1}^{1}d\tau\frac{\partial}{\partial\tau}(\tau\cdot K(\phi+\tau\xi,\bar{\phi}))\ . \tag{16}$$

Then (13) can be rewritten in a compact form:

$$\begin{aligned}\mathcal{L}_\star= &\ \bar{W}_{\bar{1}}\bar{F}+\bar{W}_{\bar{2}}\bar{\kappa}^2+F\mathcal{W}_1^{(0)}+\kappa^2\mathcal{W}_2^{(0)}+(\bar{F}F+i\kappa^\alpha\partial_\alpha^{\dot{\alpha}}\bar{\kappa}_{\dot{\alpha}})\mathcal{K}_{1\bar{1}}^{(0)}\\ &+\kappa^2\bar{F}\mathcal{K}_{2\bar{1}}^{(0)}+\bar{\kappa}^2F\mathcal{K}_{1\bar{2}}^{(0)}+\Box\bar{\phi}\mathcal{K}_{\bar{1}}^{(-1)}+\sqrt{\lambda}\kappa^2\Box\bar{\phi}\mathcal{K}_{2\bar{1}}^{(1)}+\tfrac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi}\mathcal{K}_{\bar{2}}^{(-1)}\\ &+i\kappa^\alpha(\partial_\alpha^{\dot{\alpha}}\bar{\phi})\bar{\kappa}_{\dot{\alpha}}\mathcal{K}_{1\bar{2}}^{(0)}+\sqrt{\lambda}\kappa^2\tfrac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi}\mathcal{K}_{2\bar{2}}^{(1)}+\kappa^2\bar{\kappa}^2\mathcal{K}_{2\bar{2}}^{(0)}\ .\end{aligned} \tag{17}$$

It is quite remarkable that the deformation encoded by new geometric quantities which look like the "metric" $\mathcal{K}_{1\bar{1}}^{(0)}$, "connection" $\mathcal{K}_{2\bar{1}}^{(0)}$ and the "curvature" $\mathcal{K}_{2\bar{2}}^{(0)}$ in the smearing target space. But there is no any certainty that this quantities are really consistent among themselves and correspond to some geometrical structure of the target space manifold. It is easy to see that we can rewrite (17) in the canonical form with a proper kinetic term for the scalars $\partial^{\alpha\dot{\alpha}}\phi\partial_{\alpha\dot{\alpha}}\bar{\phi}\mathcal{K}_{1\bar{1}}^{(0)}$ but, due to the extra dependence of $\mathcal{K}^{(0)}(\phi,F,\bar{\phi})$ of the auxiliary field F, there will be new terms containing one derivative of the auxiliary field $\partial^{\alpha\dot{\alpha}}F$. At the limit $\lambda\to 0$ this terms will vanish. This is the great difference between (15) and (17).

Now consider generic nonanticommuting supersymmetric sigma-model (i.e. the model without superpotential W but with arbitrary Kahlerian potential K). It was shown in Ref. [7] that for $D=2$, $\mathcal{N}=2$ nonanticommuting sigma-model the component action infinite series can be resummed to a very simple and clear form. Let's consider such possibility for $D=4$, $\mathcal{N}=\frac{1}{2}$ nonanticommuting sigma-model. In the linear approximation on λ the Lagrangian (13) after introducing a new metric $\tilde{g}=g+\frac{\lambda}{6}F^2K_{3\bar{1}}$ can be rewritten as follows

$$\mathcal{L}_\star=-\frac{1}{2}\partial^{\alpha\dot{\alpha}}\phi\partial_{\alpha\dot{\alpha}}\bar{\phi}(g+\frac{\lambda}{2}F^2K_{3\bar{1}}+\frac{\lambda}{3}F\kappa^2K_{4\bar{1}})+(F\bar{F}+i\kappa^\alpha\partial_\alpha^{\dot{\alpha}}\bar{\kappa}_{\dot{\alpha}})\tilde{g}+i\kappa^\alpha(\partial_\alpha^{\dot{\alpha}}\bar{\phi})\bar{\kappa}_{\dot{\alpha}}\tilde{g}_{\bar{1}} \tag{18}$$

$$+\bar{F}\kappa^2\tilde{g}_1+F\bar{\kappa}^2\tilde{g}_{\bar{1}}+\kappa^2\bar{\kappa}^2\tilde{g}_{1\bar{1}}+\frac{\lambda}{6}F\kappa^2\cdot\partial^{\alpha\dot{\alpha}}(\partial_{\alpha\dot{\alpha}}\bar{\phi}K_{3\bar{1}})+\frac{1}{4}F^2\cdot\partial^{\alpha\dot{\alpha}}(\partial_{\alpha\dot{\alpha}}\bar{\phi}K_{2\bar{1}})\ .$$

The equation of motion for field F following from this Lagrangian is $F\tilde{g}+\kappa^2\tilde{g}_1=0$ and at that time two last terms $\sim\kappa^4$ that vanish. This allows to note that the expression

$(g + \frac{\lambda}{2}F^2K_{3\bar{1}} + \frac{\lambda}{3}F\kappa^2K_{4\bar{1}})$ become equal $\tilde{g}$. Thus we see that Lagrangian (18) in the first order on λ is one to one correspond to the Zumino Lagrangian with the metric $\tilde{g}$. We point out that such a consideration is true only $W = 0$ and for a singlet fermionic field. In accordance to Ref. [7] one can verify that the action given by Eq. (13) at $W = 0$ and $\bar{W} = 0$ in all orders on λ can be rewritten in the form Eq. (18).

Next we discuss elimination of the auxiliary fields $F, \bar{F}$ from the component Lagrangian (13) keep in mind the task investigate the structure of classical vacua. The Lagrangian (13) is linear in $\bar{F}$ but strongly nonlinear in F. Therefore it is difficult to expect that we obtain the exact solution on F and $\bar{F}$ but we can perturbatively find several first corrections to the scalar potential and to the scalar - fermion interaction terms. In particular, the scalar potential is the most important object for studying the possible vacua of the theory and a symmetry breaking mechanism. Let's consider only space-time independent vacuum expectation values for the scalar and fermionic physical fields. We suppose that $F = F_0 + F_1 + \cdots\ ,\quad \bar{F} = \bar{F}_0 + \bar{F}_1 + \cdots$, where F_0 and $\bar{F}_0$ are the solutions for auxiliary fields equations of motion in undeformed model and $F_n \sim \lambda^n$, $\bar{F}_{\bar{n}} \sim \lambda^{\bar{n}}$ are the corrections. Substituting this expansion into the Lagrangian (13) and keeping only linear in λ terms without derivatives we obtain first corrections to the auxiliary fields. This gives us, in addition to the ordinary potential U, a linearly dependent on λ correction

$$\Delta U_1(\lambda) = g(F_1\bar{F}_0 + \bar{F}_1F_0) + F_1(W_1 + K_{1\bar{2}}\bar{\kappa}^2) + \bar{F}_1(\bar{W}_{\bar{1}} + K_{2\bar{1}}\kappa^2) + \frac{\lambda}{6}F_0^2K_{4\bar{2}}\kappa^2\bar{\kappa}^2 \qquad (19)$$

$$+\frac{\lambda}{6}F_0^2\bar{F}_0(K_{4\bar{1}}\kappa^2 + F_0K_{3\bar{1}}) + \frac{\lambda}{6}F_0^2(W_4\kappa^2 + F_0W_3) + \frac{\lambda}{6}F_0^3K_{3\bar{2}}\bar{\kappa}^2\ .$$

As a result, we finally obtain that the potential U is given by a series of additional terms dependent on λ. Considering the expressions (15, 19) one can see that the full potential as a function of the scalar fields and fermionic condensate $\langle\kappa^2\rangle$ can be as positive as negative defined depending on concrete forms of the Kählerian and chiral superpotentials. It means that at nonvanishing λ the potential possesses a possibility to get a minimum, though the initial potential (15) has none minimum. Therefore one can expect some kind of symmetry breaking in the model under consideration.

To summarize, we have considered the supersymmetric generic chiral superfield model on $\mathcal{N} = \frac{1}{2}$ nonanticommutative superspace. This model is given in terms of arbitrary Kählerian potential, chiral and antichiral superpotentials. We have developed a general procedure for deriving the component structure of the model and obtained the component action in the explicit form as a infinite series in the nonanticommutativity parameter. This series is summed up into compact expression using the specific integral representations. It was shown that the additional "deformed" part of the action allows a perturbative translation invariant solution for the auxiliary fields equations of motion. Leading corrections to nondeformed potential are calculated. The results obtained can be applied to studying a wide class of various $\mathcal{N} = \frac{1}{2}$ chiral superfield models including supersymmetric sigma-models and models with different chiral and antichiral superpotentials.

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Twist quantization generated by maximal Jordanian r-matrix for the exceptional Lie algebra g_2

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Abstract

The formulae for twist quantization of g_2 are given, corresponding to the maximal Jordanian $r-$matrix described by the solution of classical YB equation with support in the 8-dimensional Borel subalgebra of g_2. We present the chain of twists consists of the four factors describing the four steps of quantization: Jordanian twist, the two twist factors extending Jordanian twist and the deformed Jordanian or in second variant additional Abelian twist. The explicit formulae for twisted coproducts and the choice of proper nonlinear basis are given in [1].

1 Introduction

In this paper we shall consider the basic nonstandard quantum deformations of complex exceptional Lie algebra g_2. There are four complex semisimple Lie algebras of rank 2, given by $A_2 \simeq sl(3)$, $D_2 \simeq o(4) = o(3) \oplus o(3)$, $B_2 \simeq C_2 \simeq o(5) \simeq sp(4)$ and g_2, with 8, 6, 10 and 14 generators respectively. The 8-dimensional carrier of classical r-matrices which describe our deformations is equal to the Borel subalgebra $\mathfrak{b}_+(g_2)$ of g_2.

We shall consider in Sect. 2 the Lie algebra g_2 in Cartan-Weyl basis (see e.g. [2]). We present firstly, the important class of triangular r-matrices for g_2, satisfying the classical Yang-Baxter equation (CYBE). It appears that the two-parameter families of such r-matrices have as its carrier algebra the whole 8-dimensional Borel subalgebra $\mathfrak{b}_+(g_2) \subset g_2$. In Sect. 3 we shall recall the general formulae which describe the twist quantization method [3]–[8], and we shall introduce the general twisting function, describing the twist

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quantization procedure for g_2 with the 8-dimensional carrier space for its r-matrix. The explicit formulae for coproducts and the choice of the suitable nonlinear basis of twisted g_2 algebra are given in [1].

The motivation for our work is mainly to present a new mathematical result - interesting class of quantum deformations for an important Lie algebra. On the other side it should be stressed that g_2 algebra recently has attracted attention of physicists in the domain of elementary particle physics and fundamental interactions theory. In particular we recall that:

i) In eleven-dimensional M-theory there were proposed the internal manifolds with g_2 holonomy as a base for the grand unification describing extension of the standard model in particle physics (see e.g. [10]–[13]).

ii) There are four Hurwitz algebras (real numbers R, complex numbers C, quaternions H and octonions O); G_2 acts on seven imaginary octonionic units and describes the automorphism group of the octonion algebra. All applications of exceptional and octonions groups to the description of symmetries in elementary particle physics (see e.g. [14]) is strongly linked therefore with the appearance of G_2 symmetry.

2 Cartan-Weyl Basis of g_2 and Jordanian Type Classical r-Matrices

2.1 Cartan-Weyl basis of g_2

In order to describe Cartan–Weyl basis of g_2 let us introduce the Dynkin diagram for its simple roots $\Pi = \{\alpha_1, \alpha_2\}$:

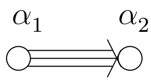

Fig. 1. Dynkin diagram of the Lie algebra g_2.

The corresponding standard $A = (a_{ij})(i, j = 1, 2)$ and symmetric $A^{sym} = (a_{ij}^{sym})_{i,j}$ Cartan matrices are given by

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad A^{sym} = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}. \tag{2.1}$$

The Lie algebra g_2 is generated by the six Chevalley elements e_{α_i}, $e_{-\alpha_i}$, h_{α_i} $(i = 1, 2)$ with the defining relations (see e.g. [2])

$$\begin{aligned} [h_{\alpha_i}, h_{\alpha_j}] &= 0\,, \\ [h_{\alpha_i}, e_{\pm\alpha_j}] &= \pm a_{ij}^{sym}\, e_{\pm\alpha_j}\,, \\ [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij}\, h_{\alpha_i}\,, \\ [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_2}]] &= 0\,, \\ [[[[e_{\pm\alpha_1}, e_{\pm\alpha_2}], e_{\pm\alpha_2}], e_{\pm\alpha_2}], e_{\pm\alpha_2}] &= 0\,. \end{aligned} \tag{2.2}$$

The positive $\Sigma_+(g_2)$ (and total $\Sigma(g_2) = \Sigma_+(g_2)\bigcup(-\Sigma_+(g_2)))$ root systems of g_2 is presented in terms of an orthonormalized basis $\{\epsilon_1,\, \epsilon_2\}$ of a 2-dimensional Euclidian space as follows

$$\Sigma_+(g_2) \;=\; \Big\{\sqrt{3}\epsilon_1,\, \epsilon_2,\, \frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{1}{2}\epsilon_2,\, \frac{\sqrt{3}}{2}\epsilon_1 \pm \frac{3}{2}\epsilon_2\Big\} \tag{2.3}$$

where the simple roots are given by $\alpha_1 = \frac{\sqrt{3}}{2}\epsilon_1 - \frac{3}{2}\epsilon_2$ and $\alpha_2 = \epsilon_2$.

For construction of the composite root vectors e_γ, $(\gamma \neq \pm\alpha_1,\, \pm\alpha_2)$, we fix the following normal ordering of the positive root system $\Sigma_+(g_2)$ (see [2])

$$\alpha_1\,, \alpha_1+\alpha_2\,, 2\alpha_1+3\alpha_2\,, \alpha_1+2\alpha_2\,, \alpha_1+3\alpha_2\,, \alpha_2\,, \tag{2.4}$$

which corresponds to "clockwise" ordering for positive roots in Fig. 2 if we start from the root α_1 to the root α_2. For convenience we introduce the short notations

$$e_{k,l} := e_{k\alpha_1+l\alpha_2}\,, \qquad h_{k,l} := kh_{\alpha_1} + lh_{\alpha_2} \tag{2.5}$$

for $k, l = 0, \pm1, \ldots$. According to the ordering (2.4) we set the composite roots generators with suitably chosen numerical coefficients as follows

$$\begin{array}{ll} e_{1,1} = [e_{1,0}, e_{0,1}]\,, & e_{-1,-1} = -[e_{-1,0}, e_{0,-1}]\,, \\ e_{1,2} = [e_{1,1}, e_{0,1}]\,, & e_{-1,-2} = -\frac{3}{4}[e_{0,-1}, e_{-1,-1}]\,, \\ e_{1,3} = [e_{1,2}, e_{0,1}]\,, & e_{-1,-3} = -\frac{3}{4}[e_{0,-1}, e_{-1,-2}]\,, \\ e_{2,3} = [e_{1,3}, e_{1,0}]\,, & e_{-2,-3} = -\frac{3}{4}[e_{-1,0}, e_{-1,-3}]\,. \end{array} \tag{2.6}$$

The complete set of relations for Cartan-Weyl basis of g_2 can be calculated from (2.2) and (2.6).

2.2 Jordanian type classical r-matrices for g_2

Let us consider in the Lie algebra g_2 the maximal root generator $e_{2,3} = e_{2\alpha_1+3\alpha_2}$. The extended Jordanian matrix of maximal order is provided by formula:

$$r_{2,3;2}(\xi) = \xi\,(h_{2,3}\wedge e_{2,3} + e_{1,1}\wedge e_{1,2} + e_{1,3}\wedge e_{1,0})\;. \tag{2.7}$$

In order to obtain the generalizations of the r-matrix (2.7) one can use the theorem by Belavin and Drinfeld which states that the sum of two r-matrices r_1, r_2 is again a classical r-matrix [15] if r_2 has a carrier $L \in g_2$ $(r_2 \in L\otimes L)$ which cocommutes with r_1 (i.e. it is a kernel of the bialgebra cobracket).

The maximal subalgebra in g_2 which is kernel of the Lie bialgebra cobracket determined by the r-matrix (2.7) has the following linear basis

$$L = (h_{0,1}, e_{0,1}, e_{0,-1}, e_{2,3})\,. \tag{2.8}$$

i.e. $[r_{2,3;2}(\xi),\, l\otimes 1 + 1\otimes l] = 0$ $(l \in L)$. From the generators of the subalgebra L one can construct the following five classical r-matrices:

a) $h_{0,1}\wedge e_{0,1}$, b) $h_{0,1}\wedge e_{2,3}$, c) $e_{0,1}\wedge e_{2,3}$, d) $h_{0,1}\wedge e_{0,-1}$, e) $e_{0,-1}\wedge e_{2,3}$.

The r-matrices which we shall consider below are obtained as the linear combination of (2.7) and the r-matrices a) and b). One can show that the results of addition of the

r-matrix (2.7) and the r-matrices c)–e) can be obtained from the previous two cases by suitable automorphisms of the algebra g_2.

It follows that we can consider two r-matrices as basic ones, or more explicitly:

$$r_1 = \alpha\, h_{0,1} \wedge e_{0,1} + \xi\,(h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0})\ , \tag{2.9}$$

$$r_2 = \beta\, h_{0,1} \wedge e_{2,3} + \xi\,(h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0})\ , \tag{2.10}$$

where ξ, α, β are arbitrary.

One can raise the question whether the classical r-matrices (2.9,b) can be extended to carrier space containing also the generators belonging to $\mathfrak{b}_-$. Unfortunately such an extension, which can not be eliminated by the inner automorphism of g_2, is not possible from purely algebraic reason. One can show that there does not exist an even dimensional subalgebra of g_2, with dimension ten (two extra generators from $\mathfrak{b}_-$), which extends the full Borel subalgebra $\mathfrak{b}_+$. In fact, the consideration of classical r-matrices with the carrier in both Borel subalgebras of g_2 which however are not simultaneously the classical r-matrices for $sl(3)$ subalgebra is an interesting problem to study, going beyond the scope of the present paper.

Below we shall consider the quantization of g_2 in the four steps, corresponding to the quantization of the following sequence of r-matrices:

i) Jordanian twist quantization

$$r_J = \xi\, h_{2,3} \wedge e_{2,3}\,. \tag{2.11}$$

ii) Two extended Jordanian twist quantizations

$$r_{EJ} = \xi\,(h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2})\,, \tag{2.12}$$

$$r_{E'EJ} = \xi\,(h_{2,3} \wedge e_{2,3} + e_{1,1} \wedge e_{1,2} + e_{1,3} \wedge e_{1,0})\,. \tag{2.13}$$

The r-matrix r_{EJ} describes the extended Jordanian twist quantization of the $sl(3)$ subalgebra.

iii) Full twist quantization with additional twist factors describing deformed Jordanian twist (classical r-matrix (2.9)) and the Abelian twist (classical r-matrix (2.10)).

It should be observed that the parameters α, β and ξ occurring in the classical r-matrices (2.9,b) can be rescaled by inner automorphisms of g_2 algebra as well as by the overall scaling of the r-matrices. In particular performing the two-parameter rescaling by Cartan generators (we use the notation $(ad^{\otimes}\, a)A \otimes B \equiv [a, A] \otimes B + A \otimes [a, B]$).

$$\exp[ad^{\otimes}(c_1 h_{1,0} + c_2 h_{0,1})]r_1 = e^{\frac{1}{2}c_1} r_1\,, \tag{2.14}$$

$$\tag{2.15}$$

$$\exp[ad^{\otimes}(c_1 h_{1,0} + c_2 h_{0,1})]r_2 = \tag{2.16}$$

$$= e^{(-\frac{1}{2}c_1+\frac{1}{3}c_2)}\,\beta\, h_{0,1} \wedge e_{0,1} + e^{\frac{1}{2}c_1} r_{E'EJ} \tag{2.17}$$

we see that while the parameter α remains unchanged, the parameters β and ξ can be rescaled e.g. to unity. In order to modify the parameter α we can employ the overall scaling of the r-matrix. We see therefore, that similarly like in the case of Jordanian deformation of $sl(2)$ or κ-deformation of Poincaré algebra, the deformations with different values of the parameters α, β and ξ are mathematically equivalent (provided $\alpha \neq 0, \beta \neq 0, \xi \neq 0$) but distinguishable if applied to physical models.

3 Twist Quantization Method and the General Twist Functions for $\mathbf{g}_2$

3.1 Quantum deformations by twisting coproducts of universal enveloping algebras

Consider the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as a Hopf algebra with the comultiplication $\Delta^{(0)}$ generated by the primitive coproduct in $\mathfrak{g}$. The parametric invertible solution $\mathcal{F}(\xi) = \sum f_i^{(1)} \otimes f_i^{(2)} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ of the twist equations [4]

$$\mathcal{F}_{12}(\Delta^{(0)} \otimes 1)(\mathcal{F}) = \mathcal{F}_{23}(1 \otimes \Delta^{(0)})(\mathcal{F}), \tag{3.1}$$

$$(\epsilon \otimes \mathrm{id})(\mathcal{F}) = (\mathrm{id} \otimes \epsilon)(\mathcal{F}) = 1 \otimes 1\,, \tag{3.2}$$

defines the deformed (twisted) Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$ with the unchanged multiplication, unit and counit (as in $U(\mathfrak{g})$), the twisted comultiplication and antipode defined by the relations

$$\Delta_{\mathcal{F}}(u) \;=\; \mathcal{F}\Delta^{(0)}(u)\mathcal{F}^{-1}\,, \quad u \in U(\mathfrak{g}), \tag{3.3}$$

$$\tag{3.4}$$

$$S_{\mathcal{F}}(u) \;=\; v\, S^{(0)}(u)\, v^{-1}, \quad v = \sum f_i^{(1)} S^{(0)}(f_i^{(2)})\,. \tag{3.5}$$

The twisted algebra $U_{\mathcal{F}}(\mathfrak{g})$ is triangular, with the universal $\mathcal{R}$-matrix

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{F}^{-1}\,, \tag{3.6}$$

which belongs to some extension of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. When $\mathcal{F}$ is a smooth function of ξ and $\lim_{\xi\to 0} \mathcal{F} = 1 \otimes 1$ then in the neighborhood of the origin the $\mathcal{R}$-matrix can be presented as

$$\mathcal{R}_{\mathcal{F}} = 1 \otimes 1 + \xi r_{\mathcal{F}} + o(\xi)\,, \tag{3.7}$$

where $r_{\mathcal{F}}$ is the skewsymmetric classical r-matrix corresponding to the twist $\mathcal{F}$. Let us write explicitly the r-matrix as follows:

$$r_{\mathcal{F}} = a^{ij}\, I_i \wedge I_j \;. \tag{3.8}$$

Then we obtain

$$\mathcal{F} = 1 \otimes 1 + \xi\, \tilde{a}^{ij} I_i \otimes I_j + \mathcal{O}(\xi)\,, \tag{3.9}$$

where $a^{ij} = \frac{1}{2}(\tilde{a}^{ij} - \tilde{a}^{ji})$.

By a nonlinear change of basis in $U(\mathfrak{g})$ one can modify the twisted coproducts and locate part of the deformation in the algebraic sector.

3.2 Twist deformations for $U(g_2)$ Hopf algebra

Our aim is to construct explicitly such a sequence of the twist deformations $U_{\mathcal{F}}(g_2)$ of the algebra $U(g_2)$ that will lead to the largest possible carrier subalgebra for the corresponding classical r-matrices. The final element of the corresponding twists will be the full chain of extended twists whose carrier coincides with the Borel subalgebra of g_2. The peculiarity of the chain twist deformation is that the deformed algebra can be twisted step by step by the consecutive twisting factors with their specific properties. One of the important aims will be also the construction of proper nonlinear basis in $U(g_2)$. Indeed, on each step we shall construct the nonlinear basis in which the costructure of the Hopf algebra $U_{\mathcal{F}}(g_2)$ becomes more transparent.

In Sect. II we have presented the sequence of classical r-matrices for $U(g_2)$ (see (2.11), (2.12,b) and (2.9,b)). The quantization of these classical r-matrices is performed as follows.

a) Firstly we introduce the standard Jordanian twist quantizing the classical r-matrix (2.11), corresponding to the long root $2\alpha_1 + 3\alpha_2$ in g_2. We have the following twisting element [16]

$$\mathcal{F}_J = e^{h_{2,3}\otimes\sigma_{2,3}} = e^{H\otimes\sigma}\,, \tag{3.10}$$

where

$$H = h_{2,3} = 2h_{1,0} + 3h_{0,1}\,, \qquad \sigma = \ln(1 + e_{2,3})\,. \tag{3.11}$$

b) There are four types of the extension twisting factors that can be applied to $U_J(g_2)$ [8]:

$$\mathcal{F}_{E_+} = e^{e_{1,1}\otimes e_{1,2}e^{-\frac{1}{2}\sigma}}\,, \tag{3.12}$$

$$\mathcal{F}_{E_-} = e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}\,, \tag{3.13}$$

$$\widetilde{\mathcal{F}}_{E_+} = e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}\,, \tag{3.14}$$

$$\widetilde{\mathcal{F}}_{E_-} = e^{-e_{1,0}\otimes e_{1,3}e^{-\frac{1}{2}\sigma}}\,. \tag{3.15}$$

They can be composed to provide the following four types of the two-element extensions of (3.10)

$$\mathcal{F}_{E_{++}} = e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}e^{e_{1,1}\otimes e_{1,2}e^{-\frac{1}{2}\sigma}}\,, \tag{3.16}$$

$$\mathcal{F}_{E_{+-}} = e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}\,, \tag{3.17}$$

$$\mathcal{F}_{E_{-+}} = e^{-e_{1,0}\otimes e_{1,3}e^{-\frac{1}{2}\sigma}}e^{e_{1,1}\otimes e_{1,2}e^{-\frac{1}{2}\sigma}}\,, \tag{3.18}$$

$$\mathcal{F}_{E_{--}} = e^{-e_{1,0}\otimes e_{1,3}e^{-\frac{1}{2}\sigma}}e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}\,. \tag{3.19}$$

One can note that exponential factors in the twists (3.16) commute with each other, and do not describe themselves the solutions of twist equations (3.1,3.2) with primitive coproduct $\Delta^{(0)}$. The four twists (3.16) lead to the equivalent Hopf algebras however their coalgebra relations differ considerably. The most elegant result is obtained when the extension is chosen as follows

$$\mathcal{F}_E := \mathcal{F}_{E_{-+}} = \mathcal{F}_{E_-}\mathcal{F}_{E_+} = e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}\,, \tag{3.20}$$

with the extended twist

$$\mathcal{F}_{EJ} := e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}e^{H\otimes\sigma}\,. \tag{3.21}$$

It should be added that the products of twists $\mathcal{F}_{E_\pm}\mathcal{F}_J$ describe the twist quantization of $sl(3)$ subalgebra.

c) The additional Abelian twist factor ($h \equiv 3h_{0,1}$)

$$\mathcal{F}_A = e^{h\otimes\sigma}\,, \tag{3.22}$$

that produces a kind of a "rotation" in the root space of g_2, can enlarge the extended twist (3.21):

$$\mathcal{F}_{AEJ} := e^{h\otimes\sigma} e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}\, e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}\, e^{H\otimes\sigma}\,. \tag{3.23}$$

In such a way we obtain the quantization of the classical r-matrix (2.10).

d) We can construct the chain of twists (see e.g. [5, 6]) for g_2 by additionally deforming the twisted $U_{EJ}(g_2)$ by the second link of the chain, which is the Jordanian factor:

$$\mathcal{F}_{J'} = e^{h\otimes\omega} \tag{3.24}$$

with

$$\omega = \ln\left(1 + e_{0,1} + \frac{1}{2}\left(e_{1,2}\right)^2\right)\,. \tag{3.25}$$

This gives the quantization with the largest carrier

$$\mathcal{F}_{J'EJ} := e^{h\otimes\omega} e^{-e_{1,2}\otimes e_{1,1}e^{-\frac{1}{2}\sigma}}\, e^{e_{1,3}\otimes e_{1,0}e^{-\frac{1}{2}\sigma}}\, e^{H\otimes\sigma}. \tag{3.26}$$

The twist function (3.26) describes the quantization of the classical r-matrix (2.9). The twist (3.23) can also form the chain with $\mathcal{F}_{J''} = e^{h\otimes\omega''}$. But the Abelian twist factors $\mathcal{F}_A$ and this new Jordanian factor are related by the formula $\mathcal{F}_{J''}\mathcal{F}_A = \mathcal{F}_{J'}$. This means that for any "rotated" extended twist $\mathcal{F}_{AEJ}$ we get the unique chain (3.26).

The explicit calculations of the twisted coproducts (formulae (3.3-b)) for the twists (3.23) and (3.26) are given in [1]. One obtains very complicated coproduct formulae. In order to simplify them we introduced nonlinear basis, with deformed classical Lie algebra g_2 relations. The general scheme how to introduce nonlinear basis for twisted simple Lie algebras has been presented in [9]. The calculations were quite involved and, in order to simplify them, there were introduced some new formulae for the similarity transformations of tensor products (see Sect. III D in [1]).

4 Final Remarks

The aim of this note is to present the general quantization scheme and to announce the calculations, provided in [1], of explicit formulae describing maximal twist quantization of g_2 Lie algebra. Due to the relation $sl(3) \subset g_2$ these formulae extend the most general ones for twisted $sl(3)$ algebra (see [8]). The Lie algebra generators described by the roots (2.3) belong to 14−dimensional adjoint representation $\{\underline{14}\}$ of g_2 which decomposes under $sl(3)$ (or $su(3)$) as follows

$$\{\underline{14}\} = \{\underline{18}\} + \{\underline{3}\} + \{\overline{\underline{3}}\} \tag{4.1}$$

In particular, the coset space $\frac{G_2}{SU(3)}$ describes the sphere S^6 with torsion. As one of possible applications of our paper could be the new quantum deformation of such six-dimensional sphere.

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Non-singlet Q-deformations of $\mathcal{N}=2$ Gauge Theories

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Abstract

We study a non-anticommutative chiral non-singlet deformation of the $\mathcal{N}=(1,1)$ abelian gauge multiplet in Euclidean harmonic superspace. We present a closed form of the gauge transformations and the unbroken $\mathcal{N}=(1,0)$ supersymmetry transformations preserving the Wess-Zumino gauge, as well as the bosonic sector of the $\mathcal{N}=(1,0)$ invariant action. This contribution is a summary of our main results in `hep-th/0510013`.

Extensions of gauge theories to non-commutative and non-(anti)commutative superspaces are currently of remarkable interest within the high energy physics community, mainly due to their relevance to subjects like string theory (see for example [2, 3] and references therein). Here we focus in a subclass of non-(anti)commutative Euclidean supersymmetric field theories called *Q-deformed*, realized via a Weyl-Moyal product with a bilinear nilpotent Poisson operator, which is constructed in terms of the supercharges,

$$P = -\overleftarrow{Q}^i_\alpha C^{\alpha\beta}_{ik} \overrightarrow{Q}^k_\beta \,. \tag{1}$$

The Moyal product of two superfields is then defined by

$$A \star B = A e^P B \,. \tag{2}$$

The deformation parameters $C^{\alpha\beta}_{ij}$ form a constant tensor which is symmetric under the simultaneous permutation of the Latin and Greek indices, $C^{\alpha\beta}_{ij} = C^{\beta\alpha}_{ji}$. Generically, it breaks the full automorphism symmetry $Spin(4)\times$ O(1,1)$\times$ SU(2) $\equiv$ SU(2)$_\mathrm{L}\times$ SU(2)$_\mathrm{R}\times$ O(1,1)$\times$ SU(2) of the $\mathcal{N}=(1,1)$ superalgebra —O(1,1) and SU(2) being the R-symmetry groups— down to SU(2)$_\mathrm{R}$. An important feature of Q-deformations is the nilpotent nature of the Poisson operator ($P^5=0$) which makes the Moyal product polynomial, ensuring local actions. In virtue of the commutation properties of P with respect to the spinor covriant derivatives,

$$\left[D^\pm_\alpha, P\right] = 0 \,, \qquad \left[\bar{D}^\pm_{\dot\alpha}, P\right] = 0 \,, \qquad \left[D^{\pm\pm}, P\right] = 0, \tag{3}$$

the product (2) breaks $\mathcal{N}=(1,1)$ supersymmetry down to $\mathcal{N}=(1,0)$ while preserving both chirality and Grassmann harmonic analyticity of the involved superfields, as well as the harmonic conditions[1] $D^{\pm\pm}A=0$. Operator (1) can be split as follows,

$$P=-I\overleftarrow{Q}^{i}_{\alpha}\varepsilon^{\alpha\beta}\varepsilon_{ik}\overrightarrow{Q}^{k}_{\beta}-\overleftarrow{Q}^{i}_{\alpha}\hat{C}^{\alpha\beta}_{ik}\overrightarrow{Q}^{k}_{\beta}\,. \tag{4}$$

The first term is $Spin(4)\times$SU(2)-preserving while the second term involves a SU(2) $_{\rm L}\times$ SU(2) constant tensor which is symmetric under the independent permutations of Latin and Greek indices, $\hat{C}^{\alpha\beta}_{ij}=\hat{C}^{\beta\alpha}_{ij}=\hat{C}^{\alpha\beta}_{ji}$. For the generic choice, it fully breaks Euclidean symmetry, SU(2)$_{\rm L}$ and R-symmetry SU(2). Q-deformations induced only by the first term are called *singlet* or *QS-deformations*, whereas those associated with the second term, *non-singlet* or *QNS-deformations*. In this contribution we report important results on dynamical aspects of *QNS-deformations* of the $N=(1,1)$ U(1) vector multiplet in harmonic superspace. The talk is based on paper [1], where detailed calculations are performed and a complete list of references is given.

Gauge transformations The residual gauge transformations of the component fields of the Abelian $\mathcal{N}=(1,1)$ vector multiplet in the WZ gauge can be found from the Q-deformed superfield transformation [8]

$$\delta_\Lambda V^{++}_{\rm WZ}={\rm D}^{++}\Lambda+[V^{++}_{\rm WZ},\Lambda]_\star\,, \tag{5}$$

with $V^{++}_{\rm WZ}$ being the analytic harmonic U(1) superfield gauge connection and Λ the analytic residual gauge parameter satisfying $D^+_\alpha\Lambda=\bar{D}^+_{\dot\alpha}\Lambda=0$. In the left-chiral basis, where $x^{\alpha\dot\alpha}_A=x^{\alpha\dot\alpha}_L-4{\rm i}\theta^{-\alpha}\bar\theta^{+\dot\alpha}$ [9], $V^{++}_{\rm WZ}$ has the following θ-expansion

$$\begin{aligned}V^{++}_{\rm WZ}&=(\theta^+)^2\bar\phi+\bar\theta^+_{\dot\alpha}\left[2\theta^{+\alpha}A^{\dot\alpha}_{\alpha}+4(\theta^+)^2\bar\Psi^{-\dot\alpha}-2{\rm i}(\theta^+)^2\theta^{-\alpha}\partial^{\dot\alpha}_{\alpha}\bar\phi\right]\\&+(\bar\theta^+)^2\Big[\phi+4\theta^+\Psi^-+3(\theta^+)^2D^{--}-{\rm i}(\theta^+\theta^-)\partial^{\alpha\dot\alpha}A_{\alpha\dot\alpha}+\theta^{-\alpha}\theta^{+\beta}F_{\alpha\beta}\\&-(\theta^+)^2(\theta^-)^2\Box\bar\phi+4{\rm i}\,(\theta^+)^2\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha}\Big].\end{aligned} \tag{6}$$

The superparameter $\Lambda_0={\rm i}a+2\theta^{-\alpha}\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}a-{\rm i}(\theta^-)^2(\bar\theta^+)^2\Box\,a$ (being a an arbitrary function of x_L) found for the undeformed and singlet cases, breaks the WZ gauge in the non-singlet case, due to the appearance of an unwanted dependence on the harmonic variables $u^\pm_i$ in the expresions for the gauge variations. It is clear that is imperative to choose a gauge parameter Λ that preserves the WZ gauge, that is, for non-singlet deformations, some correction terms $\Delta\Lambda$ must be added to Λ_0, where

$$\begin{aligned}\Delta\Lambda&=\theta^+_\alpha\bar\theta^+_{\dot\alpha}\,\partial^{\dot\alpha}_{\beta}\,a\,B^{--\alpha\beta}_1+(\bar\theta^+)^2\partial_{\beta\dot\beta}\,a\,A^{\dot\beta}_{\alpha}G^{--\alpha\beta}+(\theta^+)^2(\bar\theta^+)^2\,\Box aP^{-4}\\&+(\bar\theta^+)^2\theta^+_\alpha\,\left[\bar\Psi^{-\dot\beta}\partial_{\beta\dot\beta}\,a\,B^{--\alpha\beta}_2+\bar\Psi^{+\dot\beta}\partial_{\beta\dot\beta}\,a\,G^{-4\alpha\beta}\right]+(\theta^+)^2(\bar\theta^+)^2\,\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_{\beta}\,\bar\phi\,B^{-4\alpha\beta}_3\\&+{\rm i}\,\theta^+_\alpha\theta^-_\beta\,(\bar\theta^+)^2\,\Box\,a\,B^{--\alpha\beta}_1+{\rm i}\,\theta^+_\alpha\theta^-_\gamma\,(\bar\theta^+)^2\,\partial_{\beta\dot\lambda}a\,\partial^{\gamma\dot\lambda}\bar\phi\,\frac{d}{d\bar\phi}B^{--\alpha\beta}_1\,.\end{aligned} \tag{7}$$

The coefficients in (7) are some undetermined functions of harmonics, the field $\bar\phi$ and deformation parameters, calculated by requiring

$$\partial^{++}\delta A_{\alpha\dot\alpha}=0,\quad \partial^{++}\delta\phi=0,\quad (\partial^{++})^2\delta\Psi^-_\alpha=0,\quad (\partial^{++})^3\delta D^{--}=0\,. \tag{8}$$

[1] Giving up chirality and analiticity, it is also possible to use spinor covariant derivatives to construct a nilpotent Poisson operator. We recommend ref.[5] for a deeper treatment of the subject.

The correction term to $\delta_0 V^{++}_{\rm WZ}$ is,

$$\hat{\delta} V^{++}_{\rm WZ} = {\rm D}^{++}\Delta\Lambda + [V^{++}_{\rm WZ}, \Delta\Lambda]_\star \,, \tag{9}$$

and the full WZ preserving gauge transformations are given by $\delta V^{++}_{\rm WZ} = \delta_0 V^{++}_{\rm WZ} + \hat{\delta} V^{++}_{\rm WZ}$. Unfortunately, it is very difficult to find closed solutions of these equations for general deformation parameters, though their perturbative solutions always exist as series expansion. For the general choice of $\hat{C}^{ij}_{\alpha\beta}$, the gauge and susy transformations and the corresponding action are known only to few first orders in the parameters of deformation [6]. Nevertheless, exact solutions can be found for the product structure

$$\hat{C}^{ij}_{\alpha\beta} = b^{ij} c_{\alpha\beta} \,,$$

which correspons to the maximally symmetric non-singlet deformations. The full set of non trivial QNS-deformed gauge transformations laws for the $\mathcal{N}=(1,1)$ vector multiplet in WZ gauge are then

$$\begin{array}{rl} \delta\, A_{\alpha\dot\alpha} = & X \coth X \partial_{\alpha\dot\alpha} a, \quad \delta\,\phi = 2\sqrt{c^2\, b^2}\left(\frac{1 - X\coth X}{X}\right) A^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} a, \\ \delta D_{ij} = & 2{\rm i} b_{ij} c^{\alpha\beta}\partial_{\alpha\dot\alpha}\bar\phi\, \partial^{\dot\alpha}_{\beta} a \,, \quad \delta\Psi^i_\alpha = 2\sqrt{c^2 b^2}\left[2\left(\coth X - \frac{1}{X}\right) - X\right]\bar\Psi^{i\dot\alpha}\partial_{\alpha\dot\alpha} a \,, \end{array} \tag{10}$$

where

$$X = 2\bar\phi\sqrt{b^{ij} b_{ij}\, c^{\alpha\beta}_{\alpha\beta}}. \tag{11}$$

Detailed calculations of these transformations laws are carried out in [1]. Having the explicit QNS-deformed gauge transformations, one can deduce a minimal Seiberg-Witten-like map which puts these transformations into the standard undeformed form

$$\begin{array}{l} \Psi^i_\alpha = \widetilde{\Psi}^i_\alpha + 2\sqrt{c^2 b^2}\left[2\left(\coth X - \dfrac{1}{X}\right) - X\right]\bar\Psi^{i\dot\alpha}\widetilde{A}_{\alpha\dot\alpha} \,, \quad D_{ij} = \widetilde{D}_{ij} + 2{\rm i} b_{ij} c^{\alpha\beta}\partial_{\alpha\dot\alpha}\bar\phi\, \widetilde{A}^{\dot\alpha}_{\beta} \,, \\ A_{\alpha\dot\alpha} = \widetilde{A}_{\alpha\dot\alpha}\, X\coth X, \quad \phi = \widetilde{\phi} + \widetilde{A}^2\,\sqrt{c^2\, b^2}\; X\coth X\left(\dfrac{1 - X\coth X}{X}\right). \end{array} \tag{12}$$

For the fields with tilde we obtain the standard transformations

$$\delta\widetilde{A}_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} a \,, \qquad \delta\widetilde{\phi} = 0 \,, \qquad \delta\widetilde{D}^{ij} = 0 \,, \qquad \delta\widetilde{\Psi}^k_\alpha = 0 \,.$$

The gauge field strength $F_{\alpha\beta} = 2{\rm i}\partial_{(\alpha\dot\alpha}A^{\dot\alpha}_{\beta)}$ which is non-covariant with respect to the deformed transformations is redefined under the transformation $A_{\alpha\dot\alpha} \to \widetilde{A}_{\alpha\dot\alpha}$ as

$$F_{\alpha\beta} = \widetilde{F}_{\alpha\beta} X\coth X + 4{\rm i}\sqrt{b^2\, c^2}\widetilde{A}_{(\beta\dot\alpha}\partial^{\dot\alpha}_{\alpha)}\bar\phi\left(\coth X - \frac{X}{\sinh^2 X}\right), \quad \text{where} \quad \widetilde{F}_{\alpha\beta} = 2{\rm i}\partial_{(\alpha\dot\alpha}\widetilde{A}^{\dot\alpha}_{\beta)}.$$

Unbroken susy transformations Unbroken supersymmetry is realized on $V^{++}_{\rm WZ}$ as

$$\delta V^{++}_{\rm WZ} = \left(\epsilon^{+\alpha}\partial_{+\alpha} + \epsilon^{-\alpha}\partial_{-\alpha}\right) V^{++}_{\rm WZ} - {\rm D}^{++}\Lambda_c - \left[V^{++}_{\rm WZ}\,,\, \Lambda_c\right]_\star \,, \tag{13}$$

where the star bracket, like in the previous consideration, is defined via the non-singlet Poisson structure with the deformation matrix $\hat{C}^{ij}_{\alpha\beta} = b^{ij} c_{\alpha\beta}$ and Λ_c is the compensating gauge parameter which is necessary for preserving WZ gauge. As in the QNS-deformed gauge transformations case, variations obtained using the original Λ_ϵ for undeformed and singlet cases (see[8]), violate the WZ gauge due to an unbalanced apperance of harmonic and Grassmann variables [1], so one is led to properly modify Λ_ϵ . Thus we define

$$\Lambda_c = \Lambda_\epsilon + F_\epsilon \tag{14}$$

We denote by $\check{\delta} V^{++}_{\mathrm{WZ}}$ the lowest-order non-singlet part of the transformations coming from the star commutator in (13) using Λ_ϵ, and rewrite (13) in the following way

$$\delta V^{++}_{\mathrm{WZ}} = \check{\delta} V^{++}_{\mathrm{WZ}} - \mathrm{D}^{++} F_\epsilon - \left[V^{++}_{\mathrm{WZ}} \,,\, F_\epsilon \right]_\star \tag{15}$$

with

$$\begin{aligned} \Lambda_\epsilon = \;& 2(\epsilon^-\theta^+)\bar\phi + \bar\theta^-_{\dot\alpha}\left[4\mathrm{i}(\epsilon^-\theta^+)\theta^-_\alpha\partial^{\alpha\dot\alpha}\bar\phi - 2\epsilon^-_\alpha A^{\alpha\dot\alpha} + 4(\epsilon^-\theta^+)\bar\Psi^{-\dot\alpha}\right] + \\ & (\bar\theta^+)^2\Big[2(\epsilon^-\Psi^-) + 2\mathrm{i}\epsilon^{-\alpha}\theta^{-\beta}\partial^{\dot\alpha}_\beta A_{\alpha\dot\alpha} - 2(\epsilon^-\theta^+)(\theta^-)^2\Box\bar\phi \\ & \quad +4\mathrm{i}(\epsilon^-\theta^+)\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha} + 2(\epsilon^-\theta^+)D^{--}\Big]\,. \end{aligned} \tag{16}$$

The additional compensating gauge parameter intended for restoring the WZ gauge with the minimal set of terms needed to eliminate the improper harmonic and Grassmann dependence amounts to the following form

$$\begin{aligned} F_\epsilon = \;& \theta^{+\alpha} f^-_\alpha + \bar\theta^+_{\dot\alpha}\left[\bar g^{-\dot\alpha} + 2\mathrm{i}\theta^-_\alpha\,\theta^{+\beta}\partial^{\alpha\dot\alpha} f^-_\beta + \theta^{+\alpha} b^{--\dot\alpha}_\alpha + (\theta^+)^2\,\bar g^{(-3)\dot\alpha}\right] \\ + \;& (\bar\theta^+)^2\Big[g^{--} - (\theta^-)^2\theta^{+\alpha}\Box f^-_\alpha + \mathrm{i}\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar g^{-\dot\alpha} + \mathrm{i}\theta^{+\alpha}\theta^{-\beta}\partial^{\dot\alpha}_\beta b^{--}_{\alpha\dot\alpha} + \theta^{+\alpha} f^{(-3)}_\alpha \\ & +\mathrm{i}(\theta^+)^2\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar g^{(-3)\dot\alpha} + (\theta^+)^2\,X^{(-4)}\Big]\,. \end{aligned} \tag{17}$$

Requiring the elimination of terms with unbalanced Grassmann variables and

$$\partial^{++}\delta\bar\phi = 0\,,\; \left(\partial^{++}\right)^2\delta\bar\Psi^-_{\dot\alpha} = 0\,,\; \partial^{++}\delta A_{\alpha\dot\alpha} = 0\,,\; (\partial^{++})^2\delta\Psi^-_\alpha = 0\,,\; (\partial^{++})^3\delta D^{--} = 0\,, \tag{18}$$

we can explicitly find components of F_ϵ and restore the correct $\mathcal{N} = (1,0)$ supersymmetry transformations preserving WZ gauge. The full set of these transformations together with the full supersymmetric action will be given[2] in [10]. Here we show the simplest subalgebra

$$\begin{aligned} \delta\bar\phi \;&= 0\,, \quad \delta A_{\alpha\dot\alpha} = 8\bar\phi\epsilon^{i\beta}\bar\Psi^j_{\dot\alpha}\,b_{ij}c_{\alpha\beta} + 2\epsilon^k_\alpha\bar\Psi_{k\dot\alpha}\,X\coth X\,, \\ \delta\bar\Psi^i_{\dot\alpha} \;&= \left[\tfrac{2\mathrm{i}}{\sqrt{b^2c^2}}\cosh X\,\sinh X\,c^{\alpha\beta}b^{ij} - \mathrm{i}\cosh^2 X\,\varepsilon^{\alpha\beta}\varepsilon^{ij}\right]\epsilon_{j\beta}\partial_{\alpha\dot\alpha}\bar\phi\,. \end{aligned} \tag{19}$$

These variations form an algebra which is closed modulo a gauge transformation with the composite parameter $a_c = -2\mathrm{i}(\epsilon\cdot\eta)\bar\phi$:

$$[\delta_\epsilon, \delta_\eta]\,\bar\phi = 0\,, \quad [\delta_\epsilon, \delta_\eta]\,\bar\Psi^j_{\dot\alpha} = 0\,, \quad [\delta_\epsilon, \delta_\eta]\,A_{\alpha\dot\alpha} = -2\mathrm{i}(\epsilon\cdot\eta)\,(X\coth X)\,\partial_{\alpha\dot\alpha}\bar\phi\,.$$

Bosonic action Now we present the bosonic sector of the $\mathcal{N} = (1,0)$ gauge invariant action in components. The QNS-deformed action for the $\mathcal{N} = (1,1)$ U(1) gauge theory in harmonic superspace [9], in the form most appropriate for our purposes, is written in the same way as in the QS-deformed case [8]

$$S = \frac{1}{4}\int d^4x_L\,d^4\theta\,du\,\mathcal{W}\star\mathcal{W} = \frac{1}{4}\int d^4x\,d^4\theta\,du\,\mathcal{W}^2\,. \tag{20}$$

[2] In fact, it is of no actual necessity to explicitly know these transformations, since our procedure of deriving the action is manifestly $\mathcal{N} = (1,0)$ supersymmetric by construction [1].

Here $\mathcal{W}$ is the covariant superfield strength

$$\mathcal{W} = -\frac{1}{4}(\bar{D}^+)^2 V^{--} \equiv \mathcal{A}(x_L, \theta^+, \theta^-) + \bar{\theta}^+_{\dot{\alpha}} \tau^{-\dot{\alpha}}(x_L, \theta^+, \theta^-) + (\bar{\theta}^+)^2 \tau^{--}(x_L, \theta^+, \theta^-)\,, \quad (21)$$

and V^{--} is the non-analytic harmonic connection related to $V^{++}_{\rm WZ}$ by the harmonic flatness condition

$$\mathrm{D}^{++} V^{--} - \mathrm{D}^{--} V^{++}_{\rm WZ} + \left[V^{++}_{\rm WZ}\,,\, V^{--} \right]_\star = 0\,. \quad (22)$$

The whole effect of the considered deformation in the above action comes from the structure of $\mathcal{W}$ due to the presence of the star commutator in the equation (22) defining V^{--}. As a consequence of the latter, (21) satisfies the condition

$$\mathrm{D}^{++}\mathcal{W} + \left[V^{++}_{\rm WZ}\,,\, \mathcal{W} \right]_\star = 0\,. \quad (23)$$

It is not hard to prove that the only contribution to the entire action is the superfield $\mathcal{A}$ in (21) (see [8]). Thus, the invariant action is reduced to

$$S = \frac{1}{4} \int d^4x\, d^4\theta\, du\, \mathcal{A}^2\,. \quad (24)$$

Once again we refer to [1] for details of the calculations leading to the relevant components of $\mathcal{A}$. Finally the bosonic limit of the action, after performing the minimal SW map (12), is

$$\begin{aligned} S_{bos} = \int d^4x \Big[& -\tfrac{1}{2}\tilde{\phi}\Box\bar{\phi} - \tfrac{1}{2}(b^2\, c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}}\bar{\phi}\partial^{\alpha\dot{\alpha}}\bar{\phi}\Box\bar{\phi} + \tfrac{1}{4}\tfrac{\widetilde{D}^2}{\cosh^2 X} \\ & -\tfrac{1}{16}\widetilde{F}^2 \cosh^2 X + \tfrac{1}{4} b^2 (c\cdot\widetilde{F})^2 \bar{\phi}^2 \tfrac{\sinh^2 X}{X^2} + \tfrac{1}{2}\bar{\phi}(b\cdot\widetilde{D})(c\cdot\widetilde{F})\tfrac{\tanh X}{X}\Big]. \end{aligned} \quad (25)$$

This action is invariant under the standard abelian gauge transformations. Turning off the deformation parameters we are left with the usual bosonic sector of the undeformed action. Performing the further field redefinition

$$d^{ij} = \frac{1}{\cosh^2 X}\widetilde{D}^{ij} + \bar{\phi}(c\cdot\widetilde{F})b^{ij}\frac{\tanh X}{X}, \quad \varphi = \frac{1}{\cosh^2 X}\left[\tilde{\phi} + (b^2c^2)^{3/2}(\partial\bar{\phi})^2 \tanh X\right],$$

the bosonic action can be transformed into a simple form

$$S_{bos} = \int d^4x\, \cosh^2 X \left[-\frac{1}{2}\varphi\Box\bar{\phi} + \frac{1}{4} d^{ij} d_{ij} - \frac{1}{16}\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta} \right]. \quad (26)$$

From this expression it is obvious that we cannot disentangle the interaction between the gauge field and $\bar{\phi}$ by any field redefinition. This is similar to the singlet case [8, 7], where a scalar factor $(1+4I\bar{\phi})^2$ appears instead of $\cosh^2 X$. Note that the bosonic action (26) involves only squares c^2 and b^2, so it preserves space-time $Spin(4)$= $\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}$ symmetry and SU(2) R-symmetry as in the singlet case. This property is similar to what happens in the deformed Euclidean $\mathcal{N} = (1/2, 1/2)$ Wess-Zumino model where the deformation parameter $C^{\alpha\beta}$ also appears squared [3]. However, we know that the fermionic completion of (26) will explicitly include both $c^{\alpha\beta}$ and b^{ik} [10], so these two symmetries are broken in the total action. This feature also compares with the breaking of Lorentz symmetry in the deformed $\mathcal{N} = (1/2, 1/2)$ gauge theory action, due to fermionic terms [3]. Though the string interpretation of the QS-deformation is known [8], the possible stringy origin of the non-singlet case —e.g. as some special $\mathcal{N} = 4$ superstring background— is still unclear.

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A gravity theory on noncommutative spaces

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Abstract

A deformation of the algebra of diffeomorphisms is constructed for canonically deformed spaces with constant θ. The algebra remains the same, while the comultiplication rule (Leibniz rule) changes. Based on this deformed algebra a covariant tensor calculus is derived and the concepts like metric, covariant derivatives, curvature and torsion are introduced. This enables one to construct a deformation of the commutative Einstein-Hilbert action which is invariant under the deformed diffeomorphisms.

1 Introduction

The talk given by the author is based on the common work with Paolo Aschieri, Christian Blohmann, Frank Meyer, Peter Schupp and Julius Wess [1].

The concept of symmetry is very important in physics. Classically, symmetries are described by Lie groups or Lie algebras and the physical space is the representation space of the symmetry algebra. Therefore, the question arises if one can introduce the noncommutative (deformed)[1] spaces as representation spaces of some symmetry algebras. It turns out that this is possible in the framework of Hopf algebras and quantum groups [2].

Here we analyse one special example of noncommutative spaces, the θ-deformed space. It was generally believed until recently that this space has no quantum group symmetry acting on it. However, in [3], [4] the quantum group symmetry (given in terms of the θ-deformed Poincaré Hopf algebra) was constructed. Going one step further, one can analyse the θ-deformed diffeomorphism symmetry [1], [5]. Then the θ-deformed Poincaré Hopf algebra is a sub(Hopf)algebra of this larger symmetry algebra.

The θ-deformed space is defined by

$$[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is real antisymmetric constant. The star product ($\star$-product) is the deformation of the of the usual pointwise multiplication and it encodes the information

[1]"Noncommutative" and "deformed" will be used as synonyms from now on, whereas "classical", "undeformed" and "usual" will be synonyms for "commutative".

about the noncommutativity (deformation). In the case of θ-deformed space, the $\star$-product is given by the Moyal-Weyl $\star$-product [7]

$$
\begin{aligned}
f \star g\,(x) &= e^{\frac{i}{2}\frac{\partial}{\partial x^\rho}\theta^{\rho\sigma}\frac{\partial}{\partial y^\sigma}} f(x)g(y)\Big|_{y\to x} && (2)\\
&= \sum_{n=1}^{\infty}\Big(\frac{i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}f(x)\Big)\Big(\partial_{\sigma_1}\dots\partial_{\sigma_n}g(x)\Big)\\
&= fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) - \frac{1}{8}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}(\partial_{\rho_1}\partial_{\rho_2}f)(\partial_{\sigma_1}\partial_{\sigma_2}g) + \dots . && (3)
\end{aligned}
$$

The derivatives consistent with the algebra (1) are given by the $\star$-derivatives $\partial^\star_\lambda$

$$\partial^\star_\lambda = \partial_\lambda, \tag{4}$$

where ∂_λ are the usual partial derivatives. In the following we will mainly write ∂_λ, only when we want to stress something we write explicitly $\partial^\star_\lambda$. Because of (4) this makes no difference to our results. The Leibniz rule for the derivatives (4) is the classical (undeformed) one

$$\partial^\star_\lambda \star (f\star g) = (\partial^\star_\lambda \star f)\star g + f\star(\partial^\star_\lambda \star g). \tag{5}$$

2 Deformed diffeomorphisms

In this section we introduce the deformed diffeomorphism symmetry.

We define the transformation law of a noncommutative scalar field $\phi(x)$[2] to be

$$\delta_\xi\phi(x) = \phi'(x) - \phi(x) = -\xi^\mu\partial_\mu\phi(x) = -(X^\star_\xi \star \phi(x)), \tag{6}$$

where $\xi^\mu(x)$ is an arbitrary function of coordinates. The higher order differential operator $X^\star_\xi$ is constructed perturbatively from the above requirement using the $\star$-product (3)

$$
\begin{aligned}
X^\star_\xi &= X^{\star 0}_\xi + X^{\star 1}_\xi + \dots\\
(X^\star_\xi \star\phi) &= (X^{\star 0}_\xi\star\phi) + (X^{\star 1}_\xi\star\phi) + \dots\\
&= (X^{\star 0}_\xi\phi) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho X^{\star 0}_\xi)(\partial_\sigma\phi) + (X^{\star 1}_\xi\phi) + \dots\\
&\stackrel{\text{def}}{=} \xi^\mu(\partial_\mu\phi).
\end{aligned}
$$

This leads to the solution up to first order in the deformation parameter θ

$$X^\star_\xi = \xi^\mu\partial_\mu - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\xi^\mu)\partial_\sigma\partial_\mu. \tag{7}$$

It is not difficult to generalise this to all orders

$$X^\star_\xi = \sum_n \Big(\frac{-i}{2}\Big)^n\frac{1}{n}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}\xi^\mu\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n}\partial_\mu. \tag{8}$$

[2]In the following we will usually omit explicitly writing x dependence of the field.

To see if this transformations close in the algebra, one calculates

$$\delta_\xi \delta_\eta \phi = \delta_\xi(- X^\star_\eta \star \phi) = \Big(X^\star_\eta \star (X^\star_\xi \star \phi)\Big).$$

Form here it follows

$$\delta_\xi \delta_\eta - \delta_\eta \delta_\xi = \delta_{[\xi,\eta]}, \tag{9}$$

the deformed transformations (6) close in the undeformed algebra. However, this result was expected since what has been done so far is just rewriting the classical transformation law of a scalar field in a rather complicated way (so no reason to call it "deformed"). But now we remember that under the classical diffeomorphisms the pointwise product of two scalar fields transforms as a scalar field. This we generalise by demanding that the $\star$-product of two scalar fields is a scalar field again

$$\delta_\xi(\phi_1 \star \phi_2) = -(X_\xi \star (\phi_1 \star \phi_2)). \tag{10}$$

The right-hand side of (10), written more explicitly using (6), reads

$$\begin{aligned} \delta_\xi(\phi_1 \star \phi_2) &= -\xi^\mu(\partial_\mu(\phi_1 \star \phi_2)) = -\xi^\mu\Big((\partial_\mu\phi_1) \star \phi_2 + \phi_1 \star (\partial_\mu \phi_2)\Big) \\ &\neq -(\xi^\mu(\partial_\mu\phi_1)) \star \phi_2 - \phi_1 \star (\xi^\mu(\partial_\mu\phi_2)), \end{aligned}$$

since the $\star$-product is noncommutative. Commuting ξ^μ through the $\star$-product gives additional terms

$$\delta_\xi(\phi_1 \star \phi_2) = (\delta_\xi\phi_1) \star \phi_2 + \phi_1 \star (\delta_\xi\phi_2) - \frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{(\partial_\rho\xi)}\phi_1)\partial_\sigma\phi_2 + (\partial_\rho\phi_1)(\delta_{(\partial_\sigma\xi)}\phi_2)\Big), \tag{11}$$

with $\delta_{(\partial_\rho\xi)}\phi_1 \stackrel{\mathrm{def}}{=} -(\partial_\rho\xi^\mu)\partial_\mu\phi_1$. We see that the transformations (6) have a deformed Leibniz rule[3] and this justifies calling them "deformed" transformations.

In order to construct the full Hopf algebra of deformed diffeomorphisms one has to check if the Leibniz rule (11) leads to a good coproduct (coassociative, consistent with the algebra (9),...). All this can be done to all orders in θ [1]. For completeness we cite here the full θ-deformed Hopf algebra of diffeomorphisms

$$\begin{aligned} \delta_\xi\delta_\eta - \delta_\eta\delta_\xi &= \delta_{[\xi,\eta]}, \\ \Delta\delta_\xi &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\Big(\delta_\xi \otimes 1 + 1 \otimes \delta_\xi\Big)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} \\ &= \delta_\xi \otimes 1 + 1 \otimes \delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\Big(\delta_{(\partial_\rho\xi)} \otimes \partial_\sigma + \partial_\rho \otimes \delta_{(\partial_\sigma\xi)}\Big) + \dots, \\ \varepsilon(\delta_\xi) &= 0, \quad S(\delta_\xi) = -\delta_\xi. \end{aligned} \tag{12}$$

Once again, we mention that the algebra sector of this Hopf algebra remains undeformed, while the comultiplication changes.

[3] For the classical transformations we have

$$\delta^{cl}_\xi\Big(\phi_1(x)\phi_2(x)\Big) = \Big(\delta^{cl}_\xi\phi_1(x)\Big)\phi_2(x) + \phi_1(x)\Big(\delta^{cl}_\xi\phi_2(x)\Big).$$

In analogy with (6), the transformation laws of a covariant vector field and a contravariant vector field are given by

$$\begin{aligned}
\delta_\xi V_\mu &= -\xi^\lambda(\partial_\lambda V_\mu) - (\partial_\mu \xi^\lambda)V_\lambda = -(X^\star_\xi \star V_\mu) - (X^\star_{(\partial_\mu\xi^\lambda)} \star V_\lambda) \qquad (13)\\
&= -\xi^\lambda \star (\partial_\lambda V_\mu) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \xi^\lambda) \star (\partial_\sigma\partial_\lambda V_\mu)\\
&\quad -(\partial_\mu\xi^\lambda)\star V_\lambda + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\partial_\mu\xi^\lambda)\star(\partial_\sigma V_\lambda) + \ldots,\\
\delta_\xi V^\mu &= -(X^\star_\xi \star V^\mu) + (X^\star_{(\partial_\lambda\xi^\mu)} \star V^\lambda), \qquad (14)
\end{aligned}$$

where in the second and the third line $X^\star_\xi$ and $X^\star_{(\partial_\mu\xi^\lambda)}$ are expanded. This can be generalised to the transformation law of an arbitrary tensor.

Using the deformed coproduct (12) one can show that the $\star$-product of two arbitrary tensors transforms like a tensor of the appropriate rank again. Also, having covariant and contravariant vectors and tensors one can construct invariants. For example,

$$\begin{aligned}
\delta_\xi(V_\mu \star V^\mu) &= (\delta_\xi V_\mu)\star V^\mu + V_\mu \star(\delta_\xi V^\mu) - \frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{(\partial_\rho\xi)}V_\mu)(\partial_\sigma V^\mu) + (\partial_\rho V_\mu)(\delta_{(\partial_\sigma\xi)}V^\mu)\Big)\\
&= \ldots\\
&= -\xi^\lambda\partial_\lambda(V_\mu\star V^\mu) = -(X^\star_\xi\star(V_\mu\star V^\mu)). \qquad (15)
\end{aligned}$$

3 Curvature and torsion

Having constructed the deformed diffeomorphism symmetry, we proceed as in the commutative case, by observing that the partial derivative of a vector field transforms as

$$\begin{aligned}
\delta_\xi(\partial_\mu V_\nu) &= (\partial_\mu\delta_\xi V_\nu) \qquad (16)\\
&= -(X^\star_\xi\star(\partial_\mu V_\nu)) - (X^\star_{(\partial_\mu\xi^\lambda)}\star(\partial_\lambda V_\nu)) - (X^\star_{(\partial_\nu\xi^\lambda)}\star(\partial_\mu V_\lambda)) - (X^\star_{(\partial_\mu\partial_\nu\xi^\lambda)}\star V_\lambda).
\end{aligned}$$

Here we have used

$$(\partial_\mu X^\star_\xi) = \sum_n \Big(\frac{-i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\ldots\partial_{\rho_n}\partial_\mu\xi^\lambda\Big)\partial_{\sigma_1}\ldots\partial_{\sigma_n}\partial_\lambda = X_{(\partial_\mu\xi^\lambda)}\partial_\lambda \qquad (17)$$

and similarly $\partial_\mu X_{(\partial_\nu\xi^\lambda)} = X_{(\partial_\mu\partial_\nu\xi^\lambda)}$. Because of the last term in (16) this is not the transformation law of a tensor. To repair this we introduce the covariant derivative

$$D_\mu V_\nu = (\partial_\mu V_\nu) - \Gamma^\alpha_{\mu\nu}\star V_\alpha, \qquad (18)$$

where $\Gamma^\alpha_{\mu\nu}$ is the noncommutative connection. From the demand that (18) transforms as a tensor of rank two

$$\delta_\xi(D_\mu V_\nu) = -(X^\star_\xi\star(D_\mu V_\nu)) - (X^\star_{(\partial_\mu\xi^\lambda)}\star(D_\lambda V_\nu)) - (X^\star_{(\partial_\nu\xi^\lambda)}\star(D_\mu V_\lambda)) \qquad (19)$$

one calculates the transformation law of the connection $\Gamma^\alpha_{\mu\nu}$

$$\delta_\xi\Gamma^\alpha_{\mu\nu} = -(X^\star_\xi\star\Gamma^\alpha_{\mu\nu}) - (X^\star_{(\partial_\mu\xi^\lambda)}\star\Gamma^\alpha_{\lambda\nu}) - (X^\star_{(\partial_\nu\xi^\lambda)}\star\Gamma^\alpha_{\mu\lambda}) + (X^\star_{(\partial_\lambda\xi^\alpha)}\star\Gamma^\lambda_{\mu\nu}) - \partial_\mu\partial_\nu\xi^\alpha. \quad (20)$$

Note that, after expanding all the $\star$-products and $X^\star$ operators, this transformation law reduces to the classical one.

In analogy with (18) one defines the covariant derivative of a contravariant vector and of an arbitrary tensor

$$\begin{aligned} D_\mu V^\nu &= (\partial_\mu V^\nu) + \Gamma^\nu_{\mu\alpha} \star V^\alpha, \qquad (21)\\ D_\lambda T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_p} &= (\partial_\lambda T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_p}) - \Gamma^\alpha_{\lambda\mu_1} \star T^{\nu_1\dots\nu_r}_{\alpha\mu_2\dots\mu_p} - \dots - \Gamma^\alpha_{\lambda\mu_p} \star T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_{p-1}\alpha}\\ &\quad + \Gamma^{\nu_1}_{\lambda\alpha} \star T^{\alpha\nu_2\dots\nu_r}_{\mu_1\dots\mu_p} + \dots + \Gamma^{\nu_r}_{\lambda\alpha} \star T^{\nu_1\dots\nu_{r-1}\alpha}_{\mu_1\dots\mu_p}. \qquad (22)\end{aligned}$$

The $\star$-commutator of two covariant derivatives applied on a vector field gives the curvature tensor and torsion

$$[D_\mu \stackrel{\star}{,} D_\nu] \star V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T^\alpha_{\mu\nu} \star D_\alpha V_\rho, \qquad (23)$$

with

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma &= (\partial_\nu \Gamma^\sigma_{\mu\rho}) - (\partial_\mu \Gamma^\sigma_{\nu\rho}) + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \qquad (24)\\ T^\alpha_{\mu\nu} &= \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}. \qquad (25)\end{aligned}$$

From (24) it follows

$$R_{\mu\nu\rho}{}^\sigma = -R_{\nu\mu\rho}{}^\sigma \qquad (26)$$

like in the commutative case, but

$$\begin{aligned} R_{\mu\nu\rho\sigma} &\stackrel{\text{def}}{=} R_{\mu\nu\rho}{}^\alpha \star G_{\alpha\sigma} \neq R_{\mu\nu\sigma\rho}, \qquad (27)\\ R_{\mu\nu\rho\sigma} &\neq R_{\rho\sigma\mu\nu}. \qquad (28)\end{aligned}$$

This is a consequence of having the $\star$-product in (24).

In (27) we have introduced the noncommutative metric tensor $G_{\mu\nu}$. By definition, it is a symmetric tensor of rank two

$$\hat\delta_\xi G_{\mu\nu} = -(X^\star_\xi \star G_{\mu\nu}) - (X^\star_{(\partial_\mu \xi^\rho)} \star G_{\rho\nu}) - (X^\star_{(\partial_\nu \xi^\rho)} \star G_{\mu\rho}), \qquad (29)$$

with the condition that it reduces to the classical metric tensor in the $\theta \to 0$ limit,

$$G_{\mu\nu}\Big|_{\theta=0} = g_{\mu\nu}. \qquad (30)$$

However, these conditions do not determine $G_{\mu\nu}$ uniquely and in the following we present a few different solutions.

Looking at the transformation law of $G_{\mu\nu}$ we see that the choice $G_{\mu\nu} = g_{\mu\nu}$, that is the noncommutative metric equals the classical metric, is consistent with (29). The condition (30) is automatically fulfilled and we obtain the θ-independent metric tensor. However, this metric tensor becomes θ-dependent after solving the equations of motion coming from the deformed Einstein-Hilbert action.

One can also choose to start from a θ-dependent metric tensor. Then one expands it in orders of the deformation parameter θ

$$G_{\mu\nu} = g_{\mu\nu} + G^1_{\mu\nu} + \dots, \qquad (31)$$

where $G^1_{\mu\nu}$ is the first order correction which one calculates again by solving the equations of motion.

On the other hand, we remember that the classical metric tensor can be expressed in terms of the vierbein $e_\mu{}^a$

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b, \tag{32}$$

where η_{ab} is the flat Minkowski metric and a and b are local Lorentz indices. This we generalise to the noncommutative metric tensor

$$G_{\mu\nu} = \frac{1}{2}\Big(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^a \star E_\mu{}^b\Big)\eta_{ab}, \tag{33}$$

where $E_\mu{}^a$ is the noncommutative vierbein. In order to fulfil (29), $E_\mu{}^a$ has to transform as a vector field (13) and the coproduct (12) has to be used. Because of (30) in the limit $\theta \to 0$ it has to reduce to the classical vierbein

$$E_\mu{}^a = e_\mu{}^a + E_\mu{}^{a\,1} + \dots. \tag{34}$$

Note that one can also start with the classical vierbein (it is consistent with both (29) and (30)) and after solving the equations of motion obtain that it becomes θ-dependent.

Starting with the noncommutative metric tensor $G_{\mu\nu}$, one can introduce two inverses. The inverse with respect to the pointwise multiplication (classical inverse) we denote by $G^{\mu\nu}$

$$G_{\mu\nu} \cdot G^{\nu\rho} = \delta_\mu^\rho, \tag{35}$$

and the inverse with respect to the $\star$-multiplication with $G^{\mu\nu\star}$

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta_\mu^\rho. \tag{36}$$

Expanding $G^{\nu\rho\star}$ in the deformation parameter θ and inserting the expansion in (36) gives the $\star$-inverse in terms of the classical inverse

$$\begin{aligned} G^{\mu\nu\star} &= G^{\mu\nu} + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho G^{\mu\alpha})(\partial_\sigma G_{\alpha\beta})G^{\beta\nu} \\ &= 2G^{\mu\nu} - G^{\mu\alpha} \star G_{\alpha\beta} \star G^{\beta\nu}. \end{aligned} \tag{37}$$

This result is valid up to first order in θ. The exact result will of course depend on the choice of $G_{\mu\nu}$. From (36), using the comultiplication (12), it follows that $G^{\mu\nu\star}$ transforms like a tensor of rank two

$$\delta_\xi G^{\mu\nu\star} = -(X_\xi^\star \star G^{\mu\nu\star}) + (X^\star_{(\partial_\rho \xi^\mu)} \star G^{\rho\nu\star}) + (X^\star_{(\partial_\rho \xi^\nu)} \star G^{\mu\rho\star}). \tag{38}$$

Note that, although $G_{\mu\nu}$ is a symmetric tensor, its $\star$-inverse is not symmetric

$$G^{\mu\nu\star} \neq G^{\nu\mu\star}. \tag{39}$$

The Ricci tensor is defined as

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma. \tag{40}$$

Contracting the first and the fourth index gives the same result because of (26). Unlike in the classical case, here it is also possible to contract the third and the fourth index since the curvature tensor is not antisymmetric with respect to these two indices. However, the commutative limit of this result[4] will not give the commutative Ricci tensor, so we do not

[4]In the deformed case from (27) we have $R_{\mu\nu\sigma}{}^\sigma = \mathcal{O}(\theta)$ and in the limit $\theta \to 0$, $R_{\mu\nu\sigma}{}^\sigma \to 0$.

consider this possibility. From this analysis it follows that we can define the Ricci tensor uniquely. One should also note that it is not symmetric

$$R_{\mu\nu} \neq R_{\nu\mu}. \tag{41}$$

However, there are more possible definitions of the scalar curvature. Some of them are

$$R = G^{\mu\nu\star} \star R_{\nu\mu}, \tag{42}$$
$$R = R_{\nu\mu} \star G^{\mu\nu\star}, \tag{43}$$
$$R = \frac{1}{2}(G^{\mu\nu\star} \star R_{\nu\mu} + R_{\nu\mu} \star G^{\mu\nu\star}). \tag{44}$$

We choose (42) to be our working definition, but one should keep in mind that there are other possibilities.

Finally, from (25) we see that if the connection is symmetric, the torsion vanishes. In the following we analyse only the torsion-free case, that is

$$\Gamma^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\mu\nu}. \tag{45}$$

In order to relate the connection with the metric tensor in the commutative case one imposes the metricity condition. We generalise this construction to the θ-deformed case. Namely, we demand that the covariant derivative of the metric tensor vanishes

$$D_\alpha G_{\beta\gamma} = (\partial_\alpha G_{\beta\gamma}) - \Gamma^{\rho}_{\alpha\beta} \star G_{\rho\gamma} - \Gamma^{\rho}_{\alpha\gamma} \star G_{\beta\rho} = 0. \tag{46}$$

Then the unique result for the connection follows

$$\Gamma^{\sigma}_{\alpha\beta} = \frac{1}{2}\Big((\partial_\alpha G_{\beta\gamma}) + (\partial_\beta G_{\alpha\gamma}) - (\partial_\gamma G_{\alpha\beta})\Big) \star G^{\gamma\sigma\star}. \tag{47}$$

To obtain this result we have used that the metric tensor and connection are symmetric. In analogy with the commutative case, we call the connection (47) Christoffel symbol. Using the transformation properties of $G_{\mu\nu}$ and $G^{\mu\nu\star}$, (29) and (38) respectively, and the coproduct (12), from (47) the transformation law (20) of the Christoffel symbol follows.

Using the result (47) one expresses the curvature tensor, Ricci tensor and scalar curvature in terms of the metric tensor and its $\star$-inverse.

4 Deformed Einstein-Hilbert action

Our aim is to construct an action invariant under the deformed diffeomorphisms which in the zeroth order limit reduces to the classical Einstein-Hilbert action. To do this, we first need an integral with the cyclic property,

$$\int \mathrm{d}^4x\, (f_1 \star f_2 \star \ldots \star f_k) = \int \mathrm{d}^4x\, (f_k \star f_1 \star f_2 \star \ldots \star f_{k-1}). \tag{48}$$

Fortunately, the θ-deformed space is simple enough and the usual commutative integral has this property.

We also need a $\star$-density $E^\star$ that transforms like

$$\delta_\xi E^\star = -(X^\star_\xi \star E^\star) - (X^\star_{(\partial_\lambda \xi^\lambda)} \star E^\star), \tag{49}$$

such that

$$\delta_\xi (E^\star \star R) = -\partial_\mu \Big(X^\star_{\xi^\mu} \star (E^\star \star R) \Big). \tag{50}$$

Then the action

$$S = \int \mathrm{d}^4 x\, E^\star \star R \tag{51}$$

is invariant under the deformed diffeomorphisms

$$\delta_\xi \Big(\int \mathrm{d}^4 x\, E^\star \star R \Big) = 0. \tag{52}$$

The problem with this so far undetermined $\star$-density is that the transformation law (49) does not give enough conditions to fix $E^\star$ uniquely. Adding the requirement of the proper commutative limit does not help, so we have to make a choice once again. We remember that the classical Einstein-Hilbert action can be written as

$$S = \int \mathrm{d}^4 x\, e R^0, \tag{53}$$

where $e = \det e_\mu{}^a$ is the determinant of the classical vierbein and R^0 is the classical scalar curvature. In the previous section we have already introduced the noncommutative vierbein and we have to generalise the concept of a determinant. This is not too difficult, we define the $\star$-determinant as

$$E^\star = \det{}_\star E_\mu{}^a = \frac{1}{4!} \varepsilon^{\mu_1 \dots \mu_4} \varepsilon_{a_1 \dots a_4} E_{\mu_1}^{\ a_1} \star \dots \star E_{\mu_4}^{\ a_4}, \tag{54}$$

where $\varepsilon^{\mu_1 \dots \mu_4}$ is the totally antisymmetric tensor of rank 4. By using the comultiplication (12) one checks that (54) has the right transformation property (49) and (34) ensures the good commutative limit.

Finally, the deformed Einstein-Hilbert action we define as

$$\begin{aligned} S &= \int \mathrm{d}^4 x\, (E^\star \star R + \text{ c.c. }), \\ &= \int \mathrm{d}^4 x\, (E^\star \star R + \bar{R} \star \overline{E^\star}). \end{aligned} \tag{55}$$

In order to have a real action we added the complex conjugated part also. The action (55) can be varied with respect to $E_\mu{}^a$ to give the equations of motion. Of course, this fixes our choice of the noncommutative metric tensor to (33) and all the quantities like $R_{\mu\nu}$, R, ... have to be expressed in terms of $E_\mu{}^a$.

5 Expansion in the deformation parameter

In this section we expand some of the results from the previous sections up to first order in the deformation parameter θ and in terms of the classical fields, vierbein $e_\mu{}^a$, metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$[5] . We start with the basic object, the vierbein. It is given by

$$E_\mu{}^a = e_\mu{}^a + E_\mu{}^{a\,1} + \dots, \tag{56}$$

[5]Since one can express $g_{\mu\nu}$ and $g^{\mu\nu}$ in terms of $e_\mu{}^a$, what we obtain might not be the final form for the results.

where ${E_\mu}^{a\,1}$ is linear in $\theta^{\rho\sigma}$. Note that this differs from the approach that was taken in [1]. There the vierbein is taken to be the classical object, keeping in mind that after solving the equations of motion it becomes θ-dependent. Here we start from the beginning with the θ-dependent object. Using (56) and (54) one calculates $E^\star$

$$E^\star = e + \frac{1}{3!}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{a_1a_2a_3a_4}{E_{\mu_1}}^{a_1\,1}{e_{\mu_2}}^{a_2}{e_{\mu_3}}^{a_3}{e_{\mu_4}}^{a_4}. \tag{57}$$

From (33) and (36) it follows

$$\begin{aligned} G_{\mu\nu} &= g_{\mu\nu} + \eta_{ab}({E_\mu}^{a\,1}{e_\nu}^b + {e_\mu}^a{E_\nu}^{b\,1}), &(58)\\ G^{\mu\nu\star} &= g^{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}g^{\mu\alpha}(\partial_\rho g_{\alpha\beta})(\partial_\sigma g^{\beta\nu}) \\ &\quad -\eta_{ab}g^{\mu\alpha}({E_\alpha}^{a\,1}{e_\beta}^b + {e_\alpha}^a{E_\beta}^{b\,1})g^{\beta\nu}. &(59)\end{aligned}$$

For the Christoffel symbol from (47) we obtain

$$\begin{aligned} \Gamma_{\mu\nu}^\alpha &= \Gamma_{\mu\nu}^{0\,\alpha} + \frac{i}{4}\theta^{\rho\sigma}(\partial_\rho\partial_\mu g_{\gamma\nu} + \partial_\rho\partial_\nu g_{\gamma\mu} - \partial_\rho\partial_\gamma g_{\mu\nu})(\partial_\sigma g^{\gamma\alpha}) \\ &\quad -\frac{i}{4}\Gamma_{\mu\nu}^{0\,\gamma}\Big(\theta^{\rho\sigma}(\partial_\rho g_{\gamma\omega})(\partial_\sigma g^{\omega\alpha}) - 2i\eta_{ab}({E_\gamma}^{a\,1}{e_\omega}^b + {e_\gamma}^a{E_\omega}^{b\,1})g^{\omega\alpha}\Big) \\ &\quad +\eta_{ab}\Big(\partial_\mu({E_\nu}^{a\,1}{e_\gamma}^b + {e_\nu}^a{E_\gamma}^{b\,1}) \\ &\qquad +\partial_\nu({E_\mu}^{a\,1}{e_\gamma}^b + {e_\mu}^a{E_\gamma}^{b\,1}) - \partial_\gamma({E_\mu}^{a\,1}{e_\nu}^b + {e_\mu}^a{E_\nu}^{b\,1})\Big)g^{\gamma\alpha} \\ &\stackrel{\text{def}}{=} \Gamma_{\mu\nu}^{0\,\alpha} + {\Gamma_{\mu\nu}}^{\alpha\,1}. &(60)\end{aligned}$$

We see that already this result is long and not very readable. Therefore, we just give the implicit result for the curvature tensor

$$\begin{aligned} {R_{\mu\nu\alpha}}^\beta &= {R^0_{\mu\nu\alpha}}^\beta + (\partial_\nu\Gamma_{\mu\alpha}^{\beta\,1}) - (\partial_\mu\Gamma_{\nu\alpha}^{\beta\,1}) + \frac{i}{2}\theta^{\rho\sigma}((\partial_\rho\Gamma_{\nu\alpha}^{0\,\lambda})(\partial_\sigma\Gamma_{\lambda\mu}^{0\,\beta}) - (\partial_\rho\Gamma_{\mu\alpha}^{0\,\lambda})(\partial_\sigma\Gamma_{\lambda\nu}^{0\,\beta})) \\ &\quad +\Gamma_{\nu\alpha}^{\lambda\,1}\Gamma_{\lambda\mu}^{0\,\beta} + \Gamma_{\nu\alpha}^{0\,\lambda}\Gamma_{\lambda\mu}^{\beta\,1} - \Gamma_{\mu\alpha}^{\lambda\,1}\Gamma_{\lambda\nu}^{0\,\beta} - \Gamma_{\mu\alpha}^{0\,\lambda}\Gamma_{\lambda\nu}^{\beta\,1}. &(61)\end{aligned}$$

One can continue like this and calculate $R_{\mu\nu}$ and R in terms of the classical fields and corrections. This results can be inserted into the equation of motion obtained by varying the action (55). Solving that equation one finds the corrections to the classical vierbein and sees how the noncommutativity influences the classical solutions. However, we are not going to do these calculations here, they will be the subject of future research.

6 Conclusions

We have seen how the deformed diffeomorphism symmetry can be constructed[6]. The method used is a rather general one and can be applied to other deformed spaces as well. As the final result we presented the deformed Einstein-Hilbert action. In the next step the equations of motion should be calculated and solved to see how does the noncommutativity effect the classical solutions.

[6]For another approach to this problem see [5].

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Representations of A-type Hecke algebras

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Abstract

We review some facts about the representation theory of the Hecke algebra. We adapt for the Hecke algebra case the approach of [1] which was developed for the representation theory of symmetric groups. We justify an explicit construction of the idempotents in the Hecke algebra in terms of Jucys-Murphy elements. Ocneanu's traces for these idempotents (which can be interpreted as q-dimensions of corresponding irreducible representations of quantum linear groups) are presented.

1 Introduction

Main statements of the representation theory of Hecke algebras are known mostly due to the works by V.Jones, I.V.Cherednik, G.Murphy, R.Dipper and G.James, H.Wenzl, a.o. (see, e.g., [2] – [5]). In this report the approach of [1], developed for the representation theory of symmetric groups, is generalized to the case of the A-type Hecke algebras. Certain propositions below are given without proofs due to lack of space and, also, because the corresponding statements for Hecke algebras are proved like those for symmetric groups.

The importance of the theory of the A-type Hecke algebra H_M is that H_M is the centralizer of the action of general linear quantum groups $U_q(gl(N))$ in the tensor powers $V^{\otimes M}$ of the vector representation V of $U_q(gl(N))$. We have shown recently [6] that an arbitrary representation of the Hecke algebra H_M defines an integrable model on a chain with M sites. This fact demonstrates the importance of the representation theory of the Hecke algebra in the theory of integrable models also.

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2 A-Type Hecke algebras and Jucys - Murphy elements

A braid group $\mathcal{B}_{M+1}$ is generated by Artin elements σ_i $(i = 1, \dots M)$ subject to relations:

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1} \; , \quad \sigma_i \, \sigma_j = \sigma_j \, \sigma_i \; \text{ for } \; |i-j| > 1 \; . \tag{2.1}$$

An A-Type Hecke algebra $H_{M+1}(q)$ (see e.g. [2] and Refs. therein) is a quotient of the group algebra of the braid group $\mathcal{B}_{M+1}$ by an additional relation

$$\sigma_i^2 - 1 = (q - q^{-1}) \, \sigma_i \; , \quad (i = 1, \dots, M) \; . \tag{2.2}$$

Here $q \in \mathbf{C} \backslash \{0\}$ is a parameter. The group algebra of $\mathcal{B}_{M+1}$ (2.1) has an infinite dimension while its quotient H_{M+1} is finite dimensional. It can be shown (see e.g. [5]) that H_{M+1} is spanned linearly by $(M+1)!$ elements, e.g., those which appear in the expansion of the special operator

$$\Sigma_{1 \to M+1} = f_{1 \to M+1} \, f_{1 \to M} \cdots f_{1 \to 2} \, f_{1 \to 1} \; ,$$

where $f_{1 \to n}$ are 1-shuffles defined inductively by $f_{1 \to 1} = 1$, $f_{1 \to n+1} = 1 + f_{1 \to n} \, \sigma_n$. Below we assume that $q \neq \exp(2\pi i n/m)$, $n, m \in \mathbf{Z}$ (q is "generic"); for these values of q, there exists an isomorphism between the algebra $H_{M+1}(q)$ and the group algebra of the symmetric group S_{M+1} (the case $q = \pm 1$ is exceptional, in this case H_{M+1} = group algebra of S_{M+1}).

An essential information about a finite dimensional semisimple algebra $\mathcal{A}$ is contained in the structure of its regular bimodule which decomposes into direct sums: $\mathcal{A} = \bigoplus_{\alpha=1}^{s} \mathcal{A} \cdot e_\alpha$, $\mathcal{A} = \bigoplus_{\alpha=1}^{s} e_\alpha \cdot \mathcal{A}$ of left and right submodules (ideals), respectively (left- and right-Peirce decompositions). Here the elements $e_\alpha \in \mathcal{A}$ $(\alpha = 1, \dots, s)$ are mutually orthogonal idempotents: $e_\alpha \, e_\beta = \delta_{\alpha\beta} \, e_\alpha$, resolving the identity operator: $1 = \sum_{\alpha=1}^{s} e_\alpha$. There are two important decompositions of the identity operator and correspondingly two sets of the idempotents in $\mathcal{A}$:
(1) *Primitive idempotents.* An idempotent e_α is *primitive* if it can not be further resolved into a sum of nontrivial mutually orthogonal idempotents.
(2) *Primitive central idempotents.* An idempotent e'_β is *primitive central* if it is primitive in the class of central idempotents.

For the A-type Hecke algebra $H_{M+1}(q)$ a set of elements $\{y_i\}$ $(i = 1, \dots, M+1)$ is defined inductively: $y_1 = 1$, $y_{i+1} = \sigma_i y_i \sigma_i$. These elements are called *Jucys - Murphy elements* and can be written (using the Hecke condition (2.2) and the braid relation (2.1)) in the form

$$y_i = \sigma_{i-1} \dots \sigma_2 \, \sigma_1^2 \, \sigma_2 \dots \sigma_{i-1} = (q - q^{-1}) \sum_{k=1}^{i-1} \sigma_k \dots \sigma_{i-2} \, \sigma_{i-1} \, \sigma_{i-2} \dots \sigma_k + 1. \tag{2.3}$$

Sometimes it is more convenient to use elements $(y_i - 1)/(q - q^{-1})$ which, due to (2.3), have a nontrivial classical limit $(q \to 1)$. The elements y_i pairwise commute. The following statement explains the importance of the set $\{y_i\}$.

Proposition 1. *The set of Jucys - Murphy elements* $\{y_i\}$ $(i = 1, \dots, M+1)$ *generates a maximal commutative subalgebra* Y_{M+1} *in* H_{M+1}.

We construct primitive orthogonal idempotents $e_\alpha \in H_{M+1}$ as functions of the elements $y_i \in Y_{M+1}$; they are common *eigenidempotents* of y_i: $y_i e_\alpha = e_\alpha y_i = a_i^{(\alpha)} e_\alpha$ $(i = 1, \dots, M +$

1). We denote (as in [1], for symmetric groups) by $\mathrm{Spec}(y_1,\ldots,y_{M+1})$ the set $\{\Lambda(e_\alpha)\}$ $(\forall\alpha)$ of strings of eigenvalues: $\Lambda(e_\alpha)=(a_1^{(\alpha)},\ldots,a_{M+1}^{(\alpha)})$. In view of the following inclusions of the subalgebras Y_i and $H_i(q)$:

$$\begin{array}{ccc} H_i(q) & \subset & H_{i+1}(q) \\ \cup & & \cup \\ Y_i & \subset & Y_{i+1} \end{array}$$

one can describe the idempotents $\in H_{i+1}$ by considering the branching of the idempotents of H_i in H_{i+1}. It can be shown that the multiplicity of this branching is equal to one and y_i are semi-simple for generic q.

We need important intertwining operators [8] (presented in another form in [3])

$$U_{n+1}=\sigma_n y_n - y_n\sigma_n \;\;(1\le n\le M)\;. \tag{2.4}$$

Elements U_i satisfy relations[1] $U_n\,U_{n+1}\,U_n=U_{n+1}\,U_n\,U_{n+1}$ and

$$U_{n+1}y_n=y_{n+1}U_{n+1},\; U_{n+1}y_{n+1}=y_nU_{n+1},\; [U_{n+1},\,y_k]=0\;(k\neq n,n+1), \tag{2.5}$$

$$U_{n+1}^2=(qy_n-q^{-1}\,y_{n+1})\,(q\,y_{n+1}-q^{-1}\,y_n)\;. \tag{2.6}$$

The operators U_{n+1} "permute" elements y_n and y_{n+1} (see (2.5)) which supports a statement that the center Z_{M+1} of the Hecke algebra H_{M+1} is generated by symmetric functions in $\{y_i\}$ $(i=2,\ldots,M+1)$ (to prove this fact it is enough to check relations: $[\sigma_k,\,y_n+y_{n+1}]=0=[\sigma_k,\,y_ny_{n+1}]$ for all $k<n+1$).

Proposition 2. *One has*

$$\mathrm{Spec}(y_j)\subset\{q^{2\mathbf{Z}_j}\}\quad\forall j=1,2,\ldots,M+1\;, \tag{2.7}$$

where $\mathbf{Z}_j$ *denotes the set of integers* $\{1-j,\ldots,-2,-1,0,1,2,\ldots,j-1\}$.

Proof. We prove (2.7) by induction. Obviously, $\mathrm{Spec}(y_1)$ satisfies (2.7). Assume that the spectrum of y_{j-1} satisfies (2.7) for some $j\ge 2$. Consider a characteristic equation for y_{j-1} $(j\ge 2)$:

$$f(y_{j-1}):=\prod_\alpha(y_{j-1}-a_{j-1}^{(\alpha)})=0\quad(a_{j-1}^{(\alpha)}\in\mathrm{Spec}(y_{j-1}))\;.$$

Using properties (2.5)-(2.6) of operators U_j, we deduce

$$0=U_jf(y_{j-1})U_j=f(y_j)U2_j=f(y_j)(q^2y_{j-1}-y_j)(y_j-q^{-2}y_{j-1})\;. \tag{2.8}$$

which means that $\mathrm{Spec}(y_j)\subset(\mathrm{Spec}(y_{j-1})\cup q^{\pm2}\cdot\mathrm{Spec}(y_{j-1}))$. •

3 Generalization of the approach of [1] to the Hecke algebra case

Consider a subalgebra $\hat{H}_2^{(i)}$ in H_{M+1} with generators y_i, y_{i+1} and σ_i (for fixed $i\le M$). We investigate representations of $\hat{H}_2^{(i)}$ with diagonalizable y_i and y_{i+1}. Let e be a common

[1] The definition (2.4) of intertwining elements is not unique. One can multiply U_{n+1} by a function $f(y_n,y_{n+1})$: $U_{n+1}\to U_{n+1}f(y_n,y_{n+1})$. Then eqs. (2.5)-(2.6) are valid if $f(y_n,y_{n+1})f(y_{n+1},y_n)=1$.

eigenidempotent of y_i, y_{i+1}: $y_i e = a_i e$, $y_{i+1} e = a_{i+1} e$. Then the left action of $\hat{H}_2^{(i)}$ closes on elements $v_1 = e$ and $v_2 = \sigma_i e$ and is given by matrices:

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & -(q-q^{-1})a_{i+1} \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & (q-q^{-1})a_{i+1} \\ 0 & a_i \end{pmatrix}; \tag{3.1}$$

$a_i \neq a_{i+1}$ otherwise y_i, y_{i+1} are not diagonalizable. The matrices y_i, y_{i+1} (3.1) can be simultaneously diagonalized by a similarity transformation $y \to V^{-1} y V$, where

$$V = \begin{pmatrix} 1 & \frac{(q-q^{-1})\,a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -\frac{(q-q^{-1})\,a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}.$$

As a result we obtain

$$\sigma_i = \begin{pmatrix} -\frac{(q-q^{-1})\,a_{i+1}}{a_i - a_{i+1}} & 1 - \frac{(q-q^{-1})2\,a_i a_{i+1}}{(a_i - a_{i+1})2} \\ 1 & \frac{(q-q^{-1})\,a_i}{a_i - a_{i+1}} \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}. \tag{3.2}$$

When $a_{i+1} = q^{\pm 2} a_i$, the 2-dimensional representation (3.2) reduces to a 1-dimensional representation with $\sigma_i \cdot e = \pm q^{\pm 1}\, e$, respectively. We summarize the above results as (cf. Proposition 4.1 [1]):

Proposition 3. *Let $\Lambda = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_{M+1}) \in \mathrm{Spec}(y_1, \ldots, y_{M+1})$ be a possible spectrum of the set $(y_1, \ldots, y_{M+1})$ which corresponds to a primitive idempotent $e_\Lambda \in H_{M+1}$. Then $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (see Prop. 2) and (a) $a_i \neq a_{i+1}$ for $i \leq M$; (b) if $a_{i+1} = q^{\pm 2} a_i$ then $\sigma_i \cdot e_\Lambda = \pm q^{\pm 1} e_\Lambda$; (c) if $a_{i+1} \neq q^{\pm 2} a_i$ then*

$$\Lambda' = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_{M+1}) \in \mathrm{Spec}(y_1, \ldots, y_{M+1}) \tag{3.3}$$

and the left action of the elements σ_i, y_i, y_{i+1} in the linear span of $v_\Lambda = e_\Lambda$ and $v_{\Lambda'} = \sigma_i\, e_\Lambda + \frac{(q-q^{-1})a_{i+1}}{a_i - a_{i+1}}\, e_\Lambda$ is given by (3.2).

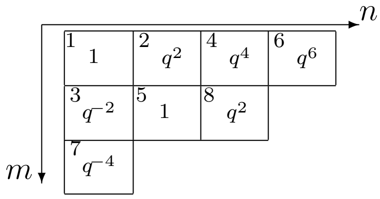
Proposition 4. *Consider the string $\Lambda = (a_1, \ldots, a_n)$ of numbers $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (see Prop. 2). Then $\Lambda = (a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(y_1, y_2, \ldots, y_n)$ iff Λ satisfies the following conditions ($z \in \mathbf{Z}$)*

$$\begin{array}{ll} (1) & a_1 = 1\,; \\ (2) & a_j = q^{2z} \Rightarrow \{q^{2(z+1)}, q^{2(z-1)}\} \cap \{a_1, \ldots, a_{j-1}\} \neq \emptyset \quad \forall j > 1\,, \;\; z \neq 0; \\ (3) & a_i = a_j = q^{2z}\ (i < j) \Rightarrow \ \{q^{2(z+1)}, q^{2(z-1)}\} \subset \{a_{i+1}, \ldots, a_{j-1}\}\,. \end{array} \tag{3.4}$$

Proof. The condition (1) is the identity $y_1 = 1$. Conditions (2),(3) can be proven by induction (see the proof of analogous Theorem 5.1 in [1]). To prove the condition (3) we need the fact that the combinations $(\ldots, a_{i-1}, a_i, a_{i+1}, \ldots) = (\ldots, a, q^{\pm 2} a, a, \ldots)$ cannot appear in Λ: the braid relation $\sigma_i \sigma_{i\pm 1} \sigma_i = \sigma_{i\pm 1} \sigma_i \sigma_{i\pm 1}$ is incompatible with the values $\sigma_i = \pm q^{\pm 1}$, $\sigma_{i+1} = \mp q^{\mp 1}$ (see the condition (b) of Proposition 3). •

Consider a Young diagram with $M+1$ nodes. We place the numbers $1, \ldots, M+1$ into the nodes of the diagram in such a way that these numbers are arranged along rows and columns in ascending order in right and down directions. Such diagram is called a standard Young tableau $[\nu]_{M+1}$. The standard Young tableau $[\nu]_{M+1}$ defines an ascending set of standard tableaux: $[\nu]_1 \subset [\nu]_2 \subset \ldots \subset [\nu]_{M+1}$. In addition we associate a number

$q^{2(n-m)}$ (the "content") to each node of the standard Young tableau, where (n, m) are coordinates of the node. Example:



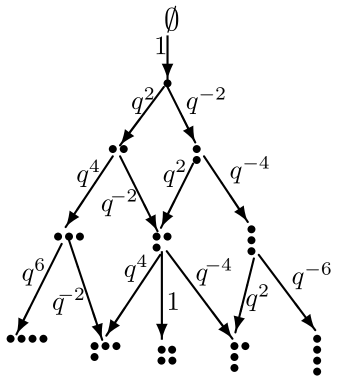
$$(3.5)$$

In general, for the tableau $[\nu]_{M+1}$, the i-th node $[\nu]_i \backslash [\nu]_{i-1}$ with coordinates (n, m) looks like: $\boxed{{}^{i}\, q^{2(n-m)}}$. Thus, to each standard Young tableau $[\nu]_n$ one can associate a string $(a_1, \ldots, a_n)$ with $a_i = q^{2(n-m)}$. E.g., a standard Young tableau (3.5) corresponds to a string $(1, q^2, q^{-2}, q^4, 1, q^6, q^{-4}, q^2)$. This string satisfies conditions of Prop. 3 and therefore $(1, q^2, q^{-2}, q^4, 1, q^6, q^{-4}, q^2) \in \mathrm{Spec}(y_1, \ldots, y_8)$. This relation between contents of $[\nu]_n$ and elements of $\mathrm{Spec}(y_1, \ldots, y_n)$ can be formulated as (cf. Prop. 5.3 [1]):

Proposition 5. *There is a bijection between the set $T(n)$ of the standard Young tableaux with n nodes and the set $\mathrm{Spec}(y_1, \ldots, y_n)$.*

4 Coloured Young graph and explicit construction of idempotents e_α

The above results can be visualized in a different form, in terms of a Young graph. By definition, a Young graph is a graph whose vertices are Young diagrams and edges indicate inclusions of diagrams. We put the eigenvalues a_i (colours) of the Jucys-Murphy elements y_i on the edges in such a way that the string $(a_1, a_2, \ldots, a_n)$ along the path from the top $\emptyset$ of the Young graph to the diagram λ with n nodes gives the content string of the tableau of shape λ. For example, the coloured Young graph for H_4 is:



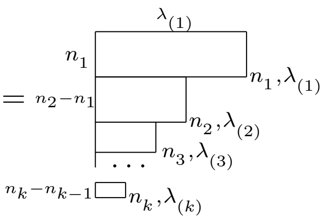
The path $\{\emptyset \xrightarrow{1} \bullet \xrightarrow{q^2} \bullet\bullet \xrightarrow{q^{-2}} \begin{smallmatrix}\bullet & \bullet \\ \bullet & \end{smallmatrix} \xrightarrow{1} \begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\}$ corresponds to the tableau $[\nu]_4 := \begin{array}{|c|c|}\hline {}^1\,1 & {}^2\,q^2 \\ \hline {}^3\,q^{-2} & {}^4\,1 \\ \hline\end{array}$ with content string $(1, q^2, q^{-2}, 1)$: the shape of the tableau is given by the shape of the last vertex of the path while the labels of nodes of the tableau shows in which sequence the points $\bullet$ appear in the vertices along the path. The edge indices of the path are eigenvalues of the Jucys-Murphy elements: $(1, q^2, q^{-2}, 1) \in \mathrm{Spec}(y_1, y_2, y_3, y_4)$ corresponding to the values of y_i on the primitive idempotent $e\left([\nu]_4\right)$. Thus, we associate a standard Young tableau with n nodes (related to a string in $\mathrm{Spec}(y_1, \ldots, y_n)$ and, correspondingly, to the primitive orthogonal idempotent of H_n) with a path which starts from the vertex $\emptyset$ and goes down to the vertex with Young diagram with n nodes (the path with n edges in the coloured Young graph). Denote by $X(n)$ the set of all such paths and by $\mathrm{Str}(n)$ the set of the strings $\Lambda = (a_1, \ldots, a_n)$ of numbers $a_i = q^{2m_i}$ satisfying conditions (3.4). We collect the above construction in the following statement.

Proposition 6. *There is a bijection between the set $T(n)$ of the standard Young tableaux with n nodes, the set $\mathrm{Spec}(y_1, \ldots, y_n)$, the set $\mathrm{Str}(n)$ and the set $X(n)$ of the paths of length n in the Young graph: $T(n) \leftrightarrow \mathrm{Spec}(y_1, \ldots, y_n) \leftrightarrow \mathrm{Str}(n) \leftrightarrow X(n)$.*

The dimension of the irreducible representation of $H_n(q)$ (corresponding to the Young diagram λ with n nodes) is equal to the number of standard tableaux $[\nu]_n$ of shape λ or, as we saw, to the number of paths which lead to this Young diagram from the top vertex $\emptyset$. This number is given by a Frobenius formula $d_\lambda = n!(h_1! \ldots h_k!)^{-1} \prod_{i<j}(h_i - h_j)$, where k is the number of rows in λ and h_i are hook lengths of the nodes in the first column of λ (see, e.g., [7]).

Since the coloured Young graph for H_{M+1} contains the whole information about the spectrum of y_k, we can deduce the expressions (in terms of the elements y_k) of all orthogonal primitive idempotents for the Hecke algebra using the inductive procedure proposed in [7]. This special set of primitive orthogonal idempotents has also been described in [4].

Let λ be a Young diagram with $n = n_k$ rows: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $|\lambda| := \sum_{i=1}^n \lambda_i$ be the number of its nodes. Consider the case when $\lambda_1 = \ldots = \lambda_{n_1} = \lambda_{(1)} > \lambda_{n_1+1} = \lambda_{n_1+2} = \ldots = \lambda_{n_2} = \lambda_{(2)} > \ldots > \lambda_{n_k - n_{k-1}+1} = \ldots = \lambda_{|\lambda|} = \lambda_{(n_k)}$:

$\lambda =$ 

(4.1)

Here $(n_i, \lambda_{(i)})$ are coordinates of the nodes corresponding to the corners of the diagram λ. Consider any standard Young tableau $[\nu]_{|\lambda|}$ of shape (4.1). Let $e([\nu]_{|\lambda|}) \in H_{|\lambda|}$ be a primitive idempotent corresponding to the tableau $[\nu]_{|\lambda|}$. Taking into account the branching rule implied by the coloured Young graph for $H_{|\lambda|+1}$ we conclude that the following identity holds

$$e([\nu]_{|\lambda|}) \prod_{r=1}^{k+1} \left(y_{|\lambda|+1} - q^{2(\lambda_{(r)} - n_{r-1})} \right) = 0 \, ,$$

where $\lambda_{(k+1)} = n_0 = 0$. Thus, for a new tableau $[\nu_j]_{|\lambda|+1}$ which is obtained by adding to the tableau $[\nu]_{|\lambda|}$ of shape (4.1) a new node with coordinates $(n_{j-1}+1, \lambda_{(j)}+1)$ we obtain the following primitive idempotent (after a normalization)

$$e([\nu_j]_{|\lambda|+1}) := e([\nu]_{|\lambda|}) \prod_{\substack{r=1 \\ r \neq j}}^{k+1} \frac{\left(y_{|\lambda|+1} - q^{2(\lambda_{(r)} - n_{r-1})} \right)}{\left(q^{2(\lambda_{(j)} - n_{j-1})} - q^{2(\lambda_{(r)} - n_{r-1})} \right)} = e([\nu]_{|\lambda|}) \, \Pi_j \, . \tag{4.2}$$

Using this formula and "initial data" $e\left(\boxed{1}\right) = 1$, one can deduce step by step explicit expressions for all primitive orthogonal idempotents for Hecke algebras.

5 q-dimensions for Young diagrams

Consider a linear map $Tr_{d(m+1)}$: $H_{m+1}(q) \to H_m(q)$ from the Hecke algebra $H_{m+1}(q)$ to its subalgebra $H_m(q)$ such that ($\forall X, Y \in H_m(q)$, $Z \in H_{m+1}(q)$)

$$\begin{gathered} Tr_{d(m+1)}(X) = z_d\, X \;, \;\; Tr_{d(m+1)}(X\, Z\, Y) = X\, Tr_{d(m+1)}(Z)\, Y \;\;, \\ Tr_{d(m+1)}(\sigma_m^{\pm 1} X \sigma_m^{\mp 1}) = Tr_{d(m)}(X) \;, \quad Tr_{d(m+1)}(\sigma_m) = 1 \;, \\ Tr_{d(m)} Tr_{d(m+1)}(\sigma_m Z) = Tr_{d(m)} Tr_{d(m+1)}(Z \sigma_m) \;, \end{gathered} \tag{5.1}$$

where z_d is a constant which we fix as $z_d = \frac{1-q^{-2d}}{q-q^{-1}}$ for later convenience. Then one can define an Ocneanu's trace $\mathcal{T}r^{(m+1)}$: $H_{m+1}(q) \to \mathbf{C}$ as a sequence of maps $\mathcal{T}r^{(m+1)} := Tr_{d(1)} Tr_{d(2)} \cdots Tr_{d(m+1)}$.

Proposition 7. *Ocneanu's traces of idempotents $e([\nu]_{|\lambda|})$, $e([\nu']_{|\lambda|})$ corresponding to tableaux $[\nu]_{|\lambda|}$, $[\nu']_{|\lambda|}$ of the same shape λ coincide. Thus,*

$$\mathrm{qdim}(\lambda) := \mathcal{T}r^{(|\lambda|)} e([\nu]_{|\lambda|}) = \mathcal{T}r^{(|\lambda|)} e([\nu']_{|\lambda|})$$

depends on the diagram λ only.

Using (5.1) we deduce an identity (see Appendix)

$$1 + (q-q^{-1}) Tr_{d(|\lambda|+1)} \left(\frac{y_{|\lambda|+1}\, \tau}{1 - y_{|\lambda|+1}\, \tau} \right) = \frac{(1-\tau\, q^{-2d})}{(1-\tau)} \prod_{k=1}^{|\lambda|} \frac{(1-\tau\, y_k)^2}{(1-q^2 \tau y_k)(1-q^{-2}\tau y_k)} \;, \tag{5.2}$$

where τ is a parameter. To calculate "qdim" for the diagram (4.1) we need to find the value of the element (5.2) on the idempotent $e([\nu]_{|\lambda|})$, where $[\nu]_{|\lambda|}$ is any Young tableau of shape (4.1). We take the "row-standard" tableau $[\nu]_{|\lambda|}$ corresponding to the eigenvalues of y_k arranged along the rows from left to right and from top to bottom:

$$\begin{array}{l} y_1 = 1, \; y_2 = q^2, \; y_3 = q^4, \; \dots, \; y_{\lambda_1 - 1} = q^{2(\lambda_1 - 2)}, \; y_{\lambda_1} = q^{2(\lambda_1 - 1)}, \\ y_{\lambda_1 + 1} = q^{-2}, \; y_{\lambda_1+1} = 1, \; \dots, \; y_{\lambda_1 + \lambda_2} = q^{2(\lambda_2 - 2)}, \\ \dots\dots\dots\dots\dots\dots, \\ y_{|\lambda| - \lambda_n + 1} = q^{-2(n-1)}, \; \dots, \; y_{|\lambda|} = q^{2(\lambda_n - n)} \;. \end{array}$$

The result is ($n_k = n, n_0 := 0$)

$$Tr_{d(|\lambda|+1)} \left(\sum_j P_j \frac{(q-q^{-1})\, \mu_j\, \tau}{1-\mu_j\, \tau} \right) = e([\nu]_{|\lambda|}) \left(\frac{1-\tau\, q^{-2d}}{1-\tau q^{-2n}} \prod_{j=1}^{k} \frac{1-\tau\, q^{2(\lambda_{(j)} - n_j)}}{1-\tau\, \mu_j} - 1 \right) , \tag{5.3}$$

where we have inserted into the l.h.s. the spectral decomposition of the idempotent $e([\nu]_{|\lambda|})$ (see (4.2)):

$$e([\nu]_{|\lambda|}) = e([\nu]_{|\lambda|}) \sum_j \Pi_j = \sum_j P_j \;, \quad P_j\, y_{|\lambda|+1} = P_j\, q^{2(\lambda_{(j)} - n_{j-1})} = P_j\, \mu_j \;.$$

The operator P_j projects $y_{|\lambda|+1}$ on its eigenvalue $\mu_j := q^{2(\lambda_{(j)} - n_{j-1})}$ which appeared in the denominator of the r.h.s. of (5.3). Comparing both sides of eq. (5.3) we deduce

$$Tr_{d(|\lambda|+1)}\, (P_j) = e([\nu]_{|\lambda|}) \lim_{\tau \to 1/\mu_j} \frac{(1-\mu_j\, \tau)}{(q-q^{-1})} \left(\frac{1-\tau\, q^{-2d}}{1-\tau q^{-2n}} \prod_{r=1}^{k} \frac{1-\tau\, q^{2(\lambda_{(r)} - n_r)}}{1-\tau\, \mu_r} \right)$$

$$= e([\nu]_{|\lambda|}) \cdot q^{-d} \, [q^{(\lambda_{(j)} - n_{j-1} + d)}]_q \, \frac{\prod_{n,m \in \lambda} [h_{n,m}]_q}{\prod_{n,m \in \lambda^{(j)}} [h_{n,m}]_q} \, , \tag{5.4}$$

where $h_{n,m}$ are hook lengths of nodes (n, m) of the diagrams λ or $\lambda^{(j)}$ ($\lambda^{(j)}$ is a diagram obtained by adding to the diagram λ a new node with coordinates $(n_{j-1} + 1, \lambda_{(j)} + 1)$). Applying the Ocneanu's trace $\mathcal{T}r^{(|\lambda|)}$ to eq. (5.4) we find a recurrent relation:

$$\mathrm{qdim}(\lambda^{(j)}) = \mathrm{qdim}(\lambda) \, q^{-d} \, [\lambda_{(j)} - n_{j-1} + d]_q \, \frac{\prod_{n,m \in \lambda} [h_{n,m}]_q}{\prod_{n,m \in \lambda_j} [h_{n,m}]_q} \, ,$$

which is solved by

$$\mathrm{qdim}(\lambda) = q^{-d|\lambda|} \prod_{n,m \in \lambda} \frac{[d + m - n]_q}{[h_{n,m}]_q} \, .$$

Up to a normalization factor this formula has firstly been obtained in [5].

For R-matrix representations of $H_{M+1}(q)$ (about R-matrix representations of the Hecke algebra see Refs. [9], [10]) which corresponds to the quantum supergroup $GL_q(N|M)$, the parameter d equals $N - M$. This justifies our choice of the parametrization of z_d in the first eq. of (5.1).

Proposition 7 can be generalized. Let T be a quantum matrix satisfying

$$\hat{R}_{12} \, T_1 \, T_2 = \hat{R}_{12} \, T_1 \, T_2 \tag{5.5}$$

in the notations of [10], where $\hat{R}_{12} = \rho(\sigma_1)$ is the R-matrix representation of the Hecke algebra.

Proposition 8. *The quantum traces (for the definition of the quantum trace see e.g. [10], [11], [12]) of the matrices* $[T_1 \cdots T_{|\lambda|} \, \rho(e([\nu]_{|\lambda|}))]$ *and* $[T_1 \cdots T_{|\lambda|} \, \rho(e([\nu']_{|\lambda|}))]$

$$\chi_\lambda(T) := Tr_{R(1 \to |\lambda|)} \left(T_1 \cdots T_{|\lambda|} \, \rho(e([\nu]_{|\lambda|})) \right) = Tr_{R(1 \to |\lambda|)} \left(T_1 \cdots T_{|\lambda|} \, \rho(e([\nu']_{|\lambda|})) \right) \, ,$$

corresponding to tableaux $[\nu]_{|\lambda|}$ *and* $[\nu']_{|\lambda|}$ *of the same shape* λ*, coincide. Thus,* $\chi_\lambda(T)$ *depends only on the diagram* λ*.*

Consider the $GL_q(N)$ quantum group (5.5) with a standard $GL_q(N)$ Drinfeld-Jimbo R-matrix $\hat{R}_{12}$ [10]. It is known [9], [10] that the standard $GL_q(N)$ matrix $\hat{R}_{12}$ defines the representation of the Hecke algebra. We note that the $GL_q(N)$ quantum matrix T can be realized by arbitrary numerical diagonal $(N \times N)$ matrix X. Then $\chi_\lambda(X)$ is a numerical function of the deformation parameter q and the entries of X. In the classical limit $q \to 1$ the operator $\rho(e([\nu]_{|\lambda|}))$ tends to the Young projector and the function $\chi_\lambda(X)$ coincides with a character of the element X ($X \in GL(N)$) in the representation corresponding to the diagram λ.

6 Appendix

Taking into account the definition of the generators y_m we have equations

$$\frac{1}{(t - y_{m+1})} \sigma_m^{-1} = \sigma_m^{-1} \frac{1}{(t - y_m)} + \frac{\lambda y_m}{(t - y_{m+1})} \frac{1}{(t - y_m)} \tag{6.1}$$

$$\frac{1}{(t - y_{m+1})} \sigma_m = \sigma_m^{-1} \frac{1}{(t - y_m)} + \frac{\lambda t}{(t - y_m)} \frac{1}{(t - y_{m+1})} \, , \tag{6.2}$$

where $\lambda := q - q^{-1}$. Eqs. (6.1), (6.2) and the definition of the map (5.1) give a recurrent relation

$$\frac{(t - q^2 y_m)(t - q^{-2} y_m)}{(t - y_m)^2} Z_{m+1} = Z_m + \frac{\lambda y_m}{(t - y_m)^2} \left[1 - \lambda z_d\right] \ , \qquad (6.3)$$

where the parameter z_d is introduced in (5.1) and

$$Z_m := Tr_{d(m)} \left(\frac{1}{(t - y_m)} \right) \ .$$

Eq. (6.3) is simplified by the substitution $Z_m = \tilde{Z}_m - \left[1 - \lambda z_d\right] / (\lambda t)$ and we have

$$\frac{(t - q^2 y_m)(t - q^{-2} y_m)}{(t - y_m)^2} \tilde{Z}_{m+1} = \tilde{Z}_m \ .$$

This equation can be easily solved and finally we obtain the expression

$$Z_{m+1} = \frac{1}{\lambda t} \left(1 + \frac{\lambda \, z_d}{(t-1)} \right) \prod_{k=1}^{m} \frac{(t - y_k)^2}{(t - q^2 y_k)(t - q^{-2} y_k)} - \frac{1}{\lambda t} \left[1 - \lambda z_d\right] \ ,$$

which is equivalent to (5.2) for $t = 1/\tau$.

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Modular Invariants, Graphs and Quantum Symmetries

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Abstract

We propose a computational method allowing us to find the entire set of toric matrices associated to a modular invariant and to recover the corresponding Ocneanu graph, using as input data the modular splitting equation, the algebra of characters and the modular invariant.

Introduction and general aspects

The original ADE classification of modular invariant partition functions of affine $SU(2)$ conformal field theories [2] has become the starting point of a rich development in the field of mathematical physics. The original identification was mostly justified by the fact that exponents of the corresponding ADE Lie algebra were in correspondence with the diagonal entries of the modular invariant partition function, but the diagram itself was not an ingredient of the model. Later the occurrence of ADE diagrams in the classification of affine $SU(2)$ models changes when V.B. Pasquier stated [12] that the diagrams actually participate in the construction of the symmetry algebra of the field theory.

About ten years ago, the occurrence of ADE diagrams in the affine $SU(2)$ classification was understood in a rather different way. The observation (already present in [12]) was that the vector space spanned by the vertices of the diagram A_n possesses an associative and commutative algebra structure encoded by the diagram. this "graph algebra" is the truncation at certain level of the Weyl alcoves of $\widehat{g}_k$ and is isomorphic to the fusion algebra of irreps of $\widehat{g}_k$.

The algebra of quantum symmetries: Associated to every ADE Dynkin diagram G there exist a special kind of weak Hopf algebra (or quantum groupoid) $\mathcal{B}G$ which is finite dimensional and semi-simple for its two associative structures [9]. Existence of a coproduct on the underlying vector space (and on its dual) allows us to define two -usually distinct- algebras of characters living on the same vector space. The first one, called the fusion algebra, and denoted $A(G)$, is identified with the graph algebra of A_n in the sense of Pasquier [12]. The second algebra of characters is called the **algebra of quantum symmetries** denoted by $Oc(G)$; it is an associative – but not necessarily commutative – algebra with two generators. This algebra comes with a particular basis, and the multiplication of its basis elements by the two generators is encoded by the so called Ocneanu graph which is also denoted by $Oc(G)$. The quantum groupoid $\mathcal{B}G$ has several interesting properties, for instance, $Oc(G)$ is a bimodule on the graph algebra $A(G)$, the double action of $A(G)$ on $\mathcal{B}G$ $\lambda\, x\, \mu = \sum_y (W_{xy})_{\lambda\mu}\, y$ is encoded by the set of "toric matrices" W_{xy}. Toric matrices have positive and integer entries and establish the relation with CFT: in effect the toric matrix corresponding to $x = y = 0$ is modular invariant, and when contracted with the characters of an affine algebra, gives the partition function of the corresponding CFT, $\mathcal{Z} = \sum_{ij} \chi_i (W_{00})_{ij} \bar{\chi}_j$. The others matrices have been interpreted [13] as giving CFT with twisted boundary conditions and defect lines labelled by x, y.

A. Ocneanu showed ([10][11]) that this construction can be generalized for highther Coxeter-Systems, in particular he showed the resulting graphs giving the classification of affine $SU(3)$ and $SU(4)$ theories and suggested that the construction (the quantum grupoid structure) was straightforward generalizable for higher levels. However the explicit construction of the quantum grupoid $\mathcal{BG}$ is a very complicate problem, and except for $SU(2)$ and to some extent $SU(3)$, the Coxeter-Dynkin system itself is not a priory known. Whereas the modular invariants are provided by the algorithms of T. Gannon [6], this allows to explore up to rather high levels the toric structures associated to it.

The objective of our work (see [4] and [8]) is to propose a computational method allowing to find the entire set of toric matrices associated to a modular invariant and to recover the corresponding Ocneanu graph, using as input data the modular splitting equation, the algebra of character or fusion algebra and the modular invariant itself. In most of the cases this information is enough to explicitly construct the algebra of quantum symmetries associated to the modular invariant.

Modular Splitting

Let $\lambda, \mu, \nu, \ldots$ and $x, y, z, \ldots$ denote the vertices of $A(G)$ and $Oc(G)$ respectively, associativity in these two algebras imposes conditions which result in a constraint equation called the double fusion equation, this equation can be written in terms of double annular matrices [13] [3] or in terms of toric matrices W_{xy} [15]

$$\sum_{\lambda''\mu''} (N_\lambda)_{\lambda'\lambda''} (N_\mu)_{\mu'\mu''} (W_{xy})_{\lambda''\mu''} = \sum_z (W_{0z})_{\lambda\mu} (W_{z0})_{\lambda'\mu'} \tag{1}$$

Take x and y equal to zero, the l.h.s. of (1) involves only known quantities, namely, the modular matrix $M = W_{0,0}$ and the fusion coefficients $N_{\lambda,\mu}^{\nu}$ of the $A(G)$ algebra . For each pair $(\lambda, \mu) \in \{1, 2, \ldots d_A = dimA(G)\}$ the l.h.s. of (1) is a known $d_A \times d_A$ matrix $K_{\lambda\mu} = \left(N_\lambda \cdot M \cdot N_\mu^T\right)$ containing only positive and integer entries, and the r.h.s. involves the set of toric matrices W_{z0} and W_{0z} to be determined.

$$K_{\lambda\mu} = \sum_{z \in Oc(G)} (W_{0z})_{\lambda,\mu} W_{z0} \tag{2}$$

This is the so called **modular splitting** (MS) equation, its solution gives the set of toric matrices associated to M, and the method used to compute the matrices in the r.h.s of (2) is called Modular Splitting (MS) algorithm.

Non degenerate case

Consider the case in which all the matrices W_{z0} are different, then the toric matrices form a basis of the vectorial space K spanned by the matrices $K_{\lambda\mu}$, the dimension of K is equal to the number of points in the Ocneanu graph $d_O = Tr(M \cdot M^\dagger)$ [1]. In order to solve equation (2) we use the following relation [14] giving the norm of a matrix $K_{\lambda\mu}$ $(K_{\lambda\mu})_{\lambda^*\mu^*} = \sum_z \mid (W_{0z})_{\lambda\mu} \mid^2$. Because coefficients $(W_{0z})_{(\lambda\mu)}$ are positive integers, the number of terms in the expansion (2) can be deduced from its norm as follows:

- If $(N_\lambda \cdot M \cdot N_\mu)_{\lambda^*\mu^*} = 1 = 1^2 \Longrightarrow K_{\lambda\mu} = W_{z_{(\lambda\mu)}0}$. Then $K_{(\lambda\mu)}$ is a toric matrix.
- If $(N_\lambda \cdot M \cdot N_\mu)_{\lambda^*\mu^*} = 2 = 1^2 + 1^2 \Longrightarrow K_{\lambda\mu} = W_{z_{(\lambda\mu)}0} + W_{z'_{(\lambda\mu)}0}$.
- If $(N_\lambda \cdot M \cdot N_\mu)_{\lambda^*\mu^*} = 3 = 1^2 + 1^2 + 1^2 \Longrightarrow K_{\lambda\mu} = W_{z_{(\lambda\mu)}0} + W_{z'_{(\lambda\nu)}0} + W_{z"_{(\lambda\mu)}0}$.

Consider the pairs (λ, μ) s.t. $(N_\lambda \cdot W_{00} \cdot N_\mu^T)_{(\lambda^* u^*)} = 1$, then $W_{z_{\lambda\mu}} = K_{\lambda\mu}$ is a toric matrix for each $z_{\lambda\mu}$. We call $N \ni z_{\lambda\mu}$ the set of already known toric matrices.

Next consider the list of pairs (λ, μ) s.t. $(N_\lambda \cdot W_{00} \cdot N_\mu^T)_{(\lambda^*\mu^*)} = 2$. In this case the corresponding matrix $K_{\lambda\mu}$ is the sum of two toric matrices, and there are three possibilities: Either $K_{\lambda\mu}$ is the sum of two already known toric matrices, or it is the sum of an already known toric matrix and a new one, or it is equal to twice a new toric matrix. In any case it is enough to calculate the set of differences $K_{\lambda\mu} - W_{z,0}$ with $z \in N$, and impose that all the components of such differences should be positive integers to recuperate a new toric matrix. The resulting matrices are added to N.

The next step is to consider the list of pairs (λ, μ) s.t. $(N_\lambda \cdot W_{00} \cdot N_\mu^T)_{(\lambda^*\mu^*)} = 3, 4, 5$, etc. and to generalize the previous discussion.

The process stops, ultimately, since the dimension of K is finite.

* * *

Once the set of toric matrices has been computed, they are introduced in the double fusion equation in order to obtain the generalized MS equation:

$$N_\lambda \cdot W_{x0} \cdot N_\mu^T = \sum_z (W_{x,z})_{\lambda\mu} W_{z,0} \tag{3}$$

This new equation involves toric matrices on the l.h.s. and twisted toric matrices W_{xy} on the r.h.s., for each $x \in Oc(G)$ equation (3) is the expansion of the set of matrices $K^x_{\lambda,\mu} = N_\lambda \cdot W_{x0} \cdot N_\mu^T$ in the basis of K. In the present case the vectors giving the expansion of a matrix $K^x_{\lambda,\mu}$ are know, and only the coefficients remain to be determined. The solution is then straightforward, but depending on the size and the number of the matrices it can be a heavy computational task, more details about this computation can be found in [4] and [8]. Fixing $(\lambda, \mu) = \{(1,0), (0,1)\}$ the solution of equation (3) gives the pair of matrices $(V_{10})_{xy} = (W_{xy})_{10}$ and $(V_{01})_{xy} = (W_{xy})_{o1}$

The Ocneanu graph and the algebra of quantum symmetries. Denoting by 0 and 1 the identity and the fundamental vertices of of the graph $A(G)$, the matrices $V_{1,0}$ and $V_{0,1}$ are respectively the left and right chiral adjacency matrices of the left and right chiral parts of the Ocneanu graph [9],[14],[3].

The Ocneanu graph is the Cayley graph of the algebra of quantum symmetries, this algebra comes with two algebraic generators called chiral generators and denoted by 1_L and 1_R. The two adjacency matices V_{10} and V_{01} are associated to these generators, and encodes the multiplication of the vertices of $Oc(G)$ by 1_L and 1_R respectively. We denote this two matrices as $O_{1_L} = V_{(1,0)}$ and $O_{1_R} = V_{(0,1)}$, and multiplication of a vertex by the chiral generators is given by:

$$x\,\underline{1} = \textstyle\sum_y (O_{\underline{1}})_{x,y}\, y \qquad\qquad x\,\underline{1}' = \textstyle\sum_y (O_{\underline{1}'})_{x,y}\, y \tag{4}$$

In all known Ocneanu graphs, this contains all the graphs associated to the $SU(2)$, $SU(3)$ and $SU(4)$ systems, each chiral part of $Oc(G)$ is composed of a direct sum of a chiral graph sub algebra and one or several modules over this sub algebra. The two chiral sub algebras, denoted by $\mathcal{G}_R$ and $\mathcal{G}_L$, are generated by the chiral generators 1_R and 1_L and have the same Coxeter number as $A(G)$.

The multiplication between vertices of $Oc(G)$ reads $x \cdot y = \sum_z (O_x)_{y,z}\, z$ and is encoded by a set of matrices, called Ocneanu matrices, containing positive and integer coefficients. The compatibility of this product with the associativity forces the matrices O_x to form a representation of the quantum symmetries algebra

$$O_x \cdot O_y = \sum_z (O_x)_{y,z} O_z \tag{5}$$

For some cases (for example: A_n, E_6, E_8 and D_{2n} of the $SU(2)$ family, or $\mathcal{A}_n, \mathcal{A}_n^*, \mathcal{D}_3, \mathcal{E}_5$) a $d_O \times d_O$ representation of $Oc(G)$ can be constructed from equations (4) by substituting the vertices x by their corresponding matrix O_x. In these cases the matrices have the form of polynomials $O_x = Pol_x(I_{d_O \times d_O}, O_1, O_{1'})$ of the identity matrix and the two chiral generators. In other more complicated examples ($D_{2n-1}, \mathcal{E}_9$ and $\mathcal{E}_9$ of $SU(3)$) a direct, and more complex, solution of equation (5) has to be implemented in order to recover the structure of the algebra of quantum symmetries.

Degenerate case:

In general, the dimension of K can be strictly smaller that the number d_O of vertices of the Ocneanu graph, this happens when the same toric matrix is associated to more than one vertex of $Oc(G)$. This is for example the case for the graph D_4 which has rank equal to 5 but where $d_O = 8$. In this case toric matrices do not form a basis of K, and the dimension of the vector space is given by the rank of the matrix with entries $K_{(\lambda\mu);(\lambda'\mu')} = (N_\lambda \cdot M \cdot N_\mu)_{\lambda'\mu'}$. This dimension gives the number of different toric matrices to be recuperated from the MS equation , the dimension of $Oc(G)$ d_O provides information about the multiplicity of the toric matrices.

In this degenerate case the analysis of the decomposition of each vector $K_{\lambda\mu}$ is slightly more complicate. Actually for matrices $K_{\lambda\mu}$ s.t. $K_{(\lambda\mu)(\lambda^*\mu^*)} \geq 2$ the decomposition of the $K_{\lambda\mu}$ can contain one or more repeated matrices and the corresponding toric matrices with multiplicities have to be separated by hand.

Once the set of toric matrices has been determined, the set of twisted toric matrices can be recovered using the same method described for the non degenerate case. The only remark concerns the solution of equation (3) which comes with undetermined coefficients corresponding to the different possibilities for identifying the repeated toric matrices with the vertices of $Oc(G)$. The solution to this problem is provided by the structure of the

$Oc(G)$ itself: from all the possible choices obtained after solving equation (3), in general only one defines well defined graph-modules on the two chiral sub algebras $\mathcal{G}_R$ and $\mathcal{G}_L$.

Some explicit cases

First of all, the $SU(2)$ cases have all been recovered using the modular splitting method, even if this examples do not add any new information about the ADE classification, they help to develop the method and gives useful information about the bialgebra $\mathcal{B}G$. The results for these cases can be found in [15] and references therein, and the detailed solution of the case E_6 in [4].

For $SU(3)$ the results for the (already known) cases A_n and the exceptional $\mathcal{E}_5$ have also been recovered. The solution of the case $\mathcal{E}_9$ of the $SU(3)$ system is a new result, although the graph $Oc(\mathcal{E}_9)$ was presented by A. Ocneanu in [11] the set of toric matrices has never been presented, as well as the structure of the algebra of Quatum Symmetries. There are 48 different toric matrices with multiplicities s.t. $Oc(\mathcal{E}_9)$ has dimension 72. The algebra of quantum symmetries is composed of three copies of the graph $\mathcal{E}_9$ and three copies of the orbifold $\mathcal{E}_9/3$ denoted by M_9. The structure is as follows $Oc(\mathcal{E}_9) = E_9 \oplus e_9 \oplus e_9 \oplus M_9 \oplus M_9 \oplus M_9$. The first one is a graph sub algebra corresponding to the graph $\mathcal{E}_9$, the others are graph module on both $Oc(\mathcal{E}_9)$ and the sub graph E_9. The details of this problem as well as the solution of the orbifold case appears in [8].

Another development which this approach allows is the exploration of the structure for non ADE cases. For instance the analysis of the quantum symmetries of the F_4 case has been realized in [4], using as starting point a partition function Z_{F_4} obtained as restriction of the E_6 modular invariant, which has the interesting characteristic of being invariant not under the action of the modular invariant group, but of the congruence sub group $\Gamma_0^{(2)}$. With this problem one actually recover a graph of type F_4 as sub graph of the candidate for the graph of quantum symmetries associated to the partition function. Nevertheles incongruence with the structure of $\mathcal{B}G$ has been founded, suggesting that a new kind of structure should be explored.

Conclusions

We propose a method allowing to compute candidates for the quantum symmetry algebra associated to modular invariants of affine $SU(N)$ models. The objectives are two: first, to complete the already known researche, which often uses the list of graphs presented by A. Ocneanu, by proposing a method for generating these graphs and to establish the identification with the corresponding partition function. Second, to obtain, by means of the study of examples, information which will be useful for the formal study of the quantum grupoid structure associated to a given modular invariant. This last is an algebraic formalism which is far from being completely understood, and that participates as the basis of many physical models and theories. The immediate next step consist of solving some higher level examples ($SU(4)$ for instance), and to construct the quantum groupoids of some examples like E_6 or another non A_n $SU(2)$ example. Another more ambitious objective consist in learning to explicitly compute the bialgebra $\mathcal{B}G$ for higher level problems, and to develop useful applications to physical theories like String Theories and Brane Theories.

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$\mathcal{N} = (1, 1/2)$ Supersymmetric $U(N)$ Gauge Theory

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Abstract

We study deformed supersymmetry in $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in non(anti)commutative $\mathcal{N} = 1$ superspace. Using the component formalism, we construct deformed $\mathcal{N} = (1, 1/2)$ supersymmetry explicitly. We also discuss central extension of the deformed supersymmetry.

1 Introduction

Supersymmetric field theories in non(anti)commutative superspace [1, 2] has been attracted much interests from the viewpoint of effective field theories on D-branes in the graviphoton background [3, 4, 5]. Superstrings in this background provide some interesting low-energy physics in $\mathcal{N} = 2$ supersymmetric field theories [6] and their $\mathcal{N} = 1$ deformations [7]. It would be important to study $\mathcal{N} = 2$ supersymmetric gauge theories in non(anti)commutative superspace in order to understand graviphoton effects in the low-energy effective theories from the microscopic point of view.

It is convenient to use $\mathcal{N} = 2$ extended non(anti)commutative superspace for studying non(anti)commutative gauge theories where supersymmetry is manifestly realized [8, 9, 10, 11, 12, 13, 14]. In particular, $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory in non(anti)commutative $\mathcal{N} = 2$ harmonic superspace has been studied [11, 15, 16]. The authors discussed the deformed Lagrangian up to the first order in the deformation parameter C of the superspace and examined their deformed symmetries. It is, however, difficult to calculate higher order C-corrections and extend the $U(1)$ gauge group to $U(N)$.

There exist two cases such that the deformed Lagrangian of $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory becomes simple. One is the case of the singlet deformation where the deformation parameter belongs to the singlet representation of the R-symmetry group $SU(2)$ [9, 10, 17, 13, 14]. The other is the case that one introduces only deformation into $\mathcal{N} = 1$ subsuperspace of $\mathcal{N} = 2$ superspace. In a recent paper [16], it is shown that the $O(C)$ Lagrangian of the $U(1)$ theory defined in non(anti)commutative $\mathcal{N} = 2$ harmonic superspace leads to the theory in the non(anti)commutative $\mathcal{N} = 1$ superspace [2] by the reduction of deformation parameters and some field redefinitions. It is also shown that the theory has $\mathcal{N} = (1, 1/2)$ supersymmetry consistent with the Poisson structure of the theory. Here $\mathcal{N} = (1, 1/2)$ means that there are two chiral and one antichiral supercharges, as in [10].

In this article we study deformed supersymmetry in $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in no(anti)commutative $\mathcal{N} = 1$ superspace. Using component formalism, we construct deformed $\mathcal{N} = (1, 1/2)$ supersymmetry explicitly. We also discuss central extension of the deformed supersymmetry. This article is based on the papers[24].

2 Non(anti)commutative $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory

We begin with reviewing the $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in the deformed $\mathcal{N} = 1$ superspace [18]. Let $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ $(m = 0, \ldots, 3,\ \alpha, \dot{\alpha} = 1, 2)$ be supercoordinates of $\mathcal{N} = 1$ superspace and $\sigma^m_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{m\dot{\alpha}\alpha}$ Dirac matrices. We will study Euclidean space-time so that chiral and antichiral fermions transform independently under the Lorentz transformations. $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$ and $\bar{Q}^{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\theta_\alpha\bar{\sigma}^{m\dot{\alpha}\alpha}\partial_m$ are supercharges. $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$ and $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta_\alpha\bar{\sigma}^{m\dot{\alpha}\alpha}\partial_m$ are the supercovariant derivatives. $\sigma^{mn} = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)$, and $\bar{\sigma}^{mn} = \frac{1}{4}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)$ are the Lorentz generators. Here we will follow the conventions of Wess and Bagger [19].

The non(anti)commutativity in $\mathcal{N} = 1$ superspace is introduced by the $*$-product:

$$f * g(x, \theta, \bar{\theta}) = f(x, \theta, \bar{\theta}) \exp\left(-\frac{1}{2}\overleftarrow{Q}_\alpha C^{\alpha\beta}\overrightarrow{Q}_\beta\right) g(x, \theta, \bar{\theta}). \tag{1}$$

Using this $*$-product, the anticommutation relations for θ become

$$\left\{\theta^\alpha, \theta^\beta\right\}_* \;=\; C^{\alpha\beta} \tag{2}$$

while the chiral coordinates $y^m = x^m + i\theta\sigma^m\bar{\theta}$ and $\bar{\theta}$ are still commuting and anticommuting coordinates, respectively.

$\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in this deformed supespace can be constructed by vector superfields V, chiral superfields Φ and an anti-chiral superfields $\bar{\Phi}$, where Φ and $\bar{\Phi}$ belong to the adjoint representation of $U(N)$. We introduce the basis t^a $(a = 1, \cdots, N^2)$ of Lie algebra of $U(N)$, normalized as $\mathrm{tr}(t^a t^b) = k\delta^{ab}$. The Lagrangian is

$$\mathcal{L} = \frac{1}{k}\int d^2\theta d^2\bar{\theta}\, \mathrm{tr}(\bar{\Phi} * e^V * \Phi * e^{-V}) + \frac{1}{16kg^2}\mathrm{tr}\left(\int d^2\theta W^\alpha * W_\alpha + \int d^2\bar{\theta}\bar{W}_{\dot{\alpha}} * \bar{W}^{\dot{\alpha}}\right) \tag{3}$$

where g denotes the coupling constant. $W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$ and $\bar{W}_{\dot{\alpha}} = \frac{1}{4}D^2 e^{-V}\bar{D}_{\dot{\alpha}} e^V$ are the chiral and antichiral field strengths. Note that multiplication of superfields are defined by the $*$-product.

This Lagrangian is invariant under the gauge transformations $\Phi \to e^{-i\Lambda} * \Phi * e^{i\Lambda}$, $\bar{\Phi} \to e^{-i\bar{\Lambda}} * \bar{\Phi} * e^{i\bar{\Lambda}}$ and $e^V \to e^{-i\bar{\Lambda}} * e^V * e^{i\Lambda}$. To write down the Lagrangian in terms of component fields, it is convenient to take the Wess-Zumino(WZ) gauge as in the commutative case. Since the $*$-product deforms the gauge transformation, it is necessary to redefine the component fields such that these transform canonically under the gauge transformation[2, 18]. For $\mathcal{N}=2$ $U(N)$ theory, these superfields in the WZ gauge are

$$\begin{aligned}\Phi(y,\theta) &= A(y)+\sqrt{2}\theta\psi(y)+\theta\theta F(y),\\ \bar{\Phi}(\bar{y},\bar{\theta}) &= \bar{A}(\bar{y})+\sqrt{2}\bar{\theta}\bar{\psi}(\bar{y})+\bar{\theta}\bar{\theta}\left(\bar{F}+iC^{mn}\partial_m\left\{v_n,\bar{A}\right\}-\frac{1}{4}C^{mn}\left[v_m,\left\{v_n,\bar{A}\right\}\right]\right)(\bar{y}),\\ V(y,\theta,\bar{\theta}) &= -\theta\sigma^\mu\bar{\theta}v_\mu(y)+i\theta\theta\bar{\theta}\bar{\lambda}(y)-i\bar{\theta}\bar{\theta}\theta^\alpha\left(\lambda_\alpha+\frac{1}{4}\varepsilon_{\alpha\beta}C^{\beta\gamma}\left\{(\sigma^\mu\bar{\lambda})_\gamma,v_\mu\right\}\right)(y)\\ &\quad+\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D-i\partial^\mu v_\mu)(y).\end{aligned}\qquad(4)$$

Here $\bar{y}^m=x^m-i\theta\sigma^m\bar{\theta}$ are the antichiral coordinates and $C^{mn}=C^{\alpha\beta}\varepsilon_{\beta\gamma}(\sigma^{mn})_\alpha{}^\gamma$. Since σ^{mn} is self-dual, C^{mn} is also self-dual. Substituting (4) into the Lagrangian (3), we obtain the deformed Lagrangian written in terms of component fields. In this expression, however, normalizations of two fermions ψ and λ are different. In order to see symmetries between two fermions manifestly, it is useful to rescale V to $2gV$ and $C^{\alpha\beta}$ to $\frac{1}{2g}C^{\alpha\beta}$. Then the Lagrangian takes the form $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1$. Here $\mathcal{L}_0$ is the undeformed Lagrangian with the topological term:

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{k}\mathrm{tr}\Big(-\frac{1}{4}F^{mn}F_{mn}-\frac{1}{4}F^{mn}\tilde{F}_{mn}-i\bar{\lambda}\bar{\sigma}^m D_m\lambda+\frac{1}{2}\tilde{D}^2\\ &\quad-(D^m\bar{A})D_mA-i\bar{\psi}\bar{\sigma}^m D_m\psi+\bar{F}F-i\sqrt{2}g[\bar{A},\psi]\lambda-i\sqrt{2}g[A,\bar{\psi}]\bar{\lambda}-\frac{g^2}{2}[A,\bar{A}]^2\Big),\end{aligned}\quad(5)$$

where $F_{mn}=\partial_m v_n-\partial_n v_m+ig[v_m,v_n]$, $\tilde{F}_{mn}=\frac{1}{2}\epsilon_{mnpq}F^{pq}$ and $D_m\lambda=\partial_m\lambda+ig[v_m,\lambda]$ etc. We have also introduced an auxiliary field $\tilde{D}$ defined by $\tilde{D}=D+g[A,\bar{A}]$ in order to see undeformed $\mathcal{N}=2$ supersymmetry in a symmetric way. $\mathcal{L}_1$ is the C-dependent part of the Lagrangian:

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{k}\mathrm{tr}\left(-\frac{i}{2}C^{mn}F_{mn}\bar{\lambda}\bar{\lambda}+\frac{1}{8}|C|^2(\bar{\lambda}\bar{\lambda})^2\right.\\ &\quad\left.+\frac{i}{2}C^{mn}F_{mn}\{\bar{A},F\}-\frac{\sqrt{2}}{2}C^{\alpha\beta}\{D_m\bar{A},(\sigma^m\bar{\lambda})_\alpha\}\psi_\beta-\frac{1}{16}|C|^2[\bar{A},\bar{\lambda}][\bar{\lambda},F]\right).\end{aligned}\quad(6)$$

Here $|C|^2=C^{mn}C_{mn}$.

3 Deformed $\mathcal{N}=(1,1/2)$ supersymmetry

For $C=0$, the action is invariant under $\mathcal{N}=2$ supersymmetry transformations, where only $\mathcal{N}=1$ supersymmetry generated by Q_α and $\bar{Q}^{\dot{\alpha}}$ are manifestly realized in $\mathcal{N}=1$ superspace. Other $\mathcal{N}=1$ supersymmetry would be realized manifestly when we use $\mathcal{N}=2$ extended superspace. In particular $\mathcal{N}=2$ harmonic superspace [20] provides very efficient tools to study off-shell $\mathcal{N}=2$ supersymmetric field theories. The most general non(anti)commutative deformations are studied by using extended superspace.

In [11, 15, 16], the component formalism of $\mathcal{N}=2$ supersymmetric $U(1)$ gauge theory has been studied. In particular, for generic deformation, $\mathcal{N}=(1,0)$ deformed supersymmetry has been constructed up to the first order of the deformation parameters. When the deformation parameters are reduced such that only $\mathcal{N}=1$ subspace becomes non(anti)commutative, the deformed supersymmetry is enhanced to $\mathcal{N}=(1,1/2)$ supersymmetry. This is because supersymmetries other than $\bar{Q}^{\dot{\alpha}}$ is consistent with the Poisson structure of the deformed superspace [10]. In this reduced case, it is shown that the $O(C)$ Lagrangian defined in the $\mathcal{N}=2$ harmonic superspace is equal to that of the deformed $\mathcal{N}=1$ superspace by the field redefinitions.

We now study deformed supersymmetry in the $U(N)$ gauge theory. The undeformed superfield action is invariant under $\mathcal{N}=1$ supersymmetry generated by $\xi Q+\bar{\xi}\bar{Q}$. Since this transformation does not preserve the WZ gauge, we need to do gauge transformation to retain the WZ gauge. Then the (undeformed) supersymmetry transformations δ^0_ξ and $\delta^0_{\bar{\xi}}$ of the component fields in the WZ gauge are

$$
\begin{aligned}
\delta^0_\xi v_m &= i\xi\sigma_m\bar{\lambda},\\
\delta^0_\xi \lambda &= i\xi\tilde{D}-ig\xi[A,\bar{A}]+\sigma^{mn}\xi F_{mn}, \quad \delta^0_\xi\bar{\lambda}=0,\\
\delta^0_\xi \tilde{D} &= -\xi\sigma^m D_m\bar{\lambda}+\sqrt{2}g[\xi\psi,\bar{A}],\\
\delta^0_\xi A &= \sqrt{2}\xi\psi, \quad \delta^0_\xi\psi=\sqrt{2}\xi F, \quad \delta^0_\xi F=0,\\
\delta^0_\xi \bar{A} &= 0, \quad \delta^0_\xi\bar{\psi}=\sqrt{2}i\bar{\sigma}^m\xi D_m\bar{A}, \quad \delta^0_\xi\bar{F}=i\sqrt{2}\xi\sigma^m D_m\bar{\psi}-2gi\xi[\bar{A},\lambda],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\delta^0_{\bar{\xi}} v_m &= i\bar{\xi}\bar{\sigma}_m\lambda,\\
\delta^0_{\bar{\xi}} \lambda &= 0, \quad \delta^0_{\bar{\xi}}\bar{\lambda}=-i\bar{\xi}\tilde{D}+ig\bar{\xi}[A,\bar{A}]+\bar{\sigma}^{mn}\bar{\xi}F_{mn},\\
\delta^0_{\bar{\xi}} \tilde{D} &= \bar{\xi}\bar{\sigma}^n D_n\lambda+\sqrt{2}g[A,\bar{\xi}\bar{\psi}],\\
\delta^0_{\bar{\xi}} A &= 0, \quad \delta^0_{\bar{\xi}}\psi=\sqrt{2}i\sigma^m\bar{\xi}D_m A, \quad \delta^0_{\bar{\xi}}F=\sqrt{2}i\bar{\xi}\bar{\sigma}^m D_m\psi+2gi\bar{\xi}[\bar{\lambda},A],\\
\delta^0_{\bar{\xi}} \bar{A} &= \sqrt{2}\bar{\xi}\bar{\psi}, \quad \delta^0_{\bar{\xi}}\bar{\psi}=\sqrt{2}\bar{\xi}\bar{F}, \quad \delta^0_{\bar{\xi}}\bar{F}=0.
\end{aligned}
\tag{8}
$$

The remaining $\mathcal{N}=1$ supersymmetry denoted by δ^0_η and $\delta^0_{\bar{\eta}}$ can be obtained from (7) by using the R-symmetry: $\xi\to\eta,\quad \lambda\to-\psi,\quad \psi\to\lambda,\quad \tilde{D}\to-\tilde{D},\quad F\to\bar{F}$.

Now we will construct the deformed $\mathcal{N}=(1,1/2)$ supersymmetry which keeps the $U(N)$ Lagrangian $\mathcal{L}$ invariant up to the total derivatives. The term $\mathcal{L}_1$ is not invariant under the undeformed supersymmetry transformations δ^0_ξ, δ^0_η and $\delta^0_{\bar{\eta}}$. Since the deformed term $\mathcal{L}_1$ is a polynomial in C, we denote $\mathcal{L}_1^{(n)}$ $(n\geq 1)$ by its n-th order term in C. The deformed supersymmetry transformations can be expanded in the form $\delta=\delta^0+\delta^1+\cdots$. Here δ^n is the n-th order term in C. δ^n is determined recursively by solving the conditions $\delta^1\mathcal{L}_0+\delta^0\mathcal{L}_1^{(1)}=0$ and $\delta^2\mathcal{L}_0+\delta^1\mathcal{L}_1^{(1)}+\delta^0\mathcal{L}_2^{(1)}=0$ and so on.

The deformed transformation δ_ξ, which was calculated in [18], takes the form $\delta_\xi=\delta^0_\xi+\delta^1_\xi$ and is given by

$$
\begin{aligned}
\delta_\xi v_m &= i\xi\sigma_m\bar{\lambda},\\
\delta_\xi \lambda_\alpha &= i\xi_\alpha\tilde{D}-ig\xi_\alpha[A,\bar{A}]+(\sigma^{mn}\xi)_\alpha\left(F_{mn}+\frac{i}{2}C_{mn}\bar{\lambda}\bar{\lambda}\right), \quad \delta_\xi\bar{\lambda}=0,\\
\delta_\xi \tilde{D} &= -\xi\sigma^m D_m\bar{\lambda}+\sqrt{2}g[\xi\psi,\bar{A}],\\
\delta_\xi A &= \sqrt{2}\xi\psi, \quad \delta_\xi\psi=\sqrt{2}\xi F, \quad \delta_\xi F=0,\\
\delta_\xi \bar{A} &= 0,\\
\delta_\xi \bar{\psi} &= \sqrt{2}i\bar{\sigma}^m\xi D_m\bar{A},\\
\delta_\xi \bar{F} &= i\sqrt{2}\xi\sigma^m D_m\bar{\psi}-2gi[\bar{A},\xi\lambda]+C^{mn}D_m\left\{\bar{A},\xi\sigma_n\bar{\lambda}\right\}.
\end{aligned}
\tag{9}
$$

Note that the transformation of Φ is undeformed. The deformed transformation δ_η, which relate the gauge field v_m to chiral fermion ψ, can be calculated in a similar way. But as in the analysis of $U(1)$ case, it is necessary to calculate up to the order $O(C^2)$. The result is

$$\begin{aligned}
\delta_\eta v_m &= -i\eta\sigma_m\bar\psi - \frac{\sqrt2}{2}C^{\alpha\beta}\eta_\alpha\left\{\bar A,(\sigma_m\bar\lambda)_\beta\right\},\\
\delta_\eta\lambda^\alpha &= \sqrt2\eta^\alpha\bar F\\
&\quad -\frac{\sqrt2}{2}C^{\alpha\beta}\eta_\beta\left\{\tilde D,\bar A\right\} - \frac{\sqrt2 i}{2}C^{\alpha\beta}(\sigma^{mn}\eta)_\beta\left\{F_{mn},\bar A\right\} - \frac{\sqrt2 g}{2}C^{\alpha\beta}\eta_\beta\left\{\bar A,[\bar A,A]\right\}\\
&\quad +\frac{\sqrt2}{4}\det C\left(\{\bar\lambda\bar\lambda,\bar A\} + 2\bar\lambda_{\dot\alpha}\bar A\bar\lambda^{\dot\alpha}\right)\eta^\alpha,\\
\delta_\eta\bar\lambda &= \sqrt2 i\bar\sigma^m\eta D_m\bar A,\\
\delta_\eta\tilde D &= -\eta\sigma^m D_m\bar\psi - \sqrt2 g[\eta\lambda,\bar A] - \frac{\sqrt2}{2}iC^{\alpha\beta}\eta_\beta D_m\left\{\bar A,(\sigma^m\bar\lambda)_\alpha\right\},\\
&\quad -igC^{\alpha\beta}\eta_\beta\left\{\bar A,[\bar A,\psi_\alpha]\right\},\\
\delta_\eta A &= \sqrt2\eta\lambda + iC^{\alpha\beta}\eta_\beta\left\{\psi_\alpha,\bar A\right\},\\
\delta_\eta\psi^\alpha &= i\eta^\alpha\tilde D + ig\eta^\alpha[A,\bar A] - \varepsilon^{\alpha\beta}(\sigma^{mn}\eta)_\beta F_{mn} - iC^{\alpha\beta}\eta_\beta\left\{(\bar\lambda\bar\lambda) - \left\{\bar A,F\right\}\right\},\\
\delta_\eta F &= i\sqrt2\eta\sigma^m D_m\bar\lambda + 2gi[\bar A,\eta\psi],\\
\delta_\eta\bar A &= 0,\\
\delta_\eta\bar\psi_{\dot\alpha} &= C^{\alpha\beta}\eta_\beta\sigma^m_{\alpha\dot\alpha}\left\{\bar A,D_m\bar A\right\},\\
\delta_\eta\bar F &= \sqrt2 gC^{\alpha\beta}\eta_\beta\left\{\bar A,[\bar A,\lambda_\alpha]\right\} + \frac{\sqrt2 i}{4}\det C\Big[3\left\{\bar A,\left\{\eta\sigma^m\bar\lambda,D_m\bar A\right\}\right\}\\
&\quad +2D_m\bar A\bar A\eta\sigma^m\bar\lambda + 2\eta\sigma^m\bar\lambda\bar A D_m\bar A + 2\left\{\bar A,\left\{\eta\sigma^m D_m\bar\lambda,\bar A\right\}\right\}\Big]
\end{aligned}\tag{10}$$

Here we have used the formula $\det C = |C|^2/4$. Note that there is an ambiguity to determine the δ_η transformation as noticed in the $U(1)$ case [16]. In fact, for arbitrary functions $f_1(\bar A)$ and $f_2(\bar A)$ of $\bar A$, the transformation

$$\begin{aligned}
\tilde\delta_\eta\lambda^\alpha &= \eta^\alpha f_1 F + F f_2\eta^\alpha,\\
\tilde\delta_\eta\bar F &= i(\eta\sigma^n D_n\bar\lambda)f_1 + if_2(\eta\sigma^n D_n\bar\lambda) + i\sqrt2 g[\bar A,\eta\psi]f_1 + \sqrt2 if_2[\bar A,\eta\psi]
\end{aligned}\tag{11}$$

leaves the action invariant. In formulas (10), we have chosen f_1 and f_2 such that we recover the $U(1)$ result. This ambiguity would be fixed if we use non(anti)commutative $\mathcal N = 2$ harmonic superspace.

The deformed transformation $\delta_{\bar\eta}$ is found to be

$$\begin{aligned}
\delta_{\bar\eta} v_m &= -i\bar\eta\bar\sigma_m\psi\\
\delta_{\bar\eta}\lambda^\alpha &= \sqrt2 i\varepsilon^{\alpha\beta}(\sigma^m\bar\eta)_\beta D_m A + iC^{\alpha\beta}\left\{\bar\eta\bar\lambda,\psi_\beta\right\}, \quad \delta_{\bar\eta}\bar\lambda = \sqrt2\bar\eta F,\\
\delta_{\bar\eta}\tilde D &= \bar\eta\bar\sigma^m D_m\psi - \sqrt2 g[A,\bar\eta\bar\lambda],\\
\delta_{\bar\eta} A &= 0, \quad \delta_{\bar\eta}\psi = 0, \quad \delta_{\bar\eta}F = 0,\\
\delta_{\bar\eta}\bar A &= \sqrt2\bar\eta\bar\lambda,\\
\delta_{\bar\eta}\bar\psi &= -i\bar\eta\tilde D - ig\bar\eta[A,\bar A] - \bar\sigma^{mn}\bar\eta F_{mn},\\
\delta_{\bar\eta}\bar F &= \sqrt2 i\bar\eta\bar\sigma^m D_m\lambda - 2gi[\bar\eta\bar\psi,A] + C^{\alpha\beta}(\sigma^m\bar\eta)_\alpha D_m\left\{\psi_\beta,\bar A\right\}\\
&\quad -\frac{\sqrt2}{4}\det C\left\{3\{\bar\eta\bar\lambda,\bar\lambda\bar\lambda\} + \bar\eta\bar\lambda\bar A F + \bar A\bar\eta\bar\lambda F - 2\bar\eta\bar\lambda F\bar A + 2\bar\lambda_{\dot\alpha}(\bar\eta\bar\lambda)\bar\lambda^{\dot\alpha}\right\}.
\end{aligned}\tag{12}$$

Note that if we set $N = 1$, the cubic terms in $\bar\lambda$ and the commutators vanish. We then recover the $U(1)$ results obtained in [16].

4 Deformed Central Charge

We now compute the Noether currents associated with deformed $\mathcal{N} = (1,0)$ supersymmetry transformations δ_ξ and δ_η. Let X^m_ξ be the total derivative term obtained from the variation of the Lagrangian associated with the transformation δ_ξ:

$$\delta_\xi \mathcal{L} = \partial_m X^m_\xi .$$

Then the supercurrent $N^m_{1\alpha}$ is defined by

$$\xi^\alpha N^m_{1\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_m \varphi_A)} \delta_\xi \varphi_A - X^m_\xi \tag{13}$$

where φ_A are component fields in the WZ gauge. The other supercurrent $N^m_{2\alpha}$ associated with the transformation δ_η is defined in a similar way. From the transformations (9), we get

$$\begin{aligned} \xi N^m_1 \;=\;& \frac{1}{k}\mathrm{tr}\Big\{ -i(F^{mn} + \tilde F^{mn})\xi\sigma_n\bar\lambda + \sqrt{2} D_n\bar A \xi\sigma^n\bar\sigma^m\psi + g\xi\sigma^m\bar\lambda[A,\bar A] \\ & +(\xi\sigma_n\bar\lambda)C^{mn}\bar\lambda\bar\lambda - (\xi\sigma_n\bar\lambda)C^{mn}\left\{\bar A, F\right\}\Big\}. \end{aligned} \tag{14}$$

The supercurrent N^m_2 is given by

$$\begin{aligned} \eta N^m_2 \;=\;& \frac{1}{k}\mathrm{tr}\Big\{ i(F^{mn} + \tilde F^{mn})\eta\sigma_n\bar\psi + \sqrt{2} D_n\bar A \eta\sigma^n\bar\sigma^m\lambda - g\eta\sigma^m\bar\psi[A,\bar A] \\ & -\frac{\sqrt{2}}{2}C^{\alpha\beta}\left\{F^{mn} + \tilde F^{mn}, \bar A\right\}\eta_\alpha(\sigma_n\bar\lambda)_\beta - C^{mn}\eta\sigma_n\bar\lambda\left(\bar\lambda\bar\lambda - \left\{\bar A, F\right\}\right) \\ & +iC^{\alpha\beta}\left\{\bar A, D_n\bar A\right\}\eta_\alpha(\sigma^n\bar\sigma^m\psi)_\beta + ig\frac{\sqrt{2}}{2}C^{mn}\eta\sigma_n\bar\lambda\left\{\bar A,[\bar A, A]\right\} \\ & -i\frac{\sqrt{2}}{2}\det C\eta\sigma^m\bar\lambda\left(\left\{\bar A, \bar\lambda\bar\lambda\right\} - \left\{\bar A,\left\{\bar A, F\right\}\right\}\right)\Big\}, \end{aligned} \tag{15}$$

which contains $O(C^2)$ corrections. For $C = 0$, we recover the undeformed supercurrents [21, 22]. The supercharge $Q_{i\alpha}$ is defined by

$$Q_{i\alpha} = \int d^3x N_{i\alpha}(x).$$

We now examine the anticommutation relations for supercharges $Q_{i\alpha}$. We will use the equal-time anticommutation relations for fermions

$$\left\{\psi_\alpha(x), \bar\psi_{\dot\alpha}(y)\right\} = \delta_{\alpha\dot\alpha}\delta^3(x-y), \quad \left\{\lambda_\alpha(x), \bar\lambda_{\dot\alpha}(y)\right\} = \delta_{\alpha\dot\alpha}\delta^3(x-y). \tag{16}$$

From (14), (15) and (16), we find that $\{Q_{1\alpha}, Q_{1\beta}\}$ and $\{Q_{1\alpha}, Q_{2\beta}\}$ are undeformed:

$$\{Q_{1\alpha}, Q_{1\beta}\} = 0, \tag{17}$$

$$\{Q_{1\alpha}, Q_{2\beta}\} = 2\sqrt{2} i\varepsilon_{\alpha\beta}\int d^3x \frac{1}{k}\mathrm{tr}\Big[(F_{0\ell} + \tilde F_{0\ell})D^\ell \bar A\Big]. \tag{18}$$

The r.h.s. of (18) comes from the 1st and 2nd terms in (14) and (15) and we have eliminated auxiliary fields by using the equations of motion. Eq. (18) is nothing but the central charge obtained by Witten and Olive [21].

The C-deformation arises in the anticommutation relation $\{Q_{2\alpha}, Q_{2\beta}\}$, which is given by

$$\{Q_{2\alpha}, Q_{2\beta}\} = 4C_{\alpha\beta} \int d^3x \frac{1}{k} \mathrm{tr}\Big[(F_{0\ell} + \tilde{F}_{0\ell}) D^\ell \bar{A}^2\Big]. \tag{19}$$

The r.h.s. of (19) is obtained from the anticommutation relation among the 1st, 2nd, 4th and 7th terms in the current (15). Eq. (19) gives still the topological charge but its dependence on the vacuum expectation value of the Higgs fields is different from the undeformed topological charge (18).

5 Conclusions and Discussion

In this paper we have discussed the deformed $\mathcal{N} = (1, 1/2)$ supersymmetry in $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in the non(anti)commutative $\mathcal{N} = 1$ superspace. We have found the $\mathcal{N} = (1, 0)$ supersymmetry algebra admits non-trivial central extension which depend on the deformation parameter C.

It is an interesting problem to find monopole and dyon solutions and study how the BPS structure is modified by the non(anti)commutativity. It is also interesting to study nonperturbative effects of this non(anti)commutativity in the strong coupling region of the theory[23, 24].

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Various Noncommutativities from Twisted Hopf Algebra

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Abstract

We investigate the non(anti)commutative superspace in terms of Drinfel'd twisted Hopf algebra. We find that a twisted super Poincaré algebra causes some valuable non(anti)commutativity in superspace. It is realized in twisted Lorentz and twisted supersymmetric way clearly by construction.

1 Introduction

Field theory in noncommutative space-time is old subject and becomes common in theoretical physics. In particular, noncommutative field theory draws recently our attention in relation to superstring theory. In superstring theory with some configuration of background fields, coordinates x^μ on D3-brane become noncommutative[1];

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}. \tag{1}$$

It was pointed out that non-anticommutativity of fermionic coordinate θ^α in $\mathcal{N} = 1$ four-dimensional superspace can also arise[2, 3],

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \tag{2}$$

though that is formulated only in Euclidean space. Here $\Theta^{\mu\nu}$ and $C^{\alpha\beta}$ are constant parameters with antisymmetric and symmetric indices respectively. These from superstring theory indicate that the true description of our world may be non(anti)commutative field theory in four-dimensional (super)space, in some energy region.

Apart from higher theory, noncommutative theory is defined and investigated practically in the language of effective quantum field theory. However if we treat a noncommutative theory within quantum field theory, inevitably noncommutative parameters, often dimensionful, are introduced into the theory. To make matters worse, it commonly causes symmetry breaking. For example, it is well known that a noncommutative relation (1) breaks Lorentz symmetry.

Recently an idea to improve the situation is suggested [4, 5]. Chaichian *et. al* claimed that a original symmetry of a theory is broken by introducing noncommutativity indeed, but the deformed symmetry can remain. Our work [6] is essentially an extension to a supersymmetric case of their work.

2 Twisted Super Poincaré Algebra

The strategy of [4, 5] is to realize the noncommutative space (1) as the representation of a deformed Poincaré algebra.

Instead of Poincaré algebra we start with super Poincaré algebra, then deform it after the fashion of [4] to obtain the non(anti)commutative superspace.

Universal enveloping super Poincaré algebra $\mathcal{U}(\mathcal{SP})$ can become a Hopf algebra over $\mathcal{K}$, where $\mathcal{K}$ is the base field or ring of the Hopf algebra, by defining certain maps. The definitions of the maps for $X \in \mathcal{SP}$ and a unit element $\mathbf{1}$ of Hopf algebra are as follows:

$$\begin{aligned}
\text{product} &: m(X \otimes Y) = XY, \\
\text{unit} &: i(k) = k\mathbf{1}, \\
\text{coproduct} &: \Delta(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X,\ \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \\
\text{counit} &: \epsilon(X) = 0,\ \epsilon(\mathbf{1}) = 1, \\
\text{antipode} &: \gamma(X) = -X,\ \gamma(\mathbf{1}) = \mathbf{1}, \quad \text{for} X, Y \in \mathcal{SP}\ k \in \mathcal{K}
\end{aligned} \tag{3}$$

These definitions are extended to whole $\mathcal{U}(\mathcal{SP})$ recursively.

Because of fermionic generators in super Poincaré algebra, we should slightly change the multiplication rule of the Hopf algebra into $\mathbf{Z}_2$ graded one, such that:

$$(A \otimes B)(C \otimes D) = (-1)^{|B||C|}(AC \otimes BD). \tag{4}$$

Here $|A|$ stands for the fermion number of A.

A Hopf algebra is deformed to another Hopf algebra by the twist operation systematically. We choose a twist element $\mathcal{F}$ which is a invertible biproduct element in the Hopf algebra. A Hopf algebra is deformed by the twist element and such twisted Hopf algebra is redefined only by changing the coproduct and antipode in such a way that:

$$\begin{aligned}
\Delta_t(H) &= \mathcal{F}\Delta(H)\mathcal{F}^{-1}, \\
\gamma_t(H) &= U\gamma(H)U^{-1} \qquad U = \mathcal{F}_{(1)}\gamma(\mathcal{F}_{(2)}).
\end{aligned} \tag{5}$$

We use Sweedler's notation:

$$\mathcal{F} = \sum_i \mathcal{F}_1^{(i)} \otimes \mathcal{F}_2^{(i)} \equiv \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)}. \tag{6}$$

Twist element $\mathcal{F}$ must satisfy two conditions. First is the twist equation,

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F}, \tag{7}$$

which provides coassociativity of the twisted Hopf. Second is the counit condition

$$(\epsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \epsilon)\mathcal{F}. \tag{8}$$

Note that the twist operation do not deform the algebra structure and other maps.

In concert with the twisting, product of the representation on which the Hopf algebra acts is modified for compatibility:

$$m(a \otimes b) = ab \quad \rightarrow \quad m(\mathcal{F}^{-1} a \otimes b) = a \star b. \tag{9}$$

The twist equation (7) guarantees the associativity of this star product.

A proper twist element is easily constructed from the elements of the Abelian subalgebra. In super Poincaré algebra, an Abelian subalgebra is made up of translation generator P^μ and supercharge Q^α or P^μ and anti-supercharge $\bar{Q}^{\dot{\alpha}}$. You cannot choose both Q^α and $\bar{Q}^{\dot{\alpha}}$ because they do not (anti)commute.

P-*P* twist element

$$\mathcal{F}^{PP} = \exp\left(\frac{i}{2}\Theta^{\mu\nu} P_\mu \otimes P_\nu\right), \tag{10}$$

which is the same as [4] provides the noncommutative relation (1) in the coordinate representation;

$$\begin{aligned} [x_\mu, x_\nu]_\star &= x_\mu \star x_\nu - x_\nu \star x_\mu = m(\mathcal{F}^{-1}(x_\mu \otimes x_\nu - x_\nu \otimes x_\mu)) \\ &= i\Theta_{\mu\nu} \end{aligned}$$

Next we consider *Q*-*Q* twist for the non-anticommutativity of superspace.

$$\mathcal{F}^{QQ} = \exp\left(-\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta\right) \tag{11}$$

This element satisfy the condition (7),(8) and gives following commutators.

$$\begin{aligned} \{\theta^\alpha, \theta^\beta\}_\star &= C^{\alpha\beta}, \\ [x^\mu, x^\nu]_\star &= C^{\alpha\beta} \sigma^\mu_{\alpha\dot{\gamma}} \sigma^\nu_{\beta\dot{\delta}} \bar{\theta}^{\dot{\gamma}} \bar{\theta}^{\dot{\delta}}, \\ [x^\mu, \theta^\alpha]_\star &= i C^{\alpha\beta} \sigma^\mu_{\beta\dot{\gamma}} \bar{\theta}^{\dot{\gamma}}. \end{aligned}$$

This results are in accord with $\mathcal{N} = 1/2$ SUSY noncommutative deformation by Seiberg [3].

A noncommutativity between x^μ and θ^α is considered too. *P*-*Q* twist

$$\mathcal{F}^{PQ} = \exp\left[\frac{i}{2}\lambda^{\mu\alpha}(P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu)\right] \tag{12}$$

gives

$$\begin{aligned} [x^\mu, x^\nu]_\star &= \lambda^{\mu\alpha} \sigma^\nu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} - \lambda^{\nu\alpha} \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}}, \\ [x^\mu, \theta^\alpha]_\star &= i\lambda^{\mu\alpha}, \\ \{\theta^\alpha, \theta^\beta\}_\star &= 0. \end{aligned} \tag{13}$$

Where $\lambda^{\mu\alpha}$ is a Grassmann constant[1].

We can use more general twist element;

$$\mathcal{F} = \exp\left[\frac{i}{2}\Theta^{\mu\nu} P_\mu \otimes P_\nu + \frac{i}{2}\lambda^{\mu\alpha}(P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu) - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta\right]. \tag{14}$$

[1]In this case, we have to regard $\mathcal{K}$ as not the complex field but the Grassmann ring

The results are:

$$\begin{aligned}
[x^\mu, x^\nu]_\star &= i\Theta^{\mu\nu} + C^{\alpha\beta}\sigma^\mu_{\alpha\dot\gamma}\sigma^\nu_{\beta\dot\delta}\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta} + \lambda^{\mu\alpha}\sigma^\nu_{\alpha\dot\beta}\bar\theta^{\dot\beta} - \lambda^{\nu\alpha}\sigma^{mu}_{\alpha\dot\beta}\bar\theta^{\dot\beta}, \\
[x^\mu, \theta^\alpha]_\star &= i\lambda^{\mu\alpha} + iC^{\alpha\beta}\sigma^\mu_{\beta\dot\gamma}\bar\theta^{\dot\gamma}, \\
\{\theta^\alpha, \theta^\beta\}_\star &= C^{\alpha\beta}.
\end{aligned}$$

It is not straightforward to apply the method to extended SUSY, because in general anti-commutator $\{Q^I_\alpha, Q^J_\beta\}$ is non-zero central charge. Insteadly we try to do that in some peculiar way. We introduce central charge coordinate z^I and consider the noncommutativity in it. The twist element is

$$\mathcal{F}^{ZZ} = \exp\left(\frac{i}{2}\Xi^{IJ} Z^I \otimes Z^J\right), \quad Z^I = \frac{\partial}{\partial z_I} \tag{15}$$

and gives

$$[z_I, z_J]_\star = i\Xi_{IJ}. \tag{16}$$

This is meaningful only in the region $\mathcal{N} \geq 3$. Some other works in extended SUSY case are [7].

Although we omit the explanation here, we would emphasize that all realizations of non(anti)commutative superspace above are consistent with twisted algebra, and so as to preserve twisted super Poincaré symmetry. For more details, please see [6].

3 Summary and Discussion

We have constructed the twisted super Poincaré algebra with proper twist elements and obtain corresponding commutator relations between coordinates in super space. These non(anti)commutativity is realized to maintain the twisted super Poincaré symmetry. It is interesting to know what type of noncommutativity we can get from twisted Hopf. At this moment we have investigated twisted superconformal algebra and found out the twist element constructed from conformal supercharge and superconformal generators gives a exotic noncommutative superspace[8].

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Twist of quantum group and noncommutative field theory

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Abstract

The role of quantum universal enveloping algebras of symmetries in constructing a noncommutative geometry of space-time and corresponding field theory is discussed. It is shown that in the framework of the twist theory of quantum groups, the noncommutative space-time defined by coordinates with Heisenberg commutation relations is Poincaré invariant, as well as the corresponding field theory. Noncommutative parameters of global transformations are introduced.

One of attempts to study the structure of spacetime at Planck scale is related with a possible noncommutative nature of spacetime, hence, with a noncommutative geometry (see [1] and references therein). In this paper we would like to draw attention to interrelations between noncommutative quantum field theories and quantum groups [2]. Recently, an active research takes place in noncommutative field theory related to noncommutative geometry (see the reviews [3, 4] and references therein). One source of examples of noncommutative geometry is the theory of quantum groups [5, 6]. The reason for this is that the latter are, loosely speaking, deformations of Lie groups, which provide numerous geometric structures. There are corresponding structures in quantum groups (QG), where the commutative algebra of functions $F(G)$ on a Lie group G is deformed into an appropriate noncommutative algebra $F_q(G)$, which is defined e.g. by generators and relations [7]. Homogeneous spaces are also subject to deformation, for example $SL(2) \to SL_q(2)$ or $SU(2) \to SU_q(2)$ and two-dimensional plane $(x, y) \to$ "quantum plane" $(x, y)_q$, or Podlez q-sphere $(x, y, z) \to (x, y, z)_q$. It has been observed by several authors (see e.g. [8, 9, 10, 2]) that the twist theory of quantum groups provides a very useful tool for constructing noncommutative geometry of space-time, including vector bundles, measure, and equations of motion and their solutions.

The most important space of relativistic theory is four-dimensional Minkowski space-time $\mathcal{M}$, with coordinates x^μ, and with the Poincaré algebra acting on x^μ. To construct NC field theory, the commutative algebra of functions $C(\mathcal{M})$ on $\mathcal{M}$ is deformed to a noncommutative (NC) algebra $C_\theta(\mathcal{M})$. This algebra is generated by NC coordinates x^μ, and probably the simplest relations among the x^μ are

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu = i\theta^{\mu\nu}, \qquad (1)$$

with a constant antisymmetric matrix θ (see [3, 4]).

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There are many possible commutation relations (CR) for x^μ, x^ν with the right hand side linear or quadratic in x^μ (see [11, 12]). However, those written above follow from a special limit of string theory [13] and have attracted substantial interest.

To construct a field theory on noncommutative space-time with CR (1) for the coordinates, one has to substitute the commutative algebra of fields (functions on $\mathcal{M}$) by the noncommutative algebra C_θ. In the case of the CR (1) there is a Weyl-Moyal correspondence between these algebras through the Fourier transform. It maps a smooth function $\varphi(x) \in C(\mathcal{M})$ to an element of the algebra C_θ,

$$\varphi(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4k \tilde{\varphi}(k) \exp(ik\hat{x}), \tag{2}$$

with $\tilde{\varphi}(k)$ being the Fourier transform of the function $\varphi(x)$,

$$\tilde{\varphi}(k) = \int d^4x \varphi(x) e^{-ikx}.$$

Then the noncommutative product in the algebra C_θ is

$$\begin{aligned} \varphi(\hat{x}) g(\hat{x}) &= \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k_2) e^{ik_1\hat{x}} e^{ik_2\hat{x}} \\ &= \int \frac{dk_1}{(2\pi)^4} \frac{dk_2'}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k_2' - k_1) e^{-i\theta(k_1, k_2')} e^{ik_2'\hat{x}}, \end{aligned} \tag{3}$$

where the notation $\theta(k,p) := \frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu$ is introduced for the antisymmetric quadratic form.

Interpreting the convolution of $\tilde{\varphi}(k_1)$ and $\tilde{g}(k_2)$ with the weight function $\exp(-i\theta(k_1, k_2))$ as the Fourier transform of a new product ($*$-product) of the elements $\varphi(x)$, $g(x) \in C(\mathcal{M})$ one gets

$$\varphi(x) * g(x) = \int \frac{dk_1}{(2\pi)^4} \frac{dk_2'}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k_2' - k_1) \sum_n \frac{1}{n!} (-i\theta(k_1, k_2'))^n e^{ik_2'x}. \tag{4}$$

It is not difficult to check that this $*$-product on $C(\mathcal{M})$ is still associative, albeit noncommutative. The exponential function $\exp(ik\hat{x})$ generates symmetrized $*$-products of $\hat{x}^\nu$, which coincide with the usual products of commutative x^ν. Let us point out that the "$*$-product" is a general notion of deformation quantization (see the review [14]).

It follows that a field theory on the NC space-time can be constructed using fields $\varphi(x) \in C(\mathcal{M})$, but with multiplication given by the $*$-product. To fix an action one needs a linear functional on C_θ, and it is represented as an integral on $C(\mathcal{M})$ of the usual form, e.g.

$$S[\varphi] = \int dx \{ \frac{1}{2} (\partial_\mu \varphi(x))^2 + \frac{m^2}{2} (\varphi(x))^2 + \frac{g^2}{4!} (\varphi(x))^4_* \}. \tag{5}$$

The integral of the $*$-product of several functions is invariant only under the cyclic permutations, similarly to the trace of operators:

$$\int dx f_1(x) * f_2(x) * \ldots * f_n(x) = \int dx\, f_2(x) * \ldots * f_n(x) * f_1(x).$$

For this reason $\int dx f_1(x) * f_2(x) = \int dx f_1(x) \cdot f_2(x)$, and NC field theory and ordinary field theory conincide on the free field level (the action with quadratic terms only). However, the interaction term being written as the $*$-product of the fields, describes a nonlocal interaction, e.g. for the φ^3_*-theory

$$\begin{aligned}\int dx\,(\varphi(x))^3_* &= \int \prod_a \left(\frac{dk_a}{(2\pi)^4}\tilde{\varphi}(k_a)\right)\exp(-i\sum_{b<c}\theta(k_b,k_c))\delta(\sum_j k_j)\\ &= \int \prod_a dx_a\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\exp(2i(x_1-x_3)\theta^{-1}(x_2-x_3)),\end{aligned}$$

provided that the matrix θ is invertible (or one has to restrict the arguments to those x^j for which θ^{ij} has an inverse).

Quantization of the scalar field theory with the action $S[\varphi]$ by path integral methods yields the standard perturbation theory, but the interaction vertices include an extra oscillating factor,

$$V(k_1,\ldots,k_4) = \frac{g^2}{4!}\delta(\sum_a k_a)\prod_{b<c} e^{-\frac{i}{2}\theta_{\mu\nu}k_b^\mu k_c^\nu}.$$

This factor has only cyclic symmetry (due to the delta-function) and results in different contributions as compared to local QFT, and even in a different structure of the Feynman diagrams (planar versus non-planar graphs). The diagrammatic analysis of unitarity yields a condition on $\theta^{\mu\nu}$: $\theta^{0j}=0$. Thus the time coordinate commutes with the space coordinates, and one can apply the Hamiltonian formalism for the action (5).

Reformulating NC space-time field theory as a usual one (5) with a nonlocal interaction, it is possible to apply standard techniques to quantize it. An obvious drawback is the appearence of the set of constants $\theta^{\mu\nu}$ breaking the Lorentz invariance: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, $\theta^{\mu\nu} \to \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\,\theta^{\alpha\beta} = \widetilde{\theta}^{\mu\nu} \neq \theta^{\mu\nu}$. To cure this problem we propose to use a quantum group technique.

In this discussion we need such objects from the theory of quantum groups as a Hopf algebra $\mathcal{H}$, its $\mathcal{H}$-module algebra $\mathcal{A}$, $\mathcal{H}$-modules and $\mathcal{A}$-modules V, W (linear spaces for $\mathcal{H}$- and $\mathcal{A}$-representations). At the same time these objects have a physical interpretation: $\mathcal{H}$ is the symmetry algebra of the system under consideration, $\mathcal{A}$ is the algebra of observables, and their representation space is the space of states of the system. There are also additional structures, such as a $*$-operation (real form), a scalar product etc., which will be introduced later.

The symmetry of the relativistic field theory is described by the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré Lie algebra $\mathcal{P}$ with generators P_μ of translations and $M_{\mu\nu}$ of rotations:

$$\begin{aligned}[][P_\mu, P_\nu] &= 0,\\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}),\\ [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu).\end{aligned} \tag{6}$$

The essential part of the Hopf algebra structures $\mathcal{H}(m,\Delta,\gamma,\epsilon)$ (see [5, 6] for details) is given by the associative product (with the commutation relations (6) for $\mathcal{U}(\mathcal{P})$ in our case) and by a coproduct map $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ defining an action of the Hopf algebra $\mathcal{H}$ in the tensor product of two (or more) of its representations. The action of the generators $Y \in \mathcal{P}$ in a tensor product $V\otimes W$ is given by the symmetric map (coproduct) $\Delta(Y) = Y\otimes 1 + 1\otimes Y$, or

$$\Delta(Y)(v\otimes w) = (\hat{Y}v)\otimes w + v\otimes(\hat{Y}w), \tag{7}$$

where the hat means the action of a Hopf algebra element in the corresponding representation space. There are two other maps in the definition of the Hopf algebra: the counit $\epsilon : \mathcal{H} \to \mathcal{C}$ (a one-dimensional representation of $\mathcal{H}$) and the antipode $\gamma : \mathcal{H} \to \mathcal{H}$, which is an algebra antihomomorphism. These maps are subject to quite a few axioms, of course [5, 6]. On the generators of $\mathcal{U}(\mathcal{P})$ the antipode and counit are: $\gamma(Y) = -Y$, $\epsilon(Y) = 0, \epsilon(1) = 1$.

There is a useful transformation (twist) of the structure maps of a Hopf algebra, which is an equivalence relation among Hopf algebras, preserving their category of representations. This transformation $\mathcal{H} \to \mathcal{H}_t$ is realized by an invertible twist element $\mathcal{F} = \sum_i f_1^i \otimes f_2^i \in \mathcal{H} \otimes \mathcal{H}$ [15]. It does not change the multiplication in $\mathcal{H}$, but transforms the coproduct according to

$$\Delta(h) \to \Delta_t(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad h \in \mathcal{H}.$$

This similarity transformation preserves the coassociativity of the twisted coproduct if $\mathcal{F}$ satisfies the following twist equation (two-cocycle condition) in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ [15]

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}, \qquad (\epsilon \otimes id)\mathcal{F} = 1 \otimes 1, \tag{8}$$

where $\mathcal{F}_{23}$ means $\sum_i 1 \otimes f_1^i \otimes f_2^i \in \mathcal{H}^{\otimes 3}$, and $(\Delta \otimes id)\mathcal{F} := \sum_i \Delta(f_1^i) \otimes f_2^i \in \mathcal{H}^{\otimes 3}$. The twist does not change the counit homomorphism, but similarity-transforms the antipode:

$$\gamma(Y) \to \gamma_t(Y) = u\gamma(Y)u^{-1}, \quad \text{where} \quad u = \sum_i f_1^i \cdot \gamma(f_2^i) \in \mathcal{H}. \tag{9}$$

Usually the twist element is not symmetric under the permuation of tensor factors: $\mathcal{F} \neq \mathcal{F}_{21} = \sum_i f_2^i \otimes f_1^i$. Hence, the twisted coproduct $\Delta_t(h) := \sum h_{(1)} \otimes h_{(2)}$ is also non-symmetric

$$\Delta_t(h) \neq \Delta_t^{op}(h) = \sum h_{(2)} \otimes h_{(1)}.$$

However, for the quantum group case these coproducts are related by a similarity transformation with the $\mathcal{R}$-matrix:

$$\mathcal{R}\Delta_t = \Delta_t^{op}\mathcal{R}, \quad \mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{H} \otimes \mathcal{H}.$$

In our case, starting with the symmetric coproduct (7) the $\mathcal{R}$-matrix is given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

There are well-known statements from the theory of quantum groups which will be used in our discussion of a particular case of noncommutative space-time. Having an action of $\mathcal{H}$ on an associative algebra $\mathcal{A}$ with consistency of the coproduct of $\mathcal{H}$ and multiplication of $\mathcal{A}$ (a Leibniz rule),

$$\hat{h}(a \cdot b) = \sum (\hat{h}_{(1)}a) \cdot (\hat{h}_{(2)}b),$$

the multiplication in $\mathcal{A}$ has to be changed after twisting $\mathcal{H} \to \mathcal{H}_t$. The new product in $\mathcal{A}_t$ is

$$a * b = \sum (\hat{\bar{f}}{}_1^i a) \cdot (\hat{\bar{f}}{}_2^i b), \quad a, b \in \mathcal{A}_t, \tag{10}$$

where a notation was introduced for $\mathcal{F}^{-1} := \sum \bar{f}_1^i \otimes \bar{f}_2^i$, and the action (representation) of elements from $\mathcal{H}$ on elements from $\mathcal{A}_t$ is the same as before twisting.

The product $\varphi(x_1^\mu) * \varphi(x_2^\nu)$ of quantum fields with independent arguments belongs to the tensor product of two copies of the algebra $C_\theta(\mathcal{M})$. After twisting of $\mathcal{H}$ the elements of different copies of $\mathcal{A} \otimes \mathcal{A}$ will not commute:

$$(a_1 \otimes 1)(1 \otimes a_2) = (a_1 \otimes a_2), \qquad \text{but}$$
$$(1 \otimes a_2)(a_1 \otimes 1) = (\hat{\mathcal{R}}_2 a_1) \otimes (\hat{\mathcal{R}}_1 a_2) \neq (a_1 \otimes a_2), \quad \forall a_1, a_2 \in \mathcal{A}. \tag{11}$$

(Recall that the hat indicates the action of Hopf algebra elements on the relevant representation spaces.)

It is important that real forms survive a twist. Recall that a $*$-operation (real form) on a Hopf algebra $\mathcal{H}$ means an antilinear involutive algebra anti-automorphism and coalgebra automorphism. Due to the uniqueness of the antipode, the identity $\gamma * = * \gamma^{-1}$ is always valid, and one can re-define the real form as $\gamma^{2n}*$ for any integer number n. We can also consider homomorphic and anti-cohomomorphic antilinear operations of the kind $\xi = \gamma^{2n+1}*$.

To ensure consistency between real forms and the action of $\mathcal{H}$ on some $\mathcal{H}$-module algebra $\mathcal{A}$ with anti-involution $a \to \bar{a}$, one has to require $\overline{(ha)} = \gamma(h^*)\bar{a}$, for $h \in \mathcal{H}$ and $a \in \mathcal{A}$. So by the real form of a quantum algebra we will mean a homomorphic and anti-cohomomorphic antilinear involution $\xi = \gamma \circ *$.

Twisting a Hopf algebra $\mathcal{H} \to \mathcal{H}_t$ the same $*$-operation is defined on $\mathcal{H}_t$ if the twist $\mathcal{F}$ satisfies the condition

$$\mathcal{F}^* = \sum f_1^* \otimes f_2^* = \mathcal{F}^{-1} = \sum \bar{f}_1 \otimes \bar{f}_2. \tag{12}$$

For the involution ξ the analogous natural requirement is [8]

$$(\xi \otimes \xi)\mathcal{F} = \tau(\mathcal{F}) := \mathcal{F}_{21} = \sum f_2 \otimes f_1, \tag{13}$$

where τ is the permutation of the factors in $\mathcal{H} \otimes \mathcal{H}$.

Suppose now that $\mathcal{A}$ possesses a measure μ, i.e. a linear functional positive on elements of the form $a \cdot \bar{a}$ (like the function algebra on a locally compact topological space does). The same measure is valid for $\mathcal{A}_t$, for these $\mathcal{H}$-module algebras $\mathcal{A}$ and $\mathcal{A}_t$ coincide as linear spaces [8]. Indeed, we find $a * \bar{a} = \bar{f}_1 a \cdot \bar{f}_2 \bar{a} = \bar{f}_1 a \cdot \overline{(\xi(\bar{f}_2)a)}$. If identity (13) is fulfilled, the relation $\bar{f}_1 \otimes \xi(\bar{f}_2) = \xi(\bar{f}_2) \otimes \bar{f}_1$ holds as well and, consequently, $\bar{f}_1 \otimes \xi(\bar{f}_2)$ can be represented by a sum $\sum \varphi_i \otimes \varphi_i$. Further, we have $a * \bar{a} = \sum \varphi_i a \cdot \overline{\varphi_i a}$, and therefore $\mu(a * \bar{a}) \geq 0$. In case that (12) is true, one can extend the Hopf algebra by adding the square root of the element u that was introduced in (9). It is straightforward that the composition of the coboundary twist with the element $\Delta(u^{-\frac{1}{2}})(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ and successive twist with the element $(u^{-\frac{1}{2}} \otimes u^{-\frac{1}{2}})\mathcal{F}^{-1}(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ obeys (13). This double transformation is carried out by means of the 2-cocycle $\Delta(u^{-\frac{1}{2}})\mathcal{F}^{-1}(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$, and the required property (13) readily follows from (12) and the identity $(u \otimes u)\tau(\gamma \otimes \gamma)(\mathcal{F}^{-1}) = \mathcal{F}\Delta(u)$ fulfilled for any solution to the twist equation [15] (the element u is exactly the same as the one taking part in the definition of the twisted antipode (9)). So we can apply all the previous considerations to this composite twist, which differs from initial one by an inner automorphism only.

Let's deform the Poincaré algebra $\mathcal{U}(\mathcal{P})$ as a Hopf algebra by a simple twist element depending only on the generators of translations P_μ (an abelian subalgebra of $\mathcal{P}$) [2]:

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right) \tag{14}$$

with a constant matrix $\theta^{\mu\nu}$ (we take it to be real and antisymmetric). As an associative algebra $\mathcal{U}_t(\mathcal{P})$ is not changed (we have the same commutation relations of generators $M_{\mu\nu}$, P_α) nor is the coproduct of $P_\alpha : \Delta_t(P_\alpha) = \Delta(P_\alpha)$. However, the coproduct of $M_{\mu\nu}$ is changed:

$$\begin{aligned}\Delta_t(M_{\mu\nu}) &= \mathrm{Ad}\left(\exp\left(\frac{i}{2}\theta^{\alpha\beta}P_\alpha \otimes P_\beta\right)\right)\Delta(M_{\mu\nu}) \\ &= \Delta(M_{\mu\nu}) - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu)\otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right).\end{aligned} \tag{15}$$

It was already mentioned that the coproduct defines an action of the Hopf algebra on the product of elements from $\mathcal{A}$, and the product of $\mathcal{A}$ is also changed accordingly, to be consistent with Δ_t. The algebra $\mathcal{C}(\mathcal{M})$ is generated by the x^μ, and after twisting $\mathcal{C}(\mathcal{M}) \to \mathcal{C}_t(\mathcal{M})$ the new product is

$$\begin{aligned}x^\mu * x^\nu &= \sum(\hat{\bar{f}}_1 x^\mu)(\hat{\bar{f}}_2 x^\nu) \\ &= \textstyle\sum_{k=0}^{\infty}\frac{(i/2)^k}{k!}\prod_{j=1}^{k}\theta^{\mu_j,\nu_j}\left(\partial_{\mu_1}\dots\partial_{\mu_k}x^\mu\right)\left(\partial_{\nu_1}\dots\partial_{\nu_k}x^\nu\right) \\ &= x^\mu x^\nu + \frac{i}{2}\theta^{\mu\nu}.\end{aligned} \tag{16}$$

Hence,

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}, \tag{17}$$

and this yields $\mathcal{C}_t(\mathcal{M}) = \mathcal{C}_\theta$. One can check that with the deformed coproduct (15) these CR are invariant under the action of $M_{\mu\nu}$ [2].

The action of momentum generators P_μ on clanical and quantum fields $\varphi(x)$ is supposed to be the same

$$P_\mu\varphi(x) = i\frac{\partial}{\partial x^\mu}\varphi(x).$$

However, in classical theory fields are given by different smooth functions as elements of $\mathcal{C}(\mathcal{M})$ with Fourier expension (2) and the generators are realized as partial derivatives $P_\mu = i\partial/\partial x^\mu$. In quantum theory $\varphi(x)$ and P_μ are fixed operators as elements of the algebra of observables $\mathcal{A}$. The action of P_μ on $\varphi(x)$ is defined by the commutator

$$P_\mu \cdot \varphi(x) = [P_\mu, \varphi(x)],$$

and having in mind the expension of $\varphi(x)$ in terms of the creation and annihilation operators $a(k)$, $a^\dagger(p)$, one gets

$$[P_\mu, a(k)] = -k_\mu a(k).$$

Using the twist element $\mathcal{F} = \exp(\frac{i}{2}\theta P_\mu \otimes P_\nu)$, we have to change product of observables according to the general rule

$$\begin{aligned}a * b &= m \circ (e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu})(a \otimes b) \\ &= m \circ \Big(\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{-i}{2}\Big)^n\prod_{j=1}^{n}\theta^{\mu_j\nu_j}adP_{\mu_j}\otimes adP_{\nu_j}\Big)(a \otimes b).\end{aligned} \tag{18}$$

Hence, the twisted products of the creation and annihilation operators are

$$a(k) * a(p) = a(k)a(p)e^{-i\theta(k,p)} \tag{19}$$

$$a(k) * a^\dagger(p) = a(k)a^\dagger(p)e^{i\theta(k,p)}. \tag{20}$$

Being expressed in terms of the deformed $*$-product, the commutation relations are

$$\begin{aligned} a(k) * a(p) &= a(p) * a(k) e^{-2i\theta(k,p)} \\ & a(k) * a^\dagger(p) - e^{2i\theta(k,p)} a^\dagger(p) * a(k) = \delta(k-p), \end{aligned} \tag{21}$$

where $\theta(k,p) = -\theta(p,k) = \frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu$. The relations (21) reproduce a scalar Zamolodchikov–Faddeev algebra (cf [16]).

The parameters $\Lambda^\mu_{\ \nu}(\omega)$, a^μ of the global Poincaré transformations generate the algebra of functions $F(G)$ on the Poincaré group G. This commutative algebra $F(G) \simeq (\mathcal{U}(\mathcal{P}))^*$ is dual to $\mathcal{U}(\mathcal{P})$, and after twisting $\mathcal{U}(\mathcal{P})$ the product of the dual Hopf algebra $(\mathcal{U}(\mathcal{P}))^*$ is changed.

An important object connecting a pair of dual Hopf algebras is the canonical element (a bicharacter) [6]

$$\mathcal{T} = \sum e_k \otimes e^k, \quad e_k \in \mathcal{H}^*,\ e^k \in \mathcal{H},$$

where e_k and e^m are dual linear bases of $\mathcal{H}^*$ and $\mathcal{H}$. Here we have

$$\mathcal{T} = \exp(ia^\mu P_\mu)\exp(i\omega^{\mu\nu} M_{\mu\nu}).$$

In the case of the twist (14) the generators $\omega^{\mu\nu}$ or $\Lambda^\mu_{\ \nu}(\omega)$ are the same (commutative), but the a^μ become noncommutative (see [9, 17]),

$$[a^\mu, a^\nu] = i\theta^{\mu\nu} - i\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\theta^{\alpha\beta}. \tag{22}$$

This can be obtained from the RTT-relations [7] using the matrix representation of $\mathcal{U}(\mathcal{P})$ and the R-matrix, or from the general recipe (10) using the $\mathcal{U}(\mathcal{P})$-bimodule structure of $(\mathcal{U}(\mathcal{P}))^*$. Due to the commutativity of $\Lambda(\omega)$, if there are representations V with $\Lambda^\mu_{\ \alpha} = \delta^\mu_{\ \alpha}$, then the a^μ are commutative in such V.

The transformation of the coordinates x^μ is given by the coaction $\delta : C_\theta \to F_\theta(G) \otimes C_\theta$,

$$\tilde{x}^\mu := \delta(x^\mu) = \Lambda^\mu_{\ \alpha} \otimes x^\alpha + a^\mu \otimes 1\,. \tag{23}$$

The transformed generators satisfy the same relations, $[\tilde{x}^\mu, \tilde{x}^\nu] = i\theta^{\mu\nu}$. Hence one can conclude that the noncommutative space-time (1) is invariant under the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$.

Tensoring two copies of the NC space-time algebra, $\mathcal{C}_\theta \otimes \mathcal{C}_\theta$ with generators $x_1^\mu = x^\mu \otimes 1$ and $x_2^\nu = 1 \otimes x^\nu$, one gets their commutation relations according to (11) with R-matrix $\mathcal{R} = \exp\left(-i\theta^{\mu\nu} P_\mu \otimes P_\nu\right)$ [18]:

$$\begin{aligned} & x_1^\mu x_2^\nu - x_2^\nu x_1^\mu := x^\mu \otimes x^\nu - (1 \otimes x^\nu)(x^\mu \otimes 1) \\ &= x^\mu \otimes x^\nu - \sum_{k=0}^{\infty} \frac{(i)^k}{k!} \prod_{j=1}^{k} \theta^{\mu_j,\nu_j} \left(\partial_{\nu_1} \ldots \partial_{\nu_k} x^\mu\right) \otimes \left(\partial_{\mu_1} \ldots \partial_{\mu_k} x^\nu\right) \\ &= x^\mu \otimes x^\nu - x^\mu \otimes x^\nu - i\theta^{\nu\mu} = i\theta^{\mu\nu}. \end{aligned} \tag{24}$$

This property results in an extra factor in the Fourier transform of the vacuum expectation value $\langle \varphi(x_1) * \varphi(x_2) * \cdots * \varphi(x_n) \rangle$ of quantum fields [9].

Similar arguments can be applied in the case of (extended) supersymmetry and of the Poincaré superalgebra $s\mathcal{P}$ with additional supercharges (odd generators) $Q_\alpha, \bar{Q}_{\dot\beta}$ to get a noncommutative superspace as in [19]. The Poincaré Lie superalgebra commutation relations (the commutators below are Z_2-graded, i.e. if both elements are odd it is the anticommutator) are

$$\begin{aligned} [P_\mu, Q_\alpha] &= 0, & [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta, \\ [Q_\alpha, Q_\beta] &= 0, & [M_{\mu\nu}, \bar{Q}_{\dot\beta}] &= i(\bar\sigma_{\mu\nu})_{\dot\beta}^{\ \dot\alpha} \bar{Q}_{\dot\alpha}, \\ [\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}] &= 0, & [Q_\alpha, \bar{Q}_{\dot\beta}] &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu. \end{aligned}$$

The generators P_μ, Q_α define an abelian (supercommutative) subalgebra, and abelian twists depending on odd generatorscan be constructed as in the non-graded case, e.g. $\mathcal{F} = exp(C^{\alpha\beta} Q_\alpha \otimes Q_\beta)$ with symmetric matrix $C^{\alpha\beta} = C^{\beta\alpha}$. The exponent reproduces a Poisson tensor defining superbrackets (see e.g. [20]), and can be used to construct noncommutative superspace preserving super-Poincaré covariance [21, 22].

The algebraic sector of the twisted Hopf superalgebra $\mathcal{U}_t(s\mathcal{P})$ is not changed, as well as the coproduct of the abelian subalgebra of (super)translations with the generators P_μ, Q_α. However, the coproducts of $M_{\mu\nu}$ and $\bar{Q}_{\dot\beta}$ are changed:

$$\begin{aligned} \Delta_t(M_{\mu\nu}) &= \mathcal{F}\Delta(M_{\mu\nu})\mathcal{F}^{-1} \\ &= \Delta(M_{\mu\nu}) - i\{C^{\alpha\beta}(\sigma_{\mu\nu})_\alpha^\gamma + C^{\alpha\gamma}(\sigma_{\mu\nu})_\alpha^\beta\} Q_\gamma \otimes Q_\beta, \end{aligned}$$

$$\Delta_t(\bar{Q}_{\dot\gamma}) = \Delta(\bar{Q}_{\dot\gamma}) + 2C^{\alpha\beta}\sigma^\mu_{\alpha\dot\gamma}\left(Q_\beta \otimes P_\mu - P_\mu \otimes Q_\beta\right).$$

The standard realization of the supercharges $Q_\alpha = \partial/\partial\theta^\alpha - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial/\partial x^\mu$ and $\bar{Q}_{\dot\beta}$, yields noncommutative generators of Minkowski superspace $s\mathcal{M}_t$,

$$[\theta^\alpha, \theta^\beta] = -2C^{\alpha\beta}, \qquad [x^\mu, \theta^\alpha] = 2iC^{\alpha\beta}\sigma^\mu_{\beta\dot\gamma}\bar\theta^{\dot\gamma}, \qquad [x^\mu, x^\nu] = -2C^{\alpha\beta}\sigma^\mu_{\alpha\dot\gamma}\sigma^\nu_{\beta\dot\delta}\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta}.$$

It is important to point out that generators (parameters) of the deformed Poincaré supergroup dual to $M_{\mu\nu}, P_\mu, Q_\alpha$ will not be supercommutative. However, their commutation relations will be different from those of $s\mathcal{M}_t$.

Representing the canonical element $\mathcal{T}$ of the twisted Poincaré superalgebra $\mathcal{U}_t(sP)$ and its dual quantum Poincaré supergroup in the form

$$\mathcal{T} = \exp(\lambda^\alpha Q_\alpha + \bar\lambda^{\dot\alpha}\bar{Q}_{\dot\alpha})\exp(ia^\mu P_\mu)\exp(i\omega^{\mu\nu}M_{\mu\nu}),$$

one gets e.g. from the RTT-relation that

$$[\lambda^\alpha, \lambda^\beta] = -2C^{\alpha\beta} + 2(S(\omega))^\alpha_{\ \gamma}(S(\omega))^\beta_{\ \delta}C^{\gamma\delta},$$

where $(S(\omega))^\beta_{\ \delta}$ are the Lorentz transformation matrices acting on the chiral spinor indices. The commutation relations of the NC superspace $s\mathcal{M}_t$ are invariant with respect to the twisted Poincaré superalgebra $\mathcal{U}_t(sP)$.

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Anyons and the Landau problem in the noncommutative plane

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Abstract

The Landau problem in the noncommutative plane is discussed in the context of realizations of the two-fold centrally extended planar Galilei group and the anyon theory.

In 2+1 dimensions, Galilei group admits a two-fold central extension [1, 2] characterized by the algebra with the nonzero Poisson bracket relations

$$\{\mathcal{K}_i, \mathcal{P}_j\} = m\delta_{ij}, \qquad \{\mathcal{K}_i, \mathcal{K}_j\} = -\kappa\epsilon_{ij}, \tag{1}$$

$$\{\mathcal{K}_i, \mathcal{H}\} = \mathcal{P}_i, \qquad \{\mathcal{J}, \mathcal{P}_i\} = \epsilon_{ij}\mathcal{P}_j, \qquad \{\mathcal{J}, \mathcal{K}_i\} = \epsilon_{ij}\mathcal{K}_j, \tag{2}$$

where m and κ are the central charges. The algebra has the two Casimir elements

$$\mathcal{C}_1 = m\mathcal{J} + \kappa\mathcal{H} - \epsilon_{ij}\mathcal{K}_i\mathcal{P}_j, \qquad \mathcal{C}_2 = m\mathcal{H} - \frac{1}{2}\mathcal{P}_i^2, \tag{3}$$

which correspond to the (multiplied by the mass m) internal angular momentum (spin) and energy.

There are two possibilities to realize this algebra as a symmetry of a free particle on a plane: the *minimal* realization and the *extended* one [cf. the two formulations for a free relativistic anyon [3]]. Requiring that the particle coordinate X_i forms a Galilei covariant object with respect to the action of the generators J, $\mathcal{P}_i$ and $\mathcal{K}_i$, treating the Galilei generators as integrals of motion and identifying the $\mathcal{P}_i$ as the canonical momentum p_i, and, finally, putting the spin and internal energy to be equal to zero ($\mathcal{C}_1 = \mathcal{C}_2 = 0$), we arrive at the following realization of the generators:

$$\mathcal{P}_i = p_i, \qquad K_i = mX_i - tp_i + m\theta\epsilon_{ij}p_j, \qquad \mathcal{J} = \epsilon_{ij}X_ip_j + \frac{1}{2}\theta\vec{p}^{\,2}, \qquad \mathcal{H} = \frac{1}{2m}\vec{p}^{\,2}, \tag{4}$$

$\theta = \kappa/m^2$. As a result, the X_i has a usual free particle evolution, $\dot{X}_i = \frac{1}{m}p_i$. The price we pay for such a minimal realization of the exotic Galilei algebra is the non-commutativity of the coordinate components

$$\{X_i, X_j\} = \theta\epsilon_{ij}, \tag{5}$$

and the non-canonical form of the associated symplectic structure

$$\sigma_0 = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij}dp_i \wedge dp_j. \tag{6}$$

One can define another sort of the coordinate [4, 5],

$$Y_i = X_i + \theta\epsilon_{ij}p_j. \tag{7}$$

It has the same bracket with p_j,

$$\{Y_i, p_j\} = \delta_{ij}, \tag{8}$$

and, hence, the same evolution law as the coordinate X_i. In terms of the Y_i and X_i, the symplectic structure and angular momentum are diagonal,

$$\sigma_0 = \frac{1}{2\theta}\epsilon_{ij}\left(dY_i \wedge dY_j - dX_i \wedge dX_j\right), \qquad \mathcal{J} = \frac{1}{2\theta}\left(Y_i^2 - X_i^2\right).$$

On the other hand, in terms of the Y_i and p_i the boost generator is represented in the usual form $\mathcal{K}_i = mY_i - tp_i$. However, the Y_i, unlike the X_i, is not covariant with respect to the Galilei boosts, $\{\mathcal{K}_i, Y_j\} = t\delta_{ij} - m\theta\epsilon_{ij}$. As we shall see below, the importance of the coordinate (7) reveals under coupling the system to the external electric and magnetic fields.

Due to the noncommutative nature of the both X_i and Y_i, there is no coordinate representation associated with them. But since

$$\{Y_i, Y_j\} = -\theta\epsilon_{ij}, \qquad \{X_i, Y_j\} = 0, \tag{9}$$

one can define the third sort of the coordinate,

$$\mathcal{X}_i = \frac{1}{2}(X_i + Y_i). \tag{10}$$

It has commuting components and reduces the symplectic structure and angular momentum to a canonical form,

$$\sigma_0 = dp_i \wedge d\mathcal{X}_i, \qquad \mathcal{J} = \epsilon_{ij}\mathcal{X}_i p_j.$$

Like the Y_i, the coordinate $\mathcal{X}_i$ is not covariant with respect to the Galilean boosts, $\{\mathcal{K}_i, \mathcal{X}_j\} = t\delta_{ij} - \frac{1}{2}m\theta\epsilon_{ij}$. The importance of this third coordinate is that at the quantum level it provides us with the Schrödinger representation, $\hat{\mathcal{X}}_i\Psi(\mathcal{X}) = \mathcal{X}_i\Psi(\mathcal{X})$, $\hat{p}_i = -i\partial_i\Psi(\mathcal{X})$. In this representation in accordance with Eqs. (10), (7) the action of the covariant coordinate operator is reduced to the star multiplication [6]:

$$\hat{X}_i\Psi(\mathcal{X}) = \left(\mathcal{X}_i - \frac{i}{2}\theta\epsilon_{ij}\partial_j\right)\Psi(\mathcal{X}) \equiv \mathcal{X}_i \star \Psi(\mathcal{X}).$$

We conclude that in the minimal realization of the exotic Galilei group the coordinate of the free particle cannot be commutative and covariant simultaneously, cf. the case of the anyons [3]. There exist at least three sorts of the coordinate, each of which has definite advantages and disadvantages.

Duval and Horvathy showed [2] that within the minimal realization, the coupling of the particle to the arbitrary external electric and magnetic fields can be achieved via a simple generalization of the free symplectic structure and Hamiltonian for

$$\sigma_{em} = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij} dp_i \wedge dp_j + \frac{1}{2}eB(X)\epsilon_{ij} dX_i \wedge dX_j, \qquad H_{em} = \frac{1}{2m}\vec{p}^{\,2} + eV(X), \quad (11)$$

where $V(X)$ is a scalar potential associated with the electric field $E_i = -\partial_i V(X)$. The Poisson brackets corresponding to the σ_{em} are

$$\{X_i, X_j\} = \frac{\theta}{1 - e\theta B}\,\epsilon_{ij}, \qquad \{X_i, p_j\} = \frac{1}{1 - e\theta B}\,\delta_{ij}, \qquad \{p_i, p_j\} = \frac{eB}{1 - e\theta B}\,\epsilon_{ij}, \quad (12)$$

and the equations of motion for X_i and p_i take the form similar to the $\theta = 0$ case but with the mass m changed for the effective mass $m^* = m(1 - e\theta B)$. The essential property of the coordinate Y_i defined by Eq. (7) is that it has the same brackets (8), (9) in the presence of any magnetic field $B(X)$ [4].

It is obvious that in the case of the critical value of the magnetic field $B = B_c \equiv (e\theta)^{-1}$, for which symplectic form (11) degenerates while brackets (12) blow up and the effective mass m^* disappears, has to be treated separately [2, 4]. In [4] it was shown that in this case the system realizes a Hall-like motion, which is described by the coordinate Y_i. On the other hand, it is clear that in a generic case of the inhomogeneous magnetic field there is a problem with realization of the operators satisfying the quantum analogs of the Poisson bracket relations (12).

The simultaneous commutativity and covariance of the coordinate can be incorporated into the theory via the extended realization of the exotic Galilei group [7, 4]. This is achieved by supplying the phase space with the two additional canonically conjugate translation-invariant variables v_i associated with an infinite-component Majorana-type representation of the exotic planar Galilei group, being analogous to the Dirac α matrices. The symplectic structure is given here by

$$\sigma = dp_i \wedge dx_i + \frac{1}{2}\kappa\epsilon_{ij} dv_i \wedge dv_j, \quad (13)$$

and the rotation and the boost generators are realized in the form

$$\mathcal{J} = \epsilon_{ij} x_i p_j + \frac{1}{2}\kappa v_i^2, \qquad \mathcal{K}_i = m x_i - t p_i + \kappa\epsilon_{ij} v_j, \quad (14)$$

while as before, the translation generator is identified with p_i. Require that the first Casimir element from (3) takes zero value. Then, with taking into account (14), we fix the form of the Hamiltonian,

$$\mathcal{H} = \vec{p}\vec{v} - \frac{1}{2}m\vec{v}^{\,2}, \quad (15)$$

and find the equations of motion generated by it,

$$\dot{x}_i = v_i, \qquad \dot{p}_i = 0, \qquad \dot{v}_i = \omega\epsilon_{ij}(v_j - m^{-1}p_j), \quad (16)$$

where $\omega = m/\kappa$. Like in the case of the Dirac equation, Hamiltonian (15) is linear in momenta, the velocities are noncommuting, $\{v_i, v_j\} = -\kappa^{-1}\epsilon_{ij}$, and in the evolution of the covariant coordinate x_i, $\{x_i, x_j\} = 0$, there appears a Zitterbewegung-like term:

$$x_i(t) = X_i(0) + \frac{1}{m}p_i t - \omega^{-1}\epsilon_{ij} V_j(t),$$

where

$$X_i = x_i + \frac{\kappa}{m}\epsilon_{ij}V_j, \tag{17}$$

$$V_i = v_i - m^{-1}p_i, \tag{18}$$

and $V_i(t) = (\cos\omega t \cdot \delta_{ij} + \sin\omega t \cdot \epsilon_{ij})V_j(0)$. The quantities V_i form a planar vector invariant with respect to the space translations and boosts, $\{\mathcal{K}_i, V_j\} = 0$, $\{p_i, V_j\} = 0$, and can be associated with the internal rotation.

The quantity (17) has the same transformation properties under the action of $\mathcal{P}_i$, $\mathcal{K}_i$ and $\mathcal{J}$ as the coordinate x_i. Unlike the x_i, it is Zitterbewegung-free, $\dot{X}_i = m^{-1}p_i$, and has the non-commuting components, $\{X_i, X_j\} = \theta\epsilon_{ij}$ [cf. the properties of the covariant coordinate X_i within the minimal realization]. The X_i is analogous to the Foldy-Wouthuysen coordinate for the Dirac particle. The combination $\mathcal{X}_i = X_i - \frac{1}{2}\theta\epsilon_{ij}p_j$ (with X_i given by (17)) is also Zitterbewegung-free, it has commuting components, but is not covariant under the action of the Galilei boosts [cf. the properties of the coordinate (10)]. It is analogous to the Newton-Wigner coordinate for the Dirac particle [8].

It is interesting to note that the dynamical picture of the extended formulation turns out to be exactly the same as that for the usual planar particle ($\theta = 0$) subjected to the external homogeneous magnetic and electric fields [5].

The Hamiltonian and the rotation generator are represented equivalently in the form

$$H = \frac{1}{2m}\vec{p}^{\,2} - \frac{1}{2}m\vec{V}^2, \tag{19}$$

$$\mathcal{J} = \epsilon_{ij}X_i p_j + \frac{1}{2}\theta\vec{p}^{\,2} + \frac{1}{2}\kappa\vec{V}^2,$$

while the boost generator takes the same form as in (4) with X_i given by Eq. (17). We have not fixed yet the second Casimir element, which is reduced here to the integral of motion associated with the Zitterbewegung (circular motion), $\mathcal{C}_2 = m^2\vec{V}^2$. Such a Hamiltonian system corresponds to a special non-relativistic limit applied to the model of relativistic particle with torsion [9] associated with the (2+1)-dimensional analog of the Majorana equation and underlying the theory of relativistic anyons [8]. Like the relativistic analog, the present system is described by the higher-derivative Lagrangian

$$L = \frac{1}{2}m\dot{x}_i^2 + \theta\epsilon_{ij}\dot{x}_i\ddot{x}_j, \tag{20}$$

which was analysed by Lukierski, Stichel and Zakrzewski [10] (ignoring its relation to the relativistic higher-derivative model [9]). In accordance with the Ostrogradski theory of higher-derivative systems, at the Hamiltonian level the velocity components $\dot{x}_i$ are identified as independent phase space variables v_i.

From the structure of the Hamiltonian (19) and equivalent form of the symplectic structure (13),

$$\sigma = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij}dp_i \wedge dp_j + \frac{\kappa}{2}\epsilon_{ij}dV_i \wedge dV_j, \tag{21}$$

it is clear that the system (20) describes not a free particle in the noncommutative plane but a sort of rotator with degrees of freedom of the ghost nature since they contribute a negative kinetic term into the Hamiltonian. In order to reduce this system to a free exotic particle of Duval and Horvathy [2] (which corresponds to a minimal realization of the two-fold centrally extended Galilei group), it is sufficient to fix the second Casimir element by introducing the second class constraints $V_i = 0$, $i = 1, 2$ [4]. From the point of view of such a reduction, the coordinate (17) is the extension of the initial coordinate x_i commuting with the second class constraints [5].

There is also another possibility to reduce the system (20), preserving the linear in the momentum Hamiltonian structure (15) similar to that of the Dirac equation. Instead of the two second class constraints, the physical subspace of the system can be singled out by imposing a complex polarization condition given by one first class complex constraint

$$V_- = 0, \tag{22}$$

$V_- = V_1 - iV_2$. Then at the quantum level a state of the system can be decomposed into the series in the Fock space states associated with the velocity variables $\hat{v}_\pm = \hat{v}_1 \pm i\hat{v}_2$, $|\Psi\rangle = \sum_{k=0}^{\infty} \psi_k |k\rangle_v$, where $\hat{v}_-|0\rangle_v = 0$. As a result, the quantum system will be described by the pair of the infinite-component wave equations [7]

$$i\partial_t \psi_k + \sqrt{\frac{k+1}{2\theta}}\,\frac{\hat{p}_+}{m}\,\psi_{k+1} = 0, \tag{23}$$

$$\hat{p}_- \psi_k + \sqrt{\frac{2(k+1)}{\theta}}\,\psi_{k+1} = 0, \tag{24}$$

where $k = 0, 1, \ldots$, and $\hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2$. Eq. (23) is the Schrödinger equation corresponding to the classical Hamiltonian (15), while Eq. (24) is the quantum analog of the classical constraint (22), whose role is to separate effectively only one independent physical field degree of freedom. The set (23), (24) has the sense of the infinite-component wave equations of the Dirac-Majorana-Levy-Leblond type for the exotic particle, associated with the two-fold central extension of the planar Galilei group. It was obtained in [7] by applying a special Jackiw-Nair non-relativistic limit [11] to the spinor set of the equations proposed earlier in [12] for the description of relativistic anyons.

Having in mind the discussed nature of the coordinates which appear in the minimal realization of the exotic Galilei group, it is clear that the coupling prescription (11) in the case of the Dirac theory would correspond to the minimal coupling in terms of the Foldy-Wouhtuysen coordinates. Since the extended formulation of a free exotic particle results in the free wave equations (23), (24) realized in terms of the commuting covariant coordinates x_i, it is natural to expect that the coupling of the system to external electric and magnetic fields proceeding from the extended formulation would be more close in nature to the usual minimal coupling prescription of the Dirac theory.

The coupling of the exotic particle to external electric and magnetic fields in the extended formulation can be realized as follows [4]. Modify the complex polarization condition (22) via the minimal coupling prescription, $p_i \to P_i = p_i - eA_i(x)$, $\epsilon_{ij}\partial_i A_j = B$. Then the generalization of the Hamiltonian (15) can be fixed from the requirement of its (weak) commutativity with the changed polarization condition. The essential feature of such a coupling scheme is that the two real constraints

$$\Lambda_i = v_i - \frac{1}{m}P_i = 0, \qquad \{\Lambda_i, \Lambda_j\} = -\kappa^{-1}(1-\beta)\epsilon_{ij}, \tag{25}$$

$\beta = \beta(x) = e\theta B(x)$, corresponding to one complex polarization condition, change their nature from the second class into the first class constraints at the critical value of the magnetic field, $B = B_c$. As a result, at $B = B_c$, the constraints (25) eliminate not one but two degrees of freedom, leaving only one degree described effectively by the noncommutative coordinate Y_i [4]. In a generic case, the classical Hamiltonian weakly commuting with constraints (25) and reducing to the Hamiltonian (15) in the free case, has the form $\tilde{\mathcal{H}} = H_B + U$, with

$$H_B = \frac{1}{1-\beta}(P_i - \beta v_i)v_i - \frac{1}{2}mv_i^2, \tag{26}$$

and U being an arbitrary function of X_i, or Y_i.

In the case of homogeneous magnetic field different from the critical one and for zero electric field ($U = 0$), the obtained system describes the Landau problem in the noncommutative plane. It is necessary to distinguish the cases of subcritical and overcritical magnetic fields. Assume that $e\theta > 0$. Then the physical states for $B < B_c$ are separated by the quantum polarization condition

$$\hat{\Lambda}_-|\Psi\rangle = 0. \tag{27}$$

The solutions of Eq. (27) describe the physical states of the form

$$|\Psi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m \hat{P}_- \hat{v}_+\right)(|0\rangle_v |\psi\rangle), \tag{28}$$

where $|0\rangle_v$, $\hat{v}_-|0\rangle_v = 0$, is the vacuum state of the Fock space generated by the velocity operators, and $|\psi\rangle$ is a velocity-independent state associated with other degrees of freedom. The action of the Hamiltonian operator corresponding to (26) is reduced on the states (28) to

$$\hat{H}_B|\Psi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m \hat{P}_- \hat{v}_+\right)\left(|0\rangle_v \hat{H}_* |\psi\rangle\right), \qquad \hat{H}_* = \frac{1}{2m^*}\hat{P}_+\hat{P}_-. \tag{29}$$

For $B < 0$, the spectrum of the system is characterized by the energy values $E_N = e|B|N/m^*$, $N = 0, 1, \ldots$, and by the angular momentum values $j = N, N-1, \ldots$. For $0 < B < B_c$, $E_N = e|B|(N+1)/m^*$, $N = 0, 1, \ldots$, and $j = -N, -N+1, \ldots$ [4]. The structure of the physical states is essentially different for $B < 0$ and $0 < B < B_c$: in the former case, the finite number of the velocity Fock space states $|n\rangle_v$, $n = 0, \ldots, N$, contribute to a physical state, while in the latter case all the infinite tower of the velocity Fock states ($n = 0, 1, \ldots$) contributes to it. It is essential, however, that in the both cases the common eigenstates of the energy and angular momentum are normalisable. In the critical case, due to the first class nature of the constraints (25), equation(27) should be supplemented with the quantum condition $\hat{\Lambda}_+|\Psi\rangle = 0$. The solutions of these two equations are given by the wave functions proposed by Laughlin to describe the ground states in the fractional quantum Hall effect [13], and coincide with the solutions of the equation (27) taken in the limit $B \to B_c$, for the details see ref. [4].

In the case of overcritical magnetic field $B > B_c$ the solutions of the quantum equation (27) are not normalisable [4]. The reason of this is rooted in a simple observation. In accordance with Eq. (25), the brackets between constraints Λ_i, $i = 1, 2$, for $B > B_c$ have an opposite sign in comparison with the subcritical case $B < B_c$. It means that the operator $\hat{\Lambda}_-$ being an annihilation-like operator for $B < B_c$, transforms into the creation-like operator having no nontrivial kernel for $B > B_c$. Therefore, in the overcritical case, the physical states have to be separated by the quantum condition $\hat{\Lambda}_+|\Psi\rangle = 0$ instead of the condition (27). This change has to be accompanied by the change of the direction of time, $t \to -t$ [4, 14].

It was observed in [4] that in a generic case of inhomogeneous magnetic field the quantum analog of the classical Hamiltonian (26) commuting with the quantum condition (27) has a nonlocal nature. On the other hand, one notes that there exists a class of the quantum systems with coordinate-dependent mass related to some quasi-exactly solvable systems [15]. This, probably, indicates that for inhomogeneous magnetic field of a special form the problem of non-locality of the quantum Hamiltonian can be solved using some ideas related to quasi-exact solvability and supersymmetry [16].

Since the exotic particle system in the noncommutative plane is related via a special non-relativistic limit to the relativistic anyon, this means that the phenomenon similar to the existence of the critical magnetic field should also exist if one couples the latter system to the external electromagnetic field. The problem of non-locality should also reveal itself there for electromagnetic field of a generic form.

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$\mathcal{N} = 1/2$ Supersymmetric Non-Linear Sigma-Models from Non-anticommutative Superspace

Shin Sasaki

1 Construction of component Lagrangians

In this talk, we showed that the component structure of four-dimensional $\mathcal{N} = 1/2$ non-anticommutative (NAC) deformed supersymmetric non-linear sigma models (NLSMs) and their applications. The non-anticommutativity of the four-dimensional $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ superspace, $\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$, originates from the non-trivial background supergravity field (anti self-dual graviphoton) $F_{\alpha\beta}$ [1]. The non-anticommutative deformed gauge model ($\mathcal{N} = 1/2$ super Yang-Mills) and Wess-Zumino model have been intensively investigated in the literature.

Recently, we found the compact form of the superpotential deformation caused by the non-anticommutativity [2]. In [2], for four-dimensional single chiral model, the non-anticommutativity can be re-interpreted as the splitting effect on the target space of the superpotential;

$$\begin{aligned}\int d^2\theta\ W_\star(\Phi) &= \frac{1}{2c}\left[W(A+cF) - W(A-cF)\right] - \\ &\quad \frac{\psi^2}{4cF}\left[\frac{\partial W(A+cF)}{\partial A} - \frac{\partial W(A-cF)}{\partial A}\right]. \end{aligned} \tag{1}$$

Where, $\Phi(y) = A(y) + i\sqrt{2}\theta\psi(y) + \theta^2 F(y)$ is a chiral superfield and $c \equiv \sqrt{-\det C}$ is the deformation parameter. This result was extended to the multi-chiral case and to the Kähler potential $K(\Phi, \overline{\Phi})$ in the context of two-dimensional non-linear sigma model [3].

On the other hand, the structure of component Lagrangian for $\mathcal{N} = 1/2$ NAC deformed four-dimensional sigma-models were evaluated as the infinite power series of the deformation parameter c [7, 8]. Soon after the work [2], the compact form of the deformed Lagrangian for full-symmetric and other symmetric ordered case were found [7, 8]. For example, full-symmetrization of chiral and anti-chiral superfield ordering result is

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\ \mathcal{K}(A^i, F^i, \overline{\Phi}^{\bar{j}}) + \int d^2\theta\ \mathcal{W}(A^i, F^i) + \int d^2\bar{\theta}\ \overline{\mathcal{W}}(\overline{\Phi}^{\bar{j}}) \\ &\quad -\Box\bar{A}^{\bar{j}}\mathcal{K}_{,\bar{j}} - \frac{c}{2}\Box\bar{A}^{\bar{k}}\psi^i\psi^j\mathcal{K}'_{,ij\bar{k}} + \Box\bar{A}^{\bar{j}}\mathcal{K}''_{,\bar{j}}\,, \end{aligned} \tag{2}$$

where, $i, j, \cdots = 1, 2, \cdots, N$ and

$$\mathcal{W}(A^i, F^i) \equiv \frac{1}{2}\int_{-1}^{1} d\tau\ W(A^i + \tau c F^i),$$

$$
\begin{aligned}
\mathcal{K}(A^i, F^i, \bar{A}^{\bar{j}}) &\equiv \frac{1}{2}\int_{-1}^{1} d\tau\ K(A^i + \tau c F^i, \bar{A}^{\bar{j}}),\\
\mathcal{K}'(A^i, F^i, \bar{A}^{\bar{j}}) &\equiv \frac{1}{2}\int_{-1}^{1} d\tau\ \tau K(A^i + \tau c F^i, \bar{A}^{\bar{j}}),\\
\mathcal{K}''(A^i, F^i, \bar{A}^{\bar{j}}) &\equiv \frac{1}{2}\int_{-1}^{1} d\tau\ \frac{\partial}{\partial\tau}\left(\tau K(A^i + \tau c F^i, \bar{A}^{\bar{j}})\right). \qquad (3)
\end{aligned}
$$

The subscript $,i\bar{j}\cdots$ means the differentiation with respect to $A_i, \bar{A}_{\bar{j}}$. For the multi-chiral superfields case, the non-anticommutativity of the fermionic coordinates is re-interpreted as the "fuzziness" of the non-linear sigma model target space. The fuzziness is controlled by the auxiliary fields F^i.

Another possible way of the non-anticommutative deformation of the Kähler potential is to interpret it as some kind of the *effective chiral superpotential* - which we call chiral reduced model [4]. In [4], we first performed the anti-chiral integration $\int d^2\bar{\theta}$ in the kinetic part,

$$
\begin{aligned}
S_K &= \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi^i, \overline{\Phi}^j) = \underbrace{\int d^4y d^2\theta d^2\bar{\theta}\ K(\Phi^i, \overline{\Phi}^j)}_{\text{chiral base}}\\
&= -\frac{1}{4}\int d^4y d^2\theta\ \overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha} K(\Phi^i, \overline{\Phi}^j)\Big|_{\bar\theta=0}\\
&= -\frac{1}{4}\int d^4y d^2\theta\ \left(\frac{\partial^2 K}{\partial\overline{\Phi}^{\bar{i}}\partial\overline{\Phi}^{\bar{j}}}\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{j}}\overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{i}} + \frac{\partial K}{\partial\overline{\Phi}^{\bar{i}}}\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{i}}\right)\Big|\\
&= -\frac{1}{4}\int d^4y d^2\theta\ \left[\frac{\partial^2 K}{\partial\bar{A}^{\bar{i}}\partial\bar{A}^{\bar{j}}}\ \overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{j}}\Big|\ \overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{i}}\Big| + \frac{\partial K}{\partial\overline{\Phi}^{\bar{i}}}\left(\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{i}}\right)\Big|\right]. \qquad (4)
\end{aligned}
$$

Be careful that all superfields appearing in this form, *i.e.*

$$
\begin{aligned}
\frac{\partial^2 K(\Phi, \overline{\Phi})}{\partial\overline{\Phi}^{\bar{i}}\partial\overline{\Phi}^{\bar{j}}}\Big|_{\bar\theta=0} &= \frac{\partial^2 K(\Phi, \bar{A})}{\partial\bar{A}^{\bar{i}}\partial\bar{A}^{\bar{j}}},\\
\frac{\partial K(\Phi, \bar{\Phi})}{\partial\overline{\Phi}^{\bar{i}}}\Big|_{\bar\theta=0} &= \frac{\partial K(\Phi, \bar{A})}{\partial\bar{A}^{\bar{i}}},\\
\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{i}}\Big|_{\bar\theta=0} &= \sqrt{2}\bar{\psi}^{\bar{i}}_{\dot\alpha} - 2i\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\partial_m\bar{A}^{\bar{i}},\\
\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{i}}\Big|_{\bar\theta=0}\ \overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{j}}\Big|_{\bar\theta=0} &= 2\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} - 4\sqrt{2}i\left(\theta\sigma_m\bar{\psi}^{(\bar{i}}\right)\partial_m\bar{A}^{\bar{j})} - 4\theta^2\partial_m\bar{A}^{\bar{i}}\partial^m\bar{A}^{\bar{j}}, \qquad (5)
\end{aligned}
$$

are all chiral. This is *identity* (up to total derivative) in this stage. Next, let us introduce the non-anticommutativity, that is, replace all the products with star and symmetrize the ordering. We would like to note that because we adapted non-supersymmetric Q-deformation in this model, *i.e.* the star product contains only supercharge Q, supercovariant derivative is intact under the non-anticommutativity and can pass through the star product.

After treating the integrated Kähler potential as some kind of the effective chiral superpotential and calculating the Grassmann integration explicitly, we see the result is

$$
\begin{aligned}
\mathcal{L}_{\text{chiral reduced}} &= F^i Y_{,i} + \partial^m\bar{A}^{\bar{p}}\partial_m\bar{A}^{\bar{q}}\mathcal{K}_{,\bar{p}\bar{q}} + \Box\bar{A}^{\bar{p}}\mathcal{K}_{,\bar{p}} - \frac{1}{2}(\psi^i\psi^j)Y_{,ij}\\
&\quad - i(\psi^i\sigma^m\bar{\psi}^{\bar{p}})\partial_m\bar{A}^{\bar{q}}\mathcal{K}_{,i\bar{p}\bar{q}} - i(\psi^i\sigma^m\partial_m\bar{\psi}^{\bar{p}})\mathcal{K}_{,i\bar{p}}, \qquad (6)
\end{aligned}
$$

where

$$Y(A,\bar{A},F,\bar{F}) = \bar{F}^{\bar{p}}\mathcal{K}_{,\bar{p}} - \frac{1}{2}(\bar{\psi}^{\bar{p}}\bar{\psi}^{\bar{q}})\mathcal{K}_{,\bar{p}\bar{q}} + c\partial^m\bar{A}^{\bar{p}}\partial_m\bar{A}^{\bar{q}}\mathcal{K}'_{,\bar{p}\bar{q}} + c\Box\bar{A}^{\bar{p}}\mathcal{K}'_{,\bar{p}}\,. \tag{7}$$

The symmetrization play a role of neglecting the bare contributions of non-anticommutative parameter $C^{\alpha\beta}$. Because the Grassmann even superfield contribution $C^{\alpha\beta}\frac{\partial\Phi_1}{\partial\theta^\alpha}\frac{\partial\Phi_2}{\partial\theta^\beta}\Big|_{\mathrm{sym}}$ always cancel out. So, this deformation doesn't introduce Lorentz symmetry violating terms[1].

We would like to stress that the chiral-reduced result [4], full-symmetric ordering [8], chiral-antichiral ordering [7] and also the quotient construction [6] all give different component structures. Which implies that there exists some ambiguity to construct the component Lagrangian of the deformed non-linear sigma models. The relations among various non-anticommutative deformation of 4D non-linear sigma models are summarized in fig.[1].

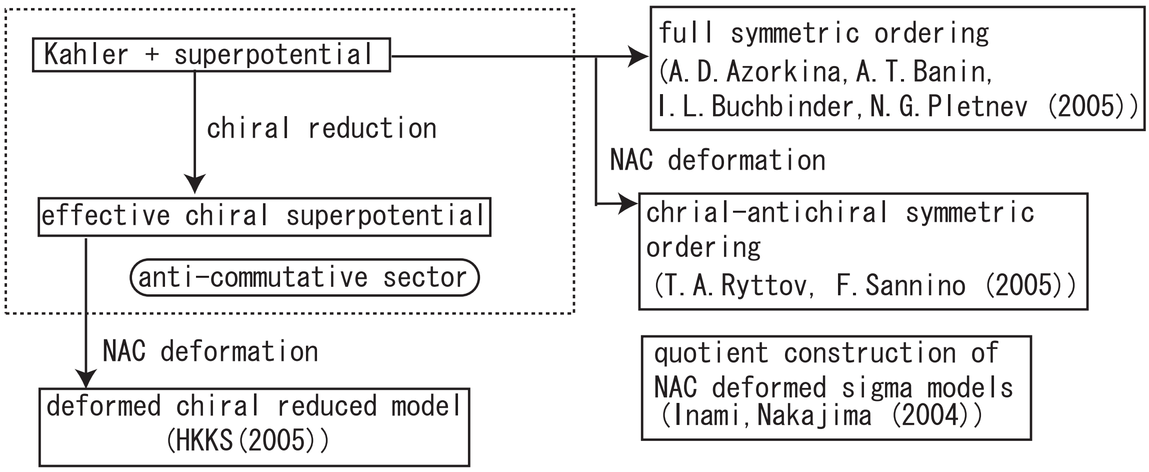

Figure 1: Various possibility of the non-anticommutative deformed 4D NLSMs

Note that in any case, it is highly non-trivial to solve the equation of motion for the auxiliary fields because in general, it becomes non-linear equation.

2 Kähler invariance

The commutative supersymmetric non-linear sigma model has Kähler invariance, *i.e.* the action (and also the metric $g_{i\bar{j}} = \partial_{A^i}\partial_{\bar{A}^{\bar{j}}}K$) is invariant under the transformation

$$K(A,\bar{A}) \longrightarrow K(A,\bar{A}) + f(A) + \bar{f}(\bar{A}). \tag{8}$$

What is the corresponding invariance in the case of non-anticommutative theory? The naive counterpart is the star deformed version of the equation (8)

$$K_\star(\Phi,\bar{\Phi}) \longrightarrow K_\star(\Phi,\bar{\Phi}) + f_\star(\Phi) + \bar{f}_\star(\bar{\Phi}). \tag{9}$$

[1]Relating this fact, we would like to note about the previous work [6] in which they constructed non-anticommutative deformed CP^n sigma model by the quotient construction. The component result of [6] contains a Lorentz violating term $\mathcal{L}_C = 2g_{a\bar{b}}g_{c\bar{d}}C^{\alpha\beta}(\sigma^{mn})_\beta{}^\gamma\psi^a_\alpha\psi^c_\gamma(\partial_m\bar{A}^{\bar{b}})(\partial_n\bar{A}^{\bar{d}})$.

The invariance under the transformation (9) can be checked easily at the stage of superfield Lagrangian

$$\delta\mathcal{L} = \int d^2\theta d^2\bar{\theta}\ f_\star(\Phi^i) + \int d^2\theta d^2\bar{\theta}\ \bar{f}_\star(\overline{\Phi}^j). \tag{10}$$

The first term is exactly zero because the star product (in the Q-deformation) doesn't break the chirality, so $f_\star(\Phi)$ is always chiral superfield and gives zero after $\bar{\theta}$ integration. The second term is essentially undeformed anti-chiral superpotential, therefore it is anti-chiral superfield and gives zero contribution after θ integration. So, the NAC deformed action is star Kähler invariant obviously.

On the other hand, we can interpret the non-anticommutative theory as the deformed (anti)commutative theory after evaluating the nilpotent star product. It is important to study the *original* Kähler invariance, namely, under the transformation (8). In [5], we showed the Kähler invariance is preserved in the chiral reduced model. Here, let us check the invariance of the single chiral ($N = 1$) full symmetric ordering result. First, Kähler potential with mixed derivatives with respect to $A, \bar{A}$ is always Kähler invariant[2]. It is easily found that the terms that have the structure $K_{,\bar{A}\bar{A}\cdots}(A+cF,\bar{A}) - K_{,\bar{A}\bar{A}\cdots}(A-cF,\bar{A})$ is always invariant. The only nontrivial part is the scalar kinetic term which gives

$$\begin{aligned}
&\delta\left[\frac{\Box\bar{A}}{2}\big(K_{,\bar{A}}(A+cF,\bar{A}) + K_{,\bar{A}}(A-cF,\bar{A})\big) + \frac{1}{2}\partial_m\bar{A}\partial^m\bar{A}\int_{-1}^{1}d\tau\ K_{,\bar{A}\bar{A}}(A+\tau cF,\bar{A})\right] \\
&= \Box\bar{A}\cdot\bar{f}'(\bar{A}) + \frac{1}{2}\partial_m\bar{A}\partial^m\bar{A}\int_{-1}^{1}d\tau\ \bar{f}''(\bar{A}) \\
&= \Box\bar{A}\cdot\bar{f}'(\bar{A}) + \partial_m\bar{A}\partial^m\bar{A}\cdot\bar{f}''(\bar{A}) = (\text{total derivative}).
\end{aligned} \tag{11}$$

Then, full-symmetric ordering result also preserves Kähler invariance. It is easy task to check the Kähler invariance of the chiral-antichiral ordering Lagrangian and multi-chiral case.

3 Application

Next, we focus on the specific model, especially, the $\mathcal{N} = 1/2$ non-anticommutative deformed CP^1 model. First, we are going to study the on-shell structure of this model. For the chiral-reduced, full-symmetric, chiral-antichiral symmetric models, the deformed equation of motion for the auxiliary field becomes the same structure;

$$\begin{aligned}
&\frac{1}{2c}\left[K_{,\bar{A}}(A+cF,\bar{A}) - K_{,\bar{A}}(A-cF,\bar{A})\right] \\
&-\frac{1}{4cF}\psi^2\left[K_{,A\bar{A}}(A+cF,\bar{A}) - K_{,A,\bar{A}}(A-cF,\bar{A})\right] + \overline{W}_{,\bar{A}} = 0.
\end{aligned} \tag{12}$$

For simplicity, we here drop the superpotential part. In the CP^1 case $K(A,\bar{A}) = \alpha\ln(1+\kappa^{-2}A\bar{A})$, - α and κ are some constants - we see the equation (12) reduces to the form

$$F^3 - (\bar{A}c)^{-2}(\kappa^2 + A\bar{A})^2F - (\bar{A}c)^{-2}\psi^2(\kappa^2 + A\bar{A}) = 0. \tag{13}$$

[2]Be careful that we have to consider Kähler transformation *before* solving the equation of motion for the auxiliary field.

Thanks to the nilpotent property of ψ^2, we can simplify the solutions to the equation (13). The result is

$$\begin{aligned} F_0 &= -\frac{\bar{A}\psi^2}{\kappa^2 + A\bar{A}} \quad \text{(undeformed phase)}, \\ F_\pm &= \frac{1}{2}\frac{\bar{A}\psi^2}{\kappa^2 + A\bar{A}} - \frac{\kappa^2 + A\bar{A}}{(\bar{A}c)^2} \quad \text{(deformed phase)}. \end{aligned} \tag{14}$$

Consistency with the commutative limit $c \to 0$ requires only the undeformed solution F_0 (Deformed phase solution becomes singular). In this case, after putting the solution $F = F_0(A, \bar{A}, \psi, \bar{\psi})$ back into the deformed Lagrangian, we can say that this undeformed solution gives undeformed ($c = 0$) Lagrangian on-shell [3], *i.e.*

$$\mathcal{L}^{CP^1}_{\text{deformed}}\Big|_{F=F_0} = \mathcal{L}^{CP^1}_{c=0}\,. \tag{15}$$

In this case, the half of supersymmetry that was broken by the non-anticommutativity is recovered. This is the special situation for the CP^1 model.

Next task is to find the deformed structure of this target space geometry. To find the deformed structure, we first want to calculate the deformed metric on this target space. The structure of the metric is different corresponding to which ordering or chiral reduced model we choose. The explicit calculation shows the metric is controlled by the given superpotential $\overline{W}(\bar{\Phi})$ in each case. To see this fact, let us concentrate on the superpotential contribution to the auxiliary field. For simplicity, put the fermion to be zero. Then the equation of motion for the auxiliary field in the case of the CP^1 is

$$\frac{A + cF}{1 + \kappa^{-2}(A + cF)\bar{A}} - \frac{A - cF}{1 + \kappa^{-2}(A - cF)\bar{A}} + \frac{2\kappa^2 c}{\alpha}\overline{W}_{,\bar{A}} = 0. \tag{16}$$

Solution to this equation is

$$F = \frac{\alpha\kappa^2 \pm \sqrt{(\alpha\kappa)^2 - \left[2c\bar{A}\overline{W}_{,\bar{A}}\,(\kappa^2 + A\bar{A})\right]^2}}{2c^2\bar{A}^2\overline{W}_{,\bar{A}}}. \tag{17}$$

This apparently depends on the given superpotential $\overline{W}(\bar{\Phi})$ through the NAC parameter c. The explicit calculation allows us to see that this dependence is succeeded to the metric. This fact is the special property for the non-anticommutative deformed theory and it is interesting to investigate the deformed geometry of the target space of $\mathcal{N} = 1/2$ non-linear sigma models.

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Lorentz invariant deformations of superspaces

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Abstract

Lorentz invariant supersymmetric deformations of superspaces based on the Moyal star product parametrized by a Majorana spinor λ_a are proposed. The constructed invariant Moyal brackets are found to be in a one to one correspondence with the well known field dependent Lorentz noninvariant (anti)commutators of supercoordinates. The correspondence is fixed by the map: $B^{-1}_{mn} \leftrightarrow i\psi_m\psi_n$, $\quad C_{ab} \leftrightarrow \lambda_a\lambda_b$, $\quad \Psi^a_m \leftrightarrow \psi_m\lambda^a$ which is valid up to the second order corrections in the deformation parameter h, where $\psi_m = -\frac{1}{2}(\bar\theta\gamma_m\lambda)$ is a composite Grassmannian vector.

1 Introduction

Studying noncommutative geometry attracts a great interest [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Much attention has been paid to the role of the constant background fields of supergravity - B_{mn}, the graviphoton C_{ab} and the gravitino Ψ^a_m - as the souce of the superspace deformations [11], [16], [17], [18], [19], [20]. The presence of the background in the (anti)commutators of the (super)coordinate operators has raised the problem of the Lorentz symmetry breaking introduced by the deformations. The proposal to overcome this problem by the transition to a twisted Hopf algebra interpretation was recently advanced [21] and its supersymmetric generalization was developed in [22],[23], [24]. Another way was observed in [25], where the Hamiltonian structure of a twistor-like model [26] of super p-brane embedded in $N = 1$ superspace extended by tensor central charge coordinates was studied. The Lorentz covariant and supersymmetric non(anti)commutative Dirac bracket relations among the brane (super)coordinates with the r.h.s. parametrized by auxiliary spinor variables were derived there. It attracts to think on a hidden spinor structure possibly associated with the Penrose twistor picture [27, 28, 29, 30, 31] behind the non(anti)commutativity . With this purpose we start here with a spinor extension of the $N = 1\, D = 4$ superspace (x_m, θ_a) by one commuting Majorana spinor λ_a and construct Lorentz invariant supersymmetric Poisson and Moyal brackets generating non(anti)commutative relations of the (super)coordinates. The r.h.s of the brackets of x_m among themselves and with θ_a contain the real (or complex) Grassmannian vector ψ_m known from the theory of spinning strings and particles [32]. The odd vector ψ_m appears here as an effective variable $\psi_m = -\frac{1}{2}(\bar\theta\gamma_m\lambda)$ [33] composed of the two Majorana spinors λ_a, θ_a and encoding the initial degrees of freedom described by θ_a. We found a correspondence between the presented here Lorentz invariant Moyal brackets and the above mentioned (anti)commutators depending of the supergravity background and the string tension parameter α'. This correspondence is schematically illustrated by the map: $B^{-1}_{mn} \leftrightarrow i\psi_m\psi_n, \quad C_{ab} \leftrightarrow \lambda_a\lambda_b, \quad \Psi^a_m \leftrightarrow \psi_m\lambda^a$ transforming the field dependent (anti)commutators into the Moyal brackets. We found that the map is valid up to the second order corrections in the deformation parameter h and it works in more sophisticated cases considered below. We studied the null twistor realization of the brackets and found correlation between supersymmetry and non(anti)commutative deformations. The Lorentz invariant supersymmetric brackets where θ-nonanticommutativity occurs only for the components of θ^a with opposite chirality were constructed. The possibilities for the generalizations to higher $D = 2, 3, 4(mod 8)$, extended supersymmetry with $N > 1$ and for the presence of additional auxiliary spinors are proposed.

2 Lorentz invariant splitting of SUSY algebra

The $D = 4\, N = 1$ supersymmetry transformations in the presence of the twistor-like Majorana spinor $(\nu_\alpha, \bar{\nu}_{\dot\alpha})$ are given by the relations [25]

$$\delta\theta_\alpha = \varepsilon_\alpha, \quad \delta x_{\alpha\dot\alpha} = 2i(\varepsilon_\alpha\bar{\theta}_{\dot\alpha} - \theta_\alpha\bar{\varepsilon}_{\dot\alpha}), \quad \delta\nu_\alpha = 0, \tag{1}$$

and the correspondent supersymmetric derivatives $\partial^{\alpha\dot\alpha} \equiv \frac{\partial}{\partial x_{\alpha\dot\alpha}}$ and $D^\alpha, \bar{D}^{\dot\alpha}$ are

$$D^\alpha = \tfrac{\partial}{\partial\theta_\alpha} - 2i\bar{\theta}_{\dot\alpha}\partial^{\alpha\dot\alpha}, \quad \bar{D}^{\dot\alpha} \equiv -(D^\alpha)^* = \tfrac{\partial}{\partial\bar{\theta}_{\dot\alpha}} - 2i\theta_\alpha\partial^{\alpha\dot\alpha}, \quad [D^\alpha, \bar{D}^{\dot\beta}] = -4i\partial^{\alpha\dot\alpha}. \tag{2}$$

The spinor coordinates $(\nu_\alpha, \bar{\nu}_{\dot\alpha})$ and the light-like vector $\varphi_{\alpha\dot\alpha} = \nu_\alpha\bar{\nu}_{\dot\alpha}$ composed from them may be used to construct the Lorentz invariant differential operators $D, \bar{D}, \partial$

$$D = \nu_\alpha D^\alpha, \quad \bar{D} = \bar{\nu}_{\dot\alpha}\bar{D}^{\dot\alpha}, \quad \partial = \varphi_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} \tag{3}$$

which form a supersymmetric subalgebra of the algebra of the invariant derivatives

$$[D, \bar{D}]_+ = -4i\partial, \quad [D, D]_+ = [\bar{D}, \bar{D}]_+ = 0, \quad [D, \partial] = [\bar{D}, \partial] = [\partial, \partial] = 0. \tag{4}$$

The superalgebra (4) may be splitted into two invariant and (anti)commuting subalgebras (D_-, ∂) and (D_+, ∂)

$$[D_\pm, D_\pm]_+ = \mp 8i\partial, \quad [D_+, D_-]_+ = 0, \quad [D_\pm, \partial] = [\partial, \partial] = 0 \tag{5}$$

formed by the supersymmetric derivatives ∂ and $D_\pm$

$$D_\pm \equiv D \pm \bar{D}. \tag{6}$$

The addition of the dilatation operator Δ

$$\Delta = \nu_\alpha\tfrac{\partial}{\partial\nu_\alpha} + \bar{\nu}_{\dot\alpha}\tfrac{\partial}{\partial\bar{\nu}_{\dot\alpha}} \tag{7}$$

changing the scale of the spinor $(\nu_\alpha, \bar{\nu}_{\dot\alpha})$ extends the supersubalgebras (5) to the superalgebras formed by the invarint derivatives (D_-, ∂, Δ) and (D_+, ∂, Δ)

$$\begin{gathered}[D_\pm, D_\pm]_+ = \mp 8i\partial, \quad [\Delta, D_\pm] = D_\pm, \quad [\Delta, \partial] = 2\partial, \\ [D_+, D_-]_+ = [D_\pm, \partial] = [\partial, \partial] = [\Delta, \Delta] = 0.\end{gathered} \tag{8}$$

Our proposal is to use the Lorentz invariant supersymmetric differential operators (8) as building blocks for the construction of Lorentz invariant supersymmetric Poisson and Moyal brackets among the (super)coordinates associated with the (anti)commutators of the supercoordinate operators in quantum theory.

3 Supersymmetric Lorentz invariant Poisson bracket

At first let us to study a simple example of the Lorentz invariant and supersymmetric Poisson bracket producing non(anti)commutative relations among the superspace coordinates $x_{\alpha\dot\alpha}, \theta_\alpha, \bar\theta_{\dot\alpha}$. Such a Poisson bracket may be constructed from the three differential operators (D_-, ∂, Δ) generating the (-)- superalgebra (8)

$$\{F, G\} = F\,[\,-\tfrac{i}{4}\overleftarrow{D}_-\overrightarrow{D}_- + (\overleftarrow{\partial}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\partial})\,]\,G, \tag{9}$$

where $\{,\}_{P.B.} \equiv \{,\}$ and $F(x,\theta,\bar\theta,\nu,\bar\nu), G(x,\theta,\bar\theta,\nu,\bar\nu)$ are generalized superfields depending on the superspace coordinates $(x,\theta,\bar\theta)$ and the commuting spinors $\nu, \bar\nu$.

As a result of (9), the twistor-like coordinates get zero P.B's. among themselves

$$\{\nu_\alpha, \nu_\beta\} = \{\nu_\alpha, \bar\nu_{\dot\beta}\} = \{\bar\nu_\alpha, \bar\nu_{\dot\beta}\} = 0 \tag{10}$$

and with the Grassmannian spinors $\theta_\alpha, \bar\theta_{\dot\alpha}$

$$\{\nu_\alpha, \theta_\beta\} = \{\nu_\alpha, \bar\theta_{\dot\beta}\} = \{\bar\nu_{\dot\alpha}, \theta_\beta\} = \{\bar\nu_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0. \tag{11}$$

However, they have non zero P.B's. with the space-time coordinates $x_{\alpha\dot\alpha}$

$$\{x_{\alpha\dot\alpha}, \nu_\beta\} = \varphi_{\alpha\dot\alpha}\nu_\beta, \quad \{x_{\alpha\dot\alpha}, \bar\nu_{\dot\beta}\} = \varphi_{\alpha\dot\alpha}\bar\nu_{\dot\beta}, \tag{12}$$

The P.B's. among the super coordinates $x_{\alpha\dot\alpha}$ and $(\theta_\alpha, \bar\theta_{\dot\alpha})$ are as follows

$$\begin{gathered}\{x_{\alpha\dot\alpha}, x_{\beta\dot\beta}\} = -i\psi_{\alpha\dot\alpha}\psi_{\beta\dot\beta}, \\ \{x_{\alpha\dot\alpha}, \theta_\beta\} = \tfrac{i}{2}\psi_{\alpha\dot\alpha}\nu_\beta, \quad \{x_{\alpha\dot\alpha}, \bar\theta_{\dot\beta}\} = -\tfrac{i}{2}\psi_{\alpha\dot\alpha}\bar\nu_{\dot\beta}, \\ \{\theta_\alpha, \theta_\beta\} = \tfrac{i}{4}\varphi_{\alpha\beta}, \quad \{\theta_\alpha, \bar\theta_{\dot\beta}\} = -\tfrac{i}{4}\varphi_{\alpha\dot\beta}, \quad \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \tfrac{i}{4}\bar\varphi_{\dot\alpha\dot\beta},\end{gathered} \tag{13}$$

where $\psi_{\alpha\dot\alpha}$ is a Grassmannian vector and $\varphi_{\alpha\beta}, \bar\varphi_{\dot\alpha\dot\beta}$ are composed symmetric spin-tensors

$$\psi_{\alpha\dot\alpha} \equiv i(\nu_\alpha\bar\theta_{\dot\alpha} - \theta_\alpha\bar\nu_{\dot\alpha}), \quad \psi_{\alpha\dot\alpha}\varphi^{\alpha\dot\alpha} = 0, \quad \varphi_{\alpha\beta} \equiv \nu_\alpha\nu_\beta, \quad \bar\varphi_{\dot\alpha\dot\beta} \equiv \bar\nu_{\dot\alpha}\bar\nu_{\dot\beta}, \tag{14}$$

with the following transformation rules under the supersymmetry (1)

$$\delta\varphi_{\alpha\beta} = \delta\bar\varphi_{\dot\alpha\dot\beta} = 0, \quad \delta\psi_{\alpha\dot\alpha} = -i(\varepsilon_\alpha\bar\nu_{\dot\alpha} - \bar\varepsilon_{\dot\alpha}\nu_\alpha). \tag{15}$$

The appearance in (13) of the odd vector $\psi_{\alpha\dot\alpha}$ (14) associated with the spin degrees of freedom hints on a spin structure behind the coordinate's non(anti)commutativity. The bilinear spinor representation (14) for $\psi_{\alpha\dot\alpha}$ was revealed in [33] as the general solution of the Dirac constraints $p^{\alpha\dot\alpha}\psi_{\alpha\dot\alpha} = 0 = p^{\alpha\dot\alpha}p_{\alpha\dot\alpha}$ for massless spinning particle [34],[35]. This spinor representation has established equivalence between spinning and Brink-Schwarz superparticles. Thus, we constructed the desired Poisson brackets (10-13) which are covariant by construction under the Lorentz and supersymmetry transformations.

These P.B's. satisfy the graded Jacobi identities having the standard form

$$\{\{A,B\},C\} + (-1)^{(b+c)a}\{\{B,C\},A\} + (-1)^{c(a+b)}\{\{C,A\},B\} = 0, \tag{16}$$

where $a, b, c = 0, 1$ denote the Grassmannian gradings of A, B and C respectively.

The P.B's. among the supercoordinates and the composite objects ψ and φ are

$$\begin{gathered}
\{\psi_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\} = -i\varphi_{\alpha\dot{\alpha}}\varphi_{\beta\dot{\beta}}, \\
\{x_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\} = \varphi_{\alpha\dot{\alpha}}\psi_{\beta\dot{\beta}} + \varphi_{\beta\dot{\beta}}\psi_{\alpha\dot{\alpha}}, \\
\{\psi_{\alpha\dot{\alpha}}, \theta_\beta\} = \tfrac{1}{2}\varphi_{\alpha\dot{\alpha}}\nu_\beta, \quad \{\psi_{\alpha\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = -\tfrac{1}{2}\varphi_{\alpha\dot{\alpha}}\bar{\nu}_{\dot{\beta}}, \\
\{x_{\alpha\dot{\alpha}}, \varphi_{\beta\dot{\gamma}}\} = 2\varphi_{\alpha\dot{\alpha}}\varphi_{\beta\dot{\gamma}}, \quad \{x_{\alpha\dot{\alpha}}, \varphi_{\beta\dot{\gamma}}\} = 2\varphi_{\alpha\dot{\alpha}}\varphi_{\beta\dot{\gamma}}, \quad \{x_{\alpha\dot{\alpha}}, \bar{\varphi}_{\dot{\beta}\dot{\gamma}}\} = 2\varphi_{\alpha\dot{\alpha}}\bar{\varphi}_{\dot{\beta}\dot{\gamma}}.
\end{gathered} \tag{17}$$

Using these Poisson brackets together with the P.B's. (10-13) we obtain

$$\begin{gathered}
\{\{\psi_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\}, \psi_{\gamma\dot{\gamma}}\} = 0, \\
\{\{\theta_\alpha, \theta_\beta\}, \theta_\gamma\} = ... = \{\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\}, \bar{\theta}_{\dot{\gamma}}\} = 0
\end{gathered} \tag{18}$$

proving the graded Jacobi identity for the 3ψ and 3θ Jacobi cycles:

$$Cycle\{\{\psi_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\}, \psi_{\gamma\dot{\gamma}}\} = Cycle\{\{\theta_\alpha, \theta_\beta\}, \theta_\gamma\} = ... = Cycle\{\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\}, \bar{\theta}_{\dot{\gamma}}\} = 0.$$

The vanishing of the $3x$ Jacobi cycle: $Cycle\{\{x_{\alpha\dot{\alpha}}, x_{\beta\dot{\beta}}\}, x_{\gamma\dot{\gamma}}\} = 0$ follows from the relation

$$\{\{x_{\alpha\dot{\alpha}}, x_{\beta\dot{\beta}}\}, x_{\gamma\dot{\gamma}}\} = 2i(\psi_{\alpha\dot{\alpha}}\psi_{\beta\dot{\beta}})\varphi_{\gamma\dot{\gamma}} + i(\psi_{\alpha\dot{\alpha}}\varphi_{\beta\dot{\beta}} - \psi_{\beta\dot{\beta}}\varphi_{\alpha\dot{\alpha}})\psi_{\gamma\dot{\gamma}}. \tag{19}$$

The same result are preserved for other Jacobi cycles proving selfconsistency of the introduced P.B. (9) that opens a way for the corresponding invariant Moyal bracket.

4 Lorentz invariant supersymmetric Moyal bracket

A transition to quantum picture based on the P.B. (9) may be done using the Weyl-Moyal correspondence establishing one to one correspondence among quantum field operators and their symbols acting on the commutative space-time. Then the quantum dynamics encodes itself in the change of usual product of Weyl symbols to their star product

$$F\star G = F\, e^{\{\frac{-ih}{8}[\overleftarrow{D}_-\overrightarrow{D}_- + (\overleftarrow{\nabla}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\nabla})]\}}\, G, \tag{20}$$

where $\nabla \equiv 4i\partial$ and h is a quantum deformation parameter associated with the expansion

$$\begin{gathered}
F\star G = FG + (\tfrac{-ih}{8})\, F\, [\overleftarrow{D}_-\overrightarrow{D}_- + (\overleftarrow{\nabla}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\nabla})\,]\, G \\
+ \tfrac{1}{2!}(\tfrac{-ih}{8})^2\, F\, [\overleftarrow{D}_-\overrightarrow{D}_- + (\overleftarrow{\nabla}\overrightarrow{\Delta}_- - \overleftarrow{\Delta}\overrightarrow{\nabla})]^2\, G + ...
\end{gathered} \tag{21}$$

The power series expansion in h (21) is presented in the orderd form as

$$\begin{gathered}
F\star G = FG + (\tfrac{-ih}{8})\, F\, [\overleftarrow{D}_-\overrightarrow{D}_- + (\overleftarrow{\nabla}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\nabla})\,]\, G \\
+ \tfrac{1}{2!}(\tfrac{-ih}{8})^2\, F\, [\,-11\, \overleftarrow{\nabla}\overrightarrow{\nabla} + 3\overleftarrow{D}_-\, \overleftrightarrow{\nabla}\, \overrightarrow{D}_- + 2(\overleftarrow{\nabla}\, \overleftarrow{D}_-\overrightarrow{D}_-\, \overrightarrow{\Delta} - \overleftarrow{\Delta}\, \overleftarrow{D}_-\overrightarrow{D}_-\, \overrightarrow{\nabla}) \\
-3(\overleftarrow{\Delta}\overleftarrow{\nabla}\overrightarrow{\nabla} + \overleftarrow{\nabla}\overrightarrow{\nabla}\overrightarrow{\Delta}) + \overleftarrow{\nabla}^2\, (\overrightarrow{\Delta}^2 + 2\, \overrightarrow{\Delta}) + (\overleftarrow{\Delta}^2 + 2\, \overleftarrow{\Delta})\overrightarrow{\nabla}^2 - 2\, \overleftarrow{\Delta}\overleftarrow{\nabla}\overrightarrow{\nabla}\overrightarrow{\Delta}\,]\, G + ...,
\end{gathered} \tag{22}$$

where we omit the higher order terms in h. Using the expansion (22) we find the second order corrections to be vanishing for the following $\star$-products of the supercoordinates:

$$\begin{gathered} x_{\alpha\dot\alpha}\star\nu_\beta = x_{\alpha\dot\alpha}\nu_\beta + \tfrac{h}{2}\varphi_{\alpha\dot\alpha}\nu_\beta + \mathcal{O}(h^3), \quad x_{\alpha\dot\alpha}\star\bar\nu_{\dot\beta} = x_{\alpha\dot\alpha}\bar\nu_{\dot\beta} + \tfrac{h}{2}\varphi_{\alpha\dot\alpha}\bar\nu_{\dot\beta} + \mathcal{O}(h^3), \\ x_{\alpha\dot\alpha}\star\theta_\beta = x_{\alpha\dot\alpha}\theta_\beta + \tfrac{ih}{4}\psi_{\alpha\dot\alpha}\nu_\beta + \mathcal{O}(h^3), \quad x_{\alpha\dot\alpha}\star\bar\theta_{\dot\beta} = x_{\alpha\dot\alpha}\bar\theta_{\dot\beta} - \tfrac{ih}{4}\psi_{\alpha\dot\alpha}\bar\nu_{\dot\beta} + \mathcal{O}(h^3), \\ \theta_\alpha\star\theta_\beta = \theta_\alpha\theta_\beta + \tfrac{ih}{8}\varphi_{\alpha\beta} + \mathcal{O}(h^3), \quad \theta_\alpha\star\bar\theta_{\dot\beta} = \theta_\alpha\bar\theta_{\dot\beta} - \tfrac{ih}{8}\varphi_{\alpha\dot\beta} + \mathcal{O}(h^3), \\ \bar\theta_{\dot\alpha}\star\bar\theta_{\dot\beta} = \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} + \tfrac{ih}{8}\bar\varphi_{\dot\alpha\dot\beta} + \mathcal{O}(h^3). \end{gathered} \tag{23}$$

Moreover, the star products of the Majorana spinor $(\nu_\alpha, \bar\nu_{\dot\alpha})$ components coincide with their usual products in all orders in h. We assume that the higher order corrections in the star products (23) can be also equal zero. On the contrary, the second order corrections in the star products of the $x_{\alpha\dot\alpha}$ components are nonzero

$$x_{\alpha\dot\alpha}\star x_{\beta\dot\beta} = x_{\alpha\dot\alpha}x_{\beta\dot\beta} - \frac{ih}{2}\psi_{\alpha\dot\alpha}\psi_{\beta\dot\beta} - \frac{1}{2!}\frac{1}{4}\frac{h^2}{}\varphi_{\alpha\dot\alpha}\varphi_{\beta\dot\beta} + \mathcal{O}(h^3), \tag{24}$$

but their contributions in the corresponding Moyal brackets are zero, because of the commutativity $\varphi_{\alpha\dot\alpha}\varphi_{\beta\dot\beta} = \varphi_{\beta\dot\beta}\varphi_{\alpha\dot\alpha}$. Consequently, the second order corections in the Lorentz invariant and supersymmetric Moyal brackets (23-24) are equal to zero

$$\begin{aligned} &[x_{\alpha\dot\alpha}, x_{\beta\dot\beta}]_\star \equiv x_{\alpha\dot\alpha}\star x_{\beta\dot\beta} - x_{\beta\dot\beta}\star x_{\alpha\dot\alpha} = -ih\psi_{\alpha\dot\alpha}\psi_{\beta\dot\beta} + \mathcal{O}(h^3), \\ &[x_{\alpha\dot\alpha}, \nu_\beta]_\star = h\varphi_{\alpha\dot\alpha}\nu_\beta + \mathcal{O}(h^3), \quad [x_{\alpha\dot\alpha}, \bar\nu_{\dot\beta}]_\star = h\varphi_{\alpha\dot\alpha}\bar\nu_{\dot\beta} + \mathcal{O}(h^3), \\ &[x_{\alpha\dot\alpha}, \theta_\beta]_\star = \tfrac{ih}{2}\psi_{\alpha\dot\alpha}\nu_\beta + \mathcal{O}(h^3), \quad [x_{\alpha\dot\alpha}, \bar\theta_{\dot\beta}]_\star = -\tfrac{ih}{2}\psi_{\alpha\dot\alpha}\bar\nu_{\dot\beta} + \mathcal{O}(h^3), \\ &[\theta_\alpha, \theta_\beta]_{\star+} = \tfrac{ih}{4}\varphi_{\alpha\beta} + \mathcal{O}(h^3), \quad [\theta_\alpha, \bar\theta_{\dot\beta}]_{\star+} = -\tfrac{ih}{4}\varphi_{\alpha\dot\beta} + \mathcal{O}(h^3), \\ &[\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}]_{\star+} = \tfrac{ih}{4}\bar\varphi_{\dot\alpha\dot\beta} + \mathcal{O}(h^3). \end{aligned} \tag{25}$$

These Moyal brackets generated by the P.B's. (10-13) replace the (anti)commutators of the coordinate operators used in the standard quantum picture for field models in non(anti)commutative superspaces.

5 Brackets and twistors

The unification of the Weyl spinors $\nu_\alpha, \bar\nu_{\dot\alpha}$ with the spinors $\omega^\alpha, \bar\omega^{\dot\alpha}$ defined as

$$\omega_\alpha = x_{\alpha\dot\alpha}\bar\nu^{\dot\alpha}, \quad \bar\omega_{\dot\alpha} = x_{\alpha\dot\alpha}\nu^\alpha \tag{26}$$

yields the null twistor $Z^{\mathcal{A}} = (i\omega^\alpha, \bar\nu_{\dot\alpha})$ and its complex conjugate $\bar Z_{\mathcal{A}} = (\nu_\alpha, -i\bar\omega^{\dot\alpha})$ connected by the condition $Z^{\mathcal{A}}\bar Z_{\mathcal{A}} = 0$ [27]. The Eqs. (10) and (12) result in the P.B. commutativity among the twistor components ω_α and $\nu_\alpha, \bar\nu_{\dot\alpha}$

$$\{\omega_\alpha, \nu_\beta\} = \{\omega_\alpha, \bar\nu_{\dot\beta}\} = \{\bar\omega_{\dot\alpha}, \nu_\beta\} = \{\bar\omega_{\dot\alpha}, \bar\nu_{\dot\beta}\} = 0, \tag{27}$$

because of the orthogonality conditions

$$\varphi_{\alpha\dot\alpha}\nu^\alpha = \varphi_{\alpha\dot\alpha}\bar\nu^{\dot\alpha} = 0. \tag{28}$$

The P.B's. between ω_α and ω_β having the same chirality, as well as between their complex conjugate, are vanishing

$$\{\omega_\alpha, \omega_\beta\} = i\bar{\eta}^2\varphi_{\alpha\beta} \equiv 0, \quad \{\bar{\omega}_{\dot{\alpha}}, \bar{\omega}_{\dot{\beta}}\} = i\eta^2\bar{\varphi}_{\dot{\alpha}\dot{\beta}} \equiv 0 \tag{29}$$

because $\eta^2 = \bar{\eta}^2 = 0$, where η and $\bar{\eta}$ are Grassmannian scalars defined by

$$\eta \equiv \theta_\alpha \nu^\alpha, \quad \bar{\eta} \equiv \bar{\theta}_{\dot{\alpha}}\bar{\nu}^{\dot{\alpha}}. \tag{30}$$

These anticommuting scalars have zero P.B's. between themselves, with ν, ω, θ

$$\{\eta, \nu_\alpha\} = \{\eta, \omega_\alpha\} = \{\eta, \theta_\alpha\} = 0 \tag{31}$$

and with $\bar{\nu}, \bar{\omega}, \bar{\theta}$. The only nonzero Poisson bracket among the components of the null twistors $Z^{\mathcal{A}} = (i\omega^\alpha, \bar{\nu}_{\dot{\alpha}})$, $\bar{Z}_{\mathcal{A}} = (\nu_\alpha, -i\bar{\omega}^{\dot{\alpha}})$ is the following P.B.

$$\{\omega_\alpha, \bar{\omega}_{\dot{\beta}}\} = i\eta\bar{\eta}\varphi_{\alpha\dot{\beta}}. \tag{32}$$

which may be written down in the equivalent form $\{\omega_\alpha, \bar{\omega}_{\dot{\beta}}\} = 8\eta\bar{\eta}\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\}$ showing the coupling of the $(\omega, \bar{\omega})$ noncommutativity with the $(\theta, \bar{\theta})$ nonanticommutativity. It establishes correlation of the twistor structure deformation with supersymmetry encoded in the Lorentz invariant P.B. (9). This correlation becomes apparent under the reduction of the original superspace to the null supertwistor subspace formed by $Z^{\tilde{\mathcal{A}}}, \bar{Z}_{\tilde{\mathcal{A}}}$ connected by the relation: $Z^{\tilde{\mathcal{A}}}\bar{Z}_{\tilde{\mathcal{A}}} = 0$ [28]. The null supertwistors are formed by the triads $Z^{\tilde{\mathcal{A}}} = (iq^\alpha, \bar{\nu}_{\dot{\alpha}}, 2\bar{\eta}), \bar{Z}_{\tilde{\mathcal{A}}} = (\nu_\alpha, -i\bar{q}^{\dot{\alpha}}, 2\eta)$, where $q_\alpha = \omega_\alpha - 2i\bar{\eta}\theta_\alpha$, whose supersubspace is closed under the supersymmetry transformations. Because of this reduction we find the counterpart of the P.B. (32) to be vanishing

$$\{q_\alpha, \bar{q}_{\dot{\beta}}\} = 0 \tag{33}$$

together with any other P.B's. among the components of $Z^{\tilde{\mathcal{A}}}, \bar{Z}_{\tilde{\mathcal{A}}}$. It means that the supersubspace of null supertwistors $Z^{\tilde{\mathcal{A}}}, \bar{Z}_{\tilde{\mathcal{A}}}$ is inert under the deformation associated with the P.B. (9). So, we meet interesting coupling of twistor structure with supersymmetry, Lorentz invariance and Poisson structure which sheds light on general structure of non(anti)commutative superspaces.

6 The Lorentz invariant bracket in higher dimensions

The passage to the Majorana representation in the Poisson brackets (10-13)

$$\nu_a = \begin{pmatrix} \nu_\alpha \\ \bar{\nu}^{\dot{\alpha}} \end{pmatrix}, \quad \theta_a = \begin{pmatrix} \theta_\alpha \\ \bar{\theta}^{\dot{\alpha}} \end{pmatrix}, \quad C^{ab} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad \chi^a = C^{ab}\chi_b, \tag{34}$$

where C^{ab} is the charge conjugation matrix, presents them in the form suitable for the generalization to higher dimensions

$$\{\nu_a, \nu_b\} = 0, \quad \{\theta_a, \nu_b\} = 0, \quad \{x_m, \nu_a\} = \varphi_m \nu_a. \tag{35}$$

The real vectors x_m and φ_m in (35) are defined by the relations [36]

$$x_m = -\tfrac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\beta}x_{\beta\dot{\alpha}}, \quad x_{\alpha\dot{\beta}} = (\sigma^m)_{\alpha\dot{\beta}}x_m, \\ \varphi_m = -\tfrac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\beta}\varphi_{\beta\dot{\alpha}} \equiv \tfrac{1}{4}(\bar{\nu}\gamma_m\nu), \tag{36}$$

where γ_m are the Dirac matrices in the Majorana representation.

To rewrite the rest of the P.B's. in the Majorana representation it is convenient to change the Majorana spinor ν_a by other Majorana spinor λ_a

$$\lambda_a = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \equiv (\gamma_5\nu)_a, \quad (\gamma_5)_a{}^b = \begin{pmatrix} -i\delta_\alpha^\beta & 0 \\ 0 & i\delta_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix} \tag{37}$$

preserving the form of the P.B's. (35). In terms of the real Majorana spinor λ_a and the composed vectors φ_m and ψ_m

$$\varphi_m = \frac{1}{4}(\bar{\lambda}\gamma_m\lambda), \quad \psi_m = -\frac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\alpha}\psi_{\alpha\dot{\alpha}} \equiv -\frac{1}{2}(\bar{\theta}\gamma_m\lambda) \tag{38}$$

the P.B's. (10-13) of the primordial coordinates x_m, θ_a, λ_a are presented as follow

$$\{\lambda_a, \lambda_b\} = 0, \quad \{\theta_a, \lambda_b\} = 0, \quad \{x_m, \lambda_a\} = \varphi_m\lambda_a, \\ \{x_m, x_n\} = -i\psi_n\psi_m, \quad \{x_m, \theta_a\} = -\tfrac{1}{2}\psi_m\lambda_a, \quad \{\theta_a, \theta_b\} = -\tfrac{i}{4}\lambda_a\lambda_b. \tag{39}$$

The P.B's. of the composite vectors ψ_m and φ_m (38) among themselves and with the primordial coordinates take the form

$$\{x_m, \psi_n\} = \varphi_m\psi_n + \varphi_n\psi_m, \quad \{\psi_m, \theta_b\} = \tfrac{i}{2}\varphi_m\lambda_b, \quad \{\psi_m, \lambda_a\} = 0, \\ \{\psi_m, \psi_n\} = -i\varphi_m\varphi_n, \quad \{\psi_m, \varphi_n\} = 0 \tag{40}$$

and respectively

$$\{x_m, \varphi_n\} = 2\varphi_m\varphi_n, \quad \{\theta_a, \varphi_m\} = \{\lambda_a, \varphi_m\} = \{\varphi_m, \varphi_n\} = 0. \tag{41}$$

The P.B's. (39-41) originally derived for $D = 4$ are valid in D-dimensional space with $D = 2, 3, 4(mod 8)$, where the Majorana spinors exist. This procedure restores the vector form of the Moyal brackets (25) in the higher dimensions.

7 Other supersymmetric Lorentz invariant brackets

Using the Majorana spinor ν_a one can constuct one more supersymmetric and Lorentz invariant Poisson bracket in the addition to the P.B. (9) which is given by

$$\{F, G\} = F\,[\,\tfrac{i}{4}\,(\overleftarrow{D}\overrightarrow{\bar{D}} + \overleftarrow{\bar{D}}\overrightarrow{D}) + \tfrac{1}{2}(\overleftarrow{\partial}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\partial})\,]\,G \tag{42}$$

and yields different invariant Poisson brackets for the supercoordinates x and θ

$$\{x_{\alpha\dot{\alpha}}, x_{\beta\dot{\beta}}\} = -i(\varphi_{\alpha\dot{\beta}}\bar{\theta}_{\dot{\alpha}}\theta_\beta - \varphi_{\beta\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\theta_\alpha), \\ \{x_{\alpha\dot{\alpha}}, \theta_\beta\} = \tfrac{1}{2}\varphi_{\beta\dot{\alpha}}\theta_\alpha, \quad \{x_{\alpha\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \tfrac{1}{2}\varphi_{\alpha\dot{\beta}}\bar{\theta}_{\dot{\alpha}}, \\ \{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0, \quad \{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = -\tfrac{i}{4}\varphi_{\alpha\dot{\beta}} \tag{43}$$

We see that the new deformation (42) generates the zero P.B's. for the θ_a components with the same chirality in contrast to the deformation (9). The P.B's. (43) are added by

$$\begin{gathered}
\{\nu_\alpha, \nu_\beta\} = \{\nu_\alpha, \bar{\nu}_{\dot\beta}\} = \{\bar{\nu}_\alpha, \bar{\nu}_{\dot\beta}\} = 0, \\
\{\nu_\alpha, \theta_\beta\} = \{\nu_\alpha, \bar{\theta}_{\dot\beta}\} = \{\bar{\nu}_{\dot\alpha}, \theta_\beta\} = \{\bar{\nu}_{\dot\alpha}, \bar{\theta}_{\dot\beta}\} = 0, \\
\{x_{\alpha\dot\alpha}, \nu_\beta\} = \tfrac{1}{2}\varphi_{\alpha\dot\alpha}\nu_\beta, \quad \{x_{\alpha\dot\alpha}, \bar{\nu}_{\dot\beta}\} = \tfrac{1}{2}\varphi_{\alpha\dot\alpha}\bar{\nu}_{\dot\beta},
\end{gathered} \tag{44}$$

The P.B. (42) satisfies the Jacobi identities and produces the corresponding Moyal bracket

$$F \star G = F\, e^{\{\frac{ih}{8}[\overleftarrow{D}\overrightarrow{\bar{D}} + \overleftarrow{\bar{D}}\overrightarrow{D}) - \frac{1}{2}(\overleftarrow{\nabla}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\nabla})]\}}\, G, \tag{45}$$

where $\nabla \equiv 4i\partial$ and h is a quantum deformation parameter.

Using the conversion formulae from Sect. 6 gives the vector form for the P.B's. (43)

$$\begin{gathered}
\{x_m, x_n\} = -\tfrac{i}{4}(\chi_m\bar{\chi}_n - \chi_n\bar{\chi}_m), \\
\{x_m, \theta_\beta\} = -\tfrac{1}{4}\bar{\chi}_m\nu_\beta, \ \ \{x_m, \bar{\theta}_{\dot\beta}\} = -\tfrac{1}{4}\chi_m\bar{\nu}_{\dot\beta}, \\
\{\theta_a, \theta_b\} = -\tfrac{i}{8}(\nu_a^{(+)}\nu_b^{(-)} + \nu_b^{(+)}\nu_a^{(-)}),
\end{gathered} \tag{46}$$

where we introduced the complex Grasssmannian vector χ_m with the real and imaginary parts presented by ψ_{1m}, ψ_{2m} and the chiral components $\theta^{(\pm)}$ and $\nu^{(\pm)}$

$$\begin{gathered}
\chi_m \equiv (\nu\sigma_m\bar{\theta}) \equiv -\bar{\nu}\gamma_m\tfrac{1+i\gamma_5}{2}\theta \equiv \psi_{1m} + i\psi_{2m}, \\
\bar{\chi}_m \equiv (\chi_m)^* = -\bar{\nu}\gamma_m\tfrac{1-i\gamma_5}{2}\theta, \quad \psi_{1m} \equiv -\tfrac{1}{2}(\bar{\theta}\gamma_m\nu), \quad \psi_{2m} \equiv -\tfrac{1}{2}(\bar{\theta}\gamma_m\gamma_5\nu), \\
\theta^{(\pm)} \equiv \tfrac{1}{2}(1 \pm i\gamma_5)\theta, \ \ \nu^{(\pm)} \equiv \tfrac{1}{2}(1 \pm i\gamma_5)\nu.
\end{gathered} \tag{47}$$

Then the P.B's. (46) are presented in the form directly generalizing the P.B's. (39)

$$\begin{gathered}
\{x_m, x_n\} = -\tfrac{i}{2}(\psi_{1m}\psi_{1n} + \psi_{2m}\psi_{2n}), \\
\{x_m, \theta_a\} = -\tfrac{1}{4}(\psi_{1m}\nu_a + \psi_{2m}\lambda_a), \\
\{\theta_a, \theta_b\} = -\tfrac{i}{8}(\nu_a\nu_b + \lambda_a\lambda_b),
\end{gathered} \tag{48}$$

where $\lambda_a \equiv (\gamma_5\nu)_a$ as in (37). Comparing (48) with (39) we observe that the change of the P.B. (9) by (42) is equivalent to the complexification of the real Grassmannian vector ψ_m (38) accompanied by the appearance of the spinors ν_a and $(\gamma_5\nu)_a$) in the r.h.s. of (48).

The P.B. (42) and respectively the Moyal bracket (45) may be generalized to the case of extended supersymmetries with $N > 1$. The corresponding P.B. may be chosen as

$$\{F, G\} = F\,[\,\tfrac{i}{4}(\overleftarrow{D}_i\overrightarrow{\bar{D}}^i + \overleftarrow{\bar{D}}^i\overrightarrow{D}_i) + \tfrac{1}{2}(\overleftarrow{\partial}\overrightarrow{\Delta} - \overleftarrow{\Delta}\overrightarrow{\partial})\,]\,G, \tag{49}$$

where $D_i \equiv \nu_\alpha D_i^\alpha$ and $\bar{D}^i \equiv \bar{\nu}_{\dot\alpha}\bar{D}^{\dot\alpha i}$ with i=1,2,..,N. The P.B's. (49) generate the following brackets for the primordial (super)coordinates

$$\begin{gathered}
\{x_{\alpha\dot\alpha}, x_{\beta\dot\beta}\} = -i(\varphi_{\alpha\dot\beta}\bar{\theta}_{\dot\alpha i}\theta^i_\beta - \varphi_{\dot\alpha\beta}\bar{\theta}_{\dot\beta i}\theta^i_\alpha), \\
\{x_{\alpha\dot\alpha}, \theta^i_\beta\} = \tfrac{1}{2}\varphi_{\dot\alpha\beta}\theta^i_\alpha, \quad \{x_{\alpha\dot\alpha}, \bar{\theta}_{\dot\beta i}\} = \tfrac{1}{2}\varphi_{\alpha\dot\beta}\bar{\theta}_{\dot\alpha i}, \\
\{\theta^i_\alpha, \theta^k_\beta\} = \{\bar{\theta}_{\dot\alpha i}, \bar{\theta}_{\dot\beta k}\} = 0, \quad \{\theta^i_\alpha, \bar{\theta}_{\dot\beta k}\} = \tfrac{i}{4}\varphi_{\alpha\dot\beta}\delta^i{}_k.
\end{gathered} \tag{50}$$

The rest of the P.B's. for the supercoordinates $x_{\alpha\dot{\alpha}}, \nu_a, \theta^i_\alpha$ coincides with the P.B's. (44).

Other Lorentz invariant supersymmetric brackets may include more spinor coordinates. For $D = 4$ it is enough to add only one new spinor coordinate μ_α, because μ_α and ν_α form the complete spinorial basis identified with the Newman-Penrose dyad [27]

$$\mu^\alpha \nu_\alpha \equiv \mu^\alpha \varepsilon_{\alpha\beta} \nu^\beta = 1, \quad \mu_\alpha \nu_\beta - \mu_\beta \nu_\alpha = \varepsilon_{\alpha\beta}. \tag{51}$$

Then one can seek Lorentz invariant supersymmetric generalization of the P.B. (9) as

$$\{F, G\} = F \, [-\tfrac{i}{4} (\overleftarrow{D}^{(\nu)}_- \overrightarrow{D}^{(\nu)}_- + \overleftarrow{D}^{(\mu)}_- \overrightarrow{D}^{(\mu)}_-) + c(\overleftarrow{\partial}^{(\nu)} + \overleftarrow{\partial}^{(\mu)}) \overrightarrow{\Delta}' - \overleftarrow{\Delta}' (\overrightarrow{\partial}^{(\nu)} + \overrightarrow{\partial}^{(\mu)}) \,] \, G, \tag{52}$$

where the independent Lorentz invariant supersymmetric derivatives are defined as

$$\begin{gathered} D^{(\nu)}_\pm \equiv D^{(\nu)} \pm \bar{D}^{(\nu)}, \quad D^{(\mu)}_\pm \equiv D^{(\mu)} \pm \bar{D}^{(\mu)}, \quad \partial^{(\nu)} \equiv (\nu_\alpha \bar{\nu}_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}), \quad \partial^{(\mu)} \equiv (\mu_\alpha \bar{\mu}_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}), \\ D^{(\nu)} = \nu_\alpha D^\alpha, \quad \bar{D}^{(\nu)} = \bar{\nu}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}, \quad D^{(\mu)} = \mu_\alpha D^\alpha, \quad \bar{D}^{(\mu)} = \bar{\mu}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}, \\ \Delta' = (\nu_\alpha \tfrac{\partial}{\partial \nu_\alpha} + \bar{\nu}_{\dot{\alpha}} \tfrac{\partial}{\partial \bar{\nu}_{\dot{\alpha}}}) - (\mu_\alpha \tfrac{\partial}{\partial \mu_\alpha} + \bar{\mu}_{\dot{\alpha}} \tfrac{\partial}{\partial \bar{\mu}_{\dot{\alpha}}}). \end{gathered} \tag{53}$$

Studying these generalizations is under investigation.

8 Discussion

It was shown that an extension of the $N = 1$ superspace (x_m, θ_a) by one commuting Majorana spinor ν_a, or equivalently $\lambda \equiv \gamma_5 \nu$, yields the Poisson and Moyal brackets desribing Lorentz invariant supersymmetric deformations of the $N = 1$ superspace. Some examples of new brackets were constructed and their selfconsistency was proved.

We found that the noncommutativity of x_m with x_n and Grassmannian spinor θ_a is parametrized by the real or complex Grassmannian vectors ψ_m. These vectors are composed from θ_a and ν_a and describe the spin degrees of freedom in the models of spinning particle and string. At the same time, the nonanticommutativity of the θ_a componets depends only on the spin tensors constructed from ν_a which may be associated with a twistor component. It points out that a hidden spinorial structure of the space-time associated with the Penrose twistor picture could be an alternative source of the (super)coordinate non(anti)commutativity. We found a one to one correspondence between the Lorentz invariant Moyal brackets (25) and the well known (anti)commutators dependent on the constant supergravity background including the antisymmetric field B_{mn}, the graviphoton C_{ab} and the gravitino Ψ^a_m which may be schematically presented as

$$B^{-1}_{mn} \leftrightarrow i\psi_m \psi_n, \quad C_{ab} \leftrightarrow \lambda_a \lambda_b, \quad \Psi^a_m \leftrightarrow \psi_m \lambda^a \tag{54}$$

up to the second order corrections in the deformation parameter h which were proved to be zero. The map (54) transforms the field dependent Lorentz noninvariant (anti)commutators into the invariant Moyal brackets (25) and restores the required Lorentz invariance of the deformation. The map gets a natural explanation in the frame of the Feynman–Wheeler action at-a-distance theory and its superymmeric generalization [37], where the (super)fields were constructed from the primary (super)space coordinates. We outlined further generalizations of the studied invariant brackets to the cases of N extended supersymmetry and additional spinor coordinates based on the possibility to construct additional Lorentz invariant supersymmetric derivatives. Studying these generalizations and the corresponding deformations of superspaces are in progress.

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On Reality in Noncommutative Gravity

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Abstract

We study the problem of reality in the geometric formalism of the 4D noncommutative gravity using the known deformation of the diffeomorphism group induced by the twist operator with the constant deformation parameters ϑ^{mn}. It is shown that the real covariant derivatives can be constructed via $\star$-anticommutators of the real connection with the corresponding fields. The minimal noncommutative generalization of the real Riemann tensor contains only ϑ^{mn}-corrections of the even degrees in comparison with the undeformed tensor. The gauge field h_{mn} describes a gravitational field on the flat background. All geometric objects are constructed as the perturbation series using $\star$-degrees of h_{mn}. We consider the nonminimal tensor and scalar functions of h_{mn} of the odd degrees in ϑ^{mn} and remark that these pure noncommutative objects can change the geometry of the noncommutative gravity.

1 Introduction

The simplest noncommutativity in the 4-dimensional space is based on the following relation for operators of coordinates $\hat{x}^m$:

$$\hat{x}^m \star \hat{x}^n - \hat{x}^n \star \hat{x}^m = i\vartheta^{mn}, \tag{1.1}$$

where ϑ^{mn} are some constants and $m, n = 0, 1, 2, 3$. We shall consider the Weyl ordering in the algebra of noncommutative (NC) fields $A_\star$ using operator polynomials symmetrized in all indices

$$\hat{\phi}(\hat{x}^m) = f^0 + f^1_m \hat{x}^m + \sum_{k=2}^{\infty} f^k_{(m_1 \ldots m_k)} \hat{x}^{(m_1} \star \ldots \star \hat{x}^{m_k)}, \tag{1.2}$$

where $f^k_{(m_1 \ldots m_k)}$ are some numerical coefficients. One can analyze the commutative image of this operator function

$$\hat{\phi}(\hat{x}^m) \to \phi(x^m) = f^0 + f^1_m x^m + \sum_{k=2}^{\infty} f^k_{(m_1 \ldots m_k)} x^{m_1} \ldots x^{m_k}, \tag{1.3}$$

We treat $\phi(x)$ as an element of the commutative algebra A of smooth functions in the space R^4 with coordinates x^m.

As it has been shown recently [1, 2] the basic properties and symmetries of the NC field theories in $A_\star$ are connected with the quantum-group structures induced by the twist operator

$$\mathcal{F} = \exp(\mathcal{P}), \quad \mathcal{P} = \tfrac{i}{2}\vartheta^{mn}\partial_m \otimes \partial_n \tag{1.4}$$

which acts on the tensor products of functions $\phi \otimes \chi$. In particular, the Moyal-Weyl representation of the noncommutative product has the following form:

$$\begin{aligned}&\phi \star \chi = \mu \circ \mathcal{F}(\phi \otimes \chi) = \phi(x)\chi(x) + \tfrac{i}{2}\vartheta^{mn}(\partial_m\phi)(x)(\partial_n\chi)(x)\\ &-\tfrac{1}{8}\vartheta^{mn}\vartheta^{rs}(\partial_m\partial_r\phi)(x)(\partial_n\partial_s\chi)(x) + O(\vartheta^3),\end{aligned} \tag{1.5}$$

where $\mu(\phi \otimes \chi) = \phi\chi$ is the bilinear multiplication map in the commutative algebra A. The corresponding bilinear map in the NC algebra $A_\star$ is defined as $\mu_\star = \mu \circ \mathcal{F}$.

The complex conjugation of this $\star$-product satisfies the relation

$$\overline{\phi \star \chi} = \bar{\chi} \star \bar{\phi}, \tag{1.6}$$

where $\bar{\phi}$ and $\bar{\chi}$ are complex conjugated functions. The noncommutative product of the real functions ϕ and χ is not a real element of $A_\star$

$$\begin{aligned}&\phi \star \chi = \tfrac{1}{2}(\phi \star \chi + \chi \star \phi) + \tfrac{1}{2}(\phi \star \chi - \chi \star \phi) \equiv \tfrac{1}{2}\{\phi \overset{\star}{,} \chi\} + \tfrac{1}{2}[\phi \overset{\star}{,} \chi],\\ &\overline{\{\phi \overset{\star}{,} \chi\}} = \{\phi \overset{\star}{,} \chi\}, \quad \overline{[\phi \overset{\star}{,} \chi]} = -[\phi \overset{\star}{,} \chi].\end{aligned} \tag{1.7}$$

Using Eq.(1.5) one can check that deformation corrections in the $\star$-anticommutator contain only even degrees in ϑ starting from ϑ^2: $\{\phi \overset{\star}{,} \chi\} = 2\phi\chi + O_+(\vartheta^2)$, while the $\star$-commutator has the odd-degree ϑ-decomposition $[\phi \overset{\star}{,} \chi] = O_-(\vartheta)$.

The authors of refs. [1, 2] considered the quantum-group deformation of the Poincaré group using the twist operator $\mathcal{F}$ (1.4) and proved that the NC algebra $A_\star$ is covariant with respect to this quantum group. We will consider this twist deformation in Section 2.

The new approach to noncommutative gravity theory was proposed in [3] (see, also, a more deep discussion of the noncommutative mathematical formalism in [4]). This approach is based on the twist deformation of the diffeomorphism group in the real 4-dimensional space. In Section 3, we review the basic principles of this approach: 1) The diffeomorphism transformations of the primary matter fields and the metric tensor are not deformed; 2) The twisted diffeomorphism group acts covariantly on the $\star$-products of fields in the noncommutative algebra.

In Section 4 we construct the real NC generalizations of the Christoffel symbols and the corresponding Riemann tensor which have the standard transformation properties in the twisted diffeomorphism group. The condition of reality is used at all stages of our geometric formalism, for instance, the real covariant derivative of the metric tensor vanishes by definition. We derive the deformed Bianchi identity for the covariant derivatives of the NC Riemann tensor. By analogy with the ordinary gravity, it is convenient to analyze all nonlinearities of the NC formalism using the gravitational gauge field h_{mn} on the flat-space background. Note that the NC geometric quantities of ref.[3] are complex; however, they can be reduced to real quantities plus some complex tensors or scalars constructed from the gravitational field.

Our minimal version of the real gauge invariant NC-gravity action is constructed in Section 5. This action contains the perturbation series in the field h_{mn} starting from the standard free spin 2 term. Each interaction term of the action is also invariant with respect to the background twisted Poincaré group. Varying this action in the field h_{mn} one obtains the real NC gravity equations transforming as the contravariant tensor density. In our geometric formalism, the reality of the NC deformation of the classic equations of gravity seems natural, and the real NC field h_{mn} describes a standard number of the physical degrees of freedom. It is shown that the minimal NC-gravity equations has no odd ϑ^{mn}-corrections in comparison with the Einstein equations.

In Section 6, we prove that the noncommutative geometry is much more flexible than the geometry of ordinary gravity, in particular, one can construct nonminimal tensor or scalar functions of the gravitational field which have no classical analogues and vanish in the commutative limit $\vartheta^{mn} \to 0$. The nonminimal scalars could be added to the NC-gravity action with additional arbitrary constants. Note that the original NC-gravity action in the complex geometric formalism is real by definition [3]. In comparison with our action of Section 5, this action gives some nonminimal corrections.

2 Twisted Poincaré symmetry

Let us consider the infinitesimal transformations of the scalar field $\phi(x)$ in the Poincaré group

$$\delta\phi(x) = -(P_c + M_\omega)\phi(x) = -(c^m P_m + \tfrac{1}{2}\omega^{mn} M_{mn})\phi(x),$$
$$P_m = \partial_m, \quad M_{mn} = x_n\partial_m - x_m\partial_n, \tag{2.1}$$

where c^m, P_m and ω^{mn}, M_{mn} are the parameters and generators of translations and Lorentz rotations, respectively.

The coproduct in the Poincaré group is trivial on the group generators

$$\Delta(P_c) = P_c \otimes 1 + 1 \otimes P_c, \quad \Delta(M_\omega) = M_\omega \otimes 1 + 1 \otimes M_\omega. \tag{2.2}$$

This coproduct determines the action of generators on the tensor and local products of fields.

The twist-deformed Poincaré group has the same transformations of the primary local fields (2.1). Twist deformations appear in the coproduct of the deformed Lorentz transformations

$$\Delta_t(\partial_m) = \exp(-\mathcal{P})\Delta(\partial_m)\exp(\mathcal{P}) = \Delta(\partial_m), \tag{2.3}$$
$$\Delta_t(M_\omega) = \exp(-\mathcal{P})\Delta(M_\omega)\exp(\mathcal{P}) = \Delta(M_\omega)$$
$$+\tfrac{i}{2}\omega^{mn}\vartheta_{ms}P_n \otimes P^s - \tfrac{i}{2}\omega^{mn}\vartheta_{rn}P^r \otimes P_m. \tag{2.4}$$

This coproduct acts on the tensor product of functions

$$-\Delta_t(M_\omega) \circ \phi \otimes \chi = (\delta_\omega\phi) \otimes \chi + \phi \otimes (\delta_\omega\chi)$$
$$+\tfrac{i}{2}(\omega^{mn}\vartheta_{ns} - \vartheta^{mn}\omega_{ns})\partial_m\phi \otimes \partial^s\chi. \tag{2.5}$$

Applying the map $\mu_\star = \mu \circ \mathcal{F}$ to this relation one can obtain the covariant formula of the deformed Lorentz transformations on the $\star$-product of the primary scalar fields

$$\hat{\delta}_\omega(\phi \star \chi) = -\mu_\star \circ \Delta_t(M_\omega) \circ \phi \otimes \chi = (\delta_\omega\phi) \star \chi + \phi \star (\delta_\omega\chi)$$
$$+\tfrac{i}{2}(\omega^{mn}\vartheta_{ns} - \vartheta^{mn}\omega_{ns})\partial_m\phi \star \partial^s\phi = -M_\omega(\phi \star \chi). \tag{2.6}$$

The covariant deformed Lorentz transformations of $\star$-products of tensor fields will be used in Section 5.

3 Twisted diffeomorphism group

We shall consider the active form of the infinitesimal diffeomorphism transformations of the real scalar field

$$\delta_\xi\phi(x) = -[\xi,\phi](x) \equiv -\mathcal{L}_\xi\phi(x) = -(\xi^m\partial_m\phi)(x), \tag{3.1}$$

where $\xi^m(x)$ are arbitrary smooth functions and $\mathcal{L}_\xi$ is a Lie derivative corresponding to a differential operator $\xi = \xi^m\partial_m$. The infinite-dimensional Lie algebra of diffeomorphisms is isomorphic to the set of the first-order differential operators $\Xi = Vect(R^4)$. The commutator of two operators ξ_1 and ξ_2 gives the Lie bracket formula for Ξ

$$[\xi_1,\xi_2] = \xi^m_{br}\partial_m, \quad \xi^m_{br} = \xi^n_1\partial_n\xi^m_2 - \xi^n_2\partial_n\xi^m_1. \tag{3.2}$$

The finite diffeomorphisms belong to the universal enveloping algebra $U\Xi$, for instance, the active diffeomorphism of the scalar field has the following form:

$$\phi'(x) = (e^{-\mathcal{L}_\xi}\phi)(x) = e^{-\xi}\phi e^{\xi}. \tag{3.3}$$

The gradient of the scalar field $\partial_m\phi$ transforms as the covariant vector field

$$\delta_\xi(\partial_m\phi) = -(\xi\partial_m\phi) - (\partial_m\xi^p)\partial_p\phi = -\mathcal{L}_\xi\partial_m\phi. \tag{3.4}$$

The contravariant vector field transforms as follows

$$\delta_\xi V^m = (-\xi + \partial_p\xi^m)V^p = -\mathcal{L}_\xi V^m. \tag{3.5}$$

It is convenient to introduce the generators L^p_q of the group GL(n,R)

$$L^p_q T^{mr}_{ns} = \delta^m_q T^{pr}_{ns} + \delta^r_q T^{mp}_{ns} - \delta^p_n T^{mr}_{qs} - \delta^p_s T^{mr}_{nq}, \tag{3.6}$$

then the compact form of the standard tensor transformation can be defined via these generators and the multiplication of local parameters $\xi^q_p(x) = \partial_p\xi^q$

$$\delta_\xi T^{mr}_{ns} = -\mathcal{L}_\xi T^{mr}_{ns} = (-\xi + \xi^q_p L^p_q)T^{mr}_{ns}. \tag{3.7}$$

Let us consider the active transformation of the metric tensor

$$\delta_\xi g_{mn} = -\mathcal{L}_\xi g_{mn} = (-\xi + \xi^q_p L^p_q)g_{mn}. \tag{3.8}$$

In the perturbation theory on the flat background metric η_{mn}, one can analyze all nonlinearities in terms of the gauge gravitational field h_{mn}

$$\begin{aligned} &g_{mn} = \eta_{mn} + \kappa h_{mn}, \\ &\delta_\xi h_{mn} = -2\kappa^{-1}\xi_{(mn)} + (-\xi + \xi^q_p L^p_q)h_{mn}, \end{aligned} \tag{3.9}$$

where κ is a gravitational constant and

$$\xi_{(mn)} \equiv \tfrac{1}{2}(\xi_{mn} + \xi_{nm}), \quad \xi_{mn} = \eta_{np}\partial_m\xi^p. \tag{3.10}$$

We use also the equivalent representations of the gauge field $h^m_n = \eta^{mp}h_{pn}, \quad h^{mn} = \eta^{mp}\eta^{nq}h_{pq}$.

In the twisted diffeomorphism group $U\Xi_\star$ [3, 4], one considers the undeformed transformations of the primary fields (3.1),(3.8)

$$\hat{\delta}_\xi \phi = \delta_\xi \phi, \quad \hat{\delta}_\xi g_{mn} = \delta_\xi g_{mn} \tag{3.11}$$

and the same Lie brackets (3.2). However, the coproduct in $U\Xi_\star$ is deformed by the twist operator $\mathcal{F}$ (1.4)

$$\begin{aligned}&\Delta_t(\xi) = \exp(-\mathcal{P})(\xi \otimes 1 + 1 \otimes \xi)\exp(\mathcal{P}) = \xi \otimes 1 + 1 \otimes \xi\\ &-\tfrac{i}{2}\vartheta^{mn}([\partial_m, \xi] \otimes \partial_n + \partial_m \otimes [\partial_n, \xi])\\ &-\tfrac{1}{8}\vartheta^{mn}\vartheta^{rs}([\partial_m, [\partial_r, \xi]] \otimes \partial_n\partial_s + \partial_m\partial_r \otimes [\partial_n, [\partial_s, \xi]]) + O(\vartheta^3)\end{aligned} \tag{3.12}$$

The corresponding twisted transformations of the noncommutative products are defined via this coproduct, for instance,

$$\hat{\delta}_\xi(\phi \star \chi) = -\mu_\star \circ \Delta_t(\xi)(\phi \otimes \chi) = -\mathcal{L}_\xi(\phi \star \chi) = -(\xi(\phi \star \chi)). \tag{3.13}$$

The twist deformation guarantees the covariance of $\star$-products of any primary tensors with respect to the transformations of $U\Xi_\star$

$$\begin{aligned}&\hat{\delta}_\xi(\partial_m\phi \star V^m) = -(\xi(\partial_m\phi \star V^m)), \quad \hat{\delta}_\xi(g_{mn} \star \phi) = (-\xi + \xi^q_p L^p_q)(g_{mn} \star \phi),\\ &\hat{\delta}_\xi(g_{mn} \star \ldots \star g_{rs}) = (-\xi + \xi^q_p L^p_q)(g_{mn} \star \ldots \star g_{rs})\end{aligned} \tag{3.14}$$

where the last formula contains an arbitrary number of fields. Note that the operators ∂_m and L^p_q satisfy the undeformed Leibniz rules, while the functions $\xi^m(x)$ and $\xi^p_q(x)$ do not commute with partial derivatives in the formula of the $\star$-product. The deformed Leibniz rules for $\hat{\delta}_\xi$ can be derived directly from these relations.

The twisted diffeomorphism group acts noncovariantly on the commutative products of fields, for instance,

$$\begin{aligned}&\hat{\delta}_\xi(g_{mn}g_{rs}) = -\tfrac{1}{2}\mu \circ \Delta_t(\xi)(g_{mn} \otimes g_{rs} + g_{rs} \otimes g_{mn}) = \delta_\xi(g_{mn}g_{rs})\\ &-\tfrac{1}{8}\vartheta^{pq}\vartheta^{ut}[(\partial_p\partial_u\xi^v)(\partial_v g_{mn})(\partial_q\partial_t g_{rs}) + (\partial_q\partial_t g_{mn})(\partial_p\partial_u\xi^v)(\partial_v g_{rs})] + O_+(\vartheta^4)\end{aligned} \tag{3.15}$$

where $\delta_\xi(g_{mn}g_{rs})$ is the undeformed transformation, and all odd degrees of ϑ are canceled due to the symmetry of indices $m, n \leftrightarrow r, s$. It should be remarked that this formula is completely compatible with (3.14).

4 Real geometry in the noncommutative space

Noncommutative products of real tensors or scalars are not real, in general, but it is not difficult to construct the real combinations in algebra $A_\star$, for instance, $\{g_{mn} \overset{\star}{,} g_{rs}\}$. We shall use the reality condition in all constructions of the noncommutative geometry.

The noncommutative relation for the $\star$-inverse tensor $g_{mn} \star G^{np} = \delta^p_m$ was analyzed in ref.[3]. It is evident that this complex tensor satisfies the conditions $\overline{G^{mn}} = G^{nm} \neq G^{mn}$. We prefer to use the real symmetric contravariant tensor satisfying the following relation:

$$\tfrac{1}{2}\{g_{mn} \overset{\star}{,} g^{np}_\star\} = \eta_{mn}g^{np}_\star + \tfrac{1}{2}\kappa\{h_{mn} \overset{\star}{,} g^{np}_\star\} = \delta^p_m. \tag{4.1}$$

It is not difficult to construct the κ-decomposition of $g_\star^{mn}$ in terms of $\star$-products of the gauge gravitational fields

$$g_\star^{mn} = \eta^{mn} - \kappa h^{mn} + \tfrac{1}{2}\kappa^2\{h_p^m \overset{\star}{,} h^{pn}\} - \tfrac{1}{4}\kappa^3\{h_p^m \overset{\star}{,} \{h_s^p \overset{\star}{,} h^{sn}\}\} + O(h^4)$$
$$= g^{mn}(\kappa) + O_+(\vartheta^2) \qquad (4.2)$$

where $g^{mn}(\kappa)$ is the inverse metric of the ordinary gravity $g_{mn}g^{np}(\kappa) = \delta_m^p$. To check properties of the $\star$-perturbative expressions one should use the inhomogeneous transformations (3.9) which connect $\star$-degrees of the fields h_{mn}

$$\hat{\delta}_\xi(h_{mn} \star h_{rs}) = -2\kappa^{-1}\xi_{(mn)}h_{rs} - 2\kappa^{-1}\xi_{(rs)}h_{mn} - \mathcal{L}_\xi(h_{mn} \star h_{rs}),$$
$$\hat{\delta}_\xi(h_{mn} \star h_{rs} \star h_{pq}) = -2\kappa^{-1}\xi_{(mn)}(h_{rs} \star h_{pq}) - 2\kappa^{-1}\xi_{(rs)}(h_{mn} \star h_{pq})$$
$$-2\kappa^{-1}\xi_{(pq)}(h_{mn} \star h_{rs}) - \mathcal{L}_\xi(h_{mn} \star h_{rs} \star h_{pq}). \qquad (4.3)$$

These transformations are equivalent to the homogeneous transformations of the tensor $\star$-degrees (3.14).

Note that noncommutative generalizations of raising and lowering of vector indices are not uniquely defined. One can consider, for instance, the reality preserving relation between the contravariant and covariant vectors

$$L_1: \quad V^m \to V_m = \tfrac{1}{2}\{g_{mn} \overset{\star}{,} V^n\} = \eta_{mn}V^n + \tfrac{1}{2}\kappa\{h_{mn} \overset{\star}{,} V^n\}. \qquad (4.4)$$

The inverse map $V_m \to V^m$ is defined via the h_{mn} decomposition, however, the alternative raising procedure can use the anticommutator $\frac{1}{2}\{g_\star^{mn} \overset{\star}{,} V_n\}$. It is not difficult to construct reality-preserving raising or lowering maps for tensors.

The real NC determinant of the metric is defined via the 4th rank antisymmetric symbols

$$g^\star \equiv D_4^\star(g_{mn}) = \frac{1}{24}\varepsilon^{m_1m_2m_3m_4}\varepsilon^{n_1n_2n_3n_4}g_{m_1n_1} \star g_{m_2n_2} \star g_{m_3n_3} \star g_{m_4n_4}. \qquad (4.5)$$

It transforms as a density of the weight -2 in $U\Xi_\star$

$$\hat{\delta}_\xi g^\star = -(\xi g^\star) - 2\partial_p\xi^p g^\star. \qquad (4.6)$$

The $\star$-polynomial κ-decomposition of this NC determinant is

$$g^\star \equiv -1 - \kappa h_m^m - \tfrac{1}{2}\kappa^2 h_m^m \star h_n^n + \tfrac{1}{2}\kappa^2 h_n^m \star h_m^n + O(h^3) = g(\kappa) + O_+(\vartheta^2), \quad (4.7)$$

where higher terms are omitted for brevity, and $g(k)$ is a determinant of the classical metric. The absence of the odd-degree ϑ-corrections can be easily proved in $\star$-monomials of this decomposition.

The real NC generalization of the 4D volume density $e^\star$ can be also calculated as the perturbative $\star$-series

$$\hat{\delta}_\xi e^\star = -(\xi e^\star) - \partial_p\xi^p e^\star,$$
$$e^\star(\kappa,\vartheta) \equiv \sqrt{-g^\star} = 1 + \tfrac{1}{2}\kappa h_m^m + \tfrac{1}{8}\kappa^2 h_m^m \star h_n^n - \tfrac{1}{4}\kappa^2 h_n^m \star h_m^n + O(h^3)$$
$$= \sqrt{-g(\kappa)} + O_+(\vartheta^2). \qquad (4.8)$$

We shall use a subsidiary condition of reality of the NC connection $\overline{\Gamma^{\star r}_{mn}} = \Gamma^{\star r}_{mn}$ which is compatible with a standard definition of the gauge transformation of the connection

$$\hat{\delta}_\xi \Gamma^{\star r}_{mn} = -\partial_m \xi^r_n + (-\xi + \xi^q_p L^p_q)\Gamma^{\star r}_{mn}. \tag{4.9}$$

Left and right products of the NC-connections and vectors obey the similar inhomogeneous transformation laws

$$\begin{aligned}
&\hat{\delta}_\xi(\Gamma^{\star r}_{mn} \star V_t) = -(\partial_m \xi^r_n)V_t - \mathcal{L}_\xi(\Gamma^{\star r}_{mn} \star V_t),\\
&\hat{\delta}_\xi(V_t \star \Gamma^{\star r}_{mn}) = -(\partial_m \xi^r_n)V_t - \mathcal{L}_\xi(V_t \star \Gamma^{\star r}_{mn}).
\end{aligned} \tag{4.10}$$

The symmetrized definition of the covariant S-derivatives is based on anticommutators

$$\begin{aligned}
&\nabla^S_m \star V_n = \partial_m V_n - \tfrac{1}{2}\{\Gamma^{\star r}_{mn} \overset{\star}{,} V_r\},\\
&\nabla^S_m \star V^r = \partial_m V^r + \tfrac{1}{2}\{\Gamma^{\star r}_{pm} \overset{\star}{,} V^p\},
\end{aligned} \tag{4.11}$$

it allows us to ensure reality and the correct tensor transformation properties of these quantities. Note that the commutator term $[\Gamma^{\star r}_{mn} \overset{\star}{,} V_r]$ is an independent $U\Xi_\star$ tensor. The covariant S-derivatives of tensors contain analogous anticommutator terms with connections for any index. The complex covariant derivative of ref. [3] is defined via a left multiplication of the corresponding connection.

To obtain a real NC generalization of the Christoffel symbols, we use a symmetrized version of the covariant constancy condition

$$\partial_m g_{nr} = \tfrac{1}{2}\{\Gamma^{\star p}_{mn} \overset{\star}{,} g_{pr}\} + \tfrac{1}{2}\{\Gamma^{\star p}_{mr} \overset{\star}{,} g_{np}\}. \tag{4.12}$$

For the symmetric connection $\Gamma^{\star p}_{mn} = \Gamma^{\star p}_{nm}$, this condition yields the simple relation

$$\kappa\gamma_{mnr} = \tfrac{1}{2}\kappa(\partial_m h_{rn} + \partial_n h_{rm} - \partial_r h_{mn}) = \eta_{rp}\Gamma^{\star p}_{mn} + \tfrac{1}{2}\kappa\{h_{rp} \overset{\star}{,} \Gamma^{\star p}_{mn}\} \tag{4.13}$$

which can be solved by iterations. The $\star$-perturbative solution for the real Christoffel symbols is

$$\begin{aligned}
&\Gamma^{\star r}_{mn} = \kappa\eta^{rp}\gamma_{mnp} - \tfrac{1}{2}\kappa^2\{h^{rp} \overset{\star}{,} \gamma_{mnr}\} + \tfrac{1}{4}\kappa^3\{h^r_s \overset{\star}{,} \{h^{sp} \overset{\star}{,} \gamma_{mnp}\}\}\\
&-\tfrac{1}{8}\kappa^4\{h^r_t \overset{\star}{,} \{h^t_s \overset{\star}{,} \{h^{sp} \overset{\star}{,} \gamma_{mnp}\}\}\} + O(h^5).
\end{aligned} \tag{4.14}$$

In Section 6 we analyze nonminimal constructions of the NC connection which can be represented as the sum of our real connection plus some tensor functions of the metric. In comparison with the classical Christoffel symbols $\Gamma^r_{mn}(\kappa)$ our minimal NC generalization has only even-degree terms in the ϑ-decomposition

$$\Gamma^{\star r}_{mn}(\kappa, \vartheta) = \Gamma^r_{mn}(\kappa) + O_+(\vartheta^2) \tag{4.15}$$

The complex NC generalization of the Christoffel symbols [3] contains terms odd in ϑ^{mn} including the linear term.

The trace of our connection

$$\Gamma^{\star p}_{mp} = \tfrac{1}{2}\kappa\partial_m h^r_r - \tfrac{1}{4}\kappa^2\partial_m(h^{pr} \star h_{pr}) + \tfrac{1}{4}\kappa^3 A^3_m(h) + \ldots$$

cannot be represented as a total derivative of some density starting from the 3rd order term $A^3_m(h)$.

The real NC generalization of the Riemann tensor has the following form:

$$R^{\star s}_{mnr} = \partial_m \Gamma^{\star s}_{nr} - \partial_n \Gamma^{\star s}_{mr} + \frac{1}{2}\{\Gamma^{\star p}_{nr} \overset{\star}{,} \Gamma^{\star s}_{mp}\} - \frac{1}{2}\{\Gamma^{\star p}_{mr} \overset{\star}{,} \Gamma^{\star s}_{np}\}. \tag{4.16}$$

Note that the trace tensor $R^{\star r}_{mnr} = \partial_m \Gamma^{\star r}_{nr} - \partial_n \Gamma^{\star r}_{mr}$ does not vanish in this representation. The tensor transformation properties

$$\hat{\delta}_\xi R^{\star s}_{mnr} = (-\xi + \xi^p_q L^q_p) R^{\star s}_{mnr} \tag{4.17}$$

follow from the basic inhomogeneous transformations

$$\hat{\delta}_\xi (\Gamma^{\star p}_{nr} \star \Gamma^{\star s}_{mt}) = -(\partial_n \xi^p_r) \Gamma^{\star s}_{mt} - (\partial_m \xi^s_t) \Gamma^{\star p}_{nr} - \mathcal{L}_\xi (\Gamma^{\star p}_{nr} \star \Gamma^{\star s}_{mt}). \tag{4.18}$$

The deformed Bianchi identity for this NC Riemann tensor

$$\begin{aligned} &\varepsilon^{upmn} \nabla^S_p R^{\star s}_{mnr} = \frac{1}{2} \varepsilon^{upmn} (\{\Gamma^{\star s}_{mz} \overset{\star}{,} \{\Gamma^{\star t}_{pr} \overset{\star}{,} \Gamma^{\star z}_{nt}\}\} - \{\Gamma^{\star t}_{pr} \overset{\star}{,} \{\Gamma^{\star z}_{nt} \overset{\star}{,} \Gamma^{\star s}_{mz}\}\}) \\ &= \frac{1}{2} \varepsilon^{upmn} [\Gamma^{\star z}_{nt} \overset{\star}{,} [\Gamma^{\star t}_{pr} \overset{\star}{,} \Gamma^{\star s}_{mz}]] \end{aligned} \tag{4.19}$$

contains the unusual tensor function of connections vanishing in the commutative limit.

The real NC Ricci tensor is symmetric by definition

$$R^\star_{(mr)} = \frac{1}{2}(\partial_m \Gamma^{\star n}_{rn} + \partial_r \Gamma^{\star n}_{mn}) - \partial_n \Gamma^{\star n}_{mr} + \frac{1}{2}\{\Gamma^{\star p}_{nr} \overset{\star}{,} \Gamma^{\star n}_{mp}\} - \frac{1}{2}\{\Gamma^{\star p}_{mr} \overset{\star}{,} \Gamma^{\star n}_{pn}\}. \tag{4.20}$$

The contraction with $g^{mn}_\star$ (4.1) yields the corresponding real scalar curvature

$$R^\star = \frac{1}{2}\{g^{mr}_\star \overset{\star}{,} R^\star_{(mr)}\} = \frac{1}{2}\{g^{mr}_\star \overset{\star}{,} R^{\star n}_{mnr}\}. \tag{4.21}$$

Thus, we have formulated the minimal NC generalization of the Riemann geometry which preserves reality and gives only even-degree deformation corrections to the classical geometric objects.

5 Minimal noncommutative gravity action and equations

The minimal real NC gravitational action can be constructed as a direct analogue of the gravity action ignoring possible ambiguities of the NC geometry

$$S_\star = \frac{1}{2\kappa^2} \int d^4x \{e^\star \overset{\star}{,} R^\star\} = \frac{1}{\kappa^2} \int d^4x e^\star \star R^\star, \tag{5.1}$$

where $e^\star$ is the NC density (4.8). The NC-gravity-matter action $A_\star = \int d^4x e^\star \star L_\star$ contains the real scalar Lagrangian of additional matter fields $L_\star$. We treat $S_\star + A_\star$ as the perturbation $\star$-series in κh_{mn}. In the next Section we analyze the possible nonminimal versions of the NC-gravity action using the pure noncommutative tensors and scalars vanishing in the commutative limit.

To derive equations of motion in the noncommutative field theory one should use the variation operator δ_v satisfying the usual Leibniz rule for $\star$-products

$$\delta_v(h_{mn} \star \ldots \star h_{pq}) = (\delta_v h_{mn} \star \ldots \star h_{pq}) + \ldots + (h_{mn} \star \ldots \star \delta_v h_{pq}), \tag{5.2}$$

where $\delta_v h_{mn}$ are arbitrary infinitesimal tensor functions. Using the variational principle and the cyclicity property of integrals on $A_\star$ one can obtain the NC gravitational equations

$$\delta_v(S_\star + A_\star) = \frac{1}{\kappa^2}\int d^4x\, \delta_v h_{mn} \star \mathcal{E}^{mn} = 0 \Rightarrow$$
$$\mathcal{E}^{mn} = \mathcal{G}^{mn}(h) - \kappa^2 \mathcal{T}^{mn} = 0, \tag{5.3}$$

where contributions of all terms with h_{mn} should be taken into account. The contravariant real tensor density $\mathcal{E}^{mn}$ contains the pure gravitational part $\mathcal{G}^{mn}$ and the NC generalization of the matter energy-momentum tensor $\mathcal{T}^{mn}$. This tensor-density equation can be transformed, for instance, to a more familiar covariant tensor form of the NC equation using the NC procedure of lowering of the tensor indices.

The perturbative $\star$-expansion of the gravitational action $S_\star(\kappa h)$ on the flat background contains the standard free spin 2 action $S^{(2)}(h)$ and the sum of higher-order interaction terms $\kappa^{k-2} S_\star^{(k)}(h)$. The reality condition for h_{mn} allows us to preserve a standard number of the physical degrees of freedom in the NC gravity.

The twisted gauge transformation of h_{mn} (3.9) connects terms of different order. In addition, each term of the action is invariant with respect to the background twisted Poincaré transformations with constant parameters c^r and ω^{rs} considered in section 2

$$\hat{\delta}_b(h_{mn} \star \ldots \star h_{pq}) = [-c^r \partial_r + \omega^{rs}(x_r \partial_s + L_{rs})](h_{mn} \star \ldots \star h_{pq}), \tag{5.4}$$

where the last operator generates Lorentz transformations of indices $(\omega^{rs} L_{rs}) h_{mn} = -\omega_{mr} h^r_n - \omega_{nr} h^r_m$.

In our treatment, the minimal NC-gravity action and equations have only even-degree ϑ-corrections in comparison with the corresponding classical action and equations

$$(S_\star + A_\star)(\kappa, \vartheta) = (S + A)(\kappa) + O_+(\vartheta^2),$$
$$\mathcal{G}^{mn} - \kappa^2 \mathcal{T}^{mn} = G^{mn}(\kappa) - \kappa^2 T^{mn} + O_+(\vartheta^2) = 0, \tag{5.5}$$

where $G^{mn}(\kappa) - \kappa^2 T^{mn} = 0$ is the contravariant tensor-density representation of the Einstein equations. Note that the $U\Xi_\star$ covariance of the ϑ-decompositions can be restored via the representation (3.15) of the deformed transformations on ordinary products of fields.

Let us consider the ϑ-linear approximate solution of this minimal deformed gravity equations

$$g_{mn} = g^{(0)}_{mn}(\kappa) + \tfrac{1}{2}\vartheta^{rs}(\Delta g)_{mn,rs}(\kappa), \tag{5.6}$$

where $g^{(0)}_{mn}(\kappa)$ is some classical gravity solution. The $O_+(\vartheta^2)$ terms in Eq.(5.5) do not contribute to this approximation, so $(\Delta g)_{mn,rs}$ can be found from the undeformed Einstein equations.

It is well known that the reality of the classical field action is directly connected with the problem of unitarity in the corresponding quantum theory. We do not analyze here the quantization problems, but hope that noncommutativity could help solving some difficulties of the quantum gravity.

6 Pure noncommutative tensors in NC-gravity

It should be remarked that the geometry of the noncommutative space based on the twisted diffeomorphism group $U\Xi_\star$ is much more flexible than the Riemann geometry in the undeformed space. We have shown that the connection and the curvature tensor have the minimal real analogs in the NC-geometry; however, it is possible to construct pure noncommutative corrections to geometrical objects vanishing in the commutative limit $\vartheta^{mn} \to 0$. These pure NC-tensors constructed from the basic metric field can be independently added to the basic NC connection or taken into account in the analysis of the NC gravity action. A priori it is not clear how to fix ambiguities arising from this NC flexibility, so one should classify all pure NC tensor and scalar terms in this approach.

The $\star$-commutator of the gravitational fields $i[g_{mn} \overset{\star}{,} g_{rs}]$ is the simplest example of the pure NC tensor term. The antisymmetric 2nd rank NC-tensor can be constructed by a real contraction of this commutator with tensor $g_\star^{nr}$ (4.1)

$$
\begin{aligned}
N_{[ms]} &= \tfrac{i}{2}\{g_\star^{nr} \overset{\star}{,} [g_{mn} \overset{\star}{,} g_{rs}]\} = i\kappa^2[h_{mn} \overset{\star}{,} h^n_s] + O(h^4) \\
&= -\vartheta^{pq}\kappa^2\partial_p h_{mn}\partial_q h^n_s + O_-(\vartheta^3).
\end{aligned} \tag{6.1}
$$

It is not difficult to construct the pure NC-tensor of a length dimension $l = -1$ using various $\star$-combinations of the connection (4.14) with the tensors g_{mn} and (4.2)

$$
S^{\star r}_{(mn)} = \tfrac{ia}{2}\{g_\star^{rp} \overset{\star}{,} [g_{rs} \overset{\star}{,} \Gamma^{\star s}_{mn}]\} + \tfrac{ib}{2}\{g_{mn} \overset{\star}{,} [g_\star^{pq} \overset{\star}{,} \Gamma^{\star r}_{pq}]\} = O_-(\vartheta), \tag{6.2}
$$

where a and b are arbitrary constants. The torsion-type NC tensor is

$$
T^{\star r}_{[mn]} = \tfrac{ic}{2}(\{g_\star^{rp} \overset{\star}{,} [g_{ms} \overset{\star}{,} \Gamma^{\star s}_{np]}\} - \{g_\star^{rp} \overset{\star}{,} [g_{ms} \overset{\star}{,} \Gamma^{\star s}_{np]}\}) = O_-(\vartheta), \tag{6.3}
$$

where c is an additional constant. These pure NC tensors can be added to the flexible NC connection

$$
\hat{\Gamma}^{\star r}_{mn} = \Gamma^{\star r}_{mn} + S^{\star r}_{(mn)} + T^{\star r}_{[mn]} = \Gamma^r_{mn}(\kappa) + O(\vartheta) \tag{6.4}
$$

which admits the appearance of odd degrees of deformation constants. The corresponding correction in the Riemann tensor contains the covariant derivatives of $S^{\star r}_{(mn)}$ and $T^{\star r}_{[mn]}$.

One can also consider the following pure NC symmetric tensor function:

$$
Q_{mn} = i[\Gamma^{\star s}_{mn} \overset{\star}{,} \Gamma^{\star p}_{sp}] = -\vartheta^{ut}\partial_u\Gamma^s_{mn}(\kappa)\partial_t\Gamma^p_{sp}(\kappa) + O_-(\vartheta^3) \tag{6.5}
$$

and its scalar contraction with $g_\star^{mn}$.

Nonminimal versions of the NC gravity admit the additional odd ϑ-corrections to the minimal action $S_\star + A_\star$. The commutator terms in the quadratic action are proportional to total derivatives and do not deform the free spin 2 equations, however, additional pure NC scalar terms could appear in the nonlinear gravity interaction. Nonminimal NC gravity equations have the linear ϑ-corrections, and these terms can essentially change the deformed gravity solutions. The $\star$-commutators of matter and gravitational fields are important in the nonminimal NC action of the matter interactions. For instance, the pure NC tensor $N_{[mn]}$ (6.1) can interact directly with the electromagnetic field-strength or $\star$-commutators like $[\partial_m\phi \overset{\star}{,} \partial_n\phi]$.

Note that the original action of the noncommutative gravity [3] was constructed via the real part of the corresponding complex scalar curvature in a tetrad representation of the NC metric. In our interpretation, the complex geometry formalism yields a nonminimal NC gravity action with linear and higher ϑ-corrections to the classical gravity action.

7 Conclusions

In this paper, we addressed the problem of reality in the ϑ^{mn}-noncommutative gravity theory based on the twist-deformed diffeomorphism group. The reality of the metric tensor is the important physical condition and the deformed geometry formalism can preserve this property. The real NC generalizations of the connection and Riemann tensor are constructed in this approach. It is shown that the Bianchi identity for the NC Riemann tensor contains the pure noncommutative tensor term. We considered the perturbative expansions of all real NC geometric quantities in $\star$-degrees of the gravitational field h_{mn} on the flat background. The corresponding minimal real action of the NC gravity and the real deformed gravity equations are presented. We combine expansions of the geometric objects in the gravitational constant κ and the deformation constants, and control symmetry properties of these perturbative methods. By definition, the minimal NC-gravity equations have only even-degree ϑ-corrections in comparison with the classical Einstein equations. The geometry of the NC gravity based on the twisted diffeomorphism group is flexible: it admits the existence of nonminimal pure noncommutative tensor and scalar functions of h_{mn} which have no classical analogues. It is not excluded that advantages of the NC gravity are connected just with the nonminimal odd-degree deformation terms in the gravity interaction. The reality-preserving flexible geometric formalism and the combined (κ, ϑ) perturbation methods seem convenient for studying physical effects of the noncommutative gravity theory. We hope that arbitrary constants of nonminimal NC interactions could be fixed analyzing possible short-distance gravitational effects.

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3 INTEGRABLE SYSTEMS

Remarks on Dimensional Reduction of Gravity

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Abstract

We discuss some problems related to dimensional reductions of (super)gravity theories to two-dimensional and one-dimensional dilaton gravity models. We consider here the most general cylindrical reductions and derive the corresponding (1+1)-dimensional dilaton gravity, paying a special attention to a possibility of producing nontrivial cosmological potentials from purely geometric variables (so to speak, from 'nothing'). Then we discuss further reductions of two-dimensional theories to the dimension one by a general procedure of separating the space and time variables (we illustrate this on the example of the spherically reduced gravity). This procedure is more general than the usual 'naive' reduction and, possibly, more general than the reductions using group theoretical methods.

1 Introduction

The procedure of dimensional reduction in classical physics is a well known matter but, when one is working with gravity, some subtle points appear, because geometric characteristics of the space-time become dynamical variables. This is most obvious in the Kaluza - Mandel - Klein - Fock reduction (KMKF reduction that usually but unjustly is called KK reduction), in which the metric coefficients become physical fields. This may look less clear in further reductions using cylindrical or spherical symmetries. Then the effective space-time becomes (1+1)-dimensional and some higher-dimensional metric coefficients become dynamical fields, which mix with the original matter fields produced by reductions from higher dimensions.

Low-dimensional models can be obtained by different chains of dimensional reductions from higher-dimensional supergravity or gravity theories (see, e.g., [1] - [5]). For instance, we may consider toroidal compactifications and KMKF reductions from eleven-dimensional theory to a four-dimensional gravity coupled to Abelian gauge fields and scalar fields. Using spherical or cylindrical symmetry we can further reduce it to a one-dimensional dilaton gravity coupled to scalar matter fields produced by the reductions. Schematically, such a chain looks like

$$(1+10) \rightarrow (1+D) \rightarrow (1+3) \rightarrow (1+1)\,(\text{spherical or cylindrical}).$$

The two-dimensional theories describe inhomogeneous cosmologies, evolution of black holes, and various types of waves (spherical, cylindrical, and plane waves). Their further reductions give both the standard (or generalized) cosmological models and static states (in particular, static black holes):

$$(1+1) \to (1+0)\,(\text{cosmological}) \;\; \text{or} \;\; (1+1) \to (0+1)\,(\text{static})$$

It is also useful to keep in mind static chains:

$$(1+3) \to (0+3)\,(\text{general static}) \; \to (0+2)\,(\text{axial}) \to (0+1)$$

We omit here some other reductions, like the general axial reduction

$$(1+3) \to (1+2)\,(\text{axial}) \; \to (1+1)\,(\text{spherical} \;\; \text{or} \;\; \text{cylindrical}).$$

Note also that it is not necessary to use step-by-step reductions. For instance, the (1+0)-dimensional homogeneous isotropic cosmologies and (0+1)-dimensional static black holes are usually derived by direct symmetry reductions from higher dimensions.

This is quite legitimate, if you are not interested in relations between these reductions and are not trying to immerse them in a more general formulation allowing for their dynamical treatment. In addition, when you have many matter fields, first considering the (1+1)-dimensional dilaton gravity allows us to obtain other interesting solutions, through which the static states, cosmologies, and waves may be interrelated (about relations between various types of solutions see [5]-[7]). Not less important is the fact that lower-dimensional dilaton gravity theories may be regarded as Lagrangian or Hamiltonian systems that are often integrable (in some sense) and thus we may hope to study them in detail and even quantize in spit of the fact that the general quantum solutions of the higher-dimensional theories can not be constructed. If we make the reductions with the due care, we may possibly find important information about solutions of higher-dimensional theories. To succeed in this one should follow a few important rules which must be used in the process of dimensional reductions.

First, one should not make 'excessive' gauge fixings before writing all the equations of motion. For example, the number of independent fields in the reduced theory must be not less than the number of the independent Einstein equations for the Ricci tensor plus the number of the equations for the matter fields. Otherwise, some solutions of the reduced theory will not satisfy (and often do not satisfy) the higher-dimensional equations of motion. Second, by analogy with the usual ('naive') reduction, it may be tempting to make all the fields to depend only on one variable (space or time, if we consider reducing (1+1)-dimensional dilaton gravity). By doing so one can loose some solutions that can be restored with the aid of more general dimensional reductions (e.g., by separating variables). We would like to also emphasize that the concept of dimensional reductions should be understood in a broader sense. An example of a more general dimensional reduction is given in [6], [7]: the solutions of a (1+1)-dimensional integrable model depend on arbitrary functions of one variable; if these functions reduce to constants, we obtain essentially one-dimensional theory. This reduction may be called a 'dynamical dimensional reduction' or a 'moduli space reduction'. In this example, a class of reduced solutions of the two-dimensional theory consists of those that essentially depend on two space-time variables (say, t and r) which nevertheless should be regarded as 'one-dimensional solutions' in a well defined but somewhat unusual sense. Unfortunately, at the moment we can introduce this new dimensional reduction only for explicitly integrable dilaton gravity theories.

To avoid misunderstanding, let us formulate the practical 'philosophy' behind our approach to dimensional reduction of gravity. As distinct from the common tendency to concentrate on geometric and symmetry properties, we follow the Arnowitt - Deser - Misner approach to treatment of gravity by using Lagrangian and Hamiltonian dynamics with constraints. Thus, the 'geometric' variables are treated on the same footing with other dynamical variables and the aim is not only to derive the metric and other geometric properties of the space-time but to construct Lagrangians and Hamiltonians and to solve the dynamical equations which, eventually, should be quantized. To quantize such a complex nonlinear theory as gravity one should first find some simple explicitly integrable approximation, like the oscillator approximation in the standard QFT. Natural candidates for such 'gravitational oscillators' may be black holes, cosmological models and some simple gravitational waves. One may argue that all these objects are somehow related to the Liouville equation rather than to the oscillator equation.

Although one should not expect that such simplified models can give completely realistic description of gravity, cosmology or gravitational waves, they may serve as a tool for developing a new intuition, which is so needed for understanding new data on the structure of our Universe. They can also give reasonable first approximations for constructing more realistic solutions as well as some hints of how the three main gravitational objects are related physically (at the moment we find only mathematical relations). Using explicitly integrable models one can clearly see a duality between black holes and cosmologies as well as observe that they both are limiting cases of certain gravitational waves. The duality can also be seen in nonintegrable models (e.g., when we use a separation of variables), while the 'triality' including some gravitational waves was up to now observed only in integrable theories.

2 1+1 Dimensional Dilaton Gravity

It is well known that there exist (1+1)-dimensional dilaton gravity theories coupled to scalar matter fields, which are reliable models for some aspects of high-dimensional black holes, cosmological models, and branes. The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory and in some cases allowed one to find general solution or some special classes of solutions in high-dimensional theories. Here, we only discuss reductions of the four-dimensional gravity theory coupled to scalar fields. In fact, after reducing to the dimension (1+1) all the matter fields are essentially equivalent to the scalar ones.

For example, spherically symmetric gravity coupled to Abelian gauge fields and massless scalar matter fields exactly reduces to a (1+1)-dimensional dilaton gravity coupled to scalar fields and can be explicitly solved if the scalar fields are constants independent of coordinates. Such solutions may describe some interesting physical objects – spherical static black holes, simplest cosmologies, etc. However, when the scalar matter fields, which presumably play a significant cosmological role, are not constant, few exact analytical solutions of high-dimensional theories are known. Correspondingly, the generic two-dimensional models of dilaton gravity nontrivially coupled to scalar matter are usually not integrable.

Some other important four-dimensional space-times, having symmetries defined by two commuting Killing vectors, may also be described by two-dimensional dilaton gravity. For example, the simplest (Einstein - Rosen) cylindrical gravitational waves are described by a (1+1)-dimensional dilaton gravity coupled to one scalar field. The simplest stationary axially symmetric pure gravity may be described by a (0+2)-dimensional dilaton gravity coupled to one scalar field (this may be related to the previous cylindrical case by the analytic continuation of one space variable to imaginary values). Similar but more general dilaton gravity models were also obtained in string theory. Some of them may be solved by using modern mathematical methods developed in the soliton theory (see, e.g., [9] - [12]).

After briefly reminding a fairly general formulation of the (1+1)-dimensional dilaton gravity, we introduce the most general cylindrical reductions that, most probably, are not integrable but are usually reduced to rather rich integrable models. The spherical reductions are generally not integrable but, after further reductions to the dimensions (1+0) and (0+1), they generate interesting (integrable and not integrable) static and cosmological models related by a duality relation. We only briefly outline the main features of our approach leaving presenting the details to future publications.

The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields ψ_n, which can be obtained by-dimensional reductions of a higher-dimensional spherically symmetric (super)gravity, may usually be (locally) transformed to the following form:

$$L = \sqrt{-g}\,[U(\varphi)R(g) + V(\varphi,\psi) + W(\varphi)(\nabla\varphi)^2 + \sum_n Z_{nm}\nabla\psi_n\nabla\psi_m]. \tag{1}$$

Here $g_{ij}(x^0,x^1)$ is a generic (1+1)-dimensional metric with signature (-1,1), $g \equiv \det|g_{ij}|$ and $R \equiv R(g)$ is the Ricci curvature of the two-dimensional space-time,

$$ds^2 = g_{ij}\,dx^i\,dx^j\,, \quad (i,j = 0,1). \tag{2}$$

The effective potentials V and Z_{nm} depend on the dilaton $\varphi(x^0,x^1)$ and on $(N-2)$ scalar fields $\psi_n(x^0,x^1)$ [1]. They may depend on other parameters characterizing the parent higher-dimensional theory, e.g. on charges introduced by solving the equations for the Abelian gauge fields, etc. There are two important simple cases: 1. $Z_{nm}(\varphi,\psi) = \delta_{nm}Z_n(\varphi)$, and 2. constant Z_n, independent of the fields. The dilaton function $U(\varphi)$ is usually monotonic and one can put (at least locally) $U(\varphi) = \varphi$ or $U(\varphi) = \exp(-2\varphi)$, etc. We also may use in Eq. (1) a Weyl transformation to exclude the gradient term for the dilaton, i.e. to make $W \equiv 0$. Under the transformations to this frame (we may call it the Weyl frame) the metric and the potential transform as

$$g_{ij} \to \tilde{g}_{ij} \equiv w(\varphi)g_{ij}\,, \;\; V \to \tilde{V} \equiv V/w(\varphi)\,, \;\; Z \to \tilde{Z} \equiv Z, \tag{3}$$

where $w(\varphi)$ is defined by the equation $w'(\varphi)/w(\varphi) = W(\varphi)/U'(\varphi)$.

As we mentioned above, in two-dimensional space-times all matter fields can eventually be reduced to different scalar fields although, for keeping traces of different symmetries, it may be convenient to retain gauge fields, spinor fields, etc. The Lagrangian Eq. (1) should be considered as an effective Lagrangian. In general, it is equivalent to the original

[1]The potentials Z_{nm} define a negative definite quadratic form.

one on the 'mass shell' but the solutions of the original equations may be completely recovered to construct the solutions of the higher-dimensional 'parent' theory. For a detailed motivation and specific examples see [5], where references to other related papers can be found. Because of the space limitations, only absolutely necessary references are given here.

To simplify derivations we will use the equations of motion in the light-cone metric, $ds^2 = -4f(u,v)\,du\,dv$ and with $U(\phi) \equiv \varphi$, $Z_{nm} = \delta_{nm} Z_n$, $W \equiv 0$. By first varying the Lagrangian in generic coordinates and then going to the light-cone ones we obtain the equations of motion

$$\partial_u \partial_v \varphi + f\,V(\varphi, \psi) = 0, \tag{4}$$

$$f \partial_i (\partial_i \varphi / f) = \sum Z_n (\partial_i \psi_n)^2 , \qquad (i = u, v) \; . \tag{5}$$

$$\partial_v (Z_n \partial_u \psi_n) + \partial_u (Z_n \partial_v \psi_n) + f V_{\psi_n}(\varphi, \psi) = \sum_m Z_{m,\psi_n} \, \partial_u \psi_m \, \partial_v \psi_m \; , \tag{6}$$

$$\partial_u \partial_v \ln |f| + f V_\varphi(\varphi, \psi) = \sum Z_{n,\varphi} \, \partial_u \psi_n \, \partial_v \psi_n \; , \tag{7}$$

where $V_\varphi = \partial_\varphi V$, $V_{\psi_n} = \partial_{\psi_n} V$, $Z_{n,\varphi} = \partial_\varphi Z_n$, and $Z_{m,\psi_n} = \partial_{\psi_n} Z_m$. These equations are not independent. Actually, (7) follows from (4) $-$ (6). Alternatively, if (4), (5), (7) are satisfied, one of the equations (6) is also satisfied.

If the Lagrangian (1) was obtained by a consistent reduction of some high-dimensional theory (i.e. not using gauge fixings, which reduce the number of independent equations, and not applying non-invertible transformations to the coordinates or unknown functions), the solutions of these equations can be reinterpreted as special solutions of the parent higher-dimensional equations.

If the scalar fields are constant, $\psi = \psi_0$, these equations can be solved with practically arbitrary potential V that should satisfy only one condition: $V_\psi(\varphi, \psi_0) = 0$, see Eq.(6). The constraints (5) then can be solved because their right-hand sides are identically zero. It is a simple exercise to prove that there exist chiral fields $a(u)$ and $b(v)$ such that $\varphi(u,v) \equiv \varphi(\tau)$ and $f(u,v) \equiv \varphi'(\tau)\,a'(u)\,b'(v)$, where $\tau \equiv a(u) + b(v)$ (the primes denote derivatives with respect to the corresponding argument). Using this result it is easy to prove that (4) has the integral $\varphi' + N(\varphi) = M$, where $N(\varphi)$ is defined by the equation $N'(\varphi) = V(\varphi, \psi_0)$ and M is the constant (integral) of motion. The horizon, defined as a zero of the metric $h(\tau) \equiv M - N(\varphi)$, exists because the equation $M = N(\varphi)$ has at least one solution in some interval of values of M. These solutions are actually one-dimensional ('automatically' dimensionally reduced) and can be interpreted as black holes (Schwarzschild, Reissner Nordstrøm, etc.) or as cosmological models. These facts are known for a long time and were derived by many authors using different approaches (for references see e.g the recent review [8]).

With this example in mind, it looks , at first sight, natural to introduce the following reduction to one-dimensional theory: let φ and ψ depend only on $\tau \equiv a(u) + b(v)$, where τ may be interpreted either as the space or the time variable. Then we obtain both the (0+1)-dimensional theory of static distributions of the scalar matter (including black holes) and (1+0)-dimensional cosmological models. However, analyzing their solutions (see simple examples in [3]) one can find that not all standard Friedmann cosmologies may be obtained in this way [5], [6]. In view of the symmetry ('duality') between the (1+0) and (0+1)-dimensional reductions one may conclude that not all static solutions are obtained in this way. In other words, this simple (naive) procedure of dimensional reduction is not complete! The same conclusion can be made if we use the space-time variables (t, r). Before discussing this phenomenon, we consider another simple source of incompleteness in the process of reductions.

3 Generalized Cylindrical Reductions

The last remark in the previous section signals that we should apply more care when using dimensional reductions in gravity. To illustrate how more general reductions may emerge we first discuss cylindrically symmetric reductions in the (1+3)-dimensional pure gravity. For acquiring a feeling of connections between the two-dimensional Lagrangian (1) and higher-dimensional theories let us consider the four-dimensional cylindrically symmetric gravity coupled to one scalar field:

$$S_4 = \int d^4x \, \sqrt{-g_4} \, [R_4 + V_4(\psi) + Z_4(\psi)(\nabla\psi)^2]. \tag{8}$$

Here the most general cylindrically symmetric metric should be used. It can be derived by applying the general KMKF reduction. The corresponding metric may be written as $(i, j = 0, 3,\ m, n = 1, 2$

$$ds_4^2 = (g_{ij} + h_{mn} A_i^m A_j^n) dx^i dx^j + 2A_{im} dx^i dy^m + h_{mn} dy^m dy^n \,, \tag{9}$$

where all the metric coefficients depend only on the x-coordinates (t, r) and y^m (z, ϕ) are some coordinates on the two-dimensional cylinder (torus).

Usually, in the four-dimensional reduction the coordinate functions A_i^m are supposed to vanish [13], but we will see in a moment that this drastically changes the resulting two-dimensional dilaton gravity theory. To see this we also suppose that ψ depends only on x and integrate out of Eq.(9) the dependence on y. Extracting the dilaton from the cylinder metric by writing

$$h_{mn} \equiv \varphi \sigma_{mn} \,, \quad \det(\sigma_{mn}) = 1, \tag{10}$$

and neglecting an inessential numeric factor, we find the two-dimensional Lagrangian (in what follows we will omit the V_4 and Z_4 terms):

$$L = \sqrt{-g} \, \{\varphi[R(g) + V_4 + Z_4(\nabla\psi)^2] + \frac{1}{2\varphi}(\nabla\varphi)^2 - \frac{\varphi}{4}\mathrm{tr}(\nabla\sigma\sigma^{-1}\nabla\sigma\sigma^{-1}) - \frac{\varphi^2}{4}\sigma_{mn} F_{ij}^m F^{nij}\} \,, \tag{11}$$

where $F_{ij}^m \equiv \partial_i A_j^m - \partial_j A_i^n$ $(i, j = 0, 3)$. These Abelian gauge fields are not propagating and their contribution is usually neglected. We propose to take them into account by solving their equations of motion and writing the corresponding effective potential. Let us first introduce a very convenient parametrization of the matrix σ_{mn}:

$$\sigma_{22} = e^{\eta} \cosh\xi, \quad \sigma_{33} = e^{-\eta} \cosh\xi, \quad \sigma_{23} = \sigma_{32} = \sinh\xi. \tag{12}$$

After simple derivations (see, e.g., [3], [5]) we exclude the gauge fields and find the effective potential

$$V_{\text{eff}} = -\frac{1}{2\varphi^2} \sum_{mn} Q_m (\sigma^{-1})_{mn} Q_n = -\frac{\cosh\xi}{2\varphi^2} [Q_1^2 e^{-\eta} - 2Q_1 Q_2 \tanh\xi + Q_2^2 e^{\eta}], \tag{13}$$

where Q_m are arbitrary constants having pure geometric origin, although they look like charges of the Abelian gauge fields F_{ij}^m. Expressing the trace in the Lagrangian (11) in terms of the variables ξ and η, we derive the Lagrangian in our standard form (1):

$$L = \sqrt{-g}\,\{\varphi R(g) + \frac{1}{2\varphi}(\nabla\varphi)^2 - V_{\rm eff} - \frac{\varphi}{2}[\cosh^2\xi\,(\nabla\eta)^2 + (\nabla\xi)^2]\}. \qquad (14)$$

This representation is convenient for writing the equations of motion (5)-(6), for further reductions to dimensions $(1+0)$ and $(0+1)$ and for analyzing special cases (such as $Q_1Q_2 = 0$, $\xi\eta = 0$). This form is also closer to the original Einstein and Rosen equations, which can be obtain by putting $Q_1 = Q_2 = 0$ and $\eta = 0$), and it is also more convenient for analyzing the physical meaning of the solutions.

The equations of motion (6) for the Lagrangian (13) are

$$2\phi\,\partial_u\partial_v\xi + [\partial_u\phi\,\partial_v\xi + (\partial_u \Leftrightarrow \partial_v)] - 2f\,\partial_\xi V_{\rm eff} - \phi\sinh 2\xi\,\partial_u\eta\,\partial_v\eta = 0, \qquad (15)$$

$$2\phi\,\partial_u\partial_v\eta + [\partial_u\phi\,\partial_v\eta + 2\phi\tanh\xi\,\partial_u\xi\,\partial_v\eta + (\partial_u \Leftrightarrow \partial_v)] - 2f\cosh^{-2}\xi\,\partial_\eta V_{\rm eff} = 0. \qquad (16)$$

If $\partial_\xi V_{\rm eff} = 0$ and $\partial_\eta V_{\rm eff} = 0$, these equations have solutions with constant η and ξ ('scalar vacuum'). However, for $Q_1Q_2 \neq 0$ we find that the solution of these two equation is

$$\exp 2\eta = Q_1^2/Q_2^2, \quad \tanh\xi = \rm{sgn}(Q_1Q_2), \quad \xi = \pm\infty.$$

If $Q_1Q_2 = 0$, there exists the solution of the first equation, $\xi \equiv 0$, while $\partial_\eta V_{\rm eff} \neq 0$. Both ξ and η can be constant if and only if $Q_1 = Q_2 = 0$.

When the potential $V_{\rm eff}$ identically vanishes, Eqs.(15, 16) as well as Eq.(4) drastically simplify and we get the Einstein-Rosen equations if $\xi \equiv 0$. Otherwise we have a nontrivially integrable system of nonlinear equations belonging to the type considered in [9] - [12]. With nonvanishing Q_1 and/or Q_2, even the further reduced (one-dimensional) equations are nontrivial and it is not quite clear whether they are integrable or not.

In summary of this section we stress once more that the two-dimensional theories (11) (and closely related static axial reductions) with vanishing gauge fields were extensively used in cosmology (see, e.g. [14], [15]) and they are integrable with the aid of modern mathematical technique (see, e.g., [12]). However, the effective potential of the geometric gauge fields most probably destroys the integrability, even if we further reduce the theory to one dimension. Nevertheless, emerging of the potential (13) that under certain circumstances can imitate effects of cosmological constant potential may be of significant interest for the present day cosmology.

4 Reducing by separating

The spherical reduction apparently does not allow appearance of the gauge fields[2] described in the previous section. Correspondingly, the general spherically symmetric metric can be written in a simpler form:

$$ds_4^2 = e^{2\alpha}dr^2 + e^{2\beta}d\Omega^2(\theta,\phi) - e^{2\gamma}dt^2 + 2e^{2\delta}dr dt\,, \qquad (17)$$

[2]A very careful discussion of the spherically symmetric space-times and of more general space-times, having subspaces of maximal symmetry, may be found in [16] (see also [17]).

where $\alpha, \beta, \gamma, \delta$ depend on (t, r) and $d\Omega^2(\theta, \phi)$ is the metric on the 2-dimensional sphere $S^{(2)}$. Substituting this into the action (8) and integrating over the variables θ, ϕ we find the reduced action[3] with the Lagrangian Eq. (1), where

$$U \Rightarrow e^{2\beta}, \;\; V \Rightarrow 2 + e^{2\beta} V_4, \;\; W(\nabla\phi)^2 \Rightarrow 2e^{2\beta}(\nabla\beta)^2, \;\; Z_{mn} \Rightarrow Z_4(\psi)e^{2\beta}, \tag{18}$$

and the 2-dimensional metric is given by $e^{2\alpha}, e^{2\gamma}, e^{2\delta}$ (see e.g. [5]). Actually, the effective two-dimensional Lagrangian also contains total derivatives that may be important in some problems but we will not discuss them here.

The equations of motion for this effective action can easily be derived and they coincide with Eqs. (4) - (7) if we pass to the light cone coordinates. It is not difficult to see (in fact, it is almost evident) that these EOM are identical to the Einstein equations (see e.g. [1]) To simplify the equations we write them in the limit of the diagonal metric (formally, one may take the limit $\delta \to -\infty$). Varying the action in α, β, γ and neglecting the δ - terms we obtain the Einstein equations for the diagonal components of the Einstein tensor. Varying the action in ψ we find the equation for ψ. Finally, by varying in δ we find one more equation corresponding to the non diagonal component of the Einstein tensor; it is not a consequence of other equations and is a combination of the two constraints (5).

The simplest way to write all necessary equations is to write the 2-dimensional effective action in the coordinates (17). First making variations in δ we find (in the limit $\delta \to -\infty$) the constraint

$$4\dot{\beta}' - 2\dot{\beta}\beta' - 2\dot{\beta}\gamma' - 2\dot{\alpha}\beta' \;=\; Z_4 \dot{\psi}\psi', \tag{19}$$

where $\dot{\psi} \equiv \partial_t \psi$, $\psi' \equiv \partial_r \psi$, etc. The other equations can be derived (in the diagonal limit) from the effective Lagrangian

$$L_{\rm eff} \;=\; V_{\rm eff} + L_{\rm t} + L_{\rm r}, \tag{20}$$

where we omitted the δ-dependence and total derivative terms. The sum of the 'r-Lagrangian'

$$L_{\rm r} \;=\; e^{-\alpha+2\beta+\gamma}(2\beta'^2 + 2\beta'\gamma' + Z_4\psi'^2), \tag{21}$$

with the 't-Lagrangian'

$$L_{\rm t} \;=\; -e^{\alpha+2\beta-\gamma}(2\dot{\beta}^2 + 2\dot{\beta}\dot{\gamma} + Z_4\dot{\psi}^2), \tag{22}$$

as well as the constraint (19) are invariant under the substitution $\partial_r \Leftrightarrow i\partial_t$ and $\alpha \Leftrightarrow \gamma$. This means that the EOM are invariant under this transformation, as the effective potential[4],

$$V_{\rm eff} \;=\; V_4\, e^{\alpha+2\beta+\gamma} + 2k\, e^{\alpha+\gamma}, \;\;\; k = 0, \pm 1, \tag{23}$$

is naturally invariant. At first sight, this invariance may look trivial but one should recall that in higher dimensions there is no complete symmetry between space and time. Thus the simple relation between static and cosmological solutions suggested by this symmetry may give some new insight into both classes of objects. Even apart from any

[3] This derivation can easily be generalized to any dimension and any number of the scalar fields with more complex coupling potentials. One may similarly treat the pseudospherical and flat symmetries as well as any symmetry given by two Killing vectors. Here $e^{2\beta}$ is the spherical dilaton denoted in the cylindrical case above by φ.

[4] Here, in addition to the case of the spherical symmetry ($k = 1$) we include the cases of pseudospherical ($k = -1$) and flat ($k = 0$) symmetries.

physical interpretation, this symmetry allows us to economize writing equations and it is extremely useful in considering separation of variables outlined below. In particular, these transformations allow one to derive cosmological solutions corresponding to static (black hole) solutions and vice versa. Although this is a special case of the formulated duality relation we call it 'static - cosmological' (SC) duality.

To illustrate how the separation of the variables looks like we write the three remaining equations (in addition to Eq. (18)):

$$[2e^{-2\alpha}(\beta'' + 2\beta'^2 - \beta'\alpha' + \beta'\gamma')] - [\alpha \Leftrightarrow \gamma,\ \partial_r \Rightarrow \partial_t] \;=\; V_{\rm eff}\, e^{-\alpha-2\beta-\gamma}\,, \tag{24}$$

$$[2e^{-2\alpha}(\beta'' + \beta'^2 - \beta'\alpha' - \beta'\gamma')] + [\alpha \Leftrightarrow \gamma,\ \partial_r \Rightarrow \partial_t] \;=\; \frac{1}{2}Z_4(E_+ + E_-)\,, \tag{25}$$

where we denote

$$E_\pm \;\equiv\; e^{-2\alpha}\psi'^2 \;\pm\; e^{-2\gamma}\dot\psi^2. \tag{26}$$

The third equation has a similar structure

$$[e^{-2\alpha}(\gamma'' + \gamma'^2 - \gamma'\alpha' - \beta'^2] - [\alpha \Leftrightarrow \gamma,\ \partial_r \Rightarrow \partial_t] \;=\; -ke^{-2\beta} + \frac{1}{2}Z_4E_-\,, \tag{27}$$

In practical procedures of the separation it may be convenient to also use the forth (dependent) equation and work with some other linear combinations of all equations.

To make a separation of the space and the time variables possible we should try to write all the equations in the form

$$\sum_{n=1}^{N} T_n(t)R_n(r) = 0, \tag{28}$$

where T_n depends only on functions (and their derivatives) of the time variable, while R_n depends only on space functions. Then, dividing by one of the functions and differentiating w.r.t. t or r, we finally find equations for functions of one variable depending on constants, which functionally depend on functions of the other variable. For $N = 2$ this is obvious: $T_1/T_2 = R_1/R_2 = C$. For $N = 3$ we may write, for instance,

$$(T_1/T_3)(R_1/R_3) + (T_2/T_3)(R_2/R_3) + 1 = 0$$

and then differentiate this equation w.r.t. t or r, thus reducing the equation to the $N = 2$ case with new arbitrary constants appearing due to differentiations.

It is evident that to write the equations in the form (28) we should make some Ansatzes allowing us to write all the terms as products of functions of one variable. It is clear that to separate the variables r and t in the metric we should require that

$$\alpha = \alpha_0(t) + \alpha_1(r),\;\; \beta = \beta_0(t) + \beta_1(r),\;\; \gamma = \gamma_0(t) + \gamma_1(r), \tag{29}$$

Then, the potentials V_4 and Z_4 must be either constant or have the necessary multiplicative form. Depending on the analytic form of the potentials, this is possible if

$$\psi = \psi_0(t) + \psi_1(r), \;\; \text{(a)} \;\; \text{or} \;\; \psi = \psi_0(t)\psi_1(r), \;\; \text{(b)}\,. \tag{30}$$

Here we will not try to find and classify all possible cases of separation and mention only typical ones. If $\dot\psi\psi' = 0$, a separation is possible for generic potentials. If $\dot\psi\psi' \neq 0$, there are three obvious classes of the potentials that are favorable to the separation: 1. constant potentials V_4 and Z_4; 2. exponential V_4 and constant Z_4 (with the ansatz (30a); 3. power dependence of V_4 on ψ and constant Z_4 (with the ansatz (30b). Note that the case of the constant Z_4 and exponential V_4 is often met in standard reductions from higher dimensions.

Inserting ansatzes (29), (30) into the equations (24), (25), (27) one may find conditions for the separation (when all the equations can be rewritten in the form of Eq.(28)). We regard possible separation as dimensional reductions in the sense that we try to clarify by the following example. Let us consider the case $\dot{\psi}\psi' = 0$. Then it is clear that the constraint (19) is rather restrictive. In fact, one can see that it allows the separation (29) in the following cases:

$$1.\ \dot{\alpha_0} = \dot{\beta_0} = 0\,; \quad 2.\ \dot{\beta_0} = \beta_1{}' = 0\,; \quad 3.\ \beta_1{}' = \gamma_1{}' = 0\,; \tag{31}$$

$$4.\ \dot{\alpha_0} = 0\,, \quad \beta_1{}' + \gamma_1{}' = 0; \quad 5.\ \gamma_1{}' = 0\,, \quad \dot{\alpha_0} + \dot{\beta_0} = 0\,. \tag{32}$$

When no one of this conditions is satisfied, the separation is possible if

$$6.\ \ 1 + \gamma_1{}'/\beta_1{}' + \dot{\alpha_0}/\dot{\beta_0} = 0\,. \tag{33}$$

One can similarly treat the case $\dot{\psi}\psi' \neq 0$. Other equations give further restrictions on the metric coefficient but they may be of somewhat different strength. Some of the restrictions may be so strong that they almost fix some parameters.

To make this point clearer suppose that we have chosen the third condition of (31) that says that β_1 and γ_1 are some constants. Then the reduction idea is to consider α_0, β_0, γ_0, ψ_0 as dynamical functions of cosmological model, while strongly restricted functions α_1, β_1, γ_1, (ψ_1 is a constant) define parameters of the background metric. Let us illustrate this idea by considering Eq. (24) with constant V_4

$$-ke^{-2\beta_1+2\alpha_0-2\beta_0} - e^{-2\gamma_1+2\alpha_0-2\gamma_0}(\ddot{\beta_0} + ...) + e^{-2\alpha_1}(\beta_1'' + ...) = \frac{1}{2}V_4 e^{2\alpha_0}, \tag{34}$$

where we omitted some terms in brackets that may be recovered using Eq. (24). As β_1, γ_1 are constants, all terms in this equation depend on t only. It is not difficult to similarly treat two remaining equations and thus to find the complete system of equations for α_0, β_0, γ_0, ψ_0. Although α_1 remains an arbitrary function of r it can be transformed to any constant by an appropriate r-transformations. Of course one may see that we recovered the 'naive' cosmological reduction. The 'naive' static reduction may be reproduced simply by using our SC-duality. With less restrictive ansatzes we may construct other cosmological models and static configurations that are dual to them. It is interesting that there may exist some 'intermediate' cases that are more symmetric under the duality transformation. It would be premature to call them 'self - dual' before a detailed study of them will be completed.

Here we considered the separating of variables approach for the spherically reduced gravity. With due care, it can be applied to the generalized cylindrical theory (14).

5 Conclusion

In a separate publication I will present the list of all these reductions and their relation to black holes, cosmologies, and waves (especially, the cylindrical ones). Here I only mention the wave - like solution obtained in the integrable (1+1)-dimensional gravity theory coupled to $N-2$ scalar matter fields [5], [6]. The general solution of the model depends on the chiral moduli fields $\xi_n(u)$, $\eta_n(v)$ that move on the surfaces of the spheres $S^{(N-2)}$. The naive reduction to one-dimensional theories emerges when the moduli fields are constant and equal, $\xi_n = \eta_n$. When they are constant but otherwise arbitrary, we have a new class of reduced solutions that correspond to waves of scalar matter coupled to gravity. Under certain conditions these waves may be localized in space and time and thus may be regarded as a sort of solitary gravitational waves[5]. The very origin of these waves signals existence of a close relation between main gravitational objects - black holes, cosmologies and waves. This relation was studied in some detail for static states and cosmologies that may be considered as a sort of 'static – cosmological' duality (SC-duality). In the integrable models the transitions between static and cosmological states are possible and, moreover, the waves play a significant role in these transitions. This observation, which does not actually require integrability, may open a way to studies of real physical connections between these apparently diverse objects.

In summary, one may identify at least three types of dimensional reduction: the 'standard' or 'naive' reduction supposes that functions of two variables depend on one variable only, the reduction by separating the variables, and the reduction in moduli spaces supposing that the moduli functions become constants. In all cases the important problem is to find the Lagrangians and Hamiltonians for the reduced systems. This is not difficult for the naive reduction and for the simple reductions based on separating the variables. It is not clear how to do this with the last, so to speak, 'moduli reduction'. In addition, it is not clear how to do such a reduction for not integrable systems.

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Integrable string models of hydrodynamical type in terms of chiral currents

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Abstract

String models of the hydrodynamic type are constructed in the bi-Hamiltonian approach to integrable systems by using local and non-local Poisson brackets of hydrodynamic type. The chiral currents of string are the local coordinates of the Riemann space. New local chiral currents was constructed from the initial chiral currents. In the case of null torsion, new nonlocal tensor chiral currents was constructed. New equations of motion in variables of currents was obtained.

1. Introduction

String theory is a very promising candidate for a unified quantum theory of gravity and all the other forces of nature. For quantum description of string model we must to have classical solutions of the string in the background fields. String theory in suitable space-time backgrounds can be considered as σ- model. The integrability of the classical σ-model is manifested through an infinite set of conserved charges, which can form non-abelian algebra. There two ways to construct these charges are known: such-named Backlund transformations, inverse scattering problem method. Any charge from the commuting subset of charges and any Casimir operators of charge algebra can be considered as Hamiltonian in the bi-Hamiltonian approach to integrable models. Let us consider a bi-Hamiltonian approach to the integrable models. The bi-Hamiltonian approach to the integrable systems was initiated by Magri [1] and it was generalized by Das, Okubo [2]. Two Poisson brackets (PBs)

$$\begin{aligned}\{u^a(\sigma), u^b(\sigma')\}_1 &= P_1^{ab}(\sigma,\sigma')(u),\\ \{u^a(\sigma), u^b(\sigma')\}_2 &= P_2^{ab}(\sigma,\sigma')(u)\end{aligned} \quad (1)$$

are called compatible if any linear combination of these PBs

$$c_1\{*,*\}_1 + c_2\{*,*\}_2$$

is PB also. The functions $u^a(\tau,\sigma)$, $a = 0, 1, ..., D-1$ are local coordinates on a certain given smooth D-dimension manifold M. The Hamiltonian operators $P_1^{ab}(\sigma,\sigma')(u), P_2^{ab}(\sigma,\sigma')(u)$

are the functions of local coordinates $u^a(\sigma)$. It is possible to construct recursion operators

$$R^a_c(\sigma,\sigma')(u) = \int\limits_0^{2\pi} P_2^{ab}(\sigma,\sigma'')P_{bc1}(\sigma'',\sigma')d\sigma'', \quad (2)$$

$$((R(\sigma,\sigma')(u))^{-1})^a_c = \int\limits_0^{2\pi} P_1^{ab}(\sigma,\sigma'')P_{2bc}(\sigma'',\sigma')d\sigma''. \quad (3)$$

Any of degrees of recursion operator is Hamiltonian operator of new PB. It is possible to find two Hamiltonians H_1 and H_2 which are satisfy to bi-Hamiltonian condition:

$$\frac{du^a(\sigma)}{d\tau} = \{u^a(\sigma), H_1\}_1 = \{u^a(\sigma), H_2\}_2. \quad (4)$$

The local PBs of hydrodynamic type was introduced by Dubrovin, Novikov [3] for Hamiltonian description of the equations of hydrodynamics. They was generalized by Ferapontov [4] and Mokhov, Ferapontov [5] on the non-local PBs of hydrodynamic type. It was shown that to construct family of integrable systems enough to consider pencil from local PB and non-local PB. Integrable systems of hydrodynamic type are described by Hamiltonians of hydrodynamic type, which are not depending of derivatives of local coordinates. Integrable bi-Hamiltonian systems of hydrodynamic type was considered by Maltsev [6], Ferapontov [7], Mokhov [9], Pavlov [10], Maltsev, Novikov [11], Dubrovin, Novikov [3].

2. Nonlocal Poisson brackets

Local PB of Dubrovin, Novikov in the flat coordinates is form:

$$\{u^a(\sigma), u^b(\sigma')\} = P_1^{ab}(\sigma,\sigma')(u) = \eta^{ab}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'), \quad (5)$$

where η^{ab} is constant metric tensor. Ferapontov [4] constructed non-local PB of following form:

$$\{u^a(\sigma), u^b(\sigma')\}_F = g^{ab}(u(\sigma))\frac{d}{d\sigma}\delta(\sigma-\sigma') - g^{ac}(u(\sigma))\Gamma^b_{ck}(u(\sigma))u'^k(\sigma)\delta(\sigma-\sigma') + \sum_{\alpha=1}^{L}(w_\alpha(u(\sigma)))^a_k u'^k(\sigma)\nu(\sigma'-\sigma)(w_\alpha(u(\sigma')))^b_l u'^l(\sigma'), \quad (6)$$

where $det(g^{ab}(u)) \neq 0$, the coefficients $g^{ab}(u)$,$\Gamma^a_{cb}(u)$,$(w_\alpha(u))^a_b$ are the smooth functions of local coordinates, $a,b,c = 0,1,...,D-1$; $\alpha = 1,2,...,L$ and $\nu(\sigma-\sigma') = sgn(\sigma-\sigma') = (\frac{d}{d\sigma})^{-1}\delta(\sigma-\sigma')$. The bracket (6) is skew-symmetric if and only if:
1. $g^{ab}(u) = g^{ba}(u)$ is symmetric contra-variant metric,
2. $\Gamma^a_{cb}(u)$ is the Riemann connection generated by metric $g^{ab}(u)$.
The bracket (6) satisfies Jacobi identity if and only if:
3. $\Gamma^a_{bc}(u) = \Gamma^a_{cb}(u)$ is symmetric connection;
metric $g^{ab}(u)$ and tensors $(w_\alpha(u))^a_b$ satisfy the Gauss-Codazzi equations for the submanifold M^D with flat normal bundle embedded in pseudo-Euclidean space E^{D+L}:
4. $g^{ab}(u)(w_\alpha(u))^c_b = g^{cb}(u)(w_\alpha(u))^a_b$,
5. $\nabla_a(w_\alpha(u))^b_c = \nabla_c(w_\alpha(u))^b_a$,
where ∇_a is the operator of covariant differentiation generated by connection $\Gamma^a_{bc}(u)$.

6. $R^{ab}_{cd}(u) = \sum_{\alpha=1}^{L}[(w_\alpha(u))^a_d(w_\alpha(u))^b_c - (w_\alpha(u))^b_d(w_\alpha(u))^a_c]$,
where $R^{ab}_{cd}(u) = g^{ak}(u)R^b_{kcd}(u)$ is the Riemann curvature tensor of metric $g^{ab}(u)$.
7. The tensors $(w_\alpha(u))^a_b$ are satisfy the Ricci equation for embedded surface:
$(w_\alpha(u))^a_b(w_\beta(u))^b_c = (w_\beta(u))^a_b(w_\alpha(u))^b_c$.
The PB (6) exactly corresponds to an D- dimensional surface with flat normal bundle embedded in a pseudo-Euclidean space E^{D+L}. The covariant tensor $g_{ab}(u)$ is the first fundamental form. The tensors $(w_\alpha(u))^a_b$ are corresponding Weingarten operators of this embedded surface and tensors $g_{ab}(w_\alpha(u))^b_c$ are the corresponding second fundamental forms.

3. Compatible pairs of Poisson brackets

As was shown by Mokhov in [9] that Ferapontov non-local PB (6) compatible with constant local Dubrovin, Novikov PB (5) under Magri condition [1] if and only if it has following form:

$$\{u^a(\sigma), u^b(\sigma')\}_2 = P^{ab}_2(\sigma,\sigma')(u) = [\eta^{ac}\frac{\partial F^b(u(\sigma))}{\partial u^c} + \eta^{bc}\frac{\partial F^a(u(\sigma))}{\partial u^c} - \eta^{ak}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^k}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^l}]\frac{d}{d\sigma}\delta(\sigma-\sigma') + [\eta^{ac}\frac{\partial^2 F^b(u(\sigma))}{\partial u^c\partial u^k} - \eta^{ac}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial^2\psi^\alpha(u(\sigma))}{\partial u^k\partial u^c}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^l}]u'^k(\sigma)\delta(\sigma-\sigma') + \eta^{ac}\eta^{bk}\sum_{\alpha=1}^{L}\frac{\partial^2\psi^\alpha(u(\sigma))}{\partial u^c\partial u^l}u'^l(\sigma)\nu(\sigma'-\sigma)\frac{\partial^2\psi^\alpha(u(\sigma))}{\partial u^k\partial u^n}u'^n(\sigma'), \tag{7}$$

where functions $F^a(u),\ \psi^\alpha(u),\ a = 0,1,...,D-1,\ \alpha = 1,2,...,L$ are the smooth functions of local coordinates. The PB (7) is PB if and only if the following equations are satisfied [9]:

$$\frac{\partial^2 F^a}{\partial u^k\partial u^n}\eta^{nl}\frac{\partial^2 F^b}{\partial u^l\partial u^c} = \frac{\partial^2 F^b}{\partial u^k\partial u^n}\eta^{nl}\frac{\partial^2 F^a}{\partial u^l\partial u^c}, \tag{8}$$

$$\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^n}\eta^{nl}\frac{\partial^2\psi^\beta}{\partial u^l\partial u^c} = \frac{\partial^2\psi^\beta}{\partial u^k\partial u^n}\eta^{nl}\frac{\partial^2\psi^\alpha}{\partial u^l\partial u^c}, \tag{9}$$

$$g^{ak}\eta^{bl}\frac{\partial^2 F^a}{\partial u^k\partial u^l} = g^{bk}\eta^{al}\frac{\partial^2 F^a}{\partial u^k\partial u^l}, \tag{10}$$

$$g^{ak}\eta^{bl}\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^l} = g^{bk}\eta^{al}\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^l}, \tag{11}$$

$$g^{ab} = \eta^{ac}\frac{\partial F^b}{\partial u^c} + \eta^{bc}\frac{\partial F^a}{\partial u^c} - \eta^{ak}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial\psi^\alpha}{\partial u^k}\frac{\partial\psi^\alpha}{\partial u^l}. \tag{12}$$

4. String model

The string model in the background constant gravity field is described by the system of equations:

$$\ddot{x}^a - x''^a = 0,\ \eta_{ab}(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0,\ \eta_{ab}\dot{x}^a x'^b = 0, \tag{13}$$

where $\dot{x}^a = \frac{dx^a}{d\tau}$, $x'^a = \frac{dx^a}{d\sigma}$, $a = 0,1,...,D-1$ and $x^a(\tau,\sigma)$ are local target space coordinates of world sheet coordinates τ, σ. The constant metric tensor η_{ab} is symmetric tensor. The closed string model is described by first kind constraints in the Hamiltonian formalism:

$$h_1 = \frac{1}{2}(\eta^{ab} p_a p_b + \eta_{ab} x^{'a} x^{'b}) \approx 0, \;\; h_2 = p_a x^{'a} \approx 0, \tag{14}$$

where $x^a(\tau, \sigma), \; p_a(\tau, \sigma)$ are the periodical functions on σ with the period on 2π. The original PBs are the canonical PBs:

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta^a_b \delta(\sigma - \sigma'), \tag{15}$$
$$\{x^a(\sigma), x^b(\sigma')\}_1 = \{p_a(\sigma), p_b(\sigma')\}_1 = 0,$$

where $\delta(\sigma - \sigma')$ is periodical function. Let us introduce the local coordinates $u^a(\sigma), \; v^a(\sigma)$:

$$u^a = \frac{1}{\sqrt{2}}(\eta^{ab} p_b + x^{'a}), \;\; v^a = \frac{1}{\sqrt{2}}(\eta^{ab} p_b - x^{'a}). \tag{16}$$

First kind constraints have following form:

$$h_1 = \frac{1}{2}\eta_{ab}(u^a u^b + v^a v^b), \;\; h_2 = \frac{1}{2}\eta_{ab}(u^a u^b - v^a v^b). \tag{17}$$

Canonical PBs are:

$$\{u^a(\sigma), u^b(\sigma')\}_1 = P^{ab}_1(\sigma, \sigma') = \eta_{ab}\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma'), \tag{18}$$
$$\{v^a(\sigma), v^b(\sigma')\}_1 = -\eta_{ab}\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma'), \;\; \{u^a(\sigma), v^b(\sigma')\}_1 = 0.$$

The Hamiltonian equations of motion under Hamiltonian $H_1 = \int\limits_0^\pi h_1(u(\sigma))d\sigma$ are described two independent left and right movers: $\dot{u}^a = u^{'a}, \; u^a(\tau, \sigma) = u^a(\tau + \sigma), \;\; \dot{v}^a = -v^{'a}, \; \dot{v}^a(\tau, \sigma) = v^a(\tau - \sigma)$. We will consider a motion of string, which is described by local coordinates $u^a(\tau, \sigma)$.

5. Integrable bi-Hamiltonian string models of hydrodynamic type

We apply the hydrodynamic approach to the integrable string models in the terms of chiral currents $u^a(\tau, \sigma)$, which are the local coordinates of the Riemann space with metric $g^{ab}(u)$. Following Mokhov [9], we consider the recursion operator generated by compatible PB (5) and PB (7):

$$R^a_b(\sigma, \sigma') = \int\limits_0^{2\pi} P^{ac}_2(\sigma.\sigma'')(u) P_{1cb}(\sigma'', \sigma')(u) d\sigma'', \tag{19}$$

where

$$P_{1ab}(\sigma, \sigma') \equiv (P^{ab}_1(\sigma, \sigma'))^{-1} = \eta_{ab}\nu(\sigma - \sigma')$$

and $\nu(\sigma - \sigma')$ is periodical function. Let us apply recursion operator (19) to the closed string model in the constant background gravity field (13). Hamiltonian equations of motion with the Hamiltonian

$$H_1 = \int\limits_0^{2\pi} \eta_{ab} u^a(\sigma) u^b(\sigma) d\sigma \tag{20}$$

are the equations of hydrodynamical type: $\dot{u}^a(\tau,\sigma) = u'^a(\tau,\sigma)$. As was shown by Mokhov [9], any system from the hierarchy

$$\frac{\partial u^a(\tau,\sigma)}{\partial \tau_n} = \int\limits_0^{2\pi}\int\limits_0^{2\pi}...\int\limits_0^{2\pi}(R(\sigma,\sigma_1)R(\sigma_1,\sigma_2)...R(\sigma_{n-1},\sigma_n))^a_b(u)\frac{\partial u^b(\tau_n,\sigma_n)}{\partial \sigma_n}d\sigma_1 d\sigma_2...d\sigma_n, \tag{21}$$

is a multi- Hamiltonian integrable system. In particular, any closed string model of the form

$$\frac{\partial u^a(\sigma)}{\partial \tau_1} = \int\limits_0^{\pi} R^a_b(\sigma,\sigma')\frac{\partial u(\sigma')}{\partial \sigma'}d\sigma' \tag{22}$$

is integrable system [9]:

$$\frac{\partial u^a(\sigma)}{\partial \tau_1} = \frac{d}{d\sigma}[F^a(u) + \eta^{ak}\eta_{bc}\frac{\partial F^b}{\partial u^k}u^c - \eta^{ak}\sum_{\alpha=1}^{L}\frac{\partial \psi^\alpha}{\partial u^k}\psi^\alpha], \tag{23}$$

where functions $F^a(u)$, $\psi^\alpha(u)$ are arbitrary solutions of the equations (8)-(12). Let us note, that there is infinite algebra of chiral currents $u^{(n)a}(\sigma)$ under non-local PB for every solution functions $F^{(n)a}(u)$, $\psi^{(n)\alpha}(u)$. The equation (23) can be obtain as Hamiltonian equation of motion with the Hamiltonian (see [9])

$$H_2 = \int\limits_0^{2\pi}[\eta_{ab}F^b(u(\sigma))u^a(\sigma) - \frac{1}{2}\sum_{\alpha=1}^{L}\psi^\alpha(u(\sigma))\psi^\alpha(u(\sigma))]d\sigma. \tag{24}$$

The PBs (5), (7) and Hamiltonians (20), (24) satisfy bi-Hamiltonian condition (4).

Let us consider the constant curvature manifold. Mokhov, Ferapontov [5] constructed non-local PB which exactly corresponds to an D-dimensional surface with constant Riemann curvature K, embedded in a pseudo-Euclidean space E^{D+1}. The Mokhov, Ferapontov non-local PB compatible with constant local Dubrovin, Novikov PB has following form [8]:

$$\begin{aligned}\{u^a(\sigma),u^b(\sigma')\}_{MF} = P^{ab}_{MF}(\sigma,\sigma')(u) = [\eta^{ac}\frac{\partial F^b(u(\sigma))}{\partial u^c} + \eta^{bc}\frac{\partial F^a(u(\sigma))}{\partial u^c} - \\ -Ku^a(\sigma)u^b(\sigma)]\frac{d}{d\sigma}\delta(\sigma-\sigma') + [\eta^{ac}\frac{\partial^2 F^b(u(\sigma))}{\partial u^c\partial u^k} - K\delta^a_c u^b(\sigma)]u'^c(\sigma)\delta(\sigma-\sigma') + \\ Ku'^a(\sigma)\nu(\sigma'-\sigma)u'^b(\sigma'),\end{aligned} \tag{25}$$

where functions $F^a(u(\sigma))$ satisfy the equations (8), (10) and metric tensor $g^{ab}(u)$ has following form:

$$g^{ab} = \eta^{ac}\frac{\partial h^b}{\partial u^c} + \eta^{bc}\frac{\partial h^a}{\partial u^c} - K\,u^a u^b. \tag{26}$$

These equations can be obtained from the equations (8)-(12) in the case of only Weingarten operator $(\omega_1(u))^a_b$

$$(\omega_1(u))^a_b = \eta^{ak}\frac{\partial^2\psi^1(u(\sigma))}{\partial u^k\partial u^b} = \sqrt{K}\delta^a_b \tag{27}$$

The recursion operator, constructed from the Hamiltonian operators $P^{ab}_{MF}(\sigma,\sigma')(u)$ and $P^{ab}_1(\sigma,\sigma')(u)$ led to a multi-Hamiltonian integrable system (21). In particular, any system of the form (22) is system of hydrodynamic type with Hamiltonian H_2 of the following form [8]:

$$H_2 = \int_0^\pi [\eta_{ab} F^b(u(\sigma))u^b(\sigma) - \frac{K}{8}\eta_{ab}u^a(\sigma)u^b(\sigma)\eta_{kl}u^k(\sigma)u^l(\sigma)]d\sigma, \tag{28}$$

where functions $F^a(u(\sigma))$ are arbitrary solutions of the equations (8), (10).

6. String models of σ-model type

Let us consider string model of σ-model type.

$$L = \frac{1}{2} g_{ab}(x(\sigma))\partial_\alpha x^a(\sigma)\partial_\alpha x^b(\sigma), \tag{29}$$

$$g_{ab}(x(\sigma))(\dot{x}^a(\sigma)\dot{x}^b(\sigma) + x'^a(\sigma)x'^b(\sigma)) = 0,\ g_{ab}(x(\sigma))\dot{x}^a(\sigma)x'^b(\sigma) = 0,$$

where $g_{ab}(x) = e^\mu_a(x)e_{\mu b}(x);\ a,b = 0,1,...,D-1;\ \alpha = 0,1$ and $\mu,\nu = 0,1,...,D-1$ are indices of the tangent space in the point x^a on the Riemann–Cartan space with metric $g_{ab}(x)$. The veilbein $e^\mu_a(x)$ and its inverse $e^a_\mu(x)$ satisfy conditions:

$$e^\mu_a(x)e^b_\mu = \delta^a_b,\ e^\mu_a(x)e^{a\nu}(x) = \eta^{\mu\nu}. \tag{30}$$

In the terms of currents

$$J^\mu_\alpha = e^\mu_a \partial_\alpha x^a,\ J_{0\mu} = e^a_\mu p_a,\ J^\mu_1 = e^\mu_a x'^a,\ x'^a = \frac{\partial x^a}{\partial \sigma}, \tag{31}$$

Hamiltonian and equations of motion are following form:

$$H = \frac{1}{2}\int_0^\pi [\eta_{\mu\nu}J^\mu_0 J^\nu_0 + \eta_{\mu\nu}J^\mu_1 J^\nu_1]d\sigma,\ \partial_\alpha J^\mu_\alpha = 0,\ \partial_0 J^\mu_1 - \partial_1 J^\mu_0 + C^\mu_{\nu\lambda}J^\nu_0 J^\lambda_1 = 0, \tag{32}$$

$$\ddot{x}^a - x''^a + \Gamma^a_{bc}(x)(\dot{x}^b\dot{x}^c - x'^b x'^c) = 0,\ \Gamma^a_{bc} = \frac{1}{2}e^a_\mu\left(\frac{\partial e^\mu_b}{\partial x^c} + \frac{\partial e^\mu_c}{\partial x^b}\right),$$

where structure functions

$$C^\lambda_{\nu\mu}(x) = \frac{\partial e^\lambda_a}{\partial x^b}(e^a_\nu e^b_\mu - e^a_\mu e^b_\nu) \tag{33}$$

is the torsion generated by metric $g^{ab}(x)$ and the veilbein $e^\mu_a(x)$ is satisfies Maurer–Cartan relation:

$$\frac{\partial e^\mu_a}{\partial x^b} - \frac{\partial e^\mu_b}{\partial x^a} = C^\mu_{\nu\lambda}(x)e^\nu_a e^\lambda_b. \tag{34}$$

The canonical commutation relations of currents are:

$$\{J_{0\mu}(x(\sigma)), J_{0\nu}(x(\sigma'))\} = C^\lambda_{\nu\mu}(x(\sigma))J_{0\lambda}(x(\sigma))\delta(\sigma - \sigma'), \tag{35}$$

$$\{J_{0\mu}(x(\sigma)), J^\nu_1(x(\sigma'))\} = C^\nu_{\mu\lambda}(x(\sigma))J^\lambda_1(x(\sigma))\delta(\sigma-\sigma') - \delta^\nu_\mu \frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma),$$

$$\{J^\mu_1(x(\sigma)), J^\nu_1(x(\sigma'))\} = 0.$$

Jacobi identity for commutation relation of currents $J_{0\mu}(x(\sigma))$ is led to following equation:

$$[D_\lambda C^\rho_{\nu\mu} + D_\nu C^\rho_{\mu\lambda} + D_\mu C^\rho_{\lambda\nu}]J^\rho_0 = 0, \tag{36}$$

where $D_\lambda C^\rho_{\nu\mu} = \partial_\lambda C^\rho_{\nu\mu} + C^\rho_{\lambda\gamma} C^\gamma_{\nu\mu}$, $\partial_\lambda = e^a_\lambda \frac{\partial}{\partial x^a}$. Under condition

$$C^\lambda_{\nu\mu} = 0, \ \frac{\partial e^\mu_a(x)}{\partial x^b} - \frac{\partial e^\mu_b(x)}{\partial x^a} = 0 \tag{37}$$

torsion and Riemann curvature are vanish. In the terms of chiral currents

$$u^\mu(x(\sigma)) = \frac{1}{\sqrt{2}}[J^\mu_0(x(\sigma)) + J^\mu_1(x(\sigma))], \ v^\mu(x(\sigma)) = \frac{1}{\sqrt{2}}[J^\mu_0(x(\sigma)) - J^\mu_1(x(\sigma))] \tag{38}$$

commutation relations are the same as commutation relations of string model of hydrodynamic type with flat metric (18). After this, we repeat procedure of compatible pairs of local Dubrovin, Novikov and non-local PBs to obtain string σ-model of hydrodynamic type. New Hamiltonians are Hamiltonians with high degrees of 1-differential forms compare with initial string model of σ-model type.

There is a problem, how are consistency conditions (9)-(13) on functions $F^a(u)$, $\psi^\alpha(u)$ solve? In the case of compatible pairs of local PBs: $\psi^\alpha(u) = 0$ or $K = 0$, Dubrovin [12] give a great number of solutions of consistency conditions for $F^a(u) = \frac{\partial F(u)}{\partial u^a}$. In this case consistency conditions reduced to WDVV [14], [15] equations of associativity. As was shown by Dubrovin [16], [12], the local coordinates u^a must to belong to Lie group space (as particular case of more general Frobenius manifold) to solve of WDVV consistency conditions. Nonlinear integrable equations of motion for local coordinates on the group space can describe by two-dimensional σ-models.

7. Local and non-local currents of 2d relativistic models

Equations of motion of hydrodynamic type for integrable models are continuity equations of chiral currents. Principal problem is: how are new chiral currents to construct from initial chiral currents and how are to construct invariant operators from the new chiral currents? The hydrodynamical approach based on the hierarchy of PBs of hydrodynamical type and it leds to consistency conditions (8)-(12). Another way based on hierarchy of Hamiltonians of hydrodynamical type (as function of right (left) chiral currents and on the hidden symmetry of σ-models (Polmeyer [17], Luscher [18], Volkov, Gershun, Tkach [19], Evans, Hassan, MacKay, Mountain [20]), of $AdS_5 \otimes S^5$ string models (Bena, Polchinski, Roiban [21], Hatsuda [22]), of flat string model (Polmeyer [23], Meusburger, Rehren [24], Thiemann [25]).

Let us consider principal chiral model Lagrangian:

$$L = \frac{1}{2} g_{ab}(\phi) \partial_\alpha \phi^a(t,x) \partial_\alpha \phi^b(t,x), \ a,b = 1,2,...,D-2, \ \alpha = 0,1 \tag{39}$$

and $\phi^a(t,x)$ is local coordinates of compact group G, of factor group in the AdS group case, of Riemann–Cartan space with torsion. The metric tensor is:

$$g_{ab}(\phi) = e^\mu_a(\phi) e^\nu_b(\phi) \eta_{\mu\nu}, \ \mu,\nu = 1,2,...,D-2. \tag{40}$$

Currents, Hamiltonian and equations of motion are following:

$$J^\mu_\alpha(\phi) = e^\mu_b \partial_\alpha \phi^b, \ H = \frac{1}{2} \int\limits_0^{2\pi} (\eta_{\mu\nu} J^\mu_0 J^\nu_0 + \eta_{\mu\nu} J^\mu_1 J^\nu_1) dx, \tag{41}$$

$$\partial_\alpha J^\mu_\alpha = 0, \ \partial_\alpha J^\mu_\beta - \partial_\beta J^\mu_\alpha + c^\mu_{\nu\lambda} J^\nu_\alpha J^\lambda_\beta = 0, \tag{42}$$

where $c^{\mu}_{\nu\lambda}$ is structure constant for compact Lie group and constant torsion for Riemann–Cartan space. Constant torsion is satisfy the Jacobi identity. It is antisymmetric tensor and it is structure constant of some Lie algebra. Therefore we can introduce generators of Lie algebra T^{μ} for each of group spaces and for Riemann–Cartan space too

$$[T^{\mu}, T^{\nu}] = c^{\mu\nu}_{\lambda} T^{\lambda},\ Tr(T^{\mu}T^{\nu}) = -\delta^{\mu\nu}$$

and can use notation $J_{\alpha} = J^{\mu}_{\alpha} T_{\mu}$ for description of product of currents. In the terms of chiral currents

$$U^{\mu} = \frac{1}{\sqrt{2}}(J^{\mu}_{0} + J^{\mu}_{1}),\ V^{\mu} = \frac{1}{\sqrt{2}}(J^{\mu}_{0} - J^{\mu}_{1}) \tag{43}$$

and light-cone two-dimension variables $x^{+} = \frac{1}{\sqrt{2}}(t+x)$, $x^{-} = \frac{1}{\sqrt{2}}(t-x)$ equations of motion have following form:

$$\frac{\partial U}{\partial x^{-}} = -\frac{\partial V}{\partial x^{+}} = \frac{1}{2}[U, V], \tag{44}$$

where $U(t,x) = U^{\mu}(t,x)T_{\mu}, V(t,x) = V^{\mu}(t,x)T_{\mu}$. The canonical commutation relations of chiral currents have form:

$$\{U^{\mu}(x), U^{\nu}(y)\} = c^{\lambda\mu\nu}[\frac{3}{2\sqrt{2}}V_{\lambda}(x) - \frac{1}{2\sqrt{2}}U_{\lambda}(x)]\delta(x-y) - \eta^{\mu\nu}\frac{\partial}{\partial y}\delta(y-x), \tag{45}$$

$$\{V^{\mu}(x), V^{\nu}(y)\} = c^{\lambda\mu\nu}[\frac{3}{2\sqrt{2}}U_{\lambda}(x) - \frac{1}{2\sqrt{2}}V_{\lambda}(x)]\delta(x-y) + \eta^{\mu\nu}\frac{\partial}{\partial y}\delta(y-x), \tag{46}$$

$$\{U^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2\sqrt{2}}c^{\lambda\mu\nu}[U_{\lambda}(x) + V_{\lambda}(x)]\delta(x-y). \tag{47}$$

As result of equations (44) we can not divide equations of motion on the independent right and left movers to obtain hydrodynamic type equations in general case. Nevertheless, as was shown in [20] (and references therein), [23] principal chiral model has additional nonlinear local and nonlocal conservation laws in terms of initial chiral currents $U(t,x)$ or $V(t,x)$. As follow from equation (44) $\frac{\partial}{\partial x^{-}}TrU = \frac{\partial}{\partial x^{+}}TrV = 0$. New conservation laws can be constructed by two different ways: 1) totally symmetric tensor currents as totally symmetric functions of product of initial chiral currents and 2) production of totally symmetric, invariant tensor and product of initial chiral currents. First way was used in the description of hidden symmetry of flat string model and tensor nonlocal chiral charges are named as Pohlmeyer charges [23], [25]. Second way was used in the description of local conserved charges in principal chiral models ([20] and references therein). The local conserved currents $U^{(n)}(U(x))$, $V^{(n)}(V(x))$ have following form:

$$\begin{aligned} U^{(n)}(U(x)) &= d_{\mu_1\mu_2...\mu_n}U^{\mu_1}U^{\mu_2}...U^{\mu_n},\ \frac{\partial U^{(n)}(U)}{\partial x^{-}} = 0, \\ V^{(n)}(V(x)) &= d_{\mu_1\mu_2...\mu_n}V^{\mu_1}V^{\mu_2}...V^{\mu_n},\ \frac{\partial V^{(n)}(V)}{\partial x^{+}} = 0, \end{aligned} \tag{48}$$

where $d_{\mu_1\mu_2...\mu_n}$ is totally symmetric, invariant tensor. It can be represented by following formula:

$$d_{\mu_1\mu_2...\mu_n} = STr(T^{\mu_1}T^{\mu_2}...T^{\mu_n}), \tag{49}$$

where STr is the trace of a completely symmetric product of matrix representation of Lie algebra generators. As was shown in [20] canonical commutation relation of new local chiral currents for the $su(l)$ model has form:

$$\{U^{(n)}(x), U^{(m)}(y)\} = nmU^{(n+m-2)}(x)\frac{\partial}{\partial x}\delta(x-y) - \frac{nm}{l}U^{(n-1)}(x)U^{(m-1)}(x)\frac{\partial}{\partial x}\delta(x-y)+$$
$$\frac{nm(m-1)}{n+m-2}\frac{\partial U^{(n+m-2)}(x)}{\partial x}\delta(x-y) - \frac{nm}{l}U^{(n-1)}(x)\frac{\partial U^{(m-1)}(x)}{\partial x}\delta(x-y). \tag{50}$$

First of fifth densities of conserved charges, which can be considered as Hamiltonians of hydrodynamical type are:

$$h^{(2)}(U) = U^{(2)}(x),\ \ h^{(3)}(U) = U^{(3)}(x),\ \ h^{(4)}(U) = U^{(4)}(x) - \frac{3}{2l}U^{(2)}(x)U^{(2)}(x),$$
$$h^{(5)} = U^{(5)}(x) - \frac{10}{3l}U^{(3)}(x)U^{(2)}(x),$$
$$h^{(6)} = U^{(6)}(x) - \frac{5}{3l}U^{(3)}(x)U^{(3)}(x) - \frac{15}{4l}U^{(4)}(x)U^{(2)}(x) + \frac{25}{8l^2}U^{(2)}(x)U^{(2)}(x)U^{(2)}(x). \tag{51}$$

The canonical commutation relation of the local chiral currents for the $so(l)$ and $sp(l)$ model has form [20]:

$$\{U^{(n)}(x), U^{(m)}(y)\} = nmU^{(n+m-2)}(x)\frac{\partial}{\partial x}\delta(x-y) + \frac{nm(m-1)}{n+m-2}\frac{\partial U^{(n+m-2)}(x)}{\partial x}\delta(x-y). \tag{52}$$

The densities of local conserved charges have form:

$$h^{(2)} = U^{(2)}(x),\ \ h^{(4)} = U^{(4)}(x) - \frac{3\alpha}{2}U^{(2)}(x)U^{(2)}(x),$$
$$h^{(6)} = U^{(6)}(x) - \frac{15\alpha}{4}U^{(4)}(x)U^{(2)}(x) + \frac{(5\alpha)^2}{8}U^{(2)}(x)U^{(2)}(x)U^{(2)}(x),$$
$$h^{(8)} = U^{(8)}(x) - \frac{2(7\alpha)}{3}U^{(6)}(x)U^{(2)}(x) - \frac{7\alpha}{4}U^{(4)}(x)U^{(4)}(x) +$$
$$\frac{(7\alpha)^2}{4}U^{(4)}(x)U^{(2)}(x)U^{(2)}(x) - \frac{(7\alpha)^3}{48}U^{(2)}(x)U^{(2)}(x)u^{(2)}(x)U^{(2)}(x), \tag{53}$$

where α is free parameter. Let us return to $SU(l)$ model. A basis $\{T_\mu\}$ of the $su(n)$ algebra the (l^2-1) traceless and hermitian matricies of the defining representation of $su(n)$ satisfy the following relations:

$$[T_\mu, T_\nu] = c^\lambda_{\mu\nu}T_\lambda,\quad \{T_\mu, T_\nu\} = -\frac{1}{l}\delta\mu\nu - id^\lambda_{\mu\nu}T_\lambda,$$
$$Tr(T_\mu) = 0,\quad T_\mu T_\nu = -\frac{1}{2l}\delta\mu\nu - \frac{i}{2}d^\lambda_{\mu\nu}T_\lambda + \frac{1}{2}c^\lambda_{\mu\nu}T_\lambda. \tag{54}$$

Therefore first of third Hamiltonians of hydrodynamical type are:

$$h^{(2)} = U^\mu U^\mu,\ \ h^{(3)} = d_{\mu\nu\lambda}U^\mu U^\nu U^\lambda,$$
$$h^{(4)} = \frac{1}{8l}(d^\lambda_{\mu\nu}U^\mu U^\nu)(d_{\rho\sigma\lambda}U^\rho U^\sigma) + \frac{1-6l}{4l^2}(U^\mu U^\mu)(U^\nu U^\nu). \tag{55}$$

In the case of $su(3)$ algebra the Hamiltonian $h^{(4)}$ reduces to the following: $h^{(4)} \rightarrow (U^\mu U^\mu)(U^\nu U^\nu)$. Using formula (24) for Hamiltonian H_2 we can receive of function $F^a(u)$ satisfying the (8),(10) equations.

Let us return to Riemann space with null torsion $C^\mu_{\nu\lambda} = 0$. We can consider left mover only, which is described by local coordinates $U^\mu(x)$. There are infinite set of tensor conserved non-local charges (Polmeyer charges [23]), which was obtained in the flat target space and curved world sheet string model. In the conformal gauge, world sheet is plane and non-local tensor charges have following form:

$$Z^{\mu_1\mu_2\cdots\mu_n} = R^{\mu_1\mu_2\cdots\mu_n} + R^{\mu_2\cdots\mu_n\mu_1} + ... + R^{\mu_n\mu_1\cdots\mu_{n-1}}, \tag{56}$$

where tensors $R^{\mu_1\mu_2\cdots\mu_n}$ are:

$$R^{\mu_1\mu_2\cdots\mu_n} = \int\limits_0^{2\pi} U^{\mu_1}(x_1)dx_1 \int\limits_0^{x_1} U^{\mu_2}(x_2)dx_2 ... \int\limits_0^{x_{n-1}} U^{\mu_n}(x_n)dx_n. \tag{57}$$

Non-local tensor chiral currents are:

$$J^{(n)} \equiv J^{\mu_1\mu_2\cdots\mu_n}(U(x)) = \Theta^{\mu_1\mu_2\cdots\mu_n} + \Theta^{\mu_2\cdots\mu_n\mu_1} + ... + \Theta^{\mu_n\mu_1\cdots\mu_{n-1}},$$

where $\Theta^{\mu_1\mu_2\cdots\mu_n}$ are:

$$\Theta^{\mu_1\mu_2\cdots\mu_n} = U^{\mu_1}(x) \int\limits_0^{x} U^{\mu_2}(x_2)dx_2 ... \int\limits_0^{x_{n-1}} U^{\mu_n}(x_n)dx_n. \tag{58}$$

To obtain the new dynamical system we use Sugawara construction for Hamiltonian [26], [27]:

$$H^{(n)} = \frac{1}{2}\int\limits_0^{\pi} J^{(n)} M J^{(n)} dx, \tag{59}$$

where M is totally symmetric, invariant, constant tensor, which can be constructed from Kronecker deltas. Let us consider first nonlocal current.

$$J^{\mu\nu}(x) = \frac{1}{2}[U^\mu(x)\int\limits_0^x U^\nu(y)dy + U^\nu(x)\int\limits_0^x U^\mu(y)dy]. \tag{60}$$

New Hamiltonian $H^{(1)}$ is

$$H^{(1)} = \frac{1}{2}\int\limits_0^{2\pi}[U^\mu(x)U^\mu(x)\int\limits_0^x U^\nu(y)dy\int\limits_0^x U^\nu(z)dz + U^\mu(x)U^\nu(x)\int\limits_0^x U^\mu(y)dy\int\limits_0^x U^\nu(z)dz]dx. \tag{61}$$

Hamiltonian $H^{(1)}$ commutes with Hamiltonian $H^{(0)} = \frac{1}{2}\int\limits_0^{2\pi} U^\mu(x)U^\mu(x)dx$ and it commutes with generator $\int\limits_0^{2\pi} U^\mu(x)dx$. The equation of motion under Hamiltonian $H^{(1)}$ is:

$$\frac{\partial U^\mu(x)}{\partial t} = \frac{\partial}{\partial x}[U^\mu(x)\int\limits_0^x U^\nu(y)dy\int\limits_0^x U^\nu(z)dz + U^\nu(x)\int\limits_0^x U^\mu(y)dy\int\limits_0^x U^\nu(z)dz]$$

$$-U^{\nu}(x)U^{\nu}(x)\int\limits_{0}^{x} U^{\mu}(y)dy - U^{\mu}(x)U^{\nu}(x)\int\limits_{0}^{x} U^{\nu}(y)dy. \tag{62}$$

In the variables $S^{\mu}(x) = \int\limits_{0}^{x} U^{\mu}(y)dy$ equation (60) can be written as follows:

$$\frac{\partial S^{\mu}}{\partial t} = \frac{\partial}{\partial x}[S^{\mu}(S^{\nu}S^{\nu})] + \int\limits_{0}^{x} S^{\mu}(S^{\nu}\frac{\partial^2 S^{\nu}}{\partial^2 y})dy,\ \mu,\nu = 1,2,...,D-2. \tag{63}$$

Last term of equation (63) can be rewritten as total derivative on x.

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On weakly non-local, nilpotent, and super-recursion operators for $N = 1$ super-equations

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Abstract

We consider nonlinear, scaling-invariant $N = 1$ boson+fermion supersymmetric systems whose right-hand sides are homogeneous differential polynomials and satisfy some natural assumptions. We select the super-systems that admit infinitely many higher symmetries generated by recursion operators; we further restrict ourselves to the case when the dilaton dimensions of the bosonic and fermionic super-fields coincide and the weight of the time is half the weight of the spatial variable. We discover five systems that satisfy these assumptions; one system is transformed to the purely bosonic Burgers equation. We construct local, nilpotent, triangular, weakly non-local, and super-recursion operators for their symmetry algebras.
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Introduction. We consider the problem of a complete description of $N = 1$ nonlinear, scaling invariant evolutionary super-equations $\{f_t = \phi^f,\ b_t = \phi^b\}$ that admit infinitely many symmetries $\{f_s = F,\ b_s = B\}$ proliferated by recursion operators $\mathcal{R}$; here b is the bosonic super-field and f is the fermionic super-field. The axioms for selecting $N = 1$ nonlinear homogeneous polynomial evolutionary systems with higher symmetries were suggested [8] by V. V. Sokolov and A. S. Sorin; the axioms are discussed in [2].

By construction, the equations are scaling invariant: their right-hand sides are differential polynomials homogeneous w.r.t. a set of (half-)integer weights $[\theta] \equiv -\frac{1}{2}$, $[x] \equiv -1$, $[t] < 0$, $[f]$, $[b] > 0$; we also assume that the negative weight $[s]$ is (half-)integer. Here we denote by θ the super-variable and we put $\partial_\theta \equiv D_\theta + \theta\, D_x$ such that $\partial_\theta^2 = D_x$; here D_θ and D_x are the total derivatives w.r.t. θ and x, respectively. All notions and notation follow [4], see also [2] for details.

In this paper, we investigate the properties of systems such that the weight of the time t is $[t] = -\frac{1}{2}$. We also assume $[f] = [b] = \frac{1}{2}$ (the weights may not be uniquely defined).

The first version of SsTools package [7] for Reduce was used for finding the systems that satisfy the above axioms and possess higher symmetries under the bound $-5 \leqslant [s] \leqslant -\frac{1}{2}$. Five systems were thus discovered, see Table I below. Later, we used the second version of SsTools [3] for symmetry analysis of the super-systems in [8] and for constructing conservation laws and recursion operators for their symmetry algebras. The method of Cartan forms [4] for the recursion operators was applied. Within this approach, the recursions are regarded as symmetries of the linearized equations. Namely, we 'forget' the internal structure of the symmetry flows $f_s = F$, $b_s = B$ and operate with F and B as we do with the components of solutions of the linearized equations. The expressions $\mathcal{R} = R(F, B)$ are the recursion operators if each $\mathcal{R}$ satisfies the linearized equation again and if they are linear w.r.t. F, B, and their derivatives.

Let us introduce some notation. Assume $\mathcal{R}$ is a recursion for an equation and consider the symbol ${}^{\text{layers}}_{\text{ord}}\mathcal{R}^{\sharp}_{\text{weight}}$. The subscripts 'ord' and 'weight' denote the differential order and the weight of the recursion $\mathcal{R}$, respectively, and the superscript 'layers' (if non-empty) indicates the required number of layers of the nonlocal variables assigned to conservation laws. The symbol '$\sharp$' denotes the number of recursions for a given differential order, weight, and the nonlocalities. Further, we denote by L the local recursion operators, by N the nonlocal or weakly non-local [1, 6] recursions, the symbol Z denotes a nilpotent recursion whose powers equal zero except for a finite set, and *Sigma* is a super-recursion that swaps the parities of the flows.

Now we list the five new super-equations and indicate their recursions. The weights of the recursion operators are calculated w.r.t. the standard values $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$.

(1) $\begin{cases} f_t = \partial_\theta b, \\ b_t = b^2 + \partial_\theta f \end{cases}$	${}^{1}_{1}N^{1}_{-1}$
(5) $\begin{cases} f_t = \partial_\theta b + fb, \\ b_t = \partial_\theta f \end{cases}$	${}^{2}_{0}N^{1}_{-1\frac{1}{2}}$, ${}^{2}_{\frac{1}{2}}N^{1}_{-2}$, ${}^{2}_{\frac{1}{2}}N^{1}_{-2\frac{1}{2}}$, ${}^{2}_{\frac{1}{2}}N^{1}_{-3}$
(6) $\begin{cases} f_t = -\alpha\, fb, \\ b_t = b^2 + \partial_\theta f \end{cases}$	$\alpha = 2:$ ${}_{\frac{1}{2}}L^{1}_{-2}$, ${}_{\frac{1}{2}}L^{1}_{-2\frac{1}{2}}$, ${}^{0}_{0}Z^{1}_{-2}$, ${}^{0}_{0}\Sigma^{1}_{-2}$, ${}^{0}_{0}\Sigma^{1}_{-2\frac{1}{2}}$, ${}^{0}_{0}\Sigma^{1}_{-2\frac{1}{2}}$; $\alpha = 1:$ ${}^{0}_{0}Z^{1}_{-2\frac{1}{2}}$, ${}_{\frac{1}{2}}Z^{1}_{-3}$; $\alpha = 4:$ ${}_{1}L^{1}_{-3\frac{1}{2}}$

Table I.

It turns out that these equations exhibit practically the whole variety of properties that superPDE of mathematical physics possess. Let us discuss the properties of the equations present in Table I in more detail.

1. The Burgers equation. First we construct an $N = 1$ supersymmetric representation of the Burgers equation and investigate its properties. We consider the system

$$f_t = \partial_\theta b, \qquad b_t = b^2 + \partial_\theta f. \tag{1}$$

There is a unique set of weights $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$ in this case. Hence we conclude that the above system precedes the invariance w.r.t. the translation along x. Equation (1) admits the continuous sequence (3) of higher symmetries $f_s = \phi^f$, $b_s = \phi^b$ at all (half-)integer weights $[s] \leq -\frac{1}{2}$. Also, there is the continuous sequence (4) of supersymmetries for Eq. (1) at all (half-)integer weights $[\bar{s}] \leq -\frac{1}{2}$ of the fermionic 'time' $\bar{s}$.

System (1) is obviously reduced to the purely bosonic Burgers equation $b_x = b_{tt} - 2bb_t$. We emphasize that the role of the independent coordinates x and t is reversed w.r.t. the standard interpretation of t as the time and x as the spatial variable. The Cole–Hopf substitution $b = -u^{-1}u_t$ from the heat equation $u_x = u_{tt}$ is thus the solution for the bosonic component of (1).

Further, we introduce the bosonic nonlocality w of weight $[w] = 0$ by the rules $\partial_\theta w = -f$, $w_t = -b$. The variable w is a potential for both fields f and b. The nonlocality satisfies the potential Burgers equation $w_x = w_{tt} + w_t^2$ such that the formula $w = \ln u$ gives the solution; the relation $f = -\partial_\theta w$ determines the fermionic component in system (1).

Now we extend the set of dependent variables f, b, and w by the symmetry generators F, B, and W that satisfy the linearized relations upon the flows of the initial super-fields, respectively. In this setting, we obtain the recursion

$$\mathcal{R}_{[1]} = \begin{pmatrix} F_x - \partial_\theta f\, F + f_x\, W \\ B_x - \partial_\theta f\, B + b_x\, W \end{pmatrix} \Longleftrightarrow R = \begin{pmatrix} D_x - \partial_\theta f + f_x\, \partial_\theta^{-1} & 0 \\ b_x\, \partial_\theta^{-1} & D_x - \partial_\theta f \end{pmatrix} \tag{2}$$

of weight $[s_R] = -1$. In agreement with [1], the above recursion is weakly non-local [6]. We recall that a recursion operator R is *weakly non-local* if each nonlocality ∂_θ^{-1} is preceded with a (shadow [4] of a nonlocal) symmetry φ_α and is followed by the gradient ψ_α of a conservation law: $R = \text{local part} + \sum_\alpha \varphi_\alpha \cdot \partial_\theta^{-1} \circ \psi_\alpha$. From [1] it follows that this property is automatically satisfied by all recursion operators which are constructed using one layer of the nonlocal variables assigned to conservation laws.

Recursion (2) generates two sequences of higher symmetries for system (1):

$$\begin{pmatrix} f_t \\ b_t \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta b_x - \partial_\theta f \partial_\theta b - f_x b \\ \partial_\theta f_x - (\partial_\theta f)^2 - b^2 \partial_\theta f + b b_x \end{pmatrix} \mapsto \cdots, \begin{pmatrix} f_x \\ b_x \end{pmatrix} \mapsto \begin{pmatrix} f_{xx} - 2\partial_\theta f f_x \\ b_{xx} - 2\partial_\theta f b_x \end{pmatrix} \mapsto \cdots. \tag{3}$$

Also, recursion (2) produces two infinite sequences of supersymmetries for (1):

$$\begin{pmatrix} \partial_\theta f \\ \partial_\theta b \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta f_x - (\partial_\theta f)^2 - f_x f \\ \partial_\theta b_x - \partial_\theta f\, \partial_\theta b - b_x f \end{pmatrix} \mapsto \cdots, \begin{pmatrix} f \partial_\theta b - b\, \partial_\theta f + b_x \\ b \partial_\theta b - f\, \partial_\theta f + f_x - f b^2 \end{pmatrix} \mapsto \cdots. \tag{4}$$

Remark 1. System (1) is not a supersymmetric extension of the Burgers equation; it is a supersymmetric representation of the Burgers equation. However, symmetries (3) and (4) are *not* reduced to the purely bosonic (x, t)-independent symmetries [5] of the Burgers equation (particularly, owing to the interchanged role of the variables x and t). We finally recall that the Burgers equation has infinitely many higher symmetries that depend explicitly on the base coordinates x, t but exceed the set of axioms [2] we use.

Two supersymmetric generalizations ($N = 0$ and $N = 2$) of the Burgers equation are constructed in [2]. The $N = 0$ extension relates it with integrable flows on associative algebras. The $N = 2$ Burgers equation contains a KdV-type component and admits an $N = 2$ modified KdV equation as a symmetry flow.

2. A system with nonlocal recursions. The second system,

$$f_t = \partial_\theta b + fb, \qquad b_t = \partial_\theta f, \tag{5}$$

is also homogeneous w.r.t. a unique set of weights $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$. Similarly to the supersymmetric representation (1) for the Burgers equation, Eq. (5) admits symmetries (f_s, b_s) for all weights $[s] \le -\frac{1}{2}$.

We conjecture that system (5) has only one conservation law that defines the fermionic variable w of weight 0 by $w_t = f$, $\partial_\theta w = b$. Then, many nonlocal conservation laws and hence many new variables appear. We use the fermionic variable v whose weight $[v] = \frac{3}{2}$ is minimal: we set $v_t = \partial_\theta b \cdot wfb + f_x wf$ and $\partial_\theta v = -\partial_\theta b \cdot fb + \partial_\theta f \cdot \partial_\theta b \cdot w + b_x wf$. Now, there are nontrivial solutions to the determining equations for recursion operators. First, we obtain the recursion of zero differential order with nonlocal coefficients:

$$R_{[-1\frac{1}{2}]} = \begin{pmatrix} -\partial_\theta b \cdot wfB + wvF + v \cdot B \\ \partial_\theta b\, wfF - vF + vw \cdot B \end{pmatrix}.$$

Also, we get a nonlocal operator with nonlocal coefficients,

$$R_{[-2]} = \begin{pmatrix} \partial_\theta b\, Vw - \partial_\theta f\, \partial_\theta B\, wf - \partial_\theta f\, \partial_\theta b\, Wf + \partial_\theta f\, \partial_\theta b\, Fw + \partial_\theta f\, V + Vwfb \\ \partial_\theta B\, \partial_\theta b\, wf + \partial_\theta b\, V - \partial_\theta b\, Fwfb + \partial_\theta f\, \partial_\theta b\, Vwf + \partial_\theta f\, Vw - Vfb \end{pmatrix}.$$

The coefficients of the recursions found for $[s_R] = -2\frac{1}{2}$ and $[s_R] = -3$ are also nonlocal.

3. A triplet of super-systems. Finally, we consider the three systems

$$f_t = -\alpha fb, \qquad b_t = b^2 + \partial_\theta f \tag{6}$$

which differ by the values $\alpha = 1$, 2, and 4 of the parameter α and therefore exhibit rather different properties. The weights for the above equation are multiply defined, and we choose the tuple $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$ to be the primary 'reference system.'

Case $\alpha = 2$**.** First, we fix $\alpha = 2$ and consider Eq. (6): we get $f_t = -2fb$, $b_t = b^2 + \partial_\theta f$. The weights for symmetries are $[s] = -\frac{1}{2}$, $[s] = -1$, and then Eq. (6) admits a continuous chain of symmetry flows for all (half-)integer weights $[s] \le -2\frac{1}{2}$. Surprisingly, no nonlocalities are needed to construct the recursion operators, although there are many conservation laws for this system. We obtain purely local recursion operators $\mathcal{R}$ that proliferate the symmetries: $\varphi = (F, B) \mapsto \varphi' = \mathcal{R}$ for any φ. The recursion

$$\mathcal{R}_{[-2]} = \begin{pmatrix} \frac{11}{2}\partial_\theta F\, \partial_\theta f\, f + 11\partial_\theta F\, fb^2 + \frac{3}{2}(\partial_\theta f)^2 F + 3\partial_\theta f\, Fb^2 + \frac{1}{2} f_x F f \\ 11\partial_\theta B\, fb^2 + 8\partial_\theta b\, Fb^2 + 22\partial_\theta b\, fBb + 7(\partial_\theta f)^2 B + \\ 14\partial_\theta f\, Bb^2 + \frac{11}{2}\partial_\theta f\, \partial_\theta B\, f + \frac{5}{2}\partial_\theta f\, \partial_\theta b\, F + \frac{1}{2} b_x F f + f_x F b + 5 f_x f B \end{pmatrix},$$

of weight $[s_R] = -2$ is triangular since R^f does not contain B. Also, we obtain the recursion of weight $2\frac{1}{2}$; its components are

$$\begin{aligned}
\mathcal{R}^f_{[-2\frac{1}{2}]} &= -2\partial_\theta b\, Ffb^2 - \partial_\theta F\, \partial_\theta f\, fb - \partial_\theta F\, fb^3 - \tfrac{1}{2} f_x F fb - 2\partial_\theta f\, fBb^2, \\
\mathcal{R}^b_{[-2\frac{1}{2}]} &= \partial_\theta B\, fb^3 + \partial_\theta b\, Fb^3 + \partial_\theta b\, fBb^2 + \tfrac{1}{8}\partial_\theta f_x F f + \\
&+ \tfrac{1}{2}\partial_\theta F b^4 + \tfrac{1}{2}\partial_\theta F(\partial_\theta f)^2 + \partial_\theta F\, \partial_\theta f\, b^2 + \tfrac{1}{8}\partial_\theta F\, f_x f + (\partial_\theta f)^2 Bb + \\
&+ \partial_\theta f\, Bb^3 + \partial_\theta f\, \partial_\theta B\, fb + \partial_\theta f\, \partial_\theta b\, Fb + \partial_\theta f\, \partial_\theta b\, fB + \tfrac{3}{8}\partial_\theta f\, F_x f + \\
&+ \tfrac{1}{4}\partial_\theta f\, f_x F + \tfrac{1}{2} b_x F fb + \tfrac{1}{2} F_x fb^2 + \tfrac{1}{4} f_x F b^2 + \tfrac{1}{2} f_x f Bb.
\end{aligned}$$

Further, we get a triangular nilpotent operator of weight -3 such that $\mathcal{R}^f_{[-3]} = 0$ and $\mathcal{R}^b_{[-3]} = (\partial_\theta f)^3 fF + 6(\partial_\theta f)^2 fb^2F + 12\partial_\theta f\, fb^4F + 8fb^6F$. The above recursion is a recurrence relation [2] which is well-defined for all symmetries of Eq. (6). Another local recursion for $[s] = -3$ is huge and therefore omitted.

For $\alpha = 2$, system (6) admits at least three super-recursions ${}^t(R^f,\, R^b)$ such that the parities of R^f and R^b are opposite to the odd parity for f (and hence for F) and to the even parity of b and B. This property is possible owing to the presence of the *odd* variable s_R. The triangular zero-order super-recursions are $\bar{\mathcal{R}}^f_{[-2]} = 4\partial_\theta f\, Ffb + 8Ffb^3$, $\bar{\mathcal{R}}^b_{[-2]} = -4\partial_\theta b\, Ffb + 2(\partial_\theta f)^2F + 6\partial_\theta f\, Fb^2 + 4\partial_\theta f\, fBb - f_xFf + 4Fb^4 + 8fBb^3$ and

$$\bar{\mathcal{R}}_{[-2\frac{1}{2}]} = \begin{pmatrix} -\partial_\theta f\, f_xF - 2f_xFb^2 \\ \partial_\theta b\, f_xF - \partial_\theta f\, b_xF + \partial_\theta f\, f_xB - 2b_xFb^2 + 2f_xBb^2 \end{pmatrix}$$

for weights $[s_R] = -2$ and $[s_R] = -2\frac{1}{2}$, respectively; the third super-recursion found for $[s_R] = -2\frac{1}{2}$ is very large. Quite naturally, system (6) has infinitely many supersymmetries if $\alpha = 2$.

Case $\alpha = 1$**.** For $\alpha = 1$ from (6) we obtain the system $f_t = -fb$, $b_t = b^2 + \partial_\theta f$. The default set of weights is the same as above: $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, and $[x] = -1$. The sequence of symmetries is not continuous and starts later than for the chain in the case $\alpha = 2$. We find out that there are symmetry flows if either $[s] = [t] = -\frac{1}{2}$ (the equation itself), $[s] = [x] = -1$ (the translation along x), or $[s] \le -3\frac{1}{2}$ such that a continuous chain starts for all (half-)integer weights $[s]$.

Similarly to the previous case, no nonlocalities are needed to construct the recursions, which therefore are purely local. The recursion operator $\mathcal{R}^f_{[-2\frac{1}{2}]} = 0$, $\mathcal{R}^b_{[-2\frac{1}{2}]} = (\partial_\theta f)^2\, Ff + 3\partial_\theta f\, Ffb^2 + \frac{9}{4}Ffb^4$ of maximal weight $[s_R] = -2\frac{1}{2}$ is nilpotent: $\mathcal{R}^2 = 0$. For the succeeding weight $[s_R] = -3$, we obtain a triangular local recursion with components

$$\begin{aligned}
\mathcal{R}^f_{[-3]} &= \tfrac{5}{3}\partial_\theta F\,(\partial_\theta f)^2 f + \tfrac{5}{2}\partial_\theta F\,\partial_\theta f\, fb^2 - \tfrac{5}{3}(\partial_\theta f)^3F - \tfrac{5}{2}(\partial_\theta f)^2Fb^2 + \\
&\quad + 5\partial_\theta f\,\partial_\theta b\, Ffb + \tfrac{20}{3}\partial_\theta f\, f_xFf + \tfrac{15}{2}f_xFfb^2, \\
\mathcal{R}^b_{[-3]} &= \partial_\theta f_x\, Ffb - \tfrac{105}{2}\partial_\theta F\,\partial_\theta b\, fb^2 - \tfrac{160}{3}\partial_\theta F\,\partial_\theta f\,\partial_\theta b\, f + 11\partial_\theta F\, f_xfb + \\
&\quad + \tfrac{5}{3}(\partial_\theta f)^2\partial_\theta Bf + \tfrac{5}{3}(\partial_\theta f)^2\partial_\theta b\, F + \tfrac{5}{2}\partial_\theta f\,\partial_\theta B\, fb^2 + \tfrac{5}{2}\partial_\theta f\,\partial_\theta b\, Fb^2 - \\
&\quad - 55\partial_\theta f\,\partial_\theta b\, fBb + \tfrac{17}{3}\partial_\theta f\, b_xFf + \partial_\theta f\, f_xfB + \tfrac{23}{2}b_xFfb^2 + \tfrac{183}{2}f_xfBb^2.
\end{aligned}$$

It generates symmetries of system (6); the differential order of $\mathcal{R}_{[-3]}$ is positive.

Case $\alpha = 4$**.** Finally, let $\alpha = 4$; then system (6) acquires the form $f_t = -4fb$, $b_t = b^2 + \partial_\theta f$. Again, the basic set of weights is $[f] = [b] = \frac{1}{2}$, $[t] = -\frac{1}{2}$, $[x] = -1$, and system (6) admits the symmetries $(f_s,\, b_s)$ such that their weights are $[s] = -\frac{1}{2}$, -1 or $[s] \le -3\frac{1}{2}$ w.r.t. the basic set. This situation coincides with the case $\alpha = 1$. Again, no nonlocalities are needed for constructing the recursion of minimal weight $[s_R] = -3\frac{1}{2}$:

$$
\begin{aligned}
\mathcal{R}^f_{[-3\frac{1}{2}]} &= -12\partial_\theta b\, F f b^4 - \partial_\theta F\,(\partial_\theta f)^2 f b - 4\partial_\theta F\,\partial_\theta f\, f b^3 - 3\partial_\theta F\, f b^5 - \\
&\quad -4(\partial_\theta f)^2 f B b^2 - 4\partial_\theta f\,\partial_\theta b\, F f b^2 - \tfrac{2}{3}\partial_\theta f\, f_x F f b - 12\partial_\theta f\, f B b^4 - 2 f_x F f b^3, \\
\mathcal{R}^b_{[-3\frac{1}{2}]} &= 3\partial_\theta B f b^5 + 3\partial_\theta b\, F b^5 + 9\partial_\theta b\, f B b^4 + \tfrac{1}{9}\partial_\theta f_x\,\partial_\theta f\, F f - \tfrac{1}{3}\partial_\theta f_x\, F f b^2 + \tfrac{3}{4}\partial_\theta F\, b^6 + \\
&\quad +\partial_\theta F\,\partial_\theta b\, f b^3 + \tfrac{1}{4}\partial_\theta F\,(\partial_\theta f)^3 + \tfrac{5}{4}\partial_\theta F\,(\partial_\theta f)^2 b^2 + \tfrac{7}{4}\partial_\theta F\,\partial_\theta f\, b^4 + \partial_\theta F\,\partial_\theta f\,\partial_\theta b\, f b + \\
&\quad +\tfrac{5}{18}\partial_\theta F\,\partial_\theta f\, f_x f + \tfrac{1}{2}\partial_\theta F\, f_x f b^2 + (\partial_\theta f)^3 B b + 4(\partial_\theta f)^2 B b^3 + (\partial_\theta f)^2\partial_\theta B\, f b + \\
&\quad +(\partial_\theta f)^2\partial_\theta b\, F b + (\partial_\theta f)^2\partial_\theta b\, f B + \tfrac{2}{9}(\partial_\theta f)^2 F_x f + \tfrac{1}{6}(\partial_\theta f)^2 f_x F + 3\partial_\theta f\, B b^5 + \\
&\quad +4\partial_\theta f\,\partial_\theta B\, f b^3 + 4\partial_\theta f\,\partial_\theta b\, F b^3 + 10\partial_\theta f\,\partial_\theta b\, f B b^2 + \tfrac{2}{3}\partial_\theta f\, b_x F f b + \partial_\theta f\, F_x f b^2 + \\
&\quad +\tfrac{2}{3}\partial_\theta f\, f_x F b^2 + \tfrac{5}{3}\partial_\theta f\, f_x f B b + 2 b_x F f b^3 + F_x f b^4 + \tfrac{1}{2} f_x F b^4 + f_x f B b^3.
\end{aligned}
$$

No nilpotent recursion operators were found for system (6) if $\alpha = 4$.

Remark 2. We discovered that an essential part of recursion operators for supersymmetric PDE are nilpotent. At present, it is not clear how the nilpotent recursion operators contribute to the integrability of supersymmetric systems and what invariants they describe or symptomize. Further, we emphasize that this property does not always originate from the rule '$f \cdot f = 0$', but this is an immanent feature of the symmetry algebras. More generally, the nilpotent recursions are quite natural in the bosonic sector, too. We have

Example (I. S. Krasil'shchik, private communication). Consider a system of linear ordinary differential equations $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$. Then any nilpotent constant matrix R that commutes with the matrix A is a recursion.

It would be of interest to construct an equation $\mathcal{E}$ that admits nilpotent differential recursion operators $\{R_1, \ldots \mid R_i^{n_i} = 0\}$ which generate an infinite sequence of symmetries φ, $R_{i_1}(\varphi)$, $R_{i_2} \circ R_{i_1}(\varphi)$, $\ldots$ for $\mathcal{E}$. Here we assume that at least two operators (without loss of generality, R_1 and R_2) do not commute and hence the flows never become zero.

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Integrable Noncommutative Sine-Gordon Model

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Abstract

As I briefly review, the sine-Gordon model may be obtained by dimensional and algebraic reduction from 2+2 dimensional self-dual U(2) Yang-Mills through a 2+1 dimensional integrable U(2) sigma model. I argue that the noncommutative (Moyal) deformation of this procedure should relax the algebraic reduction from U(2) → U(1) to U(2) → U(1)×U(1). The result are novel noncommutative sine-Gordon equations for a *pair* of scalar fields. The dressing method is outlined for constructing its multi-soliton solutions. Finally, I look at tree-level amplitudes to demonstrate that this model possesses a factorizable and causal S-matrix in spite of its time-space noncommutativity.

1 Classical sine-Gordon model ...

Extremizing the sine-Gordon action

$$S \;=\; \tfrac{1}{2}\int \mathrm{d}t\,\mathrm{d}y\big[(\partial_t\phi)^2-(\partial_y\phi)^2+8\alpha^2(\cos\phi-1)\big] \tag{1}$$

for a scalar field $\phi(t,y)$ on $\mathbb{R}^{1,1}$ with mass $=2\alpha$ yields the sine-Gordon equation,

$$(\partial_t^2-\partial_y^2)\,\phi \;+\; 4\alpha^2\sin\phi \;=\; 0\ . \tag{2}$$

This famous equation has many remarkable features, such as a Lax-pair or zero-curvature representation, infinitely many conserved local charges, a factorizable S-matrix without particle production, as well as soliton and breather solutions. The simplest soliton configuration (with velocity v) is kink-like,

$$\phi_{\text{kink}}(t,y) \;=\; 4\arctan\ \mathrm{e}^{\,-2\alpha\eta} \qquad \text{with} \quad \eta=\tfrac{y-\mathrm{v}t}{\sqrt{1-\mathrm{v}^2}}\ . \tag{3}$$

For later use I introduce light-cone coordinates

$$u \;:=\; \tfrac{1}{2}(t+y)\ ,\quad v \;:=\; \tfrac{1}{2}(t-y) \qquad \Longrightarrow \qquad \partial_u \;=\; \partial_t+\partial_y\ ,\quad \partial_v \;=\; \partial_t-\partial_y\ . \tag{4}$$

2 ... via dimensional and algebraic reduction

In 4d Yang-Mills and 3d Yang-Mills-Higgs systems the field equations are implied by first-order equations:

$$
\begin{array}{lccc}
d{=}2{+}2: & D^\mu F_{\mu\nu}=0 & \Longleftarrow & F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda} \\
\downarrow & \downarrow & & \downarrow \\
d{=}2{+}1: & \begin{array}{rcl} D^aF_{ab} &=& HD_bH \\ D^aD_aH &=& 0 \end{array} & \Longleftarrow & F_{ab}=\epsilon_{abc}D^cH\ .
\end{array}
\tag{5}
$$

One may gauge-fix and "solve" the 3d Bogomolny equations via

$$A_v \;=\; 0 \qquad \text{and} \qquad A_x{+}H \;=\; 0\ , \tag{6}$$

$$A_u \;=\; \Phi^{-1}\,\partial_u\Phi \qquad \text{and} \qquad A_x{-}H \;=\; \Phi^{-1}\,\partial_x\Phi\ , \tag{7}$$

with $\Phi(u,v,x)\in \mathrm{SU}(2)$ subject to the "Yang equation"

$$\partial_v(\Phi^\dagger\partial_u\Phi)\;-\;\partial_x(\Phi^\dagger\partial_x\Phi)\;=\;0\ . \tag{8}$$

A dimensional reduction to the 2d WZW model is achieved by letting

$$\Phi(u,v,x) \qquad\longrightarrow\qquad \mathcal{E}\ \mathrm{e}^{\,\mathrm{i}\alpha x\sigma_1}\,g(u,v)\ \mathrm{e}^{-\mathrm{i}\alpha x\sigma_1}\,\mathcal{E}^\dagger \tag{9}$$

with a constant matrix $\mathcal{E}$ and $g(u,v)\in\mathrm{SU}(2)$. The Yang equation (8) then becomes

$$\partial_v(g^\dagger\partial_u g)\;+\;\alpha^2(\sigma_1 g^\dagger\sigma_1 g - g^\dagger\sigma_1 g\,\sigma_1)\;=\;0\ . \tag{10}$$

Finally, an algebraic reduction of $g(u,v)$ to $\mathrm{U}(1)$ yields the sine-Gordon equation:

$$g \;=\; \mathrm{e}^{\,\frac{\mathrm{i}}{2}\sigma_3\,\phi} \qquad\Longrightarrow\qquad \partial_v\partial_u\phi+4\alpha^2\sin\phi\;=\;0\ . \tag{11}$$

3 Noncommutative deformation

The Moyal deformation of $\mathbb{R}^{1,1}$ replaces the ordinary pointwise product of functions, $(f\cdot g)(t,y)=f(t,y)\,g(t,y)$, by the "star product"

$$(f\star g)(t,y) \;=\; f(t,y)\,\exp\{\tfrac{\mathrm{i}\theta}{2}(\overleftarrow{\partial}_t\overrightarrow{\partial}_y-\overleftarrow{\partial}_y\overrightarrow{\partial}_t)\}\,g(t,y) \tag{12}$$

$$\;=\; f\,g\;+\;\tfrac{\mathrm{i}\theta}{2}(\partial_t f\,\partial_y g-\partial_y f\,\partial_t g)\;+\;\dots \tag{13}$$

with a constant noncommutativity parameter $\theta\in\mathbb{R}_+$. This product satisfies, in particular,

$$(f\star g)\star h \;=\; f\star(g\star h) \qquad\text{and}\qquad \textstyle\int \mathrm{d}t\,\mathrm{d}y\ f\star g \;=\; \int \mathrm{d}t\,\mathrm{d}y\,f\,g\ , \tag{14}$$

and the coordinate functions obey the commutation relations

$$t\star y-y\star t \;=\; \mathrm{i}\,\theta \qquad\Longrightarrow\qquad u\star v-v\star u \;=\; -\tfrac{\mathrm{i}}{2}\theta\ . \tag{15}$$

Additional coordinates (for d=2+1 or d=2+2) commute.

4 Poor deformations of the sine-Gordon model

Naive $\star$-ing of the sine-Gordon equation (2) yields

$$\partial_v\,\partial_u\,\phi \;=\; -4\alpha^2\,\sin_\star\phi\;, \tag{16}$$

which does not allow for conserved charges. More promising is the Moyal deformation of the above reduction procedure (9), now with $\Phi(u,v,x)\in\mathrm{U}(2)$ and

$$g(u,v) \;=\; \exp_\star\{\tfrac{\mathrm{i}}{2}\sigma_3\,\phi(u,v)\} \quad \in \mathrm{U}(1)\;. \tag{17}$$

Inserting this into the deformed version of the Yang equation (10) produces *two* equations,

$$\partial_v(\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\phi}\star\partial_u\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\phi} - \mathrm{e}_\star^{\frac{\mathrm{i}}{2}\phi}\star\partial_u\,\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\phi}) \;=\; -4\,\mathrm{i}\,\alpha^2\sin_\star\phi\;, \tag{18}$$

$$\partial_v(\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\phi}\star\partial_u\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\phi} + \mathrm{e}_\star^{\frac{\mathrm{i}}{2}\phi}\star\partial_u\,\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\phi}) \;=\; 0\;, \tag{19}$$

of which the first one becomes the standard sine-Gordon equation when $\theta\to0$ while the second one may be interpreted as a constraint that disappears in the commutative limit. These equations indeed feature infinitely many conserved local charges, but the corresponding S-matrix is acausal (containing $\sin^2(pE\theta)$ terms) and yields particle production ($2\to3$ and $2\to4$). Hence, this model does not yet represent a satisfactory deformation of the sine-Gordon theory.

5 A proposal: algebraic reduction to $\mathrm{U}(1)\times\mathrm{U}(1)$

The extension of SU(2) to U(2) for the Yang-Mills gauge group was enforced by the noncommutativity. It is therefore natural to keep the additional U(1) factor also in the algebraic reduction. Hence, let me relax the reduction

$$\text{from}\qquad g\;=\;\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\sigma_3\,\phi}\qquad\text{to}\qquad g\;=\;\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\mathbb{1}\,\rho}\star\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\sigma_3\,\varphi}\;, \tag{20}$$

i.e. take $g(u,v)\in\mathrm{U}(1)\times\mathrm{U}(1)$ and work with *two* scalar fields $\varphi(u,v)$ and $\rho(u,v)$. The Yang equation (10) in this case yields

$$\partial_v(\,\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\varphi}\star\partial_u\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\varphi}) + 2\,\mathrm{i}\,\alpha^2\sin_\star\varphi \;=\; -\partial_v[\,\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\varphi}\star R\star\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\varphi}] \tag{21}$$

$$\partial_v(\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\varphi}\star\partial_u\,\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\varphi}) - 2\,\mathrm{i}\,\alpha^2\sin_\star\varphi \;=\; -\partial_v[\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\varphi}\star R\star\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\varphi}] \tag{22}$$

$$\text{with}\qquad R\;:=\;\mathrm{e}_\star^{-\frac{\mathrm{i}}{2}\rho}\star\partial_u\,\mathrm{e}_\star^{\frac{\mathrm{i}}{2}\rho}\;. \tag{23}$$

Note that for $\rho=0$ one finds that $R=0$ and recovers (18) and (19). In the commutative limit $\theta\to0$ the system (21) plus (22) behaves as it should and decouples to

$$\partial_v\,\partial_u\,\rho\;=\;0\qquad\text{and}\qquad\partial_v\,\partial_u\,\varphi\;+\;4\alpha^2\sin\varphi\;=\;0\;. \tag{24}$$

6 Linear system

In order to unclutter my notation, I suppress all $\star$ products for the remainder of the talk but assume their implicit presence if not said otherwise. Therefore, despite appearance even scalar fields do not commute. Like in the commutative case, also the deformed version of the Yang equation (10) can be seen as the compatibility condition for a (now noncommutative) linear system

$$
\begin{aligned}
(\partial_u + \mathrm{i}\,\alpha\,\zeta\,\mathrm{ad}\sigma_3)\,\psi &= -(g^\dagger \partial_u g)\,\psi\ , && (25)\\
(\zeta\,\partial_v + \mathrm{i}\,\alpha\,\mathrm{ad}\sigma_3)\,\psi &= \mathrm{i}\,\alpha(g^\dagger \sigma_3\, g)\,\psi && (26)
\end{aligned}
$$

with $\psi(u,v,\zeta) \in \mathrm{U}(2)$ and limits

$$
\psi(\zeta{\to}0) \;=\; g^\dagger \;+\; O(\zeta) \qquad \text{and} \qquad \psi(\zeta{\to}\infty) \;=\; \mathbb{1} \;+\; O(\zeta^{-1})\ . \tag{27}
$$

Please note that, due to the Moyal deformation, the entries of all these matrices are non-commuting themselves. In a moment, I am going to exploit the holomorphic dependence on the spectral parameter $\zeta \in \mathbb{C}P^1$ in the following three equations:

$$
\begin{aligned}
\mathbb{1} &= \psi(u,v,\zeta)\,[\psi(u,v,\bar\zeta)]^\dagger\ , && (28)\\
g^\dagger \partial_u g &= \psi\,(\partial_u + \mathrm{i}\,\alpha\,\zeta\,\mathrm{ad}\sigma_3)\,\psi^\dagger\ , && (29)\\
-\,\mathrm{i}\,\alpha\, g^\dagger \sigma_3\, g &= \psi\,(\zeta\,\partial_v + \mathrm{i}\,\alpha\,\mathrm{ad}\sigma_3)\,\psi^\dagger\ . && (30)
\end{aligned}
$$

Since $\mathbb{C}P^1$ is compact, a nontrivial (i.e. non-constant) matrix function $\psi(\zeta)$ has to be meromorphic. However, the left hand sides of the above equations are independent of ζ, and so must be their right hand sides. This fact implies, in particular, that the residues of all poles in the right-hand-side expressions of (28), (29) and (30) better vanish, imposing strong conditions on the auxiliary matrix function $\psi(u,v,\zeta)$.

7 Single-pole ansatz

The simplest ansatz beyond a constant matrix reads[1]

$$
\psi_1 \;=\; \big(\mathbb{1} + \tfrac{2\,\mathrm{i}\,\mu_1}{\zeta - \mathrm{i}\,\mu_1}\,P_1\big)\,\psi_1^0 \;=\; \big(\mathbb{1} + \tfrac{\Lambda_{11} S_1^\dagger}{\zeta - \mathrm{i}\,\mu_1}\big)\,\psi_1^0 \tag{31}
$$

with $\mu_1 \in \mathbb{R}$ (an imaginary pole) and a constant matrix $\psi_1^0 \in \mathrm{U}(2)$. To be determined are the U(2) valued noncommutative functions $P_1(u,v)$ and $\Lambda_{11} S_1^\dagger(u,v)$. Inserting the ansatz (31) into (28) and isolating the residues one gets

$$
\operatorname*{res}_{\zeta=-\mathrm{i}\,\mu_1}(28) = 0 \quad\Longrightarrow\quad \left\{\begin{array}{rclcrcl} P_1^\dagger &=& P_1 \;=\; P_1^2 & \Longrightarrow & P_1 &=& T_1\,\frac{1}{T_1^\dagger T_1}\,T_1^\dagger \\ (\mathbb{1}{-}P_1)\,S_1\Lambda_{11}^\dagger &=& 0 & \Longrightarrow & T_1 &=& S_1 \end{array}\right. \tag{32}
$$

which qualifies P_1 as a hermitian projector built from a $2{\times}1$ matrix T which spans $\mathrm{im}P_1$. Next, exploiting (29) and (30) yields

$$
\operatorname*{res}_{\zeta=-\mathrm{i}\,\mu_1}(29,30) = 0 \quad\Longrightarrow\quad (\mathbb{1}{-}P_1)\,\bar L_1\,(S_1\Lambda_{11}^\dagger) \;=\; 0 \quad\Longrightarrow\quad \bar L_1\,S_1 \;=\; S_1\,\Gamma_1 \tag{33}
$$

[1]The reason for the seemingly redundant notation becomes clear in the next section.

with a constant Γ_1 and

$$\bar{L}_i \;:=\; \begin{cases} \partial_u + \alpha\,\mu_i\,\mathrm{ad}\sigma_3 & \text{for} \quad (29) \\ -\mu_i^2\,\partial_v + \alpha\,\mu_i\,\mathrm{ad}\sigma_3 & \text{for} \quad (30) \end{cases} \qquad (\text{here} \quad i=1)\;. \tag{34}$$

The residues at $\zeta = \mathrm{i}\,\mu_1$ merely lead to the complex conjugated conditions. The solution to (33) has the form (I choose $\gamma_{11}, \gamma_{12} \in \mathbb{R}$)

$$S_1(u,v) \;=\; \widehat{S}_1(\eta_1) \;=\; \mathrm{e}^{-\alpha\,\eta_1\sigma_3}\,(\,\gamma_{\,11}\,\mathrm{i}\,\gamma_{12}) \tag{35}$$

and combines the u, v dependence in a single "co-moving coordinate"

$$\eta_i \;=\; \mu_i u - \tfrac{1}{\mu_i} v \;=\; \tfrac{y - \mathrm{v}_i t}{\sqrt{1-\mathrm{v}_i^2}} \qquad (\text{here} \quad i=1)\;. \tag{36}$$

8 Dressing method

I proceed to the two-pole ansatz, in a multiplicative and an additive form:

$$\psi_2 \;=\; (\mathbb{1} + \tfrac{2\,\mathrm{i}\,\mu_2}{\zeta - \mathrm{i}\,\mu_2}\,P_2)(\mathbb{1} + \tfrac{2\,\mathrm{i}\,\mu_1}{\zeta - \mathrm{i}\,\mu_1}\,P_1)\,\psi_2^0 \tag{37}$$

$$=\; (\mathbb{1} + \tfrac{\Lambda_{21} S_1^\dagger}{\zeta - \mathrm{i}\,\mu_1} + \tfrac{\Lambda_{22} S_2^\dagger}{\zeta - \mathrm{i}\,\mu_2})\,\psi_2^0\;, \tag{38}$$

generalizing the one-pole notation of (31) in an obvious way. A look at the residues at $\zeta = -\,\mathrm{i}\,\mu_1$ reveals that P_1 and S_1 are subject to the same equations (32) and (33) as in the one-pole case and thus can be taken over from there, e.g. via (35). The analysis of the residues at $\zeta = -\,\mathrm{i}\,\mu_2$ is more involved however. First, the two forms (37) and (38) yield

$$\operatorname*{res}_{\zeta=-\,\mathrm{i}\,\mu_2}(28) = 0 \quad\Longrightarrow \begin{cases} (\mathbb{1}{-}P_2)\,P_2 \;=\; 0 \quad\Longrightarrow\quad P_2 \;=\; T_2\,\frac{1}{T_2^\dagger T_2}\,T_2^\dagger \\ \psi_2(\mu_2)\,S_2\Lambda_{22}^\dagger \;=\; (\mathbb{1}{-}P_2)\,\underbrace{(\mathbb{1} - \tfrac{2\mu_1}{\mu_1+\mu_2}P_1)\,S_2}_{T_2}\,\Lambda_{22}^\dagger \;=\; 0 \end{cases} \tag{39}$$

and, second, the additive variant (38) produces

$$\operatorname*{res}_{\zeta=-\,\mathrm{i}\,\mu_2}(29,30) = 0 \quad\Longrightarrow\quad \psi_2(\mu_2)\,\bar{L}_2\,(S_2\Lambda_{22}^\dagger) \;=\; 0 \quad\Longrightarrow\quad \bar{L}_2\,S_2 \;=\; S_2\,\Gamma_2 \tag{40}$$

with a constant Γ_2. Like before, the solution to the latter equation reads (take $\gamma_{21}, \gamma_{22} \in \mathbb{R}$)

$$S_2(u,v) \;=\; \widehat{S}_2(\eta_2) \;=\; \mathrm{e}^{-\alpha\,\eta_2\sigma_3}\,(\,\gamma_{\,21}\,\mathrm{i}\,\gamma_{22}) \tag{41}$$

with a second co-moving coordinate η_2 already defined in (36).

The iteration of this dressing procedure to the construction of higher-pole solutions ψ_N is now straightforward. The strategy is to choose pole locations μ_i (or velocities v_i) and real constants γ_{ik} and then rebuilt recursively in the order

$$\mu_i, \gamma_{ik} \;\to\; S_i \;\to\; T_i \;\to\; P_i \;\to\; \psi_N \;\to\; g_N \qquad \text{for} \quad i = 1, \ldots, N\;. \tag{42}$$

9 Noncommutative kinks

The N-pole solutions produced with the dressing method just outlined turn out to be noncommutative multi-solitons, i.e. they possess finite energy and approach their commutative cousins for $\theta\to 0$. Let me elaborate on the simplest case, $N=1$:

$$S(u,v) \;=\; \mathrm{e}^{-\alpha\,\eta\,\sigma_3}\,\big(\,\gamma_1\,\mathrm{i}\,\gamma_2\big) \;=\; \sqrt{|\gamma_1\gamma_2|}\;\mathrm{e}^{-\alpha\,(\eta-\eta_0)\sigma_3}\,\big(\,1\,\mathrm{i}\,\big)\;, \tag{43}$$

where $\eta_0=\frac{1}{2\alpha}\ln|\frac{\gamma_1}{\gamma_2}|$ determines the center of mass at t=0. For simplicity I put $\eta_0=0$ and calculate

$$T=\begin{pmatrix}\mathrm{e}^{-\alpha\eta}\\ \mathrm{i}\,\mathrm{e}^{\alpha\eta}\end{pmatrix} \quad\Rightarrow\quad P=\frac{1}{2\,\mathrm{ch}2\alpha\eta}\begin{pmatrix}\mathrm{e}^{-2\alpha\eta} & -\mathrm{i}\\ \mathrm{i} & \mathrm{e}^{+2\alpha\eta}\end{pmatrix} \quad\Rightarrow\quad g=\begin{pmatrix}\mathrm{th}2\alpha\eta & \frac{\mathrm{i}}{\mathrm{ch}2\alpha\eta}\\ \frac{\mathrm{i}}{\mathrm{ch}2\alpha\eta} & \mathrm{th}2\alpha\eta\end{pmatrix}. \tag{44}$$

Since the u, v dependence resides only in the single coordinate η, all $\star$ products trivialize and one effectively falls back on the θ=0 (and hence ρ=0) situation. Comparing g of (44) with the form introduced in (20), for ρ=0 and modulo an admissible constant rotation,

$$g \;=\; \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma_1\,\varphi} \;=\; \mathcal{E}\;\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma_3\,\varphi}\,\mathcal{E}^\dagger \qquad\text{for}\quad \mathcal{E}=\mathrm{e}^{-\mathrm{i}\,\frac{\pi}{4}\sigma_2}\;, \tag{45}$$

one reads off that

$$\cos\tfrac{\varphi}{2} \;=\; \mathrm{th}2\alpha\eta \qquad\text{and}\qquad \sin\tfrac{\varphi}{2} \;=\; \tfrac{1}{\mathrm{ch}2\alpha\eta} \qquad\Longrightarrow\qquad \tan\tfrac{\varphi}{4} \;=\; \mathrm{e}^{-2\alpha\eta}\;, \tag{46}$$

which indeed yields the standard sine-Gordon kink (3) with velocity $\mathrm{v}=\frac{1-\mu^2}{1+\mu^2}$ but no deformation. Note, however, that breathers and multi-solitons will get deformed because different co-moving coordinates do not commute,

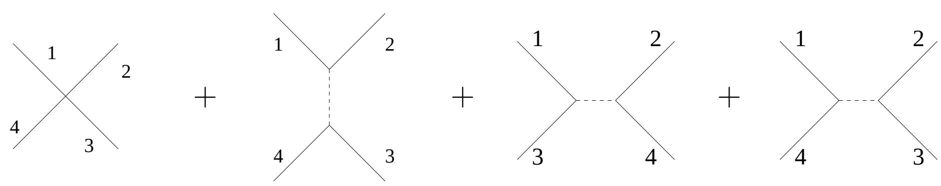
$$[\eta_i\,,\eta_k] \;=\; -\,\mathrm{i}\,\theta\,\frac{\mathrm{v}_i-\mathrm{v}_k}{\sqrt{(1-\mathrm{v}_i^2)(1-\mathrm{v}_k^2)}}\;. \tag{47}$$

10 Tree-level S-matrix

The noncommutative sine-Gordon equations (21) and (22) also follow from an action principle, which allows for a quick derivation of the Feynman rules. The two scalars φ and ρ have masses 2α and 0, respectively, and are coupled via an infinite sequence of higher-derivative interactions. As a constructive example I consider the $\varphi\varphi\to\varphi\varphi$ tree-level scattering amplitude, with the kinematics ($E^2-p^2=4\alpha^2$)

$$k_1=(E,p)\;,\quad k_2=(E,-p)\;,\quad k_3=(-E,p)\;,\quad k_4=(-E,-p)\;. \tag{48}$$

The sum of the relevant four-point diagrams



$$2\mathrm{i}\alpha^2\cos^2(\theta Ep) \;-\; \tfrac{\mathrm{i}}{2}p^2\sin^2(\theta Ep) \;+\; \tfrac{\mathrm{i}}{2}E^2\sin^2(\theta Ep) \;+\; 0$$

add up to $A_{\varphi\varphi\to\varphi\varphi} = 2\,\mathrm{i}\,\alpha^2$ which is causal! Likewise, one can show that all other 2→2 amplitudes vanish. Also, $\varphi\varphi \to \varphi\varphi\varphi\varphi$ and $\varphi\varphi\varphi \to \varphi\varphi\varphi$ do not occur, indicating the absence of particle production. The S-matrix appears to be causal and factorizable at tree level.

Most results presented here and all relevant references can be found in [1].

References

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Bosonization in Presence of an Impurity: Some New Features of Vertex Representations

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Abstract

Bosonization is extended to the case of a massless scalar field, interacting with a dissipationless point-like defect in two dimensional space-time. The main features of the corresponding vertex algebra are established. The vertex representation of the $\widehat{sl}(2)$ Kac-Moody algebra is constructed, and the interplay between the left and right sectors due to the interaction with the defect is studied in some detail.

1 Introduction

In [1] an algebraic-field theoretical treatment for quantum integrable systems in (1+1) dimensions with a reflecting and transmitting impurity has been proposed. Inspired by the factorized scattering, we had rather naturally been led to add a counterpart to the Zamolodchikov-Faddeev (ZF) algebra [2] in order to deal with such a kind of impurity. We called such a generalization of the ZF algebra Reflection-Transmission algebra, or RT algebra, in which distinguished elements, called reflection and transmission generators encode the particle-impurity interactions.

The underlying algebraic structure is described in detail in [1], where the relative Fock representations are explicitly constructed and a general factorized scattering theory developed in this framework. The quantum Yang-Baxter equations satisfied by the reflection and transmission generators are explored in more detail in [3], where their physical origin is explicited and general families of solutions are described with explicit representatives in each case. These results allowed to establish a direct relationship with previous works on the subject, making evident the advantages of the RT algebra as an universal approach to integrable systems with impurities. The particularly interesting case of the non-linear Schrödinger equation with point-like impurity has been considered along these lines in [4]: for this exactly solvable example with non-trivial bulk scattering matrix, a family of point-like impurities preserving integrability has been established and the construction of the exact second quantized solution of the model has been described in terms of an appropriate RT algebra.

The case of the interaction of a scalar field with impurities has also received a lot of attention. First we generalized our framework to finite temperature quantum field theory. Second, we remarked that RT algebras are well adapted for treating impurity problems in higher dimensional space-time and studied the interaction of a scalar field in $(s+1)+1$ dimensions with impurities localized on s-dimensional hyperplanes with $s \geq 0$. Such generalizations are obviously essential for realistic applications to condensed matter physics. As a direct application, we derived the energy density at finite temperature and established the correction to the Stefan-Boltzmann law generated by the impurity [5]. Let us add that, still in spaces of any dimension, the construction of quantum fields induced on a $(s+1)$-dimensional dissipationless defect by bulk fields propagating in a $(s+1)+1$ dimensional space has been achieved and the universal critical behavior of the underlying system determined [6].

But, coming back to (1+1) dimensions, the apparatus of RT algebra looks well adapted for studying bosonization when a dissipationless point-like defect is present. And that is the purpose of our contribution, the plan of which is the following. We start by gathering necessary tools about the massless scalar field and its dual interacting with a generic defect of the above type. Then, we construct the relative vertex operators, showing that they generally obey anyon statistics, thus describing anyon fields in the presence of defects. We go farther in our investigations by considering some aspects of non-abelian bosonization with impurities, describing the vertex operator construction of the $\widehat{sl}(2)$ Kac-Moody algebra, and establishing the interplay between the left and right sectors due to the interaction with the defect. The impact of the defect on the Virasoro algebra, constructed in the Sugawara representation, is also investigated. A more detailed presentation of these results can be found in [7] where, as an application of this framework, the vertex algebra with a defect is discussed and the massless Thirring model with impurity solved.

2 Massless scalar field with defect

Bosonization has a long history (see e. g. [8]). Since the main building blocks are the massless scalar field $\varphi(t,x)$ and its dual $\widetilde{\varphi}(t,x)$, our first step will be to establish the basic properties of $\{\varphi,\, \widetilde{\varphi}\}$ when a point-like dissipationless defect is present in space. Without loss of generality one can localize the defect at $x=0$ and consider thus the following equation of motion

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(t,x) = 0\,, \qquad x \neq 0\,, \tag{2.1}$$

with standard initial conditions fixed by the equal-time canonical commutation relations

$$[\varphi(0,x_1)\,,\, \varphi(0,x_2)] = 0\,, \qquad [(\partial_t\varphi)(0,x_1)\,,\, \varphi(0,x_2)] = -i\delta(x_1-x_2)\,. \tag{2.2}$$

The most general dissipationless interaction of $\varphi(t,x)$ with the defect at $x=0$ is described [9] by the boundary condition

$$\begin{pmatrix} \varphi(t,+0) \\ \partial_x\varphi(t,+0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi(t,-0) \\ \partial_x\varphi(t,-0) \end{pmatrix}\,, \quad \forall t \in \mathbb{R}\,, \tag{2.3}$$

where

$$ad - bc = 1\,, \qquad a,...,d \in \mathbb{R}\,. \tag{2.4}$$

We observe that a and d are dimensionless, whereas b and c have a non-trivial and opposite dimension.

The dual field $\widetilde{\varphi}(t,x)$ also satisfies

$$\left(\partial_t^2 - \partial_x^2\right) \widetilde{\varphi}(t,x) = 0\,, \qquad x \neq 0\,, \tag{2.5}$$

and as usual is related to $\varphi(t,x)$ by

$$\partial_t \widetilde{\varphi}(t,x) = -\partial_x \varphi(t,x)\,, \quad \partial_x \widetilde{\varphi}(t,x) = -\partial_t \varphi(t,x)\,, \qquad x \neq 0\,. \tag{2.6}$$

Eqs. (2.1)-(2.6) have a unique solution $\{\varphi\,,\widetilde{\varphi}\}$, which represents the basis for bosonization with a point-like defect. In the case when impurity bound states are absent[1], the solution $\{\varphi\,,\widetilde{\varphi}\}$ can be written in the form

$$\varphi(t,x) = \varphi_+(t,x) + \varphi_-(t,x)\,, \qquad \widetilde{\varphi}(t,x) = \widetilde{\varphi}_+(t,x) + \widetilde{\varphi}_-(t,x)\,, \tag{2.7}$$

where

$$\varphi_\pm(t,x) = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2|k|}} \left[a^{*\pm}(k) \mathrm{e}^{i|k|t - ikx} + a_\pm(k) \mathrm{e}^{-i|k|t+ikx}\right]\,, \tag{2.8}$$

$$\widetilde{\varphi}_\pm(t,x) = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk\, \varepsilon(k)}{2\pi\sqrt{2|k|}} \left[a^{*\pm}(k) \mathrm{e}^{i|k|t - ikx} + a_\pm(k) \mathrm{e}^{-i|k|t+ikx}\right]\,. \tag{2.9}$$

These expressions have the familiar form of superpositions of creation $a^{*\pm}(k)$ and annihilation $a_\pm(k)$ operators. The interaction with the impurity deforms only their commutation relations, which read now

$$a_{\xi_1}(k_1)\, a_{\xi_2}(k_2) - a_{\xi_2}(k_2)\, a_{\xi_1}(k_1) = 0\,, \tag{2.10}$$

$$a^{*\xi_1}(k_1)\, a^{*\xi_2}(k_2) - a^{*\xi_2}(k_2)\, a^{*\xi_1}(k_1) = 0\,, \tag{2.11}$$

$$\begin{gathered} a_{\xi_1}(k_1)\, a^{*\xi_2}(k_2) - a^{*\xi_2}(k_2)\, a_{\xi_1}(k_1) = \\ \left[\delta_{\xi_1}^{\xi_2} + T_{\xi_1}^{\xi_2}(k_1)\right] 2\pi\delta(k_1 - k_2)\, \mathbf{1} + R_{\xi_1}^{\xi_2}(k_1) 2\pi\delta(k_1+k_2)\, \mathbf{1}\,, \end{gathered} \tag{2.12}$$

where

$$R_+^+(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 + i(a+d)k - c}\,, \qquad R_-^-(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 - i(a+d)k - c}\,, \tag{2.13}$$

$$T_+^-(k) = \frac{2ik}{bk^2 + i(a+d)k - c}\,, \qquad T_-^+(k) = \frac{-2ik}{bk^2 - i(a+d)k - c}\,, \tag{2.14}$$

are the *reflection* and *transmission coefficients* from the impurity. The associated *reflection* and *transmission matrices*

$$R(k) = \begin{pmatrix} R_+^+(k) & 0 \\ 0 & R_-^-(k) \end{pmatrix}\,, \qquad T(k) = \begin{pmatrix} 0 & T_+^-(k) \\ T_-^+(k) & 0 \end{pmatrix}\,, \tag{2.15}$$

satisfy hermitian analyticity

$$R(k)^\dagger = R(-k)\,, \qquad T(k)^\dagger = T(k)\,, \tag{2.16}$$

[1]For the general case we refer to [5].

and unitarity

$$T(k)T(k) + R(k)R(-k) = \mathbb{I}\,, \tag{2.17}$$
$$T(k)R(k) + R(k)T(-k) = 0\,. \tag{2.18}$$

The exchange relations (2.10-2.12) deserve some comments. We observe first of all that (2.10-2.12) preserve the conventional initial conditions (2.2). In a slightly more general form the relations (2.10-2.12) appeared for the first time [1] in the context of integrable models with impurities. The associative algebra generated by $\{a^{*\pm}(k),\, a_\pm(k),\, \mathbf{1}\}$, satisfying (2.10-2.12) and the constraints

$$a_\xi(k) \;=\; T_\xi^\eta(k)a_\eta(k) + R_\xi^\eta(k)a_\eta(-k)\,, \tag{2.19}$$
$$a^{*\xi}(k) \;=\; a^{*\eta}(k)T_\eta^\xi(k) + a^{*\eta}(-k)R_\eta^\xi(-k)\,, \tag{2.20}$$

has been called reflection-transmission (RT) algebra because it translates the analytic boundary conditions (2.3) in algebraic terms, directly related to the physical reflection and transmission amplitudes (2.13, 2.14). For this reason RT algebras represent a natural and universal tool for studying QFT with defects [4, 5, 6] and it is not at all surprising that they appear also in the process of bosonization with impurities.

The derivation of the correlation functions of $\{\varphi\,,\widetilde{\varphi}\}$ in the Fock representation [1] of the RT algebra (2.10-2.12) is straightforward. It is convenient to change basis introducing the right and left chiral fields

$$\varphi_{_R}(t,x) = \varphi(t,x) + \widetilde{\varphi}(t,x)\,, \qquad \varphi_{_L}(t,x) = \varphi(t,x) - \widetilde{\varphi}(t,x)\,. \tag{2.21}$$

Inserting (2.7-2.9) in (2.21) one gets

$$\varphi_{_R}(t,x) = \theta(x)\varphi_{+R}(t-x) + \theta(-x)\varphi_{-R}(t-x)\,, \tag{2.22}$$
$$\varphi_{_L}(t,x) = \theta(x)\varphi_{+L}(t+x) + \theta(-x)\varphi_{-L}(t+x)\,, \tag{2.23}$$

where

$$\varphi_{\pm R}(\xi) = \int_0^{+\infty} \frac{dk}{\pi\sqrt{2k}} \left[a^{*\pm}(k)\mathrm{e}^{ik\xi} + a_\pm(k)\mathrm{e}^{-ik\xi}\right]\,, \tag{2.24}$$
$$\varphi_{\pm L}(\xi) = \int_0^{+\infty} \frac{dk}{\pi\sqrt{2k}} \left[a^{*\pm}(-k)\mathrm{e}^{ik\xi} + a_\pm(-k)\mathrm{e}^{-ik\xi}\right]\,. \tag{2.25}$$

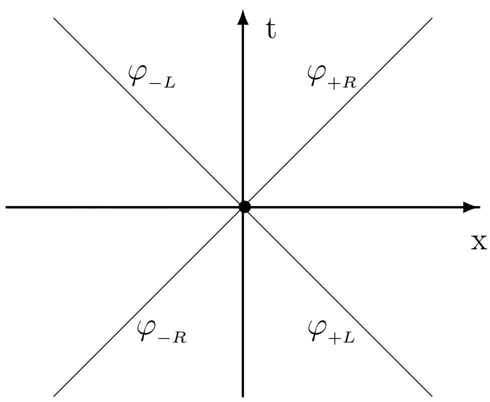

Fig. 1. The localization of $\varphi_{\pm R}$ and $\varphi_{\pm L}$ on the light cone.

The four components $\varphi_{\pm R}$ and $\varphi_{\pm L}$, whose localization is displayed on Fig. 1, couple each other through the defect at $x = 0$. This characteristic feature of our system is captured by the correlation functions of $\varphi_{\pm R}$ and $\varphi_{\pm L}$, we are going to derive now. Using (2.12) and the fact that $a_\pm(k)$ annihilate the Fock vacuum, one gets the following two-point functions

$$\langle \varphi_{+R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \langle \varphi_{-R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \\ \langle \varphi_{+L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{-L}(\xi_2)\rangle = u(\mu\xi_{12})\,, \tag{2.26}$$

where $\xi_{12} \equiv \xi_1 - \xi_2$, μ is a parameter with dimension of mass having a well-known infrared origin and

$$u(\xi) = -\frac{1}{\pi}\ln(|\xi|) - \frac{i}{2}\varepsilon(\xi) = -\frac{1}{\pi}\ln(i\xi + \epsilon)\,, \quad \epsilon > 0\,. \tag{2.27}$$

The correlators (2.26) do not depend on the defect and coincide with the familiar defect-free ones. This conclusion obviously holds also for the commutators

$$[\varphi_{+R}(\xi_1)\,,\, \varphi_{+R}(\xi_2)] = [\varphi_{-R}(\xi_1)\,,\, \varphi_{-R}(\xi_2)] = -i\varepsilon(\xi_{12})\,, \tag{2.28}$$

$$[\varphi_{+L}(\xi_1)\,,\, \varphi_{+L}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\, \varphi_{-L}(\xi_2)] = -i\varepsilon(\xi_{12})\,, \tag{2.29}$$

which follow directly from eqs. (2.26,2.27).

The defect shows up in the mixed correlation functions in the following way. The transmission relates the plus and minus components with the same chirality. One has for instance

$$\langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}T_+^-(k)\,. \tag{2.30}$$

The reflection instead relates different chiralities on the same half-line, for example

$$\langle \varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}R_+^+(k)\,, \tag{2.31}$$

Finally,

$$\begin{aligned}\langle \varphi_{+R}(\xi_1)\varphi_{-L}(\xi_2)\rangle &= \langle \varphi_{-L}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \\ \langle \varphi_{-R}(\xi_1)\varphi_{+L}(\xi_2)\rangle &= \langle \varphi_{+L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = 0\,.\end{aligned} \tag{2.32}$$

Although in a simpler form, the phenomenon of left-right and plus-minus mixing appears also in the case of boundary conformal field theory.

The integral representations (2.30,2.31) determine well-defined distributions which allow to analyze the locality properties of $\{\varphi,\, \widetilde{\varphi}\}$. Deriving the commutators at generic points t_1, x_1 and t_2, x_2 is quite hard. Fortunately however the computation drastically simplifies when the points are space-like separated. In fact, in the domain $t_{12}^2 - x_{12}^2 < 0$ one finds

$$[\varphi(t_1,x_1)\,,\, \varphi(t_2,x_2)] = [\widetilde{\varphi}(t_1,x_1)\,,\, \widetilde{\varphi}(t_2,x_2)] = 0\,, \tag{2.33}$$

$$[\varphi(t_1,x_1)\,,\, \widetilde{\varphi}(t_2,x_2)] = \frac{i}{2}[\varepsilon(x_{12}) + \varepsilon(\widetilde{x}_{12})]\theta(x_1x_2)\,, \tag{2.34}$$

where $\widetilde{x}_{12} \equiv x_1 + x_2$. Therefore, like in the case without defects, φ and $\widetilde{\varphi}$ are *local* fields, but *not relatively local*. As recognized already in the early sixties, this feature is the corner stone of bosonization.

Now we describe two concrete sets of parameters $\{a, b, c, d\}$, which nicely illustrate both the above general structure and the characteristic features of bosonization with defects.

a) **quasi-conformal defects;**

We start by considering the one-parameter family of defects

$$\{a = 1/\lambda,\, 0,\, 0,\, d = \lambda \neq 0\}\,. \tag{2.35}$$

Since the dimensional parameters b and c are set to 0, we call them quasi-conformal defects. These defects coincide with the permeable conformal walls introduced in [10]. From (2.13,2.14) one gets

$$R_+^+(k) = -R_-^-(k) = r(\lambda) \equiv \frac{1-\lambda^2}{1+\lambda^2}\,, \qquad T_+^-(k) = T_-^+(k) = 1 - r(\lambda)\,. \tag{2.36}$$

Accordingly, one has in addition to (2.26) the following non-trivial correlation functions

$$\begin{aligned} \langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle &= \langle \varphi_{-R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \\ \langle \varphi_{+L}(\xi_1)\varphi_{-L}(\xi_2)\rangle &= \langle \varphi_{-L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = [1-r(\lambda)]\, u(\mu\xi_{12})\,, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \langle \varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle &= -\langle \varphi_{-R}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \\ \langle \varphi_{+L}(\xi_1)\varphi_{+R}(\xi_2)\rangle &= -\langle \varphi_{-L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = r(\lambda)\, u(\mu\xi_{12})\,, \end{aligned} \tag{2.38}$$

which vanish in the conformal case. All correlators of the quasi-conformal defect are expressed in terms of the logarithm $u(\mu\xi)$ and the parameter λ. In addition to the universal (defect independent) commutators (2.28,2.29) one has:

$$[\varphi_{+R}(\xi_1)\,,\, \varphi_{-R}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\, \varphi_{+L}(\xi_2)] = -i[1-r(\lambda)]\,\varepsilon(\xi_{12})\,, \tag{2.39}$$

$$[\varphi_{+R}(\xi_1)\,,\, \varphi_{+L}(\xi_2)] = -[\varphi_{-L}(\xi_1)\,,\, \varphi_{-R}(\xi_2)] = -ir(\lambda)\,\varepsilon(\xi_{12})\,. \tag{2.40}$$

b) **δ-defects;**

As a second example we consider the impurities defined by

$$\{a = d = 1,\, b = 0,\, c = 2\eta > 0\}\,. \tag{2.41}$$

One usually refers to this one-parameter family as δ-defects, because they can be implemented by coupling φ to the external potential $U(x) = 2\eta\delta(x)$. The reflection and transmission coefficients take the form

$$R_+^+(k) = \frac{-i\eta}{k+i\eta}\,, \qquad R_-^-(k) = \frac{i\eta}{k-i\eta}\,, \tag{2.42}$$

$$T_+^-(k) = \frac{k}{k+i\eta}\,, \qquad T_-^+(k) = \frac{k}{k-i\eta}\,. \tag{2.43}$$

The two-point functions read

$$\langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = v_-(\eta\xi_{12})\,, \tag{2.44}$$

$$\langle \varphi_{-R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \langle \varphi_{+L}(\xi_1)\varphi_{-L}(\xi_2)\rangle = v_+(-\eta\xi_{12})\,, \tag{2.45}$$

$$\langle \varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = v_-(\eta\xi_{12}) - u(\mu\xi_{12})\,, \tag{2.46}$$

$$\langle \varphi_{-R}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \langle \varphi_{+L}(\xi_1)\varphi_{+R}(\xi_2)\rangle = v_+(-\eta\xi_{12}) - u(\mu\xi_{12})\,, \tag{2.47}$$

where u is defined by (2.27) and

$$v_\pm(\xi) \equiv -\frac{1}{\pi}\mathrm{e}^{-\xi}\,\mathrm{Ei}(\xi \pm i\epsilon)\,, \qquad \epsilon > 0\,, \tag{2.48}$$

Ei being the exponential-integral function. Recalling the expansion

$$\mathrm{Ei}(\xi \pm i\epsilon) = \gamma_E + \ln(\xi \pm i\epsilon) + \sum_{n=1}^{\infty} \frac{\xi^n}{n\cdot n!}\,, \tag{2.49}$$

we see that $u(\xi)$ and $v_\pm(\mp\xi)$ have the same logarithmic singularity in $\xi = 0$, confirming that the correlators (2.46,2.47) are not singular at $\xi_1 = \xi_2$.

From (2.44-2.47) one gets the commutators:

$$[\varphi_{+R}(\xi_1)\,,\,\varphi_{-R}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\,\varphi_{+L}(\xi_2)] = -2i\theta(\xi_{12})\mathrm{e}^{-\eta\xi_{12}}\,, \tag{2.50}$$

$$[\varphi_{+R}(\xi_1)\,,\,\varphi_{+L}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\,\varphi_{-R}(\xi_2)] = -2i\theta(\xi_{12})\mathrm{e}^{-\eta\xi_{12}} + i\varepsilon(\xi_{12})\,. \tag{2.51}$$

We stress that in deriving (2.44-2.47) we essentially used that $\eta > 0$. The correlators (2.44-2.47) are singular in the limit $\eta \to 0$, which forbids to recover from them the free case $\eta = 0$. Such type of discontinuity appears [11] also on the half-line between the scalar field quantized with Robin and Neumann boundary conditions.

3 Vertex operators in presence of defects

We have enough background at this point for constructing vertex operators. For any couple $\zeta \equiv (\alpha, \beta) \in \mathbb{R}^2$ we introduce the field

$$V(t,x;\zeta) =: \exp[i\sqrt{\pi}(\alpha\varphi + \beta\widetilde{\varphi})] : (t,x)\,, \tag{3.1}$$

where the normal ordering $:\ \ :$ is taken with respect to the creation and annihilation operators $\{a^{*\pm}(k),\, a_\pm(k)\}$. Like in the case without defect [12], the operators (3.1) generate an algebra $\mathcal{V}$. The exchange properties of the vertex operators $V(t,x;\zeta)$ determine their statistics. A standard calculation shows that

$$V(t_1,x_1;\zeta_1)V(t_2,x_2;\zeta_2) = \mathcal{E}(t_{12},x_1,x_2;\zeta_1,\zeta_2)\,V(t_2,x_2;\zeta_2)V(t_1,x_1;\zeta_1)\,, \tag{3.2}$$

the exchange factor $\mathcal{E}$ being

$$\mathcal{E}(t_{12},x_1,x_2;\zeta_1,\zeta_2) = \mathrm{e}^{-\pi[\alpha_1\varphi(t_1,x_1)+\beta_1\widetilde{\varphi}(t_1,x_1)\,,\,\alpha_2\varphi(t_2,x_2)+\beta_2\widetilde{\varphi}(t_2,x_2)]}\,. \tag{3.3}$$

The statistics of $V(t,x;\zeta)$ is determined by the value of (3.3) at space-like distances $t_{12}^2 - x_{12}^2 < 0$. By means of (2.33,2.34) one finds in this domain

$$\mathcal{E}(t_{12}, x_1, x_2; \zeta_1, \zeta_2) = \mathrm{e}^{\frac{i\pi}{2}[(\alpha_1\beta_2+\alpha_2\beta_1)\varepsilon(x_{12})+(\alpha_1\beta_2-\alpha_2\beta_1)\varepsilon(\widetilde{x}_{12})]\theta(x_1x_2)}\,. \tag{3.4}$$

Setting $\zeta_1 = \zeta_2 \equiv \zeta$ in (3.4) one obtains

$$\mathcal{E}(t_{12}, x_1, x_2; \zeta, \zeta) = \mathrm{e}^{i\pi\alpha\beta\varepsilon(x_{12})\theta(x_1x_2)}\,, \tag{3.5}$$

which governs the statistics of $V(t,x;\zeta)$.

It follows from (3.5) that the exchange properties of the vertex operators depend not only on the parameters $(\alpha,\ \beta)$, but also on the position. This is a new phenomenon in the context of bosonization, which has its origin in the breakdown of translation invariance by the impurity. The θ-factor in the exponent of (3.5) implies that two vertex operators localized on the opposite sides of the impurity are exchanged as bosons. However, when the vertex operators are localized on the same half-line, they behave as anyons with statistics parameter

$$\vartheta \equiv \alpha\beta\,. \tag{3.6}$$

For $\vartheta = 2k$ and $\vartheta = 2k+1$ with $k \in \mathbb{Z}$ one recovers Bose and Fermi statistics respectively. The remaining values of ϑ lead to abelian braid (anyon) statistics.

4 Non-abelian bosonization

Following the Frenkel-Kac construction [13] of the vertex representation of the affine Kac-Moody algebra $\widehat{sl}(2)$, we introduce the operators

$$H_{\epsilon Z}(\xi) = \sqrt{\pi}\partial_\xi\varphi_{\epsilon Z}(\xi)\,, \qquad E^{\pm}_{\epsilon Z}(\xi) = \mu : \mathrm{e}^{\pm i\sqrt{2\pi}\varphi_{\epsilon Z}(\xi)} :\,, \tag{4.1}$$

where $\epsilon = \pm$, $Z = L, R$. Using eqs. (2.28,2.29), *for fixed* $\{\epsilon,\ Z\}$ one gets the well-known $\widehat{sl}(2)$ commutation relations:

$$[H_{\epsilon Z}(\xi_1)\,,\ H_{\epsilon Z}(\xi_2)] = 2\pi i\,\delta'(\xi_{12})\mathbb{I}\,, \tag{4.2}$$

$$[H_{\epsilon Z}(\xi_1)\,,\ E^{\pm}_{\epsilon Z}(\xi_2)] = \pm 2\pi\,\delta(\xi_{12})\sqrt{2}\,E^{\pm}_{\epsilon Z}(\xi_2)\,, \tag{4.3}$$

$$[E^{+}_{\epsilon Z}(\xi_1)\,,\ E^{-}_{\epsilon Z}(\xi_2)] = 2\pi i\,\delta'(\xi_{12})\mathbb{I} + 2\pi\,\delta(\xi_{12})\,H_{\epsilon Z}(\xi_1)\,, \tag{4.4}$$

$$[E^{+}_{\epsilon Z}(\xi_1)\,,\ E^{+}_{\epsilon Z}(\xi_2)] = [E^{-}_{\epsilon Z}(\xi_1)\,,\ E^{-}_{\epsilon Z}(\xi_2)] = 0\,. \tag{4.5}$$

In this way one recovers four vertex representations $\{\varrho_{\epsilon Z}\,:\, \epsilon = \pm,\ Z = R,\ L\}$ of $\widehat{sl}(2)$.

We observe that ϱ_{+R} and ϱ_{-L} as well as ϱ_{-R} and ϱ_{+L} commute because of (2.32). However, since $\{\varphi_{\epsilon Z}\}$ interact among themselves, there is a non-trivial interplay among the other four pairs $\{\varrho_{+R},\ \varrho_{+L}\}$, $\{\varrho_{-R},\ \varrho_{-L}\}$, $\{\varrho_{+R},\ \varrho_{-R}\}$ and $\{\varrho_{+L},\ \varrho_{-L}\}$ of representations. For a generic defect $\{a,\ b,\ c,\ d\}$ the commutator of two generators belonging to $\varrho_{\epsilon_1 Z_1}$ and $\varrho_{\epsilon_2 Z_2}$ is in general a *bilocal* operator of the type

$$B^{\pm\ \pm}_{\epsilon_1 Z_1,\,\epsilon_2 Z_2}(\xi_1, \xi_2) \equiv : \mathrm{e}^{\pm i\sqrt{2\pi}\varphi_{\epsilon_1 Z_1}(\xi_1) \pm i\sqrt{2\pi}\varphi_{\epsilon_2 Z_2}(\xi_2)} :\ . \tag{4.6}$$

It turns out that the mixed commutators within the pairs $\{\varrho_{+R},\, \varrho_{+L}\}$, $\{\varrho_{-R},\, \varrho_{-L}\}$, $\{\varrho_{+R},\, \varrho_{-R}\}$ and $\{\varrho_{+L},\, \varrho_{-L}\}$ have all the same structure. For illustrating the latter we consider the commutators between ϱ_{+R} and ϱ_{+L}. One finds

$$[H_{+R}(\xi_1)\,,\, H_{+L}(\xi_2)] = i\partial_{\xi_1} f(\xi_{12})\mathbb{I}\,, \tag{4.7}$$

$$[H_{+R}(\xi_1)\,,\, E^{\pm}_{+L}(\xi_2)] = \pm f(\xi_{12})\sqrt{2}\, E^{\pm}_{+L}(\xi_2)\,, \tag{4.8}$$

$$[H_{+L}(\xi_1)\,,\, E^{\pm}_{+R}(\xi_2)] = \mp f(-\xi_{12})\sqrt{2}\, E^{\pm}_{+R}(\xi_2)\,, \tag{4.9}$$

$$[E^{+}_{+R}(\xi_1)\,,\, E^{+}_{+L}(\xi_2)] = g_+(\xi_{12}) B^{+\ \ +}_{+R,+L}(\xi_1, \xi_2)\,, \tag{4.10}$$

$$[E^{-}_{+R}(\xi_1)\,,\, E^{-}_{+L}(\xi_2)] = g_+(\xi_{12}) B^{-\ \ -}_{+R,+L}(\xi_1, \xi_2)\,, \tag{4.11}$$

$$[E^{+}_{+R}(\xi_1)\,,\, E^{-}_{+L}(\xi_2)] = g_-(\xi_{12}) B^{+\ \ -}_{+R,+L}(\xi_1, \xi_2)\,, \tag{4.12}$$

$$[E^{-}_{+R}(\xi_1)\,,\, E^{+}_{+L}(\xi_2)] = g_-(\xi_{12}) B^{-\ \ +}_{+R,+L}(\xi_1, \xi_2)\,, \tag{4.13}$$

where f and $g_\pm$ are some functions depending on the defect and thus on the parameters $\{a,\, b,\, c,\, d\}$. For the quasi-conformal defects (2.35) one has

$$f(\xi) = 2\pi r(\lambda)\delta(\xi)\,, \qquad r(\lambda) = \frac{1-\lambda^2}{1+\lambda^2}\,, \tag{4.14}$$

$$g_\pm(\xi) = \pm 2i\, \mu^{2\pm r(\lambda)}\, \sin[\pi r(\lambda)]\, \varepsilon(\xi)\, |\xi|^{\pm 2r(\lambda)}\,. \tag{4.15}$$

The δ-defects (2.41) lead instead to

$$f(\xi) = -2\pi\, \eta\, \theta(\xi) \mathrm{e}^{-\eta\xi}\,, \tag{4.16}$$

$$g_\pm(\xi) = \pm 2i\, \mu^2\, \sin\left(2\pi \mathrm{e}^{-\eta\xi}\right)\, \theta(\xi)\, \mathrm{e}^{\pm\gamma(\xi;\eta,\mu)}\,, \tag{4.17}$$

with

$$\gamma(\xi;\eta,\mu) = 2\left[\mathrm{e}^{-\eta\xi}\left(\gamma_E + \ln(\eta|\xi|) + \sum_{n=1}^{\infty}\frac{(\eta\xi)^n}{n\cdot n!}\right) - \ln(\mu|\xi|)\right]\,. \tag{4.18}$$

The commutators (4.7-4.11) deserve some comments. Like in (4.2), the commutator of the left and right Cartan generators is proportional to the identity operator $\mathbb{I}$. A first novelty is the central extension multiplication factor $i\partial_{\xi_1} f(\xi_{12})$, which is different and depends on the defect. The commutation of Cartan generators with step operators reproduces the latter up to a factor which, in analogy with (4.3), is the integral of the central extension factor. Finally, the commutation of step operators leads, up to the structure functions $g_\pm$, to the bilocal operators (4.6).

It is perhaps useful to recall that the representations $\{\varrho_{\epsilon Z}\}$ of $\widehat{sl}(2)$ have a direct physical application. They describe the symmetry content of the $SU(2)$-invariant massless Thirring model with a δ-impurity [7].

Let us consider now the energy-momentum tensor Θ of the quantum field φ interacting with the defect [5]. The chiral components

$$\Theta_Z(x,\xi) = \theta(-x)\Theta_{-Z}(\xi) + \theta(x)\Theta_{+Z}(\xi) \tag{4.19}$$

of Θ can be expressed in terms of the generators $H_{\epsilon Z}$ by means of

$$\Theta_{\epsilon Z}(\xi) = \frac{1}{2\pi} : H_{\epsilon Z}\, H_{\epsilon Z} : (\xi)\,, \tag{4.20}$$

which is precisely the Sugawara representation [13]. As expected, *for fixed* $\{\epsilon, Z\}$ one finds

$$[\Theta_{\epsilon Z}(\xi_1)\,,\,\Theta_{\epsilon Z}(\xi_2)] = 2i\delta^{'}(\xi_{12})\Theta_{\epsilon Z}(\xi_1) - \frac{i}{6\pi}\delta^{'''}(\xi_{12})\,\mathbb{I}\,, \tag{4.21}$$

From the properties of $H_{\epsilon Z}$ one infers that Θ_{+R} commutes with Θ_{-L} and Θ_{-R} commutes with Θ_{+L}. The remaining commutators are however non-trivial. In the quasi-conformal case one finds for instance

$$[\Theta_{+R}(\xi_1)\,,\,\Theta_{+L}(\xi_2)] = i\delta^{'}(\xi_{12})r(\lambda)\,[\Theta_{+R,\,+L}(\xi_1) + \Theta_{+L,\,+R}(\xi_1)] - \frac{ir(\lambda)^2}{6\pi}\delta^{'''}(\xi_{12})\mathbb{I}\,, \tag{4.22}$$

where

$$\Theta_{\epsilon_1 Z_1,\,\epsilon_2 Z_2}(\xi) = \frac{1}{2\pi}\,:H_{\epsilon_1 Z_1}\,H_{\epsilon_2 Z_2}:(\xi)\,. \tag{4.23}$$

The appearance of *mixed* Sugawara terms of the type (4.23) is a new feature, which has once more its origin in the left-right and plus-minus mixing due to the defect. We observe also that the commutator (4.22) has a central term, the central charge being renormalized by a factor of $r(\lambda)^2$ with respect to (4.21). One might be tempted to change the normalization of $\Theta_{\epsilon Z}$ in order to eliminate all factors $r(\lambda)$ from the right hand side of (4.22), but then the inverse of this factor will appear in (4.21).

It is worth mentioning that the operators (4.20,4.23) close actually an algebra. A straightforward but long computation using the RT algebra relations (2.10-2.12) gives in fact

$$[\Theta_{+Z}(\xi_1)\,,\,\Theta_{+R,\,+L}(\xi_2)] = i\delta^{'}(\xi_{12})\,[r(\lambda)\Theta_{+Z}(\xi_1) + \Theta_{+R,\,+L}(\xi_1)] - \frac{ir(\lambda)}{6\pi}\delta^{'''}(\xi_{12})\mathbb{I}\,, \tag{4.24}$$

and

$$\begin{gathered}[\Theta_{+R,\,+L}(\xi_1)\,,\,\Theta_{+R,\,+L}(\xi_2)] = \\ i\delta^{'}(\xi_{12})\,[\Theta_{+R}(\xi_1) + r(\lambda)\Theta_{+R,\,+L}(\xi_1) + \Theta_{+L}(\xi_1)] - \frac{i[r(\lambda)^2+1]}{12\pi}\delta^{'''}(\xi_{12})\mathbb{I}\,,\end{gathered} \tag{4.25}$$

which complete the picture in the quasi-conformal case. Like in the Kac-Moody algebra, for more general defects the commutators (4.22,4.24,4.25) involve bilocal operators which are now of the form

$$\Theta_{\epsilon_1 Z_1,\,\epsilon_2 Z_2}(\xi_1,\xi_2) = \frac{1}{2\pi}\,:H_{\epsilon_1 Z_1}(\xi_1)\,H_{\epsilon_2 Z_2}(\xi_2):\,. \tag{4.26}$$

Summarizing, we have shown in this section how some familiar structures from conformal field theory are modified by the presence of a point-like impurity, which preserves unitarity and locality. Together with the left-right mixing, a relevant characteristic feature is the appearance of bilocal operators.

We conclude by observing that the above construction of the $\widehat{sl}(2)$ Kac-Moody algebra and the Sugawara representation of the energy-momentum tensor can be extended to the case of $\widehat{sl}(n)$ with $n > 2$.

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Relativistic Coulomb Problem for Massive Charged Particles with Arbitrary Half-Integer Spin

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Abstract

Using new relativistic tensor-bispinorial equations proposed in Phys. Rev. D **64**, 125013 (2001) we solve the Coulomb problem for massive charged particles with arbitrary half-integer spin (for details see hep-th/0412214).

1 Introduction

One of the triumphs of the Dirac electron theory consists in the fact that it predicts the electron motion in the field generated by a Coulomb potential. The corresponding exact Sommerfeld formula for the associated energy levels belongs to foundation stones of the relativistic quantum theory. However the extension of this result to the case of charged particles with higher spins appeared to be very complicated. Already for spin $s = 1$ the corresponding relativistic wave equation (RWE) namely the Kemmer-Duffin equation for vector bosons is not satisfactory and predicts the orbital particle will falls down into the attractive center [3], [4]. The other serious problems with RWE for particles with higher spins are connected with violation of causality [5], ill-defined interaction with a constant and homogeneous external magnetic field [6], and so on (see [1], [7]).

In paper [1] new RWE for particles of arbitrary half-integer spin were proposed which are free of all above mentioned inconsistencies. They are causal, allow the correct value of the gyromagnetic ratio $g = 2$ and have well defined quasi-relativistic limits which admit a good physical interpretation and describe the Pauli, spin-orbit and Darwin couplings.

In the present talk we describe how the tensor-spinorial equations [1] can be used to solve the Coulomb problem for massive particles with *arbitrary* half-integer spins.

2 Relativistic wave equations for massive particles with arbitrary half-integer spins

The new RWE has the following form [1]

$$
\begin{gathered}
\left(\gamma_\lambda p^\lambda - m\right)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} \\
-\tfrac{1}{4s}\Sigma_{\mathcal{P}}\left(\gamma^{\mu_1}\gamma^{\nu_1} - \gamma^{\nu_1}\gamma^{\mu_1}\right)p_\lambda\gamma_\sigma\psi^{[\lambda\sigma][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0.
\end{gathered}
\tag{1}
$$

Here γ_ν are Dirac matrices, $p^\mu = i\frac{\partial}{\partial x_\mu}$, m is the mass of particle with arbitrary half integer spin and the symbol $\Sigma_{\mathcal{P}}$ denotes the sum over all possible permutations of subindices $(2,\cdots,n)$ with 1.

The corresponding wave function $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ is an irreducible tensor with respect to the complete Poincaré group of rank $2n = 2s-1$ antisymmetric w.r.t. permutations of indices in the square brackets and symmetric w.r.t. permutations of pairs of indices $[\mu_i,\nu_i] \iff [\mu_j,\nu_j],\ i,j = 1,2,\cdots,n$. The irreducibility requirement means also that convolutions w.r.t. any pair of indices and cyclic permutations of any triplet of indices of the wave function reduce it to zero. In addition, components of $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ are bispinors of rank 1. This means that the wave function has an additional spinorial index α (which we omit) running from 1 to 4.

In addition, the wave function $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ has to satisfy the static constraint [1]

$$
\gamma_\mu\gamma_\nu\psi^{[\mu\nu][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0,
\tag{2}
$$

which is necessary to reduce the number of independent components of the tensor-spinor from $16s$ to $4(2s+1)$. In order to obtain a theoretically required $2(2s+1)$-component wave function we impose on $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ additionally either Majorana condition or a parity-violating constraint $(1+i\gamma_5)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0$ (or $(1-i\gamma_5)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0$).

Equations (1), (2) are manifestly invariant with respect to the complete Poincaré group and admit the Lagrangian formulation. For the case of a charged particle interacting with an external electromagnetic field equation (1) is generalized to a rather complicated one which is equivalent to the following second-order equation [1]:

$$
\left(\pi_\mu\pi^\mu - m^2 - \tfrac{i(k+1)}{2s}S_{\mu\nu}F^{\mu\nu}\right)\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0
\tag{3}
$$

where

$$
\begin{gathered}
\psi_\pm^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = \psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} \\
\pm\tfrac{1}{2n}\gamma_5\Sigma_{\mathcal{P}}\varepsilon^{\mu_1\nu_1}{}_{\lambda\sigma}\psi^{[\lambda\sigma][\mu_2\nu_2][\mu_3\nu_3]\cdots[\mu_n\nu_n]},
\end{gathered}
$$

$\pi_\lambda = p_\lambda - eA_\lambda,\ A_\lambda$ and $F^{\mu\nu}$ are vector-potential and tensor of the external electromagnetic field, k is an arbitrary parameter, $S_{\mu\nu}$ are generators of the Lorentz group whose action on the antisymmetric tensor-spinor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ is given by the following formula

$$
\begin{gathered}
S^{\mu\nu}\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} \\
+i\Sigma_{\mathcal{P}}\Big(g^{\mu\mu_1}\psi_+^{[\nu\nu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]} - g^{\nu\mu_1}\psi_+^{[\mu\nu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]} \\
-g^{\mu\nu_1}\psi_+^{[\nu\mu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]} + g^{\nu\nu_1}\psi_+^{[\mu\mu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]}\Big),
\end{gathered}
\tag{4}
$$

where $g^{\mu\nu}$ is the metric tensor with signature $(+,-,-,-)$.

In accordance with its definition tensor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ transforms according the representation $D(s-1/2,0)\otimes D(1/2,0)\oplus D(0,s-1/2)\otimes D(0,1/2)\equiv D(s,0)\oplus D(s-1,0)\oplus D(0,s)\oplus D(0,s-1)$ of the Lorentz group. Moreover, condition (2) reduces this representation to $D(s,0)\oplus D(0,s)$ whose generators without loss of generality can be expressed as

$$S_{ab}=\varepsilon_{abc}S_c,\quad S_{0a}=i\hat{\varepsilon}S_a,\ a,b=1,2,3,$$

where S_a are matrices forming a direct sum of two irreducible representation $D(s)$ of algebra $so(3)$, ε_{abc} is totally antisymmetric unit tensor of rank 3 and $\hat{\varepsilon}$ is an involutive matrix distinct from the unit one and commuting with S_a. This matrix can be expressed via the Casimir operators $C_1=S_{\mu\nu}S^{\mu\nu}$ and $C_2=\frac{1}{4}\varepsilon_{\mu\mu\lambda\sigma}S^{\mu\nu}S^{\lambda\sigma}$ of the Lorentz group:

$$\hat{\varepsilon}=C_1C_2^{-1}. \tag{5}$$

Thus instead of (3) we can study the equivalent equation

$$\left(\pi_\mu\pi^\mu-m^2-\tfrac{i(k+1)}{2s}\hat{\varepsilon}S_a(iF^{0a}+\tfrac{1}{2}\varepsilon_{0abc}F^{bc})\right)\Psi=0 \tag{6}$$

where Ψ is a $2(2s+1)$-component spinor belonging to the space of irreducible representation $D(s,0)\oplus D(0,s)$ of the Lorentz group, S_a ($a=1,2,3$) are direct sums of two $(2s+1)\times(2s+1)$ matrices forming the irreducible representation $D(s)$ of algebra $so(3)$. The components of the related tensor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ can be expressed via components of spinor Ψ by using the Wigner coefficients (see [2]).

Taking into account commutativity of matrix $\hat{\varepsilon}$ with matrices S_a equation (4) can be decoupled to two subsystems

$$\left(\pi_\mu\pi^\mu-m^2-\tfrac{i(k+1)}{2s}\varepsilon S_a(iF^{0a}+\tfrac{1}{2}\varepsilon_{0abc}F^{bc})\right)\Psi_\varepsilon=0 \tag{7}$$

where Ψ_ε are eigenvectors of operator $\hat{\varepsilon}$ defined in (5) corresponding to the eigenvalues $\varepsilon=\pm1$.

Thus, equation (3) can be reduced to equation (7) which is very convenient and is easy to handle in the important case when the external field is generated by a point charge.

3 Radial equations for the Coulomb problem

Consider a charged particle with an arbitrary half-integer spin s and electric charge e interacting with an external electromagnetic field. When this field is generated by a point charge Ze the related vector-potential has the form

$$\mathbf{A}=0,\quad A_0=\frac{\alpha}{r}, \tag{8}$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$ and $\alpha=Ze^2$.

Now the equations of motion (3) up to (7) will be used. Since both equations (7) corresponding to $\varepsilon=1$ and to $\varepsilon=-1$ lead to the same energy spectrum we shall consider only the case $\varepsilon=1$ and omit index ε at function Ψ_ε from now on.

For the states with energy E the corresponding solutions Ψ of (7)an be written as

$$\Psi=\exp(-iEx_0)\psi(\mathbf{r}), \tag{9}$$

where $\psi(\mathbf{r})$ is a $(2s+1)$-component function depending on spatial variables and satisfying the following second order equation

$$\left(E - \frac{\alpha}{r}\right)^2 \psi = \left(m^2 - \Delta + ik\alpha\frac{\mathbf{S}\cdot\mathbf{r}}{r^3}\right)\psi. \tag{10}$$

Taking into account the rotational invariance of equation (10) it is convenient to expand its solutions in terms of spherical spinors $\Omega^s_{j\ l\ m}$:

$$\psi = \xi_\lambda(r)\Omega^s_{j\ j-\lambda\ m}, \tag{11}$$

where $\Omega^s_{j\ l\ m}$ are orthonormalized joint eigenvectors of the following four commuting operators: of total angular momentum square J^2, orbital momentum square L^2, spin square S^2 and of the third component of the total angular momentum J_3, whose eigenvalues are $j(j+1),\ l(l+1),\ s(s+1)$ and m respectively. Denoting $l = j - \lambda$ we receive

$$m = -j, -j+1, \cdots, j,$$

and

$$\lambda = -s, -s+1, \cdots - s + 2m_{sj},$$

where $m_{sj} = s$ if $s \leq j$ and $m_{sj} = j$ if $s > j$.

The expressions for spherical spinors via spherical functions are given in the Appendix of [2].

We note that the action of the scalar matrix $\mathbf{S}\cdot\mathbf{r}$ to the spinors $\Omega_{j\ j-\lambda\ m}$ is well defined and given by the formula

$$\mathbf{S}\cdot\mathbf{r}\ \Omega_{j\ j-\lambda\ m} = rK^{sj}_{\lambda\lambda'}\Omega_{j\ j-\lambda'\ m}, \tag{12}$$

where $K^{sj}_{\lambda\lambda'}$ are numerical coefficients whose values are presented in the Appendix of [2].

Substituting (11) into (10), using (12) and the following representation for the Laplace operator Δ

$$\Delta = \frac{1}{r^2}\left(\frac{\partial}{\partial x}\left(r^2\frac{\partial}{\partial x}\right) - L^2\right), \tag{13}$$

where L^2 is the square of the orbital momentum operator $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, we receive the following equations for the radial functions

$$F\xi_\lambda = \frac{1}{r^2}M_{\lambda\lambda'}\xi_{\lambda'}, \tag{14}$$

where F is the second order differential operator

$$F = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (E + \alpha/r)^2 - m^2 - \frac{j(j+1)}{r^2} \tag{15}$$

and M is a matrix whose elements are

$$M_{\lambda\lambda'} = \lambda(\lambda - 2j - 1)\delta_{\lambda\lambda'} + ig\alpha K^{sj}_{\lambda\lambda'}. \tag{16}$$

Formula (14) presents the equation for the radial wave function of a particle with arbitrary half-integer spin interacting with the Coulomb field.

4 Energy spectrum

Matrix M is *normal*, i.e., it satisfies the condition $MM^\dagger = M^\dagger M$. Thus it is possible to diagonalize it using some invertible matrix U:

$$M \to \tilde{M} = UMU^{-1}, \quad \tilde{M}_{\lambda\lambda'} = \delta_{\lambda\lambda'}\nu_\lambda \tag{17}$$

(where ν_λ are eigenvalues of M) thus system (14) is reduced to the sequence of decoupled equations

$$F\tilde{\xi}^\lambda = \frac{1}{r^2}\nu_\lambda\tilde{\xi}^\lambda \ \text{ (no sum over } \lambda) \tag{18}$$

where $\tilde{\xi}^\lambda$ is a λ component of vector $\tilde{\xi} = U\xi$.

Changing the variables $r \to \rho = 2\sqrt{m^2 - E^2}r$, $\tilde{\xi} \to f = \sqrt{\frac{\rho}{m^2-E^2}}\tilde{\xi}$ equation (18) is transformed to the well known form

$$\rho\frac{d^2 f}{d\rho^2} + \frac{df}{d\rho} + \left(\beta - \frac{\rho}{4} - \frac{k_\lambda^2}{4\rho}\right) f = 0, \tag{19}$$

where

$$\beta = \frac{\alpha E}{\sqrt{m^2 - E^2}}, \quad k_\lambda^2 = (2j+1)^2 + 4\nu_\lambda - 4\alpha^2. \tag{20}$$

For the bound states, i.e., for $m^2 > E^2$ solutions of (19) can be expressed via degenerated hypergeometric function $\mathcal{F}(\tilde{n}, d, \rho)$ as

$$f = C\rho^{\frac{k_\lambda}{2}} \exp\left(-\frac{\rho}{2}\right)\mathcal{F}\left(\frac{k_\lambda + 1}{2} - \beta, k_\lambda + 1, \rho\right), \tag{21}$$

where C is an integration constant.

Solutions (21) are bounded at infinity provided the argument $\tilde{n} = \frac{k_\lambda+1}{2} - \beta$ is a non-positive integer, i.e., $\tilde{n} = -n' = 0, -1, -2, \cdots$. Then from (20) we obtain the possible values of energy for bound states:

$$E = m\left(1 + \frac{\alpha^2}{\left((n' + 1/2 + k_\lambda)^2 - \alpha^2\right)^{\frac{1}{2}}}\right)^{-\frac{1}{2}}. \tag{22}$$

Here k_λ are parameters defined in expression (20), where ν_λ takes the values which coincide with the roots of the characteristic equation for matrix M:

$$\det(M - \nu_\lambda I) = 0, \tag{23}$$

where I is the unit matrix of the appropriate dimension $D = 2s+1$ for $s \le j$ or $D = 2j+1$ for $j \le s$.

Thus we have found the exact values of energy levels for the Coulomb system for the orbital particle having arbitrary half-integer spin. However, formulae (22), (20) include parameter ν_λ defined with the help of the algebraic equation (23) of order D which can be solved in radicals for $j \le 3/2$ or $s = 3/2$. For other values of s and j the formula (23) defines an algebraic equation whose order is larger than 4, which does not have exact analytic solutions. The related possible values of ν_λ should be calculated numerically.

5 Discussion

We have obtained the generalized Sommerfeld formula (22) for energy levels of particles with arbitrary half-integer spins s interacting with the Coulomb potential. In contrast with the formula generated by the Dirac equation our expression (22) includes parameter g whose value is not fixed *a priori*. In accordance with the analysis present in [1] and [2] this parameter is associated with the gyromagnetic ratio of a particle described by equation (3).

Analyzing formula the approximation of (22) up to order $1/m^2$ according to [2] we conclude that in addition to the most popular values $g = 1/s$ and $g = 2$ there exist one more intriguing value, namely $g = \sqrt{2/s}$ which corresponds to a specific degeneracy of the related energy spectrum. We notice that in the case $s = 1/2$ all mentioned privileged values of the gyromagnetic ratio coincide while for $s > 1/2$ the relation $\frac{1}{s} < \sqrt{\frac{2}{s}} < 2$ is valid. In other words the degeneracy related value of g lies between the recognized values $1/s$ and 2.

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Elements of Fedosov Geometry in Lagrangian BRST Quantization

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Abstract

A Lagrangian formulation of the BRST quantization of generic gauge theories in general irreducible non-Abelian hypergauges is proposed on the basis of the multilevel Batalin–Tyutin formalism and a special BV–BFV dual description of a reducible gauge model on the symplectic supermanifold $\mathcal{M}_0$ locally parameterized by the antifields for Lagrangian multipliers and the fields of the BV method. The quantization rules are based on a set of nilpotent anticommuting operators $\Delta^{\mathcal{M}}, \mathcal{V}^{\mathcal{M}}, \mathcal{U}^{\mathcal{M}}$ defined through both odd and even symplectic structures on a supersymplectic manifold $\mathcal{M}$ locally representable as an odd (co)tangent bundle over $\mathcal{M}_0$ provided by the choice of a flat Fedosov connection and a non-symplectic metric on $\mathcal{M}_0$ compatible with it. The generating functional of Green's functions is constructed in general coordinates on $\mathcal{M}$ with the help of operators of contracting homotopies for $\mathcal{V}^{\mathcal{M}}$ and $\mathcal{U}^{\mathcal{M}}$. We prove the gauge independence of the S-matrix and derive the Ward identity.

1. Introduction The conventional form of the Lagrangian [1] ([2]) and Hamiltonian [3] ([4]) quantization schemes for general gauge theories realizing the BRST (BRSTantiBRST) symmetry was developed about 15–20 years ago and is sufficient, as a whole, for a perturbative quantization of gauge models formulated on the basis of the variational principle. Despite this fact, it is still of interest to investigate a number of additional problems related, first of all, to geometrically covariant formulations of quantization procedures reflecting global and invariant properties of the manifold of field variables in specific models. This activity was initiated by the Lagrangian multilevel formalism [5] as well as by the (modified) triplectic [6] ([7]) BRSTantiBRST scheme. On the other hand, it is closely related, through the notion of supertime $\chi = (t, \theta)$, with the problem of an equitable representation of the dynamics and BRST transformations of a given model within the superfield Lagrangian [8, 9, 10] and Hamiltonian [11, 12] quantization schemes.

A solution to the first series of problems w.r.t. general aspects of quantization is the construction of a $\star$-product by Kontsevich [13] within the deformation quantization for an arbitrary Poisson manifold $\mathcal{M}_{\mathrm{P}}$ for which in [14] one applies a topological Poisson σ-model defined on $\mathcal{M}_{\mathrm{P}}$, whose field-antifield spectrum, following the AKSZ approach [15], in an $N = 2$ superfield formulation coincides with the BV method corresponding set of fields and antifields. One of such problems is the construction of a deformation quantization for dynamical systems with second-class constraints and symplectic manifolds [16], as well as for non(Lagrangian)Hamiltonian gauge theories [17], in which one essentially uses a symmetric connection compatible with the symplectic structure, i.e., the Fedosov connection [18].

While the Lagrangian BRSTantiBRST quantization [19] defined in general coordinates preserves the tensor character of compatible differential operations, i.e., extended antibrackets, odd operators $(\Delta^a, \mathcal{V}^a, \mathcal{U}^a), a = 1, 2$, only in the case of a *flat* Fedosov connection on the supermanifold $\widetilde{\mathcal{M}}_0$ in Darboux coordinates parameterized by the fields ϕ^A of the BV method and by the corresponding antifields $\overline{\phi}_A$, being the sources to the commutator of BRSTantiBRST transformations [2], the construction of Lagrangian BRST quantization in irreducible [5] and reducible (introduced in [8]) non-Abelian hypergauges in fact does not use the concept of Fedosov connection. It has been shown by the local superfield BRST quantization [8, 9] that using merely the ingredients of the *first-level* formalism, in view of a special presence of Lagrangian multipliers λ^a for hypergauges $G_a(\Gamma)$, is insufficient for an introduction of a covariant derivative on $\mathcal{N}$. For this purpose, one must use not only λ^a but also the antifields λ^*_a appearing in the *second-level* formalism [5].

The principal goals of this report are the following:

1. Description of the gauge algebra of a reducible gauge model (RGM) by means of a *special BV–BFV duality* between odd $\mathcal{N}_{\min}$ and even $\mathcal{M}_{0\min}$ symplectic supermanifolds underlying the quantization procedure and intersecting w.r.t. the manifold parameterized by the minimal-sector fields of the BV method.
2. Investigation of a supersymplectic structure on the quantization manifold $\mathcal{M}$[1] compatible with the requirements of anticommutation for the set of operators $\Delta^{\mathcal{M}}$, $V^{\mathcal{M}}$, $U^{\mathcal{M}}$.
3. Formulation of quantization rules for gauge models in general coordinates on $\mathcal{M}$ with an essential use of operators $\mathcal{V}^*$,$\mathcal{U}^*$ (constructed in terms of both even and odd Poisson brackets (PB)) whose sum $(\mathcal{V}^* + \mathcal{U}^*)$ is a contracting homotopy for $V^{\mathcal{M}}$ w.r.t. the operator $N^{\mathcal{M}}$ nondegenerate on nonconstant functions on $C^\infty(\mathcal{M})$.

2. Special BV-BFV dual description of a gauge model Let us recall that an L-stage RGM based on the variational principle with classical fields A^i, i=$1, ..., n$=$n_+ + n_-$ (in condensed notation and with Grassmann parity ε: $\varepsilon(A^i)$=ε_i) is defined by a classical bosonic action $\mathcal{S}_0(A)$: $C^\infty(\mathcal{M}_{\rm cl}) \to \mathbb{R}$, $\mathcal{M}_{\rm cl}$=$\{A^i\}$, invariant w.r.t. gauge transformations, δA^i=$R^i_{\alpha_0}(A)\xi^{\alpha_0}$, $\alpha_0 = 1, ..., m_0 = m_{0+} + m_{0-}$, $\varepsilon(\xi^{\alpha_0}) = \varepsilon_{\alpha_0}$, for notations $\mathcal{S}_{0,i} \equiv \delta\mathcal{S}_0/\delta A^i$,

$$\mathcal{S}_{0,i}\, R^i_{\alpha_0}(A) = 0, \text{ for rank} \,\|\mathcal{S}_{0,ij}\,(A)\|_{\mathcal{S}_{0,k}=0} = \overline{n} - \overline{m}_{-1} \equiv (n_+, n_-) - (m_{-1+}, m_{-1-}), \quad (1)$$

by means of reducibility relations in condensed notations for $s = 1, ..., L$,

$$\begin{aligned} &Z^{\alpha_{s-2}}_{\alpha_{s-1}}(A)Z^{\alpha_{s-1}}_{\alpha_s}(A) = \mathcal{S}_{0,j}\, L^{\alpha_{s-2}j}_{\alpha_s}(A)\,, \alpha_s = 1, ..., m_s = m_{s+} + m_{s-}, \varepsilon(Z^{\alpha_s}_{\alpha_{s+1}}) = \varepsilon_{\alpha_s} + \varepsilon_{\alpha_{s+1}}, \\ &\overline{m}_{s-1} > \sum\nolimits_{k=0}^{s-1}(-1)^k\overline{m}_{s-k-2} = \text{rank}\,\left\|Z^{\alpha_{s-2}}_{\alpha_{s-1}}\right\|_{\mathcal{S}_{0,k}=0},\ \overline{m}_L = \sum\nolimits_{k=0}^{L}(-1)^k\overline{m}_{L-k-1} = \\ &\text{rank}\,\left\|Z^{\alpha_{L-1}}_{\alpha_L}\right\|_{\mathcal{S}_{0,k}=0},\ Z^{\alpha_{-1}}_{\alpha_0} \equiv R^i_{\alpha_0},\ L^{\alpha_{-1}j}_{\alpha_1} \equiv K^{ij}_{\alpha_1} = -(-1)^{(\varepsilon_i+1)(\varepsilon_j+1)}K^{ji}_{\alpha_1}. \end{aligned} \quad (2)$$

Definitions (1), (2), partially determining the gauge algebra first-order structure relations and functions, are encoded, following the BV method via an odd Hamiltonian formulation of the model in $\Pi T^*\mathcal{M}_{\min}$=$\{\Gamma^{p_k}_k$=$(\phi^{A_k}, \phi^*_{A_k})|\phi^{A_k}$=$(A^i, C^{\alpha_s}, s = 0, ..., L), A_k = 1, ..., n_k =$

[1] $\mathcal{M}$ locally represents a vector bundle over the manifold $\mathcal{M}_0$, $\mathcal{M} \to \mathcal{M}_0$, so that $\mathcal{M}_0 \supset \mathcal{M}_{0\min}; \mathcal{N} \supset \mathcal{N}_{\min}; \mathcal{M}_0, \mathcal{N} \subset \mathcal{M}$.

$n+\sum_{r=0}^{L} m_r, \varepsilon(C^{\alpha_s}) = \varepsilon_{\alpha_s}+s+1, \varepsilon(\phi^*_{A_k}) = \varepsilon(\phi^{A_k})+1 = \varepsilon_{A_k}+1; k=\min\}$, by means of a bosonic functional and a classical master-equation in the minimal sector [1]

$$S_k(\Gamma_k) = \mathcal{S}_0(A) + \sum\nolimits_{s=0}^{L} \left(C^*_{\alpha_{s-1}} Z^{\alpha_{s-1}}_{\alpha_s}(A) + o(\phi^*_k)\right) C^{\alpha_s} + o(C^{\alpha_s}),$$

$$(S_k, S_k)^k = \frac{\delta_r S_k}{\delta \Gamma^p_k} \omega^{pq}_k(\Gamma_k) \frac{\delta_l S_k}{\delta \Gamma^q_k} = 0,\ \ \|\omega^{pq}_k\| = \mathrm{antidiag}\,(-\mathbf{1}_{n_k}, \mathbf{1}_{n_k})\,. \tag{3}$$

The quantum action $S^{\psi}(\Gamma_k, \hbar)$ of BV method (in what follows, $k=\mathrm{ext}$) is constructed by an extension of $S_{\min}$ by the pyramids of ghosts and auxiliary fields up to $S_k(\Gamma_k)$ given on $\Pi T^*\mathcal{M}_k{=}\{\Gamma^{p_k}_k{=}(\phi^{A_k}, \phi^*_{A_k})|\phi^{A_k}{=}(\phi^{A_{\min}}, C^{\alpha_s}_{s'}, B^{\alpha_s}_{s'}, s'=0,...,s, s{=}0,...,L), A_k{=}1,...,n_k{=}n+\sum_{r=0}^{L}(2r{+}3)m_r; k{=}\mathrm{ext}\}$, and by imposing an Abelian hypergauge for a $\hbar$-deformed $S_k(\Gamma_k,\hbar)$

$$S_k(\Gamma_k) = S_{\min} + \sum\nolimits_{s=0}^{L}\sum\nolimits_{s'=0}^{s} C^*_{s'\alpha_s} B^{\alpha_s}_{s'};\ \ S^{\psi}(\Gamma_k,\hbar) = \exp\left[\left(\psi(\phi_k,\ \ )^k\right)\right] S_k(\Gamma_k,\hbar). \tag{4}$$

The functionals $[S_k, S^{\psi}](\Gamma_k,\hbar)$ obey a quantum master-equation and provide its proper solutions in terms of a nilpotent operator Δ^k constructed via an odd PB, a trivial density function $\rho(\Gamma_k)$, $\rho=1$, and $\omega^k_{pq}(\Gamma_k)$, $\|\omega^k_{pq}\| = \mathrm{antidiag}\,(\mathbf{1}_{n_k}, -\mathbf{1}_{n_k})$, $\omega^{pq}_k \omega^k_{qr} = \delta^p_r$,

$$\Delta^k \exp\left\{\tfrac{i}{\hbar} E(\Gamma_k,\hbar)\right\} = 0,\ E\in\{S^{\psi}, S_k\},\ \Delta^k = \tfrac{1}{2}(-1)^{\varepsilon(\Gamma^q)}\rho^{-1}\omega^k_{qp}\left(\Gamma^p_k, \rho\left(\Gamma^q_k,\ \cdot\ \right)^k\right)^k. \tag{5}$$

In the second-level formalism [5], the presence of antifields λ^*_a to Lagrangian multipliers λ^a introducing the first-level hypergauges $G_a(\Gamma_k)$ to the exponent of the path integral $Z^{(1)}$[2] permits one to construct a special BV–BFV dual description of RGM on the cotangent bundle $T^*\mathcal{M}_k{=}\{x^{p_k}_k{=}(\phi^{A_k}, \lambda^*_{A_k}), \varepsilon(\lambda^*_{A_k}){=}\varepsilon_{A_k}\}$ in the case of rank 1 gauge theories [in general, on a sub-bundle $\mathcal{N}_{\mathrm{aux}} \to \mathcal{M}_k$ of the bundle $\Pi T^*(T^*\mathcal{M}_k)$, $\Pi T^*(T^*\mathcal{M}_k) \supset \mathcal{N}_{\mathrm{aux}} \supset T^*\mathcal{M}_k$ with a fiber over ϕ^{A_k}: $\mathcal{F}^{\mathcal{N}_{\mathrm{aux}}}_{\phi^A}{=}\{(\lambda^*_{A_k}, \phi^*_{A_k})\}$] by means of a *BRST-like charge.* This object is nilpotent for $\hbar{=}0$ w.r.t. an even PB defined on $C^\infty(T^*\mathcal{M}_k)$ $[C^\infty(\mathcal{N}_{\mathrm{aux}})]$ and determines a formal dynamical system subject to first-class constraints by means of an algorithm different from that of [9, 12, 15]. To this end, consider a functional $\Omega_k(x_k, \phi^*_k) \in C^\infty(\mathcal{N}_{\mathrm{aux}})$, $\varepsilon(\Omega_k){=}1$, constructed from fermionic quantities V^*_k, η, $\eta=\mathrm{const}$, as well as with use of an odd operator Δ^k_{d} dual to Δ^k and an even PB $\{\ ,\ \}^k$

$$\Omega_k = V^*_k S_k(\Gamma_k,\hbar) + \eta\mathcal{S}_0(A) \equiv \lambda^*_{A_k}\left(\delta_l S_k/\delta\phi^*_{A_k}\right) + \eta\mathcal{S}_0,\ \ (V^*_k)^2 = \eta^2 = 0, \tag{6}$$

$$\Delta^k_{\mathrm{d}} = \eta(-1)^{\varepsilon_{A_k}}\frac{\delta_l}{\delta\phi^{A_k}}\frac{\delta_l}{\delta\lambda^*_{A_k}},\ \{\ ,\ \}^k = \frac{\delta_r}{\delta x^p_k}\omega^{pq}_{\mathrm{d};k}(x_k)\frac{\delta_l}{\delta x^q_k},$$

$$\varepsilon(\omega^{pq}_{\mathrm{d};k}) = \varepsilon(\omega^{pq}_k)+1 = \varepsilon(x^p_k)+\varepsilon(x^q_k), \left\|\omega^{pq}_{\mathrm{d};k}\right\| = \mathrm{antidiag}\,(-\mathbf{1}_{n_k}, \mathbf{1}_{n_k})\,. \tag{7}$$

The operator V^*_k, being a contracting homotopy for nilpotent V_k, $V_k = \phi^*_{A_k}\left(\delta_l/\delta\lambda^*_{A_k}\right)$, w.r.t. an operator N_{V_k}, $N_{V_k} = [V_k, V^*_k]_+$, nondegenerate on $\mathcal{F}^{\mathcal{N}_{\mathrm{aux}}}_{\phi^A}$, satisfies the properties

$$[V^*_k, \Delta^k]_+ = 0,\ V^*_k\left(\mathcal{F},\mathcal{G}\right)^k = \left(V^*_k\mathcal{F},\mathcal{G}\right)^k - (-1)^{\varepsilon(\mathcal{F})}\left(\mathcal{F}, V^*_k\mathcal{G}\right)^k,$$

$$[V_k, \Delta^k_{\mathrm{d}}]_+ = 0,\ V_k\left\{F,G\right\}^k = \left\{V_kF,G\right\}^k + (-1)^{\varepsilon(F)}\left\{F, V_kG\right\}^k.^{3} \tag{8}$$

The gauge algebra relations (1)–(3), eqs.(4,5) are equivalently described by means of a correspondence among Poisson brackets of opposite parities for arbitrary functionals $\mathcal{F}_{\mathrm{t}}(\Gamma_k) \in$

[2]In the case of a fiber bundle $\Pi T^*\mathcal{M}_k$: $a = A_k, G_a = G_{A_k}(\Gamma_k) = \left(\phi^*_{A_k} - \delta\psi/\delta\phi^{A_k}\right)$ and $[\lambda^a, \lambda^*_a] = \left[\lambda^{A_k}, \lambda^*_{A_k}\right]$ that corresponds to the construction of $S^{\psi}(\Gamma_k,\hbar)$ in (4).

[3]As a whole, on $\Pi T^*(T^*\mathcal{M}_k){=}\left\{(x^p_k, (\phi^*_{A_k}, \lambda^{A_k}))\right\}$, there exist nilpotent operators $\left(U_k, U^*_k, \Pi\Delta^k_{\mathrm{d}}\right){=}\left(\lambda^{A_k}\frac{\delta_l}{\delta\phi^{A_k}}, \phi^{A_k}\frac{\delta_l}{\delta\lambda^{A_k}}, \eta(-1)^{\varepsilon_{A_k}+1}\frac{\delta_l}{\delta\lambda^{A_k}}\frac{\delta_l}{\delta\phi^*_{A_k}}\right)$ analogous to $\left(V_k, V^*_k, \Delta^k_{\mathrm{d}}\right)$, which obey the same properties as in (8) for corresponding exchange $\left(V_k, V^*_k, \Delta^k_{\mathrm{d}}\right) \leftrightarrow \left(U_k, U^*_k, \Pi\Delta^k_{\mathrm{d}}\right)$, mutually anticommute, $[E,D]_+{=}0, E \in \left\{V_k, V^*_k, \Delta^k_{\mathrm{d}}\right\}, D \in \left\{U_k, U^*_k, \Pi\Delta^k_{\mathrm{d}}\right\}$, and yield the operator $N_{U_k}{=}[U_k, U^*_k]_+$ nondegenerate on $\mathcal{F}^{\Pi T^*(T^*\mathcal{M}_k)}_{x^p_k}$.

$C^\infty(\Pi T^*\mathcal{M}_k)$ and $F_{\rm t}(x_k,\phi^*_k)=[V^*_k\mathcal{F}_{\rm t}(\Gamma_k)+\eta\mathcal{F}_{0\rm t}(\phi_k)]$, $\mathcal{F}_{0\rm t}\equiv\mathcal{F}_{\rm t}|_{\mathcal{M}_k}\in\ker N_{V_k}$, for t=1, 2,

$$V_k\{F_1,F_2\}^k=N_{V_k}(\mathcal{F}_1,\mathcal{F}_2)^k,\ \Delta^k_{\rm d}F_{\rm t}=\eta\Delta^k\mathcal{F}_{\rm t};\ \{\Omega_{\min},\Omega_{\min}\}^{\min}=0, \quad (9)$$

$$1/2\,\{E_k(x_k,\phi^*_k),E_k(x_k,\phi^*_k)\}^k=i\hbar\Delta^k_{\rm d}E_k,\ E_k\in\left\{\Omega_k,\Omega^\psi\right\},\Omega^\psi=\exp\left[\{F,\ \}\right]\Omega_k, \quad (10)$$

with a *gauge boson* $F(\phi_k)=\eta\psi(\phi_k)$. From eqs.(9, 10), it follows that for rank-1 gauge theories (i.e., for $S_k=\mathcal{S}_0+\phi^*_{A_k}H^{A_k}(\phi_k)\Leftrightarrow V^*_kS_k\in T^*\mathcal{M}_k$, k=ext) to whose class one can always reduce an initial RGM the BFV–BRST quantities in (10) are defined only on $C^\infty(T^*\mathcal{M}_{\rm ext})$.

3. Poisson brackets and triplectic-like algebra of $\Delta^{\mathcal{M}}, V^{\mathcal{M}}$ ($\mathcal{V}^*$,$\mathcal{U}^*$) Leaving aside the realization of an initial gauge model in the BV method, consider a Poisson supermanifold $(\mathcal{M}_0,\{\cdot,\cdot\}_0)$, $\mathcal{M}_0=\{x^p\}$, $\dim\mathcal{M}_0=\dim T^*\mathcal{M}_{\rm ext}$ with an even PB defined by a tensor field $\omega^{pq}(x)$ over $\mathcal{M}_0$, $\omega^{pq}=-(-1)^{\varepsilon_p\varepsilon_q}\omega^{qp}$, $\varepsilon(\omega^{pq})=\varepsilon_p+\varepsilon_q$, $\varepsilon(x^p)=\varepsilon_p$, and also with a covariant derivative ∇_p on $\mathcal{M}_0$ transforming $\mathcal{M}_0$ into a Poisson supermanifold with a symmetric connection $\Gamma^p{}_{rs}(x)$, $\Gamma^p{}_{rs}=(-1)^{\varepsilon_r\varepsilon_s}\Gamma^p{}_{sr}$ (for a nondegenerate $\omega^{pq}(x)$, i.e., if $\exists\omega_{pq}(x)|$ $\omega_{pq}=-(-1)^{\varepsilon_p\varepsilon_q}\omega_{qp},\omega^{pq}\omega_{qr}(-1)^{\varepsilon_q}=\delta^p_r$, $\mathcal{M}_0$ transforms into a Fedosov supermanifold [20])

$$\overleftarrow{\nabla}_r\omega^{pq}=1/2\left[(\delta_r\omega^{pq})/(\delta x^r)+2\omega^{ps}\Gamma^q{}_{rs}(-1)^{\varepsilon_s(\varepsilon_q+1)}\right]-(-1)^{\varepsilon_p\varepsilon_q}(p\leftrightarrow q)=0. \quad (11)$$

Let us introduce a manifold $\mathcal{M}=\{(x^p,\eta_p)\}$ locally realized by an odd (co)tangent bundle over $\mathcal{M}_0$, $\mathcal{M}=\Pi T^*\mathcal{M}_0\simeq\Pi T\mathcal{M}_0$, whose fibers are parameterized by covariantly constant vectors η_p, $(\varepsilon,{\rm gh})\eta_p=(\varepsilon_p+1,-1-{\rm gh}(x^p))$, being antifields to x^p [for $\mathcal{M}_0=T^*\mathcal{M}_{\rm ext}$, $(x^p;\eta_p)=(\phi^{A_k},\lambda^*_{A_k};\phi^*_{A_k},\lambda^{A_k})$]. We next define a scalar functional $T(x,\eta)$ w.r.t. an (extended to $\mathcal{M}$) covariant derivative $\nabla^{\mathcal{M}}_p$

$$T=(1/2)\,\eta_p\omega^{pq}(x)\eta_q,\ (\varepsilon,{\rm gh})T=(0,0),\ \overleftarrow{\nabla}^{\mathcal{M}}_pT=\delta_rT/\delta x^p+(\delta_rT/\delta\eta_q)\,\eta_r\Gamma^r{}_{qp}=0, \quad (12)$$

in view of eqs. (11) and the relations $\overleftarrow{\nabla}^{\mathcal{M}}_p\eta_q=0$. Supplementing $\mathcal{M}_0$ by a bosonic scalar density $\rho(x)$ is sufficient to determine the covariant operations characteristic for the supersymplectic manifold $\mathcal{M}$: an antibracket $(\cdot,\cdot)^{\mathcal{M}}$ and operators $\Delta^{\mathcal{M}}$, $V^{\mathcal{M}}$,

$$(\cdot,\cdot)^{\mathcal{M}}=\left(\overleftarrow{\nabla}^{\mathcal{M}}_p\cdot\right)(\delta_l\cdot/\delta\eta_p)-(\delta_r\cdot/\delta\eta_p)\,\overrightarrow{\nabla}^{\mathcal{M}}_p\cdot, \quad (13)$$

$$\Delta^{\mathcal{M}}=-(-1)^{\varepsilon_p}\,(\delta_r\ /\delta\eta_p)\left[\overleftarrow{\nabla}^{\mathcal{M}}_p+(1/2)\,(\delta_r\rho/\delta x^p)\right],\ V^{\mathcal{M}}=(T,\ )^{\mathcal{M}}=-\eta_p\omega^{pq}\overrightarrow{\nabla}^{\mathcal{M}}_q, \quad (14)$$

which (by an explicit verification for scalars on $\mathcal{M}$) can be shown to obey the relations of a *triplectic-like algebra* [19], $[E_1,E_2]_+=0$ for $E_1,E_2\in\{\Delta^{\mathcal{M}},\ V^{\mathcal{M}}\}$, compatible with the Leibniz rule of differentiating an antibracket similar to (8) by any $E_{\rm t}$ only in case of a *flat Poisson manifold* $\mathcal{M}_0$[4]. A representation of $\mathcal{M}$ as $\Pi T\mathcal{M}_0$ permits one to lift the nondegenerate PB $\{\cdot,\cdot\}_0$ to a flat Fedosov manifold $(\mathcal{M},\{\cdot,\cdot\})$ by the relation

$$\{\cdot,\cdot\}=\left(\overleftarrow{\nabla}^{\mathcal{M}}_p\cdot\right)\omega^{pq}\left(\overrightarrow{\nabla}^{\mathcal{M}}_q\cdot\right)+\alpha\,(\delta_r\cdot/\delta\eta_p)\,\omega_{pq}\,(\delta_l\cdot/\delta\eta_q)\,,\ \alpha={\rm const}\in\mathbb{R}. \quad (15)$$

The covariant definition of an nilpotent operator $U^{\mathcal{M}}$ satisfying the relations $[E_1,E_2]_+$ $=0$ for $E_1,E_2\in\{\Delta^{\mathcal{M}},\ V^{\mathcal{M}},\ U^{\mathcal{M}}\}$, is impossible as contrasted to [19] in terms of an anti-Hamiltonian vector field but is ensured by equipping $\mathcal{M}_0$ with an additional Riemann-type nondegenerate even structure $g_{pq}(x)$, $g_{pq}=(-1)^{\varepsilon_p\varepsilon_q}g_{qp}$, in the form

$$U^{\mathcal{M}}=-\eta_p\omega^{ps}g_{st}\omega^{tq}(-1)^{\varepsilon_s}\overrightarrow{\nabla}^{\mathcal{M}}_q,\ \left(U^{\mathcal{M}}\right)^2=0\Leftrightarrow\overrightarrow{\nabla}_qg_{ps}=0. \quad (16)$$

[4]That is, for a vanishing curvature tensor $R^q{}_{prs}(x)$ defined for arbitrary vectors $T^r(x)$ as follows: $[\overleftarrow{\nabla}_q,\overleftarrow{\nabla}_p]T^r(x)=-(-1)^{\varepsilon_s(\varepsilon_r+1)}T^s(x)R^r{}_{sqp}(x)$.

Since the action of ∇ is undetermined as a tensor operation on x^p, an explicit definition of an operator $V^{*\mathcal{M}}$ ($U^{*\mathcal{M}}$) of contracting homotopy for $V^{\mathcal{M}}$ ($U^{\mathcal{M}}$) w.r.t. a nondegenerate (on non-constant functions from $C^{\infty}(\mathcal{M})$) operator $N^{\mathcal{M}}$ ($N_U^{\mathcal{M}}$) is possible only for a symplectic $\mathcal{M}_0$, i.e., for $\Gamma^p{}_{rs}$=0. To this end, we consider the adjoint action of a fermionic functional $\Omega_T(x,\eta)$, $\Omega_T = -\eta_p x^p$, w.r.t. the non-tensor PB (15), and to single out operators $\mathcal{V}^*$,$\mathcal{U}^*$ analogous to V_k^*, U_k^* in Sec. 2 define an operator $U^{*\mathcal{M}}$ via the non-tensor PB (13) with bosonic $T^*(x,\eta)$, $T^* = -(1/2)x^p g_{pq}(x)x^q(-1)^{\varepsilon_q}$,

$$V^{\mathcal{M}} + \alpha V^{*\mathcal{M}} = \{\Omega_T,\ \} = -\eta_p\left(\omega^{pq}\frac{\delta_l}{\delta x^q} - (-1)^{\varepsilon_r(\varepsilon_q+1)}\frac{\delta_r\omega^{pq}}{\delta x^r}\eta_q\frac{\delta_l}{\delta\eta_r}\right) + \alpha\left(x^p\omega_{pr} - \right. \tag{17}$$
$$\left.(-1)^{\varepsilon_r(\varepsilon_q+1)}\alpha^{-1}\eta_p\frac{\delta_r\omega^{pq}}{\delta x^r}\eta_q\right)\frac{\delta_l}{\delta\eta_r},\quad U^{*\mathcal{M}} = (T^*,\ )^{\mathcal{M}} = -x^p\left(g_{pq} + \frac{1}{2}\frac{\delta_l g_{ps}}{\delta x^q}x^s(-1)^{\varepsilon_q\varepsilon_s}\right)\frac{\delta_l}{\delta\eta_q}.$$

Necessary conditions for the quantities $V^{*\mathcal{M}} = 1/\alpha\left[\{\Omega_T,\ \} - (T,\ )^{\mathcal{M}}\right]$, $U^{*\mathcal{M}}$ to be nilpotent, as well as to anticommute with each other and with $\Delta^{\mathcal{M}}$, are the fulfilment of the equations $(D,D)^{\mathcal{M}} = 0, \Delta^{\mathcal{M}}D = 0$ for $D \in \{T, T^*\}$, whereas $N^{\mathcal{M}}$ is defined w.r.t. non-tensor PB (15)

$$N^{\mathcal{M}} = 1/(2\alpha)\,\{\{\Omega_T,\Omega_T\},\ \ \} = -(x^p + o(x,\eta))\delta_l/\delta x^p - \eta_p\delta_l/\delta\eta_p. \tag{18}$$

Next define mutually anticommuting nilpotent operators $\mathcal{V}^*,\mathcal{U}^*$, so that $(\mathcal{V}^*{+}\mathcal{U}^*) = V^{*\mathcal{M}}$,

$$\mathcal{D}_a^* = (1/2)\left[(-1)^a/\alpha\left(\{\Omega_T,\ \} - (T,\ )^{\mathcal{M}}\right) - (T^*,\ )^{\mathcal{M}}\right], (\mathcal{D}_1^*,\mathcal{D}_2^*) \equiv (\mathcal{V}^*,\mathcal{U}^*)\,, a = 1,2. \tag{19}$$

In particular case of Darboux coordinates, of $g_{pq}(x)$ and the vanishing of $\Gamma^p{}_{rs}(x)$ on $\mathcal{M}_0$,

$$(x^p,\eta_p,[\omega^{pq},g_{pq}](x),\rho(x)) = \left((\phi^A,\lambda^*_A),(\phi^*_A,\lambda^A),\mathrm{antidiag}\left[\left(-\delta^A_B,\delta^A_B\right),\left(\delta^A_B,\delta^A_B\right)\right],1\right), \tag{20}$$

we obtain correspondence $\left(V^{\mathcal{M}},\mathcal{V}^*,\mathcal{U}^*,\Delta^{\mathcal{M}}\right) = \left(U^k - V^k, V_k^*, U_k^*, \Delta^k + (-1)^{\varepsilon_A+1}\frac{\delta_l}{\delta\lambda^A}\frac{\delta_l}{\delta\lambda^*_A}\right)$.

4. Quantization rules Let us define the generating functional of Green's functions $Z^{\mathcal{M}}\left[J_{\mathcal{V}}, J^{\mathcal{V}^*}\!,\eta\right]$ and the vacuum functional $Z_X^{\mathcal{M}}{=}Z^{\mathcal{M}}[0,0,0]$ in general coordinates $z^P{=}(x^p,\eta_p)$,

$$Z^{\mathcal{M}} = \int d\tilde{z}\mathcal{D}_0(\tilde{x})q^{\mathcal{M}}(\tilde{z})\exp\left\{\frac{i}{\hbar}\left[W(\tilde{z},\hbar) + [X + i\hbar H](\tilde{x},\tilde{\eta}-\eta,\hbar) + J^p_{\mathcal{V}}\tilde{\eta}_p + J_p^{\mathcal{V}^*}\tilde{x}^p\right]\right\}, \tag{21}$$

where $J_p^{\mathcal{V}^*}, J^p_{\mathcal{V}}$ are a redundant set of sources to $z^P{:}\varepsilon(J_p^{\mathcal{V}^*}){=}\varepsilon_p{=}\varepsilon(J^p_{\mathcal{V}}){+}1$; H,W,X stand for bosonic functionals providing the correct reduction, e.g., of $Z_X^{\mathcal{M}}$ to the BV partition function [1] or of $Z^{\mathcal{M}}$ to generating functional of Green's functions $\mathcal{Z}(\theta)_{|\theta=0}$ in [9], as well as the respective quantum and corresponding to irreducible hypergauges $G_a(z)$ gauge-fixing bosonic actions, satisfying the generalized master equations and an additional equation for W,

$$\Delta^{\mathcal{M}}\exp\left[(i/\hbar)\,E(z,\hbar)\right] = 0,\ \ E \in \{W, X + i\hbar H\},\ \ \mathcal{V}W(z,\hbar) = 0, \tag{22}$$

for the first of which only X is a proper solution in $\mathcal{M}$, while W is subject to a boundary condition with the classical action, $W(z,\hbar)_{\eta=\hbar=0} = \mathcal{S}_0(x)$. The density function $\mathcal{D}_0(x)$ determining the invariant measure on $\mathcal{M}$ [for $\alpha = 1$ in (15) so that functional ρ (13) can be defined as $\rho = \ln\ \mathrm{sdet}^{-1}\|\omega^{pq}(x)\|$ as in [19]] and the weight functional $q^{\mathcal{M}}[z]$ are given by

$$\mathcal{D}_0(x) = \mathrm{sdet}^{-1}\|\omega^{pq}(x)\|,\ \ q^{\mathcal{M}}(z) = \delta\left(G_{a_1}^{\mathcal{V}^*}(z)\right),\ a_1 = 1,\ldots,\ 1/2\left(\dim_+\mathcal{M}_0 + \dim_-\mathcal{M}_0\right). \tag{23}$$

In (21), (23), in accordance with the decomposition of $V^{*\mathcal{M}}$ in (19), we have used the polarization of $V^{\mathcal{M}}$ into a corresponding sum of nilpotent anticommuting operators $\mathcal{V},\mathcal{U}$

$$V^{\mathcal{M}}=\mathcal{V}+\mathcal{U}:(\mathcal{V},\mathcal{U})=(-1/2)\,\eta_p\left(\omega^{pq}+\omega^{ps}g_{st}\omega^{tq}(-1)^{\varepsilon_s},\omega^{pq}-\omega^{ps}g_{st}\omega^{tq}(-1)^{\varepsilon_s}\right)\overrightarrow{\nabla}^{\mathcal{M}}_q. \quad (24)$$

In turn, the independent functions $G^{\mathcal{V}^*}_{a_1}(z)$=0 playing, in fact, the role of *second-level hypergauge conditions* within the formalism [5], are required to retain the explicit covariant form of $Z^{\mathcal{M}}$. The independent functions $G^{\mathcal{V}^*}_{a_1}(z)$ are equivalent to an (explicitly given only in case of symplectic $\mathcal{M}_0$) set of functions $\mathcal{V}^*\eta_p$, $G^{\mathcal{V}^*}_{a_1}(z)$=$\left[Y^p_{a_1}(z)\mathcal{V}^*\eta_p\right]$ with certain $Y^p_{a_1}$ so that

$$\mathrm{rank}\left\|\overrightarrow{\nabla}^{\mathcal{M}}_p E_{\mathrm{t}}(z)\right\|_{\overrightarrow{\nabla}^{\mathcal{M}}W=\delta W/\delta\eta=\overrightarrow{\nabla}^{\mathcal{M}}X=\delta X/\delta\eta=G^{\mathcal{V}^*}=0}=\frac{1}{2}\dim\mathcal{M}_0,(E_1,E_2)=(G^{\mathcal{V}^*}_{a_1},\mathcal{V}^*\eta_p), \quad (25)$$

and define the functional $H(z,\hbar)$ in (21, 22) in the explicitly covariant form

$$H(z,\hbar)=-\frac{1}{2}\ln\left\{J(z)\mathcal{D}_0^{-1}(x)\mathrm{sdet}M\right\},M=\begin{Vmatrix}(F^{\mathcal{U}^*}_{a_1}(z),G^{c_1}_{\mathcal{V}}(z))^{\mathcal{M}} & (F^{\mathcal{U}^*}_{a_1}(z),F^{d_1}_{\mathcal{U}}(z))^{\mathcal{M}}\\ (G^{\mathcal{V}^*}_{b_1}(z),G^{c_1}_{\mathcal{V}}(z))^{\mathcal{M}} & (G^{\mathcal{V}^*}_{b_1}(z),F^{d_1}_{\mathcal{U}}(z))^{\mathcal{M}}\end{Vmatrix}, \quad (26)$$

where the functions $(F^{a_1}_{\mathcal{U}},F^{\mathcal{U}^*}_{b_1},G^{c_1}_{\mathcal{V}})$=$(\widetilde{Z}^{a_1}_p\mathcal{U}x^p,Z^p_{b_1}\mathcal{U}^*\eta_p,\widetilde{Y}^{c_1}_p\mathcal{V}x^p)$ with certain $(\widetilde{Z}^{a_1}_p,Z^p_{b_1},\widetilde{Y}^{c_1}_p)(z)$ determine an invertible change of variables, $z^P\rightarrow\overline{z}^P=(F^{\mathcal{U}^*}_{a_1},G^{\mathcal{V}^*}_{b_1},G^{c_1}_{\mathcal{V}},F^{d_1}_{\mathcal{U}})$, with J=Ber $\left\|\delta\overline{z}^Q/\delta z^P\right\|$.

The basic properties of the functionals $Z^{\mathcal{M}}_X$, $Z^{\mathcal{M}}$ are encoded by the *generalized generator* $s^{\mathcal{M}}$ of *BRST-like transformations* with an arbitrary bosonic functional $R^{\mathcal{M}}(z)$,

$$s^{\mathcal{M}}=(\hbar/i)\,T^{-1}(z)\left(\ \ ,T(z)R^{\mathcal{M}}(z)\right)^{\mathcal{M}},\ T(z)=\exp\left[(i/\hbar)\left(W-X-i\hbar H\right)(z)\right]. \quad (27)$$

For instance, the BRST transformations for a constant μ, $\delta_\mu z^P=s^{\mathcal{M}}z^P\mu$, with $Z^{\mathcal{M}}_X$ and $Z^{\mathcal{M}}[0,0,\eta]$, are derived from (27), with $R^{\mathcal{M}}$=1, and from additional equations,

$$\left(G^{\mathcal{V}^*}_{a_1}(z),T(z)\right)^{\mathcal{M}}=0\Longleftrightarrow\delta_\mu G^{\mathcal{V}^*}_{a_1}(z)=0. \quad (28)$$

The derivation of the Ward identity for the functional $Z^{\mathcal{M}}$ and the proof of gauge-independence of the S-matrix are based on the eqs.(22), on transformation (27) and on additional equations for $G^{\mathcal{V}^*}_{a_1}(z)$ (28) in a manner described in [8, 9]. For instance, after exponentiating the functional $q^{\mathcal{M}}$ in the functional integral (21), $q^{\mathcal{M}}(\tilde{z})=\int d\tilde{\lambda}_{(2)}\times\exp\{(i/\hbar)G^{\mathcal{V}^*}_{a_1}(\tilde{z})\tilde{\lambda}^{a_1}_{(2)}\}$, $\varepsilon(\tilde{\lambda}^{a_1}_{(2)})$=$\varepsilon(G^{\mathcal{V}^*}_{a_1})$=$\varepsilon_{a_1}$, the corresponding Ward identity has the form

$$\left[J^{\mathcal{V}^*}_p+\left(\frac{\delta_r W}{\delta\tilde{x}^p}+(-1)^{\varepsilon_{a_1}}\langle\tilde{\lambda}^{a_1}_{(2)}\rangle\frac{\delta_r G^{\mathcal{V}^*}_{a_1}}{\delta\tilde{x}^p}\right)\left(\frac{\hbar}{i}\frac{\delta_l}{\delta J^{\mathcal{V}^*}},\frac{\hbar}{i}\frac{\delta_l}{\delta J_{\mathcal{V}}}\right)\right]\frac{\delta_l}{\delta\eta_p}Z^{\mathcal{M}}\left[J_{\mathcal{V}},J^{\mathcal{V}^*},\eta\right]=0.(29)$$

The sign $\langle F(\tilde{z},\tilde{\lambda}_{(2)})\rangle$ here denotes the functional averaging of a quantity $F(\tilde{z},\tilde{\lambda}_{(2)})$ w.r.t. $Z^{\mathcal{M}}$.

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Higher Dimensional Vertex Algebras and Rational Conformal Field Theory Models

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Abstract

The notion of *global conformal invariance* (GCI) in Minkowski space allows to prove rationality of correlation functions and to extend the concept of vertex algebra to any number D of space-time dimensions. The case of even D, which includes a conformal stress-energy tensor with a rational 3-point function, is of particular interest. Recent progress, reviewed in the talk, includes a full account of Wightman positivity at the 4-point level for D=4, and a study of modular properties of thermal expectation values of the conformal energy operator.

1 Introduction

Invariance of Wightman functions in quantum field theory under finite conformal transformations in Minkowski space has far reaching implications: local fields commute for non-lightlike separations and have, as a result, rational correlation functions. The theory is reformulated in a complex realization of compactified Minkowski space, in which the forward tube is mapped inside the unit ball, and yields a higher dimensional extension of the notion of a vertex algebra. Thermal correlators are shown to be elliptic functions of the conformal time variable. Energy mean values in a Gibbs state for free massless fields are expressed in terms of modular forms. For large compactification radii the energy density approaches the Minkowski space thermodynamic limit, reproducing the Stefan-Boltzmann law.

The talk is based on work of Nikolay M. Nikolov, Karl-Henning Rehren, Yassen S. Stanev and the author [1, 2, 3, 4, 5, 6, 7, 8]. The present outline is chiefly meant as a general introduction to the subject.

1.1 Why should one care at all for conformally invariant quantum field theory models?

Every once in a while since the discovery (by Cunningham and Bateman in 1910) of the conformal invariance of (vacuum) Maxwell's electrodynamics the conformal group is attracting the attention of mathematical physicists by its elusive beauty. The "real world" is certainly not conformally invariant: the presence of discrete positive masses of atoms and elementary particles signals violation of even the weaker *dilation symmetry*. One is entitled to ask: does the invariance of free massless equations provide enough rationale to care for conformally invariant quantum field theory (QFT)? Let me cite two reasons why the study of conformal models may still be of interest.

The first is a negative one: the relativistic quantum theory of the real world has proven too difficult for us. The first attempts to formulate QFT date over three quarter of a century ago, but in spite of vigorous efforts by theoretical - and mathematical - physicists (as we now distinguish between the two brands) we still have no mathematically established interacting QFT in four (or higher) space-time dimensions. This justifies the study of not entirely realistic QFT models - either in low space-time dimensions or having a higher symmetry (or both).

A positive argument affirms that conformal invariance may be a meaningful approximation to a realistic QFT at very short distances (or large energies and transferred momenta) when particle masses can be neglected. A QFT with a (ultraviolet stable) renormalization group fixed point has to be dilation invariant at that point. On the other hand, a dilation invariant QFT with a stress-energy tensor is expected to be (under reasonable assumptions) also conformally invariant. (A recently discussed counterexample [9] - which violates those assumptions - also violates Wightman positivity and thus looks rather pathological.) Progress in physics often needs idealizations: without neglecting friction Galileo could not have discovered the law of inertia.

Two-dimensional (2D) conformal field theory (CFT) not only provides a rich family of soluble QFT models and thereby of universality classes of 2D critical phenomena (and string vacua), [10], it also gives rise to a new fruitful mathematical concept, the notion of a *(chiral) vertex algebra* which naturally incorporates an important class of infinite dimensional Lie algebras, [11, 12, 13, 14, 15]. Recently, a (not fully understood) intriguing relation was discovered between Zhu's vertex algebra approach [13] to rational CFT and Haag's [16] von Neumann algebra framework applied to the classification of chiral CFT models [17, 18, 19, 20]. It is all the more interesting that the concept of a vertex algebra - and the associated modular properties of thermal energy mean values - appear to admit a higher dimensional generalization, [1, 6, 7].

The paper is organized as follows. In the rest of this introduction (subsection 1.2) some background material and early developments (of [1, 2, 21]), are sketched, including a complex realization of compactified Minkowski space which prepares the ground for a higher dimensional extension of the notion of vertex algebra [6]. Sect. 2 reviews the construction of 4-point functions of scalar fields [3, 4] (providing some new formulae for the $d = 3$ case). Sect. 3 outlines the explicit construction of conformal partial wave expansions and its application to the study of Wightman positivity [8]. The final Sect. 4 gives a bird's-eye view of equilibrium states in a GCI QFT, including the appearance of elliptic thermal correlation functions and an application of modular invariance to the Gibbs energy-mean-value.

1.2 Background and early results

The concept of conformal invariance in Minkowski space involves a subtlety absent in the euclidean formulation of the theory. While the spinorial euclidean conformal group,

$Spin(5,1)$, is simply connected and so is the (one-point) conformal compactification, $\mathbb{S}^4$, of the euclidean 4-space, the corresponding Minkowski space group, $\mathcal{C} = Spin(4,2) = SU(2,2)$, and compactified space-time, $\bar{M} = \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2$, are not: in fact, each has an infinite sheeted universal cover. It follows that infinitesimal conformal transformations of euclidean Green's functions can be integrated and invariance under such transformations is thus equivalent to a global invariance property (see Sect. 1.1 of [7]). This implies, in turn, that the corresponding Minkowski space QFT can be continued to the (simply connected) universal cover of $\bar{M}$, the cylinder space $\tilde{M} = \mathbb{S}^3 \times \mathbb{R}$, and is invariant under the universal cover $\tilde{\mathcal{C}}$ of the conformal group [21]. The projection of this conformal group action on (compactified) Minkowski space itself is, in general, multivalued. The condition of global conformal invariance (GCI) in Minkowski space introduced in [2] is, therefore, stronger as it allows to continue the Wightman functions to invariant distributions on $\bar{M}$. It turns out that this rather natural condition has far reaching implications. Combined with locality GCI on M yields the *Huygens principle* - the vanishing of (observable, Bose) field commutators for non light-like separations. Together with Wightman axioms [22] this implies rationality of correlation functions (Theorem 3.1 of ref. [2]).(It has been noted long ago that such a condition can be realized in the context of (generalized) free fields - see, e.g. [23].)

Mentioning *observable* local fields is not accidental. One cannot expect that, e.g. gauge dependent quantities satisfy even (the weaker) infinitesimal conformal invariance. It is therefore important to state which fields are assumed to be observable. The stress energy tensor T is an obvious candidate for such a role [24, 25]. The general form of the 3-point function of T [26] implies that T can only be assumed GCI in even dimensional space time - in line with the validity domain of the classical Huygens principle. For the sake of definiteness (and to stay closer to reality) we shall restrict attention in what follows to the $D = 4$ case. We have proposed recently a model in which the observable algebra is generated by a GCI Lagrangian density $\mathcal{L}$ of scale dimension $D(= 4)$ [4] (and contains an infinite ladder of conserved tensor fields starting with T).

Rationality of Wightman functions signals the possibility of an algebraic formulation of the theory - in the spirit of chiral vertex algebras (or "meromorphic CFT" in the terminology of the physicist oriented early survey [12]). This is made easier by introducing appropriate complex parametrization of compactified Minkowski space and of the *future tube* - the analyticity domain of vector valued functions of the form $\phi(z)|0>$ for any local field ϕ (the counterpart of the unit circle and the unit disk in the 1D chiral case). To describe it we perform a complex conformal transformation from the Cartesian Minkowski space coordinates $x = (x^0, \mathbf{x})$ to the complex 4-vector $z = (\mathbf{z}, z_4) = z(x)$ [1, 2]:

$$\mathbf{z} = \frac{\mathbf{x}}{\omega(x)}, z_4 = \frac{1-x^2}{2\omega(x)}, 2\omega(x) = 1 + x^2 - 2ix^0, \tag{1}$$

$$z^2 = \mathbf{z}^2 + z_4^2 = \frac{\bar{\omega}}{\omega},\ x^2 = \mathbf{x}^2 - (x^0)^2 (= \frac{1+z^2-2z_4}{1+z^2+2z_4}). \tag{2}$$

In the z coordinates the image T_+ of the future tube is the connected component of the complement of compactified Minkowski space

$$\bar{M} = \{z \in \mathbb{C}^4\,;\ z = \frac{\bar{z}}{\bar{z}^2} = e^{2\pi i\zeta} u\, \zeta \in \mathbb{R}\,,\ u \in \mathbb{R}^4, z^2 = \sum_{\alpha=1}^{4} (z_\alpha)^2 = e^{4\pi i\zeta}\} \tag{3}$$

($\bar{M} = \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2$) containing the origin:

$$T_+ = (z \in \mathbb{C}^4; |z^2| < 1, |z|^2 = \sum_{i=1}^{4} |z_i|^2 < \frac{1+|z^2|^2}{2}). \tag{4}$$

Fields $\phi(z)$ are then defined as formal power series of the form

$$\phi(z) = \sum_{n\in\mathbb{Z}} \sum_{m\geq 0} (z^2)^n \phi_{nm}(z)\,, \tag{5}$$

$\phi_{nm}(z)$ being an operator valued polynomial in z that is homogeneous of degree m and harmonic. The Huygens principle admits an algebraic formulation in terms of such formal power series. If ϕ is a GCI irreducible spin- tensor of dimension d and $SU(2) \times SU(2)$ weight $(j_1, j_2; 2j_{1,2} = 0, 1, ...)$ then the strong locality condition reads

$$(z_{12}^2)^n(\phi(z_1)\phi^*(z_2) - (-1)^{2j_1+2j_2}\phi^*(z_2)\phi(z_1)) = 0, \tag{6}$$
$$for\, n \geq d + j_1 + j_2 (\in \mathbb{N}),\;\; z_{12} = z_1 - z_2. \tag{7}$$

We assume that the field algebra is spanned by *conformal* fields, transforming homogeneously under infinitesimal conformal transformations. In particular, under commutation with the *conformal Hamiltonian* H, the generator of the centre of the maximal compact subgroup $U(1) \times Spin(4)$ of $\mathcal{C}$ (whose significance has been emphasized by Segal [27]), a field ϕ of scale dimension d satisfies

$$[H,\, \phi(z)] = (z\frac{\partial}{\partial z} + d)\phi(z)\,. \tag{8}$$

H is related to the Minkowski space energy P^0 by $2H = P^0 + wP^0w^{-1}$ where w is the *Weyl inversion* - the proper conformal transformation that changes the sign of z (cf. [27]; it follows that the conformal energy H is positive whenever the Minkowski space one is. Energy positivity implies analyticity of the vector-valued function $\phi(z)|0>$ in T_+, and hence the vanishing of $\phi_{nm}|0>$ for negative n. Covariance of local fields under (complex) translations T_α allows to formulate the *state-field correspondence* as follows: to each finite-energy state v corresponds a unique local field (or *vertex operator*) $Y(v, z)$) such that

$$Y(v, 0)|0>= v\,,\; [T_\alpha,\, Y(v, z)] = \frac{\partial}{\partial z}Y(v, z). \tag{9}$$

2 General rational 4-point functions of scalar fields

The fact that GCI implies rationality of correlation functions provides a powerful tool for explicit construction of Wightman functions.

Given the conformally invariant 2-point function of a scalar field of dimension d,

$$< 12 > (=< 12 >_d) = N(z_{12}^2)^{-d}\,, \quad N = N(d) > 0, \tag{10}$$

we can write its most general rational conformal 4-point function as

$$< 1234 >=< 12 >< 34 > + < 13 >< 24 > + < 14 >< 23 > +p(z_{ij}^2)F(s, t) \tag{11}$$

where s and t are the conformally invariant cross ratios

$$s = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}, \; t = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2}; \tag{12}$$

the prefactor $p(\rho_{ij})$ is a monomial in its six arguments and their inverses homogeneous of degree $-d$ in $\rho_{ij} = \rho_{ji}$ for each fixed i and $j = 1, ..., 4, j \neq i$; the invariant amplitude F is a polynomial in s, t, s^{-1}, t^{-1} of degree restricted by Wightman positivity. For the cases $d = 2, 3, 4$ of interest the truncated 4-point function $p.F$, obeying locality and conformal invariance, depends on $2d - 3$ parameters. Using the analysis of [3, 4, 5, 8] we can choose the prefactor p and the amplitude F in these three cases as

$$\begin{aligned}
&d = 2 : p(\rho_{ij}) = (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-1}, \; F(s,t) = c(1 + s + t); \\
&d = 3 : p(\rho_{ij}) = (\Pi_{i<j}\rho_{ij})^{-1}, \; F(s,t) = \sum_{i=0,1} a_i \tilde{J}_i \; + b; \\
&d = 4 : p(\rho_{ij}) = (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-2}, \\
&stF(s,t) = \sum_{i=0}^{2} a_i J_i(s,t) \; + st(bD(s,t) + b'Q(s,t)) \, .
\end{aligned} \tag{13}$$

Here $J_i(s,t)$ are polynomials of overall degree 5 in their arguments (given in [5, 8]), $D(s,t)$ is a second degree polynomial (given in (22) below), $Q(s,t) = s + t + st$;

$$\tilde{J}_0 = s + t + \frac{t+1}{s} + \frac{s+1}{t}, \; \tilde{J}_1 = \frac{(1-t)(1-t^2)}{st} - t^{-1} - t - s(1 + t^{-1}) + \frac{s^2}{t} \tag{14}$$

are the symmetrized twist two contributions to the 4-point function which are computed as follows. We organize, following [4], the *operator product expansion* (OPE) of two hermitean scalar fields of (integer) dimension d in terms of (mutually orthogonal) bilocal fields $V_\kappa(z_1, z_2)$ of twist 2κ:

$$\phi(z_1)\phi(z_2) =< 12 > + \sum_{\kappa > 0} (z_{12}^2)^{\kappa - d} V_\kappa(z_1, z_2). \tag{15}$$

The first of them, V_1 can be expanded in an infinite series of even-rank conserved symmetric traceless tensors and, as a consequence, is harmonic in each argument (see Proposition 2.1 of [4]). This allows to compute the 4-point function of two V_1 as a finite linear combination of a standard basis of GCI solutions:

$$z_{13}^2 z_{24}^2 < 0|V_1(z_1, z_2)V_1(z_3, z_4)|0 >= \sum_{\nu=0}^{d-1} a_\nu j_\nu(s,t), j_0 = 1 + t^{-1}, ... \tag{16}$$

(We shall display the general form of j_ν in Sect. 3, below.) Then $\tilde{J}_\nu$ and J_ν are crossing symmetric expressions satisfying

$$\frac{s}{t}\tilde{J}_\nu(s,t) - j_\nu(s,t) = O(s), \; t^{-3}J_\nu(s,t) - j_\nu(s,t) = O(s) \tag{17}$$

(see [5] Sect. 5.2).

Here is also an example of a mixed 4-point function of a pair of scalar fields $t(z)$ and $W(z)$ of dimensions 2 and 3, respectively:

$$< 0|t(z_1)W(z_2)W(z_3)t(z_4)|0 > -\frac{N}{(z_{14}^2)^2(z_{23}^2)^3} = \frac{c(1+s)+c't}{z_{12}^2 z_{13}^2 z_{23}^2 z_{24}^2 z_{34}^2 t}. \tag{18}$$

The 1D reduction of Eqs. (13) for $d=2$ and $d=3$ and (18) is an extension of the 1985 Zamolodchikov's W_3 model (for a review see [28]). This model would be recovered if there were a single field of dimension four in the $1D$ restriction (which would then necessarily coincide with the normal square of $t(z)$ and yield $a_0 = c$, $6a_1 + b = \frac{16c(2-c)}{5c+22}$, $N = \frac{c^2}{6}$, $c' = 2c$). This is not the case, however, since the 4D stress-energy tensor will also appear as a chiral field of dimension $d=4$ in the 1D restriction.

In the simplest non-trivial case, $d=2$ the truncated 4-point function involves a single parameter c which can be defined by the invariant under rescaling of the basic field: $8 < 12 >< 23 >< 13 >= c(< 123 >)^2$. The following result was established in [3].

Theorem 2.1. *The scalar GCI field $t(z)$ of dimension $d=2$ with 2-point function given by (10) with $N = \frac{c}{2}$ obeys an OPE of the form (15) with a single singular term $V_1 \equiv V$ in the sum,*

$$t(z_1)t(z_2) =< 12 > + \frac{V(z_1, z_2)}{z_{12}^2} + : t(z_1)t(z_2) :, \; V(z,z) = 2t(z). \tag{19}$$

Wightman positivity implies that V generates a unitary vacuum representation of a central extension of the infinite dimensional real symplectic Lie algebra with positive integral central charge c. As a corollary t can be presented as (half) the sum of normal squares of c free commuting massless scalar fields.

The *proof* involves two essential steps. First, one finds the $2n$-point functions of V as sums of 1-loop graphs (see Sect. 2 of [3] and Appendix A.1 of [4]) and derives on this basis the commutation relations of the (extended) symplectic algebra for the V's. Secondly, one obtains (Sect. 5.1 of [3]) an analogue of Kac's determinant formula [29] for this infinite dimensional Lie algebra.

Remark 2.1. In a 1982 paper [30](which contains an early proof of the fact that the Huygens principle implies rationality) Baumann has proven that all massless scalar fields with a trivial S-matrix are Wick polynomials of free fields. It is, in fact, clear that the state-field correspondence and the presence of (zero-mass) asymptotically complete set of particle states implies that the field algebra is generated by free local fields.[1] Thus a CFT (even if it only obeys infinitesimal conformal invariance) can never have a (nontrivial) scattering theory. One should therefore use a more subtle criterion to distinguish nontrivial conformal models which are known to exist (at least in two space-time dimensions).

We view the case $d=4$ [4, 6] corresponding to a (gauge invariant) Lagrangian density, whose dimension is expected to be protected, as the most promising one for providing a non-trivial GCI model. As discussed in [6] (see, in particular, Eq. (1.10) there, at the end of the Introduction) the systematic study of this theory also requires the knowledge of a system of scalar fields of lower dimensions ($d = 2, 3$).

[1] I owe this remark to Nikolay Nikolov.

3 Conformal partial wave expansion: a tool for verifying Wightman positivity

A powerful tool in studying Wightman positivity is provided by operator product expansions [31, 32] that yield *conformal partial wave expansions* [33]. The positive definite 4-point function of a pair of hermitean scalar fields A and B of dimensions d and $d+\delta$, respectively, admits a conformal partial wave expansion of the form

$$< 0|A(z_1)B(z_2)B(z_3)A(z_4)|0 >= \sum_{\kappa L} < 0|A(z_1)B(z_2)\Pi_{\kappa L}B(z_3)A(z_4)|0 >=$$
$$[(z_{12})^2(z_{34})^2]^{-d}[(z_{23})^2]^{-\delta}\sum_{\kappa L} B_{\kappa L}\beta^{\delta}_{\kappa L}(s,t) \tag{20}$$

where $\Pi_{\kappa L}$ is the projection on the positive energy irreducible representation (IR) of the conformal group of $u(1)\times su(2)\times su(2)$ weight $(2\kappa+L, \frac{1}{2}L, \frac{1}{2}L)$; the *conformal partial waves* $\beta_{\kappa L}$ are universal functions, only depending on the above IR. Following [34] they can be expressed in terms of hypergeometric functions by using a higher dimensional analogue u, v of the 2D chiral coordinates:

$$(u-v)\beta_{\kappa L} = uv(G^{\delta}_{\kappa+L-\frac{\delta}{2}}(u)G^{\delta}_{\kappa-1-\frac{\delta}{2}}(v) - (u\leftrightarrow v))$$
$$G^{\delta}_{\nu}(z) = z^{\nu}F(\nu,\nu;2\nu+\delta;z), \tag{21}$$

where u and v are related to s and t (12) by

$$s = uv,\ t = (1-u)(1-v),\ (u-v)^2 = (1-t)^2 - 2s(1+t) + t^2 =: D(s,t). \tag{22}$$

The full dynamical information is carried by the coefficients $B_{\kappa L}$. Wightman positivity for the 4-point function is equivalent to the requirement that all these coefficients are non-negative. The computation of $B_{\kappa L}$ [8] uses the expression for $(u-v)F(s,t)$ (13) as a sum of products of non-negative powers of u or $\frac{u}{1-u}$ and similar monomials in v. For the twist two contributions we have, in particular,

$$(u-v)j_{\nu}(s,t) = f_{\nu}(u) - f_{\nu}(v),\ f_{\nu}(u) = \frac{u^{\nu}}{(1-u)^{\nu}} + (-1)^{\nu}u^{\nu}. \tag{23}$$

To compare the expressions (13) and (18) with the expansion (20) one uses (special cases of) the identity

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n(\beta)_n}{n!(\gamma+n-1)_n}z^nF(n+\alpha,n+\beta;2n+\gamma;z) = 1. \tag{24}$$

In the most interesting case in which the basic scalar field can be interpreted as a (gauge invariant) Lagrangian $\mathcal{L}(z)$ of $d=4$ (see the last equation (13)) we find for the twist two partial waves

$$\frac{1}{2}B_{1L} = a_0 + L(L+1)a_1 + \frac{1}{4}(L-1)L(L+1)(L+2)a_2. \tag{25}$$

The (more complicated) expressions of all higher twist amplitudes are written down in (Sect. 2 of) [6] where it is established that Wightman positivity is satisfied for the closure of a non-empty open set in the five- dimensional parameter space describing the 4-point function. One finds, in particular, that $a_\nu, \nu = 0, 1, 2$ and b' should be non-negative, while if $b' = 0$ then

$$-3a_1 \leq b \leq \frac{1}{3}(2a_0 + a_1). \tag{26}$$

As discussed in [6] it is of particular interest to consider the case in which the operator product expansion of $\mathcal{L}(z_1)\mathcal{L}(z_2)$ involves no scalar field of dimension 2 or 4, so that both a_0 and b' vanish and Eq. (26 further simplifies.

4 Thermal states. Modular properties of energy mean values

The conformal Hamiltonian H satisfying (8) has a discrete (integer or half-integer) spectrum in the vacuum space $\mathcal{V}$ of a GCI theory which is assumed finitely degenerate and such that there exist a partition function $Z(\tau)$ (defined as a trace over the Boltzmann weights in $\mathcal{V}$) and *thermal mean values* $< A >_q$ for any product A of local GCI fields:

$$Z(\tau) < A >_q = tr(Aq^H),\, Z(\tau) = tr(q^H),\, q = e^{2\pi i\tau},\, Im\tau > 0\,(|q| < 1), \tag{27}$$

an assumption verified for (generalized) free fields [7]. Here τ is interpreted as the (complexified) inverse temperature; more precisely,

$$Im\tau = \frac{1}{kT}. \tag{28}$$

In order to reveal the properties of thermal correlation functions it is advantageous to use (real variable)*compact picture fields* $\phi(\zeta, u)$ related to the corresponding analytic (z-picture) vertex operators $\phi(z)$ (of dimension d) by

$$\phi(\zeta, u) = e^{2\pi i d\zeta}\phi(z)\, for\, z = e^{2\pi i\zeta}. \tag{29}$$

A compact picture field is (anti)periodic in ζ depending on its spin:

$$\phi(\zeta + 1, u) = (-1)^{2j_1+2j_2}\phi(\zeta, u). \tag{30}$$

The *Kubo-Martin-Schwinger boundary condition* [35] which says, e.g. for the 2-point function of ϕ,

$$w_q(\zeta_1 - \zeta_2 - \tau; u_1, u_2) \equiv < \phi(\zeta_1, u_1)\phi^*(\zeta_2, u_2) >_q = < \phi^*(\zeta_2, u_2)\phi(\zeta_1, u_1) >_q, \tag{31}$$

implies that the function $w_q(\zeta, u_1, u_2)$ has a second period, $\tau + 1$, related to the inverse temperature, on top of the period 1 (or 2, for Fermi fields). For a scalar field w_q depends on u_1, u_2 through their scalar product, $u_1 u_2 = cos(2\pi\alpha)$. In particular, for the free massless field φ we find

$$w_q(\zeta, \tau; \alpha) = \sum_{n\in\mathbb{Z}} w_0(\zeta + n\tau, \alpha) \tag{32}$$

where the vacuum 2-point function of φ is given by

$$w_0(\zeta, \alpha) = \frac{-1}{4sin(\pi(\zeta + \alpha))sin(\pi(\zeta - \alpha))} = \frac{1}{4sin2\pi\alpha}(cot(\pi(\zeta+\alpha)) - cot(\pi(\zeta-\alpha))). \tag{33}$$

On the other hand, for a suitable choice of the vacuum energy E_0, the thermal mean value in the theory of a free massless scalar field is given by the unique weight-two (normalized) modular form:

$$< H + E_0 >_q = G_4(\tau) = -\frac{B_4}{8} + \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, E_0 = -\frac{B_4}{8} = \frac{1}{240} \tag{34}$$

(B_n being the Bernoulli numbers). Note that, restoring the energy units, $H + E_0 \mapsto \hbar\nu(H + E_0)$, it gives for $q = exp(-\frac{\hbar\nu}{kT})$ Planck's black-body energy distribution of the harmonic oscillator (with energy eigenvalues $n\hbar\nu$). Invariance under the modular inversion, $\tau^{-4}G_4(\frac{-1}{\tau}) = G_4(\tau)$, allows to compute the high temperature expansion of $< H >_q$ in terms of its low temperature (small q) behavior. If we replace the unit 3-sphere in (3) by a sphere of radius R, substituting $z(x)$ by $Rz(\frac{x}{2R})$, and identifying the frequency ν in the definition of the resulting Hamiltonian by $\frac{c}{R}$, then the high temperature asymptotics reproduces the Stefan-Boltzmann law for the energy density in the infinite volume limit (Sect. 5.1 of [7]). This result is, in fact, valid under more general conditions (see Appendix A of [36]).

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Liouville Field Theory on a Pseudosphere

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Abstract

Liouville field theory is considered with boundary conditions corresponding to a quantization of the classical Lobachevskiy plane (i.e. euclidean version of AdS_2). We solve the bootstrap equations for the out-vacuum wave function and find an infinite set of solutions. This solutions are in one to one correspondence with the degenerate representations of the Virasoro algebra. Consistency of these solutions is verified by both boundary and modular bootstrap techniques. Perturbative calculations lead to the conclusion that only the "basic" solution corresponding to the identity operator provides a "natural" quantization of the Lobachevskiy plane.

1 Introduction

Liouville field theory (LFT) is widely considered as an appropriate field theoretic background for a certain universality class of two-dimensional quantum gravity. It has been demonstrated in numerous examples that in 2D the scaling limit of the so-called "dynamical triangulations" [1, 2, 3, 4] (which are in fact a discrete model of a two-dimensional surface with fluctuating geometry) in many cases can be described by appropriately applied LFT [5, 6, 7]

Local dynamics of LFT is determined by the action density

$$\mathcal{L}(z) = \frac{1}{4\pi}(\partial_a \phi(z))^2 + \mu e^{2b\phi(z)} \tag{1}$$

where ϕ is the Liouville field and b is a dimensionless parameter which, roughly speaking, determines the "rigidity" of a 2D surface to quantum fluctuations of the metric. Ordinarily $\exp(2b\phi(z))d^2z$ is interpreted as the quantum volume element of the fluctuating surface, parameter μ being the cosmological coupling constant. LFT is a conformal field theory with central charge

$$c_L = 1 + 6Q^2 \tag{2}$$

where Q is yet another convenient parameter called the "background charge"

$$Q = b^{-1} + b \tag{3}$$

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More details about the space of states in LFT, the set of local primary fields and local operator algebra can be found e.g. in [11].

Local equation of motion for (1)

$$\Delta\phi = 4\pi\mu b e^{2b\phi} \tag{4}$$

is the quantum version of the classic Liouville equation

$$\Delta\varphi = 2R^{-2}e^{\varphi} \tag{5}$$

It describes locally a metric

$$ds^2 = e^{\varphi(z)}\,|dz|^2 \tag{6}$$

of constant negative curvature $-2R^{-2}$ in the isothermal (or conformal) coordinates. Classical situation arises in LFT if $b \to 0$. In this limit we identify the classical field $\varphi = 2b\phi$ while $R^{-2} = 4\pi\mu b^2$.

Discrete models of quantum gravity, such as the random triangulations or matrix models, typically deal with compact fluctuating surfaces of different topologies either with or without boundaries. This type of problems is most relevant in the string theory. In this context the main problem is somewhat different from that considered usually in field theory. Namely, observables of primary interest are the “integrated” correlation functions, which bear no coordinate dependence and can be rather called the “correlation numbers”. They are used to describe certain “deformations” or “flows” caused by relevant perturbations (see e.g. the reviews [8, 9] and [10] for more details). In many problems of this kind the discrete approaches presently appear more efficient then the field theoretic description based on LFT. In the discrete schemes the correlation functions naturally arise in the “integrated” form while the field theoretic approach implies a gauge fixing and gives the correlation functions as the functions of certain moduli (invariants of the complex structure in the case of LFT). These functions, although being themselves of considerable interest, should be yet integrated over the moduli space to produce the correlation numbers. Therefore in such problems of quantum gravity LFT still lags behind matrix models or other discrete approaches. Up to now only general scaling exponents and a limited set of correlation functions in simplest compact topologies can be predicted in LFT (see e.g. [9] and references therein).

It is well known that the Liouville equation (5) admits “basic” solution, which describes the geometry of infinite constant negative curvature surface, the so-called Lobachevskiy plane, or pseudosphere. This surface can be realized as the disk $|z| < 1$ with metric (6) where

$$e^{\varphi(z)} = \frac{4R^2}{(1 - z\bar{z})^2} \tag{7}$$

Here R is interpreted as the radius of the pseudosphere. The points at the circle $|z| = 1$ are infinitely far away from any internal point and form a one-dimensional infinity called the absolute. Geometry of the pseudosphere is described in detail in standard textbooks and here we will not go into further details like $SL(2, R)$ symmetry, geodesics etc. Let us mention only the so-called Poincaré model, where the same geometry is represented in the upper half plane of complex ξ with the metric

$$e^{\varphi(\xi)} = \frac{R^2}{(\mathrm{Im}\xi)^2} \tag{8}$$

It seems natural to expect that LFT, at least in the semi-classical regime $c_L > 25$, also allows a solution corresponding to a quantization of this geometry. In this paper we present what we believe might be the solution to this problem. Surprisingly, we find an infinite set of different consistent solutions parameterized by a couple of positive integers (m, n) (which can be put in natural correspondence with the degenerate representations of the Virasoro algebra). Only the set $(1, n)$ has a smooth behavior as $b \to 0$ and therefore can be literally called the "quantization" of (7). Remarkably, all the solutions of this $(1, n)$ series are indistinguishable in the classical limit (and even at the one-loop level), dependence on n appearing only in two-loop corrections. However, actual higher loop calculations show that only the solution $(m, n) = (1, 1)$ is consistent with the standard loop perturbation theory. Therefore we are inclined to interpret this last solution as the "basic" one, corresponding to a "natural" quantization of the Lobachevskiy plane. Although the nature of other solutions is still beyond our understanding (even of the "perturbative" series $(1, n)$, $n > 1$) they probably can be speculated as describing different phases of quantum gravity.

In principle all local properties of a field theory are encoded in its operator product expansions. The latter are basically known in LFT (see [12, 11]). To have a complete description we also need certain information of what is happening "faraway" from the observer, i.e, about the boundary conditions at infinity. This information is encoded in the wave function of the state which "comes from infinity", the so-called out-vacuum. In order, the out-vacuum wave function can be described as the set of vacuum expectation values (VEV's) of all local fields in the theory. In conformal field theory, like LFT, it suffices to determine the VEV's (or one-point functions) of all primary operators. The basic Liouville primaries are the exponential fields

$$V_\alpha = \exp(2\alpha\phi) \tag{9}$$

of dimensions $\Delta_\alpha = \alpha(Q - \alpha)$. Thus the set of VEV's $\langle V_\alpha \rangle$ is just the complementary information we need to describe LFT in the pseudosphere geometry. In this paper we mainly concentrate on this characteristic.

The paper is arranged as follows. In sect.2 the bootstrap technique is applied to derive the one-point functions $\langle V_\alpha \rangle$. We observe that all the out-vacuums (m, n), if considered as conformal boundary conditions at absolute, allow only finite set of boundary operators. In particular, the basic out-vacuum $(1, 1)$ does not contain any boundary fields except the identity operator and its conformal descendents. Few simplest bulk-boundary structure constants are also derived in this section. Certain properties of the solutions are discussed in section 3. This includes perturbative expansions of the one-point functions and some evidence about the content of the boundary operators in the state (m, n). In sect.4 one- and two-loop contributions to the one-point functions are evaluated in the framework of standard Feynmann diagram technique. At two loops these calculations agree with the expansion of the "basic" vacuum state $(1, 1)$.

In sect.5 the powerful modular bootstrap technique is applied to verify the consistency of the proposed operator content at the out-vacuum states (m, n). Partition function of an annulus with "boundaries" corresponding to different out-vacua (m, n) is considered. It turns out that the modular invariance of this partition function perfectly agrees with the suggested operator content and can be further implemented for "finite" boundary conditions discussed in ref.[13].

With a finite set of boundary operators any two-point function in the bulk is constructed as a finite sum of four-point conformal blocks. In sect.6 we develop this construction explicitly for two simplest vacua $(1,1)$ and $(1,2)$ and verify numerically that it satisfies the bulk-boundary bootstrap. Some outlook and discussion is presented in sect.7.

2 One-point bootstrap

The basic assumption of the further development is that the out-vacuum state generated by the absolute of the pseudosphere is conformally invariant, i.e., consists of a superposition of the Ishibashi states [14]. The one-point functions of primary fields are nothing but the amplitudes of different Ishibashi primaries in the out-vacuum wave function.

In this section we will use the Poincaré model of the Lobachevskiy plane with complex coordinate ξ in the upper half plane. Due to the conformal invariance the coordinate dependence of any one-point function is prescribed by the dimension of the operator

$$\langle V_\alpha(\xi)\rangle = \frac{U(\alpha)}{\left|\xi-\bar{\xi}\right|^{2\Delta_\alpha}} \tag{10}$$

Thus we will call coordinate independent function $U(\alpha)$ the one-point function and normalize it in the usual in field theory way $U(0)=1$.

Of course, local properties of LFT do not depend on the boundary conditions. In particular the set of (bulk) degenerate fields

$$\Phi_{m,n} = \exp\left(((1-m)b^{-1}+(1-n)b)\phi\right) \tag{11}$$

still exists for any pair of positive m and n. Therefore one can make use of the trick applied by J.Teschner in the study of the operator algebra [15] (see also [16] for very similar discussions). Consider the following auxiliary two-point correlation function with the insertion of an operator $\Phi_{1,2}=V_{-b/2}$

$$G_{-b/2,\alpha}(\xi,\xi') = \left\langle V_{-b/2}(\xi)V_\alpha(\xi')\right\rangle \tag{12}$$

Degenerate fields have very special structure of the operator product expansions. In particular, the product $\Phi_{1,2}V_\alpha$ contains in the right hand side only two primary fields $V_{\alpha-b/2}$ and $V_{\alpha+b/2}$. Function (12) is therefore combined of two degenerate conformal blocks

$$G_{-b/2,\alpha} = \frac{\left|\xi'-\bar{\xi}'\right|^{2\Delta_\alpha-2\Delta_{12}}}{\left|\xi-\bar{\xi}'\right|^{4\Delta_a}}\left[C_+(\alpha)U(\alpha-b/2)\mathcal{F}_+(\eta)+C_-(\alpha)U(\alpha+b/2)\mathcal{F}_-(\eta)\right] \tag{13}$$

In our normalization the special structure constants $C_\pm(\alpha)$ read explicitly [15, 13]

$$\begin{aligned} C_+(\alpha) &= 1 \\ C_-(\alpha) &= -\pi\mu\frac{\Gamma(2\alpha b-b^2-1)\Gamma(1-2\alpha b)\Gamma(1+b^2)}{\Gamma(2+b^2-2\alpha b)\Gamma(2\alpha b)\Gamma(-b^2)} \end{aligned} \tag{14}$$

Degenerate conformal blocks $\mathcal{F}_\pm(\eta)$ are functions of the projective invariant

$$\eta = \frac{(\xi-\xi')(\bar{\xi}-\bar{\xi}')}{(\xi-\bar{\xi}')(\bar{\xi}-\xi')} \tag{15}$$

They are known explicitly and can be expressed in terms of hypergeometric functions

$$\begin{aligned} \mathcal{F}_+(\eta) &= \eta^{\alpha b}(1-\eta)^{-b^2/2}{}_1F_2(2\alpha b-2b^2-1,-b^2,2\alpha b-b^2,\eta) \\ \mathcal{F}_-(\eta) &= \eta^{1+b^2-\alpha b}(1-\eta)^{-b^2/2}{}_1F_2(-b^2,1-2\alpha b,2+b^2-2\alpha b,\eta) \end{aligned} \tag{16}$$

The same expression (13) can be rewritten also in terms of the cross-channel degenerate blocks $\mathcal{G}_\pm(\eta)$

$$G_{-b/2,\alpha} = \frac{\left|\xi'-\bar{\xi}'\right|^{2\Delta_\alpha-2\Delta_{12}}}{\left|\xi-\bar{\xi}'\right|^{4\Delta_\alpha}}\left[B^{(+)}(\alpha)\mathcal{G}_+(\eta)+B^{(-)}(\alpha)\mathcal{G}_-(\eta)\right] \tag{17}$$

where

$$\begin{aligned} \mathcal{G}_+(\eta) &= \eta^{\alpha b}(1-\eta)^{-b^2/2}{}_1F_2(-b^2,2\alpha b-2b^2-1,-2b^2,1-\eta) \\ \mathcal{G}_-(\eta) &= \eta^{\alpha b}(1-\eta)^{1+3b^2/2}{}_1F_2(1+b^2,2\alpha b,2+2b^2,1-\eta) \end{aligned} \tag{18}$$

The boundary structure constants $B^{(\pm)}(\alpha)$ can be determined from the relations

$$\begin{aligned} \mathcal{F}_+(\eta) &= \frac{\Gamma(2\alpha b-b^2)\Gamma(1+2b^2)}{\Gamma(1+b^2)\Gamma(2\alpha b)}\mathcal{G}_+(\eta)+\frac{\Gamma(2\alpha b-b^2)\Gamma(-1-2b^2)}{\Gamma(2\alpha b-2b^2-1)\Gamma(-b^2)}\mathcal{G}_-(\eta) \\ \mathcal{F}_-(\eta) &= \frac{\Gamma(2+b^2-2\alpha b)\Gamma(1+2b^2)}{\Gamma(1+b^2)\Gamma(2+2b^2-2\alpha b)}\mathcal{G}_+(\eta)+\frac{\Gamma(2+b^2-2\alpha b)\Gamma(-1-2b^2)}{\Gamma(1-2\alpha b)\Gamma(-b^2)}\mathcal{G}_-(\eta) \end{aligned} \tag{19}$$

The block $\mathcal{G}_-(\eta)$ is recognized as corresponding to the identity boundary operator of dimension 0 while the boundary dimension $\Delta_{13}=-1-2b^2$ corresponding to the block $\mathcal{G}_+(\eta)$ suggests to identify it as the contribution of the degenerate boundary operator $\psi_{1,3}$.

Projective invariant (15) can be interpreted in terms of the geodesic distance $s(\xi,\xi')$ on the pseudosphere. In the classical metric (8)

$$\eta = \tanh^2\frac{s}{2R} \tag{20}$$

It is important that on the pseudosphere as $\eta\to 1$ the geodesic distance becomes infinite. In a unitary field theory a two-point correlation function is expected to decay in a product of the one-point ones as the distance goes to infinity. The corresponding contribution is provided by the identity operator. Therefore in a unitary theory with the usual large-distance decay of correlations one would expect for $B^{(-)}(\alpha)$

$$B^{(-)}(\alpha) = U(\alpha)U(-b/2) \tag{21}$$

Together with (19) this gives the following non-linear functional equation for $U(\alpha)$

$$\frac{\Gamma(-b^2)U(\alpha)U(-b/2)}{\Gamma(-1-2b^2)\Gamma(2\alpha b-b^2)} = \frac{U(\alpha-b/2)}{\Gamma(2\alpha b-2b^2-1)} - \frac{\pi\mu\Gamma(1+b^2)U(\alpha+b/2)}{(2\alpha b-b^2-1)\Gamma(-b^2)\Gamma(2\alpha b)} \tag{22}$$

Of course this equation admits many solutions. The set of the solutions can be restricted largely by adding a similar "dual" functional equation where α is shifted in $b^{-1}/2$ instead of $b/2$ in (22). Dual equation arises from the same calculation as (22) but with the degenerate field Φ_{21} taken instead of Φ_{12} in the auxiliary two-point function (12). Due to the duality of LFT (see e.g., [11]) this amounts the substitution $b\to b^{-1}$, $\mu\to\tilde{\mu}$ in (22). Here

$$\pi\tilde{\mu}\gamma(b^{-2}) = \left(\pi\mu\gamma(b^2)\right)^{1/b^2} \tag{23}$$

and as usual $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

It seems that (at least for real incommensurable values of b and $1/b$) all possible solutions fall into an infinite family parameterized by two positive integers (m,n)

$$U_{m,n}(\alpha) = \frac{\sin(\pi b^{-1}Q)\sin(\pi m b^{-1}(2\alpha-Q))}{\sin(\pi m b^{-1}Q)\sin(\pi b^{-1}(2\alpha-Q))}\frac{\sin(\pi bQ)\sin(\pi nb(2\alpha-Q))}{\sin(\pi nbQ)\sin(\pi b(2\alpha-Q))}U_{1,1}(\alpha) \tag{24}$$

where the "basic" $(1,1)$ one-point function reads

$$U(\alpha) = U_{1,1}(\alpha) = \frac{[\pi\mu\gamma(b^2)]^{-\alpha/b}\,\Gamma(bQ)\Gamma(Q/b)Q}{\Gamma(b(Q-2\alpha))\Gamma(b^{-1}(Q-2\alpha))(Q-2\alpha)} \tag{25}$$

In the next section we will discuss some properties of these solutions. Now let's take a look at the contribution of the block $\mathcal{G}_-(\eta)$ to (12). This term is interpreted as the contribution of the boundary operator $\psi_{1,3}$. Combining (19) and (24) one finds

$$\begin{aligned}\frac{B^{(+)}_{m,n}(\alpha)}{U_{m,n}(\alpha)} &= (-)^{m-1}\left[\pi\mu\gamma(b^2)\right]^{1/2}\frac{\Gamma(1+2b^2)\Gamma(1-2b\alpha)\Gamma(2\alpha b-2b^2-1)}{\pi\Gamma(b^2)} \\ &\times\frac{\sin(2\pi nb(\alpha-b))\sin(2\pi b\alpha)-\sin(2\pi nb\alpha)\sin(2\pi b(\alpha-b))}{\sin(\pi nb(2\alpha-b))}\end{aligned} \tag{26}$$

In the standard CFT picture $B^{(+)}_{m,n}(\alpha)$ is composed from the bulk-boundary structure constants $R^{(1,3)}_{m,n}(\alpha)$ for the operators V_α and $V_{-b/2}$ merging to the boundary operator ψ_{13} near the (m,n) boundary

$$B^{(+)}_{m,n}(\alpha) = R^{(1,3)}_{m,n}(\alpha)R^{(1,3)}_{m,n}(-b/2)D^{(1,3)}_{m,n} \tag{27}$$

Here $D^{(1,3)}_{m,n}$ stands for the boundary two-point function of two operators $\psi_{1,3}$.

In principle the bootstrap technique allows to continue this process and calculate all bulk-boundary structure constants $R^{(p,q)}_{m,n}(\alpha)$ corresponding to any degenerate boundary operator with odd p and q. Here we will not proceed systematically along this line. What can be already seen from eq.(26) is that $R^{(1,3)}_{m,n}(\alpha)$ vanishes for the basic out-vacuum state $(m,n)=(1,1)$. The following guess (which will be further supported in the subsequent sections) seems rather natural. *The basic vacuum* $(1,1)$ *contains no primary boundary operators except the identity.* In this sense the basic vacuum is similar to the basic conformal boundary condition discovered by J.Cardy [17] in the context of rational conformal field theories.

The whole variety of vacua (m,n) in this picture is naturally associated with the boundary conditions corresponding to the degenerate fields (11) themselves. Then, the content of boundary operators acting on the vacuum (m,n) (or, more generally, of the juxtaposition operators between different vacua (m,n) and (m',n')) is determined by the fusion algebra, exactly as in the rational case. For instance, the vacuum $(1,2)$ contains only identity boundary operator ($\psi_{1,1}=I$) and the degenerate field $\psi_{1,3}$.

In principle, all these suggestions can be verified by systematic calculations of the higher structure constants. We choose to postpone this difficult problem for future studies. Instead in sect.5 we will see that the above pattern is perfectly consistent with the modular bootstrap of the annulus partition function.

3 The one-point function

The solution (24) for the one-point function bears some remarkable properties.

1. One-point Liouville equation. For all (m, n)

$$U_{m,n}(b) = \frac{Q}{\pi\mu b} \tag{28}$$

(of course the dual relation $U_{m,n}(1/b) = bQ/(\pi\tilde{\mu})$ with $\tilde{\mu}$ from (23) is also valid). In particular, this means that the quantum Liouville equation in the form (4) holds on the one-point level. Indeed, if the quantum Liouville field ϕ is defined as

$$\phi = \frac{1}{2}\left.\frac{\partial V_\alpha}{\partial\alpha}\right|_{\alpha=0} \tag{29}$$

it follows from (10) that (we take the Poincaré model (8) for the moment)

$$\langle\phi(\xi)\rangle_{m,n} = -Q\log\left|\xi-\bar{\xi}\right|^2 + \partial U_{m,n}(\alpha)/\partial\alpha\big|_{\alpha=0} \tag{30}$$

and therefore

$$\Delta\langle\phi(\xi)\rangle_{m,n} = \frac{4Q}{\left|\xi-\bar{\xi}\right|^2} = 4\pi\mu b\,\langle V_b(\xi)\rangle_{m,n} \tag{31}$$

2. Normalization. All the one-point functions $U_{m,n}(\alpha)$ are normalized by $U_{m,n}(0) = 1$ so that the expectation value of the identity operator is 1. It will prove convenient to introduce the function

$$W(\lambda) = \frac{2\pi\lambda\left(\pi\mu\gamma(b^2)\right)^{-\lambda/b}}{\Gamma(1-2\lambda/b)\Gamma(1-2b\lambda)} \tag{32}$$

It satisfies the following functional relations

$$\begin{aligned} W(\lambda)W(-\lambda) &= -\sin(2\pi b\lambda)\sin(2\pi\lambda/b) \\ \frac{W(iP)}{W(-iP)} &= S_L(P) \end{aligned} \tag{33}$$

$S_L(P)$ being the standard Liouville reflection amplitude [11]

$$S_L(P) = \left(\pi\mu\gamma(b^2)\right)^{-2iP/b}\frac{\Gamma(1+2iP/b)\Gamma(1+2ibP)}{\Gamma(1-2iP/b)\Gamma(1-2ibP)} \tag{34}$$

Notice, that $U(Q/2+\lambda)$ differs from $W(\lambda)$ only in overall normalization

$$U(\alpha) = \frac{W(\alpha-Q/2)}{W(-Q/2)} \tag{35}$$

3. Reflection relation. Apparently, all the solutions $U_{m,n}(\alpha)$ satisfy the so-called reflection relations (see e.g. ref. [11] for details)

$$U_{m,n}(\alpha) = S_L\left(\frac{2\alpha-Q}{2i}\right)U_{m,n}(Q-\alpha) \tag{36}$$

with the Liouville reflection amplitude (34). This is consistent with the local properties of the LFT primary fields suggested in ref.[11].

4. In the "basic" vacuum $(\mathbf{m},\mathbf{n})=(\mathbf{1},\mathbf{1})$ the one-point function $U_{1,1}(\alpha)$ reads

$$U_{1,1}(\alpha)=U(\alpha)=\frac{[\pi\mu\gamma(b^2)]^{-\alpha/b}\,\Gamma(2+b^2)\Gamma(1+1/b^2)}{\Gamma(2+b^2-2b\alpha)\Gamma(1+b^{-2}-2\alpha/b)} \tag{37}$$

while from eq.(26) we have

$$B_{1,1}^{(+)}(\alpha)=0 \tag{38}$$

i.e., as it has been mentioned in the previous section, boundary field $\psi_{1,3}$ does not contribute to (12) in the basic vacuum. As it has been suggested above, this is a particular instance of a more general phenomenon: In the basic vacuum the only primary boundary operator is the identity one. This feature makes the basic state rather distinguished among the whole set (24). We are inclined to identify it as the generic out-vacuum state of LFT on the Lobachevskiy plane. In the next section this statement is checked against the ordinary (loop) perturbation theory up to two-loop order.

5. Perturbative expansions. If $m>1$ the one-point function $U_{m,n}(\alpha)$ is essentially singular at $b\to 0$ and therefore admits no usual classical limit. Singular behavior in the classical limit makes it rather difficult to interpret these states as a "quantization" of any classical metric. For the moment let us restrict attention to the case $m=1$ where $U_{1,n}(\alpha)$ is smooth in the classical limit and can be expanded in an (asymptotic) series in b.

Take the disk model (7) of the pseudosphere, where the one-point function reads

$$\langle V_\alpha(z)\rangle=\frac{U(\alpha)}{(1-z\bar z)^{2\alpha(Q-\alpha)}} \tag{39}$$

First, it is convenient to take the logarithm of this function and expand it in powers of α

$$\log\langle V_\alpha(z)\rangle=\sum_{k=1}^{\infty}\frac{(2\alpha)^k}{k!}G_k(b^2) \tag{40}$$

The coefficients are readily interpreted as the VEV's of "connected powers" of the Liouville field ϕ, e.g.

$$\begin{aligned}
G_1(b)\ &=\langle\phi\rangle\\
G_2(b)\ &=\langle\phi^2\rangle-\langle\phi\rangle^2\\
G_3(b)\ &=\langle\phi^3\rangle-3\langle\phi\rangle\langle\phi\rangle^2+2\langle\phi\rangle^3\\
&\text{etc.}
\end{aligned} \tag{41}$$

In order, each $G_k(b^2)$ allows an asymptotic expansion in powers of b

$$G_k(b^2)\sim\sum_{l=k-1}^{\infty}G_k^{(l)}b^{2l-k} \tag{42}$$

For the first few coefficients we find (for the general "perturbative" solution $U_{1,n}(\alpha)$)

$$\begin{aligned}
G_1(b)\ &\sim-\frac{1}{2b}\log\left(\pi\mu b^2(1-z\bar z)^2\right)+b\left(-\log(1-z\bar z)+\frac{3}{2}\right)\\
&\quad+b^3\left(\frac{\pi^2}{6}-\frac{13}{12}+\frac{\pi^2(n^2-1)}{3}\right)+\dots\\
G_2(b)\ &\sim\log(1-z\bar z)-1+b^2\left(\frac{3}{2}-\frac{\pi^2}{6}-\frac{\pi^2(n^2-1)}{3}\right)+\dots\\
G_3(b)\ &\sim -b+\dots
\end{aligned} \tag{43}$$

Notice that the classical limit $G_1^{(0)}$ and one-loop terms $G_1^{(1)}$ and $G_2^{(1)}$ are completely unsensitive to the sort of the out-vacuum $(1,n)$, the dependence on the vacuum appearing only in the terms corresponding to two and higher loops ($l \geq 2$).

4 Loop perturbation theory

Expansions (43) can be compared against the standard loop perturbation theory around the classical solution (7). In this section we will use the most "naive" perturbation theory which treats the LFT action

$$A_L = \int \left[\frac{1}{4\pi}(\partial_a \phi)^2 + \mu e^{2b\phi} \right] d^2x \tag{44}$$

straightforwardly and leads to a diagram technique with different tadpole diagrams. In this technique the standard Liouville scaling exponents do not appear exactly from the very beginning but are the result of complete summation (however rather simple) of the tadpoles. Of course, more sophisticated diagram techniques can be developed for the Liouville field theory, which automatically take advantage of the Weil and conformal invariance of the theory to reduce the set of tadpole diagrams and start with the exact exponents [18]. These versions of LFT perturbation theory (which are of course equivalent to the "naive" one) have essential advantages at higher loop calculations where tadpole diagrams are rather numerous and their counting becomes a certain combinatorial problem.

The Weil invariance of LFT means that the theory with the action (44) is equivalent to LFT in any background metric $g_{ab}(x)$. In general the LFT action reads

$$A_L[g] = \int \left[\frac{1}{4\pi} g^{ab} \partial_a \phi \partial_b \phi + \frac{Q}{4\pi} R\phi + \mu e^{2b\phi} \right] \sqrt{g} d^2x \tag{45}$$

where R is the scalar curvature of the background metric, the theory being essentially independent on g. "Naive" form (44) of the Liouville action implies the "trivial" background metric $g_{ab} = \delta_{ab}$ in the parametric space, e.g. inside the unit disk (7). This means that all ultraviolet divergencies are regularized with respect to this trivial metric.

Of course the classical solution ϕ_{cl}

$$e^{2b\phi_{\text{cl}}} = \frac{1}{\pi\mu b^2 (1 - z\bar{z})^2} \tag{46}$$

does not depend on any background metric. Substituting

$$\phi = \phi_{\text{cl}} + \chi \tag{47}$$

into (44) we have

$$A_L = A_L^{(\text{cl})} + \int \left[\frac{1}{4\pi}(\partial_a \chi)^2 + \frac{e^{2b\chi} - 2b\chi - 1}{\pi b^2 (1 - z\bar{z})^2} \right] d^2x \tag{48}$$

At the classical (zero-loop) level only G_1 is non-zero

$$G_1 = -\frac{1}{2b} \log\left(\pi\mu b^2 (1 - z\bar{z})^2\right) + \dots \tag{49}$$

and consistent with $G_1^{(0)}$ in expansion (43). The one-loop (Gaussian) part of the action (48) has the form

$$\frac{1}{2\pi}\int\left[\frac{1}{2}(\partial_a\chi)^2+\frac{4\chi^2}{(1-z\bar{z})^2}\right]d^2x \tag{50}$$

It leads to the following "bare" propagator of the field χ

$$g(z,z')=\langle\chi(z,\bar{z})\chi(z',\bar{z}')\rangle=-\frac{1}{2}\left(\frac{1+\eta}{1-\eta}\log\eta+2\right) \tag{51}$$

The propagator depends only on the invariant

$$\eta=\frac{(z-z')(\bar{z}-\bar{z}')}{(1-z\bar{z}')(1-\bar{z}z')} \tag{52}$$

which is related to the "geodesic distance" s between the points z and z' as in eq.(20)

The simplest one-loop diagram of fig.1a contributes to $\langle\chi^2(z,\bar{z})\rangle$

$$\textbf{Fig.1a}=\lim_{z'\to z}\left(g(z,z')+\log|z-z'|\right)=\log(1-z\bar{z})-1 \tag{53}$$

With this result it is easy to evaluate the one-loop correction to $\langle\chi(z,\bar{z})\rangle$ as given by the diagram Fig.1b

$$\begin{aligned}\textbf{Fig.1b}\ &=-4b\int g(z,z')\frac{\langle\chi^2(z')\rangle}{(1-z'\bar{z}')^2}d^2z'\\ &=b\left(-\log(1-z\bar{z})+3/2)\right)\end{aligned} \tag{54}$$

Both (53) and (54) agree with the one-loop terms in (43).

In general there is no need to calculate separately the contribution to G_1 at any loop order once the corresponding contributions to higher G's are known. This is due to the following Ward identity

$$\langle\chi\rangle=\frac{2}{\pi b}\int g(z,z')\frac{\left\langle e^{2b\chi(z')}\right\rangle-2b\left\langle\chi(z')\right\rangle-1}{(1-z'\bar{z}')^2}d^2z' \tag{55}$$

which apparently holds order by order in the loop perturbation theory. Notice that this identity can be considered as a perturbative equivalent of the exact relation (28).

The simplest two-loop correction is the leading contribution to G_3. There is only one diagram Fig.1f which is readily evaluated

$$\textbf{Fig.1f}=-\frac{8b}{\pi}\int\frac{g^3(z,z')}{(1-z'\bar{z}')^2}d^2z'=-8b\int_0^1\frac{g^3(\eta)d\eta}{(1-\eta)^2}=-b \tag{56}$$

again in agreement with the exact value of $G_3^{(2)}$ in (43). Next, take the two-loop corrections to G_2. Let us first evaluate the tadpole contributions of Fig.1c and Fig.1d. Since

$$b\,(\textbf{Fig.1a})+\textbf{Fig.1b}=b/2 \tag{57}$$

we have

$$\begin{aligned}\textbf{Fig.1c}+\textbf{Fig.1d}\ &=-4b^2\int_0^1\frac{g^2(\eta)}{(1-\eta)^2}d\eta\\ &=\frac{6-\pi^2}{9}b^2\end{aligned} \tag{58}$$

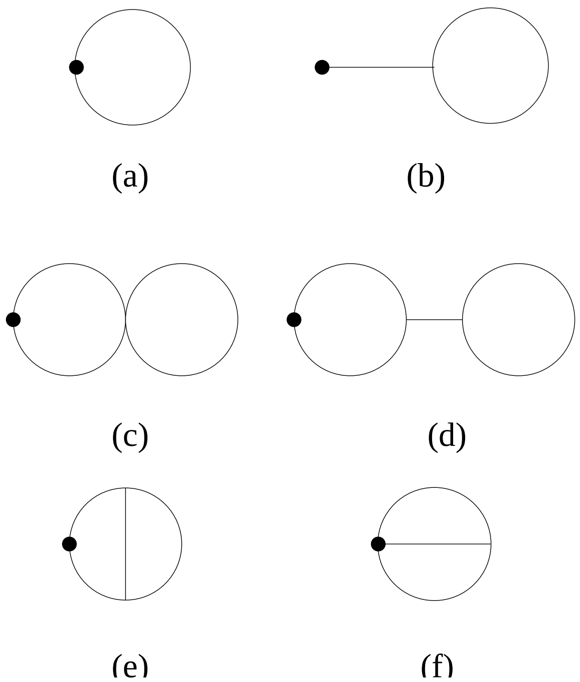
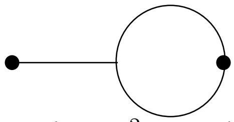


Figure 1: Diagrams contributing to the expectation values G_1, G_2 and G_3 at one- and two-loop order.

To evaluate the two-loop diagram Fig.1e we need the following two-point function

$$g_{1,2}(\eta) = \left\langle \chi(z)\chi^2(z') \right\rangle \tag{59}$$

as given by the following one-loop diagram

$g_{1,2}(z,z') = \chi$  $\chi^2 = \int \frac{g(z,z'')g^2(z'',z')}{(1-z''\bar{z}'')^2} d^2z''$

$$= -\frac{\pi}{8}\left(\frac{\eta\log^2\eta}{(1-\eta)^2} - 1\right) \tag{60}$$

For the two-loop diagram we obtain

$$\begin{aligned} \textbf{Fig.1e} &= \frac{32b^2}{\pi^2}\int \frac{g_{1,2}(z,z')g(z,z')}{(1-z'\bar{z}')^2} d^2z' \\ &= \frac{15-\pi^2}{18} b^2 \end{aligned} \tag{61}$$

Adding all the two-loop diagrams together results in

$$G_2^{(2)} = \textbf{Fig.1c} + \textbf{Fig.1d} + \textbf{Fig.1e} = \frac{9-\pi^2}{6} b^2 \tag{62}$$

and agrees with the expansion (43) if we take $n = 1$, i.e., for the "basic" vacuum state.

Here we will not develop further the loop perturbation theory for LFT on the Lobachevskiy plane. To go at higher loop diagrammatic calculations it is worth first to improve the technique to better handle the tadpole diagrams (which become rather numerous at higher orders) and second to take advantage of the space-time symmetries (the $SL(2,R)$ group) of the theory. We hope to turn at these interesting points in close future.

5 Modular bootstrap

General non-degenerate Virasoro character is written as

$$\chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)} \tag{63}$$

where

$$\eta\left(\tau\right) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = \exp(2i\pi\tau) \tag{64}$$

Here P is related to the central charge and the dimension of the representation via eqs.(2) and

$$\Delta_P = Q^2/4 + P^2 \tag{65}$$

Degenerate representations appear at [19]

$$\Delta_{m,n} = Q^2/4 - (m/b + nb)^2/4 \tag{66}$$

where (m, n) are positive integers. At general b there is only one null-vector at the level mn. Hence the degenerate character reads simply as

$$\chi_{m,n}(\tau) = \frac{q^{-(m/b+nb)^2/4} - q^{-(m/b-nb)^2/4}}{\eta(\tau)} \tag{67}$$

Applying the identity

$$\begin{aligned} \chi_P(\tau') &= e^{2i\pi P^2\tau'}\sqrt{-i\tau'}\eta^{-1}(\tau) \\ &= \sqrt{2}\int \chi_{P'}(\tau)e^{4i\pi PP'}dP' \end{aligned} \tag{68}$$

where

$$\begin{aligned} \tau' &= -1/\tau \\ q' &= \exp(2i\pi\tau') \end{aligned} \tag{69}$$

we find

$$\begin{aligned} \chi_{m,n}(\tau') &= \sqrt{2}\int \chi_P(\tau)\left(\cosh 2\pi(m/b+nb)P - \cos 2\pi(m/b-nb)P\right)dP \\ &= 2\sqrt{2}\int \chi_P(\tau)\sinh(2\pi mP/b)\sinh(2\pi nbP)dP \end{aligned} \tag{70}$$

In particular

$$\begin{aligned} \chi_{1,1}(q') &= 2\sqrt{2}\int \chi_P(q)\sinh(2\pi bP)\sinh(2\pi P/b)dP \\ &= \int \Psi_{1,1}(P)\Psi_{1,1}(-P)\chi_P(q)dP \end{aligned} \tag{71}$$

where we have set

$$\begin{aligned} \Psi_{1,1}(P) &= \frac{2^{3/4}2i\pi P}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}(\pi\mu\gamma(b^2))^{-iP/b} \\ &= 2^{3/4}W(iP) \end{aligned} \tag{72}$$

The function $W(\lambda)$ has been defined in eq.(32). $\Psi_{1,1}(P)$ is interpreted as the wave function of the basic out-vacuum state

$$\langle(1,1)\text{ outvac}| = \int \Psi_{1,1}(P)\,\langle P|\,dP \tag{73}$$

Here $\langle P|$ are the Ishibashi states [14]

$$\langle P| = \langle v_P| \left(1 + \frac{L_1 \bar{L}_1}{2\Delta_P} + \dots\right) \tag{74}$$

for different primary states v_P. The last are assumed to be normalized as follows

$$\langle v_P | v_{P'}\rangle = \delta(P - P') \tag{75}$$

Let us now take (70) and represent it in the form

$$\chi_{m,n}(q') = \int \Psi_{m,n}(P)\Psi_{1,1}(-P)\chi_P(q)dP \tag{76}$$

with

$$\Psi_{m,n}(P) = \Psi_{1,1}(P)\frac{\sinh(2\pi mP/b)\sinh(2\pi nbP)}{\sinh(2\pi P/b)\sinh(2\pi bP)} \tag{77}$$

(compare this expression with eq.(24)). This is naturally interpreted as the wave function of a general (m,n) out-vacuum state. It remains us to verify the operator content in the decomposition of the "partition function"

$$\begin{aligned} Z_{(m,n),(m',n')}(q) &= \int \Psi_{m,n}(P)\Psi_{m',n'}(-P)\chi_P(q)dP \\ &= \int \frac{\sinh(2\pi mP/b)\sinh(2\pi nbP)\sinh(2\pi m'P/b)\sinh(2\pi n'bP)}{\sinh(2\pi P/b)\sinh(2\pi bP)}\chi_P(q)dP \end{aligned} \tag{78}$$

Thanks to the identity

$$\sinh(2\pi nbP)\sinh(2\pi n'bP) = \sum_{l=0}^{\min(n,n')-1} \sinh(2\pi bP)\sinh(2\pi b(n+n'-2l-1)P) \tag{79}$$

this results in the standard character set

$$Z_{(m,n),(m',n')}(q) = \sum_{k=0}^{\min(m,m')-1}\sum_{l=0}^{\min(n,n')-1} \chi_{m+m'-2k-1,n+n'-2l-1}(q') \tag{80}$$

determined by the fusion algebra of the degenerate representations.

Consider also a general non-degenerate character with $P = s/2$ (i.e., with $\Delta = Q^2/4 + s^2/4$)

$$\chi_{s/2}(q') = \sqrt{2}\int \chi_P(q)\cos(2\pi sP)dP \tag{81}$$

This can be interpreted as

$$\chi_{s/2}(q') = \int \Psi_{1,1}(P)\Psi_s(-P)\chi_P(q)dP \tag{82}$$

if

$$\begin{aligned}\Psi_s(P) &= \frac{2^{-1/4}\Gamma(1+2ibP)\Gamma(1+2iP/b)\cos(2\pi sP)}{-2i\pi P}(\pi\mu\gamma(b^2))^{-iP/b} \\ &= \frac{2^{-1/4}W(iP)\cos(2\pi sP)}{\sinh(2\pi bP)\sinh(2\pi P/b)}\end{aligned} \tag{83}$$

This wave function has been already discussed in ref.[13] (see also [20] for modular considerations) in connection with certain local conformally invariant boundary conditions in boundary LFT. Therefore it is natural to associate such boundary state with a general non-degenerate representation with $P = s/2$.

Next, let us decompose the following overlap integral

$$\int \Psi_{m,n}(P)\Psi_s(-P)\chi_P(q)dP = \sqrt{2}\int \chi_P(q)\frac{\sinh(2\pi mP/b)\sinh(2\pi nbP)}{\sinh(2\pi P/b)\sinh(2\pi bP)}\cos(2\pi sP)dP \tag{84}$$

Again, we use the identity

$$\frac{\sinh(2\pi nbP)}{\sinh(2\pi bP)} = \sum_{l=1-n,2}^{n-1}\exp(2\pi lbP) \tag{85}$$

(here $\sum_{l=1-n,2}^{n-1}$ denotes the sum over the set $l = \{-n+1, -n+3, \ldots, n-1\}$) to obtain

$$\int \Psi_{m,n}(P)\Psi_s(-P)\chi_P(q)dP = \sum_{k=1-m,2}^{m-1}\sum_{l=1-n,2}^{n-1}\chi_{(s+i(k/b+lb))/2}(q') \tag{86}$$

i.e., the standard fusion of the degenerate representation (m, n) and a general one with $P = s/2$.

It remains us to analyze the partition function with two boundary conditions characterized by different boundary parameters s and s'. It is given by the overlap integral

$$\begin{aligned}Z_{s,s'} &= \int \Psi_s(-P)\Psi_{s'}(P)\chi_P(q)dP \\ &= \sqrt{2}\int \Psi_s(-P)\Psi_{s'}(P)e^{-4i\pi PP'}\chi_{P'}(q')dPdP' \\ &= \int_0^\infty \rho(P')\chi_{P'}(q')dP'\end{aligned} \tag{87}$$

where, according to [20] the density of states $\rho(P')$ flowing along the strip reads

$$\begin{aligned}\rho(P') &= 2\sqrt{2}\int \Psi_s(-P)\Psi_{s'}(P)e^{-4i\pi PP'}dP \\ &= \int_{-\infty}^{\infty}\frac{2\cos(st)\cos(s't)}{\sinh(bt)\sinh(t/b)}e^{-2iP't}\frac{dt}{2\pi}\end{aligned} \tag{88}$$

In this integral some regularization of the singularity at $P = 0$ is implied. Eq.(88) has to be compared with the logarithmic derivative

$$\rho(P) = -\frac{i}{2\pi}\frac{d}{dP}\log D_B(P|s, s') \tag{89}$$

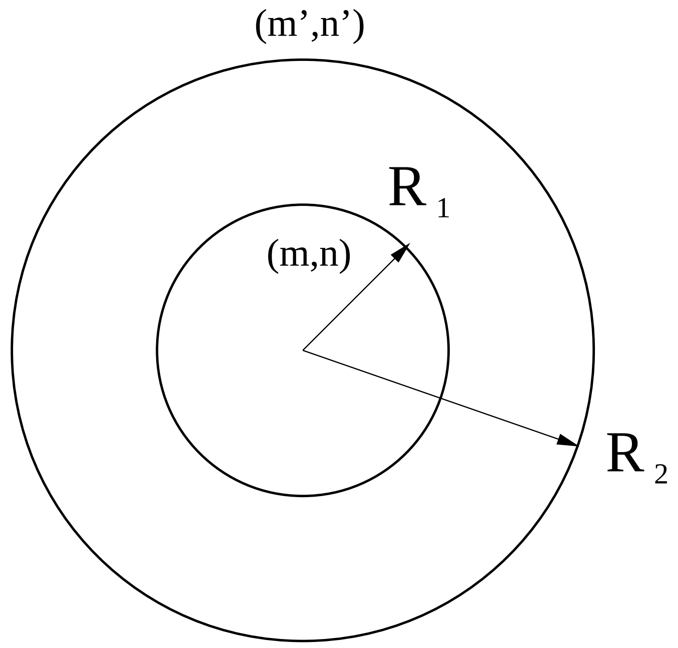


Figure 2: The annulus with two "boundary conditions" corresponding to the out-vacuum states (m, n) and (m', n').

of the boundary Liouville two-point function constructed in [13]. As it has been mentioned in [20] the two expressions match up to an s-independent quantity. There is also some specific s-independent (but still P-dependent) part of the two-point function, and consequently of the density of states on the strip, which cannot be restored from the integral (88) before the regularization at $P = 0$ is specified.

All the above calculations are quite formal. The underlying physical picture involves LFT on an annulus with a (purely imaginary) modular parameter

$$\tau = \frac{i}{\pi} \log \frac{R_2}{R_1} \tag{90}$$

(see fig.2). An annulus with the out-vacuum states at both boundaries is interpreted as finite-temperature partition function of gravitational modes in AdS_2 geometry [25]. Thus, the right-hand side of (80) exposes the state content of such theory. Note that in the case when both of the out-vacua are of $(1, 1)$ type, the corresponding space of states contains only identity operator (i.e. the $SL(2, R)$ invariant state found in [26]) and its conformal descendents. The situation is more difficult to interpret when the out-vacuum associated with one (or both) of the boundaries is of $(m, n) \neq (1, 1)$ type, in which cases (76) and (80) indicate presence of nontrivial primary states with Kac dimensions (66). Proper interpretation of these states (and the "excited" out-vacua $(m, n) \neq (1, 1)$ themselves) is one of the most interesting questions remaining open. Nevertheless, one can notice that, at least on the formal level, the above modular pattern is strikingly similar to the situation in boundary rational conformal field theories, as discussed in [17]. Much of the similarity remains there when one of the out-vacua is replaced by a local boundary condition Ψ_s; the right-hand side of (86) reveals the state content of "semi-infinite" AdS_2, with one "AdS boundaries" replaced by a local boundary condition of [13] at a finite distance. Again, true interpretation of these states still needs to be clarified.

6 Boundary bootstrap

With finite number of boundary fields any two-point function of bulk primary fields is combined of finite number of conformal blocks. In this section we use the structure constants calculated in sect.2 to construct this two-point function in some simplest cases and verify that it satisfies the boundary bootstrap relations.

We consider the general two-point function

$$G_{\alpha_1\alpha_2}(\xi_1,\xi_2) = \langle V_{\alpha_1}(\xi_1)V_{\alpha_2}(\xi_2)\rangle_{m,n} \tag{91}$$

computed in some out-vacuum (m,n). Let us study two simplest cases.

1. “Basic” vacuum $(\mathbf{m},\mathbf{n}) = (\mathbf{1},\mathbf{1})$. In the basic out-vacuum $(m,n) = (1,1)$ there is only identity operator at the boundary. The function reads

$$G_{\alpha_1\alpha_2}(\xi_1,\xi_2) = \frac{\left|\xi_2-\bar{\xi}_2\right|^{2\Delta_1-2\Delta_2} U_{1,1}(\alpha_1)U_{1,1}(\alpha_2)}{\left|\xi_1-\bar{\xi}_2\right|^{4\Delta_2}} \mathcal{F}\left(\begin{array}{cc}\alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2\end{array}, iQ/2, 1-\eta\right) \tag{92}$$

where $\mathcal{F}$ is the standard four-point conformal block with “intermediate” dimension $\Delta = 0$. To get rid of excessive multipliers it is convenient to define a “normalized” two-point function as

$$g_{\alpha_1\alpha_2}(\eta) = \frac{\langle V_{\alpha_1}(\xi_1)V_{\alpha_2}(\xi_2)\rangle_{1,1}}{\langle V_{\alpha_1}(\xi_1)\rangle_{1,1}\langle V_{\alpha_2}(\xi_2)\rangle_{1,1}} \tag{93}$$

which is simply expressed in terms of this single block

$$g_{\alpha_1\alpha_2}(\eta) = (1-\eta)^{2\Delta_1}\mathcal{F}\left(\begin{array}{cc}\alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2\end{array}, iQ/2, 1-\eta\right) \tag{94}$$

and depends only on the invariant η.

Another representation of this function comes from the bulk operator product expansion of the fields $V_{\alpha_1}(\xi_1)$ and $V_{\alpha_2}(\xi_2)$. This gives the “normalized” two-point function (up to possible discrete terms) in the form

$$g_{\alpha_1\alpha_2}(\eta) = (1-\eta)^{2\Delta_1}\int_{-\infty}^{\infty}\frac{dP}{4\pi}\frac{C(\alpha_1,\alpha_2,Q/2+iP)U_{1,1}(Q/2-iP)}{U_{1,1}(\alpha_1)U_{1,1}(\alpha_2)}\mathcal{F}\left(\begin{array}{cc}\alpha_1 & \alpha_1\\ \alpha_2 & \alpha_2\end{array}, P, \eta\right) \tag{95}$$

As the first numerical example we take a quite arbitrary value $b^2 = 0.8086\ldots$ and two “puncture” operators with $\alpha_1 = \alpha_2 = Q/2$. In this case there are no discrete terms in the expression (95). The two-point function $g_{Q/2,Q/2}$ v.s. the invariant “distance” η is plotted in fig.3. Solid line is for eq.(94) and circles are computed as the integral (95). Few numbers are presented in table 1. The first values $g^{(\text{bound})}(\eta)$ are evaluated as the vacuum conformal block (94) and $g^{(\text{bulk})}(\eta)$ stands for (95). We quote this table only to illustrate the numerical precision of our calculations. It should be noted that for $g^{(\text{bulk})}(\eta)$ only the first 10 digits are correct, the errors being due to the numerical integration over P in (95) and evaluation of the special functions entering the structure constants.

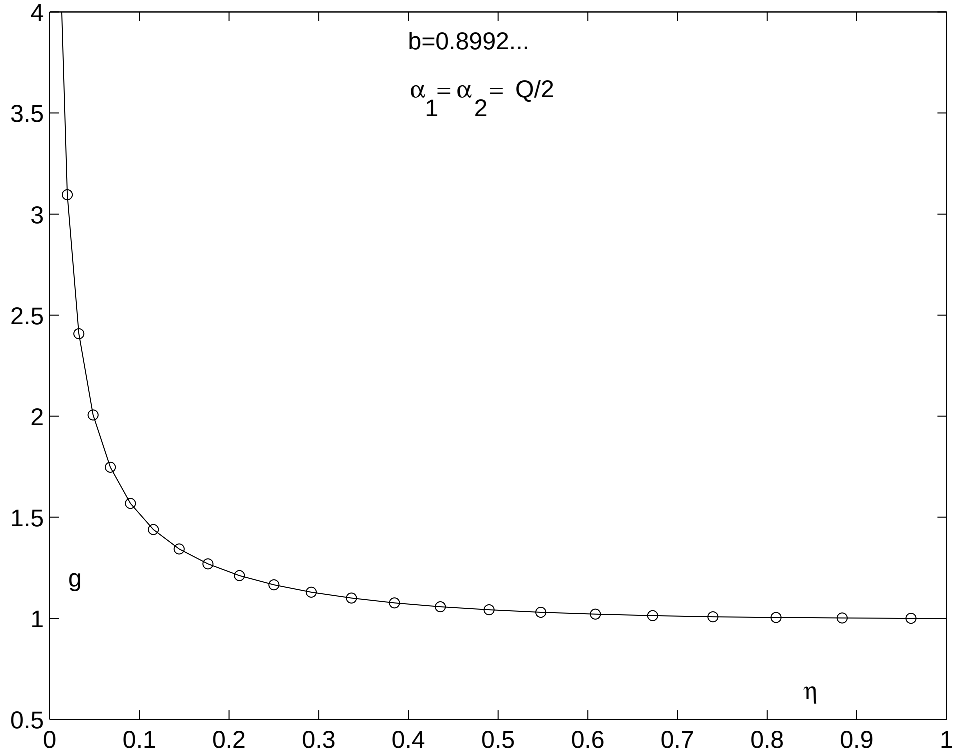


Figure 3: Normalized two-point function $g_{Q/2,Q/2}(\eta)$ evaluated as the single vacuum block (94) (solid line) and as the cross-channel integral (95) (circles).

2. Vacuum $(\mathbf{m},\mathbf{n}) = (\mathbf{1},\mathbf{2})$. In this vacuum two boundary operators contribute with dimensions $\Delta_{1,1} = 0$ and $\Delta_{1,3} = Q^2/4 - (b^{-1} + 2b)^2/4$. Taking again the "normalized" correlation function

$$g_{\alpha_1\alpha_2}(\eta) = \frac{\langle V_{\alpha_1}(\xi_1)V_{\alpha_2}(\xi_2)\rangle_{1,2}}{\langle V_{\alpha_1}(\xi_1)\rangle_{1,2}\,\langle V_{\alpha_2}(\xi_2)\rangle_{1,2}} \tag{96}$$

with respect to this vacuum, we have, instead of eq.(94)

$$\begin{aligned} g_{\alpha_1\alpha_2}(\eta) = \quad & (1-\eta)^{2\Delta_1} \times \quad \left[\mathcal{F}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}, iQ/2, 1-\eta\right) \\ & + \frac{F_{1,2}(\alpha_1)F_{1,2}(\alpha_2)}{U_{1,2}(-b/2)F_{1,2}(-b/2)}\mathcal{F}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}, i(b+b^{-1}/2), 1-\eta\Big)\Big] \end{aligned} \tag{97}$$

where

$$F_{1,2}(\alpha) = \frac{B^{(+)}_{1,2}(\alpha)}{U_{1,2}(\alpha)} \tag{98}$$

as given by expression (26) with $(m,n) = (1,2)$. The "cross channel" representation remains the same as in eq.(95) with the substitution $U_{1,1} \to U_{1,2}$. In fig.4 the numerical values of (97) and the cross-channel integral (95) are compared at $b = 0.7048\ldots$ and again for $\alpha_1 = \alpha_2 = Q/2$. Notice that in this case the two-point function $g_{\alpha_1\alpha_2}(\eta)$ is an exponentially growing function of the geodesic distance. This situation is typical for the "excited" vacua $(m,n) \neq (1,1)$ and related to the negative dimensions (66) of the degenerate boundary fields $\psi_{m,n}$ (at real b).

η	$g^{(\text{bound})}(\eta)$	$g^{(\text{bulk})}(\eta)$
0.10	1.511254162734526	1.511254162670712
0.20	1.228318394284875	1.228318394218384
0.30	1.123052815598698	1.123052815525115
0.40	1.069857238682268	1.069857238610545
0.50	1.039506854956745	1.039506854882646
0.60	1.021302866577855	1.021302866502048
0.70	1.010340291843230	1.010340291788767
0.80	1.004036952201786	1.004036952278245
0.90	1.000898855218405	1.000898855824359

Table 1: Numerical comparison of the "boundary" (94) and "bulk" (95) representations of the normalized two-point function $g_{Q/2,Q/2}(\eta)$ at $b^2 = 0.8086\ldots$

7 Discussion

We have demonstrated that the pseudosphere geometry provides a new physical picture of 2D quantum gravity. It is different from the compact problems and in fact much closer to standard physics in ordinary field theory (peculiarities of a field theory at constant negative curvature are discussed in [23]). However, many conceptual questions related to the suggested constructions remain open. Let us mention some of them.

- Unitary and non-unitary matter fields coupled to Liouville quantum gravity in this geometry present a separate and rather interesting problem, particularly in relation to the recent studies of AdS/CFT correspondence [24].

- Of course the closing of the bootstrap program requires construction of all bulk-boundary and boundary structure constants for all degenerate boundary fields associated with different out-vacua (m, n), including the juxtaposition operators. This difficult technical task remains to be done. For "ordinary" boundary conditions in boundary LFT this program has been started in [13] (see also [21, 22, 20]).

- The most intriguing point is the nature of the "excited" vacua. As we have already mentioned, in all such vacua correlation functions typically grow exponentially with the geodesic distances. This suggests that these states can be a kind of "boundary excitations" of the corresponding boundary conformal field theory. A meaning of these quantum excitations of the (physically infinite faraway) absolute remains to be comprehended. Let us mention also that these growing correlations at large distances are dominated by non-trivial degenerate boundary operators of negative dimensions. Therefore the physical "decay property" (with which we started our arguments in sect.2) doesn't hold literally in these excited vacua, being a formal requirement (21) for the contribution of the identity operator. This means that even the logic of the whole development deserves more careful examination.

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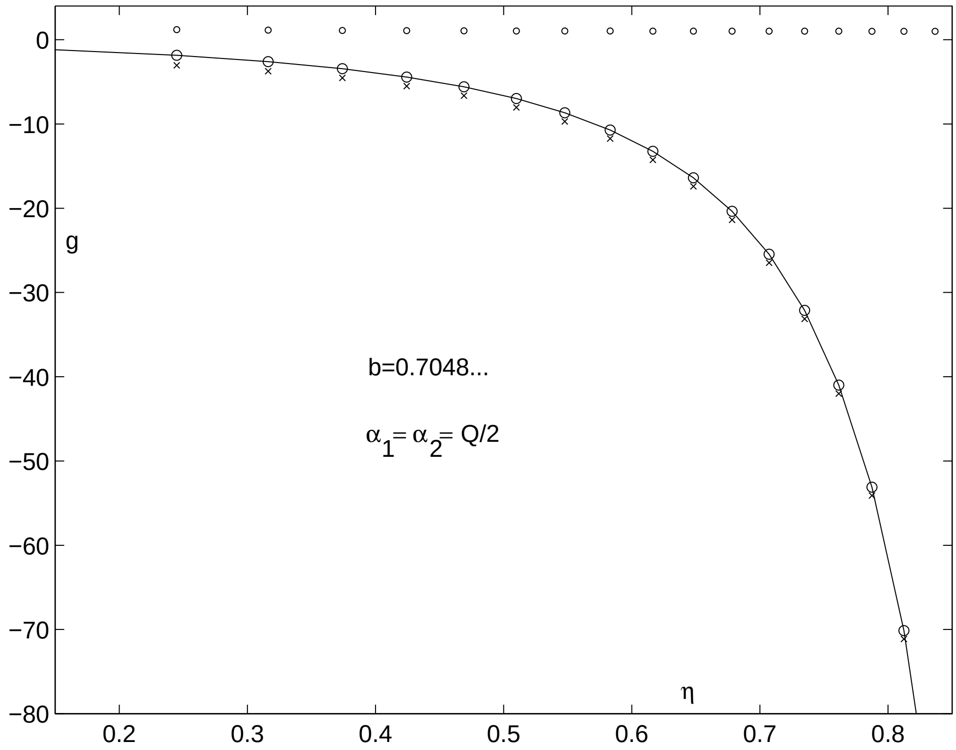

Figure 4: Boundary (97) (solid line) and bulk (95) (circles) representations of the normalized two-point function $g_{Q/2,Q/2}(\eta)$ are compared at $b = 0.7048\ldots$. Small circles and crosses are respectively the contributions of the first and second terms in eq.(97). The two-point function is almost saturated by the ψ_{13} contribution.

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4 SUPERSYMMETRY AND QUANTUM FIELD THEORY

Supersymmetric Threshold Corrections to the Higgs Sector in the Minimal Supersymmetric Model with Explicit CP Violation

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Abstract

The explicit CP violation in the Higgs sector of the minimal supersymmetric model (MSSM) can be induced by Higgs–squark–squark interactions. In this paper we calculate the supersymmetric threshold corrections with scalar quarks mass parameters $M_{\tilde{Q}}$, $M_{\tilde{U}}$, $M_{\tilde{D}}$ splitting by effective potential method.

Introduction

In the models with two doublets of scalar fields (THDM) [1] CP invariance can be broken explicitly by the terms of the potential containing $(\Phi_1^\dagger \Phi_2)$ and conjugated, with complex parameters. In the MSSM case complex parameters of the effective two-doublet potential can appear at the account of the Higgs interaction with scalar quarks, which constant of interaction generally can be complex [2, 3, 4]. This result can be received using the method of effective potential, by integration off the massive superpartners of quarks degrees of freedom [5]. The interesting problem here is to take into account different scalar quarks mass parameters $M_{\tilde{Q}}$, $M_{\tilde{U}}$, $M_{\tilde{D}}$ [6, 7] and to investigate the scenario for baryogenesis in the MSSM for first light scalar top mass.

Supersymmetric threshold corrections

In the THDM model we enter two $SU(2)_L$ doublets of complex scalar fields with nonzero real vacuum expectation values. The potential can contain invariant terms [2, 3, 4]. The parameters λ_1, ..., λ_4 and μ_1^2, μ_2^2 remain as real numbers after integration on the degrees of freedom of massive scalar quarks fields. In the supersymmetric $SU(2) \times U(1)$ gauge theories the constant of self-action λ_i are parameters of the potential and in the tree approximation are defined through $SU(2)_L$ and $U(1)_Y$ gauge couplings by the following parities at the supersymmetry scale M_{SUSY} [8]. Therefore at the tree level four masses of the Higgs bosons and the two mixing angles in the Higgs sector are defined by two independent parameters. In the tree approximation CP violation is not present.

The complex parameters $\lambda_{5,6,7}$ are induced in the effective potential, if to take into account interactions of scalar quarks $\tilde{t}$ and $\tilde{b}$ with the scalar Higgs fields, including complex parameters of mixing. The real and imaginary parts of μ_{12}^2 are defined by minimum conditions of the effective potential. However the parameters of the model, generally speaking, depend on the energy scale ($\sqrt{s}$) at which they are measured or fixed. This dependence is described by renormalization group equations (RGE).

The solution of the RGE allows to consider evolution of the complex parameters λ_5, λ_6, λ_7, having extrapolated them from the high energy area to the energies accessible today in experiments on colliders. The feature of the present analysis in comparison with the standard scheme of the summation of leading logarithms by means of RGE solution is the account in boundary conditions of effects of interaction between Higgs and the third generation of scalar quarks. Such interaction is caused by addition in the general superpotential softly breaking supersymmetry terms. The parameters A_t, A_b (trilinear couplings of interaction in scalar sector) and μ (higgsino mass parameter) in the sector of interaction of scalar quarks with Higgs fields [9] can be complex leading to CP violation in the effective scalar potential.

Using the method of functional integration it is possible to receive the following expression [5] for the one-loop renormalizable effective potential $\mathcal{V}$ (in Landau gauge with the use of the dimensional reduction and the modified scheme of the minimal subtraction ($\overline{\mathrm{MS}}$)):

$$\mathcal{V} = \mathcal{V}^0 + \frac{N_C}{32\pi^2} \hat{\mathrm{tr}} \mathcal{M}^4 \left[\ln \left(\frac{\mathcal{M}^2}{\sigma^2} \right) - \frac{3}{2} \right] , \qquad (1)$$

Where the Higgs part $\mathcal{V}^0$ coincides with potential at the tree level, $\mathcal{M}^2$ is the squared mass matrix of scalar quarks, σ – the scale of renormalization, tr means a capture of an operational trace. It is clear that the complex μ and $A_{t,b}$ can become the reason of occurrence of the complex parameters in the effective potential.

In that specific case $M_{\tilde{U}} = M_{\tilde{D}} = M_{\tilde{Q}} \equiv M_{\mathrm{SUSY}}$ (thus $\mathcal{M}_M^2 = \hat{1} M_{\mathrm{SUSY}}^2$) we can spread out the effective potential (1) on return degrees of M_{SUSY}^2 [5, 4, 10]. In this article we present results for nonequal $M_{\tilde{U}}$, $M_{\tilde{D}}$, $M_{\tilde{Q}}$, M_{SUSY}, assumed $|M_{\tilde{U}} - M_{\tilde{Q}}|$ and $|M_{\tilde{Q}} - M_{\tilde{D}}|$ less than $M_{\tilde{Q}}$. For reception λ_i it is necessary to consider from this decomposition the terms containing four degrees of Higgs doublets Φ. Taking into account, that $\mathcal{M}^2_\Gamma$ contains one degree of the field Φ, and $\mathcal{M}^2_\Lambda$ – two degrees, we obtain the effective potential of the quartic interaction:

$$V_{quartic} = \Lambda^{jl}_{ik}(\Phi^\dagger_i\Phi_j)(\Phi^\dagger_k\Phi_l) + \frac{N_C}{32\pi^2}\left\{\ln\left(\frac{M^2_{\tilde{Q}}}{m^2_{top}}\right)\hat{\mathrm{tr}}(\mathcal{M}^2_\Lambda)^2 + \right.$$

$$+\frac{1}{M^2_{\tilde{Q}}}\hat{\mathrm{tr}}(\mathcal{M}^2_\Gamma)^2\,\mathcal{M}^2_\Lambda - \frac{1}{12M^4_{\tilde{Q}}}\hat{\mathrm{tr}}(\mathcal{M}^2_\Gamma)^4 + \tag{2}$$

$$\left. +\frac{1}{M^2_{\tilde{Q}}}\hat{\mathrm{tr}}\Delta\mathcal{M}^2_{\tilde{Q}}(\mathcal{M}^2_\Lambda)^2 - \frac{1}{2M^4_{\tilde{Q}}}\hat{\mathrm{tr}}(\Delta\mathcal{M}^2_{\tilde{Q}})^2(\mathcal{M}^2_\Lambda)^2 - \frac{1}{M^4_{\tilde{Q}}}\hat{\mathrm{tr}}\Delta\mathcal{M}^2_{\tilde{Q}}(\mathcal{M}^2_\Gamma)^2\mathcal{M}^2_\Lambda\right\}. \tag{3}$$

The evaluation of the scalar mass splitting effects are in $\Delta\mathcal{M}^2_{\tilde{Q}}$: $\mathcal{M}^2_M = M^2_{\tilde{Q}}\cdot\hat{1} + \Delta\mathcal{M}^2_{\tilde{Q}}$.

Substituting to the effective potential (1) the mass matrix of scalar quarks, it is possible to receive the one-loop effective complex parameters λ_i ($i = 5, 6, 7$) on the scale below M_{SUSY}, corresponding to the boundary conditions determined running gauge coupling constants and final amendments from interactions of Higgs bosons with scalar quarks [11, 5], softly breaking supersymmetry (all λ-couplings and more common approach are presented in [12]):

$$\lambda_5 = -\Delta\lambda_5 \;\;=\;\; -\frac{3}{96\,\pi^2}\left(h_t^4\left(\frac{\mu A_t}{M^2_{\tilde{Q}}}\right)^2 + h_b^4\left(\frac{\mu A_b}{M^2_{\tilde{Q}}}\right)^2\right), \tag{4}$$

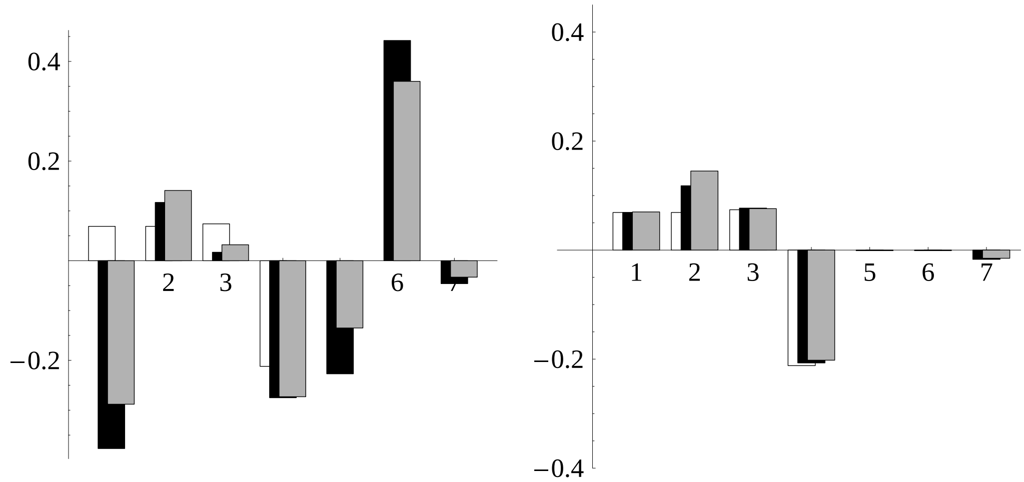
$$\begin{aligned}\lambda_6 = -\Delta\lambda_6 \;\;&=\;\; \frac{3}{96\,\pi^2}\left[h_t^4\,\frac{|\mu|^2\mu A_t}{M^4_{\rm SUSY}} - h_b^4\,\frac{\mu A_b}{M^2_{\rm SUSY}}\left(6 - \frac{|A_b|^2}{M^2_{\rm SUSY}}\right) + \right. \\ &\left. + (h_b^2A_b - h_t^2A_t)\,\frac{3\,\mu}{M^2_{\rm SUSY}}\,\frac{g_2^2 + g_1^2}{4}\right] - \Delta\lambda_6[mass-split],\end{aligned} \tag{5}$$

$$\begin{aligned}\lambda_7 = -\Delta\lambda_7 \;\;&=\;\; \frac{3}{96\,\pi^2}\left[h_b^4\,\frac{|\mu|^2\mu A_b}{M^4_{\rm SUSY}} - h_t^4\,\frac{\mu A_t}{M^2_{\rm SUSY}}\left(6 - \frac{|A_t|^2}{M^2_{\rm SUSY}}\right) + \right. \\ &\left. + (h_t^2A_t - h_b^2A_b)\,\frac{3\,\mu}{M^2_{\rm SUSY}}\,\frac{g_2^2 + g_1^2}{4}\right] - \Delta\lambda_7[mass-split].\end{aligned} \tag{6}$$

where Yukawa couplings are determined as usual $h_t = \frac{\sqrt{2}\,m_t}{v\sin\beta}$, $h_b = \frac{\sqrt{2}\,m_b}{v\cos\beta}$ and

$$\Delta\lambda_6[mass-split] = \frac{\mu}{64M^4_{\tilde{Q}}\pi^2}(A_bh_b^2(g_1^2 - 6h_b^2)(M^2_{\tilde{b}} - M^2_{\tilde{Q}}) + 2A_tg_1^2h_t^2(M^2_{\tilde{Q}} - M^2_{\tilde{t}}) \tag{7}$$

$$\Delta\lambda_7[mass-split] = -\frac{\mu}{64M^4_{\tilde{Q}}\pi^2}(A_bg_1^2h_b^2(M^2_{\tilde{b}} - M^2_{\tilde{Q}}) + 2A_th_t^2(M^2_{\tilde{Q}} - M^2_{\tilde{t}})(g_1^2 - 3h_t^2) \tag{8}$$



(a) (b)

Fig. 1. The effective parameters λ_i (the column i corresponds to λ_i, $|\lambda_5|$, $|\lambda_6|$, $|\lambda_7|$). With degenerated scalar mass parameters – (a); with splitting in masses – (b)

The two-Higgs-doublet effective potential parameters are calculated using the MSSM potential of the Higgs bosons - scalar quarks interaction and include the contributions coming from the F terms, leading and nonleading D terms, the wave-function renormalization terms, and leading two-loop Yukawa-QCD-corrections. Parameters λ_{1-7} are presented in Fig. 1(a), which contains the effective parameters λ_i (for convenience if $i = 1, 2$ then we plot $\lambda_{1,2}/2$, the column i corresponds to λ_i) at the tree-level at M_{SUSY} (white bars) and black bars – with finite threshold corrections at M_{SUSY}, one-loop level with leading two-loop corrections at m_{top} – grey bars for $m_{top} = 175$ GeV and CPX parameter space for strong dependence for $|\lambda_5|$, $|\lambda_6|$, $|\lambda_7|$ and therefor for Higgs masses and observables on phase [4, 12]. The values of fixed parameters: $m_Z = 91.19$ GeV, $m_b = 3$ GeV, $m_t = 175$ GeV, $m_W = 79.96$ GeV, $g_2 = 0.6517$, $g_1 = 0.3573$, $v = 245.4$ GeV, $G_F = 1.174 \cdot 10^{-5}$ GeV^{-2}, $\alpha_S(m_t) = 0.1072$, $\tan\beta = 5$, $M_{SUSY} = 500$ GeV, $\sigma = m_t$, $m_{H^\pm} = 300$ GeV, $A = 1000$ GeV, $|\mu| = 2000$ GeV, $\varphi \equiv \arg(\mu A_{t,b}) = 0$. Fig. 1(b) – the same but for splitting in scalar quark masses and parameter space for small complex parameters: $M_Q = 500$ GeV, $m_{\tilde{t}} = 800$ GeV, $m_{\tilde{b}} = 200$ GeV and μ=200 GeV, $A = X_t + \mu/\tan\beta$ at $X_t = 700$ GeV. The last case corresponds to CP states Higgs spectrum of MSSM.

The question about approximation in above mentioned two private cases is of particular interest [12]. The combination $\frac{|m^2_{\tilde{t}_1} - m^2_{\tilde{t}_2}|}{m^2_{\tilde{t}_1} + m^2_{\tilde{t}_2}}$ is a criteria for different approximations in the case of mass splitting and nonequal diagonal elements of scalar quarks mass matrix [13]. In the second case it is some less than 0.5, but in the first case it may be from 0.4 at small phases up to 0.8 at CP phase about π.

Summary

In this work the analysis of parameters of the two-doublet Higgs sector is done by the method of effective potential. The evolution of parameters of the Higgs self-action depends on possible interactions with scalar quarks in the model with soft breaking of the supersymmetry. The complex parameters of the soft supersymmetry breaking induce complex parameters in the effective Higgs potential of the THDM which, thus, radiatively breaks CP invariance. The supersymmetric threshold corrections with scalar quarks mass splitting are calculated with definite approximation.

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Hidden Singularities of Gluon Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills

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Abstract

After reduction techniques, two-loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory can be written in a basis of integrals containing scalar double-box integrals with rational coefficients, though the complete basis is unknown. Generically, at two loops, the leading singular behavior of a scalar double box integral with seven propagators is captured by a hepta-cut. However, it turns out that a certain class of such integrals has an additional propagator-like singularity. One can then formally cut the new propagator to obtain an octa-cut which localizes the cut integral just as a quadruple cut does at one-loop. This immediately gives the coefficient of the scalar double box integral as a product of six tree-level amplitudes. This procedure can be naturally generalized to higher loops.

1 Motivation and Review

The interest to study gluon amplitudes in $\mathcal{N} = 4$ Yang-Mills theory has recently been renewed after the discovery that twistor string theory [1] captures the perturbation theory of the maximally supersymmetric Yang-Mills theory (pMSYM). Twistor string theory inspired new ideas for the computation of tree level and one-loop amplitudes of gluons in QCD, $\mathcal{N} = 1$ and $\mathcal{N} = 4$ Yang-Mills theories. See [2] for a review and references therein. Before twistor string theory was introduced, the study of pMSYM at one-loop was mainly motivated by two facts: one is the decomposition of a QCD amplitude, A^{QCD}, with only a gluon running in the loop in terms of supersymmetric amplitudes and an amplitude with only a scalar running in the loop, A^{scalar}, (see [3] for a review),

$$A^{\mathrm{QCD}} = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1}_{\mathrm{chiral}} + A^{\mathrm{scalar}}, \tag{1}$$

where $A^{\mathcal{N}=4}$ has the full $\mathcal{N} = 4$ multiplet in the loop and $A^{\mathcal{N}=1}_{\mathrm{chiral}}$ only an $\mathcal{N} = 1$ chiral multiplet. The other motivation is a proposal of Anastasiou, Bern, Dixon, and Kosower (ABDK) that two- (and, perhaps, higher-) loop amplitudes in pMSYM can be completely determined in terms of one-loop amplitudes [4]. This proposal was explicitly verified for four-gluon two-loop amplitude in [4] and for four-gluon three-loop in [5].

In this note, we consider amplitudes of gluons in $\mathcal{N} = 4$ super-Yang-Mills. Each gluon carries the following information: momentum $p_{a\dot{a}}$, polarization vector $\epsilon_{a\dot{a}}$ and color index a. The color structure can be striped out by a color decomposition. Here we only consider the leading color or planar part of the amplitudes. The information in momentum and polarization vectors can be encoded in terms of spinors λ, $\tilde{\lambda}$ and the helicity of the gluon h. At tree-level, the leading color approximation is exact. An amplitude is given by

$$A_{(\{p_i,\epsilon_i,a_i\})} = g_{\rm YM}^{n-2} \sum_{\sigma\in S_n/Z_n} Tr(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n)}}) A_{(\{\lambda_{\sigma(1)},\tilde{\lambda}_{\sigma(1)},h_{\sigma(1)}\},\dots,\{\lambda_{\sigma(n)},\tilde{\lambda}_{\sigma(n)},h_{\sigma(n)}\})}. \tag{2}$$

It is convenient to denote the set of data $\{\lambda_i, \tilde{\lambda}_i, h_i\}$ by i^{h_i}, where $h_i = \pm$ is the helicity of the i^{th} gluon. The amplitudes on the right hand side of eq. (2) are known as leading color partial amplitudes and are computed from color-ordered Feynman rules. One can study a given order $A(1^{h_1},\dots,n^{h_n})$ and the rest can be obtained by application of permutations, σ. The partial amplitude $A(1^{h_1},\dots,n^{h_n})$ can be computed using a variety of methods (see [3] for a review on many of the techniques developed in the 80's and 90's). More recently, two new techniques became available, namely, the MHV decomposition [6] and the BCFW recursion relations [7, 8].

Amplitudes of gluons at one-loop admit a color decomposition (see [3] for a review) with single and double trace contributions. We will only concentrate on the leading color partial amplitudes. One-loop amplitudes of gluons in supersymmetric theories are four-dimensional cut-constructible [9, 10]. This means that they can be completely determined by their finite branch cuts and discontinuities. $\mathcal{N}=4$ amplitudes are even more special. Reduction techniques can be used to express these amplitudes in terms of scalar box integrals [9]. These are one-loop box Feynman integrals in a scalar field theory where a massless scalar runs in the loop,

$$\mathcal{I} = \int d^4\ell \frac{1}{(\ell^2+i\epsilon)((\ell-k_1)^2+i\epsilon)((\ell-k_1-k_2)^2+i\epsilon)((\ell+k_4)^2+i\epsilon)}, \tag{3}$$

where k_1, k_2, k_3, k_4 are the external momenta at each vertex. They are not independent since by momentum conservation $k_3 = -(k_4+k_1+k_2)$. Note that the integral (3) is singular when at least one k_i is a null vector. Therefore, we should specify a regularization procedure, like dimensional regularization. However, we will be considering cuts that are finite and do not depend on the regularization procedure. Since $A(1,\dots,n)$ is color-ordered, each k can only be the sum of consecutive momenta of external gluons. Moreover, since we only consider the planar contributions, we can define a given contribution by specifying i,j,k,l such that $k_1 = K_i+\dots+K_{j-1}$, $k_2 = K_j+\dots+K_{k-1}$ and $k_3 = K_k+\dots+K_{l-1}$. The reduction procedure then gives for the amplitude an expansion of the form [9]

$$A(1,\dots,n) = \sum_{1<i<j<k<m<n} B_{ijkl}\; \mathcal{I}_{(K_i+\dots+K_{j-1},K_j+\dots+K_{k-1},K_k+\dots+K_{l-1})}, \tag{4}$$

where the coefficients B_{ijkm} are rational functions of the spinor products. Since all scalar box integrals are known explicitly, the problem of computing $A(1,\dots,n)$ is reduced to that of computing the coefficients B_{ijkl}. A general formula for the coefficients B_{ijkl} was found in [11] in terms of products of tree level amplitudes. Let us review the derivation of the formula because the idea is useful in the analysis at higher loops. If we think of the scalar box integrals as an independent basis of some vector space we can interpret $A(1,\dots,n)$ as a general vector. All we need to do is to find a way to project $A(1,\dots,n)$ along the space spanned by a given scalar box integral $\mathcal{I}$. From the definition of $\mathcal{I}$ in (3) we see that each integral is uniquely determined once its four propagators are given.

It is natural to think that the way to determine the coefficient B is by looking at the region of integration where all four propagators become singular. In fact, the integral obtained by cutting, i.e., by dropping the principal part of all four propagators computes the discontinuity of the given scalar box integral across a singularity which is unique to it. The set of four equations that gives ℓ is the following

$$\ell^2 = 0, \qquad (\ell - k_1)^2 = 0, \qquad (\ell - k_1 - k_2)^2 = 0, \qquad (\ell + k_4)^2 = 0. \tag{5}$$

One can show that these equations do not have solutions if ℓ is a real vector in Minkowski space for general external momenta. The way out of this problem is to complexify all momenta and make a Wick rotation to $(--++)$ signature. In the new signature the delta functions are still well defined and there are always solutions to eq. (5). One can

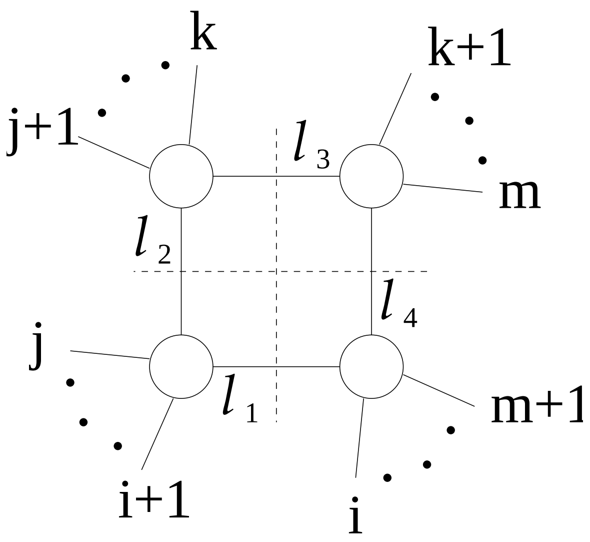

Figure 1: A quadruple cut diagram. Momenta in the cut propagators flows clockwise and external momenta are taken outgoing. The tree-level amplitude A_1^{tree}, for example, has external momenta $\{i+1, ..., j, \ell_2, \ell_1\}$

also look at the same regime on the left hand side of eq. (4) by considering only Feynman diagrams that posses the four propagators entering in (5). Summing over them one finds the following equation

$$\int d\mu \sum_J n_J A_{(1)}^{tree} A_{(2)}^{tree} A_{(3)}^{tree} A_{(4)}^{tree} = B_{ijkm} \int d\mu, \tag{6}$$

where sum over J represents a sum over all possible particles in the $\mathcal{N} = 4$ multiplet. The measure $d\mu$ is the same one both sides of the integrals,

$$d\mu = d^4\ell \; \delta^{(+)}(\ell^2) \; \delta^{(+)}((\ell - k_1)^2) \; \delta^{(+)}((\ell - k_1 - k_2)^2) \; \delta^{(+)}((\ell + k_4)^2), \tag{7}$$

and the tree-level amplitudes are defined as follows (see Fig. 1)

$$\begin{aligned} A_{(1)}^{tree} &= A(-\ell_1, i+1, i+2, \ldots, j-1, j, \ell_2), \\ A_{(2)}^{tree} &= A(-\ell_2, j+1, j+2, \ldots, k-1, k, \ell_3), \\ A_{(3)}^{tree} &= A(-\ell_3, k+1, k+2, \ldots, m-1, m, \ell_4), \\ A_{(4)}^{tree} &= A(-\ell_4, m+1, m+2, \ldots, i-1, i, \ell_1). \end{aligned} \tag{8}$$

where

$$\ell_1 = \ell,\ \ell_2 = \ell - k_1,\ \ell_3 = \ell - k_1 - k_2,\ \ell_4 = \ell + k_4,\ k_1 = K_{i+1} + \ldots + K_j,$$
$$k_2 = K_{j+1} + \ldots + K_k,\ k_3 = K_{k+1} + \ldots + K_m,\ k_4 = K_{m+1} + \ldots + K_i. \tag{9}$$

The integral $\int d\mu$ is just given by a Jacobian $1/\sqrt{\Delta}$. This Jacobian cancels on both sides since the integral is localized by the delta functions and the coefficient is given by [11]

$$B_{ijkl} = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)}. \tag{10}$$

Here $\mathcal{S}$ is the set of solutions to the conditions imposed by the delta functions, and $|\mathcal{S}|$ is the number of solutions. The sum also involves a sum over all possible particles that can propagate in the loop.

At two loops, only the four-gluon amplitude has been computed [12]. The answer is given as a linear combination of double box scalar integrals with coefficients that are rational function of the spinor variables. A double box scalar integral is the analog of the one-loop scalar box integral introduced above, more explicitly,

$$\mathcal{I}(k_1,\ldots,k_6) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+i\epsilon)((p-k_1)^2+i\epsilon)((p-k_1-k_2)^2+i\epsilon)} \times$$
$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q+k_6)^2+i\epsilon)(q^2+i\epsilon)((q-k_5)^2+i\epsilon)((q-k_4-k_5)^2+i\epsilon)}. \tag{11}$$

This integral is UV finite but it might have IR divergences when some k's are null vectors. Again, as in the one-loop case, one has to choose a regularization procedure but we do not do so because we only discuss finite cuts. The planar contribution to the four-gluon amplitude is [12]

$$A_4^{2-loop}(K_1,K_2,K_3,K_4) = A_4^{tree}\ s\, t\, (s\, \mathcal{I}(K_1,K_2,0,K_3,K_4,0) + t\, \mathcal{I}(K_4,K_1,0,K_2,K_3,0)), \tag{12}$$

where $s = (K_1+K_2)^2$ and $t = (K_2+K_3)^2$.

2 Two-Loop Amplitudes and Hidden Singularities

At two loops, a decomposition similar to (4) in terms of some given set of integrals is expected by using reduction procedures. Unfortunately, the complete basis of two-loop integrals is currently unknown. However, scalar double box integrals are a natural ingredient of such a basis. We will concentrate on the calculation of the coefficient of certain classes of planar scalar double box integrals. These are the integrals that arise in scalar field theory with a massless scalar running along internal lines and with the double-box structure depicted in Fig. 2. Here we propose a method [13] for computing the coefficient of any scalar double box integral given in Fig. 2a when at least one of the two boxes has two adjacent massless three-particle vertices. The idea is to use a hidden propagator-type singularity whose origin will be explained below. The coefficient of any double box given in Fig. 2b is much easier to compute. In fact, computations of this type of coefficients does not require use of hidden singularities. We will not discuss it here referring to the original paper [13].

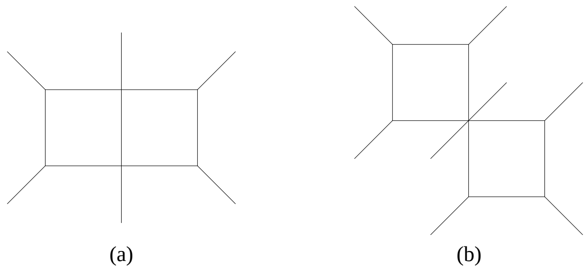

Figure 2: The two possible different structures of planar scalar double box integrals. (a) Double boxes. (b) Split double boxes. Note that the momenta of the external lines is given by the sum of the momenta of external gluons.

Let us consider the double boxes that have seven propagators. We will show that when at least one of the two boxes has two adjacent three particle vertices then there is an extra propagator-like hidden singularity that can be cut. This produces one more delta-function which together with the natural hepta-cut completely localizes the cut integral. Even though we concentrate only on the planar configurations, exactly the same logic can be applied for non-planar configurations as well. Consider an arbitrary double-box configuration in Fig. 3. The corresponding hepta-cut integral is

$$\mathcal{I} = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\delta(p^2)\delta((p-k_1)^2)\delta((p-k_1-k_2)^2)\delta((p+q+k_6)^2) \qquad (13)$$
$$\times\delta(q^2)\delta((q-k_5)^2)\delta((q-k_4-k_5)^2).$$

Let us perform, for example, the p-integration. The integral over p,

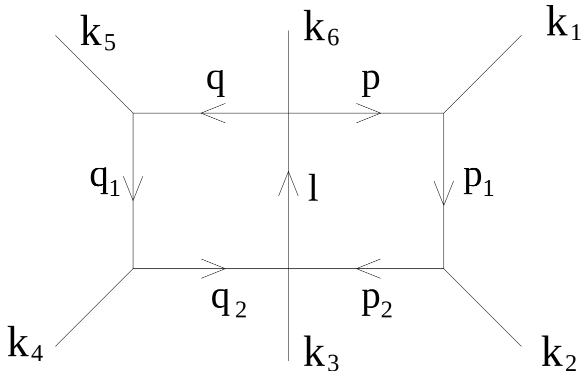

Figure 3: An arbitrary double-box configuration.

$$\mathcal{I}_p = \int d^4p\delta(p^2)\delta((p-k_1)^2)\delta((p-k_1-k_2)^2)\delta((p+q+k_6)^2), \qquad (14)$$

is localized and the answer is [11]

$$\mathcal{I}_p = \frac{2}{(k_1+k_2)^2(k_1+k_6+q)^2\rho}, \qquad (15)$$

where

$$\rho = \sqrt{1 - 2(\lambda_1 + \lambda_2) + (\lambda_1 - \lambda_2)^2},$$
$$\lambda_1 = \frac{k_1^2(k_3 + k_4 + k_5 - q)^2}{(k_1 + k_2)^2(k_1 + k_6 + q)^2}, \quad \lambda_2 = \frac{k_2^2(k_6 + q)^2}{(k_1 + k_2)^2(k_1 + k_6 + q)^2}. \tag{16}$$

The crucial observation is that when

$$\rho = 1, \tag{17}$$

$\mathcal{I}$ acquires an extra propagator-type singularity, i.e. $1/(k_1 + k_6 + q)^2$. We can formally cut the new propagator by replacing it with a delta-function creating an eighth cut. In other words, after performing the p-integration we end up with following integral over q (we omit the overall q-independent factor)

$$\mathcal{I}_q = \int d^4q\delta(q^2)\delta((q - k_4)^2)\delta((q - k_3 - k_4)^2)\frac{1}{(k_1 + k_6 + q)^2}. \tag{18}$$

This integral looks like a triple cut of the effective box in Fig. 4. Note that the momentum

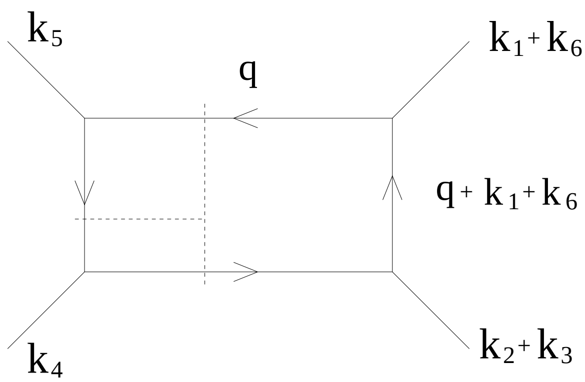

Figure 4: Effective box that arises after a quadruple cut is used to localize the p integral. The momentum flowing along the uncut line is $q + k_1 + k_6$.

flowing along the uncut line is exactly $q + k_1 + k_6$. From this viewpoint it is natural to cut the remaining propagator. Note that this procedure localizes the q-integral. Then it is straightforward to write down the coefficients of such double-box integrals. They are given by

$$c_\alpha = -\frac{i}{|\mathcal{S}|}\sum_{h,J_1,J_2,\mathcal{S}}(n_{J_1}n_{J_2}A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)}A^{tree}_{(5)}A^{tree}_{(6)})_h, \tag{19}$$

where the sum over h is the sum over all helicity configurations, the sums over J_1 and J_2 are the sums over all particles that can propagate in both loops, $\mathcal{S}$ is the set of all solutions for the internal lines of the following system of equations

$$p^2 = 0, \quad (p - k_1)^2 = 0, \quad (p - k_1 - k_2)^2 = 0, \quad (p + q + k_6)^2 = 0,$$
$$q^2 = 0, \quad (q - k_5)^2 = 0, \quad (q - k_4 - k_5)^2 = 0, \quad (k_1 + k_6 + q)^2 = 0, \tag{20}$$

and $|\mathcal{S}|$ is the number of solutions. This expression is analogous to the formula for one-loop coefficients of box integrals [11]. It is important to remember that this discussion is valid if

$$\rho = 1, \quad \lambda_1 = \lambda_2 = 0. \tag{21}$$

Otherwise, the singularity $1/(k_1 + k_6 + q)^2$ will be replaced by a more complicated one which is not propagator-like, as it can easily be seen from eq. (16). The conditions given in (21) are satisfied if a given box has two adjacent three-particle vertices. It easy to check that this is always the case if the number of gluons is less than seven. This means that every double-box coefficient of any gluon amplitude with less than seven external lines is given by eq. (19). The first double-box configuration where eq. (21) is not satisfied appears when the number of external gluons is seven and is shown in Fig. 5. However, even if

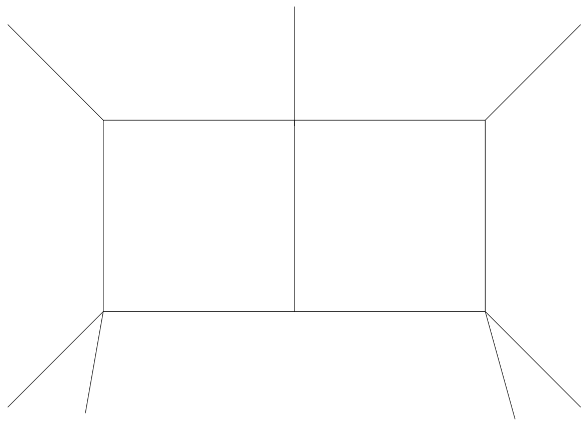

Figure 5: The simplest double-box configuration for which the conditions in (21) are not satisfied.

the number of external gluons is greater than six, there are double-box configurations for which eq. (21) is satisfied. In such cases the eighth cut still exists and eq. (19) is still correct.

In fact, eq. (19) requires some additional explanations. Note that existence of the effective box in Fig. 4 implies that either the momentum l or the momentum p_1 in Fig. 3 vanishes. In Minkowski space, this would mean that some tree level amplitudes in eq. (19) vanish. Moreover, in Minkowski space, the system of equations (20) does not have solutions, which means that we cannot see the singularities under consideration. Therefore, it is not surprising that eq. (19) at least naively, is meaningless in Minkowski space. In order to see the new kind of singularities, we have to analytically continue all momenta to signature $(--++)$. But in signature $(--++)$, the statement that a tree amplitude vanishes when one of the incoming or outgoing momentum vanishes is not correct. Each tree amplitude is constructed by using spinors. When one of the incoming or outgoing $(--++)$ momenta vanishes, it is impossible to determine its spinors components even up to rescaling. This leaves the amplitude undetermined. For example, assume that we have a helicity configuration containing a three-gluon amplitude $A(p^-, p_1^-, k_1^+)$. It is given by

$$A(p^-, p_1^-, k_1^+) = \frac{\langle p_1\ p\rangle^3}{\langle p\ k_1\rangle\langle k_1\ p_1\rangle}. \qquad (22)$$

If p_1 vanishes, the spinor λ_{p_1} cannot be uniquely determined. In fact, λ_{p_1} is not uniquely defined even for non-zero p_1 as it is defined up to rescaling. However, when $p_1 = 0$ the freedom in not being able to determine λ_{p_1} becomes much larger. One can always say that $p_1 = 0$ implies that $\tilde{\lambda}_{p_1} = 0$ and λ_{p_1} is arbitrary. This means that $A(p^-, p_1^-, k_1^+)$ becomes arbitrary. Therefore, the numerator in eq. (19) is a discontinuous function of momenta and we have to give a prescription on how to define it as l or p_1 goes to zero. The natural way to define it is as follows. Consider first the loop with momentum p. Let $A^{tree}_{(1)}, A^{tree}_{(2)}, A^{tree}_{(3)}$ and $A^{tree}_{(4)}$ be the four tree amplitudes which depend on p. Assuming that they are all non-zero, we can solve the first four p-dependent equations in (20) to determine p as a function of the external momenta and q and then evaluate the product $A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)}$ on these solutions. We claim that this product can be simplified in such a way that it is a well-defined function when the constraint $(k_1 + k_6 + q)^2 = 0$ is imposed. Having found the product $A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)}$ as a function of the external momenta and q, we then multiply it by the remaining two tree amplitudes $A^{tree}_{(5)}$ and $A^{tree}_{(6)}$ and evaluate the product on the solution of the remaining four equations in (20). We propose this as a method for calculating double-box coefficients provided conditions (21) are fulfilled.

For concrete examples and generalizations for higher loops, we will refer to the original paper [13].

3 Acknowledgements

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Finite Unified Theories and Quantum Reduction of Couplings

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Abstract

Finite Unified Theories (FUTs) are N=1 supersymmetric Grand Unified Theories, which can be made all-loop finite, both in the dimensionless (gauge and Yukawa couplings) and dimensionful (soft supersymmetry breaking terms) sectors. This remarkable property, based on the reduction of couplings at the quantum level, provides a drastic reduction in the number of free parameters, which in turn leads to an accurate prediction of the top quark mass in the dimensionless sector, and predictions for the Higgs boson mass and the supersymmetric spectrum in the dimensionful sector. Here we examine the predictions of two FUTs taking into account a number of theoretical and experimental constraints. We present the results of a detailed scanning concerning the Higgs mass prediction for both models, while for the second we present a representative prediction of its spectrum.

1 INTRODUCTION

The theoretical efforts to establish a deeper understanding of Nature have led to very interesting frameworks such as String theories and Non-commutative Geometry both of which aim to describe physics at the Planck scale. Looking for the origin of the idea that coordinates might not commute we might have to go back to the days of Heisenberg. In the recent years the birth of such speculations can be found in refs. [1, 2]. In the spirit of Non-commutative Geometry also particle models with non-commutative gauge theory were explored [3] (see also [4]), [5, 6]. On the other hand the present intensive research has been triggered by the natural realization of non-commutativity of space in the string theory context of D-branes in the presence of a constant background antisymmetric field [7]. After the work of Seiberg and Witten [8], where a map (SW map) between non-commutative and commutative gauge theories has been described, there has been a lot of activity also in the construction of non-commutative phenomenological Lagrangians, for example various non-commutative standard model like Lagrangians have been proposed [9, 10][1]. In particular in ref. [10], following the SW map methods developed in refs. [11], a non-commutative standard model with $SU(3) \times SU(2) \times U(1)$ gauge group has been presented. These non-commutative models represent interesting generalizations of the SM and hint at possible new physics. However they do not address the usual problem of the SM, the presence of a plethora of free parameters mostly related to the ad hoc introduction of the Higgs and Yukawa sectors in the theory. At this stage it is worth recalling that various schemes, with the Coset Space Dimensional Reduction (CSDR) [14, 15, 16, 17] being pioneer, were suggesting that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions. In the successes of the CSDR scheme certainly should be added the possibility to obtain chiral theories in four dimensions [18, 19, 20, 21] as well as softly broken supersymmetric or non-supersymmetric theories starting from a supersymmetric gauge theory defined in higher dimensions [22].

The original plan of this paper was to present an overview covering the following subjects:
a) Quantum Reduction of Couplings and Finite Unified Theories
b) Classical Reduction of Couplings and Coset Space Dimensional Reduction
c) Renormalizable Unified Theories from Fuzzy Higher Dimensions [23]
The aim was to present an unified description of our current attempts to reduce the free parameters of the Standard Model by using Finite Unification and extra dimensions, but due to space limitations we will cover only the first subject.

2 REDUCTION OF COUPLINGS AND FINITENESS IN $N = 1$ SUSY GAUGE THEORIES

Here we will review some of the main points and ideas concerning the *reduction of couplings* and *finiteness* in $N = 1$ supersymmetric theories. A RGI relation among couplings g_i, $\Phi(g_1, \cdots, g_N) = 0$, has to satisfy the partial differential equation $\mu \, d\Phi/d\mu = \sum_{i=1}^{N} \beta_i \, \partial\Phi/\partial g_i = 0$, where β_i is the β-function of g_i. There exist $(N-1)$ independent Φ's, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [25], $\beta_g \, (dg_i/dg) = \beta_i$, $i = 1, \cdots, N$, where g and β_g are the primary coupling and its β-function. Using all the $(N-1)\,\Phi$'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling g. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The

[1] These SM actions are mainly considered as effective actions because they are not renormalizable. The effective action interpretation is consistent with the SM in [10] being anomaly free [12]. Non-commutative phenomenology has been discussed in [13].

superpotential of the theory is given by

$$W = \frac{1}{2}\, m^{ij}\, \Phi_i\, \Phi_j + \frac{1}{6}\, C^{ijk}\, \Phi_i\, \Phi_j\, \Phi_k \ , \tag{1}$$

where m^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G.

The one-loop β-function of the gauge coupling g is given by

$$\beta_g^{(1)} \;=\; \frac{dg}{dt} = \frac{g^3}{16\pi^2}\,[\sum_i \, l(R_i) - 3\, C_2(G)\,]\ , \tag{2}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C^{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_i^j of the matter fields Φ_i as:

$$\beta_C^{ijk} = \frac{d}{dt}\, C^{ijk} = C^{ijp} \sum_{n=1} \frac{1}{(16\pi^2)^n}\, \gamma_p^{k(n)} + (k \leftrightarrow i) + (k \leftrightarrow j)\ . \tag{3}$$

At one-loop level γ_i^j is given by

$$\gamma_i^{j(1)} = \frac{1}{2} C_{ipq}\, C^{jpq} - 2\, g^2\, C_2(R_i)\delta_i^j \ , \tag{4}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C_{ijk}^*$.

All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G)\,,\ \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i)\ , \tag{5}$$

where $l(R_i)$ is the Dynkin index of R_i, and $C_2(G)$ is the quadratic Casimir invariant of the adjoint representation of G.

A very interesting result is that the conditions (5) are necessary and sufficient for finiteness at the two-loop level [31, 32].

The one- and two-loop finiteness conditions (5) restrict considerably the possible choices of the irreducible representations R_i for a given group G as well as the Yukawa couplings in the superpotential (1). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (5), due to $C_2[U(1)] = 0$. This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. The all-loop order finiteness theorem of ref.[26] is based on: (a) the structure of the supercurrent in $N = 1$ SYM and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [26]. Alternatively, similar results can be obtained [27, 33] using an analysis of the all-loop NSVZ gauge beta-function [34].

3 SOFT SUPERSYMMETRY BREAKING AND FINITENESS

The above described method of reducing the dimensionless couplings has been extended [30, 29] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. Recently very interesting progress has been made [35]-[43] concerning the renormalization properties of the SSB parameters, based conceptually and technically on the work of ref. [37]. In this work the powerful supergraph method [40] for studying supersymmetric theories has been applied to the softly broken ones by using the "spurion" external space-time independent superfields [41]. In the latter method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire "vacuum expectation values". Based on this method the relations among the soft term renormalization and that of an unbroken supersymmetric theory have been derived. In particular the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. The key point in the strategy of refs. [35]-[43] in solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators [35]. It is indeed possible to do this on the RGI surface which is defined by the solution of the reduction equations. In addition it was found that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule at one-loop [39]. This result was generalized to two-loops for finite theories [43], and then to all-loops for general Gauge-Yukawa and Finite Unified Theories [36].

In order to obtain a feeling of some of the above results, consider the superpotential given by (1) along with the Lagrangian for SSB terms

$$-\mathcal{L}_{\rm SB} = \frac{1}{6}\, h^{ijk}\, \phi_i\phi_j\phi_k + \frac{1}{2}\, b^{ij}\, \phi_i\phi_j + \frac{1}{2}\, (m^2)^j_i\, \phi^{*\,i}\phi_j + \frac{1}{2}\, M\, \lambda\lambda + \text{H.c.}, \tag{6}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since only finite theories are considered here, it is assumed that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. It is also assumed that the reduction equations admit power series solutions of the form $C^{ijk} = g\, \sum_{n=0} \rho^{ijk}_{(n)} g^{2n}$. According to the finiteness theorem [26], the theory is then finite to all-orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved by [32]

$$h^{ijk} = -MC^{ijk} + \ldots = -M\rho^{ijk}_{(0)}\, g + O(g^5)\ . \tag{7}$$

An additional constraint in the SSB sector up to two-loops [43], concerns the soft scalar masses as follows

$$\frac{(\ m_i^2 + m_j^2 + m_k^2\ )}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2}\, \Delta^{(2)} + O(g^4) \tag{8}$$

for i, j, k with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} = -2\sum_l [(m_l^2/MM^\dagger) - (1/3)]\ T(R_l), \tag{9}$$

which vanishes for the universal choice [32], i.e. when all the soft scalar masses are the same at the unification point.

If we know higher-loop β-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the β-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results, we need something else instead of knowledge of explicit β-functions, e.g. some relations among β-functions. Due to space limitations we refer the interested reader to ref. [36].

4 FINITE UNIFIED THEORIES

In this section we examine two concrete $SU(5)$ finite models, where the reduction of couplings in the dimensionless and dimensionful sector has been achieved. For other interesting Finite Unified Theories based on cross group structure see ref.[44]. A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, which obviously should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model. The model of ref. [28], which will be labeled **A**, and a slight variation of this model (labeled **B**), which can also be obtained from the class of the models suggested by Kazakov *et al.* [35] with a modification to suppress non-diagonal anomalous dimensions[2].

The superpotential which describes the two models takes the form [28, 43]

$$W = \sum_{i=1}^{3} [\ \frac{1}{2} g_i^u\, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d\, \mathbf{10}_i \overline{\mathbf{5}}_i\, \overline{H}_i\ ] + g_{23}^u\, \mathbf{10}_2 \mathbf{10}_3 H_4 + g_{23}^d\, \mathbf{10}_2 \overline{\mathbf{5}}_3$$
$$\overline{H}_4 + g_{32}^d\, \mathbf{10}_3 \overline{\mathbf{5}}_2\, \overline{H}_4 + \sum_{a=1}^{4} g_a^f\, H_a\, \mathbf{24}\, \overline{H}_a + \frac{g^\lambda}{3}\, (\mathbf{24})^3\ , \qquad (10)$$

where H_a and $\overline{H}_a$ $(a = 1, \ldots, 4)$ stand for the Higgs quintets and anti-quintets.

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ for the models $\{\mathbf{A}\ ,\ \mathbf{B}\}$ are:

$$\begin{aligned}
(g_1^u)^2 &= \{\frac{8}{5}, \frac{8}{5}\} g^2\ ,\ (g_1^d)^2 = \{\frac{6}{5}, \frac{6}{5}\} g^2\ , \\
(g_2^u)^2 &= (g_3^u)^2 = \{\frac{8}{5}, \frac{4}{5}\} g^2\ , \\
(g_2^d)^2 &= (g_3^d)^2 = \{\frac{6}{5}, \frac{3}{5}\} g^2\ ,
\end{aligned} \qquad (11)$$

[2] An extension to three families, and the generation of quark mixing angles and masses in Finite Unified Theories has been addressed in [45], where several realistic examples are given. These extensions are not considered here.

$$
\begin{aligned}
(g^u_{23})^2 &= \{0, \frac{4}{5}\} g^2 \ , \ (g^d_{23})^2 = (g^d_{32})^2 = \{0, \frac{3}{5}\} g^2 \ , \\
(g^\lambda)^2 &= \frac{15}{7} g^2 \ , \ (g^f_2)^2 = (g^f_3)^2 = \{0, \frac{1}{2}\} g^2 \ , \\
(g^f_1)^2 &= 0 \ , \ (g^f_4)^2 = \{1, 0\} g^2 \ .
\end{aligned}
$$

According to the theorem of ref. [26] these models are finite to all orders. After the reduction of couplings the symmetry of W is enhanced [28, 43].

The main difference of the models **A** and **B** is that three pairs of Higgs quintets and anti-quintets couple to the **24** for **B** so that it is not necessary to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$.

In the dimensionful sector, the sum rule gives us the following boundary conditions at the GUT scale [43]:

$$
m^2_{H_u} + 2m^2_{\mathbf{10}} = m^2_{H_d} + m^2_{\overline{\mathbf{5}}} + m^2_{\mathbf{10}} = M^2 \ \text{ for } \ \mathbf{A} \ ; \tag{12}
$$

$$
\begin{aligned}
m^2_{H_u} + 2m^2_{\mathbf{10}} &= M^2 \ , \ m^2_{H_d} - 2m^2_{\mathbf{10}} = -\frac{M^2}{3} \ , \\
m^2_{\overline{\mathbf{5}}} + 3m^2_{\mathbf{10}} &= \frac{4M^2}{3} \quad \text{for } \ \mathbf{B},
\end{aligned} \tag{13}
$$

where we use as free parameters $m_{\overline{\mathbf{5}}} \equiv m_{\overline{\mathbf{5}}_3}$ and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for the model **A**, and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for **B**, in addition to M.

5 PREDICTIONS OF LOW ENERGY PARAMETERS

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (11), the $h = -MC$ relation, and the soft scalar-mass sum rule (8) at $M_{\rm GUT}$, as applied in the two models. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones with the relevant boundary conditions. Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale M_s (which we define as the average of the stop masses) and therefore below that scale the effective theory is just the SM.

The predictions for the top quark mass M_t are ~ 183 and ~ 174 GeV in models **A** and **B** respectively. Comparing these predictions with the most recent experimental value $M_t^{exp} = (172.7 \pm 2.9)$ GeV [46], and recalling that the theoretical values for M_t may suffer from a correction of $\sim 4\%$ [47], we see that they are consistent with the experimental data (for model **A** the agreement is at the two sigma level). In addition the value of $\tan\beta$ is found to be $\tan\beta \sim 54$ and ~ 48 for models **A** and **B** respectively.

In the SSB sector, besides the constraints imposed by finiteness there are further restrictions imposed by phenomenology. In the case where all the soft scalar masses are universal at the unfication scale, there is no region of M below $O(few\ TeV)$ in which $m_{\tilde\tau} > m_{\chi^0}$ is satisfied (where $m_{\tilde\tau}$ is the lightest $\tilde\tau$ mass, and m_{χ^0} the lightest neutralino mass, which is the lightest supersymmetric particle). But once the universality condition is

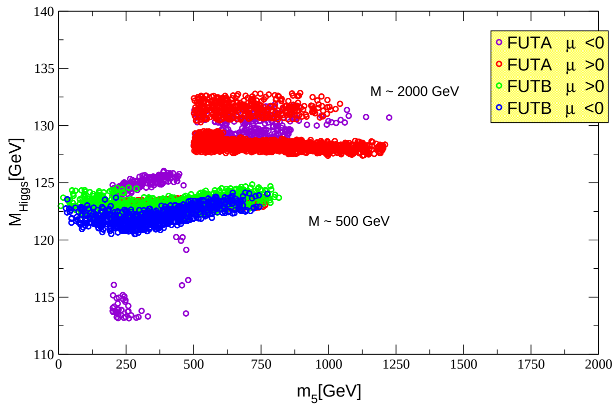

Figure 1: m_h as function of $m_{\mathbf{5}}$ for different values of M for models **FUTA** and **FUTB**, for $\mu < 0$ and $\mu > 0$.

relaxed this problem can be solved naturally (thanks to the sum rule). More specifically, using the sum rule (8) and imposing the conditions a) successful radiative electroweak symmetry breaking, b) $m^2_{\tilde{\tau}} > 0$ and c) $m_{\tilde{\tau}} > m_{\chi^0}$, a comfortable parameter space for both models (although model **B** requires large $M \sim 1$ TeV) is found.

As additional constraints, we consider the following observables: the anomalous magnetic moment of the muon, $(g-2)_\mu$, rare b decays $\text{BR}(b \to s\gamma)$ and $\text{BR}(B_s \to \mu^+\mu^-)$, as well as the density of cold dark matter in the Universe, assuming it consists mainly of neutralinos.

For the branching ratio $BR(b \to s\gamma)$ [48], we take the present experimental value estimated by the Heavy Flavour Averaging Group (HFAG) is [49]

$$\text{BR}(b \to s\gamma) = (3.54^{+0.30}_{-0.28})\, 10^{-4}, \tag{14}$$

where the error includes an uncertainty due to the decay spectrum, as well as the statistical error. In the case of the anomalous magnetic moment of the muon $a_\mu \equiv (g-2)_\mu$, we compare our different models with $a_\mu^{\text{exp}} - a_\mu^{\text{theo}} = (25.2 \pm 9.2)\, 10^{-10}$.

For the branching ratio $\text{BR}(B_s \to \mu^+\mu^-)$, the SM prediction is $(3.4 \pm 0.5) \cdot 10^{-9}$ [50], and the present experimental upper limit from the Fermilab Tevatron collider is $3.4 \cdot 10^{-7}$ at the 95% C.L. [51], providing the possibility for the MSSM to dominate the SM contribution.

The lightest supersymmetric particle (LSP) is an excellent candidate for cold dark matter (CDM) [52], with a density that falls naturally within the range

$$0.094 < \Omega_{\text{CDM}} h^2 < 0.129 \tag{15}$$

favoured by a joint analysis of WMAP and other astrophysical and cosmological data [53].

In the graph we show the **FUTA** and **FUTB** results concerning M_h, for different values of M, for the cases where $\mu < 0$ and $\mu > 0$, the LSP is a neutralino χ^0 and the constraints imposed by the cold dark matter density Eq. (15), are satisfied. The results for $\mu > 0$ and $\mu < 0$ are slightly different for **FUTA**: with $\mu < 0$ the spectrum starts around 750 GeV, whereas for $\mu > 0$ the spectrum starts around 500 GeV. The main difference, though, is in the value of the running bottom mass $m_{bot}(m_{bot})$, where we have included the corrections coming from bottom squark-gluino loops and top squark-chargino loops [54]. In the $\mu < 0$ case, $m_{bot} \sim 3.5 - 4.0$ GeV is just below the experimental value

$m_{bot}^{exp} \sim 4.0 - 4.5$ GeV [55], whereas in the $\mu > 0$ case, $m_{bot} \sim 4.8 - 5.3$ GeV, i. e. above the experimental value.

In the case of **FUTB** the spectrum starts around $300 \sim 400$ GeV, and the $m_{bot} \sim 4 - 4.3$ GeV for $\mu < 0$ and $m_{bot} \sim 4.8 - 5.1$ GeV for $\mu > 0$.

The Higgs mass prediction of the two models is, although the details of each of the models differ, in the following range

$$m_h = \quad \sim 112 - 132 \quad GeV, \tag{16}$$

where the uncertainty comes from variations of the gaugino mass M and the soft scalar masses, and from finite (i.e. not logarithmically divergent) corrections in changing renormalization scheme. The one-loop radiative corrections have been included [56] for m_h, but not for the rest of the spectrum. In making the analysis, the value of M was varied from $200 - 2000$ GeV. We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5% of the FUT boundary conditions. This small variation does not give a noticeable effect in the results at low energies. The requirement $m_h > 114.4$ GeV [57] (neglecting the theoretical uncertainties) excludes the possibility of $M = 200$ GeV for FUTA, as seen also from the graph..

A more detailed numerical analysis, where the results of our program and of the known programs FeynHiggs [58] and Suspect [59] are combined, is currently in progress [60].

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BCS and Witten's SUSY QM as Different Facets of the Same Lattice Model

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Abstract

We investigate the large-N behaviour of simple examples of supersymmetric interactions for fermions on a lattice. Witten's supersymmetric quantum mechanics and the BCS model appear as just two different aspects of one and the same model. For the BCS model, supersymmetry is only respected in a coherent superposition of Bogoliubov states. This talk is based on Ref. [1].

1 Algebraic framework

We realize the simplest supersymmetric system by a finite fermion lattice. The basic structural elements of supersymmetry are a graded C^*-algebra $\mathcal{A}$ (the Fermi algebra) and an odd nilpotent element $Q \in \mathcal{A}$ (the supercharge):

$$Q^2 = 0 \Rightarrow \{Q^\dagger\}^2 = 0, \qquad Q + Q^\dagger := G, \qquad QQ^\dagger + Q^\dagger Q =: H = G^2. \tag{1.1}$$

H is supposed to be the generator of the time evolution and, by (1.1), has the properties

$$\begin{array}{rl} (i) & [Q, H] = 0 \Leftrightarrow [Q^\dagger, H] = 0 \\ (ii) & H|0\rangle = 0 \Leftrightarrow Q|0\rangle = Q^\dagger|0\rangle = 0 \\ (iii) & E: \ H|e\rangle = E|e\rangle, \ E > 0 \Rightarrow \qquad \text{is at least twofold-degenerate.} \end{array} \tag{1.2}$$

For *(iii)*, note that either $Q|e\rangle$ or $Q^\dagger|e\rangle$ must be different from zero and also belongs to the eigenvalue E. Since $Q|e\rangle$ cannot be $\sim |e\rangle$, there must be a degeneracy.

Already at this level of generality, the Hilbert space $\mathcal{H}$ assumes a structure: Eq.(1.2) implies that it can be written as a sum of a zero-space $\mathcal{H}_0 = P_0\mathcal{H}$ and a tensor product of a degeneracy space $\mathcal{C}^2$ and the rest, $\mathcal{H}_H$: $\mathcal{H} = \mathcal{H}_0 \bigoplus (\mathcal{C}^2 \otimes \mathcal{H}_H)$. In $\mathcal{H}_H$, H has strictly positive eigenvalues, while the generator of the supertransformation G has eigenvalues $\pm\sqrt{H}$ and leaves these spaces invariant. Thus the supertransformation and the time evolution are linked. This closeness is lost in the limit of an infinite lattice, where it can even happen that the supertransformation is not well defined on local operators whereas the time evolution is.

Defining $\eta = Q/\sqrt{H}, P_0\eta = 0$, we have $\eta\eta^\dagger + \eta^\dagger\eta = 1 - P_0$. Here we identify $H = H(1-P_0)$; $\sqrt{H}$ denotes the positive square root of $H \in \mathcal{B}(\mathcal{H}_H)$, but there are others: if $G_\alpha = e^{i\alpha}Q + e^{-i\alpha}Q^\dagger, \alpha \in (0, 2\pi)$, then $G_\alpha^2 = H\ \forall\alpha$. The gauge transformation $G = G_0 \to G_\alpha$ is effected by $F = [\eta, \eta^\dagger]$, so that $e^{i\alpha F}G_0e^{-i\alpha F} = G_\alpha$.

In $\mathcal{C}^2 \otimes \mathcal{H}_H$ every element can be written as $A\eta\eta^\dagger + B\eta^\dagger\eta + C\eta + D\eta^\dagger, A, B, C, D \in \mathcal{B}(\mathcal{H}_H)$. This gives the algebra a grading, the first two terms being even and the others odd. All this emerges from a single nilpotent operator, namely Q. The G_α's generate a supertransformation, which mixes even and odd elements of $\mathcal{A}$:

$$a_k \ \to\ a_k(s,\alpha) = e^{isG_\alpha}a_ke^{-isG_\alpha}. \tag{1.3}$$

Thus we associate with the supersymmetry a Clifford variable. If η were Grassman, this would mean $\{\eta, \eta^\dagger\} = 0$, which is not possible in a $\mathcal{C}^*$-algebra; but our η is nilpotent and anticommutes with the other odd elements. We start with a finite-dimensional $\mathcal{A}$, $i, k = 1, 2, ..., N$, and later investigate the limit $N \to \infty$.

The supertransformation is a non-linear transformation of the a's that preserves their algebraic relations. In the simplest case,

$$N = 1,\ Q = a,\ G = e^{i\alpha}a + e^{-i\alpha}a^\dagger,$$

H has to be trivial since it is twofold-degenerate. The supertransformation reads

$$a(s,\alpha) = (\cos s)^2a + e^{2i\alpha}(\sin s)^2a^\dagger + ie^{i\alpha}\cos s\sin s(a^\dagger a - aa^\dagger). \tag{1.4}$$

It is a two-dimensional generating subset (but not subgroup) of the automorphism group, which is isomorphic to $SU(2)$. Equation (1.4) tells us that in its embryonic form the supertransformation is a Bogoliubov transformation plus a quadratic term.

To obtain such an explicit expression for higher N, we impose a locality condition. We think in terms of a lattice and assume Q to be the sum of charges at the lattice sites:

$$Q = \sum_{i=1}^{N} q_i. \tag{1.5}$$

Equation (1.1) requires $\{q_i, q_j^\dagger\} = 0\ \forall i \neq j$, in addition to $\{q_i, q_j\} = 0$.

I. <u>One kind of fermions at each lattice site</u>

The most general Q — and thus the generators G and H — are of the form

$$\begin{aligned} &Q = \textstyle\sum_i z_ia_i, \quad z_i \in \mathbf{C}, \\ &G = \textstyle\sum_i z_i(a_i + a_i^\dagger)\,, \qquad H = \textstyle\sum_i z_i^2, \end{aligned} \tag{1.6}$$

so, with H being a c-number, the time evolution is trivial. The $\mathcal{H}_0$ space is empty, the η variable becomes $\sum_i z_ia_i/(\sum_k z_k)^{-1/2}$ and is a collective fermion coordinate. Since it obeys the CAR-relations we have $||\eta|| = 1$, although this is a sum of N operators with a norm $N^{-1/2}$.

However, the supertransformation e^{isG} is not trivial:

$$a_k(s) = e^{isG}a_ke^{-isG} = (a_k - z_k/2G)\,e^{-2isG} + z_k/2G, \quad a_k(0) = a_k. \tag{1.7}$$

Also, it is not just a tensor product of the unitaries of the baby model since q_i and $q_i^\dagger$ in (1.5) anticommute at different points. One readily verifies that this is an automorphism of $\mathcal{A}$, but it is not a local transformation — $a_k(s)$ depends on all the other a's and $a^\dagger$'s.

II. Two kinds of fermions at each lattice site

These might be electrons with spin up and down. $\mathcal{A}$ is now generated by $a_{\uparrow,i}$ and $a_{\downarrow,i}$ and a typical set — a local supercharge and symmetry generators — reads

$$\begin{aligned} &Q = \textstyle\sum_i a_{\uparrow,i}^\dagger a_{\uparrow,i} a_{\downarrow,i}\, z_i\,, \quad z_i \in \mathbf{R}^+, \\ &G = \textstyle\sum_i z_i a_{\uparrow,i}^\dagger a_{\uparrow,i}(a_{\downarrow,i} + a_{\downarrow,i}^\dagger), \quad H = \textstyle\sum_i z_i^2\, a_{\uparrow,i}^\dagger a_{\uparrow,i}. \end{aligned} \tag{1.8}$$

Thus a free time evolution where half of the fermions are quiet is supersymmetric with a local supercharge. In this case the vacuum $|\uparrow 0\rangle, a_{\uparrow,i}|\uparrow 0\rangle$ satisfies $Q|\uparrow 0\rangle = Q^\dagger|\uparrow 0\rangle = 0$, irrespective of the down spins. All these vectors belong to the eigenvalue zero of H and span $\mathcal{H}_0$. In fact the eigenvalues are at least 2^N-fold degenerate.

The supertransformation generated by e^{isG}:

$$a_{\uparrow,k}(s) = a_{\uparrow,k} e^{-is(G-(a_{\downarrow,k}+a_{\downarrow,k}^\dagger)z_k)} e^{-isG}, \tag{1.9}$$

is indeed an automorphism group of $\mathcal{A}$ that mixes spin up and down as well as even and odd elements; however, the time evolution leaves $a_{\downarrow,k}$ invariant up to a phase and mixes neither between even and odd elements nor between elements at different sites.

III. Three kinds of fermions at each lattice site

We start again with the CAR algebra $\mathcal{A}$ generated by $\{a_i^\alpha\}$. However, we drop locality, since the charge Q in (1.5) creates a non-local supertransformation even with strictly local q_i. Instead we impose translation invariance of the q_i's in such a way that Q becomes translation- and even permutation- invariant. Furthermore we think of $\{a_i^1\, a_i^2\}$ as Cooper pairs, and thus consider the subalgebra $\mathcal{C} \subset \mathcal{A}$, generated by $b_i = a_i^1 a_i^2$ and a_k^3. Although the b_i's commute for different sites, they do not form a *bona fide* Bose field since there is at most one pair per site, $b^2 = 0$. However in $\mathcal{C}$ the anticommutator $\{b_i^\dagger, b_i\}$ is a projection of the centre and, in an irreducible representation, it equals unity. These are the representations we are interested in, so we can think of the b's as of spin variables:

$$b_i = (\sigma_i^x - i\sigma_i^y)/2, \qquad 1 - 2b_i^\dagger b_i = \sigma_i^z.$$

Thus our algebra $\mathcal{C}$ is defined by

$$\begin{aligned} &\{a_i^3, a_j^{3\dagger}\} = \delta_{ij},\ \{a_i^3, a_j^3\} = 0,\ [a_i^3, b_k] = 0 \qquad && i, k = 1, \ldots, N, \\ &[b_i, b_k^\dagger] = \delta_{ik}(1 - 2b_i^\dagger b_i), \quad \{b_i, b_i^\dagger\} = 1 && \alpha, \beta = 1, 2, 3. \end{aligned} \tag{1.10}$$

The supertransformation, and thus the dynamics, is defined by the fluctuation variables

$$M_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N b_k, \qquad \eta_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N a_k^3, \tag{1.11}$$

for which the following holds:

(i) $\eta_N \eta_N^\dagger + \eta_N^\dagger \eta_N = 1, \quad \eta_N^2 = 0;$

(ii) $[M_N, M_N^\dagger] = 1 - \frac{2}{N}\sum_{k=1}^{N} b_k^\dagger b_k$;

(iii) $[M_N, \eta_N] = [M_N, \eta_N^\dagger] = 0$.

Remarks

1. η represents a collective Fermi mode and will serve as a Clifford variable. The fact that the number of single fermion modes s equals the number of pairs is not essential;

2. M is a collective Bose mode and, in a representation based on the "vacuum" $|0\rangle : \quad b_i|0\rangle = 0$ (all spins down), it assumes for $N \to \infty$ the properties of $(x + ip)/\sqrt{2}$ in quantum mechanics; we thus arrive at Witten's supersymmetric quantum mechanics [2];

3. By anticommutativity, a_k^3 are so correlated that $\|\eta_N\| = 1 \ \forall N$. On the contrary, $\|M_N\| = \sqrt{N}/2$ since we can think of M_N as of $(S^x - iS^y)/(2\sqrt{N})$, with $S = \sum_{k=1}^{N} \sigma_k$.

In agreement with our desideratum, Eq. (2.2), we take $q_i = b_i\eta/\sqrt{N}$ and thus obtain

$$Q_N = M_N\eta_N, \qquad G = Q_N + Q_N^\dagger$$
$$H_{SS} = G^2 = \{Q_N, Q_N^\dagger\} = M_N^\dagger M_N + \eta_N\eta_N^\dagger\left(1 - \frac{2}{N}\sum_{k=1}^{N} b_k^\dagger b_k\right). \tag{1.12}$$

Note that, in the BCS model, $H_{\mathrm{BCS}} = -M_N^\dagger M_N$; it thus differs from $-H_{\mathrm{SS}}$ only by $O(1)$. The energies per particle H/N coincide for $N \to \infty$, so with three fermions per site we can construct a supersymmmetric version of the BCS model. This is mathematically well explored and we can behold the many vistas that the limit $N \to \infty$ offers.

2 The limit $N \to \infty$

Investigation of the limit $N \to \infty$ for three different states — the ground state (GS) of H, its ceiling state (CS), and the Bogoliubov state (BS), which in this limit has the same energy per particle as the CS but remains pure on the quasi-local algebra — shows it as state-dependent. We also distinguish between three types of limiting observables:

a) local operators, i.e. polynomials in the operators A_i, the elements of $\mathcal{A}$, which are localized at the site i;

b) mesoscopic observables, i.e. limits of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} A_i$;

c) macroscopic observables, i.e. limits of $\frac{1}{N}\sum_{i=1}^{N} A_i$.

For the limiting procedure we impose only minimal requirements: we assume that a state for arbitrary N is given and if the expectation values converge, we demand that the limits can be interpreted as expectation values of a limiting algebra. The latter has to be big enough for the mesoscopic observables to still reflect some quantum features.

In Tables 1 and 2 our confusing results for the time evolution and for the supertransformation for this variety of operators are collected (for details, see [1]).

The automorphisms turn out to be different in all cases. It is not even true that the microscopic time evolution determines the mesoscopic one. The supertransformation is finite and non-trivial only for the ground state, where Witten's supersymmetric quantum mechanics emerges in the limiting procedure. In the coherent superposition of Bogoliubov states (i.e. the ceiling state) it escapes our observation on the local level. There, the mesoscopic algebra is stable under the emerging time evolution, whereas it is not so in the Bogoliubov state, which also breaks supersymmetry.

	Local	Mesoscopic	Macroscopic
GS	$\overrightarrow{\sigma}^{(j)}(t) = \overrightarrow{\sigma}^{(j)}$	$q(t) + ip(t) = e^{it}(q + ip)$ $\eta(t) = e^{it}\eta$	constant
BS	$\sigma_x^j(t) = \sigma_x^j(0)$ $\sigma_y^j(t) + i\sigma_z^j(t) = e^{it}(\sigma_y^j + i\sigma_z^j)$	$q(t) = q - pt, \quad p(t) = p$ $\eta(t) = \eta$	constant
CS	$\overrightarrow{\sigma}$ rotates around $(\cos\varphi, \sin\varphi, 0)$	p_φ, φ constant	constant

Table 1: The time evolution

	Local	Mesoscopic	Macroscopic
GS	$\overrightarrow{\sigma}^{(j)}(s) = \overrightarrow{\sigma}^{(j)}(0)$	$q' + ip' = i\eta^\dagger$ $\eta' = i(q - ip)[\eta, \eta^\dagger]$	$\overrightarrow{S}(s) = \text{constant}$
BS	$\overrightarrow{\sigma}^{(j)}(s) = \overrightarrow{\sigma}^{(j)}(0)$	$\eta' \sim \sqrt{N}$ becomes infinite	$\overrightarrow{S}(s) = \text{constant}$
CS	σ' is ill-defined		S'_z becomes infinite

Table 2: The supertransformation

3 Comparison with the BCS theory

Without referring to supersymmetry, we consider the Hamiltonian

$$H'_{\mathrm{BCS}} = -\frac{S_+S_-}{N} = -\frac{S^2 - S_z^2}{4N} = -\frac{S_x^2 + S_y^2}{4N}\,. \tag{3.13}$$

It differs from the previous one by only S_z/N — a bounded operator that therefore converges to an operator in the centre and does not affect the dynamics. The two Hamiltonians can thus be thought of as describing one and the same physical situation. In this case the ground-state energy can be approximated by the rotated ground state of H_{SS}:

$$|\psi_{\alpha,N}\rangle = \prod \left(\frac{1}{\sqrt{2}}\right)^N \left| \begin{array}{c} e^{i\alpha} \\ e^{-i\alpha} \end{array} \right\rangle. \tag{3.14}$$

The limits of the expectations of the local operators give a pure product state

$$\lim\langle\psi_{\alpha,N}|\prod \sigma_{\alpha_j}^{i_j}|\psi_{\alpha,N}\rangle = \prod_j \left(\frac{1}{2}\right)^N \left\langle \begin{array}{c} e^{i\alpha} \\ e^{-i\alpha} \end{array} \right| \sigma_{\alpha_j}^{ij} \left| \begin{array}{c} e^{i\alpha} \\ e^{-i\alpha} \end{array} \right\rangle. \tag{3.15}$$

Similarly to the ground state of H_{SS}, we can interpret

$$\begin{aligned} &\lim e^{ir\,\Sigma\,\sigma_y^k/\sqrt{N}} = e^{irq}, \qquad \lim e^{is\,\Sigma\,\sigma_z^k/\sqrt{N}} = e^{isp} \\ &\lim e^{ir\,\Sigma\,\sigma_y^k/\sqrt{N}} e^{is\,\Sigma\,\sigma_z^k/\sqrt{N}} e^{-ir\,\Sigma\,\sigma_y^k/\sqrt{N}} = e^{-irs}e^{isp} \end{aligned} \tag{3.16}$$

as a non-trivial global algebra, so that e^{irq} and e^{isp} satisfy the Weyl relations in $L^2(R, dq)$, and q and p can be interpreted as space and momentum operators, respectively. This is nothing else but the fluctuation algebra discussed in [3].

The time evolution on the quasi-local algebra corresponds to a rotation of the individual spins around the x-axis [4]. For the global algebra we get something quite different:

$$\left[\frac{S_+S_-}{N}, \frac{S_x - N}{\sqrt{N}}\right] = -i\frac{S_y}{2N\sqrt{N}}, \quad \left[\frac{S_+S_-}{N}, \frac{S_y}{\sqrt{N}}\right] = i\frac{S_x}{2N\sqrt{N}}, \quad \left[\frac{S_+S_-}{N}, \frac{S_z}{\sqrt{N}}\right] = 0$$

in leading order in N. This corresponds, for the Weyl algebra, to $[H, q] = -ip\,,\ [H, p] = 0$, and thus to a free evolution, $q \to q + pt$, $p = \text{const}$. Therefore no invariant state exists and the state over the fluctuation algebra has to change in time.

4 Summary

We have studied three sets of operators — local, mesoscopic and macroscopic — in representations based on three different states of H_{SS}: the ground state, the Bogoliubov state ω_α and the ceiling state ω_c, which is the ground state of H_{BCS}. The states ω_c and ω_α lead to the same energy per particle. In the BCS theory, there is some discussion [5] on which one is better. Below, we list a few arguments to compare the ensuing representations π_c and π_α.

(i) ω_c satisfies ODLRO (off-diagonal long-range order), ω_α does not: for $k \neq j$

$$|\omega_c(\sigma_x^k \sigma_x^j) - \omega_c(\sigma_x^k)\omega_c(\sigma_x^j)| = 1/2$$
$$|\omega_\alpha(\sigma_x^k \sigma_x^j) - \omega_\alpha(\sigma_x^k)\omega_c\alpha(\sigma_x^j)| = 0;$$

(ii) In π_c the time evolution mixes local and mesoscopic quantities, in π_α it is strictly local and corresponds to a rotation around the α-axis;

(iii) In π_α the Josephson phase is fixed to be α, in π_c it is a dynamical variable;

(iv) π_α represents the quasi-local variables irreducibly, in π_c the weak closure contains the non-trivial commuting macroscopic observables.

Remarks:

1. ODLRO is the basis of the Meissner effect [6]. The spectrum of p_φ corresponds to the quantization of the magnetic flux;

2. The absolute phase α has no physical meaning. What can be measured is the phase difference between superconductors. This means either staying in the mesoscopic algebra of π_c or comparing the inequivalent representations π_α and $\pi_{\alpha'}$.

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Conformal Sigma Models in Three Dimensions

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Abstract

Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry is formulated in the framework of Wilson renormalization group in three dimensional space-time. Interesting conformal field theories are found as fixed points of the renormalization group equation. Any Einstein-Kähler manifold corresponds to a conformal field theory when the anomalous dimension is $\gamma = -1/2$.

1 Introduction

Possible candidates for interacting field theories are severely restricted by the requirement of renormalizability in the perturbation theory. Interacting field theories cease to exist in the space-time whose dimension is larger than a critical value. The critical dimension of self-interacting scalar field theories or non-Abelian gauge theories is four, whereas the critical dimension is two in the gravity or non-linear sigma models. It is an important and interesting problem if these critical values remain unchanged even in non-perturbative treatment of field theories.

Although gauge and gravity theories are more important than non-linear sigma models, non-linear sigma models offer the theoretical playground for these models since there is deep similarity between gauge theories and sigma models. In this work, therefore, we study the non-perturbative renormalization property of three dimensional non-linear sigma models, non-renormalizable in the perturbation theory.

For that purpose, we use the Wilson's renormalization group[1]. In this method, field variables are divided into two parts, field ϕ_Λ and with wavelength shorter than $1/\Lambda$ and $\phi_>$ of higher frequency modes. Then, fields with frequencies higher than Λ are integrated out in the path integral formulation in order to obtain the effective action S_Λ

describing the field dynamics with the ultraviolet cutoff Λ. Since it is difficult to integrate out all the field variables of higher frequencies, we integrate out field variables in an infinitesimal momentum range $\Lambda \cdot e^{-\delta t} < k \leq \Lambda$ to calculate the infinitesimal change of the effective action when the ultraviolet cutoff is changed infinitesimally from Λ to $\Lambda \cdot e^{-\delta t}$. The infinitesimal change of the effective action is described by the renormalization group equation(RGE) [1, 2].

The effective action, expanded in powers of space-time derivatives, contains all higher derivative terms, and the RGE is a set of coupled differential equations of infinite dimensions [3]. In order to solve these equations, we have to introduce some kind of truncations. In the simplest truncation method, one retains only terms without derivative, and is called the local potential approximation. In the next approximation, we retain all terms with two derivatives. We may call this approximation as the sigma model approximation, since the typical Lagrangian with two derivatives are written as

$$\mathcal{L} = \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j$$

which is nothing but the Lagrangian of non-linear sigma model, where $g_{ij}(\phi)$ describes the metric of the target manifold $\mathcal{M}$ where field variables ϕ reside. In this work, we use this sigma model approximation. Furthermore, we confine ourselves to theories with $\mathcal{N} = 2$ supersymmetry, in order to forbid the appearance of local potential terms.

The renormalization group equation for nonlinear sigma models describes the deformation of the metric of the target manifold $\mathcal{M}$ when the ultraviolet cutoff has been changed. The fixed points of the RGE correspond to conformal field theories which remain unchanged under the change of the mass scale. In this work, we study the fixed point theories of the RGE. It should be emphasized that RGE obtained in the perturbation theory has the similar form with the RGE obtained in the Wilson's renormalization method can only be used in the vicinity of the free field theory, whereas the Wilsonian RGE can be used to study even nontrivial conformal field theories located far away from the free field theory.

2 Nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry in three dimensions

Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in three dimensions are defined by the so-called Kähler potential $K(\phi, \bar{\phi})$, which is a function of the chiral and anti-chiral superfields, ϕ^i and $\bar{\phi}^{\bar{j}}$. A chiral superfield $\phi^i(x, \theta)$ consists of a complex scalar field $\varphi^i(x)$ and a complex fermion $\psi^i(x)$. The bosonic fields $\varphi^i(x)$ play the role of the coordinates of the target manifold $\mathcal{M}$. The metric, characterizing the target manifold $\mathcal{M}$, is obtained by the second derivative of this Kähler potential

$$g_{i\bar{j}} = \frac{\partial^2 K(\varphi, \bar{\varphi})}{\partial \varphi^i \partial \bar{\varphi}^{\bar{j}}} \equiv K_{,i\bar{j}} \,.$$

The manifold defined by a Kähler potential is called the Kähler manifold.

3 Renormalization group equation

Renormalization group equation for the metric of the target manifold $\mathcal{M}$ in three dimensional sigma models has been derived in [4, 5, 6]

$$-\frac{d}{dt}g_{i\bar{j}} = \frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i\xi_{\bar{j}} + \nabla_{\bar{j}}\xi_i \tag{3.1}$$

where t parametrize the change of the cutoff $\Lambda \to \mathrm{e}^{-t}\Lambda$. The vector field ξ^i can be written

$$\xi^i = \left(\frac{1}{2} + \gamma\right)\varphi^i \tag{3.2}$$

in the Kähler normal coordinate[7]. The anomalous dimension γ of the field ϕ is introduced to normalize the field at the origin

$$g_{i\bar{j}}\,|_{\varphi=0} = \delta_{i\bar{j}}. \tag{3.3}$$

The renormalization group equation (3.1), called the Ricci flow in mathematical literature [8], describes the deformation of the target manifold of the effective theory when the renormalization mass scale is changed from Λ to $\mathrm{e}^{-t}\Lambda$.

The fixed point, invariant under the change of the mass scale, is obtained by solving an equation

$$\frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i\xi_{\bar{j}} + \nabla_{\bar{j}}\xi_i = 0. \tag{3.4}$$

The metric $g_{i\bar{j}}$ satisfying this equation defines a conformal field theory.

4 Fixed point theory for $\gamma = -\frac{1}{2}$

When the anomalous dimension of the field takes a specific value $-\frac{1}{2}$, the fixed point of the renormalization group equation has an extremely simple form

$$\frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} = 0. \tag{4.5}$$

By comparing with the equation for the Einstein-Kähler manifolds [1]

$$R_{i\bar{j}} - h\lambda^2 g_{i\bar{j}} = 0, \tag{4.6}$$

with a positive cosmological constant $h\lambda^2 > 0$, we find the coupling constant λ (inverse radius of the Einstein-Kähler manifold) of the fixed point theory is given by

$$\lambda^2 = \frac{2\pi^2}{h}. \tag{4.7}$$

We found that any Einstein-Kähler manifold corresponds to the conformally invariant field theory, when the radius, the inverse coupling constant, takes a specific value (4.7).

A special class of the Kähler-Einstein manifolds is provided by the hermitian symmetric space(HSS) of the form G/H. The compact HSS is completely classified and listed in the following table, where h denotes the dual coxeter number of the group G. [2]

[1] A parameter λ has been introduced to satisfy the renormalization condition (3.3) at the origin.
[2] For S^2, λ^2 is related to the radius a^2 of the sphere by $\lambda^2 = 1/2a^2$ and $h = 2$.

G/H	$D = dim_{\mathbb{C}}(G/H)$	h
$SU(N)/SU(N-1) \times U(1)$	$N-1$	N
$U(N)/U(N-M) \times U(M)$	$M(N-M)$	N
$SO(N)/SO(N-2) \times U(1)$	$N-2$	$N-2$
$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$	$N+1$
$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)$	$N-1$
$E_6/SO(10) \times U(1)$	16	12
$E_7/E_6 \times U(1)$	27	18

The metric of HSS is explicitly constructed by using the gauge theory technique in ref.[9], therefore it is possible to write down the Lagrangian of conformal field theories explicitly.

5 Two-dimensional manifold

Although it is difficult to solve eq.(3.4) explicitly for $\gamma \neq -\frac{1}{2}$, it can be solved for two-dimensional target space $\mathcal{M}$ by using a graphical method. In this section, we use real variables to describe the target manifold $\mathcal{M}$, and choose a special gauge where the line element of $\mathcal{M}$ takes the following form

$$ds^2 = dr^2 + e^2(r)d\phi^2 \tag{5.8}$$

where we have assumed the rotational symmetry in the ϕ direction. Here $e(r)$ denotes the radius of a circle for a fixed value of r. In this coordinate system, the Ricci tensor and the vector field ξ^i takes the following form

$$\begin{aligned} R_{rr} &= R^{\phi}_{r\phi r} = -\frac{e''}{e}, \quad R_{\phi\phi} = R^{r}_{\phi r\phi} = -ee'', \\ \xi^r &= ce(r), \quad \xi^{\phi} = 0, \qquad (c = \frac{1}{2} + \gamma). \end{aligned} \tag{5.9}$$

Corresponding to the renormalization condition (3.3), we impose a boundary condition for $e(r)$

$$lim_{r\to 0}\frac{e(r)}{r} = 1. \tag{5.10}$$

The RG equation reads in this gauge

$$-a^2e'' - e + 2cee' = 0. \tag{5.11}$$

When $c \neq 0$, it is convenient to rewrite the second order differential equation to a set of the first order differential equations

$$\begin{aligned} e' &= p \\ p' &= -\frac{1}{a^2}e(1-2cp) \end{aligned} \tag{5.12}$$

with the boundary condition

$$e(0) = 0, \quad p(0) = 1 \tag{5.13}$$

The vector field of the flow (5.12) is shown in fig.1. When $0 \leq 2c < 1$, this equation defines a compact manifold since the trajectory starting from the initial point (5.13) comes back to $e = 0$ at a finite r implying the circumference of the circle at that r vanishes. On the other hand, the solution corresponds to a noncompact manifold for $2c \geq 1$.

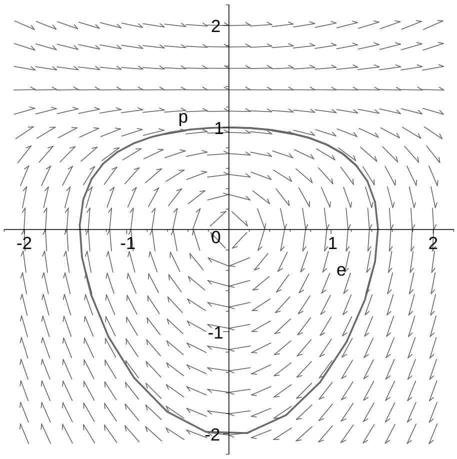

Figure 1: Flow of the first order differential equations (5.12) for $2c < 1$ in the "phase space" $(e(r), p(r))$. The solid line represents the solution specified by the boundary condition.

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Lorentz Gauge Gravity and Induced Effective Theories

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Abstract

We develop the gauge approach based on the Lorentz group to the gravity with torsion. With a Lagrangian quadratic in curvature we show that the Einstein-Hilbert action can be induced from a simple gauge model due to quantum corrections of torsion via formation of a gravito-magnetic condensate. An effective theory of cosmic knots at Planckian scale is proposed.

1. Utiyama-Kibble-Sciama gauge approach to gravity

The gauge approach to gravity based on Lorentz and Poincare group was proposed in [1] and later was developed in many studies (see refs. in [2]). The Lorentz gauge models were further studied by Carmeli [3]. The possibility of inducing the Einstein gravity via quantum corrections was considered by many physicists in various models [4]. In most of these models the Einstein-Hilbert term is induced by quantum corrections due to interaction with matter fields.

In the present article we propose a simple gauge model of quantum gravity based on the Lorentz group as a structural group. In the framework of this gauge model we demonstrate that even in a pure quantum gravity case with torsion the Einstein-Hilbert action can be induced due to the quantum dynamics of torsion via formation a non-trivial vacuum with a gravito-magnetic condensate. We develop the gauge approach to the gravity by suggesting that *the torsion represents exactly the dynamical variable of quantum gravity. Moreover, we conjecture that the torsion can be confined and exists intrinsically as a quantum object, and its quantum dynamics manifests itself by inducing the Einstein-Hilbert theory as an effective theory of quantum gravity.*

We start with the formalism of the Lorentz gauge model along the lines proposed in [1]. The vielbein e_a^μ is treated as a fixed background field which obtains the dynamical content after inducing the Hilbert-Einstein term in the effective theory. The covariant derivative with respect to the Lorentz structural group is defined in a standard manner

$$D_a = e_a^\mu(\partial_\mu + \mathbf{A}_\mu), \tag{1}$$

where $\mathbf{A}_\mu \equiv A_{\mu cd}\Omega^{cd}$ is a general affine connection taking values in Lorentz Lie algebra. The original Lorentz gauge transformation is the following

$$\delta e^\mu_a = \Lambda^b_a e^\mu_b, \qquad \delta \mathbf{A}_\mu = -\partial_\mu \mathbf{\Lambda} - [\mathbf{A}_\mu, \mathbf{\Lambda}], \tag{2}$$

where $\mathbf{\Lambda} \equiv \Lambda_{cd}\Omega^{cd}$. One can split a general gauge connection $A_{\mu cd}$ into two parts, the background (classical) part and the quantum one. In what follows we will specify the classical background as one corresponding to Riemanian space-time geometry

$$A_{\mu c}{}^d = \varphi_{\mu c}{}^d(e) + K_{\mu c}{}^d, \tag{3}$$

where $K_{\mu c}{}^d$ is a contorsion, and $\varphi_{\mu c}{}^d$ is Levi-Civita spin connection given in terms of the vielbein. In the presence of contorsion we have two types of local symmetry transformations:
(I) the classical, or background, gauge transformation

$$\delta\varphi_\mu = -\partial_\mu \mathbf{\Lambda} - [\varphi_\mu, \mathbf{\Lambda}], \qquad \delta \mathbf{K}_\mu = -[\mathbf{K}_\mu, \mathbf{\Lambda}], \tag{4}$$

(II) the quantum gauge transformation

$$\delta\varphi_\mu = 0, \qquad \delta \mathbf{K}_\mu = -\hat{D}_\mu \mathbf{\Lambda} - [\mathbf{K}_\mu, \mathbf{\Lambda}], \tag{5}$$

where the background covariant derivative is defined with the help of Levi-Civita connection $\hat{D}_\mu = \partial_\mu + \varphi_\mu$, $\varphi_\mu \equiv \varphi_{\mu cd}\Omega^{cd}$.

Following the gauge principle as a guiding rule we postulate the gauge symmetry in the model under these two types of transformations. The postulate restricts strongly the admissible gauge invariants as possible candidates for the Lagrangian. For instance, all terms quadratic in torsion (like contact terms) are forbidden since they spoil the type II gauge invariance and by this the renormalizability of the theory.

Let us remind the main lines of Rieman-Cartan geometry (see, for ex., [2]). To define the derivative D_μ covariant under the space-time diffeomorphisms one should include a general Cristoffel symbol $\Gamma^\nu_{\mu\rho}$

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{6}$$

The Cristoffel symbol is related to a general Lorentz connection $\gamma_{\mu a}{}^b$ through the equation $D_\mu e^{\rho a} = \partial_\mu e^{\rho a} + \Gamma^\rho_{\mu\nu} e^{\nu a} - e^{\rho b}\gamma_{\mu b}{}^a = 0$. This allows to convert space-time indices into Lorentz ones and vise versa by using the vielbein. The contorsion is connected with torsion as follows

$$K_{abc} = e^\mu_a K_{\mu bc} = -\frac{1}{2}(T_{abc} - T_{bca} + T_{cab}). \tag{7}$$

Upon making the decomposition (3) the curvature tensor is splitted into two parts

$$\begin{aligned} R_{abcd} &= \hat{R}_{abcd} + \tilde{R}_{abcd}, \\ \hat{R}_{abcd} &= -\hat{D}_{[\underline{a}}\varphi_{\underline{b}]\underline{c}}{}^d - \varphi_{[a|c}{}^e \varphi_{b]e}{}^d, \\ \tilde{R}_{abcd} &= -\hat{D}_{[\underline{a}} K_{\underline{b}]\underline{c}}{}^d - K_{[a|c}{}^e K_{b]e}{}^d, \end{aligned} \tag{8}$$

where the underlined indices stand for indices over which the covariantization is performed, and here we introduce a short notation for the antisymmetrization over indices $[a, b] = ab - ba$.

The classical action for a pure quantum gravity in our approach contains the Maxwell type term quadratic in curvature

$$S_{cl} \;=\; \frac{1}{4g^2}\int\sqrt{-g}d^4xtr\mathbf{R}^2_{\mu\nu} = -\frac{1}{4g^2}\int\sqrt{-g}d^4xR_{\mu\nu cd}R^{\mu\nu cd}, \tag{9}$$

where $\mathbf{R}_{\mu\nu}\equiv R_{\mu\nu cd}\Omega^{cd}$, and we have written down explicitly a new gravitational gauge coupling constant g corresponding to the Lorentz gauge group. For brevity of notations we will use a redefined contorsion which absorbs the coupling constant. The same Lagrangian with the general Lorentz connection constructed from $SL(2,C)$ dyads and vielbeins was considered by Carmeli [3]. It was demonstrated that the corresponding equations of motion after projection with vielbein result in Newman-Penrose form of Einstein-Hilbert equation in the vacuum. Later Martellini and Sodano considered Carmeli's model treating the connection as an independent quantity on vielbein and proved the renormalizability of the model [5].

One should mention, since the Lorentz group is not compact the classical Lagrangain leads to the Hamiltonian which is not positively definite. We adopt the point of view that even though the classical action (9) does not lead to a positively definite Hamiltonian, nevertheless, a consistent quantum theory can be formulated. Since the canonical quantization method fails to handle our model we will apply the quantization scheme based on continual functional integration in Euclidean space-time. Within this quantization scheme the quantum theory can be constructed since in the Euclidean space-time the Lorentz group is locally isomorphic to the product of compact unitary groups $SU(2)\times SU'(2)$.

2. Effective action

The general approach to derivation of the effective theory is to integrate out all high energy (heavy mass) modes while keeping light modes (massless or light particles). Starting with the classical action (9) and imposing the gauge fixing condition $\hat{D}_\mu\mathbf{K}^\mu=0$ one can write down the effective action

$$\exp\Big[iS_{eff}\Big] = \int\mathcal{D}K_{\mu cd}\mathcal{D}\mathbf{c}\mathcal{D}\bar{\mathbf{c}}\exp\Big\{\,i\int\sqrt{-g}d^4x\mathrm{tr}\Big[\frac{1}{4}\hat{\mathbf{R}}^2_{\mu\nu}$$
$$+\;\frac{1}{2}\mathbf{K}_\mu(g_{\mu\nu}\hat{D}\hat{D}-4\hat{\mathbf{R}}_{\mu\nu})\mathbf{K}_\nu+\bar{\mathbf{c}}(\hat{D}\hat{D})\mathbf{c}\;\Big]\Big\}, \tag{10}$$

where $\mathbf{c},\bar{\mathbf{c}}$ are Faddeev-Popov ghosts. The formal expression for the one-loop effective action can be written in the form

$$S_{eff} \;=\; S_{cl}-\frac{i}{2}Tr\ln[(g_{\mu\nu}(\hat{D}\hat{D})^{cd}_{ab}-2\hat{R}_{\mu\nu}{}^{ef}(f_{ef})^{cd}_{ab})]+iTr\ln[(\hat{D}\hat{D})^{cd}_{ab}], \tag{11}$$

where $(f_{ef})^{cd}_{ab}$ are the structural constants of Lorentz Lie algebra. The functional determinants in (11) are not well-defined in Minkowski space-time. As is known, the adding of infinitesimal number factor $-i\epsilon$ to the bare Laplace operator in $\hat{D}\hat{D}$ is conditioned by the requirement of causality. The infinitesimal addition $-i\epsilon$ defines uniquely the Wick rotation from Minkowski space-time to Euclidean one. In our case we should perform the Wick rotation in the base space-time and in the tangent space-time both, so that the Lorentz group in Euclidean sector turns into the compact group $SO(4)\simeq SU(2)\times SU'(2)$. With this the functional integral becomes well-defined. Certainly, there remains a problem of analytical continuation of the final expressions from Euclidean space-time back to Minkowski space-time.

We have the following factorization for the Lie algebra valued curvature tensor

$$R_{\mu\nu cd}\Omega^{cd} = -i(R^i_{\mu\nu}T^i + R'^i_{\mu\nu}T'^i), \tag{12}$$

where T^i, T'^i are generators of the group $SU(2) \times SU(2)'$.

The functional determinants (11) are factorized into the direct product of $SU(2)$ determinants, and the effective action takes a simple form

$$\begin{aligned} S_{eff} &= S_{cl} - \frac{i}{2} Tr \ln[(g_{\mu\nu}(\hat{D}\hat{D})^{ij} - 2\hat{R}^k_{\mu\nu}\epsilon^{kij})] - \frac{i}{2} Tr \ln[(g_{\mu\nu}(\hat{D}'\hat{D}')^{ij} - 2\hat{R}'^k_{\mu\nu}\epsilon^{kij})] \\ &\quad + iTr \ln[(\hat{D}\hat{D})^{ij}] + iTr \ln[(\hat{D}'\hat{D}')^{ij}]. \end{aligned} \tag{13}$$

where all quantities corresponding to the group $SU(2)'$ are marked with apostrophe.

Notice that the curvature squared term contains a dual tensor $\tilde{\hat{R}}^{\mu\nu cd}$, for instance,

$$\begin{aligned} (\hat{R}^i_{\mu\nu})^2 &= \frac{1}{8}(\hat{R}_{\mu\nu cd}\hat{R}^{\mu\nu cd} + \hat{R}_{\mu\nu cd}\tilde{\hat{R}}^{\mu\nu cd}) \equiv \frac{1}{8}(\hat{R}^2 + \hat{R}\tilde{\hat{R}}), \\ (\hat{R}'^i_{\mu\nu})^2 &= \frac{1}{8}(\hat{R}_{\mu\nu cd}\hat{R}^{\mu\nu cd} - \hat{R}_{\mu\nu cd}\tilde{\hat{R}}^{\mu\nu cd}) \equiv \frac{1}{8}(\hat{R}^2 - \hat{R}\tilde{\hat{R}}). \end{aligned} \tag{14}$$

For a constant background one can apply the Schwinger's proper time method and ζ-function regularization in full analogy with the case of $SU(2)$ chromodynamics (QCD) [6]. We will consider a constant homogeneous gravito-magnetic background field $H = \sqrt{\hat{R}^2_{\mu\nu cd}/2}$ which assumes that $\hat{R}_{\mu\nu cd}\tilde{\hat{R}}^{\mu\nu cd} = 0$ in an appropriate coordinate frame. For such gravito-magnetic background the final expression for the one-loop effective Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{eff} &= -\frac{1}{2}H^2 - \frac{11g^2}{48\pi^2}H^2(\ln\frac{gH}{\mu^2} - c), \\ c &= 1 - \frac{1}{2} - \frac{24}{11}\zeta(-1, \frac{3}{2}) = 1.29214\ldots. \end{aligned} \tag{15}$$

With a proper renormalization condition $\frac{\partial^2 V}{\partial H^2}\Big|_{H=\bar{\mu}^2} = \frac{1}{\bar{g}^2}$ one can obtain the renormalized effective potential

$$V = \frac{1}{2\bar{g}^2}H^2 + \frac{11}{48\pi^2}H^2(\ln\frac{H}{\bar{\mu}^2} - \frac{3}{2}). \tag{16}$$

One can check that the effective potential satisfies a renormalization group equation with the same β-function as in a pure $SU(2)$ Yang-Mills theory.

The minimum of the effective potential leads to a gravito-magnetic condensate

$$< H >= \bar{\mu}^2 \exp[-\frac{24\pi^2}{11\bar{g}^2} + 1]. \tag{17}$$

The presence of the minimum of the effective potential does not guarantee that the corresponding new vacuum is stable. The stability of the vacuum condensate even in the pure $SU(2)$ model of QCD presents a long-standing problem, and its solution have passed through several controversal results since of the first paper on that by Nielsen and Olesen [7]. Without clear evidence or at least a strong indication to the vacuum stability one can not make any serious statement based on existence of a non-trivial vacuum condensate. Recently the progress in resolving that problem in favor of stability of the magnetic vacuum has been achieved in [6]. Moreover, it has been found recently [8] that a stable classical configuration made of monopole-antimonopole strings does exist in $SU(2)$ model of QCD providing a strong argument that a stable magnetic vacuum can exist in QCD and, therefore, in our model of quantum gravity as well.

3. Effective induced theories

Due to two types of gauge symmetries the condensate of torsion must vanish $< T_{abc} >= 0$. It is possible that there is a deep analogy with QCD, and the torsion plays a role of the off-diagonal (valence) gluon in QCD, so that one can expect that the torsion can be confined. The presence of a stable gravito-magnetic condensate generates a new scale in the theory, and one can also expect a non-vanishing vacuum averaged value for the curvature containing the torsion part

$$< \tilde{R}_{abcd} >= M^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \tag{18}$$

The sign of the number factor M^2 is chosen positive since it corresponds to the positive curvature space-time which can only be created due to quantum fluctuations.

Expanding the original Lagrangian near the vacuum one obtains the Einstein-Hilbert Lagrangian and the cosmological constant term in lower order approximation (in units $\hbar = c = 1$)

$$\mathcal{L} = -\frac{1}{4g^2}(\hat{R}_{abcd} + \tilde{R}_{abcd})^2 = -\frac{1}{4g^2}\hat{R}^2_{abcd} - \frac{1}{16\pi G}(\hat{R} + 2\lambda) + ... \tag{19}$$

where the Newton constant G and the cosmological constant λ are defined by only one parameter, the renormalized coupling constant $\bar{g}$ at some scale $\bar{\mu}^2$ which supposed to be of order of Planckian scale $10^{19} Gev$.

Certainly, the assumption (18) leads straightforward to a desired induced Einstein-Hilbert term what was known very well before. The most important point is how to make foundation to that hypothesis. In our approach we put this assumption on the real ground by the explicit calculation of the effective potential and derivation of a stable classical vacuum configuration in $SU(2)$ Yang-Mills theory [8].

Till now the numeric value of the gauge coupling $\bar{g}$ was not fixed, and it is a free parameter in the theory. It is possible that there are two phases corresponding to the strong and weak coupling constant. The existence of two phases in gravity was suggested in [9] in a different approach. In the strong coupling phase we can adjust the coupling constant $\bar{g}$ to $\bar{g}^2 \simeq 19$ to obtain the value for G close to the experimental value of the Newton constant. This provides also a large value for the cosmological constant which is consistent with cosmological models containing the initial inflation at very early universe.

It is interesting to consider the possibility of existence of a weak coupling phase with $\bar{g}^2/4\pi < 1$. Using the experimental data for the vacuum energy density $\rho_v = \frac{2\lambda}{16\pi G} = 2\cdot 10^{-47}(Gev)^4$ one can find an appropriate value for the structure constant $\alpha_{\bar{g}} = \bar{g}^2/4\pi = 0.0123$. This value can be compared with the value $\alpha_{SSGUT} \simeq 1/24$ of the structure constant in supersymmetric $SO(10)$ GUT model at unification scale $2 \times 10^{16} Gev$. The same order of the structure constants α_g and α_{SSGUT} might be a hint to the origin of the supersymmetry and its relation to quantum gravity.

Since the Lorentz group $SO(1,3)$ contains a maximal compact subgroup $SO(3) \simeq SU(2)$ we have the same homotopy structure as in $SU(2)$ QCD, in particular, the Hopf mapping $\pi_3(SO(1,3)/SO(2)) = \mathbb{Z}$. This suggests the existence of topological solitons with non-trivial Hopf numbers like the knots in Faddeev-Niemi-Skyrme model [10]. It has been shown that the generalized Faddeev-Niemi-Skyrme model appears as an effective theory of QCD [6]. The derivation was based on Abelian decomposition of the $SU(2)$ gauge potential [11]. The essential part of this decomposition is represented by a topological triplet $\hat{n}^i$ which parameterizes the coset $S^2 \simeq SU(2)/U(1)$. We can apply the results obtained in $SU(2)$ QCD to our model in a case of gravito-magnetic background H. With this the effective Lagrangian corresponding to the generalized Faddeev-Niemi-Skyrme model in gravity is given by [6]

$$\mathcal{L}_{eff} = -\frac{\mu^2}{2}(\partial_\mu \hat{n})^2 - \frac{1}{4}(\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 - \frac{\alpha_1}{4}(\partial_\mu \hat{n} \cdot \partial_\nu \hat{n})^2 - \frac{\alpha_2}{2}(\hat{n} \times \partial^2 \hat{n})^2, \tag{20}$$

where μ, α_1, α_2 are parameters proportional to vacuum averaging values of operator products of the magnetic potential $\tilde{C}_\mu(\hat{n})$. When the parameters $\alpha_{1,2}$ vanish the Lagrangian coincides with one of Faddeev-Niemi-Skyrme model [10], so that we expect the existence of cosmic knot solutions at Planckian scale with the mass of order M_{Planck}. Recently the knot-like cosmic strings were considered in [12].

The detailed consideration of our results will be presented elsewhere.

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The Radiation Reaction Effects in the BMT Model of Spinning Charge and the Radiation Polarization Phenomenon

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Abstract

The effect of radiation polarization (RP) attended with the motion of spinning charge in the magnetic field could be viewed through the classical theory of self-interaction. The quantum expression for the polarization time follows from semi-classical relation $T_{QED} \sim \hbar c^3/\mu_B^2\omega_c^3$, and needs quantum explanation neither for the orbit nor for the spin motion. In our approach the polarization emerges as a result of natural selection in the ensemble of elastically scattered electrons, among which the group of particles that bear their spins in the 'right' directions has the smaller probability of radiation.

1. Introduction

The rise in popularity of the classical spin models was stimulated by the difficulties with high spin wave equations accounting for the interaction of particle with external EM field [1]. The close relation of the (pseudo)classical models of spinning particles to the string theory raises a new phase of interest in this topic [2, 3]. The criterion used by different authors to check the spin degrees of freedom are described correctly, is the possibility for one to obtain, at least with some approximations involved, the Bargmann-Michel-Telegdi (BMT) or the Frenkel-Nyborg equations determining the spin evolution (see e.g. [2, 4]). The reason for this is the universal character of the BMT equation and its well established experimental applicability. Here we consider the problem of self-interaction of the BMT particle and its relation to the RP phenomenon. [1)]

The effect of preferable polarization emerges when the relativistic ($\sim 1\,GeV$) electrons execute the motion in magnetic field during the polarization time [2)]

$$T_{QED} = \frac{8\sqrt{3}}{15}\frac{a_B}{c}\gamma^{-2}\left(\frac{H_c}{H}\right)^3 \tag{1}$$

[1)]It is worth noting that the account for radiation through the local ALD-type equation for spinning charge seems hardly to have a practical meaning (see the extensive study of that topic in [5])

[2)]With some obvious exceptions we use the system of units with $c = 1$, $\hbar = 1$, $\alpha = e^2/4\pi\hbar c$, $a_B = 4\pi\hbar^2/me^2$ and $H_c = m^2c^3/e\hbar$ (α, a_B and H_c being the fine structure constant, the Bohr radius and critical QED field strength).

on the laboratory clocks [6]. RP manifests itself through the asymmetry of the probability of the spontaneous spin-flip transitions

$$w^{\uparrow\downarrow} = \frac{1}{T_{QED}}\left(1 + \zeta_3 \frac{8\sqrt{3}}{15}\right) \tag{2}$$

w.r.t. the value of the initial polarization $\zeta_3 = \pm 1$ [7]. The physical ground for that asymmetry is, of course, radiation process so that the phenomenon itself could be considered as a back-reaction effect. The latter one can describe with the help of the semiclassical elastic scattering probability [8]

$$\exp\left(-\frac{2}{\hbar}\Im\Delta W\right) \; < 1 \tag{3}$$

where the classical self-action of the charge [3)]

$$\Delta W = \frac{1}{2}\int\int \; J_\mu(x)\Delta_c(x,x')J_\mu(x')\, dx\, dx' \tag{4}$$

should have a positive imaginary part ($\Im\Delta W \; > 0$) pointing to the presence of radiation. The photon Green function $\Delta_c(x,x') = i(2\pi)^{-2}/[(x-x')^2 + i0]$ and the source

$$J_\mu = j_\mu + \partial_\nu M_{\mu\nu} \tag{5}$$

includes the orbit (j_μ) and the spin ($\partial_\nu M_{\mu\nu}$) contributions.
Below we clarify in short technical points of calculations performed and discuss the results and some differences from the original considerations in [6, 11].

2. The mass shift and the internal geometry of the world lines

With the help of eqn.(5) the self-action ΔW could be decomposed as follows:

$$\Delta W = \Delta W_{or} + \Delta W_{so} + \Delta W_{ss}\,. \tag{6}$$

The orbit part ΔW_{or} does not contain the spin degrees of freedom and for the case of constant homogeneous EM field (which is the matter of interest to us here)was studied in [8]. The spin-orbit part

$$\Delta W_{so} = -\frac{\mu e}{2\pi^2}\int d\tau \int d\tau' \frac{\varepsilon_{\alpha\beta\gamma\delta}(x-x')_\alpha \dot{x}_\beta(\tau)\dot{x}_\gamma(\tau')S_\delta(\tau')}{[(x-x')^2]^2}\,, \tag{7}$$

was discussed in [9, 10], and

$$\Delta W_{ss} = \frac{\mu^2}{2}\int d\tau \int d\tau' \; u_{[\beta'}S_{\rho]}\, u'_{[\beta}S'_{\rho]}\, \partial'_{\beta'}\partial_\beta\, \Delta_c(x,x') \tag{8}$$

[3)] The subtraction of UV divergences corresponding to the definition of the observable mass is implied in (4) [8, 9].

(unlike ΔW_{ss} self-action ΔW_{so} contains no UV divergences). We use the following notations in (5), (7) and (8):

$$M_{\alpha\beta}(x) = \int d\tau \mu_{\alpha\beta}(\tau)\, \delta^{(4)}(x - x(\tau)) \tag{9}$$

is the polarization density; Frenkel polarization tensor

$$\mu_{\alpha\beta} = i\mu\varepsilon_{\alpha\beta\gamma\delta}\dot{x}_\gamma(\tau)S_\delta(\tau) \tag{10}$$

with $\mu = \frac{1}{2}\, g\, \mu_B$, and g, $\mu_B = e/2m$ being the g factor and Bohr magneton correspondingly. 4-velocities $u \equiv \dot{x}(\tau)$, $u' \equiv \dot{x}(\tau')$ and spin 4-vectors $S \equiv S(\tau)$, $S' \equiv S(\tau')$ are determined from Lorentz and BMT equations:

$$\dot{u} = \kappa \hat{F} \cdot u\,, \tag{11}$$

$$\dot{S} = \frac{1}{2}\, g\, \kappa \hat{F} \cdot S + (\frac{g}{2} - 1)\, \kappa\, u\, (u \cdot \hat{F} \cdot S), \tag{12}$$

($\kappa \equiv e/m$) where the dot from above means derivative w.r.t. proper time.

For the constant homogeneous background the translational symmetry entails in

$$\Delta W = -\Delta m\, T\,, \tag{13}$$

with Δm denoting the mass shift (MS) and T corresponding to the proper time interval of the charge's stay in external field. In application of eqn.(13) it is, generally, supposed that the formation (proper) time of the non-local Δm is much less than T.

The important property of the motion in the constant field is the 'isometry' property of the world lines [8]:

$$(x(\tau) - x(\tau'))^2 = f(\tau - \tau')\,. \tag{14}$$

Here the function f is an even function of the proper time difference $\Delta\tau = \tau - \tau'$. Given this difference, the integrands in expressions (7) and (8) preserve their value along the world line so that these non-local geometrical characteristics exhibit some kind of 'rigidity' which eventually gives rise to eqn.(13).

To compute the invariants present as integrands in the self-actions ΔW_{so} and ΔW_{ss} one can exploit Frenet-Serret (FS) formalism adapted to the case of constant homogeneous EM field in [12]. Let e^A $(A = 0, ..., 3)$ be a FS tetrad with $e^{(0)} \equiv u(\tau)$. For every element of tetrad the Lorentz equation is valid:

$$\dot{e}^A = \kappa \hat{F} \cdot e^A(\tau)\,. \tag{15}$$

Combining the eqn.(15) with the basic equation of FS formalism,

$$\dot{e}^A = \Phi^A_B\, e^B(\tau)\,, \tag{16}$$

one can turn the action of the Lorentzian matrix $\hat{F}$ into the tetrad basis. Of first importance now is the constancy of Frenet matrix Φ^A_B. The non-zero elements of Φ^A_B are the curvature (k), the first (t_1) and the second (t_2) torsions, which have their representations directly in terms of electric and magnetic fields [12].

3. Results

Below we concentrate on the plane motion in the purely magnetic field and $g = 2$ – those conditions are simplest one to make possible the comparison with standard QED [6] and semiclassical QED [11] approaches. For that case $k = \kappa H v_\perp \gamma_\perp = v_\perp t_1$ and $t_2 = 0$, so that the MS Δm_{so} and Δm_{ss} corresponding to the self-actions (7) and (8) can be transformed into forms:

$$\Delta m_{so} = -i \frac{\mu e}{2\pi^2} S_3 \omega_c^2 f_{so}(v_\perp) , \tag{17}$$

$$\Delta m_{ss} = \begin{cases} k_0 + k_{13} - k_{13} \zeta_3^2 + (k_{12} - k_{13}) \zeta_{\perp v}^2 , \\ \\ k_0 + k_{12} - k_{12} \zeta_3^2 - (k_{12} - k_{13}) \zeta_v^2 . \end{cases} \tag{18}$$

The upper and lower representations in eqn.(18) are equivalent since the spin vector $\vec{\zeta}$ in the rest frame of the particle satisfies the relation

$$\vec{\zeta}^2 = \zeta_3^2 + \zeta_v^2 + \zeta_{\perp v}^2 = 1 . \tag{19}$$

Note that $S_3 = \zeta_3$, $S_v = \gamma_\perp \zeta_v$, and $S_{\perp v} = \zeta_{\perp v}$ are the (conserved) spin components parallel to the field $\mathbf{H}$, parallel to the velocity $\mathbf{v}(= \mathbf{v}_\perp)$ and perpendicular to $\mathbf{v}$ correspondingly. The following notations were used in formulas (17) and (18):

$$f_{so} = \frac{v_\perp^2}{\gamma_\perp} \int_0^\infty \frac{\sin^2 w - w \sin w \cos w}{(v_\perp^2 \sin^2 w - w^2)^2} \, dw , \tag{20}$$

$$k_{12} = \frac{-i\mu^2 \omega_c^3}{4\pi^2 \gamma_\perp^4} \int_0^\infty \left[\frac{-w^2 \sin^2 w}{(w^2 - v_\perp^2 \sin^2 w)^3} + \frac{\gamma_\perp^6}{w^2} \right] dw , \tag{21}$$

$$k_{12} - k_{13} = \frac{-i\mu^2 \omega_c^3 v_\perp^2}{4\pi^2 \gamma_\perp^2} \int_0^\infty \frac{(w \cos w - \sin w)^2}{(w^2 - v_\perp^2 \sin^2 w)^3} \, dw \tag{22}$$

with $\omega_c = eH/m$ and $\gamma_\perp$ being the Lorentz factor. The term $k_0 + k_{13}$ ($k_0 + k_{12}$) in the upper (lower) part of the eqn.(18) do not depend on the spin direction and is not of importance in explaining RP. As the numerical investigation shows, the functions k_{12} and k_{13} are rather close each other. Note also, that $i(k_{12} - k_{13})$ is positive in no dependence on the energy of the particle as well as the functions ik_{12} and ik_{13} itself.

4. Discussion

The probability of not-emitting the photons is decreasing with the proper time according to general law (see (3) and (13)):

$$\exp\left(\Im \Delta m \cdot T\right) , \qquad \Im \Delta m < 0 . \tag{23}$$

Accounting for the positivity of the integrals in the r.h.s. of expressions (21), (22) one can guess from the eqn.(18) that particles with $\zeta_v \neq 0$ would have a better chance to preserve their state whereas the particles with $\zeta_{\perp v} \neq 0$ such a possibility should lose just with the same rate.

Supposing the relativistic energies for electrons we find for the spin-dependent part of the total MS $\Delta m = \Delta m_{or} + \Delta m_{so} + \Delta m_{ss}$ the following sum

$$-i\frac{1}{4\sqrt{3}\,a_B}\,\chi^2\zeta_3+i\frac{15}{64\sqrt{3}\,a_B}\,\chi^3\zeta_3^2+i\frac{1}{64\sqrt{3}\,a_B}\,\chi^3\zeta_v^2\,, \tag{24}$$

where the first term comes from Δm_{so} (it would have an opposite sign for positrons [9]) and $\chi=\gamma_\perp H/H_c$. With the notation

$$\lambda=-\frac{2}{\hbar}\Im\Delta mc^2 \tag{25}$$

we arrive at the spin contribution to the decay rate in the form:

$$\lambda_{spin}=\frac{c}{a_B}\chi\left[\frac{1}{2\sqrt{3}}\,\chi\zeta_3-\frac{15}{32\sqrt{3}}\,\chi^2\zeta_3^2-\frac{1}{32\sqrt{3}}\,\chi^2\zeta_v^2\right]. \tag{26}$$

λ_{spin} would be negative for $\zeta_3<0$. This, of course, has no effect on the positivity of the total 'decay rate' λ since $\chi\ll 1$.

Being the probability of radiation per unit proper time, λ in (25) corresponds to either change of particle's state of motion- not only to the spin-flip transitions. The negative ζ_3 slightly reduces this probability, as well as two last terms in (26) do - in no dependence on the signs of ζ_3 and ζ_v. Note, that according to eqns.(18) and (26) spin-spin interaction itself does not give the preferable polarization for elastically scattering particles ('down' for electrons and 'up' for positrons). The RP effect emerges in conjunction of the spin-orbit and spin-spin interactions. The characteristic laboratory times extracted from (26) are

$$T_{ss}^{(1)}=\frac{32\sqrt{3}}{15}\,\frac{a_B}{c}\chi^{-3}\gamma_\perp\,,\qquad T_{ss}^{(2)}=15\,T_{ss}^{(1)}\,. \tag{27}$$

The presence of the last term in (26) corresponds to the incomplete polarization degree among the elastically scattered electrons estimated as $\sim 15/16=0.938$ (compare it with the dynamic value 0.924 of the polarization degree in QED). The relation $T_{ss}^{(1)}=4\,T_{QED}$ one finds from (1) and (27), should be addressed to the lack of the direct correspondence between λ and $w^{\uparrow\downarrow}$ in (2). The classical model of spin relaxation proposed in [13, 7] gives for the polarization time T_{QED} the wanted expression up to the factor of order unity (not four) and for the polarization degree 100%. So, in what concerns classical consideration, the 'nice' "4" and $0<T_{ss}^{(2)}<\infty$ (see (27)) are the main variation of our results from those of [13, 7].

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On Scale Invariant Generalization of $\mathcal{N} = 3$ Born-Infeld Action

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Abstract

The conventional Born-Infeld (BI) action can be represented as a series over even powers of Maxwell strength field F_{mn} starting with F^2, F^4, F^6 terms. There has been recently found a $\mathcal{N} = 3$ supersymmetric BI-action in the harmonic superspace approach reproducing the same terms in its component expansion. This action contains a dimensional parameter and is not scale invariant. We show how to put this action into a scale invariant form without any dimensional parameters.

Originally, the Born-Infeld action was invented as a non-linear generalization of classical Maxwell electrodynamics [1]. It can be written in terms of standard Maxwell strength filed $F_{mn} = \partial_m A_n - \partial_n A_m$ as

$$S_{BI} = X^2 \int d^4x \left[\sqrt{-\det(\eta_{mn} + F_{mn}/X)} - 1\right]. \tag{1}$$

Here X is a constant of mass dimension 2, η_{mn} is the Minkowski space metrics. Despite it has no direct physical application, this model was intensively studied due to its very beautiful properties: the action (1) respects the electro-magnetic duality $F_{mn} \leftrightarrow \tilde{F}_{mn}$, it describes the physical propagation of Electro-Magnetic waves and it has soliton-like solutions due to its non-linearity. For some descriptions of these properties see review papers [2]. At present the interest to this action is renewed by the modern achievements in superfield/superstring theories. For example, the Born-Infeld action is known to describe the dynamics of D-branes in string theory, its supersymmetric generalizations are important for the study if non-linear realizations of supersymmetry, its superconformaly invariant generalizations are of relevance with the problem of the low-energy effective action in $\mathcal{N} = 2, 4$ superconformal field theories. Some of these questions are reviewed in [2].

Before proceeding further, we point out another useful form of the action (1)

$$\begin{aligned} S_{BI} &= \int d^4x \left[-\frac{1}{2}(F^2 + \bar{F}^2) + \frac{1}{2}\frac{F^2\bar{F}^2}{X^2} - \frac{1}{4}\frac{F^2\bar{F}^2(F^2+\bar{F}^2)}{X^4}\right. \\ &\quad \left. + \frac{1}{8}\frac{F^2\bar{F}^2(3F^2\bar{F}^2 + F^4 + \bar{F}^4)}{X^6} + \ldots\right]. \end{aligned} \tag{2}$$

Here $F^2 = F_{\alpha\beta}F^{\alpha\beta}$, $\bar{F}^2 = \bar{F}_{\dot\alpha\dot\beta}\bar{F}^{\dot\alpha\dot\beta}$ and $F_{\alpha\beta}$, $\bar{F}_{\dot\alpha\dot\beta}$ are spinorial components of the strength $F_{\mu\nu}$, $F_{\alpha\beta} = \frac{i}{4}(\sigma_{mn})_{\alpha\beta}F^{mn}$, $\bar{F}_{\dot\alpha\dot\beta} = -\frac{i}{4}(\tilde\sigma_{mn})_{\dot\alpha\dot\beta}F^{mn}$. The matrices σ_{mn}, $\tilde\sigma_{mn}$ are standard antisymmetrized products of usual sigma-matrices, see [3] for our conventions.

The action (2) appears in the decomposition of the square root of det in (1) into a power series over strength fields. A supersymmetric generalization of the action (2) means that there exists some superfield action which has the expression (2) in the bosonic sector of the theory. For example, $\mathcal{N}=1$ supersymmetric BI action was constructed in [4]. The corresponding $\mathcal{N}=2$ supersymmetric generalization was given in [5, 6]. The $\mathcal{N}=3$ supersymmetric formulation of BI action was found in [7] within the harmonic superspace approach. Let us dwelt on some details of the work [7] since we will follow the similar steps but the aim of our work is to obtain a scale-invariant $\mathcal{N}=3$ supersymmetric BI action. This problem is probably of some relevance to the study of low-energy effective action in $\mathcal{N}=3$ SYM model.

The $\mathcal{N}=3$ harmonic superspace was introduced in the works [8], some details of this approach are given in the book [9]. By definition, the $\mathcal{N}=3$ HSS is defined as a superspace with coordinates $\{Z, u\}$, where $Z = \{x^{\alpha\dot\alpha}, \theta^\alpha_i, \bar\theta^{i\dot\alpha}\}$ [1] is a set of standard $\mathcal{N}=3$ coordinates and u are the harmonics parameterizing the coset SU(3)/U(1)×U(1). We consider the harmonics u^I_i and their conjugate $\bar{u}^i_I$ $(I = 1, 2, 3)$ as SU(3) matrices

$$u^I_i\bar{u}^i_J = \delta^I_J, \qquad u^I_i\bar{u}^j_I = \delta^j_i, \qquad \varepsilon^{ijk}u^1_i u^2_j u^3_k = 1. \tag{3}$$

The harmonic superspace $\{Z, u\}$ contains the so called analytic subspace with the coordinates $\{\zeta_A, u\} = \{x^{\alpha\dot\alpha}_A, \theta^\alpha_2, \theta^\alpha_3, \bar\theta^{1\dot\alpha}, \bar\theta^{2\dot\alpha}, u\}$ where

$$x^{\alpha\dot\alpha}_A = x^{\alpha\dot\alpha} - 2i(\theta^\alpha_1\bar\theta^{1\dot\alpha} - \theta^\alpha_3\bar\theta^{3\dot\alpha})\,, \qquad \theta^\alpha_I = \theta^\alpha_i\bar{u}^i_I\,, \qquad \bar\theta^{I\dot\alpha} = \bar\theta^{\dot\alpha i}u^I_i\,. \tag{4}$$

The analytic superspace plays an important role in harmonic superspace approach since it is closed under supersymmetry and all $\mathcal{N}=3$ actions can be written in the analytic coordinates.

The harmonic superspace is equipped with Grassmann covariant derivatives D^I_α, $\bar{D}_{I\dot\alpha}$ and harmonic covariant ones D^I_J which form the su(3) algebra (see [9] for details). The classical free action of $\mathcal{N}=3$ SYM model [8]

$$S_2[V] = -\frac{1}{4}\int d\zeta(^{33}_{11})du\,[V^2_3 D^1_3 V^1_2 + \frac{1}{2}(D^1_2 V^2_3 - D^2_3 V^1_2)^2] \tag{5}$$

is formulated in terms of analytic superfields V^1_2, V^2_3 which are nothing but the connections covariantizing the harmonic derivatives: $D^1_2 \to \nabla^1_2 = D^1_2 + V^1_2$, $D^2_3 \to \nabla^2_3 = D^2_3 + V^2_3$. The integration in (5) is performed over $\mathcal{N}=3$ analytical subspace with the measure $d\zeta(^{33}_{11})du$. Physical bosonic component fields are contained in the prepotentials as [8]

$$\begin{aligned} V^2_3 = {} & [(\bar\theta^1\bar\theta^2)u^2_k - (\bar\theta^2)^2 u^1_k]\phi^k + \theta^\alpha_3\bar\theta^{2\dot\alpha}A_{\alpha\dot\alpha} - i\theta^\alpha_2\theta^\beta_3(\bar\theta^2)^2 H_{\alpha\beta} \\ & + \text{spinors and auxiliary fields}\,, \\ V^1_2 = {} & -\widetilde{(V^2_3)} = -[(\theta_2\theta_3)\bar{u}^k_2 - (\theta_2)^2\bar{u}^k_3]\bar\phi_k + \theta^\alpha_2\bar\theta^{1\dot\alpha}A_{\alpha\dot\alpha} + i(\theta_2)^2\bar\theta^{1\dot\alpha}\bar\theta^{2\dot\beta}\bar{H}_{\dot\alpha\dot\beta} \\ & + \text{spinors and auxiliary fields}\,. \end{aligned} \tag{6}$$

Here ϕ^i, $\bar\phi_i$ are complex scalar fields, $A_{\alpha\dot\alpha}$ is a vector gauge field, $H_{\alpha\beta}$, $\bar{H}_{\dot\alpha\dot\beta}$ are the auxiliary fields which ensure the correct structure of the gauge field sector of the theory [7].

[1] We denote by small Greek symbols the SL(2,C) spinor indices, $\alpha, \dot\alpha, \ldots = 1, 2$; the small Latin letters are SU(3) indices, $i, j, \ldots = 1, 2, 3$.

Substituting eq. (6) into classical action (5) one can derive the component form of the action S_2 in the SU(3) singlet gauge field sector [7]

$$S_2 = \int d^4x[V^2 + \bar{V}^2 - 2(\bar{V}\bar{F} + VF) + \frac{1}{2}(F^2 + \bar{F}^2)], \tag{7}$$

where

$$V_{\alpha\beta} = \frac{1}{4}(H_{\alpha\beta} + F_{\alpha\beta}), \qquad \bar{V}_{\dot\alpha\dot\beta} = \frac{1}{4}(\bar{H}_{\dot\alpha\dot\beta} + \bar{F}_{\dot\alpha\dot\beta}). \tag{8}$$

The auxiliary fields $V_{\alpha\beta}$, $\bar{V}_{\dot\alpha\dot\beta}$ can be eliminated by their algebraic classical equations of motion $V_{\alpha\beta} = F_{\alpha\beta}$, $\bar{V}_{\dot\alpha\dot\beta} = \bar{F}_{\dot\alpha\dot\beta}$. As a result, the free classical action (7) takes the form of the usual Maxwell action

$$S_2 = -\frac{1}{2}\int d^4x(F^2 + \bar{F}^2)\,. \tag{9}$$

The Maxwell action (9) corresponds to the second-order term in the BI action (2).

In order to describe the higher terms in the BI action (2) in a $\mathcal{N} = 3$ supersymmetric way we need the $\mathcal{N} = 3$ superfield strengths which are defined as [10]

$$\begin{array}{ll} W_{23} = \frac{1}{4}\bar{D}_{3\dot\alpha}\bar{D}^{\dot\alpha}_3 V^3_2\,, & \bar{W}^{12} = -\frac{1}{4}D^{1\alpha}D^1_\alpha V^2_1\,, \\ W_{12} = D^3_1 W_{23}, & \bar{W}^{23} = -D^3_1\bar{W}^{12}\,, \\ W_{13} = -D^2_1 W_{23}, & \bar{W}^{13} = D^3_2\bar{W}^{12}\,. \end{array} \tag{10}$$

Here V^2_1, V^3_2 are non-analytic prepotentials which are the solutions of zero-curvature equations [7], $D^2_1 V^1_2 = D^1_2 V^2_1$, $D^3_2 V^2_3 = D^2_3 V^3_2$. The superfields (10) have the following component structure in the sector of physical bosons [10]

$$\begin{array}{rl} W_{23} = & u^1_i\phi^i(x_{A+}) + 4i\theta^\alpha_2\theta^\beta_3 V_{\alpha\beta}(x_{A+}) + \text{spinors and auxiliary fields}\,, \\ \bar{W}^{12} = & \bar{u}^i_3\bar{\phi}_i(x_{A-}) + 4i\bar{\theta}^{1\dot\alpha}\bar{\theta}^{2\dot\beta}\bar{V}_{\dot\alpha\dot\beta}(x_{A-}) + \text{spinors and auxiliary fields} \end{array} \tag{11}$$

where $x^{\alpha\dot\alpha}_{A\pm} = x^{\alpha\dot\alpha}_A \pm 2i\theta^\alpha_2\bar{\theta}^{2\dot\alpha}$.

The $\mathcal{N} = 3$ supersymmetric Born-Infeld action [7] can be represented in the following form:

$$\begin{array}{rcl} S^{\mathcal{N}=3}_{BI} & = & S_2 + S_E, \\ S_E & = & \dfrac{1}{32X^2}\displaystyle\int d\zeta\binom{33}{11}du\,(\bar{W}^{12}W_{23})^2\hat{E}(A/X^4), \end{array} \tag{12}$$

where

$$A = \frac{1}{2^{10}}(D^1)^2(\bar{D}_3)^2[D^{2\alpha}W_{12}D^2_\alpha W_{12}\bar{D}_{2\dot\alpha}\bar{W}^{23}\bar{D}^{\dot\alpha}_2\bar{W}^{23}]. \tag{13}$$

The superfield function $\hat{E}(A)$ is defined as a solution of the equation [7]

$$\hat{E}(a) = \frac{4}{a}[2t^2(a) + 3t(a) + 1], \qquad t^4 + t^3 - \frac{1}{4}a = 0, \qquad t|_{a=0} = -1. \tag{14}$$

The series expansion for the function $\hat{E}(a)$ starts with

$$\hat{E}(a) = 1 - \frac{a}{4} + \frac{3a^2}{16} + \ldots, \tag{15}$$

where the 1-st term corresponds to the 4-th order interaction S_4

$$S_4 = \frac{1}{32}\int d\zeta(^{33}_{11})du\,\frac{(\bar{W}^{12}W_{23})^2}{X^2}\,. \tag{16}$$

The action (16) produces the first nontrivial term of the BI interaction (2) (after elimination of auxiliary fields)

$$\frac{1}{2}\int d^4x\frac{F^2\bar{F}^2}{X^2}. \tag{17}$$

The action (12) clearly violates the scale invariance since it depends on the dimensional parameter X. As soon as we are going to construct a scale invariant generalization of the action (12), we need another action which has no any dimensional constants but reproducing the action (2) in the electromagnetic sector. The scale invariance of the action (2) can be achieved by replacing the constant X with the combination of scalar fields

$$\int d^4x\frac{F^{2n}\bar{F}^{2n}}{X^{4n-2}} \quad\longrightarrow\quad \int d^4x\frac{F^{2n}\bar{F}^{2n}}{(\phi^i\bar{\phi}_i)^{4n-2}}, \quad n=1,\dots,\infty. \tag{18}$$

The scalar fields ϕ^i, $\bar{\phi}_i$ are the lowest components of the strength superfields (11). Note that similar dependence on the scalar fields was pointed out in [11] for the 4-th order term in the low-energy effective action of $\mathcal{N}=4$ SYM model.

The main idea of our further considerations is to replace the constant X in the $\mathcal{N}=3$ BI action (12) by some combination of superfields with the same dimension. Moreover, in components, such action should reproduce the terms (18). The suitable superfield expression is

$$\bar{W}^{IJ}W_{IJ} = \bar{W}^{12}W_{12} + \bar{W}^{23}W_{23} + \bar{W}^{13}W_{13}\,. \tag{19}$$

Indeed, the component expansion of this superfield starts with the scalars (see [10] for details)

$$\bar{W}^{IJ}W_{IJ}|_{\theta=\bar{\theta}=0} = \phi^i\bar{\phi}_i\,. \tag{20}$$

However, the expression (19) cannot be naively inserted into the integral in (16) in place of the constant X. The point is that the superfield $(\bar{W}^{IJ}W_{IJ})$ is not analytic since the superfield strengths $\bar{W}^{23}, \bar{W}^{13}, W_{12}, W_{13}$ are not analytic, while the integration in (16) goes over the analytic superspace. Therefore we have to rewrite the action (16) in full $\mathcal{N}=3$ HSS and then to insert $\bar{W}^{IJ}W_{IJ}$ into the integral.

The action (16) in the full $\mathcal{N}=3$ HSS is written as

$$S_4 = \frac{1}{32}\int d^4x d^{12}\theta du\frac{1}{X^2}\left[\frac{(\bar{D}_1)^2}{4\Box}(W_{23})^2\right]\left[\frac{(D^3)^2}{4\Box}(\bar{W}^{12})^2\right]. \tag{21}$$

To check that the actions (16) and (21) are actually identical to each other, one should express the integration measure of the full $\mathcal{N}=3$ superspace through the analytic one $d^4xd^{12}\theta = d\zeta(^{33}_{11})\,\frac{1}{4}(D^1)^2\,\frac{1}{4}(\bar{D}_3)^2$, and then apply the anticommutation relations between spinor derivatives. Replacing the constant X by the superfield $\bar{W}^{IJ}W_{IJ}$ in (21), we arrive at the action

$$S_4^{scale-inv} = \alpha\int d^4xd^{12}\theta du\frac{1}{(\bar{W}^{IJ}W_{IJ})^2}\left[\frac{(\bar{D}_1)^2}{4\Box}(W_{23})^2\right]\left[\frac{(D^3)^2}{4\Box}(\bar{W}^{12})^2\right], \tag{22}$$

where α is some dimensionless constant which will be specified further. Since the action (22) includes no any dimensional parameters, it is scale invariant. It can be checked to be γ_5-invariant as well.

What about the higher terms of BI action, they can be described by the action

$$S_G^{scale-inv} = \int d^4x d^{12}\theta du \frac{(D^2)^2(\bar{D}_2)^2(\bar{W}^{IJ}W_{IJ})^2}{(\bar{W}^{IJ}W_{IJ})^4} G\left(\frac{A}{(\bar{W}^{IJ}W_{IJ})^4}\right), \qquad (23)$$

where A is defined in eq. (13) and G is some function which can be represented as a series

$$G(a) = \sum_{n=0}^{\infty} \beta_n a^n \qquad (24)$$

with some coefficients β_n. The action (23) is a scale invariant generalization of eq. (12) in the sense that it reproduces in components all terms in the scale invariant BI action, starting from F^8/ϕ^{12}, with definite coefficients which can be fixed by choosing β_n in the appropriate way. Therefore, this action, taken in a sum with the quadratic S_2 (2) and quartic $S_4^{scale-inv}$ (22) actions, can generate the scale invariant BI action in the bosonic sector

$$S_{BI}^{scale-inv} = S_2 + S_4^{scale-inv} + S_G^{scale-inv} \qquad (25)$$

But the coefficients α, β_n in (22,23) remain unspecified so far. Obviously, they should be fixed by the requirement that the action (25), after passing to the component form, must reproduce the action (2) with the corresponding coefficients at each order. For this purpose we substitute the prepotentials and the superfield strengths in the forms (6) and (11) (but without the spinors) into the action (25). As a result, after the integration over Grassmann and harmonic variables and careful elimination of auxiliary fields with the help of its effective equations of motion, we find

$$\begin{aligned} S_{BI}^{scale-inv} &= \int d^4x \left[-\frac{1}{2}(F^2+\bar{F}^2) + \frac{1}{2}\frac{F^2\bar{F}^2}{(\phi^i\bar{\phi}_i)^2} - \frac{1}{4}\frac{F^2\bar{F}^2(F^2+\bar{F}^2)}{(\phi^i\bar{\phi}_i)^4} \right. \\ &\quad \left. + \frac{1}{8}\frac{F^2\bar{F}^2(3F^2\bar{F}^2+F^4+\bar{F}^4)}{(\phi^i\bar{\phi}_i)^6} + \ldots \right], \end{aligned} \qquad (26)$$

on condition that $\alpha = \frac{15}{32}$, $\beta_0 = -\frac{1}{15\cdot 2^{15}}$. Note that all higher coefficients β_n, $n > 0$, can also be uniquely fixed by the comparison of the component structure of the actions (25) and (2) at higher orders.

It should be noticed that the action (25) is by no means the unique superfield expression capable to reproduce the corresponding terms of the BI action in the bosonic limit. There is a freedom in distributing the Grassmann derivatives among different factors in eq. (23). As we suppose, this freedom can be compensated by the proper choice of function $G(a)$. Here we consider just an example of such an action which is most convenient for studying the component structure. Note that a similar situation was observed in the construction of $\mathcal{N}=2$ generalization of BI action [5, 6], when different superfield structures reproduce the identical component expressions in the bosonic sector. However, it was pointed out in [6] that this ambiguity can be resolved by applying the selfduality condition of a supersymmetrized BI action. Therefore it would be very interesting to study the selfduality of $\mathcal{N}=3$ SYM and BI models in order to clarify further the question of unique construction of higher order terms in the BI action and possibly in the low-energy effective action in SYM model.

To summarize, in this work we demonstrate that the $\mathcal{N} = 3$ BI action, proposed before in the work [7], can be brought to a scale invariant form by replacing the dimensional constant X with the combination of superfield strength (19). The resulting action is written in the form (25). This action reproduces in components all terms in BI action (2) in the bosonic sector. These results were particularly published in [3].

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$6D$ Superconformal Theory as the Theory of Everything

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Abstract

We argue that the fundamental Theory of Everything is a conventional field theory defined in the *flat* multidimensional bulk. Our Universe should be obtained as a 3-brane classical solution in this theory. The renormalizability of the fundamental theory implies that it involves higher derivatives (HD). It should be supersymmetric (otherwise one cannot get rid of the huge induced cosmological term) and probably conformal (otherwise one can hardly cope with the problem of ghosts) . We present arguments that in conformal HD theories the ghosts (which are inherent for HD theories) might be not so malignant. In particular, we present a nontrivial QM HD model where ghosts are *absent* and the spectrum has a well defined ground state.

The requirement of superconformal invariance restricts the dimension of the bulk to be $D \leq 6$. We suggest that the TOE lives in six dimensions and enjoys the maximum $\mathcal{N} = (2, 0)$ superconformal symmetry. Unfortunately, no renormalizable field theory with this symmetry is presently known. We construct and discuss an $\mathcal{N} = (1, 0)$ $6D$ supersymmetric gauge theory with four derivatives in the action. This theory involves a dimensionless coupling constant and is renormalizable. At the tree level, the theory enjoys conformal symmetry, but the latter is broken by quantum anomaly. The sign of the β function corresponds to the Landau zero situation.

1 Motivation

Arguably, the most burning unresolved problem of modern theoretical physics is the absense of a satisfactory quantum theory of gravity. The *main* obstacle here is the geometric nature of gravity. Time is intertwined there with spatial coordinates and the notion of universal flat time is absent. As a result, in constrast to conventional field theory, one cannot write the (functional) Schrödinger equation, define the Hilbert space with unitary evolution operator, etc.

As a matter of fact, Einstein gravity (and any other theory where the metric is considered as a fundamental dynamic variable) has problems also at the classical level. The

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equations of motion cannot be *always* formulated as Cauchy problem. This leads to breaking of causality for some exotic configurations like Gödel universes or wormholes [1].

Even though these configurations are not realized in our world at the macroscopic level, their existence presents conceptual difficulties.

The modern paradigm is that the fundamental Theory of Everything is a form of string theory. If this is true, gravity has the status of effective theory and one is not allowed to blame it for inconsistencies. But string theory also does not provide a satisfactory answer to all these troubling questions. Actually, they cannot even be posed there: we understand more or less well what string theory is only at the perturbative level (and even there we are not sure yet whether technical difficulties preventing one now to perform calculation of string amplitudes beyond two loops can be efficiently resolved), while its non-perturbative formulation is simply absent.

This has led us to suggest [2] that the TOE is a *field theory* living in *flat* higher-dimensional space. This higher-dimensional theory should involve 3-brane classical solutions, which might be associated with our Universe in the spirit of [3]. The gravity is induced there as an effective theory living on the brane. One can imagine a thin soap bubble. Its effective hamiltonian is

$$H^{\mathrm{eff}} \;=\; \sigma \int \sqrt{g}\, d^2x \;, \tag{1.1}$$

where σ is the surface tension. The hamiltonian (1.1) is geometric, but the fundamental theory of soap is not: it is formulated in flat $3D$ space and does not know anything about the metric, etc. Of course, the analogy is not exact because the effective hamiltonian (1.1) does not have an Einstein form but looks rather as a cosmological term. The Einstein term and also the terms involving higher powers of curvature appear as corrections, however. In the observable world, the cosmological term is either zero or very small and one should think of a mechanism to get rid of it. One could succeed in that (if any) only if the fundamental theory is supersymmetric. Indeed, only supersymmetry can provide for the exact calcellation of quantum corrections to the energy density of the brane solution.

If we want the fundamental higher-dimensional theory to be renormalizable, the canonical dimension of the lagrangian should be greater than 4, i.e. it should involve higher derivatives. HD theories are known to have a problem of ghosts, which in many cases break unitarity and/or causality of the theory [4] [1]. However, a model study performed in Refs. [2, 5] indicates that in some cases, namely, when the theory enjoys *exact* conformal invariance, the ghosts are not so malignant, a well defined ground state (the vacuum) might exist and the theory might enjoy a unitary S-matrix.

We conclude that the TOE should be superconformal theory. This restricts the number of dimensions D in the flat space-time where the theory is formulated by $D \leq 6$. Indeed, all superconformal algebras involving the super-Poincare algebra as a subalgebra are classified [6]. Their highest possible dimension is six, which allows for the minimal conformal superagebra (1,0) and the extended chiral conformal superalgebra (2,0).

Our hypothesis is that the TOE lives in six dimensions and enjoys the highest possible supersymmetry (2,0).

Unfortunately, no field theory with this symmetry group is actually known now. The corresponding lagrangian is not constructed, and only indirect results concerning scaling

[1]Physically, a ghost–ridden theory is simply a theory where the spectrum has no bottom and one cannot define what vacuum is.

behavior of certain operators have been obtained so far [7]. In [8], we derived (using the formalism of harmonic superspace (HSS) [9]) the lagrangian for the $6D$ gauge theory with unextended $(1,0)$ superconformal symmetry. This theory is conformal at the classical level and renormalizable. However, it is not finite: the β function does not vanish there and conformal symmetry is broken at the quantum level by anomaly. In other words, this theory cannot be regarded as a viable candidate for the TOE. Its study represents, however, a necessary step before the problem of constructing and studying the $(2,0)$ theory could be tackled.

In the next section, we explain in more details *what* are the ghosts, why (if not dealt with) they make the theory sick, and also present a special QM HD model where the ghosts *are* tamed. In sect. 3 we derive the lagrangian of our superconformal $6D$ theory and calculate its beta function. The last section is devoted, as usual, to conclusions and speculations.

2 Ghost-free QM higher derivative model.

To understand the nature of ghosts, one does not need to study field theories. It is clearly seen in toy models with finite number of degrees of freedom. Consider e.g. the lagrangian

$$L \;=\; \frac{1}{2}\ddot{q}^2 - \frac{\Omega^4}{2}q^2 \;. \tag{2.1}$$

It is straightforward to see that four independent solutions to the corresponding classical equations of motion are $q_{1,2}(t) = e^{\pm i\Omega t}$, $q_3(t) = e^{-\Omega t}$ and $q_4(t) = e^{\Omega t}$. The exponentially rising solution $q_4(t)$ displays instability of the classical vacuum $q = 0$. The quantum hamiltonian of such a system is not hermitian and the evolution operator $e^{-i\hat{H}t}$ is not unitary.

This vacuum instability is characteristic for all *massive* HD field theories — the dispersive equation has complex solutions in this case for small enough momenta. But for intrinsically massless (conformal) field theories the situation is different. Consider the lagrangian

$$L \;=\; \frac{1}{2}(\ddot{q} + \Omega^2 q)^2 - \frac{\alpha}{4}q^4 - \frac{\beta}{2}q^2\dot{q}^2 \;. \tag{2.2}$$

Its quadratic part can be obtained from the HD field theory lagrangian $\mathcal{L} = (1/2)\phi\Box^2\phi$ involving massless scalar field, when restricting it on the modes with a definite momentum $\vec{k}$ $(\Omega^2 = \vec{k}^2)$. If neglecting the nonlinear terms in (2.2), the solutions of the classical equations of motion $q(t) \sim e^{\pm i\Omega t}$ and $q(t) \sim te^{\pm i\Omega t}$ do not involve exponential instability, but include only comparatively "benign" oscillatory solutions with linearly rising amplitude.

We showed in [2] that, when nonlinear terms in Eq.(2.2) are included, an island of stability in the neighbourhood of the classical vacuum [2]

$$q = \dot{q} = \ddot{q} = q^{(3)} = 0 \tag{2.3}$$

[2]Usually, the term classical vacuum is reserved for the point in the configuration (or phase) space with minimal energy. For HD theories and in particular for the theory (2.2) the classical energy functional is not bounded from below and by "classical vacuum" we simply mean a stationary solution to the classical equations of motion.

exists in a certain range of the parameters α, β. In other words, when initial conditions are chosen at the vicinity of this point, the classical trajectories $q(t)$ do not grow, but display a decent oscillatory behaviour. This island is surrounded by the sea of instability, however. For generic initial conditions, the trajectories become singular: $q(t)$ and its derivatives reach infinity in a finite time.

Such a singular behaviour of classical trajectories often means trouble also in the quantum case. A well-known example when it does is the problem of $3D$ motion in the potential

$$V(r) \;=\; -\frac{\gamma}{r^2} \; . \tag{2.4}$$

The classical trajectories where the particle falls to the centre (reaches the singularity $r = 0$ in a finite time) are abundant. This occurs when $l > \sqrt{2m\gamma}$, where l is the classical angular momentum. And it is also well known that, if $m\gamma > 1/4$, the quantum problem is not very well defined: the eigenstates with arbitrary negative energies exist and the hamiltonian does not have a ground state.

The bottomlessness of the quantum hamiltonian is not, however, a necessary corollary of the fact that the classical problem involves singular trajectories. In the problem (2.4), the latter are present for all positive γ, but the quantum ground state disappears only when γ exceeds the boundary value $1/(4m)$.

Our main observation here is that the system (2.2) exhibits a similar behaviour. If both α and β are nonnegative (and at least one of them is nonzero), the quantum hamiltonian has a bottom and the quantum problem is perfectly well defined even though some classical trajectories are singular.

2.1 Free theory

Before analyzing the full nonlinear system (2.2), let us study the dynamics of the truncated system with the lagrangian $L = (\ddot{q} + \Omega^2 q)^2/2$. As was observed in [10], this system displays a singular behavior. It is instructive to consider first the lagrangian

$$L \;=\; \frac{1}{2}\left[\ddot{q}^2 - (\Omega_1^2 + \Omega_2^2)\dot{q}^2 + \Omega_1^2\Omega_2^2 q^2\right] \tag{2.5}$$

and look what happens in the limit $\Omega_1 \to \Omega_2$. When $\Omega_1 > \Omega_2$, the spectrum of the theory (2.5) is

$$E_{nm} = \left(n + \frac{1}{2}\right)\Omega_1 - \left(m + \frac{1}{2}\right)\Omega_2 \tag{2.6}$$

with nonnegative integer n, m. On the other hand, when $\Omega_1 = \Omega_2 = \Omega$, the spectrum is

$$E_n = n\Omega \tag{2.7}$$

with generic integer n. In both cases, the quantum hamiltonian has no ground state, but in the limit of equal frequencies the number of degrees of freedom is apparently reduced in a remarkable way: instead of two quantum numbers n, m (the presence of two quantum numbers is natural — the phase space of the system (2.5) is 4–dimensional having two pairs $(p_{1,2}, q_{1,2})$ of canonic variables), we are left with only one quantum number n.

This deficiency of the number of eigenstates compared to natural expectations would not surprise a mathematician. A generic 2×2 matrix has two different eigenvectors. But the Jordan cell $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only *one* eigenvector $\propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The statement is therefore that in the limit $\Omega_1 = \Omega_2$ our hamiltonian represents a kind of generalized Jordan cell.

Actually, the "lost" degrees of freedom reinstall themselves when taking into account nontrivial *time dynamics* of the degenerate system (2.5) with $\Omega_1 = \Omega_2$. The situation is rather similar to what has been unravelled back in the sixties when studying degenerate systems displaying "nonexponential decay" behavior (see e.g. [11]). We will not discuss here nonstationary problem and concentrate on the *spectrum* of this system.

To begin with, let us construct the canonical hamiltonian corresponding to the lagrangian (2.5). This can be done using the general Ostrogradsky formalism [12] [3]. For a lagrangian like (2.5) involving $q, \dot{q}$, and $\ddot{q}$, it consists in introducing the new variable $x = \dot{q}$ and writing the hamiltonian $H(q, x; p_q, p_x)$ in such a way that the classical Hamilton equations of motion would coincide after excluding the variables x, p_x, p_q with the equations of motion

$$q^{(4)} + (\Omega_1^2 + \Omega_2^2)\ddot{q} + \Omega_1^2\Omega_2^2 q \;=\; 0 \tag{2.8}$$

derived from the lagrangian (2.5). This hamiltonian has the following form

$$H \;=\; p_q x + \frac{p_x^2}{2} + \frac{(\Omega_1^2 + \Omega_2^2)x^2}{2} - \frac{\Omega_1^2\Omega_2^2 q^2}{2} \; . \tag{2.9}$$

For example, the equation $\partial H/\partial p_q = \dot{q}$ gives the constraint $x = \dot{q}$, etc.

When $\Omega_1 \neq \Omega_2$, the quadratic hamiltonian (2.9) can be diagonalized by a certain canonical transformation $x, q, p_x, p_q \;\to a_{1,2}, a_{1,2}^*$ [10, 5]. We obtain

$$H \;=\; \Omega_1 a_1^* a_1 - \Omega_2 a_2^* a_2 \; . \tag{2.10}$$

The classical dynamics of this hamiltonian is simply $a_1 \propto e^{-i\Omega_1 t}$, $a_2 \propto e^{i\Omega_2 t}$. Its quantization gives the spectrum (2.6). The negative sign of the second term in (2.10) implies the negative sign of the corresponding kinetic term, which is usually interpreted as the presence of the ghost states (the states with negative norm) in the spectrum. We prefer to keep the norm positive definite, with the creation and annihilation operators $a_{1,2}, a_{1,2}^\dagger$ (that correspond to the classical variables $a_{1,2}, a_{1,2}^*$) satisfying the usual commutation relations $[a_1, a_1^\dagger] \;=\; [a_2, a_2^\dagger] \;=\; 1$. However, irrespectively of whether the metric is kept positive definite or not and the world "ghost" is used or not, the spectrum (2.6) does not have a ground state and, though the spectral problem for the free hamiltonian (2.10) is perfectly well defined, the absence of the ground state leads to a trouble, the falling to the centre phenomenon when switching on the interactions. [4]

We are interested, however, not in the system (2.5) as such, but rather in this system in the limit $\Omega_1 = \Omega_2$. As was mentioned, this limit is singular. The best way to see what happens is to write down the explicit expressions for the wave functions of the states (2.6)

[3] See e.g. [13] for its detailed pedagogical description.

[4] A characteristic feature of this phenomenon is that some classical trajectories reach singularity in a finite time while the quantum spectrum involves a *continuum* of states with arbitrary low energies [14]. In our case, the "centre" is not a particular point in the configuration (phase) space but rather its boundary at infinity, but the physics is basically the same.

and explore their behaviour in the equal frequency limit. This can be done by substituting the operators $-i\partial/\partial x$, $-i\partial/\partial q$ for p_x and p_q in Eq. (2.9) and searching for the solutions of the Schrödinger equation in the form

$$\Psi(q,x) \;=\; e^{-i\Omega_1\Omega_2 qx} \exp\left\{-\frac{\Delta}{2}\left(x^2+\Omega_1\Omega_2 q^2\right)\right\}\phi(q,x)\ , \tag{2.11}$$

where $\Delta = \Omega_1 - \Omega_2$. Then the operator acting on $\phi(q,x)$ is

$$\tilde{H} \;=\; -\frac{1}{2}\frac{\partial^2}{\partial x^2} + (\Delta x + i\Omega_1\Omega_2 q)\frac{\partial}{\partial x} - ix\frac{\partial}{\partial q} + \frac{\Delta}{2}\ . \tag{2.12}$$

It is convenient to introduce

$$z = \Omega_1 q + ix\ , \qquad u = \Omega_2 q - ix\ , \tag{2.13}$$

after which the operator (2.12) acquires the form

$$\tilde{H}(z,u) \;=\; \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial u}\right)^2 + \Omega_1 u\frac{\partial}{\partial u} - \Omega_2 z\frac{\partial}{\partial z} + \frac{\Delta}{2}\ . \tag{2.14}$$

The holomorphicity of $\tilde{H}(z,u)$ means that its eigenstates are holomorphic functions $\phi(z,u)$. An obvious eigenfunction with the eigenvalue $\Delta/2$ is $\phi(z,u) = \mathrm{const}$. Further, if assuming ϕ to be the function of only one holomorphic variable u or z, the equation $\tilde{H}\phi = E\phi$ acquires the same form as for the equation for the preexponential factor in the standard oscillator problem. Its solutions are Hermit polynomials,

$$\begin{aligned}
\phi_n(u) \;&=\; H_n(i\sqrt{\Omega_1}u) \equiv H_n^+, \qquad E_n = \frac{\Delta}{2} + n\Omega_1\ ,\\
\phi_m(z) \;&=\; H_m(\sqrt{\Omega_2}z) \equiv H_m^-, \qquad E_m = \frac{\Delta}{2} - m\Omega_2\ .
\end{aligned} \tag{2.15}$$

The solutions (2.15) correspond to excitations of only one of the oscillators while another one is in its ground state. For sure, there are also the states where both oscillators are excited. One can be directly convinced that the functions

$$\begin{aligned}
\phi_{nm}(u,z) \;&=\; \sum_{k=0}^{m}\left(\frac{i\Delta}{4\sqrt{\Omega_1\Omega_2}}\right)^k \frac{(n-m+k+1)!}{(m-k)!k!} H^+_{n-m+k}H^-_k\,, \qquad m \le n\ ,\\
\phi_{nm}(u,z) \;&=\; \sum_{k=0}^{n}\left(\frac{i\Delta}{4\sqrt{\Omega_1\Omega_2}}\right)^k \frac{(m-n+k+1)!}{(n-k)!k!} H^+_k H^-_{m-n+k}, \qquad m > n
\end{aligned} \tag{2.16}$$

are the eigenfunctions of the operator (2.14) with the eigenvalues (2.6). Multiplying the polynomials (2.16) by the exponential factors as distated by Eq.(2.11), we arrive at the normalizable wave functions of the hamiltonian (2.9).

We are ready now to see what happens in the limit $\Omega_1 \to \Omega_2$ ($\Delta \to 0$). Two important observations are in order.

- The second exponential factor in (2.11) disappears and the wave functions cease to be normalizable.

- We see that in the limit $\Delta \to 0$, only the first terms survive in the sums (2.16) and we obtain

$$\begin{aligned} \lim_{\Delta\to 0} \phi_{nm} &\sim H^+_{n-m} \quad , \quad m \le n \\ \lim_{\Delta\to 0} \phi_{nm} &\sim H^-_{m-n} \quad , \quad m > n \ . \end{aligned} \tag{2.17}$$

  In other words, the wave functions depend only on the difference $n - m$, which is the only relevant quantum number in the limit $\Omega_1 = \Omega_2$.

As this phenomenon is rather unusual and very important for us, let us spend few more words to clarify it. Suppose Ω_1 is very close to Ω_2, but still not equal. Then the spectrum includes the sets of nearly degenerate states. For example, the states $\Psi_{00}, \Psi_{11}, \Psi_{22}$, etc have the energies $\Delta/2, 3\Delta/2, 5\Delta/2$, etc, which are very close. In the limit $\Delta \to 0$, the energy of all these states coincides, but rather than having an infinite number of degenerate states, we have only one state: the wave functions $\Psi_{00}, \Psi_{11}, \Psi_{22}$, etc simply *coincide* in this limit by the same token as the eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ \Delta & 1 \end{pmatrix}$ coincide in the limit $\Delta \to 0$.

2.2 Interacting theory.

When $\Omega_1 = \Omega_2$, $u = \bar z$ and the operator (2.14) acquires the form

$$\tilde H(z, \bar z) \ = \ \frac{1}{2}\left(\frac{\partial}{\partial \bar z} - \frac{\partial}{\partial z}\right)^2 + \Omega\left(\bar z \frac{\partial}{\partial \bar z} - z\frac{\partial}{\partial z}\right) \ . \tag{2.18}$$

Its spectrum is bottomless. Let us deform (2.18) by adding there the quartic term $\alpha z^2 \bar z^2$ with positive α. Note first of all that it cannot be treated as a perturbation, however small α is: the wave functions are not normalizable and the matrix elements of $\alpha z^2 \bar z^2$ diverge. But one can use the variational approach. Let us take the Ansatz

$$|\mathrm{var}\rangle \ = \ z^n e^{-Az\bar z} \ , \tag{2.19}$$

where A, n are the variational parameters. The matrix element of the unperturbed quadratic hamiltonian (2.18) over the state (2.19) is

$$\langle \mathrm{var}|\tilde H|\mathrm{var}\rangle \ = \ \frac{A(n+1)}{2} - \Omega n \ . \tag{2.20}$$

Obviously, by choosing n large enough and A small enough, one can make it as close to $-\infty$ as one wishes. The bottom is absent and one cannot reach it. For the deformed hamiltonian, the situation is different, however. We have

$$E^{\mathrm{var}}(n, A) = \langle \mathrm{var}|\tilde H + \alpha z^2 \bar z^2|\mathrm{var}\rangle \ = \ \frac{A(n+1)}{2} - \Omega n + \frac{\alpha(n+1)(n+2)}{4A^2} \ . \tag{2.21}$$

This function has a global minimum. It is reached when

$$A - \Omega - \frac{\alpha}{4A^2} \ = \ 0 \tag{2.22}$$

and $n = A^3/\alpha - 2$.

For small $\alpha \ll \Omega^3$,

$$A \approx \Omega, \quad n \approx \frac{\Omega^3}{\alpha}, \quad \text{and} \quad E^{\text{var}} \approx -\frac{\Omega^4}{4\alpha} \,. \tag{2.23}$$

The smaller is α, the lower is the variational estimate for the ground state energy and the ground state energy itself. In the limit $\alpha \to 0$, the spectrum becomes bottomless. But for a finite α, the bottom exists. Note that in the interacting system, the spectrum is completely rearranged compared to the HD oscillator studied above and there is no reason to expect the peculiar Jordan-like degeneracy anymore. The eigenstates are conventional normalized functions and the solution of the time-dependent Schrödinger equation has the standard form.

Bearing in mind that $z = \Omega q + ix = \Omega q + i\dot{q}$, the deformation $\alpha z^2 \bar{z}^2$ amounts to a particular combination of the terms $\sim q^4$, $\sim q^2\dot{q}^2$, and $\sim \dot{q}^4$ in the hamiltonian. For the theory (2.2) with generic α, β, the algebra is somewhat more complicated, but the conclusion is the same: in the case when the form $\alpha q^4/4 + \beta q^2 x^2/2$ is positive definite, the system has a ground state.

The requirement of positive definiteness of the deformation is necessary. In the opposite case, choosing the Ansatz

$$|\text{var}\rangle \;\sim\; (\Omega q + ix)^n \exp\{-Aq^2 - Bx^2\}$$

and playing with A, B, one can always make the matrix element $\langle$ *var* $|$ *deformation* $|$ *var* $\rangle$ negative, which would add to the negative contribution $-\Omega n$ in the variational energy, rather than compensate it. The bottom is absent in this case.

3 Superconformal $6D$ theory

We start with reminding some basic facts of life for spinors in $SO(5,1)$ (or rather $Spin(5,1)$). There are two different complex 4-component spinor representations, the $(1,0)$ spinors ψ^a and the $(0,1)$ spinors ξ_a. In the familiar $Spin(3,1)$ case, there are also two different spinor representations, which are transformed to each other under complex conjugation (on the other hand, complex conjugation leaves an Euclidean $4D$ spinor in the same representation). An essential distinguishing feature of $Spin(5,1)$ is that complex conjugation does *not* change the type of spinor represenation there (while it does for Euclidean $6D$ spinors, $Spin(6) \equiv SU(4)$).

Indeed, one can show that the spinor

$$\bar{\psi}^a = -C^a_{\dot{a}} \psi^{\dot{a}}, \tag{3.1}$$

is transformed in the same way as ψ^a. We defined $\psi^{\dot{a}} = (\psi^a)^*$ and introduced a symplectic charge-conjugation matrix C satisfying

$$C^a_{\dot{a}} C^{\dot{a}}_b = -\delta^a_b. \tag{3.2}$$

The operation $\bar{}$ is the covariant conjugation. A somewhat unusual property $\overline{\bar{\psi}^a} = -\psi^a$ holds.

Bearing in mind, however, that ψ^a and $\bar{\psi}^a$ belong to the same representation, it is very convenient [15] to treat them on equal footing and introduce $\psi^a_{i=1,2} = (\psi^a, \bar{\psi}^a)$. The relation

$$\bar{\psi}^a_i = \psi^{ai} = \epsilon^{ij}\psi_{aj} \tag{3.3}$$

holds.

We choose the antisymmetric representation of the 6D Weyl matrices

$$(\gamma^M)_{ab} = -(\gamma^M)_{ba} \quad \tilde{\gamma}^{ab}_M = \tfrac{1}{2}\varepsilon^{abcd}(\gamma_M)_{cd} \tag{3.4}$$

where $M = 0, 1, \ldots, 5$ and ε^{abcd} is the totally antisymmetric symbol. The basic relations for these Weyl matrices are

$$(\gamma_M)_{ac}(\tilde{\gamma}_N)^{cb} + (\gamma_N)_{ac}(\tilde{\gamma}_M)^{cb} = -2\delta^b_a\eta_{MN}, \tag{3.5}$$

$$\varepsilon_{abcd} = \tfrac{1}{2}(\gamma^M)_{ab}(\gamma_M)_{cd}, \tag{3.6}$$

where η_{MN} is the metric of the 6D Minkowski space ($\eta_{00} = -\eta_{11} = \ldots = -\eta_{55} = 1$) and $\gamma_M = \eta_{MN}\gamma^N$.

The generators of the (1,0) spinor representation are $S^{MN} = -\frac{1}{2}\sigma^{MN}$, where

$$(\sigma^{MN})^b_a = \frac{1}{2}(\tilde{\gamma}^M\gamma^N - \tilde{\gamma}^N\gamma^M)^b_a, \quad \overline{\sigma^{MN}} = \sigma^{MN}. \tag{3.7}$$

Supersymmetric field theories are most naturally formulated in the framework of superspace approach. The $6D$ superspace is more complicated than the 4-dimensional one. A simple-minded $6D$ superspace involves, besides 6 bosonic coordinates, 8 fermionic coordinates θ^a_i. However, one can effectively reduce the number of fermionic coordinates using the *harmonic superspace* approach and working with *Grassmann analytic* superfields [9]. We are not able to dwell on this in details and refer the reader to our paper [8]. Here we only present the results.

Let us remind first the form of the conventional quadratic in derivatives SYM action in 6 dimensons. It involves the $6D$ gauge field A_M, the gluino field ψ^a_i satisfying (3.3) and the triplet of auxiliary fields $\mathcal{D}_{ik}$. The action reads

$$S = \frac{1}{f^2}\int d^6x \,\mathrm{Tr}\left\{-\frac{1}{2}F^2_{MN} - \frac{1}{2}\mathcal{D}^{ik}\mathcal{D}_{ik} + i\psi^k\gamma_M\nabla_M\psi_k\right\} \ , \tag{3.8}$$

where f is the coupling constant of canonical dimension -1 and ∇_M is the covariant derivative.

If going down to four dimensions, one reproduces the action for $\mathcal{N} = 2$ $4D$ SYM theory. A_M gives the $4D$ gauge field A_μ and the adjoint scalar, ψ^a_i gives two $4D$ gluino fields while the triplet of auxiliary fields can be decomposed into the real auxiliary field D of the 4-dimensional $\mathcal{N} = 1$ vector multiplet and the complex auxiliary field F of the adjoint chiral multiplet.

The action of the HD $6D$ gauge theory was derived in [8]. The result is

$$\begin{aligned} S \;=\; -\frac{1}{g^2}\int d^6x \,\mathrm{Tr}\Big\{ & \left(\nabla^M F_{ML}\right)^2 + i\psi^j\gamma^M\nabla_M(\nabla)^2\psi_j + \frac{1}{2}\left(\nabla_M\mathcal{D}_{jk}\right)^2 \\ & + \mathcal{D}_{lk}\mathcal{D}^{kj}\mathcal{D}_j{}^{\,l} - 2i\mathcal{D}_{jk}\left(\psi^j\gamma^M\nabla_M\psi^k - \nabla_M\psi^j\gamma^M\psi^k\right) + (\psi^j\gamma_M\psi_j)^2 \\ & + \frac{1}{2}\nabla_M\psi^i\gamma^M\sigma^{NS}[F_{NS},\psi_j] - 2\nabla^M F_{MN}\,\psi^j\gamma^N\psi_j \Big\}. \end{aligned} \tag{3.9}$$

The lagrangian has the canonical dimension 6 and the coupling constant g is dimensionless.

Let us discuss this result. Note first of all that the quadratic terms in the lagrangian are obtained from (3.8) by adding the extra box operator (it enters with negative sign, this makes the kinetic terms positive definite in Minkowski space). It is immediately seen for the terms $\propto \mathcal{D}^2$ and for the fermions. This is true also for the gauge part due to the identity

$$\mathrm{Tr}\left\{(\nabla^M F_{MN})^2\right\} = -\frac{1}{2}\mathrm{Tr}\left\{F^{MN}\nabla^2 F_{MN}\right\} - 2i\mathrm{Tr}\left\{F_M{}^N F_{NS}F^{SM}\right\}. \tag{3.10}$$

The former auxiliary fields $\mathcal{D}^{ik}$ become dynamical. They carry canonical dimension 2 and their kinetic term involves two derivatives. There is a cubic term $\propto \mathcal{D}^3$. This sector of the theory reminds the renormalizable theory $(\phi^3)_6$. Gauge and fermion fields have the habitual canonical dimensions $[A_M] = 1$, $[\psi] = 3/2$. Their kinetic terms involve, correspondingly, 4 and 3 derivatives. The lagrangian involves also other interaction terms, all of them having the canonical dimension 6.

It is instructive to evaluate the number of on–shell degrees of freedom for this lagrangian. Consider first the gauge field. With the standard lagrangian $\propto \mathrm{Tr}\{F_{MN}^2\}$, a six–dimensional gauge field A_M has 4 on–shell d.o.f. for each color index. The simplest way to see this is to note that A_0 is not dynamical and we have to impose the Gauss law constraint on the remaining 5 spatial variables. For the higher-derivative theory, however, the presence of two extra derivatives doubles the number of d.o.f. and the correct counting is $2 \times 5 = 10$ before imposing the Gauss law constraint and $10 - 1 = 9$ after that. In addition, there are 3 d.o.f. of the fields D_{ij} and we have all together 12 bosonic d.o.f. for each color index. The standard $6D$ Weyl fermion (with the lagrangian involving only one derivative) has 4 on–shell degrees of freedom. In our case, we have $4 \times 3 = 12$ fermionic d.o.f. due to the presence of three derivatives in the kinetic term. Not unexpectedly, the numbers of bosonic and fermionic degrees of freedom on mass shell coincide.

3.1 Renormalization

The lagrangian (3.9) does not involve dimensional parameters and is scale–invariant. A less trivial and rather remarkable fact is that the action is also invariant with respect to special conformal transformations and the full superconformal group. This is true at the classical level, but, unfortunately, conformal invariance of this theory is broken by quantum effects. To see this, let us calculate (at the one–loop level) the β function of our theory.

The simplest way to do this calculation is to evaluate 1–loop corrections to the structures $\sim (\partial_M \mathcal{D})^2$ and $\sim \mathcal{D}^3$. The relevant Feynman graphs are depicted in Figs. 1, 2.

For perturbative calculations, we absorb the factor $1/g$ in the definition of the fields. The relevant propagators are

$$\begin{aligned}
\langle A_M^A A_N^B \rangle &= -\frac{i\eta_{MN}\delta^{AB}}{p^4}, \\
\langle \psi^{jA}\psi^{kB} \rangle &= -\frac{i\epsilon^{jk}\delta^{AB}p_N\tilde{\gamma}^N}{p^4}, \\
\langle \mathcal{D}_{ik}^A \mathcal{D}_{jl}^B \rangle &= -\frac{i\delta^{AB}}{p^2}\left(\epsilon_{ij}\epsilon_{kl} + \epsilon_{il}\epsilon_{kj}\right),
\end{aligned} \tag{3.11}$$

where A, B are color indices, $A_M = A_M^A t^A$, etc. The vertices can be read out directly from the lagrangian.

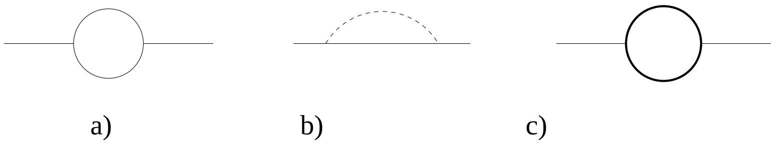

Figure 1: Graphs contributing to the renormalization of the kinetic term. Thin solid lines stand for the particle $\mathcal{D}$, thick solid lines for fermions, and dashed lines for gauge bosons.

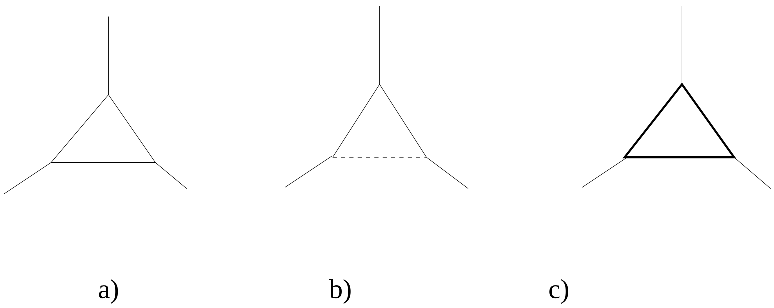

Figure 2: The same for the $\mathcal{D}^3$ vertex.

Consider first the graphs in Fig. 1. They involve logarithmic and quadratic divergences. The individual quadratically divergent contributions in the Wilsonean effective lagrangian are

$$\begin{aligned}
\Delta\mathcal{L}^{\text{eff}}_{1a} &= -\frac{9c_V}{2}\text{Tr}\,\{\mathcal{D}^2_{jk}\}\,I\,,\\
\Delta\mathcal{L}^{\text{eff}}_{1b} &= \frac{c_V}{2}\text{Tr}\,\{\mathcal{D}^2_{jk}\}\,I\,,\\
\Delta\mathcal{L}^{\text{eff}}_{1c} &= 4c_V\text{Tr}\,\{\mathcal{D}^2_{jk}\}\,I\,,
\end{aligned} \tag{3.12}$$

where c_V is the adjoint Casimir eigenvalue and

$$I = \int^\Lambda \frac{d^6p_E}{(2\pi)^6 p_E^4}\,. \tag{3.13}$$

We see that the quadratic divergences cancel out in the sum of the three graphs. The logarithmic divergences in the 2-point graphs are

$$\Delta\mathcal{L}^{\text{eff}}_{(2)} = g^2 c_V\left(-\frac{3}{2}-\frac{7}{6}+2\right)\text{Tr}\,\{(\partial_M\mathcal{D}_{jk})^2\}\,L = -\frac{2g^2c_V}{3}\text{Tr}\,\{(\partial_M\mathcal{D}_{jk})^2\}\,L\,, \tag{3.14}$$

where

$$L = \int_\mu^\Lambda \frac{d^6p_E}{(2\pi)^6 p_E^6} = \frac{1}{64\pi^3}\ln\frac{\Lambda}{\mu} \tag{3.15}$$

and three terms in the parentheses correspond to the contributions of the graphs in Fig. 1a,b,c.

The 3-point graphs in Fig. 2 involve only logarithmic divergence. We obtain

$$\Delta\mathcal{L}^{\rm eff}_{(3)} = g^3 c_V \left(-\frac{9}{2} - \frac{3}{2} + \frac{32}{3}\right) {\rm Tr}\,\{\mathcal{D}_{lk}\mathcal{D}^{kj}\mathcal{D}^l_j\}\, L = \frac{14 g^3 c_V}{3} {\rm Tr}\,\{\mathcal{D}_{lk}\mathcal{D}^{kj}\mathcal{D}^l_j\}\, L\,. \quad (3.16)$$

The full 1-loop effective lagrangian in the $\mathcal{D}$ sector is

$$\mathcal{L}^{\rm eff}_{\mathcal{D}} = -\frac{1}{2}{\rm Tr}\,\{(\partial_M \mathcal{D}_{jk})^2\}\left(1 + \frac{4g^2 c_V}{3} L\right) - g{\rm Tr}\,\{\mathcal{D}_{lk}\mathcal{D}^{kj}\mathcal{D}^l_j\}\left(1 - \frac{14 g^2 c_V}{3} L\right). \quad (3.17)$$

Absorbing the renormalization factor of the kinetic term in the field redefinition, we finally obtain

$$g(\mu) = g_0\left(1 - \frac{20 g_0^2 c_V}{3} L\right) = g_0\left(1 - \frac{5 g_0^2 c_V}{48\pi^3} \ln\frac{\Lambda}{\mu}\right) \quad (3.18)$$

for the effective charge renormalization.

The sign corresponds to the Landau zero situation, as in the conventional QED.

4 Discussion

Our study was motivated by the dream or rather by a sequence of dreams spelled out in the Introduction. By the reasons outlined there

1. We *believe* that the TOE is a conventional field theory in multidimensional bulk.

2. We *believe* that our Universe represents a thin soap bubble — a classical 3-brane solution in this theory.

3. If the theory claims to be truly fundamental, it should be renormalizable. For $D > 4$, this means the presence of higher derivatives in the action.

4. We *believe* that for superconformal theories, a way to tackle the HD ghost trouble exists.

5. We *believe* (but not so firmly, this is just the most attractive possibility) that the TOE enjoys the maximum $\mathcal{N} = 2$ superconformal symmetry in six dimensions.

Besides dreams, there are also some positive results. First, we constructed a QM HD model where the problem of ghosts *is* resolved. Second, we constructed a nontrivial example of renormalizable higher-dimensional supersymmetric gauge theory. It is $6D, \mathcal{N}{=}(1,0)$ gauge theory with four derivatives in the action and dimensionless coupling constant.

Our theory enjoys superconformal invariance at the classical level, but, unfortunately, the superconformal symmetry is anomalous in this case. As the result of this breaking, in accord with the arguments of [2], the quantum theory suffers from ghosts which can hardly be harmless.

Four-dimensional experience teaches us that though nonsupersymmetric, $\mathcal{N} = 1$, and $\mathcal{N} = 2$ supersymmetric theories are anomalous, the maximum $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is truly conformal — β function vanishes there. It is very natural therefore to *believe* that unconstructed yet Holy Grail $\mathcal{N} = (2,0)$ maximum superconformal $6D$ theory is free from anomaly.

How can it look like ? The first idea coming to mind is to ape the $4D$ construction and to couple the $6D$ gauge supermultiplet to $6D$ hypermultiplets. Adding this term to (3.9) one might hope to obtain a theory which would enjoy extended superconformal symmetry. Unfortunately, this program meets serious technical difficulties and it is not clear at the moment whether it can be carried out.

The second possibility is that the $\mathcal{N} = (2, 0)$ theory does not involve at all the gauge supermultiplet with the action (3.9), but depends on tensor rather than vector multiplets [16, 7]. Unfortunately, to describe the tensor multiplet in the framework of HSS is not a trivial task which is not solved yet. As a result, no microscopic lagrangian for interacting (2,0) tensor multiplet is known today...

Finally, one cannot exclude a disapponting possibility that the (2,0) theory does not have a lagrangian formulation whatsoever.

But the hope dies last !

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Schwinger-Dyson Equations, Ward Identities and Quantum Corrections in $N = 1$ Supersymmetric Electrodynamics, Regularized by Higher Derivatives

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Abstract

Using the Schwinger-Dyson equation and the Ward identities we calculate some contributions to the two-point Green function of the gauge superfield in N=1 supersymmetric QED exactly to all orders. The other contributions are argued to be zero, that agrees with the result of explicit calculations. Using these results we discuss the anomaly puzzle in the considered theory.

1 Introduction

The supersymmetry essentially improves the ultraviolet behavior of a theory. Thanks to that it is possible to suggest the form of the β-function exactly to all orders of the perturbation theory even in theories with unextended supersymmetry. The form of the exact β-function was proposed first in Ref. [1] from the investigation of the instanton contributions structure. For $N = 1$ supersymmetric electrodynamics, which will be considered here, such β-function (that is called the exact Novikov, Shifman, Vainshtein and Zakharov (NSVZ) β-function) is

$$\beta(\alpha) = \frac{\alpha^2}{\pi}\Big(1 - \gamma(\alpha)\Big), \tag{1}$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter superfield.

Explicit calculations, made with the dimensional reduction [2], confirm this proposal, but require a special choice of the subtraction scheme [3, 4]. Explicit calculations in two- [5, 6], three- [7] and partially four-loop [8] approximations for the $N = 1$ supersymmetric electrodynamics with the higher derivative regularization [9, 10] reveal that renormalization of the operator $W_a C^{ab} W_b$ is exhausted at the one-loop and the Gell-Mann-Low function (its definition can be found, say, in Ref. [11]) coincides with the exact NSVZ β-function and has corrections in all orders of the perturbation theory.

Here we consider the problem how to extend results of these papers to all orders of the perturbation theory [11, 12].

2 $N = 1$ supersymmetric electrodynamics and its regularization by higher derivatives

The massless $N = 1$ supersymmetric electrodynamics with the higher derivatives term in the superspace is described by the following action:

$$S = \frac{1}{4e^2}\mathrm{Re}\int d^4x\, d^2\theta\, W_a C^{ab}\Big(1+\frac{\partial^{2n}}{\Lambda^{2n}}\Big)W_b + \frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi^* e^{2V}\phi + \tilde\phi^* e^{-2V}\tilde\phi\Big). \qquad (2)$$

It is important to note, that in the Abelian case the superfield W^a is gauge invariant, so that action (2) will be also gauge invariant.

Quantization of model (2) can be made by the standard way. For this purpose it is convenient to use the supergraphs technique, described in book [13] in details, and to fix the gauge invariance by adding the following terms:

$$S_{gf} = -\frac{1}{64e^2}\int d^4x\, d^4\theta\,\Big(V D^2\bar D^2\Big(1+\frac{\partial^{2n}}{\Lambda^{2n}}\Big)V + V\bar D^2 D^2\Big(1+\frac{\partial^{2n}}{\Lambda^{2n}}\Big)V\Big). \qquad (3)$$

(In this case a part of the action, quadratic in the superfield V, will have the simplest form.) In the Abelian case, considered here, diagrams, containing ghost loops are absent.

It is well known, that adding of the higher derivative term does not remove divergences in one-loop diagrams. In order to regularize them, it is necessary to insert the Pauli-Villars determinants $\det PV(V,M_i)$ [14]. Then the generating functional can be written as

$$Z = \int DV\, D\phi\, D\tilde\phi\,\prod_i\Big(\det PV(V,M_i)\Big)^{c_i}\exp\Big(i(S_{ren}+S_{gf}+S_S)\Big), \qquad (4)$$

where S_{ren} is the renormalized action, the gauge fixing action is given by by Eq. (3) and the coefficients c_i satisfy conditions

$$\sum_i c_i = 1;\qquad \sum_i c_i M_i^2 = 0. \qquad (5)$$

Below we will assume, that $M_i = a_i\Lambda$, where a_i are some constants. Insertion of Pauli-Villars determinants allows to cancel remaining divergences in all one-loop diagrams, including diagrams, containing insertions of counterterms.

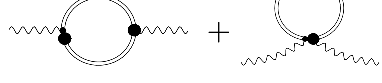
The terms with sources are written in the form

$$S_S = \int d^4x\, d^4\theta\, JV + \int d^4x\, d^2\theta\,\Big(j\,\phi + \tilde j\,\tilde\phi\Big) + \int d^4x\, d^2\bar\theta\,\Big(j^*\phi^* + \tilde j^*\tilde\phi^*\Big). \qquad (6)$$

In our notations the generating functional for the connected Green functions is written as $W = -i\ln Z$. The effective action Γ is obtained by making a Legendre transformation.

3 Schwinger-Dyson equation

From generating functional (4) it is possible to obtain [11] the Schwinger-Dyson equations, which can be graphically written as

$\Gamma^{(2)}_V =$  (7)

where $\Gamma^{(2)}_V$ is a two-point Green function of the gauge field.

The double line denotes the exact propagator

$$\left(\frac{\delta^2\Gamma}{\delta\phi^*_x\delta\phi_y}\right)^{-1} = -\frac{D^2_x\bar{D}^2_x}{4\partial^2 G}\delta^8_{xy}, \tag{8}$$

in which the function $G(q^2)$ is defined by the two-point Green function as follows:

$$\frac{\delta^2\Gamma}{\delta\phi^*_x\delta\phi_y} = \frac{D^2_x\bar{D}^2_x}{16}G(\partial^2)\delta^8_{xy}, \tag{9}$$

where $\delta^8_{xy} \equiv \delta^4(x-y)\delta^4(\theta_x-\theta_y)$, and the lower indexes denote points, in which considered expressions are taken.

The large circle denotes the effective vertex, which is written as [11]

$$\frac{\delta^3\Gamma}{\delta V_x\delta\phi_y\delta\phi^*_z}\Big|_{p=0} = \partial^2\Pi_{1/2x}\left(\bar{D}^2_x\delta^8_{xy}D^2_x\delta^8_{xz}\right)F(q^2) + \\ +\frac{1}{32}q^\mu G'(q^2)\bar{D}\gamma^\mu\gamma_5 D_x\left(\bar{D}^2_x\delta^8_{xy}D^2_x\delta^8_{xz}\right) + \frac{1}{8}\bar{D}^2_x\delta^8_{xy}D^2_x\delta^8_{xz}\,G(q^2) \tag{10}$$

due to the Ward identities. Here $\Pi_{1/2}$ is a supersymmetric transversal projector and $F(q^2)$ is a function, which can not be defined from the Ward identities.

Two adjacent circles are an effective vertex, consisting of 1PI diagrams, in which one of the external lines is attached to the very left edge. Such vertexes are given by

$$\frac{1}{4}\frac{\delta}{\delta\phi_y}\exp\left(\frac{2}{i}\frac{\delta}{\delta J_z} + 2V_z\right)\phi_z = -\frac{1}{8}G\bar{D}^2_y\delta^8_{yz} \tag{11}$$

in the case of one external V-line (the vertex in the first diagram of Eq. (7)) and

$$\frac{1}{4}\frac{\delta}{\delta V_x}\frac{\delta}{\delta\phi_y}\exp\left(\frac{2}{i}\frac{\delta}{\delta J_z} + 2V_z\right)\phi_z\Big|_{p=0} = -2\partial^2\Pi_{1/2x}\left(\bar{D}^2_x\delta^8_{xy}\delta^8_{xz}\right)F(q^2) + \frac{1}{8}D^a C_{ab}\bar{D}^2_x \times \\ \times\left(\bar{D}^2_x\delta^8_{xy}D^b_x\delta^8_{xz}\right)f(q^2) - \frac{1}{16}q^\mu G'(q^2)\bar{D}\gamma^\mu\gamma_5 D_x\left(\bar{D}^2_x\delta^8_{xy}\delta^8_{xz}\right) - \frac{1}{4}\bar{D}^2_x\delta^8_{xy}\delta^8_{xz}\,G(q^2), \tag{12}$$

in the case of two external V-lines (in the second diagram of Eq. (7)). Here $f(q^2)$ is one more function, which can not be found from the Ward identity.

Our purpose will be calculation of the expression

$$\frac{d}{d\ln\Lambda}\Gamma^{(2)}_V\Big|_{p=0}, \tag{13}$$

where p denotes the external momentum. In order to do this, the expressions for the propagators and vertexes given above are substituted into the Schwinger-Dyson equations. As a result some expressions for diagrams in Eq. (7) are obtained. The result [11] is obtained exactly to all orders of the perturbation theory:

$$\frac{d}{d\ln\Lambda}\Gamma^{(2)}_V\Big|_{p=0} = \frac{d}{d\ln\Lambda}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\,d^4\theta\,V\partial^2\Pi_{1/2}V\left(\frac{1}{2q^2}\frac{d}{dq^2}\ln(q^2G^2) - \frac{8f}{q^2G}\right) - \\ -\text{similar terms with the Pauli-Villars fields.} \tag{14}$$

(The terms with the Pauli-Villars fields are given in Ref. [11].) We see, that the second term in equation (14) contains the function f, which can not be found from the Ward identity. (In the one-loop approximation this function is equal 0. It is nontrivial only at two loops.) The terms, containing the unknown function F, are completely cancelled. However, explicit three- [7] and four-loop [8] calculations show, that the term, containing the unknown function f is equal to zero. Therefore, it is possible to propose, that the following identity take place:

$$\frac{d}{d\ln\Lambda}\int\frac{d^4q}{(2\pi)^2}\frac{f(q^2)}{q^2G(q^2)} = 0, \tag{15}$$

(Similar identity in the massive case is given in Ref. [11].) This identity was proved for a special class of diagrams in Ref. [12]. The method, proposed in it, possibly can be generalized for the other diagrams. (Actually this method is sometimes used in order to give a graphical proof of the Ward identity.) Its essence is the following:

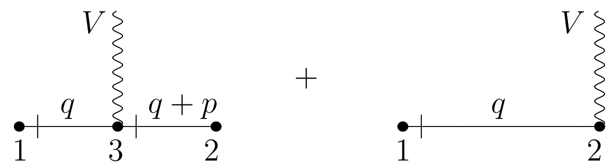


Figure 1: A typical combination of subdiagrams, which appears in the process of summation of Feynman diagrams.

We first consider subdiagrams, presented in Fig. 1 in the limit $p \to 0$. Using a result for their sum it is possible to remove vertexes with one attached external line of the field V but with no attached internal V-lines in all diagrams. After this we carry one of the external lines of the superfield V around the loop of the matter superfields until the point, to which the other external line is attached, using some special identities. Then after some involved calculations, described in Ref. [12], it is really possible to prove identity (15) for a special class of Feynman diagrams. Nevertheless, the general case is not so far considered, but the difficulties seems to be only technical.

Due to identity (15) and a similar identity for the massive case the integrals, defining the two-point Green function of the gauge field, are reduced to integrals of total derivatives in the four-dimensional spherical coordinates and can be easily calculated. As a result of this calculation [11] we obtain the following:

1. The renormalization constant Z_3 should be chosen so that

$$\frac{1}{e^2}Z_3\Big(e,\Lambda/\mu\Big) = \frac{1}{e^2} - \frac{1}{4\pi^2}\ln\frac{\Lambda}{\mu}, \tag{16}$$

where μ is a normalization point. This means, that the divergences in the two-point function of the gauge field exist only in the one-loop approximation.

2. The final expression for the corresponding part of the effective action (without the gauge fixing terms) in the massless case can be written as

$$\Gamma^{(2)}_V = \int\frac{d^4p}{(2\pi)^4}\,d^2\theta\,W_a(-p)\,C^{ab}W_b(p)\left[\frac{1}{4e^2} + \frac{1}{16\pi^2}\ln\frac{\mu}{p} - \frac{1}{16\pi^2}\ln(ZG) + \text{const}\right]. \tag{17}$$

According to this formula the Gell-Mann-Low function is written as

$$\beta(\alpha) = \frac{\alpha^2}{\pi}\Big(1-\gamma(\alpha)\Big), \quad \text{where} \quad \gamma \equiv \frac{\partial}{\partial \ln x}\,\ln(ZG)\Big|_{x=1}, \tag{18}$$

has corrections in all orders of the perturbation theory and coincides with the exact NSVZ β-function.

Note, that with the higher derivative regularization the scheme, in which the exact NSVZ β-function is obtained, is defined as follows: The renormalization of the operator $W_a C^{ab} W_b$ is made without adding finite counterterms in two- and more loops. (There are no divergencies in the corresponding Green function in that loops.) Finite counterterms, which can be added for the renormalization of the two-point Green function of the matter superfield, can be arbitrary.

4 Conclusion

Summation of Feynman diagrams, defining the two-point Green function of the gauge field in the limit $p \to 0$ in $N = 1$ supersymmetric electrodynamics, can be partially made by Schwinger-Dyson equations and Ward identities. Diagrams, which can not be summed by this way, are non-planar and give a nontrivial contribution starting from the three-loop approximation. Calculation of their sum exactly to all orders of the perturbation theory is possible, but appears to be rather complicated technically. A summation algorithm was constructed in Ref. [12].

Note, that Eq. (15), which can be easily guessed from explicit calculations, is a new identity, which is not reduced to the Ward identities. The results of Ref. [11] show, that this identity is closely connected with relation (17) for the two-point Green functions, from which the exact NSVZ β-function is obtained. Then there are some questions: What is a true reason of this identity? Is it a consequence of some symmetry of the theory? Can the obtained identity together with relation for the Green functions (17) be included in a set of identities? So far we have no answers to these questions. It seems also interesting to compare the results, presented here, with the results of Ref. [15], obtained by the background field method.

Using identity (15) it is possible to prove, that the Gell-Mann-Low function coincides with the exact NSVZ β-function. However, the renormalization of the operator $W_a C^{ab} W_b$ is exhausted at the one-loop, in accordance with the structure of the anomaly supermultiplet. This means, that the anomaly puzzle is naturally solved, if the higher derivatives regularization is used for the calculations.

Acknowledgements.

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5 TWISTORS, GEOMETRY AND ALGEBRAIC ASPECTS OF SUPERSYMMETRY

An Analysis of $\mathcal{N} = 8$ Supergravity in Supertwistor Space

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Abstract

By analogy with $\mathcal{N} = 4$ super Yang-Mills theory, the superspace constraint equations for $\mathcal{N} = 8$ supergravity are also solvable in a certain sector where the spinorial curvatures vanish. This sector can naturally be interpreted as an anti-self-dual part of $\mathcal{N} = 8$ supergravity. As in the Yang-Mills case, we find that the solvable part of these constraints arises from a Wess-Zumino-Witten (WZW) model whose target space is some extended superspace.

1 Introduction

It is well known that $\mathcal{N} = 8$ supergravity is closely related to the eleven-dimensional supergravity (for the foundation of this theory, see [1]; for the superspace formulation of it, see [2, 3]). It is shown by Howe [4] that the constraint equations for eleven-dimensional supergravity can be expressed as a simple supertorsion constraint by use of the so-called Weyl superspace. Dimensional reduction of this constraint leads to the $\mathcal{N} = 8$ supergravity constraints. This is analogous to how one obtains the constraints of $\mathcal{N} = 4$ super Yang-Mills theory [5] by those of ten-dimensional super Yang-Mills theory [6] via dimensional reduction. In [7], this analogous relation was utilized to investigate the geometrical meaning of superstring theory. Following these lines, in this paper we attempt to solve a subset of the constraint equations for $\mathcal{N} = 8$ supergravity. Our strategy is similar to the harmonic superspace approach which has been successful for $\mathcal{N} = 2$ super Yang-Mills theory [8] as well as for some supergravity theories [9, 10]. This paper can be considered as a natural extension of the previous work [11] on the maximally supersymmetric Yang-Mills theory to a theory of gravity.

2 Superspace constraints and dimensional reduction

It is known that the equations of motion for 10-dimensional super Yang-Mills theory are equivalent to its superspace constraints by use of Bianchi identities [6, 7]. The constraints can be expressed as a flatness condition, as we briefly review below. Ten-dimensional superspace is described by the coordinates (x^m, θ^α). The spinorial covariant derivative is given by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} \tag{1}$$

where Γ^m is a 10-dimensional gamma matrix ($m = 1, 2, \cdots, 10$) and θ^α is the corresponding spinor ($\alpha = 1, 2, \cdots, 32$). Gauged versions of the covariant derivatives are written as

$$\mathcal{D}_\alpha = D_\alpha + A_\alpha \ , \quad \mathcal{D}_m = \frac{\partial}{\partial x^m} + A_m \tag{2}$$

by which we can define the following field strengths on the superspace

$$\begin{aligned} F_{\alpha\beta} &= \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} + i2\Gamma^m_{\alpha\beta}\mathcal{D}_m && (3)\\ F_{\alpha m} &= [\mathcal{D}_\alpha, \mathcal{D}_m] && (4)\\ F_{mn} &= [\mathcal{D}_m, \mathcal{D}_n]. && (5) \end{aligned}$$

The constraint equations are simply expressed as

$$F_{\alpha\beta} = 0. \tag{6}$$

Under naive dimensional reduction, this constraint reduces to the superspace constraints of $\mathcal{N} = 4$ super Yang-Mills theory [5].

We would like to consider analogous constraint equations for 11-dimensional supergravity such that dimensional reduction to the $\mathcal{N} = 8$ theory is transparent. It is shown by Howe [4] that the equations of motion for 11-dimensional supergravity are described by the following (super)torsion constraints

$$T^m_{\alpha\beta} = -i2\ \Gamma^m_{\alpha\beta} \tag{7}$$

where Γ^m is now a 11-dimensional gamma matrix ($m = 1, 2, \cdots, 11$) and the torsion and curvature are defined by

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = T^M_{\alpha\beta}\mathcal{D}_M + R^{mn}_{\alpha\beta}\Sigma^{mn}. \tag{8}$$

Here $\mathcal{D}_M$ denote composite covariant derivatives $\mathcal{D}_M = (\mathcal{D}_m, \mathcal{D}_\alpha)$, while Σ^{mn} denotes the Lorentz generator on 10-dimensional (vectorial) space. Gauge potentials relevant to $\mathcal{D}_M$ can be defined as

$$A_M = e^N_M D_N + \Omega^{mn}_M \Sigma^{mn}. \tag{9}$$

(Note that one can impose $T^\gamma_{\alpha\beta} = 0$ by an analysis on the so-called Weyl superspace [4].)

Dimensional reduction of the constraint (7) can be carried out and we obtain

$$\begin{aligned} T^{\mu\ j}_{Ai\dot{B}} &= -i2\ (-1)^{i(i-1)/2}\ \delta^j_i\ \sigma^\mu_{A\dot{B}} && (10)\\ T^\mu_{AiBj} &= T^{\mu\ ij}_{\dot{A}\dot{B}} = 0 && (11) \end{aligned}$$

where $\mu = 1, 2, 3, 4$ and $\sigma^\mu = (1, -\sigma^i)$ with σ^i being the Pauli matrices. Equations (10) and (11) can be considered as the superspace constraints of $\mathcal{N} = 8$ supergravity.

3 A sector of vanishing spinorial curvature and self-duality

Eleven-dimensional supergravity has three dynamical fields in x-space, *i.e.*, the 11-bein (graviton), the Rarita-Schwinger field (gravitino) and a totally antisymmetric tensor field X_{mnr} [1]. In the superspace formulation [3], the torsions and curvatures are all described by a single superfield H_{abcd} which is defined by $H_{abcd} = \partial_a \wedge X_{bcd}$ ($a, b, c, d = 1, 2, \cdots, 11$ are vectorial indices). In the previous section, we observe that there is a direct analogy between Yang-Mills theory and general relativity in a subspace where those terms that involve the Lorentz generator Σ^{mn} are negligible. This subspace can be identified with a condition that a spinorial curvature $R^{mn}_{\alpha\beta}$ vanishes in the definition (8). In terms of H_{abcd}, the spinorial curvature is expressed by

$$R^{ab}_{\alpha\beta} = \frac{1}{6}\left[(\Gamma^{cd})_{\alpha\beta}H_{abcd} + \frac{1}{3}(\Gamma_{abcdef})_{\alpha\beta}H^{cdef}\right] \tag{12}$$

where $\Gamma^{a_1 a_2 \cdots a_n} = \Gamma^{[a_1}\Gamma^{a_2}\cdots\Gamma^{a_n]}$ are the antisymmetrized product of 11-dimensional gamma matrices (up to normalization). Under dimensional reduction, the vanishing of (12) leads to the relation

$$(\gamma^{cd})_{\alpha\beta}H_{abcd} = -\frac{1}{2}\ \epsilon_{abcd}\ (\gamma^0\gamma_{ef})_{\alpha\beta}H^{cdef} \tag{13}$$

where γ's are the usual 4-dimensional gamma matrices and $\gamma_{abcd} = \epsilon_{abcd}\gamma^0$. Notice the indices are now reduced to $1, 2, 3, 4$. This relation can be further written as

$$W_{ab} = -\frac{1}{2}\ \epsilon_{abcd}\ \gamma^0 W^{cd} \tag{14}$$

with an introduction of a matrix field $W_{ab} = \gamma^{cd}H_{abcd}$. The relation (14) can be seen as an anti-self-dual condition for W_{ab}. In this sense, the sector of vanishing $R^{ab}_{\alpha\beta}$ can be considered as 'anti-self-dual' supergravity, although self-dual supergravity is generally defined in a different manner (see, for example, [12]).

The vectorial curvature in 11-dimensions are defined by [3]

$$\begin{aligned}
R^{cd}_{ab} &= (\Gamma^{cd})_{\alpha\beta}R^{\alpha\beta}_{ab} && (15)\\
R_{ab,\gamma\delta} &= D_a T_{b\gamma\delta} - D_b T_{a\gamma\delta} + D_\gamma T_{ab\delta} + T^{\epsilon}_{a\gamma}T_{b\epsilon\delta} - T^{\epsilon}_{b\gamma}T_{a\epsilon\gamma} && (16)\\
T^{\gamma}_{\alpha\beta} &= -\frac{1}{36}\left[(\Gamma^{bcd})^{\gamma}_{\beta}H_{abcd} + \frac{1}{8}(\Gamma_{abcde})^{\gamma}_{\beta}H^{bcde}\right] && (17)\\
T_{ab\alpha} &= -\frac{i}{42}(\Gamma^{cd})^{\beta}_{\alpha}D_\beta H_{abcd}. && (18)
\end{aligned}$$

Under dimensional reduction we find $T^{\gamma}_{\alpha\beta} = 0$, using $\{\gamma^0, \gamma_a\} = 0$. The vectorial curvature then reduces to

$$\begin{aligned}
R^{cd}_{ab} &= D_\alpha D_\beta(\gamma^{cd}\gamma^{ef})^{\alpha\beta}H_{abef}\\
&= D_\alpha D_\beta(\gamma^{cd}W_{ab})^{\alpha\beta}\\
&= -i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}(\gamma^{cd}W_{ab})^{\alpha\beta}
\end{aligned} \tag{19}$$

where we use $\{D_\alpha, D_\beta\} = -i2(\Gamma^m)_{\alpha\beta}\frac{\partial}{\partial x^m}$ to obtain the last line (with $m \to \mu = 1,2,3,4$). This reduction means that the vectorial (non-supersymmetric) curvature *does* have a nontrivial value in the sector of our interest and that this sector is indeed physically sensible.

4 A solution to the constraints of $\mathcal{N} = 8$ 'anti-self-dual' supergravity

Under the 'anti-self-dual' condition (14), the constraints of $\mathcal{N} = 8$ supergravity (10), (11) reduce to the following forms

$$\begin{aligned} \{\mathcal{D}_{Ai}, \mathcal{D}_{Bj}\} &= \{\mathcal{D}^i_{\dot{A}}, \mathcal{D}^j_{\dot{B}}\} \;=\; 0, & (20)\\ \{\mathcal{D}_{Ai}, \mathcal{D}^j_{\dot{B}}\} &= -i2\;(-1)^{i(i-1)/2}\;\delta_i^j\;\mathcal{D}_{A\dot{B}}. & (21)\end{aligned}$$

In what follows, we will obtain a solution to these constraints. We introduce a complex two-component spinor u^A with scale invariance $u^A \to \lambda u^A$, with λ being a non-zero complex variable. This spinor is closely related to the harmonic variables introduced in the construction of harmonic superspace. In our case, since there is a scale invariance on u^A, the extended superspace corresponds to supertwistor space $\mathbf{CP}^{3|\mathcal{N}}$. (In the case of $\mathcal{N} = 4$, it is known that this space is a supersymmetric Calabi-Yau manifold and one can construct string theory on it.) Motivated by our previous work [11], we then introduce the following additional derivative operators

$$D_i^+ = (-1)^{i(i-1)/2}\; u^A D_{Ai} \quad , \quad D_i^- = -(-1)^{i(i-1)/2}\;\bar{\omega}^A D_{Ai} \tag{22}$$

where $\bar{\omega}^A$ is another two-component spinor that can be related to u^A by $\bar{\omega}^A = K^{A\dot{A}}\bar{u}_{\dot{A}}$ where $\bar{u}_{\dot{A}}$ is a complex conjugate of u^A, $\bar{u}_{\dot{A}} = (u^A)^*$, and $K_{A\dot{A}}$ is an arbitrary frame vector.

Note that we can express the constraints (20) and (21) as flatness conditions; $F_{AiBj} = F^{ij}_{\dot{A}\dot{B}} = 0$ and $F^{j}_{Ai\dot{B}} = 0$, respectively. Let $\mathcal{D}_i^\pm$, $\mathcal{D}^i_{\dot{A}}$ be the gauged versions of spinorial derivatives. In terms of these, the constraints can be written as $F_{ij}^{++} = F_{ij}^{+-} = F_{ij}^{-+} = F_{ij}^{--} = 0$, $F^{ij}_{\dot{A}\dot{B}} = 0$ and $F^{\pm\;j}_{i\dot{B}} = 0$, or explicitly,

$$\begin{aligned} \{\mathcal{D}_i^+, \mathcal{D}_j^+\} &= \{\mathcal{D}_i^+, \mathcal{D}_j^-\} = \{\mathcal{D}_i^-, \mathcal{D}_j^+\} = \{\mathcal{D}_i^-, \mathcal{D}_j^-\} = 0 & (23)\\ \{\mathcal{D}^i_{\dot{A}}, \mathcal{D}^j_{\dot{B}}\} &= 0 & (24)\\ \{\mathcal{D}_i^+, \mathcal{D}^j_{\dot{A}}\} &= -i2(-1)^{i(i-1)/2}\;\delta_i^j\; u^A\mathcal{D}_{A\dot{A}} & (25)\\ \{\mathcal{D}_i^-, \mathcal{D}^j_{\dot{A}}\} &= i2(-1)^{i(i-1)/2}\;\delta_i^j\;\bar{\omega}^A\mathcal{D}_{A\dot{A}}\;. & (26)\end{aligned}$$

The derivative operators on our extended superspace are expressed by $D_i^\pm$, $D^i_{\dot{A}}$, $D_{A\dot{A}} = \sigma^\mu_{A\dot{A}}\frac{\partial}{\partial x^\mu}$ along with

$$\begin{aligned} D^{++} &= u^A\frac{\partial}{\partial\bar{\omega}^A}\;, \quad D^{--} \;=\; -\bar{\omega}^A\frac{\partial}{\partial u^A}\\ D^0 &= u^A\frac{\partial}{\partial u^A} - \bar{\omega}^A\frac{\partial}{\partial\bar{\omega}^A}\;. \end{aligned} \tag{27}$$

Commutation and anticommutation relations among these derivatives (or bases) are given by

$$\begin{array}{l}
[D^{++},D_i^+]=0\ ,\quad [D^{++},D_i^-]=-D_i^+\ ,\quad [D^{++},D^i_{\dot A}]=0\\
[D^{--},D_i^+]=D_i^-\ ,\quad [D^{--},D_i^-]=0\ ,\quad [D^{--},D^i_{\dot A}]=0\\
[D^{++},D^0]=-2D^{++}\ ,\quad [D^{--},D^0]=2D^{--}\\
[D^0,D_i^+]=D_i^+\ ,\quad [D^0,D_i^-]=-D_i^-\ ,\quad [D^0,D^i_{\dot A}]=0
\end{array} \tag{28}$$

$$\begin{array}{rcl}
\{D_i^+,D_j^+\} &=& \{D_i^+,D_j^-\}=\{D_i^-,D_j^+\}=\{D_i^-,D_j^-\}=0\\
\{D^i_{\dot A},D^j_{\dot B}\} &=& 0\\
\{D_i^+,D^j_{\dot A}\} &=& -i2(-1)^{i(i-1)/2}\ \delta_i^j\ u^A D_{A\dot A}\\
\{D_i^-,D^j_{\dot A}\} &=& i2(-1)^{i(i-1)/2}\ \delta_i^j\ \bar\omega^A D_{A\dot A}\ .
\end{array} \tag{29}$$

The covariantization of anticommutators (29) are identical to the constraints in (23)-(26). The extra gauge potentials A^{++}, A^{--} do not involve in the anticommutators. The commutators in (28) mean that it is necessary to make $A^{\pm\pm}$, A^0 vanish in order to satisfy the constraints (23)-(26) in the extended superspace. These constraints are hard to solve by themselves. Our strategy to solve these is to carry out 'gauge transformaitons' in the extended superspace such that there are non-zero $A^{\pm\pm}$.

Let us look at one of the constraints, $\{\mathcal{D}_i^+,\mathcal{D}_j^+\}=\{D_i^++A_i^+,D_j^++A_j^+\}=0$ with $A_i^+=(-1)^{i(i-1)/2}\,u^AA_{Ai}$ ($A_{Ai}=e^M_{Ai}D_M+\Omega^{mn}_{Ai}\Sigma^{mn}$). Notice that A_i^+ can be expressed as a pure gauge form, $A_i^+=-D_i^+g\,g^{-1}$ such that $\{\mathcal{D}_i^+,\mathcal{D}_j^+\}=g\{D_i^+,D_j^+\}g^{-1}=0$, where $g(x^\mu,\theta^{Ai},\bar\theta^i_{\dot A};u^A,\bar\omega^A)$ is some matrix, realizing gauge transformations on the extended superspace. One can eliminate A_i^+, using such a matrix. In doing so, the additional potentials $A^{\pm\pm}$ are no longer zero, rather, in this new gauge we have

$$\begin{array}{rcl}
A'^+_i &=& 0\\
A'^-_i &=& g^{-1}A_i^-g+g^{-1}D_i^-g\\
A'^i_{\dot A} &=& g^{-1}A^i_{\dot A}g+g^{-1}D^i_{\dot A}g\\
A'^{++} &=& g^{-1}D^{++}g\\
A'^{--} &=& g^{-1}D^{--}g\\
A'^0 &=& g^{-1}D^0g.
\end{array} \tag{30}$$

Note that D^0 is a charge operator, assigning +1 charge to u^A and -1 charge to $\bar\omega^A$. Since this charge is to be preserved for any potentials under a 'gauge transformation,' we require

$$D^0g=\left(u^A\frac{\partial}{\partial u^A}-\bar\omega^A\frac{\partial}{\partial\bar\omega^A}\right)g=0. \tag{31}$$

This implies $g(x,\theta,\bar\theta;\langle u\bar\omega\rangle)$, where $\langle u\bar\omega\rangle$ is the inner product of spinors, $\langle u\bar\omega\rangle=\epsilon_{AB}u^A\bar\omega^B$ $=u^A\bar\omega_A=\bar\omega^Au_A$. The $\langle u\bar\omega\rangle$ dependence may lead to (31), however, as one can easily seen, this keep $A'^{\pm\pm}$ vanishing. We are looking for a solution of the constraints (23)-(26) by executing a 'gauge transformation' in the extended supersupace such that we have non-vanishing $A'^{\pm\pm}$. This leads us to introduce the quantities

$$u^A\theta^i_A=\xi^i\quad ,\qquad \bar\omega^A\theta^i_A=\bar\xi^i\qquad (i=1,2,\cdots,8) \tag{32}$$

We can parametrize $g = g(x, \theta, \bar{\theta}; \xi^i, \bar{\xi}^i)$ such that it is linear in ξ^i as well as in $\bar{\xi}^i$. This parametrization naturally leads $D^0 g = 0$ and non-zero $A'^{\pm\pm}$ by use of $D^{++}\bar{\xi}^i = u^A \frac{\partial}{\partial \bar{\omega}^A}\bar{\xi}^i = \xi^i$ and $D^{--}\xi^i = -\bar{\omega}^A \frac{\partial}{\partial u^A}\xi^i = -\bar{\xi}^i$.

For simplicity, let us write the gauge potentials without primes. In the new gauge, $A_i^+ = 0$, the gauged versions of the anticommutation relations (29) become

$$
\begin{aligned}
D_i^+ A_j^- = D_j^+ A_i^- &= 0 && (33)\\
D_i^- A_j^- + D_j A_i^- + \{A_i^-, A_j^-\} &= 0 && (34)\\
D_i^+ A_{\dot{A}}^j &= -i2(-1)^{i(i-1)/2}\, \delta_i^j u^A A_{A\dot{A}} && (35)\\
D_i^- A_{\dot{A}}^j + D_{\dot{A}}^j A_i^- + \{A_i^-, A_{\dot{A}}^j\} &= i2(-1)^{i(i-1)/2}\, \delta_i^j \bar{\omega}^A A_{A\dot{A}} && (36)\\
D_{\dot{A}}^i A_{\dot{B}}^j + D_{\dot{B}}^j A_{\dot{A}}^i + \{A_{\dot{A}}^i, A_{\dot{B}}^j\} &= 0 && (37)
\end{aligned}
$$

where we omit the trivial relation $\{D_i^+, D_j^+\} = 0$. The gauged versions of the commutation relations (28) become

$$
\begin{aligned}
D^{++}A^{--} - D^{--}A^{++} + [A^{++}, A^{--}] &= 0 && (38)\\
D^0 A^{++} &= 2A^{++} && (39)\\
D^0 A^{--} &= -2A^{--} && (40)\\
D_i^+ A^{++} &= 0 && (41)\\
D^{\pm\pm} A_{\dot{A}}^i - D_{\dot{A}}^i A^{\pm\pm} + [A^{\pm\pm}, A_{\dot{A}}^i] &= 0 && (42)\\
D^{\pm\pm} A_i^- - D_i^- A^{\pm\pm} + [A^{\pm\pm}, A_i^-] &= 0 && (43)\\
D_i^+ A^{--} &= -A_i^- && (44)\\
D^0 A_i^- &= -A_i^- && (45)\\
D^0 A_{\dot{A}}^i &= 0 && (46)
\end{aligned}
$$

where we also omit the trivial relation $[D^0, D_i^+] = 0$. Equations (39) and (40) imply that the group element g further has a dependence on $\xi^i\bar{\xi}^i$, that is, it is parametrized by $g(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ rather than $g(x, \theta, \bar{\theta}; \xi^i, \bar{\xi}^i)$. Parametrization of $A_{\dot{A}}^i = A_{\dot{A}}^i(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ and $A_{Ai} = A_{Ai}(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ $(A_i^- = -(-1)^{i(i-1)/2}\bar{\omega}^A A_{Ai})$ is also compatible with (45) and (46). The expression of $A'^i_{\dot{A}}$ in (30), however, includes the term $g^{-1}D_{\dot{A}}^i g$. Because of the spinorial derivative $D_{\dot{A}}^i$ acting on $g(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$, this term potentially change the degree of homogeneity of ξ's or $\bar{\xi}$'s in $A'^i_{\dot{A}}$ by the 'gauge transformations' in the extended superspace. We then further impose

$$D_{\dot{A}}^i\, g = 0\ . \tag{47}$$

Notice that we need to have $D_{Ai}\, g \neq 0$, since otherwise, $A_i^- = -D_i^+ A^{--} = -g^{-1}D_i^+ D^{--} g = g^{-1}(D^{--}D_i^+ - D_i^-)g$ vanishes and we will have a trivial solution. The chirality condition (47) means that the x^μ-dependence of g always comes in the form of $y^\mu = x^\mu - i\theta^{Ai}\sigma^\mu_{A\dot{A}}\bar{\theta}_i^{\dot{A}}$. This allows us to parametrize g as $g(y^\mu, \theta; \xi^i\bar{\xi}^i)$.

Let us recapitulate the results we have obtained so far. In terms of $g(y^\mu, \theta; \xi^i\bar{\xi}^i)$ we parametrize the gauge potentials as

$$
\begin{aligned}
A^{++} &= g^{-1}D^{++}g\\
A^{--} &= g^{-1}D^{--}g\\
A_i^+ &= 0 && (48)
\end{aligned}
$$

$$
\begin{aligned}
A_i^- &= -D_i^+ A^{--} = -D_i^+(g^{-1}D^{--}g) \\
A^i_{\dot{A}} &= g^{-1} A^{(-g)}{}^i_{\dot{A}} g
\end{aligned}
$$

where $A^{(-g)}{}^i_{\dot{A}}$ is defined in an ordinary superspace, *i.e.*, $A^{(-g)}{}^i_{\dot{A}} = A^{(-g)}{}^i_{\dot{A}}(x, \theta, \bar{\theta})$. The gauged version of this, $A^i_{\dot{A}}$, is then parametrized as $A^i_{\dot{A}}(x, \theta, \bar{\theta}; \xi^i \bar{\xi^i})$ in general. In addition to the above set of potentials, we also have $A_{A\dot{A}} = A_{A\dot{A}}(x, \theta, \bar{\theta}; u, \bar{\omega})$. With the expressions in (48), one can straightforwardly check the equations (33), (34), (38) and (42). As we have seen, the relations (39), (40), (45) and (46) are imbedded in the parametrization (48).

The rest of the constraint equations can be understood as follows. We consider the equation (41) as an analyticity condition on A^{++}. Following an idea of harmonic superspace, we regard A^{++} as an unconstrained analytic function with which every potential is to be expressed. With (38) and (41), it is easy to check one of the relations in (43) involving D^{++} and A^{++}. We may have $D_i^- A^{--} = 0$ as a consequence of (41) and, with this relation, the other equation in (43) involving D^{--}, A^{--} also holds. The equation (38) can alternatively be considered as a defining equation for A^{--} in terms of A^{++} (or an expansion of A^{++}'s) as is first shown in [13]. The equation (44) then shows A_i^- is given by A^{++}. The rest of the constraints can be considered similarly, namely, we regard the equation (37) as a defining equation for $A^i_{\dot{A}}$ and the equations (35), (36) as that of $A_{A\dot{A}}$ in terms of $A^i_{\dot{A}}$. Since $A^i_{\dot{A}}$ can be given by a function of A^{++}, all gauge potentials (in extended superspace) are then expressed by $A^{++} = g^{-1}D^{++}g$, the unconstrained analytic (chiral) function of $(y^\mu, \theta; \xi^i, \bar{\xi^i})$. The parametrization of $g^{-1}(y^\mu, \theta; \xi^i \bar{\xi^i})$ and $D^{++}g \equiv g^{++}(y^\mu, \theta; \xi^i \xi^i)$ indicates that A^{++} depends on the combinations of both $\xi^i \bar{\xi^i}$ and $\xi^i \xi^i$. This implies that A^{++} contains antiholomorphic factor in terms of the spinors (which is different from what happens in the Yang-Mills theory).

We end our discussion with the following few remarks. The equation (38) can be used to determine a proper group element g. It is possible to obtain this equation as an equation of motion for a gauged Wess-Zumino-Witten (WZW) action [11]. By use of the Polyakov-Wiegman identity, this gauged WZW action becomes a WZW action whose target space is our extended superspace $\mathbf{CP}^{3|8}$. As in the Yang-Mills case, we expect that current correlators of this WZW model describes multigraviton tree level amplitudes. It is also possible to interpret graviton amplitudes in the same manner as gluon amplitudes, with an introduction of appropriate Chan-Paton factors [14]. In this context, the graviton amplitudes can arise from the so-called super-ambitwistor space $\mathbf{CP}^{3|4} \times \mathbf{CP}^{3|4}$.

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Adelic Superanalysis

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Abstract

After a brief review of p-adic numbers, adeles and their functions, we consider real, p-adic and adelic superalgebras, superspaces and superanalyses.

1 Introduction

It is well known that supersymmetry (SUSY) relates basic properties between bosons and fermions [1]. It plays very important role in construction of new fundamental models of elementary particle physics beyond the Standard Model. In particular, SUSY plays a significant role in construction of String/M-theory, which is currently the best candidate for unification of all interactions and elementary constituents of matter.

Besides enormous success of SUSY, to our opinion it should be extended by the following adelic symmetry principle: a fundamental physical theory (like String/M-theory) has to be invariant under some interchange of real and p-adic number fields. For the first time, a similar principle was given by Volovich [2]. There are already some good illustrative examples of adelic symmetry in adelic quantum mechanics [3], [4] and adelic string product formulas [5], [6]. To extend SUSY by adelic symmetry it is natural first to find p-adic analogs of standard SUSY (over real numbers) and then to unify results in the adelic form, which takes real and all p-adic supersymmetries simultaneously and on equal footing.

In addition to SUSY a strong motivation to consider p-adic and adelic superanalysis comes also from the quest to formulate p-adic and adelic superstring theory (reviews of an early period are in [6] and [5]). Some possibilities to construct p-adic superstring amplitudes are considered in [7] (see also [8], [9], and [10]). It seems that to make further progress towards formulation of p-adic and adelic superstring theory one has previously to develop systematically the corresponding superalgebra and superanalysis.

A promising recent research in p-adic string theory has been mainly related to an extension of adelic quantum mechanics [3], [11] (see also [12]) and p-adic path integrals to string amplitudes [13] and quantum field theory [14]. Also an effective nonlinear p-adic string theory (see, e.g. [5]) with an infinite number of space and time derivatives has been recently of a great interest in the context of the tachyon condensation (for a recent review, see [15]).

It is also worth mentioning successful formulation and development of p-adic and adelic quantum cosmology (see [16] and references therein) which demonstrate discreteness of minisuperspace with the Planck length ℓ_0 as the elementary one. There are also many other models in classical and quantum physics, as well as in some related fields of other sciences, which use p-adic numbers and adeles (for a recent activity, see e.g. proceedings of conferences in p-adic mathematical physics [17], [18]).

In the sequel of this article we briefly review basic properties of p-adic numbers, adeles and their functions, and then consider p-adic and adelic superanalysis.

2 p-Adic numbers and adeles

We review here some introductory notions on p-adic numbers, their quadratic and algebraic extensions, and adeles. For further reading one can use [19], [6], [20] and [5].

Let us recall that a norm on $\mathbb{Q}$ is a map $|| \cdot || : \mathbb{Q} \to \mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$ with the following properties: (i) $||x|| = 0 \leftrightarrow x = 0$, (ii) $||x \cdot y|| = ||x|| \cdot ||y||$, and $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in \mathbb{Q}$. In addition to the absolute value, for which we use usual arithmetic notation $| \cdot |_\infty$, one can introduce on $\mathbb{Q}$ a norm with respect to each prime number p. Any rational number can be uniquely written as $x = p^\nu \, \frac{m}{n}$, where p, m, n are mutually prime and $\nu \in \mathbb{Z}$. By definition p-adic norm (or, in other words, p-adic absolute value) is $|x|_p = p^{-\nu}$ if $x \neq 0$ and $|0|_p = 0$. One can verify that $| \cdot |_p$ satisfies all the above conditions and moreover one has strong triangle inequality, i.e. $|x + y|_p \leq \; max\,(|x|_p, \, |y|_p)$. Thus p-adic norms belong to the class of non-archimedean (ultrametric) norms. According to the Ostrowski theorem any nontrivial norm on $\mathbb{Q}$ is equivalent either to the $| \cdot |_\infty$ or to one of the $| \cdot |_p$. One can easily show that $|m|_p \leq 1$ for any $m \in \mathbb{Z}$ and any prime p. The p-adic norm is a measure of divisibility of the integer m by prime p: the more divisible, the p-adic smaller. By Cauchy sequences of rational numbers one can make completions of $\mathbb{Q}$ to obtain $\mathbb{R} \equiv \mathbb{Q}_\infty$ and the fields $\mathbb{Q}_p$ of p-adic numbers using norms $| \cdot |_\infty$ and $| \cdot |_p$, respectively. The cardinality of $\mathbb{Q}_p$ is the continuum as that one of $\mathbb{Q}_\infty$. p-Adic completion of $\mathbb{Z}$ gives the ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p | \; |x|_p \leq 1\}$ of p-adic integers. Denote by $\mathbb{U}_p = \{x \in \mathbb{Q}_p | \; |x|_p = 1\}$ multiplicative group of p-adic units.

Any p-adic number $0 \neq x \in \mathbb{Q}_p$ has unique representation as the sum of a convergent series of the form

$$x = p^\nu \, (x_0 + x_1 p + x_2 p^2 + \cdots + x_n p^n + \cdots), \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \cdots, p-1\}. \tag{1}$$

It resembles expansion of a real number $y = \pm\, 10^\mu \, \sum_{k=0}^{-\infty} b_k 10^k, \; \mu \in \mathbb{Z}, \; b_k \in \{0, 1, \cdots, 9\}$, but in a sense in the opposite way. If $\nu \geq 0$, then $x \in \mathbb{Z}_p$. When $\nu = 0$ and $x_0 \neq 0$ one has $x \in \mathbb{U}_p$. Any negative integer can be easily presented starting from the representation for -1:

$$-1 = p - 1 \; + \; (p-1)p \; + \; (p-1)p^2 + \cdots + (p-1)p^n + \cdots . \tag{2}$$

By the analogy with the real case, one uses the norm $| \cdot |_p$ to introduce p-adic metric $d_p(x, y) = |x - y|_p$, which satisfies all necessary properties of metric with strong triangle inequality in the non-archimedean (ultrametric) form: $d_p(x, y) \leq max\,(\,d_p(x, z), \, d_p(z, y)\,)$. We can regard $d_p(x, y)$ as a distance between p-adic numbers x and y. Using this (ultra)metric, $\mathbb{Q}_p$ becomes ultrametric space and one can investigate the corresponding topology. Because of ultrametricity, the p-adic spaces have some exotic (from the real point of view) properties and usual illustrative examples are: a) any point of the ball $B_\mu(a) = \{x \in \mathbb{Q}_p \,| \; |x - a|_p \leq p^\mu\}$ can be taken as its center instead of a; b) any ball can be regarded as a closed as well as an open set; c) two balls may not have partial intersection, i.e. they are disjoint sets or one of them is a subset of the other; and c) all triangles are isosceles. $\mathbb{Q}_p$ is zerodimensional and totally disconnected topological space. $\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact space.

Recall that the field $\mathbb{C}$ of complex numbers can be constructed as quadratic extension of $\mathbb{R}$ by using formal solution $x = \sqrt{-1}$ of the equation $x^2 + 1 = 0$ and denoted by $\mathbb{C} = \mathbb{R}(\sqrt{-1})$. All elements of $\mathbb{C}$ have the form $z = x + \sqrt{-1}\,y$ with $x, y \in \mathbb{R}$. $\mathbb{C}$ is algebraically closed, metrically complete field, and a two-dimensional vector space.

Algebraic extensions of $\mathbb{Q}_p$ also exist and have more complex structure. Quadratic extensions have the form $\mathbb{Q}_p(\sqrt{\tau})$ with elements $z = x + \sqrt{\tau}\,y$, where $x, y, \tau \in \mathbb{Q}_p$ and τ is not square element of $\mathbb{Q}_p$. For $p \neq 2$ there are three inequivalent quadratic extensions and one can take $\tau = \varepsilon,\, \varepsilon p,\, p$, where $\varepsilon = \sqrt[p-1]{1} \in \mathbb{Q}_p$. When $p = 2$ there are seven inequivalent quadratic extensions which may be characterized by $\tau = -1, \pm 2, \pm 3, \pm 6$. Quadratic extensions are complete but not algebraically closed. For solutions of higher order algebraic equations one has to introduce some new extensions. Algebraic closer of $\mathbb{Q}_p$, denoted by $\bar{\mathbb{Q}}_p$, is an infinite dimensional vector space over $\mathbb{Q}_p$ which is not complete. Completion of $\bar{\mathbb{Q}}_p$ gives $\mathbb{C}_p$ which is algebraically closed and metrically complete.

Real and p-adic numbers are continual extrapolations of rational numbers along all possible notrivial and inequivalent metrics. To consider real and p-adic numbers simultaneously and on equal footing one uses concept of adeles. An adele x (see, e.g. [20]) is an infinite sequence $x = (x_\infty, x_2, \cdots, x_p, \cdots)$, where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $\mathcal{P}$ of primes p one has $x_p \in \mathbb{Z}_p$. Componentwise addition and multiplication endow the ring structure to the set of all adeles $\mathbb{A}$, which is the union of restricted direct products in the following form:

$$\mathbb{A} = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}), \qquad \mathbb{A}(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p\,. \tag{3}$$

A multiplicative group of ideles $\mathbb{I}$ is a subset of $\mathbb{A}$ with elements $x = (x_\infty, x_2, \cdots, x_p, \cdots)$, where $x_\infty \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $x_p \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ with the restriction that for all but a finite set $\mathcal{P}$ one has that $x_p \in \mathbb{U}_p$. Thus the whole set of ideles is

$$\mathbb{I} = \bigcup_{\mathcal{P}} \mathbb{I}(\mathcal{P}), \qquad \mathbb{I}(\mathcal{P}) = \mathbb{R}^* \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p^* \times \prod_{p \notin \mathcal{P}} \mathbb{U}_p\,. \tag{4}$$

A principal adele (idele) is a sequence $(x, x, \cdots, x, \cdots) \in \mathbb{A}$, where $x \in \mathbb{Q}$ $\quad (x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\})$. $\mathbb{Q}$ and $\mathbb{Q}^*$ are naturally embedded in $\mathbb{A}$ and $\mathbb{I}$, respectively.

Let us define an ordering on the set $\mathbb{P}$, which consists of all finite sets $\mathcal{P}_i$ of primes p, by $\mathcal{P}_1 \prec \mathcal{P}_2$ if $\mathcal{P}_1 \subset \mathcal{P}_2$. It is evident that $\mathbb{A}(\mathcal{P}_1) \subset \mathbb{A}(\mathcal{P}_2)$ when $\mathcal{P}_1 \prec \mathcal{P}_2$. Spaces $\mathbb{A}(\mathcal{P})$ have natural Tikhonov topology and adelic topology in $\mathbb{A}$ is introduced by inductive limit: $\mathbb{A} = \lim \mathrm{ind}_{\mathcal{P} \in \mathbb{P}} \mathbb{A}(\mathcal{P})$. A basis of adelic topology is a collection of open sets of the form $W(\mathcal{P}) = \mathbb{V}_\infty \times \prod_{p \in \mathcal{P}} \mathbb{V}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p$, where $\mathbb{V}_\infty$ and $\mathbb{V}_p$ are open sets in $\mathbb{R}$ and $\mathbb{Q}_p$, respectively. Note that adelic topology is finer than the corresponding Tikhonov topology. A sequence of adeles $a^{(n)} \in \mathbb{A}$ converges to an adele $a \in \mathbb{A}$ if *i*) it converges to a componentwise and *ii*) if there exist a positive integer N and a set $\mathcal{P}$ such that $a^{(n)},\, a \in \mathbb{A}(\mathcal{P})$ when $n \geq N$. In the analogous way, these assertions hold also for idelic spaces $\mathbb{I}(\mathcal{P})$ and $\mathbb{I}$. $\mathbb{A}$ and $\mathbb{I}$ are locally compact topological spaces.

3 p-Adic and adelic analysis

$\mathbb{R}, \mathbb{Q}_p, \mathbb{C}, \mathbb{Q}_p(\sqrt{\tau}), \mathbb{C}_p, \mathbb{A}, \mathbb{I}$, and the higher p-adic algebraic extensions, form a large space for realization of various mappings and the corresponding analyses. However only some of them have been used in modern mathematical physics.Thus, in addition to the classical real and complex analysis, the most important ones are related to the following mappings: $(i)\, \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, $(ii)\, \mathbb{A} \rightarrow \mathbb{A}$, $(iii)\, \mathbb{Q}_p \rightarrow \mathbb{C}$, , $(iv)\, \mathbb{A} \rightarrow \mathbb{C}$, $(v)\, \mathbb{Q}_p \rightarrow \mathbb{Q}_p(\sqrt{\tau})$ and $(vi)\ \mathbb{Q}_p(\sqrt{\tau}) \rightarrow \mathbb{C}$. We will give now an information about the first two cases.

Case $(i)\, \mathbb{Q}_p \rightarrow \mathbb{Q}_p$. All functions from the real analysis which are given by infinite power series $\sum a_n\, x^n$, where $a_n \in \mathbb{Q}$, can be regarded also as p-adic if we take $x \in \mathbb{Q}_p$. Necessary and sufficient condition for the convergence is $|a_n\, x^n|_p \rightarrow 0$ when $n \rightarrow \infty$. For example, p-adic exponential function is

$$\exp x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}\,, \tag{5}$$

where the domain of convergence is $|x|_p < |2|_p$. We see that convergence is here bounded inside $p\,\mathbb{Z}_p$. Note that

$$|n!|_p = p^{-\frac{n-n'}{p-1}}\ , \tag{6}$$

where n' is the sum of digits in the expansion of n with respect to p, i.e. $n = n_0 + n_1 p + \cdots + n_k p^k$. An interesting class of functions which domain of convergence is $\mathbb{Z}_p$ has the form $F_k(x) = \sum_{n\geq 0} n! P_k(n) x^n$, where $P_k(n) = n^k + C_{k-1} n^{k-1} + \cdots + C_0$ is a polynomial in n with $C_i \in \mathbb{Z}$ (for various properties of these functions, see [21]). Some functions can be constructed by the method of interpolation, which is based on the fact that $\mathbb{N}$ is dense in $\mathbb{Z}_p$.

In this case derivatives, antiderivatives and some definite integrals are well defined. However there is a problem to construct the p-adic valued Lebesgue measure.

Case $(ii)\, \mathbb{A} \rightarrow \mathbb{A}$. This case is an adelic collection of real and p-adic mappings which enables to consider simultaneously and on equal footing real and all p-adic aspects of a classical Lagrangian (and Hamiltonian) system. In such case parameters for a given system should be treated as rational numbers. Equations of motion must have an adelic solution, i. e. function and its argument must have the form of adeles.

In the last two cases, i. e. $(v)\, \mathbb{Q}_p \rightarrow \mathbb{Q}_p(\sqrt{\tau})$ and $(vi)\ \mathbb{Q}_p(\sqrt{\tau}) \rightarrow \mathbb{C}$, p-adic quadratic extensions are used for values of functions and as values of the argument, respectively. The analysis for $(v)\, \mathbb{Q}_p \rightarrow \mathbb{Q}_p(\sqrt{\tau})$ is developed and used for a new type of non-archimedean quantum mechanics (see, monograph [22] and references therein). Let us also mention that some mappings $\mathbb{Q}_p(\sqrt{\tau}) \rightarrow \mathbb{Q}_p(\sqrt{\tau})$ have been used for investigation of classical dynamical systems (see, book [25] and references therein)

4 p-Adic and adelic superalgebra and superspace

As a next step to superanalysis we are going to review here real and p-adic superalgebra and superspace along approach introduced by Vladimirov and Volovich [23], [24] and elaborated by Khrennikov [26] (see also [22]). Then, in the way initiated by Dragovich [27], we shall generalize this approach to the adelic case.

Let $L(\mathbb{Q}_v) = L_0(\mathbb{Q}_v) \oplus L_1(\mathbb{Q}_v)$ be Z_2-graded vector space over $\mathbb{Q}_v$, $(v = \infty, 2, \cdots, p, \cdots)$, where elements $a \in L_0(\mathbb{Q}_v)$ and $b \in L_1(\mathbb{Q}_v)$ have even $(p(a) = 0)$ and odd $(p(b) = 1)$ parities. Thus $L_0(\mathbb{Q}_v)$ and $L_1(\mathbb{Q}_v)$ are vector subspaces of different parity. Such $L(\mathbb{Q}_v)$ space becomes v-adic (i. e. real and p-adic) superalgebra, denoted by $\Lambda(\mathbb{Q}_v) = \Lambda_0(\mathbb{Q}_v) \oplus \Lambda_1(\mathbb{Q}_v)$, if it is endowed by an associative algebra with unity and multiplication with parity defined by $p(ab) \equiv p(a) + p(b)\,(mod\,2)$. Product of two elements of the same (different) parity has even (add) parity.

Supercommutator is $[a, b\} = a\,b - (-1)^{p(a)p(b)} b\,a$. Superalgebra $\Lambda(\mathbb{Q}_v)$ is called (super)commutative if $[a, b\} = 0$ for any $a\,, b$ which are elements of $\Lambda_0(\mathbb{Q}_v)$ and $\Lambda_1(\mathbb{Q}_v)$.

To obtain a Banach space from the commutative superalgebra (CSA) one has to introduce the corresponding norm

$$||fg||_v \leq ||f||_v\, ||g||_v\,, \quad f, g \in \Lambda(\mathbb{Q}_v), \tag{7}$$

which is at the end related to the absolute value $|\cdot|_\infty$ for the real case and to p-adic norms $|\cdot|_p$ for p-adic cases.

As illustrative examples of commutative superalgebras one can consider finite dimensional v-adic Grassmann algebras $G(\mathbb{Q}_v : \eta_1, \eta_2, \cdots, \eta_k)$ which dimension is 2^k and generators $\eta_1, \eta_2, \cdots, \eta_k$ satisfy anticommutative relations $\eta_i\eta_j + \eta_j\eta_i = 0$. These $\eta_i\,\eta_j$ can be realized as: 1) product of annihilation operators $a_i\,a_j$ for fermions, 2) exterior product $dx^i \wedge dx^j$, and 3) as product of some matrices. One can write

$$G(\mathbb{Q}_v : \eta_1, \cdots, \eta_k) = G_0(\mathbb{Q}_v : \eta_1, \cdots, \eta_k) + G_1(\mathbb{Q}_v : \eta_1, \cdots, \eta_k), \tag{8}$$

where $G_0(\mathbb{Q}_v : \eta_1, \eta_2, \cdots, \eta_k)$ and $G_1(\mathbb{Q}_v : \eta_1, \eta_2, \cdots, \eta_k)$ contain sums of 2^{k-1} terms with even and add number of algebra generators η_i, respectively. Note that the role of commuting and anticommuting coordinates will play these sums with even and add parity and not the coefficients in expansion over products of η_i. Infinite dimensional Grassmann algebra can be also used as CSA.

Let $\Lambda(\mathbb{Q}_v)$ be a fixed commutative v-adic superalgebra. v-Adic superspace of dimension (n, m) over $\Lambda(\mathbb{Q}_v)$ is

$$\mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_v)} = \Lambda^n_0(\mathbb{Q}_v)\, \times \Lambda^m_1(\mathbb{Q}_v)\,, \tag{9}$$

where

$$\Lambda^n_0(\mathbb{Q}_v) = \underbrace{\Lambda_0(\mathbb{Q}_v) \times\, \cdots\, \times\, \Lambda_0(\mathbb{Q}_v)}_{n}, \quad \Lambda^m_1(\mathbb{Q}_v) = \underbrace{\Lambda_1(\mathbb{Q}_v) \times\, \cdots\, \times\, \Lambda_1(\mathbb{Q}_v)}_{m}\,. \tag{10}$$

This superspace is an extension of the standard v-adic space, which has now n commuting and m anticommuting coordinates.

The points of the superspace $\mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_v)}$ are

$$\begin{aligned} X^{(v)} &= (X^{(v)}_1, X^{(v)}_2, \cdots, X^{(v)}_n, X^{(v)}_{n+1}, \cdots, X^{(v)}_{n+m}) \\ &= (x^{(v)}_1, x^{(v)}_2, \cdots, x^{(v)}_n, \theta^{(v)}_1, \cdots, \theta^{(v)}_m) = (x^{(v)}, \theta^{(v)}), \end{aligned} \tag{11}$$

where coordinates $x^{(v)}_1, x^{(v)}_2, \cdots, x^{(v)}_n$ are commuting, with $p(x^{(v)}_i) = 0$, and $\theta^{(v)}_1, \theta^{(v)}_2 \cdots, \theta^{(v)}_m$ are anticommuting (Grassmann) ones, with $p(\theta^{(v)}_j) = 1$. Since the supercommutator $[X^{(v)}_i, X^{(v)}_j\} = X^{(v)}_i X^{(v)}_j - (-1)^{p(X^{(v)}_i)p(X^{(v)}_j)} X^{(v)}_j X^{(v)}_i = 0$, the coordinates $X^{(v)}_i$, $(i = 1, 2, \cdots, n+m)$ are called supercommuting.

A norm of $X^{(v)}$ can be defined as

$$||X^{(v)}||_v = \left\{ \begin{array}{ll} \sum_{i=1}^n ||x_i^{(\infty)}||_\infty + \sum_{j=1}^m ||\theta_j^{(\infty)}||_\infty, & v=\infty, \\ \max_{1\le i\le n,\, 1\le j\le m} (\, ||x_i^{(p)}||_p \,, ||\theta_j^{(p)}||_p \,), & v = p \,, \end{array} \right. \tag{12}$$

where $||X^{(p)}||_p \,, ||x_i^{(p)}||_p$ and $||\theta_j^{(p)}||_p$ are non-archimedean norms. In the sequel, to decrease number of indices we often omit some of them when they are understood from the context.

We can now turn to the adelic case of superalgebra and superspace. Let us start with the corresponding Z_2-graded vector space over $\mathbb{A}$ as

$$L(\mathbb{A}) = \bigcup_{\mathcal{P}} L(\mathcal{P})\,, \quad L(\mathcal{P}) = L(\mathbb{R}) \times \prod_{p\in\mathcal{P}} L(\mathbb{Q}_p) \times \prod_{p\notin\mathcal{P}} L(\mathbb{Z}_p)\,, \tag{13}$$

where $L(\mathbb{Z}_p) = L_0(\mathbb{Z}_p) \oplus L_1(\mathbb{Z}_p)$ is a graded space over the ring of p-adic integers $\mathbb{Z}_p$ (and $\mathcal{P}$ is a finite set of primes p). Graded vector space (13) becomes adelic superalgebra

$$\Lambda(\mathbb{A}) = \bigcup_{\mathcal{P}} \Lambda(\mathcal{P})\,, \quad \Lambda(\mathcal{P}) = \Lambda(\mathbb{R}) \times \prod_{p\in\mathcal{P}} \Lambda(\mathbb{Q}_p) \times \prod_{p\notin\mathcal{P}} \Lambda(\mathbb{Z}_p)\,, \tag{14}$$

by requiring that $\Lambda(\mathbb{R})\,, \Lambda(\mathbb{Q}_p)\,,$ and $\Lambda(\mathbb{Z}_p) = \Lambda_0(\mathbb{Z}_p) \oplus \Lambda_1(\mathbb{Z}_p)$ are superalgebras. Adelic supercommutator may be regarded as a collection of real and all p-adic supercommutators. Thus adelic superalgebra (14) is commutative. An example of commutative adelic superalgebra is the following adelic Grassmann algebra:

$$G(\mathbb{A} : \eta_1, \eta_2, \cdots, \eta_k) = \bigcup_{\mathcal{P}} G(\mathcal{P} : \eta_1, \eta_2, \cdots, \eta_k) \tag{15}$$

$$\begin{aligned} G(\mathcal{P} : \eta_1, \cdots, \eta_k) &= G(\mathbb{R} : \eta_1, \cdots, \eta_k) \\ \times \prod_{p\in\mathcal{P}} G(\mathbb{Q}_p : \eta_1, \cdots, \eta_k) &\times \prod_{p\notin\mathcal{P}} G(\mathbb{Z}_p : \eta_1, \cdots, \eta_k). \end{aligned} \tag{16}$$

Banach commutative superalgebra for $\Lambda(\mathcal{P})$ defined in (14) obtains by taking all of $\Lambda(\mathbb{Q}_\infty)$, $\Lambda(\mathbb{Q}_p)$, $\Lambda(\mathbb{Z}_p)$ to be Banach spaces. The $\Lambda(\mathcal{P})$ will have the corresponding Tikhonov topology. Then the corresponding Banach adelic space is inductive limit $\Lambda(\mathbb{A}) = \lim\mathrm{ind}_{\mathcal{P}\in\mathbb{P}}\, \Lambda(\mathcal{P})$, and in this way it gets an adelic topology.

Adelic superspace of dimension (n,m) has the form

$$\mathbb{A}^{n,m}_{\Lambda(\mathbb{A})} = \bigcup_{\mathcal{P}} \mathbb{A}^{n,m}_{\Lambda(\mathbb{A})}(\mathcal{P})\,, \quad \mathbb{A}^{n,m}_{\Lambda(\mathbb{A})}(\mathcal{P}) = \mathbb{R}^{n,m}_{\Lambda(\mathbb{R})} \times \prod_{p\in\mathcal{P}} \mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_p)} \times \prod_{p\notin\mathcal{P}} \mathbb{Z}^{n,m}_{\Lambda(\mathbb{Z}_p)}\,, \tag{17}$$

where $\mathbb{Z}^{n,m}_{\Lambda(\mathbb{Z}_p)}$ is (n,m)-dimensional p-adic superspace over Banach commutative superalgebra $\Lambda(\mathbb{Z}_p)$.

Adelic superspace points X have the coordinate form $X = (X^{(\infty)}, X^{(2)}, \cdots, X^{(p)}, \cdots)$, where for all but a set $\mathcal{P}$ it has to be $||X^{(p)}||_p = \max_{1\le i\le n,\, 1\le j\le m} (\, ||x_i^{(p)}||_p \,, ||\theta_j^{(p)}||_p \,) \le 1$, i. e. $x_i^{(p)} \in \Lambda_0(\mathbb{Z}_p)$ and $\theta_j^{(p)} \in \Lambda_1(\mathbb{Z}_p)$.

5 Elements of p-adic and adelic superanalysis

Superanalysis is related to a map from one superspace to the other. Since we have formulated here many superspaces which are distinctly valued, therefore one can introduce many kinds of mappings between them and one can get plenty of superanalyses. For instance, one can consider the following cases: real $\longrightarrow$ real, p-adic $\longrightarrow$ p-adic, adelic $\longrightarrow$ adelic, real $\longrightarrow$ complex, p-adic $\longrightarrow$ complex, and adelic $\longrightarrow$ complex. In the sequel we will restrict our consideration to the cases without complex-valued functions. In fact we will investigate two types of maps:

$$F_v : V_v \to V_v', \qquad \Phi_{\mathbb{A}} : W(\mathcal{P}) \to W'(\mathcal{P}'), \tag{18}$$

where $V_v \subset \mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_v)}$, $V_v' \subset \mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_v)}$, and $W(\mathcal{P}) \subset \mathbb{A}^{n,m}_{\Lambda(\mathbb{A})}(\mathcal{P})$, $W'(\mathcal{P}') \subset \mathbb{A}^{n,m}_{\Lambda(\mathbb{A})}(\mathcal{P}')$.

Case F_v. The function $F_v(X)$ is continuous in the point $X \in V_v$ if

$$\lim_{||h||_v \to 0} ||F_v(X+h) - F_v||_v = 0\,, \tag{19}$$

and it is continuous in V_v if (19) is satisfied for all $X \in V_v$. This function F_v is superdifferentiable in $X \in V_v$ if it can be presented as

$$F_v(X+h) = F_v(X) + \sum_{i=1}^{n+m} f_i(X)\, h_i + g(X,h)\,, \tag{20}$$

where $f_i(X) \in V_v'$ and $||g(X,h)||_v\, ||h||_v^{-1} \to 0$ when $||h||_v \to 0$. Then $f_i(X)$ are called partial derivatives of F_v in the point X with respect to X_i and denoted by

$$f_i(X) = \frac{\partial F_v(X)}{\partial X_i} = \frac{\partial F_v(x,\theta)}{\partial x_i}\,, \quad f_{n+j}(X) = \frac{\partial F_v(X)}{\partial X_{n+j}} = \frac{\partial F_v(x,\theta)}{\partial \theta_j}\,, \tag{21}$$

where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The superdifferential is

$$DF_v(X) = \sum_{i=1}^{n+m} \frac{\partial F_v(X)}{\partial X_i}\, h_i. \tag{22}$$

If F_v is an $(n+m)$-component function then partial derivatives form $(n+m) \times (n+m)$ Jacobi matrix. The above introduced derivatives are known as the right ones. One can also introduce the left derivatives by change $f_i(X)\, h_i \to h_i\, f_i(X)$ in (20). Higher order derivatives can be introduced in the analogous way. Note that partial derivatives with odd coordinates anticommute: $\frac{\partial}{\partial \theta\, i} \frac{\partial}{\partial \theta\, j} = - \frac{\partial}{\partial \theta\, j} \frac{\partial}{\partial \theta\, i}$. Since in this approach coordinates x_i and θ_j are composed of coefficients in commutative superalgebra $\Lambda(\mathbb{Q}_v)$ there exist the corresponding Cauchy-Riemann conditions (for details, see, [23]). Note that superdifferentiability is closely related to the Frechèt differentiability in the Banach spaces.

The corresponding integral calculus is based on appropriately constructed differential forms [24]. Integration over noncommuting variables employs the standard rules

$$\int d\theta = 0\,, \quad \int \theta\, d\theta = 1\,. \tag{23}$$

When one of the commuting coordinates x_i is the time and the others are spatial in the superspace $\mathbb{Q}^{n,m}_{\Lambda(\mathbb{Q}_v)}$, the functions are called superfields in supersymmetric physical models.

Various aspects of p-adic superanalysis have been considered in detail and many of them can be found in the papers [28], [29], [30], [31], [32].

The corresponding adelic valued functions (superfields) $\Phi_{\mathbb{A}}$ must satisfy adelic structure, i.e. $\Phi_{\mathbb{A}}(X) = (F_\infty\,(X^{(\infty)}),\, F_2\,(X^{(2)}),\, \cdots,\, F_p\,(X^{(p)}), \cdots)$ with condition $||F_p||_p \leq 1$ for all but a finite set of primes $\mathcal{P}$. According to this adelic property and the above v-adic superanalysis one obtains the corresponding adelic superanalysis.

6 Concluding remarks

In this article we have given a brief review of real and p-adic analysis and superanalysis on the Banach commutative superalgebra. An introduction to adelic superanalysis is also presented. As a next step we plan to consider p-adic and adelic superanalysis with complex-valued superfields, as well as to develop adelic theory of supersymmetry and to construct p-adic analogs and adelic models of superstring and M-theory.

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New Facts about Berezinians

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Tools of supermathematics have become an essential part of the mathematical baggage of theoretical physics. On the other hand, still there are many important questions corresponding to statements well-known in the ordinary case, answers to which are not at all clear in the supercase. Here we discuss some of these problems. We consider deep relations that arise in the supercase between Berezinian (superdeterminant) and exterior powers. In particular, this allows to give a new expression for the Berezinian in terms of polynomial invariants of a matrix. Details see in our paper [3]. Recall the relations between traces and (ordinary) determinant. If A is a 2×2 matrix $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$, $\det A = ad - bc = \frac{1}{2}\left((a+d)^2 - (a^2+2bc+d^2)\right) = \frac{1}{2}\left(\mathrm{Tr}^{\,2} A - \mathrm{Tr}\, A^2\right)$ and $\det(1+Az) = 1 + z\mathrm{Tr}\, A + z^2 \det A$. This is a commonplace. In general, if A is an $n\times n$ matrix, then one can consider the following polynomial of degree n:

$$R_A(z) = \det(1+Az) = \sum_{k=0}^{n} c_k(A) z^k\,, \tag{1}$$

the characteristic polynomial of the matrix A. (For our purposes it is more convenient to consider the above polynomial instead of $\det(A-z)$.) One can easily calculate the coefficients of this polynomial by taking the derivative with respect to z. We arrive at the relation:

$$\frac{d}{dz}\det(1+Az) = \sum_{k=1}^{n} k c_k(A) z^{k-1} =$$

$$\det(1+Az)\mathrm{Tr}\,\left((1+Az)^{-1}A\right) = \sum_{k=0}^{n} c_k(A) z^k \sum_{k=0}^{\infty}(-1)^k s_{k+1} z^k\,,$$

where we denoted $s_k(A) = \mathrm{Tr}\, A^k$. This leads to recurrence relations expressing $c_k(A)$ in terms of $s_k(A)$:

$$c_0 = 1,\ c_1 = s_1,\ \ldots,\ c_{k+1} = \frac{1}{k+1}(s_1 c_k - s_2 c_{k-1} + \ldots + (-1)^k s_{k+1}),\ \ldots \tag{2}$$

In particular,

$$\det A = c_n(A) \ \text{ if } A \text{ is an operator in an } n\text{-dimensional space}, \tag{3}$$

and it can be expressed via $s_k(A)$. These are standard facts in linear algebra. What about a generalisation of the above formulae to the supercase? Let V be a $p|q$-dimensional superspace. One can describe it in the following way. Let $V_0 \oplus V_1$ be the direct sum of p-dimensional and q-dimensional vector spaces. Let $\{\mathbf{e}_i\}$ ($i = 1, \ldots, p$) and $\{\mathbf{f}_\alpha\}$ ($\alpha = 1, \ldots, q$) be bases in the spaces V_0, V_1, respectively. Consider linear combinations $\sum_{i=1}^p a^i \mathbf{e}_i + \sum_{\alpha=1}^q b^\alpha \mathbf{f}_\alpha$ where coefficients a^i are even elements of some Grassmann algebra Λ and b^α are odd elements of this Grassmann algebra. Such linear combinations are considered as points of the $p|q$-dimensional superspace V. Let A be an even linear operator on this space. The (super)matrix of the operator A has the form $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$, where A_{00}, A_{11} are $p \times p$ and $q \times q$ matrices, respectively, with even entries taken from the Grassmann algebra Λ, and A_{01}, A_{10} are $p \times q$ and $q \times p$ matrices, respectively, with odd entries from the Grassmann algebra Λ. Such a (super)matrix is called even. The Berezinian (superdeterminant) of an even matrix A is given by the famous formula due to F. A. Berezin (see [1]):

$$\operatorname{Ber} A = \frac{\det\left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right)}{\det A_{11}} . \tag{4}$$

Berezinian is a multiplicative function of matrices, $\operatorname{Ber}(AB) = \operatorname{Ber} A \cdot \operatorname{Ber} B$. Hence Ber is well-defined on operators. Berezinian is related with supertrace in the same way as the ordinary determinant, with trace: for an even supermatrix D

$$\operatorname{Ber} e^D = e^{\operatorname{Tr} D} . \tag{5}$$

We denote the supertrace of a supermatrix by the same symbol as the trace of an ordinary matrix. Recall that for an even supermatrix

$$\operatorname{Tr} D = \operatorname{Tr} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} = \operatorname{Tr} D_{00} - \operatorname{Tr} D_{11} .$$

Instead of the characteristic polynomial 1 one has to consider the *characteristic rational function* $R_A(z) = \operatorname{Ber}(1 + Az)$. We note that the straightforward use of expression 4 for the analysis of the characteristic function leads to a confusion. Let us step back and consider the geometrical meaning of coefficients $c_k(A)$ in formula 1 for the ordinary case. Suppose that $\{\mathbf{e}_i\}$ is an eigenbasis of a linear operator A on an n-dimensional space V: $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$ ($i = 1, \ldots, n$). Then

$$R_A(z) = \det(1 + Az) = \prod_{i=1}^n (1 + \lambda_i z) = \sum_{k=0}^n \left(\prod_{j_1 < j_2 < \ldots < j_k} \lambda_{j_1} \ldots \lambda_{j_k} \right) z^k = \sum_{k=0}^n c_k(A) z^k .$$

Consider the basis consisting of wedge products $\{\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_k}\}$ ($1 \leq j_1 < j_2 < \ldots < j_k \leq n$) in the exterior power $\wedge^k V$. Then $\lambda_{j_1} \ldots \lambda_{j_k}$ is the eigenvalue corresponding to the basis vector $\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_k}$. Hence we see that for the polynomial $\det(1 + Az)$,

$$c_k(A) = \operatorname{Tr} \wedge^k A,$$

where we denote by $\wedge^k A$ the operator induced by A in the exterior power $\wedge^k V$. This formula can be straightforwardly generalised to the supercase (see [4], [3]). Suppose that

$\{\mathbf{e}_i, \mathbf{f}_\alpha\}$ is an eigenbasis of a linear operator A in a $p|q$-dimensional superspace V: $A\mathbf{e}_i = \lambda_i \mathbf{e}_i, A\mathbf{f}_\alpha = \mu_\alpha \mathbf{f}_\alpha$. Here $\{\mathbf{e}_i\}$, $i = 1, \ldots, p$, are even eigenvectors and $\{\mathbf{f}_\alpha\}$, $\alpha = 1, \ldots, q$, are odd eigenvectors. Then

$$R_A(z) = \mathrm{Ber}\,(1 + Az) = \sum_{k=0}^{\infty} c_k(A) z^k = \prod_{i=1,\alpha=1}^{i=p,\alpha=q} \frac{1+\lambda_i z}{1+\mu_\alpha z} =$$

$$\sum_{r=0}^{p} \sum_{s=0}^{\infty} \left(\prod_{j_1<j_2<\ldots<j_r} \lambda_{j_1} \ldots \lambda_{j_r} \right) z^r \left(\prod_{\beta_1 \leq \beta_2 \leq \ldots \leq \beta_s} (-1)^s \mu_{\beta_1} \mu_{\beta_2} \ldots \mu_{\beta_s} \right) z^s . \tag{6}$$

Consider the basis $\{\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_r} \wedge \mathbf{f}_{\beta_1} \wedge \ldots \wedge \mathbf{f}_{\beta_s}\}$ $(1 \leq j_1 < j_2 < \ldots < j_k \leq p, 1 \leq \beta_1 \leq \ldots \leq \beta_s \leq q,\ r+s=k)$ in the exterior power $\wedge^k V$. Then $\lambda_{j_1} \ldots \lambda_{j_r} \mu_{\beta_1} \ldots \mu_{\beta_s}$ is the eigenvalue corresponding to the basis vector $\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_r} \wedge \mathbf{f}_{\beta_1} \wedge \ldots \wedge \mathbf{f}_{\beta_s}$. Hence in the same way as above the coefficients $c_k(A)$ of the expansion of the characteristic function at zero give traces of the exterior powers:

$$R_A(z) = \mathrm{Ber}\,(1 + Az) = \sum_{k=0}^{\infty} c_k(A) z^k, \text{ where } c_k(A) = \mathrm{Tr}\, \wedge^k A \ (k = 0, 1, 2, \ldots) . \tag{7}$$

Relations 2 between $c_k(A)$ and $s_k(A) = \mathrm{Tr}\, A^k$ remain the same as in the ordinary case because of 5. The essential difference is that now $R_A(z)$ is a fraction, not a polynomial as in 1; there are infinitely many terms $c_k(A)$ in the power expansion 7. Consider now the expansion of the characteristic function $R_A(z)$ at infinity. It leads to traces of the exterior powers of the inverse matrix. Indeed, $\mathrm{Ber}\,(1 + Az) = z^{p-q} \mathrm{Ber}\, A \cdot \mathrm{Ber}\,(1 + A^{-1} z^{-1})$. From 7 it follows that

$$R_A(z) = z^{p-q} \mathrm{Ber}\, A \cdot \mathrm{Ber}\,(1 + A^{-1} z^{-1}) = z^{p-q} \mathrm{Ber}\, A \sum_{k=0}^{\infty} c_k(A^{-1}) z^{-k} =$$

$$\sum_{k \leq p-q} \left(\mathrm{Ber}\, A \cdot c_{p-q-k}(A^{-1}) \right) z^k = \sum_{k \leq p-q} c_k^*(A) z^k \tag{8}$$

near infinity, where we have denoted by

$$c_k^*(A) = \mathrm{Ber}\, A \cdot c_{p-q-k}(A^{-1}) = \mathrm{Ber}\, A \cdot \mathrm{Tr}\, \wedge^{p-q-k} A^{-1}, \ (k = p-q,\ p-q-1, \ldots) . \tag{9}$$

The coefficient $c_k^*(A)$ can be interpreted as the trace of the representation on the space $\mathrm{Ber}\, V \otimes \wedge^{p-q-k} V^*$. In the ordinary case when V is an n-dimensional vector space so that $p = n, q = 0$, then both 7 and 8 are the same polynomial. Comparing them, we see that

$$c_k(A) = \mathrm{Tr}\, \wedge^k A = c_k^*(A) = \det A \cdot \mathrm{Tr}\, \wedge^{n-k} A^{-1} . \tag{10}$$

This is a well-known identity between minors of the matrix A and its inverse A^{-1}. In particular, for $k = n$ we arrive at 3. Relation 10 holds for any invertible operator A. This is due to a canonical isomorphism existing in the ordinary case between the spaces $\wedge^k V$ and $\det V \otimes \wedge^{n-k} V^*$:

$$\wedge^k V \approx \det V \otimes \wedge^{n-k} V^* . \tag{11}$$

What happens in the supercase? Both expansions 7 and 8 are infinite series. Claim: the coefficients of both series form **recurrent sequences**. Indeed, we see from 6 that the function $R_A(z)$ is the ratio of two polynomials of degrees p and q, respectively:

$$R_A(z) = \operatorname{Ber}(1 + Az) = \frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \ldots + a_p z^p}{1 + b_1 z + b_2 z^2 + \ldots + b_q z^q} \,. \tag{12}$$

Comparing this fraction with the expansion of $R_A(z)$ around zero we arrive at the recurrence relations

$$c_{k+q} + b_1 c_{k+q-1} + \ldots + b_q c_k = 0 \tag{13}$$

satisfied for all $k > p - q$. Comparing the fraction in 12 with the expansion of $R_A(z)$ around infinity we again arrive at recurrence relations:

$$c^*_k + b_1 c^*_{k-1} + \ldots + b_q c^*_{k-q} = 0$$

satisfied for all $k < 0$. We see that both sequences $\{c_k(A)\}$ and $\{c^*_k(A)\}$ satisfy the same recurrence relations of order q. It is convenient to consider these sequences for all integer k by setting $c_k = 0$ for all $k < 0$ and $c^*_k = 0$ for all $k > p-q$. Combine these two sequences in one sequence by considering the differences:

$$\gamma_k = c_k - c^*_k \,.$$

The sequence $\{\gamma_k\}$ satisfies the same recurrence relations **for all integer** k:

$$\gamma_k + b_1 \gamma_{k-1} + \ldots + b_q \gamma_{k-q} = 0, \quad \text{for all } k \,.$$

Note that in this formula the terms $c_k = \operatorname{Tr} \wedge^k A$ and $c^*_k = \operatorname{Ber} A \cdot \operatorname{Tr}^{p-q-k} A^{-1}$ are simultaneously non-zero only in a finite range where $k = 0, 1, \ldots, p - q$. Otherwise $\gamma_k = c_k - c^*_k$ equals either $c_k(A)$ for $k > p - q$ or $-c^*_k$ for $k < 0$. The condition that $\{\gamma_k\}$ is a recurrent sequence of order q can be rewritten in the following closed form:

$$\det \begin{pmatrix} \gamma_k & \cdots & \gamma_{k+q} \\ \cdots & \cdots & \cdots \\ \gamma_{k+q} & \cdots & \gamma_{k+2q} \end{pmatrix} = 0 \quad \text{for all } k \in \mathbb{Z} \,. \tag{14}$$

Using relations 13 for c_k only, one can reconstruct the function $R_A(z)$ and all rational invariants of the matrix A, including $\operatorname{Ber} A$, via the first $p + q$ traces $c_k = \operatorname{Tr} \wedge^k A$ ($k = 1, 2, \ldots, p + q$), by a recursive procedure. However, equation 14 for the differences $\gamma_k = c_k - c^*_k$ gives much more. Formula 14 stands in the supercase instead of the equality 10 holding in the ordinary case. This leads to highly non-trivial relations between exterior powers $\wedge^k V$ and $\operatorname{Ber} V \otimes \wedge^{p-q-k} V^*$ instead of the canonical isomorphism 11. Formula 14 also gives a closed expression for $\operatorname{Ber} A$ in terms of traces. Indeed, it follows from 7–9 that

$$\operatorname{Ber} A = \operatorname{Tr} \wedge^{p-q} A - \gamma_{p-q} \,. \tag{15}$$

(In the ordinary case $q = 0$, $\gamma_{p-q} = 0$ we arrive at 3.) Now, by considering relation 15 and identity 14 for $k = p - q$ we arrive at the formula

$$\operatorname{Ber} A = \frac{\det \begin{pmatrix} c_{p-q}(A) & \ldots & c_p(A) \\ \cdots & \cdots & \cdots \\ c_p(A) & \ldots & c_{p+q}(A) \end{pmatrix}}{\det \begin{pmatrix} c_{p-q+2}(A) & \ldots & c_{p+1}(A) \\ \cdots & \cdots & \cdots \\ c_{p+1}(A) & \ldots & c_{p+q}(A) \end{pmatrix}} \,. \tag{16}$$

Here as before we set $c_k = 0$ for $k < 0$. For example, let A be an even operator in a $p|1$-dimensional vector space. Then

$$\mathrm{Ber}\, A = \frac{\det \begin{pmatrix} c_{p-1}(A) & c_p(A) \\ c_p(A) & c_{p+1}(A) \end{pmatrix}}{c_{p+1}(A)} = c_{p-1}(A) - \frac{c_p^2(A)}{c_{p+1}(A)}\,.$$

The rational expression in 16 is essentially different from the original formula 4, where the numerator and denominator are not invariant functions of the matrix A. Compared to it, the numerator and denominator of the fraction in formula 16 are invariant polynomials. One can show that these invariant polynomials are the traces of the representations corresponding to certain Young diagrams. Namely, the numerator in 16 is equal to the trace of the action of the operator A on an invariant subspace in the space of tensors $V^{\otimes N}$ corresponding to the rectangular Young diagram $D_{p,q+1}$ with p rows of length $q+1$. Respectively, the denominator in 16 is equal to the trace of the action of the operator A on an invariant subspace corresponding to the Young diagram $D_{p+1,q}$ [1]. This follows from the well-known *Schur–Weyl formula* [5] which can be generalised to the supercase (see e.g. in [2]) Denote the invariant polynomials in the numerator and denominator of the fraction in 16 by $\mathrm{Ber}^+(A)$ and $\mathrm{Ber}^-(A)$, respectively. What is the meaning of $\mathrm{Ber}^+(A), \mathrm{Ber}^-(A)$ in terms of the eigenvalues of the operator A? Compare 16 with expression 12 for the characteristic function $R_A(z)$. Let $\{\lambda_1, \dots, \lambda_p\}$ and $\{\mu_1, \dots, \mu_q\}$ be the eigenvalues of the even operator A as above. Then consider the top coefficients a_p, b_q of the polynomials $P(z), Q(z)$ in 12. It follows that $a_p = \prod \lambda_i, b_q = \prod \mu_\alpha$, and $\mathrm{Ber}\, A = \frac{\prod \lambda_i}{\prod \mu_\alpha}$. Hence $\mathrm{Ber}^+(A) = R \cdot a_p$, $\mathrm{Ber}^-(A) = R \cdot b_q$, with a certain coefficient R. (Note that a_p and b_q are not polynomials in the matrix entries of A.) One can explicitly find a_p , b_q by solving straightforwardly a system of simultaneous equations corresponding to the linear recurrence relations 13. In particular, these calculations give

$$R = \det \begin{pmatrix} c_{p-q+1}(A) & \dots & c_p(A) \\ \dots & \dots & \dots \\ c_p(A) & \dots & c_{p+q-1}(A) \end{pmatrix}\,.$$

By considering 14 one can come to an important observation that $R = \prod_{i,\alpha}(\lambda_i - \mu_a)$. Up to a sign it is just the classical Sylvester's resultant for the polynomials P and Q standing at the top and bottom of the characteristic function $R_A(z)$ in 12. Thus for the invariant polynomials $\mathrm{Ber}^+(A), \mathrm{Ber}^-(A)$ we have:

$$\mathrm{Ber}^+(A) = \prod_i \lambda_i \prod_{i,\alpha}(\lambda_i - \mu_\alpha)\,, \quad \mathrm{Ber}^-(A) = \prod_\alpha \mu_\alpha \prod_{i,\alpha}(\lambda_i - \mu_\alpha)\,.$$

The polynomials $\mathrm{Ber}^+(A), \mathrm{Ber}^-(A)$ appear in the analog of the Cayley–Hamilton theorem for the supercase. In particular, the polynomial

$$\mathcal{P}_A(z) = \mathrm{Ber}^+(A - z)\mathrm{Ber}^-(A - z) \cdot \frac{1}{R}$$

is the minimal annihilating polynomial for a generic even matrix A, and its coefficients are polynomial invariants of A.

[1]If A is an ordinary $p \times p$ matrix, then by 3, $\det A = c_p(A)$ simply equals the trace $\mathrm{Tr} \wedge^p A$ on the one-dimensional space of totally antisymmetric p-tensors, corresponding to the Young diagram $D_{p,1}$ with p rows of length 1.

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Superfield Approach to Exact and Unique Nilpotent Symmetries

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Abstract

In the framework of *usual* superfield approach, we derive the exact local, covariant, continuous and off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for the $U(1)$ gauge field (A_μ) and the (anti-)ghost fields ($(\bar{C})C$) of the Lagrangian density of a four $(3+1)$-dimensional QED by exploiting the horizontality condition defined on the six $(4,2)$-dimensional supermanifold. The long-standing problem of the exact derivation of the above nilpotent symmetry transformations for the matter (Dirac) fields $(\bar{\psi}, \psi)$, in the framework of superfield formulation, is resolved by a new restriction on the $(4,2)$-dimensional supermanifold. This new gauge invariant restriction on the supermanifold, due to the *augmented* superfield formalism, owes its origin to the (super) covariant derivatives. The geometrical interpretations for all the above off-shell nilpotent transformations are provided in the framework of *augmented* superfield formalism.

The current year 2005 has been declared as the "world year of physics" to mark the 100th anniversary of the epoch-making discoveries made by Einstein in his miraculous year 1905. The year 2005 has also been a landmark year for the researchers, working in the realm of Becchi-Rouet-Stora-Tyutin (BRST) formalism, because it has celebrated the 30th birth anniversary of the discovery of BRST symmetries in the context of gauge theories [1,2]. This formalism, during its three decades of existence, has found applications in some of the frontier areas of research like topological field theories [3,4] and string field theories [5].

The key ideas of the BRST formalism have deep connections with the mathematics of differential geometry and (theoretical) physics of gauge theories as well as supersymmetries. One of its intuitive connections is with supersymmetry through the *usual* superfield formulation [6] which provides the geometrical interpretations for the nilpotent ($Q^2_{(a)b} = 0$) and anticommuting ($Q_b Q_{ab} + Q_{ab} Q_b = 0$) (anti-)BRST charges ($Q_{(a)b}$) in a beautiful manner. There exist, however, some long-standing problems in this domain of research which have defied their resolutions during the last 25 years. In our presentation, we shall touch upon one such long-standing problem (connected with the superfield approach to BRST formalism) and provide its resolution by exploiting the importance of *gauge invariance*.

Under the usual superfield approach [6], a D-dimensional Abelian gauge theory (endowed with the first-class constraints in the language of Dirac's prescription [7,8]) is considered on a $(D,2)$-dimensional supermanifold parameterized by D-number of spacetime (even) co-ordinates x^μ ($\mu = 0, 1, 2, 3....D-1$) and a couple of (odd) Grassmannian variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). In general, the $(p+1)$-form super curvature $\tilde{F}^{(p+1)} = \tilde{d}\tilde{A}^{(p)}$, constructed from the super exterior derivative $\tilde{d}$ (with $\tilde{d}^2 = 0$) and the super p-form connection $\tilde{A}^{(p)}$ (corresponding to a p-form ($p = 1, 2...$) Abelian gauge theory) is restricted to be flat along the Grassmannian directions of the $(D, 2)$-dimensional supermanifold due to the so-called horizontality condition *. Mathematically, this condition implies $\tilde{F}^{(p+1)} = F^{(p+1)}$ where $F^{(p+1)} = dA^{(p)}$ is the $(p+1)$-form curvature defined on the ordinary D-dimensional manifold through the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and ordinary p-form Abelian connection $A^{(p)} = dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p} \; A_{\mu_1 \mu_2 .... \mu_p}$.

The above horizontality condition on the six $(4, 2)$-dimensional supermanifold leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge- and (anti-)ghost fields of the (anti-)BRST invariant Lagrangian density of a given four $(3+1)$-dimensional (4D) 1- and 2-form (non-)Abelian gauge theories [6]. However, it does not shed any light on the nilpotent (anti-)BRST symmetry transformations that are associated with the matter (Dirac) fields of the interacting 1-form (non-)Abelian gauge theories where there is a coupling between the gauge field and the matter conserved current, constructed by the Dirac fields. This issue (i.e. the derivation of the nilpotent transformations for matter fields) has been a long-standing problem in the superfield approach to BRST formalism.

In a recent set of papers † [10-13], the usual superfield formalism has been consistently extended by invoking the additional restrictions on the six $(4, 2)$-dimensional supermanifold that are complimentary to the horizontality condition [6]. These additional restrictions on the supermanifold are the equality of (i) the conserved (super) matter current [10,11] (as well as other conserved quantities [11]), and (ii) the gauge invariant quantities owing their origin to the (super) covariant derivatives on the (super) matter fields [12,13].

The former set of restrictions [10,11] lead to the consistent derivation of the nilpotent symmetry transformations for the matter fields. On the other hand, the latter restrictions [12,13] lead to the exact and unique derivation of the nilpotent symmetry transformations for the matter fields. We christen these extended versions of the usual superfield approach to BRST formalism as the augmented superfield formalism. Both types of extensions have their own merits and advantages. Any further (consistent) extension of the usual superfield approach would be a welcome sign for the future of this area of research.

In our presentation, we *first* focus on the strength of the horizontality condition in the exact and unique derivation of the nilpotent symmetry transformations for the gauge and (anti-)ghost fields of a 4D interacting $U(1)$ gauge theory with the Dirac fields. This interacting Abelian system has been taken into consideration *only* for the sake of simplicity. The ideas, proposed in our presentation, can be generalized to a non-Abelian interacting gauge theory in a straightforward manner. Second, we concentrate on the consistent derivation of the nilpotent transformations for the matter (Dirac) fields by exploiting the equality of the conserved matter (super) current on the six $(4, 2)$-dimensional supermanifold. Finally, we obtain the exact and unique nilpotent symmetry transformations for the Dirac fields by exploiting the equality of the gauge invariant quantity on the above supermanifold that owes its origin to the (super) covariant derivatives on the (super) Dirac fields.

*Nakanishi and Ojima call it the "soul-flatness" condition [9]. For the 1-form non-Abelian gauge theory, $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ and $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$ in the horizontality condition $\tilde{F}^{(2)} = F^{(2)}$ [6].

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Let us begin with the (anti-)BRST invariant Lagrangian density $\mathcal{L}_b$ for the *interacting* four $(3+1)$-dimensional $U(1)$ gauge theory in the Feynman gauge [14]

$$\mathcal{L}_b = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi + B\, (\partial \cdot A) + \frac{1}{2}\, B^2 - i\, \partial_\mu \bar{C} \partial^\mu C, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric field strength tensor for the $U(1)$ Abelian gauge theory that is derived from the 2-form $dA^{(1)} = \frac{1}{2}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ [‡]. As is evident, the latter is constructed by the application of the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) on the 1-form $A^{(1)} = dx^\mu A_\mu$ which defines the Abelian vector potential A_μ. The gauge-fixing term $(\partial \cdot A)$ is derived through the operation of the co-exterior derivative δ (with $\delta = - * d *, \delta^2 = 0$) on the one-form $A^{(1)}$ (i.e. $\delta A^{(1)} = - * d * A = (\partial \cdot A)$) where $*$ is the Hodge duality operation. The fermionic Dirac fields $(\psi, \bar{\psi})$, with the mass m and charge e, couple to the $U(1)$ gauge field A_μ (i.e. $-e\bar{\psi}\gamma^\mu A_\mu \psi$) through the conserved current $J_\mu = \bar{\psi}\gamma_\mu \psi$. The anticommuting ($C\bar{C} + \bar{C}C = 0, C^2 = \bar{C}^2 = 0, C\psi + \psi C = 0$ etc.) (anti-)ghost fields $(\bar{C})C$ are required to maintain the unitarity and "quantum" gauge (i.e. BRST) invariance together at any arbitrary order of perturbation theory for a given physical process [§]. The Nakanishi-Lautrup auxiliary field B is required to linearize the quadratic gauge-fixing term $-\frac{1}{2}(\partial \cdot A)^2$, present in the Lagrangian density (1), in a subtle way.

The above Lagrangian density (1) respects the following off-shell nilpotent ($s_{(a)b}^2 = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST ($s_{(a)b}$) [¶] symmetry transformations [14]

$$\begin{array}{lll} s_b A_\mu &=& \partial_\mu C, \qquad s_b C = 0, \qquad s_b \bar{C} = iB, \qquad s_b \psi = -ieC\psi, \\ s_b \bar{\psi} &=& -ie\bar{\psi}C, \qquad s_b B = 0, \quad s_b F_{\mu\nu} = 0, \quad s_b(\partial \cdot A) = \Box C, \\ s_{ab} A_\mu &=& \partial_\mu \bar{C}, \qquad s_{ab} \bar{C} = 0, \qquad s_{ab} C = -iB, \qquad s_{ab} \psi = -ie\bar{C}\psi, \\ s_{ab} \bar{\psi} &=& -ie\bar{\psi}\bar{C}, \qquad s_{ab} B = 0, \quad s_{ab} F_{\mu\nu} = 0, \quad s_{ab}(\partial \cdot A) = \Box \bar{C}. \end{array} \tag{2}$$

The noteworthy points, at this stage, are (i) under the nilpotent (anti-)BRST transformations, it is the kinetic energy term (more precisely $F_{\mu\nu}$ itself) that remains invariant. (ii) The electric and magnetic fields E_i and B_i (that are components of $F_{\mu\nu}$) owe their origin to the operation of cohomological operator d on the one-form $A^{(1)}$. (iii) The symmetry transformations in (2) are generated by the local, conserved and nilpotent charges $Q_{(a)b}$. This statement, for the local generic field $\Sigma(x)$, can be succinctly expressed as

$$s_r\, \Sigma(x) = -i\, [\, \Sigma(x), Q_r\,]_\pm, \qquad\qquad r = b, ab, \tag{3}$$

[‡]We adopt here the conventions and notations such that the 4D flat Minkowski metric is: $\eta_{\mu\nu} =$ diag $(+1, -1, -1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_i)^2, F_{0i} = E_i = \partial_0 A_i - \partial_i A_0 = F^{i0}, F_{ij} = \epsilon_{ijk} B_k, B_i = (1/2)\epsilon_{ijk} F_{jk}, D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$ where ϵ_{ijk} is the 3D totally antisymmetric Levi-Civita tensor and electric and magnetic fields are E_i and B_i, respectively. In equation (1), γ's are the usual 4×4 Dirac matrices. Furthermore, the Greek indices: $\mu, \nu, \rho ... = 0, 1, 2, 3$ in (1), correspond to the spacetime directions and Latin indices $i, j, k ... = 1, 2, 3$ stand *only* for the space directions on the 4D spacetime manifold.

[§]The full strength of the (anti-)ghost fields turns up in the discussion of the unitarity and "quantum" gauge (i.e. BRST) invariance for the perturbative computations in the realm of non-Abelian gauge theory where, for each loop diagram of the gauge (gluon) fields corresponding to a physical process, a loop diagram consisting of *only* the (anti-)ghost fields is required to exist as its counterpart (see, e.g., [15] for details).

[¶]We adopt here the notations and conventions followed in [14]. In fact, in its full glory, a nilpotent ($\delta_B^2 = 0$) BRST transformation δ_B is equivalent to the product of an anticommuting ($\eta C = -C\eta, \eta\bar{C} = -\bar{C}\eta, \eta\psi = -\psi\eta, \eta\bar{\psi} = -\bar{\psi}\eta$ etc.) spacetime independent parameter η and s_b (i.e. $\delta_B = \eta\, s_b$) where $s_b^2 = 0$.

where $\Sigma(x) = A_\mu(x), C(x), \bar{C}(x), \psi(x), \bar{\psi}(x), B(x)$ and the $(+)-$ signs, as the subscripts on the square bracket, correspond to the (anti-)commutators for the generic local field $\Sigma(x)$ (of the Lagrangian density (1)) being (fermionic)bosonic in nature.

To derive the above anticommuting and nilpotent transformations $s_{(a)b}$ for the bosonic $U(1)$ gauge field A_μ and the fermionic (anti-)ghost fields $(\bar{C})C$, we exploit the *usual* superfield formalism, endowed with the horizontality restriction on a six $(4,2)$-dimensional supermanifold. This supermanifold is parametrized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ where x^μ $(\mu = 0, 1, 2, 3)$ are a set of four even (bosonic) spacetime coordinates and fermionic θ and $\bar{\theta}$ are a set of two odd (Grassmannian) coordinates. One can define a super 1-form $\tilde{A}^{(1)} = dZ^M \tilde{A}_M$ where the supervector superfield $\tilde{A}_M$ (with $\tilde{A}_M = (B_\mu(x,\theta,\bar{\theta}),\ \mathcal{F}(x,\theta,\bar{\theta}),\ \bar{\mathcal{F}}(x,\theta,\bar{\theta}))$ has the component multiplet superfields $B_\mu, \mathcal{F}, \bar{\mathcal{F}}$. These component superfields can be expanded in terms of the basic fields $(A_\mu, C, \bar{C})$, auxiliary field (B) of the Lagrangian density (1) and some extra secondary fields, as [6]

$$\begin{array}{rcl} B_\mu(x,\theta,\bar{\theta}) &=& A_\mu(x) + \theta\, \bar{R}_\mu(x) + \bar{\theta}\, R_\mu(x) + i\, \theta\, \bar{\theta} S_\mu(x), \\ \mathcal{F}(x,\theta,\bar{\theta}) &=& C(x) + i\, \theta \bar{B}(x) + i\, \bar{\theta}\, \mathcal{B}(x) + i\, \theta\, \bar{\theta}\, s(x), \\ \bar{\mathcal{F}}(x,\theta,\bar{\theta}) &=& \bar{C}(x) + i\, \theta\, \bar{\mathcal{B}}(x) + i\, \bar{\theta}\, B(x) + i\, \theta\, \bar{\theta}\, \bar{s}(x). \end{array} \tag{4}$$

It is straightforward to note that the local fields $R_\mu(x), \bar{R}_\mu(x), C(x), \bar{C}(x), s(x), \bar{s}(x)$ are fermionic (anticommuting) in nature and their number matches with the bosonic (commuting) local fields $A_\mu(x), S_\mu(x), \mathcal{B}(x), \bar{\mathcal{B}}(x), B(x), \bar{B}(x)$ in (4).

All the secondary fields will be expressed in terms of basic fields $(A_\mu, C, \bar{C})$ and the auxiliary field (B) due to the restrictions emerging from the application of horizontality condition. The explicit forms of $\tilde{F}^{(2)}$ and $F^{(2)}$, in the horizontality restriction, are:

$$\tilde{F}^{(2)} = F^{(2)}, \quad \tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} = \tfrac{1}{2}(dZ^M \wedge dZ^N)\tilde{F}_{MN}, \quad F^{(2)} = dA^{(1)} = \tfrac{1}{2}(dx^\mu \wedge dx^\nu)F_{\mu\nu}. \tag{5}$$

The super exterior derivative $\tilde{d}$ and the connection super one-form $\tilde{A}^{(1)}$, in (5), are

$$\begin{array}{rcl} \tilde{d} &=& dZ^M\, \partial_M = dx^\mu\, \partial_\mu \ +\ d\theta\, \partial_\theta \ +\ d\bar{\theta}\, \partial_{\bar{\theta}}, \\ \tilde{A}^{(1)} &=& dZ^M\, \tilde{A}_M = dx^\mu\, B_\mu(x,\theta,\bar{\theta}) + d\theta\, \bar{\mathcal{F}}(x,\theta,\bar{\theta}) + d\bar{\theta}\, \mathcal{F}(x,\theta,\bar{\theta}). \end{array} \tag{6}$$

Mathematically, the above condition (5) implies the "flatness" of all the components of the (anti-)symmetric super curvature tensor $\tilde{F}_{MN}$ that are directed along the θ and/or $\bar{\theta}$ directions of the supermanifold. Ultimately, the soul-flatness (horizontality) condition $(\tilde{d}\tilde{A}^{(1)} = dA^{(1)})$ of the equation (5) (with $\tilde{F}^{(2)} = F^{(2)}$), yields ‖

$$\begin{array}{rclll} R_\mu\,(x) &=& \partial_\mu\, C(x), & \bar{R}_\mu\,(x) = \partial_\mu\, \bar{C}(x), & s\,(x) = \bar{s}\,(x) = 0, \\ S_\mu\,(x) &=& \partial_\mu B\,(x) & B\,(x) + \bar{B}\,(x) = 0, & \mathcal{B}\,(x) = \bar{\mathcal{B}}(x) = 0. \end{array} \tag{7}$$

The insertion of all the above values in expansion (4) leads to the derivation of (anti-)BRST symmetries for the gauge- and (anti-)ghost fields of the theory as **

$$\begin{array}{rcl} B^{(h)}_\mu\,(x,\theta,\bar{\theta}) &=& A_\mu(x) +\ \theta\,(s_{ab}A_\mu(x)) +\ \bar{\theta}\,(s_b A_\mu(x)) +\ \theta\,\bar{\theta}\,(s_b s_{ab} A_\mu(x)), \\ \mathcal{F}^{(h)}\,(x,\theta,\bar{\theta}) &=& C(x)\ +\ \theta\,(s_{ab}C(x))\ +\ \bar{\theta}\,(s_b C(x))\ +\ \theta\,\bar{\theta}\,(s_b\, s_{ab} C(x)), \\ \bar{\mathcal{F}}^{(h)}\,(x,\theta,\bar{\theta}) &=& \bar{C}(x)\ +\ \theta\,(s_{ab}\bar{C}(x))\ +\bar{\theta}\,(s_b\bar{C}(x))\ +\ \theta\,\bar{\theta}\,(s_b\, s_{ab}\bar{C}(x)). \end{array} \tag{8}$$

‖In the explicit computation of $\tilde{d}\tilde{A}^{(1)}$, we have taken into account $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, dx^\mu \wedge d\theta = -d\theta \wedge dx^\mu, d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta$, etc., that emerge from the requirement of the nilpotency of $\tilde{d}$ (i.e. $\tilde{d}^2 = 0$).

For the non-Abelian gauge theory where $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$, the off-shell nilpotent symmetry transformations for the gauge (i.e. $s_b A_\mu = D_\mu C$) and (anti-)ghost fields (with $s_b C = \frac{1}{2} C \times C$, etc.) were found in a beautiful paper by **Bonora and Tonin with exactly the same kind of expansion as given in (8) (see, [6] for details). The horizontality condition $(\tilde{F}^{(2)} = F^{(2)})$ plays an important role in this case, too.

The above exercise provides the physical interpretation for the (anti-)BRST charges $Q_{(a)b}$ as simply the generators (cf. (3)) of translations (i.e. $\mathrm{Lim}_{\bar\theta\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar\theta)$) along the Grassmannian directions of the supermanifold. It is obvious that now $\tilde{d}\tilde{A}^{(1)}_{(h)} = dA^{(1)}$, where $\tilde{A}^{(1)}_{(h)} = dx^\mu B^{(h)}_\mu + d\theta \bar{\mathcal{F}}^{(h)} + d\bar\theta \mathcal{F}^{(h)}$ is the modified version of the 1-form super connection $\tilde{A}^{(1)}$ (cf. (6)) after the application of the horizontality (soul-flatness) condition.

We now derive the nilpotent symmetry transformations for the matter (Dirac) fields $(\psi, \bar\psi)$ due to the invariance of the conserved matter current of the theory on the supermanifold. We start off with the super expansion of the superfields $(\Psi, \bar\Psi)(x,\theta,\bar\theta))$, corresponding to the ordinary Dirac fields $(\psi,\bar\psi)(x)$ of the Lagrangian density (1), as [10,12]

$$\begin{array}{lcl} \Psi(x,\theta,\bar\theta) &=& \psi(x) + i\,\theta\,\bar{b}_1(x) + i\,\bar\theta\,b_2(x) + i\,\theta\,\bar\theta\,f(x), \\ \bar\Psi(x,\theta,\bar\theta) &=& \bar\psi(x) + i\,\theta\,\bar{b}_2(x) + i\,\bar\theta\,b_1(x) + i\,\theta\,\bar\theta\,\bar{f}(x). \end{array} \tag{9}$$

In the limit $(\theta,\bar\theta)\to 0$, from the above expansions, we get back the usual Dirac fields $(\psi,\bar\psi)$ (of the Lagrangian density (1)) and the number of bosonic fields $(b_1, \bar{b}_1, b_2, \bar{b}_2)$ match with the fermionic fields $(\psi, \bar\psi, f, \bar{f})$ for the consistency with supersymmetry.

We construct the supercurrent $\tilde{J}_\mu(x,\theta,\bar\theta)$ with the following general super expansion

$$\tilde{J}_\mu(x,\theta,\bar\theta) = \bar\Psi(x,\theta,\bar\theta)\,\gamma_\mu\,\Psi(x,\theta,\bar\theta) = J_\mu(x) + \theta\,\bar{K}_\mu(x) + \bar\theta\,K_\mu(x) + i\,\theta\,\bar\theta\,L_\mu(x), \tag{10}$$

where the above components (i.e. $\bar{K}_\mu, K_\mu, L_\mu, J_\mu$), can be expressed in terms of the components of the basic super expansions (9), as (see, e.g., [10])

$$\begin{array}{l} \bar{K}_\mu(x) = i(\bar{b}_2\gamma_\mu\psi - \bar\psi\gamma_\mu\bar{b}_1), \qquad K_\mu(x) = i(b_1\gamma_\mu\psi - \bar\psi\gamma_\mu b_2), \\ L_\mu(x) = \bar{f}\gamma_\mu\psi + \bar\psi\gamma_\mu f + i(\bar{b}_2\gamma_\mu b_2 - b_1\gamma_\mu\bar{b}_1), \qquad J_\mu(x) = \bar\psi\gamma_\mu\psi. \end{array} \tag{11}$$

To be consistent with our earlier observation that the (anti-)BRST transformations ($s_{(a)b}$) are equivalent to the translations along the $(\theta)\bar\theta$-directions of the supermanifold, it is straightforward to re-express the expansion in (10) as

$$\tilde{J}_\mu(x,\theta,\bar\theta) = J_\mu(x) + \theta\,(s_{ab}J_\mu(x)) + \bar\theta\,(s_b J_\mu(x)) + \theta\,\bar\theta\,(s_b s_{ab} J_\mu(x)). \tag{12}$$

It can be checked explicitly that, under the (anti-)BRST transformations (2), the conserved current $J_\mu(x)$ remains invariant (i.e. $s_b J_\mu(x) = s_{ab} J_\mu(x) = 0$). Thus, from (11), we have

$$b_1\gamma_\mu\psi = \bar\psi\gamma_\mu b_2, \qquad \bar{b}_2\gamma_\mu\psi = \bar\psi\gamma_\mu\bar{b}_1, \qquad \bar{f}\gamma_\mu\psi + \bar\psi\gamma_\mu f = i(b_1\gamma_\mu\bar{b}_1 - \bar{b}_2\gamma_\mu b_2), \tag{13}$$

as the conditions for $s_{(a)b}J_\mu = 0$. This, ultimately, implies: $K_\mu = L_\mu = \bar{K}_\mu = 0$ in (10).

One of the possible solutions to the above restrictions, present in (13), is [10]

$$\begin{array}{l} b_1 = -e\bar\psi C, \qquad b_2 = -eC\psi, \qquad \bar{b}_1 = -e\bar{C}\psi, \qquad \bar{b}_2 = -e\bar\psi\bar{C}, \\ f = -ie\,[\,B + e\bar{C}C\,]\,\psi, \qquad \bar{f} = +ie\,\bar\psi\,[\,B + eC\bar{C}\,]. \end{array} \tag{14}$$

It is evident that the above expressions are consistent but *not* uniquely determined by the restriction $\tilde{J}_\mu(x,\theta,\bar\theta) = J_\mu(x)$ on the supermanifold. However, it should be emphasized that, barring the constant factors, the above solutions are very logical. For instance, for the validity of $b_1\gamma_\mu\psi = \bar\psi\gamma_\mu b_2$, the pair of bosonic fields b_1 and b_2 should be proportional to the fermionic fields $\bar\psi$ and ψ, respectively. The corresponding equality can be achieved, *only* by bringing in, the (anti-)ghost fields of the theory. There is *no* other possible choice.

Thus, we judiciously choose $b_1 \sim \bar{\psi} C$ and $b_2 \sim C\psi$. Rest of the consistent choices of (14) are made on similar lines of argument with appropriate constants i and e thrown in.

The stage is now set for the exact derivation of (14). To this end in mind, we begin with the following *gauge invariant* restriction on the supermanifold [12]

$$\bar{\Psi}(x,\theta,\bar{\theta})\ (\tilde{d} + ie\tilde{A}^{(1)}_{(h)})\ \Psi(x,\theta,\bar{\theta}) = \bar{\psi}(x)\ (d + ieA^{(1)})\ \psi(x), \tag{15}$$

where the superfields Ψ and $\bar{\Psi}$ are from (9). The r.h.s. of the above equation, expressed in terms of the differential dx^μ (as $dx^\mu \bar{\psi}(\partial_\mu + ieA_\mu)\psi$), is obviously a $U(1)$ gauge invariant quantity. The l.h.s. of the above equation yields the coefficients of the differentials $dx^\mu, d\theta$ and $d\bar{\theta}$. The analogues of the latter two, as is evident from (15), do not exist on the r.h.s.

It is straightforward to note that the coefficients of $d\theta$, collected from the l.h.s., should be set equal to zero. This requirement leads to the following two independent relationships

$$-i\ \bar{\psi}\ (\bar{b}_1 + e\bar{C}\psi) = 0, \qquad \bar{\psi}\ (if + e\bar{C}b_2 - eB\psi) = 0. \tag{16}$$

Similarly, the coefficients of $d\bar{\theta}$ equal to zero, implies the following relationships [12]

$$-i\ \bar{\psi}\ (b_2 + eC\psi) = 0, \qquad \bar{\psi}\ (-if + eC\bar{b}_1 + eB\psi) = 0. \tag{17}$$

Together, the above two equations, lead to the following results (for $\bar{\psi} \neq 0$)

$$\bar{b}_1 = -e\ \bar{C}\ \psi, \qquad b_2 = -e\ C\ \psi, \qquad f = -ie\ (B + e\bar{C}C)\ \psi. \tag{18}$$

In fact, out of *exactly* four relations, only *two* in (16) and (17), are independent [12].

We shall focus now on the collection of the coefficients of $dx^\mu, dx^\mu(\theta), dx^\mu(\bar{\theta})$ and $dx^\mu(\theta\bar{\theta})$. The coefficient of the "pure" dx^μ match from the l.h.s. and r.h.s. Exploiting the inputs from (18), we set equal to zero the coefficient of $dx^\mu(\theta)$ and $dx^\mu(\bar{\theta})$. These imply

$$i\ [\ \bar{b}_2 + e\ \bar{\psi}\bar{C}\ ]\ [\ D_\mu\psi\ ] = 0, \qquad i\ [\ b_1 + e\ \bar{\psi}C\ ]\ [\ D_\mu\psi\ ] = 0. \tag{19}$$

The above conditions lead to the exact determination of b_1 and $\bar{b}_2$ as: $b_1 = -e\bar{\psi}C, \bar{b}_2 = -e\bar{\psi}\bar{C}$. Here, it will be noted that $D_\mu\psi \neq 0$ for the QED with Dirac fields. Finally, we collect the coefficients of $dx^\mu(\theta\bar{\theta})$ and set them equal to zero. This condition implies [12]

$$[\ i\bar{f} + e\bar{\psi}\ (B + eC\bar{C})\ ]\ [\ D_\mu\psi\ ] = 0, \tag{20}$$

where we have exploited the inputs from (18) and have inserted the values of b_1 and $\bar{b}_2$ that were obtained earlier. It is obvious that, for $D_\mu\psi \neq 0$, we obtain the exact value of $\bar{f}$ as: $\bar{f} = ie[B + eC\bar{C}]\bar{\psi}$. Thus, from the restriction (15), we obtain *exactly* all the values of (14). Insertions of the values of (14) into (9) leads to the following (see, [12] for details)

$$\begin{aligned} \Psi\ (x,\theta,\bar{\theta}) &= \psi(x)\ +\ \theta\ (s_{ab}\psi(x))\ +\ \bar{\theta}\ (s_b\psi(x))\ +\ \theta\ \bar{\theta}\ (s_b\ s_{ab}\psi(x)), \\ \bar{\Psi}\ (x,\theta,\bar{\theta}) &= \bar{\psi}(x)\ +\ \theta\ (s_{ab}\bar{\psi}(x))\ +\bar{\theta}\ (s_b\bar{\psi}(x))\ +\ \theta\ \bar{\theta}\ (s_b\ s_{ab}\bar{\psi}(x)). \end{aligned} \tag{21}$$

This establishes the fact that the nilpotent (anti-)BRST charges $Q_{(a)b}$ are the translations generators $(\mathrm{Lim}_{\bar{\theta}\to 0}(\partial/\partial\theta))$ $\mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta})$ along the $(\theta)\bar{\theta}$ directions of the supermanifold.

To summarize, the geometrical interpretations for (i) the (anti-)BRST transformations $s_{(a)b}$ and their corresponding generators $Q_{(a)b}$, (ii) the nilpotency property of $s_{(a)b}$ and $Q_{(a)b}$, and (iii) the anticommutativity property of $s_{(a)b}$ and $Q_{(a)b}$, for *all* the fields of

QED with Dirac fields, emerge in the framework of augmented superfield formalism. Mathematically, these can be expressed, in an explicit manner, as illustrated below

$$
\begin{aligned}
& s_b \;\Leftrightarrow\; Q_b \;\Leftrightarrow\; \mathrm{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar\theta}, \qquad s_{ab} \;\Leftrightarrow\; Q_{ab} \;\Leftrightarrow\; \mathrm{Lim}_{\bar\theta\to 0}\frac{\partial}{\partial\theta}, \\
& s^2_{(a)b} = 0 \;\;\Leftrightarrow\;\; Q^2_{(a)b} = 0 \;\;\Leftrightarrow\;\; \Big(\frac{\partial}{\partial\theta}\Big)^2 = 0,\; \Big(\frac{\partial}{\partial\bar\theta}\Big)^2 = 0, \\
& s_b s_{ab} + s_{ab} s_b = 0 \;\Leftrightarrow\; Q_b Q_{ab} + Q_{ab} Q_b = 0 \;\Leftrightarrow\; \frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta} = 0.
\end{aligned}
\tag{22}
$$

The *exact* nilpotent (anti-)BRST symmetries for the matter (Dirac) fields are obtained from the gauge *invariant* restriction (15) on the supermanifold which is different in nature than the gauge *covariant* restriction of the horizontality condition (5) (see, [13] for details).

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The Mini-Superambitwistor Space

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Abstract

We present the construction of the mini-superambitwistor space, which is suited for establishing a Penrose-Ward transform between certain bundles over this space and solutions to the $\mathcal{N}=6$ super Yang-Mills equations in three dimensions.

The essential point underlying twistor string theory [1] is the marriage of Calabi-Yau and twistor geometry in the space $\mathbb{C}P^{3|4}$. This complex projective space is a Calabi-Yau supermanifold and simultaneously the supertwistor space of the complexified, compactified Minkowski space. The interest in a twistorial string theory is related to the fact that twistor geometry allows for a very convenient description of the solution spaces of classical gauge theories [2, 3]. In such a description, spacetime is associated with a certain complex manifold, its *twistor space*. Subsequently, the space of holomorphic vector bundles over this twistor space is mapped to the solution space of the gauge theory via a so-called *Penrose-Ward transform*.

Suitable twistor spaces are well-known for four-dimensional self-dual Yang-Mills theory and its supersymmetric extensions, as well as for four-dimensional Yang-Mills theory and its $\mathcal{N}=3$ supersymmetric extension, see [4, 5, 1, 6]. Via dimensional reduction, one obtains so-called *mini-twistor spaces* upon which a description of the solution space to the Bogomolny monopole equation [7] and its supersymmetric extensions can be constructed, see e.g. [8, 9]. In this collection, the mini-twistor space suited for a Penrose-Ward transform yielding solutions to the three-dimensional Yang-Mills-Higgs equations and their $\mathcal{N}=6$ supersymmetric extension is evidently missing. This gap was filled in [10], and here we will concisely report on the results and thus review the construction of the *mini-superambitwistor space*.

Let us first recall that twistors were invented by Penrose to give a unified description of General Relativity and quantum mechanics. Consider a light ray, which is given by the set of points $x^{\alpha\dot\alpha}$ satisfying the equation $x^{\alpha\dot\alpha}=x_0^{\alpha\dot\alpha}+tp^{\alpha\dot\alpha}$. Here, $x_0^{\alpha\dot\alpha}$ is an arbitrary point on the light ray and $t\in\mathbb{R}$ is a parameter. Taking a light ray which does not pass through the origin, one can obviously choose $x_0^{\alpha\dot\alpha}$ to be null. Since one can decompose every null vector into a pair of commuting two-spinors, we can rewrite the equation defining our light ray as $x^{\alpha\dot\alpha}=c\omega^\alpha\tilde\omega^{\dot\alpha}+t\lambda^{\dot\alpha}\tilde\lambda^\alpha$. Multiplication of this equation by $\lambda_{\dot\alpha}$ together with the right choice of normalization $c=(\tilde\omega^{\dot\alpha}\lambda_{\dot\alpha})^{-1}$ gives rise to the *incidence relation*

$$\omega^\alpha \;=\; x^{\alpha\dot\alpha}\lambda_{\dot\alpha}\;. \tag{1}$$

A *twistor* Z^i is now a projective[1] pair of two-spinors $Z^i=(\omega^\alpha,\lambda_{\dot\alpha})\in\mathbb{C}P^3$, which trans-

[1] the incidence relation is invariant under scaling

forms under coordinate shifts $x^{\alpha\dot\alpha} \to x^{\alpha\dot\alpha} + r^{\alpha\dot\alpha}$ according to the incidence relation.

The space $\mathbb{C}P^3$ is the twistor space of the complexified, compactified Minkowski space S^4_c. Taking out the sphere $S^2 \cong \mathbb{C}P^1 = \{(\omega^\alpha \neq 0, \lambda_{\dot\alpha} = 0)\}$ which is incident to $x^{\alpha\dot\alpha} = \infty$, we can consider $\lambda_{\dot\alpha}$ as the homogeneous coordinates on the Riemann sphere $\mathbb{C}P^1$. Due to the incidence relation, the ω^αs are homogeneous polynomials of degree one in $\lambda_{\dot\alpha}$ and thus describe a section of the rank two complex vector bundle $\mathcal{O}(1)\oplus\mathcal{O}(1) \to \mathbb{C}P^1$. This bundle's total space, which we denote by $\mathcal{P}^3$, is the twistor space of $\mathbb{C}^4$.

The incidence relation allows us furthermore to establish the double fibration

$$
\begin{tikzcd}
 & \mathbb{C}^4 \times \mathbb{C}P^1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
\mathcal{P}^3 & & \mathbb{C}^4
\end{tikzcd}
\tag{2}
$$

where $\pi_2(x^{\alpha\dot\alpha}, \lambda_{\dot\alpha}) = (x^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \lambda_{\dot\alpha})$ and $\pi_1(x^{\alpha\dot\alpha}, \lambda_{\dot\alpha}) = (x^{\alpha\dot\alpha})$, from which one can easily read off the following twistor correspondence: A point $x^{\alpha\dot\alpha} \in \mathbb{C}^4$ defines a sphere $S^2 \cong \mathbb{C}P^1_x$ embedded in $\mathcal{P}^3$, while a point $p \in \mathcal{P}^3$ is incident to a null two-plane $\mathbb{C}^2_p$ in $\mathbb{C}^4$.

To obtain an $\mathcal{N}$-extended supertwistor space, one can simply start from the projective superspace $\mathbb{C}P^{3|\mathcal{N}}$, take out the $\mathbb{C}P^{1|\mathcal{N}}$ corresponding to the point at infinity and arrive at the supervector bundle

$$\mathcal{P}^{3|\mathcal{N}} \;:=\; \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^{\mathcal{N}} \otimes \Pi\mathcal{O}(1) \tag{3}$$

over the Riemann sphere $\mathbb{C}P^1$. The operator Π simply inverts the parity of the fibre coordinates of a vector bundle, and one has therefore $\mathcal{N}$ additional homogeneous Graßmann coordinates $\eta_1, \ldots, \eta_{\mathcal{N}}$. The incidence relation (1) is extended to

$$\omega^\alpha \;=\; x^{\alpha\dot\alpha}\lambda_{\dot\alpha} \quad \text{and} \quad \eta_i \;=\; \eta_i^{\dot\alpha}\lambda_{\dot\alpha}\ , \tag{4}$$

which naturally gives rise to the double fibration

$$
\begin{tikzcd}
 & \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
\mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}}
\end{tikzcd}
\tag{5}
$$

Here, π_2 is given by the extended incidence relations (4) and π_1 is the obvious projection.

In the special case $\mathcal{N} = 4$, the first Chern number of (the tangent bundle of) the supertwistor space vanishes. This is due to the fact that Berezin integration is equivalent to a differentiation and therefore the contribution of $\Pi\mathcal{O}(1)$ to the total first Chern number is -1. Altogether, we have a contribution of 2 from the tangent bundle to the Riemann sphere $TS^2 \cong \mathcal{O}(2)$ and 1 from each bosonic $\mathcal{O}(1)$, which is cancelled by the -4 of the fermionic line bundles $\Pi\mathcal{O}(1)$. Thus, $\mathcal{P}^{3|4}$ comes with a holomorphic measure $\Omega^{3,0|4,0}$ and this supertwistor space is a Calabi-Yau supermanifold.

One can now establish a relation between a topological string theory having $\mathcal{P}^{3|4}$ as its target space and $\mathcal{N} = 4$ self-dual Yang-Mills theory in four dimensions: The open topological B-model on $\mathcal{P}^{3|4}$ with a stack of n space-filling D5-branes is equivalent to holomorphic Chern-Simons theory on the same space, which describes the dynamics of a $\mathrm{gl}(n,\mathbb{C})$-valued connection $(0,1)$-form $\mathcal{A}^{0,1}$ on a rank n complex vector bundle $\mathcal{E} \to \mathcal{P}^{3|4}$ [1]. The action of this holomorphic Chern-Simons theory reads as [1]

$$S \;=\; \int \Omega^{3,0|4,0} \wedge \operatorname{tr}\left(\mathcal{A}^{0,1} \wedge \bar\partial\mathcal{A}^{0,1} + \frac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right)\ , \tag{6}$$

where $\Omega^{3,0|4,0}$ is the holomorphic measure on $\mathcal{P}^{3|4}$. (Some minor assumptions about the explicit form of $\mathcal{A}^{0,1}$ have to be made at this point, see [6].) The corresponding equations of motion are given by $\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0$ and their solutions describe holomorphic structures which promote the complex vector bundle $\mathcal{E}$ to holomorphic vector bundles $(\mathcal{E}, \mathcal{A}^{0,1})$. Via a generalized Penrose-Ward transform using supertwistor spaces [1, 6], one can map these holomorphic vector bundles to solutions to the $\mathcal{N} = 4$ extended SDYM equations on $\mathbb{C}^4$. These equations are the supersymmetric extensions of the self-dual Yang-Mills equations $F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, which read in spinorial notation $F_{\mu\nu} \to F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta}f_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}$ as

$$\begin{aligned}
f_{\dot{\alpha}\dot{\beta}} &= 0 \ , \\
\nabla_{\alpha\dot{\alpha}}\chi^{\alpha i} &= 0 \ , \\
\Box\phi^{[ij]} &= +\tfrac{1}{2}\{\chi^{\alpha i}, \chi^{j}_{\alpha}\} \ , \\
\varepsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\alpha\dot{\alpha}}\tilde{\chi}^{[ijk]}_{\dot{\gamma}} &= -2[\phi^{[ij}, \chi^{k]}_{\alpha}] \ , \\
\varepsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\alpha\dot{\alpha}}G^{[ijkl]}_{\dot{\gamma}\dot{\delta}} &= -\{\chi^{[i}_{\alpha}, \tilde{\chi}^{jkl]}_{\dot{\delta}}\} + [\phi^{[ij}, \nabla_{\alpha\dot{\delta}}\phi^{kl]}] \ ,
\end{aligned} \tag{7}$$

where the nontrivial fields $(f_{\alpha\beta}, \chi^{i}_{\alpha}, \phi^{[ij]}, \tilde{\chi}^{[ijk]}_{\dot{\alpha}}, G^{[ijkl]}_{\dot{\alpha}\dot{\beta}})$ have helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$. Neglecting the trivial field $f_{\dot{\alpha}\dot{\beta}}$, the field content of $\mathcal{N} = 4$ self-dual Yang-Mills theory is identical to the one of $\mathcal{N} = 4$ super Yang-Mills theory, but the interactions in the two theories are different.

Let us now turn our attention to another twistor space, the so-called *superambitwistor space*, upon which a Penrose-Ward transform for the full $\mathcal{N} = 3$ super Yang-Mills equations can be constructed. Important here is the observation that the $\mathcal{N} = 3$ supermultiplet is reducible and splits into a self-dual and an anti-self-dual part. One is thus naturally led to glue together the twistor space $\mathcal{P}^{3|3}$ for $\mathcal{N} = 3$ self-dual Yang-Mills theory with a dual copy[2] $\mathcal{P}^{3|3}_{*}$ for the anti-self-dual part. Denoting the homogeneous coordinates on these two spaces by $(\omega^{\alpha}, \lambda_{\dot{\alpha}}, \eta_i)$ and $(\omega^{\dot{\alpha}}_{*}, \lambda^{*}_{\alpha}, \eta^{i}_{*})$, we can write the appropriate gluing condition as the quadric equation

$$\kappa \ := \ \omega^{\alpha}\lambda^{*}_{\alpha} - \omega^{\dot{\alpha}}_{*}\lambda_{\dot{\alpha}} + 2\eta^{i}_{*}\eta_i \ = \ 0 \ , \tag{8}$$

which defines the superambitwistor space $\mathcal{L}^{5|6}$ as a subset of $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_{*}$.

To examine the geometry of $\mathcal{L}^{5|6}$ more closely, note that the appropriate incidence relations for the space $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_{*}$ read as

$$\omega^{\alpha} \ = \ x^{\alpha\dot{\alpha}}_{R}\lambda_{\dot{\alpha}} \ , \quad \eta_i \ = \ \eta^{\dot{\alpha}}_{i}\lambda_{\dot{\alpha}} \ , \quad \omega^{\dot{\alpha}}_{*} \ = \ x^{\alpha\dot{\alpha}}_{L}\lambda^{*}_{\alpha} \ , \quad \eta^{i}_{*} \ = \ \eta^{i\alpha}_{*}\lambda^{*}_{\alpha} \ . \tag{9}$$

The quadric condition (8) is automatically and most generally solved, if we choose

$$x^{\alpha\dot{\alpha}}_{R} \ = \ x^{\alpha\dot{\alpha}} - \eta^{\alpha i}_{*}\eta^{\dot{\alpha}}_{i} \quad \text{and} \quad x^{\alpha\dot{\alpha}}_{L} \ = \ x^{\alpha\dot{\alpha}} + \eta^{\alpha i}_{*}\eta^{\dot{\alpha}}_{i} \ , \tag{10}$$

and thus $x^{\alpha\dot{\alpha}}_{R}$ and $x^{\alpha\dot{\alpha}}_{L}$ are indeed right- and left-handed chiral coordinates on the chiral superspaces $\mathbb{C}^{4|6}_{R}$ and $\mathbb{C}^{4|6}_{L}$. These incidence relations, too, define a double fibration:

$$\begin{tikzcd}
 & \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_{*} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
\mathcal{L}^{5|6} & & \mathbb{C}^{4|12}
\end{tikzcd} \tag{11}$$

[2]The word dual here refers to the spinor indices and *not* to the line bundles underlying $\mathcal{P}^{3|3}$.

Over the superambitwistor space, one can then establish a Penrose-Ward transform which is a map between solutions to the $\mathcal{N}=3$ super Yang-Mills equations and certain holomorphic vector bundles over $\mathcal{L}^{5|6}$, see e.g. [6].

One can also find twistor spaces which describe self-dual Yang-Mills theory after a dimensional reduction $\mathbb{C}^4 \to \mathbb{C}^3$. In our conventions, one can make the following identification of vector fields[3]

$$\mathcal{T}_3 \ :=\ \frac{\partial}{\partial x^3} \ =\ -\frac{\partial}{\partial x^{1\dot{2}}}+\frac{\partial}{\partial x^{2\dot{1}}} \sim \frac{\partial}{\partial x^{[2\dot{1}]}}\ . \tag{12}$$

Dimensional reduction of the x^3-direction thus implies eliminating the modulus $x^{[\alpha\dot{\alpha}]}$. On the twistor space side, this can be done by changing the incidence relation on $\mathcal{P}^3 = \mathcal{O}(1)\oplus\mathcal{O}(1)$ from $\omega^\alpha = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}$ to $\upsilon = x^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\alpha}}\lambda_{\dot{\beta}}$. The latter equation defines sections of the line bundle $\mathcal{P}^2 := \mathcal{O}(2)$ over the Riemann sphere $\mathbb{C}P^1$. More formally, one has $(\mathcal{O}(1)\oplus\mathcal{O}(1))/\mathcal{G} = \mathcal{O}(2)$, where $\mathcal{G}$ is the abelian group generated by the holomorphic vector field on $\mathcal{P}^3$ which corresponds to $\mathcal{T}_3$ [9]. The space $\mathcal{P}^2$ is called the *mini-twistor space* [7], and the corresponding double fibration reads as

$$\begin{array}{ccc} & \mathbb{C}^3\times\mathbb{C}P^1 & \\ {}^{\nu_2}\swarrow & & \searrow^{\nu_1} \\ \mathcal{P}^2 & & \mathbb{C}^3 \end{array} \tag{13}$$

After applying this reduction to the space $\mathcal{P}^{3|3}\times\mathcal{P}^{3|3}_*$, we arrive at the space $\mathcal{P}^{2|3}\times\mathcal{P}^{2|3}_*$ together with the incidence relations

$$\upsilon \ =\ y^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\alpha}}\lambda_{\dot{\beta}}\ ,\quad \eta_i \ =\ \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}\ ,\quad \upsilon_* \ =\ y_*^{\dot{\alpha}\dot{\beta}}\lambda^*_{\dot{\alpha}}\lambda^*_{\dot{\beta}}\ ,\quad \eta^i_* \ =\ \eta_*^{i\dot{\alpha}}\lambda^*_{\dot{\alpha}}\ . \tag{14}$$

Here, we adjusted the spinor indices anticipating that there is no distinction between left- and right-handed spinors on $\mathbb{C}^3$. The quadric equation (8) is correspondingly reduced to the equation

$$\left(\upsilon-\upsilon_*+2\eta_*^i\eta_i\right)\big|_{\lambda=\lambda_*} \ =\ 0\ , \tag{15}$$

and this condition defines the *mini-superambitwistor space* as a subset of $\mathcal{P}^{2|3}\times\mathcal{P}^{2|3}$ [10]. Altogether, we have the dimensional reductions

$$\begin{array}{ccc} \mathcal{P}^{3|3}\times\mathcal{P}^{3|3}_* & \rightarrow & \mathcal{P}^{2|3}\times\mathcal{P}^{2|3}_* \\ \downarrow & & \downarrow \\ \mathcal{L}^{5|6} & \rightarrow & \mathcal{L}^{4|6} \\ \downarrow & & \downarrow \\ \mathbb{C}P^1\times\mathbb{C}P^1_* & & \mathbb{C}P^1\times\mathbb{C}P^1_* \end{array} \tag{16}$$

The reduced quadric equation (15) is solved, if we choose the "chiral coordinates"

$$y^{\dot{\alpha}\dot{\beta}} \ =\ y_0^{\dot{\alpha}\dot{\beta}}-\eta_*^{i(\dot{\alpha}}\eta_i^{\dot{\beta})} \quad\text{and}\quad y_*^{\dot{\alpha}\dot{\beta}} \ =\ y_0^{\dot{\alpha}\dot{\beta}}+\eta_*^{i(\dot{\alpha}}\eta_i^{\dot{\beta})} \tag{17}$$

in the incidence relations.

[3] See also [9], where the explicit identification is slightly different.

The incidence relations (14) determine together with the reduced quadric equation (15) yielding (17) the dimensional reduction of the double fibration (11) to be

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_* \arrow[dl, "\pi_2"'] \arrow[dd] \arrow[dr, "\pi_1"] & \\
\mathcal{L}^{5|6} \arrow[dd] & & \mathbb{C}^{4|12} \arrow[dd] \\
 & \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_* \arrow[dl, "\nu_2"'] \arrow[dr, "\nu_1"] & \\
\mathcal{L}^{4|6} & & \mathbb{C}^{3|12}
\end{tikzcd} \tag{18}$$

Although all the constructions seem to go through without difficulties, the geometry of $\mathcal{L}^{4|6}$ contains some surprises. First of all, note that the reduced quadric condition (15) is not imposed over the whole base $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ of the supervector bundle $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$, but only over its diagonal $\Delta := \mathrm{diag}(\mathbb{C}P^1 \times \mathbb{C}P^1_*)$, which is the subspace of the base for which $\lambda = \lambda_*$. Considering the projection $\pi : \mathcal{L}^{4|6} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$, we see that $\pi^{-1}(\lambda, \lambda_*) \cong \mathbb{C}^2$ for $\lambda \neq \lambda_*$, but $\pi^{-1}(\lambda, \lambda_*) \cong \mathbb{C}$ on the diagonal Δ. One can in fact show that $\mathcal{L}^{4|6}$ is a fibration [10], but since its fibre dimension varies, it is evidently not a vector bundle. However, we will see in the following, that this seemingly unpleasant property does not impose any relevant obstructions.

First, let us find an interpretation of the geometries involved in the double fibration for the mini-superambitwistor space which is contained in (18). Recall that for the well-known double fibrations (2), (5) and (11), there is a nice interpretation in terms of flag manifolds [3]. In the case of the double fibrations for the mini-supertwistor and mini-superambitwistor spaces, we find a quite similar description. For simplicity, we restrict our discussion to the bosonic subspaces, i.e. to the bodies of the considered superspaces.

After imposing reality conditions [9] on the spaces involved in the double fibration (13), we obtain

$$\begin{tikzcd}
 & \mathbb{R}^3 \times S^2 \arrow[dl, "\nu_2"'] \arrow[dr, "\nu_1"] & \\
\mathcal{P}^2_r & & \mathbb{R}^3
\end{tikzcd} \tag{19}$$

The space $\mathbb{R}^3 \times S^2$ on the top is the space of oriented lines in $\mathbb{R}^3$ with one marked point. Keeping the point and dropping the line evidently leads to an element of the space $\mathbb{R}^3$, while keeping the line and dropping the point – or, alternatively, moving the point as close as possible to the origin – leads to an element of $\mathcal{P}^2_r = \mathcal{O}(2) \cong TS^2$. The projections ν_1 and ν_2 in (19) have therefore a clear geometric meaning.

The real double fibration for the bosonic part of the mini-superambitwistor space,

$$\begin{tikzcd}
 & \mathbb{R}^3 \times S^2 \times S^2_* \arrow[dl, "\nu_2"'] \arrow[dr, "\nu_1"] & \\
\mathcal{L}^4_r & & \mathbb{R}^3
\end{tikzcd} \tag{20}$$

has a similar interpretation. The space $\mathbb{R}^3 \times S^2 \times S^2_*$ is the space of two oriented lines in $\mathbb{R}^3$ with a common marked point. Dropping the lines leads again to elements of $\mathbb{R}^3$, while moving the point on one of the lines (together with the attached second line) to its shortest distance to the origin yields an element of $\mathcal{L}^4_r$.

Ultimately, one is certainly interested in extending the discussion to the level of topological strings. Recall that the superambitwistor space is in fact a (local) Calabi-Yau supermanifold [1], and one can therefore use $\mathcal{L}^{5|6}$ as a target space for the topological B-model. Although the mini-superambitwistor space $\mathcal{L}^{4|6}$ is not a manifold, one nevertheless finds that a certain Calabi-Yau property still persists.

A Calabi-Yau manifold can be defined as a manifold whose tangent bundle has vanishing first Chern class. Chern classes of vector bundles in turn are related to certain degeneracy loci of a set of generic sections: On a rank n vector bundle, the first Chern class is Poincaré dual to the degeneracy loci of n generic sections. Straightforward calculations show, that the appropriate degeneracy loci for $\mathcal{L}^{5|6}$ and $\mathcal{L}^{4|6}$ are rationally equivalent [10]. This is a strong hint that $\mathcal{L}^{4|6}$ comes with the necessary properties for using this space as target space for a topological B-model. Furthermore, if the conjecture [11] by Aganagic, Neitzke and Vafa is correct and $\mathcal{L}^{5|6}$ is indeed the mirror symmetry partner of $\mathcal{P}^{3|4}$ then by applying dimensional reduction, it is only natural to conjecture that the mini-superambitwistor space $\mathcal{L}^{4|6}$ is the mirror of the mini-supertwistor space $\mathcal{P}^{2|4}$.

As far as the Penrose-Ward transform is concerned, the discussion over $\mathcal{L}^{4|6}$ follows essentially the lines of the discussion over $\mathcal{L}^{5|6}$. Since the mini-superambitwistor space is only a fibration and not a manifold, we have to slightly extend the notion of a vector bundle. We define an $\mathcal{L}^{4|6}$*-bundle of rank n* by a Čech 1-cocycle $\{f_{ab}\} \in \check{Z}^1(\mathcal{L}^{4|6}, \mathfrak{S})$, where $\mathfrak{S}$ is the sheaf of smooth $\mathrm{GL}(n, \mathbb{C})$-valued functions on $\mathcal{L}^{4|6}$. This 1-cocycle takes over the rôle of a transition function in an ordinary vector bundle. A *holomorphic* $\mathcal{L}^{4|6}$*-bundle* is correspondingly defined by a holomorphic such Čech 1-cocycle. We call two $\mathcal{L}^{4|6}$-bundles given by two 1-cocylces $\{f_{ab}\}$ and $\{f'_{ab}\}$ *topologically equivalent*, if there is a Čech 0-cochain $\{\psi_a\} \in \check{C}^0(\mathcal{L}^{4|6}, \mathfrak{S})$ such that $f_{ab} = \psi_a^{-1} f'_{ab} \psi_b$. In particular, an $\mathcal{L}^{4|6}$-bundle is topologically trivial (topologically equivalent to the trivial bundle), if its defining 1-cocycle can be decomposed by a Čech 0-cochain according to $f_{ab} = \psi_a^{-1} \psi_b$.

With these definitions, we can state that topologically trivial, holomorphic $\mathcal{L}^{4|6}$-bundles, which become holomorphically trivial vector bundles upon restriction to any $\mathbb{C}P^1 \times \mathbb{C}P^1_* \subset \mathcal{L}^{4|6}$ are equivalent to solutions of the $\mathcal{N} = 6$ super Yang-Mills equations on $\mathbb{C}^3$. The number of supersymmetries doubled in the dimensional reduction process, as the complex supercharges in four dimensions are converted into two real supercharges in three dimensions.

Recall at this point that the $\mathcal{N} = 3$ and $\mathcal{N} = 4$ super Yang-Mills equations are the same and only the field content differs by an additional reality condition [4] in the case $\mathcal{N} = 4$; this condition renders the fourth supersymmetry linear. The $\mathcal{N} = 6$ and $\mathcal{N} = 8$ super Yang-Mills equations in three dimensions are identical in the same sense.

There is a further Penrose-Ward transform for ordinary Yang-Mills theory in four dimensions, which can also be translated to a Yang-Mills-Higgs theory in three dimensions. Here, one considers holomorphic vector bundles over a third-order thickening of[4] $\mathcal{L}^5 \subset \mathcal{P}^3 \times \mathcal{P}^3_*$. That is, instead of demanding that κ in (8) vanishes, we only impose the condition that $\kappa^3 \sim 0$ and arrive at an infinitesimal neighborhood of $\mathcal{L}^5$ in $\mathcal{P}^3 \times \mathcal{P}^3_*$. For a recent review on such complex manifolds which have additional even nilpotent directions, see e.g. [12]. The Penrose-Ward transform, which can then be established [4, 5] maps holomorphic vector bundles over the third-order thickening of $\mathcal{L}^5 \subset \mathcal{P}^3 \times \mathcal{P}^3$ to solutions to the ordinary Yang-Mills equations on $\mathbb{C}^4$.

The corresponding *mini-ambitwistor space* $\mathcal{L}^4$ is obtained by simply dropping all fermionic coordinates of $\mathcal{L}^{4|6}$. For the Penrose-Ward transform, one has in fact to consider

[4]All the spaces in the following are derived from the corresponding superspaces by putting their fermionic coordinates to zero.

a "subthickening", i.e. the space $\mathcal{L}^4 \subset \mathcal{P}^2 \times \mathcal{P}^2_*$ with the formal third-order neighborhood of the diagonal Δ in the base $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ of the fibration $\mathcal{L}^4$ [10]. We can then establish a Penrose-Ward transform, which states that holomorphic $\mathcal{L}^4$-bundles over a third order subthickening of $\mathcal{L}^4$ which become holomorphically trivial vector bundles upon restriction to any $\mathbb{C}P^1 \times \mathbb{C}P^1_* \subset \mathcal{L}^4$ are in a one-to-one correspondence with solutions to the Yang-Mills-Higgs equations on $\mathbb{C}^3$ up to $\mathcal{L}^4$-bundle equivalence and gauge equivalence relations.

Summing up, we can state that there are twistor spaces for both Yang-Mills-Higgs theory and $\mathcal{N} = 6$ super Yang-Mills theory in three dimensions. Although these spaces are fibrations but no manifolds, they come with nice properties, and the mini-superambitwistor space $\mathcal{L}^{4|6}$ possibly plays an important rôle as the mirror manifold of the mini-supertwistor space $\mathcal{P}^{2|4}$. For future work, it remains to define a topological B-model on the mini-superambitwistor space and to substantiate the above pronounced mirror conjecture.

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Harmonic Maps into Loop Spaces of Compact Lie Groups

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1 Introduction

In this paper we study harmonic maps from Riemann surfaces M to the loop spaces ΩG of compact Lie groups G. The motivation to study such maps comes from a result by Atiyah [1], asserting that the moduli space of G-instantons on $\mathbb{R}^4$ can be identified with the space of based holomorphic maps from the Riemann sphere $\mathbb{CP}^1$ into the loop space ΩG. Thus it's natural to conjecture that the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$ can be likewise identified with the space of based harmonic maps $\mathbb{CP}^1 \to \Omega G$.

For the construction of harmonic maps from Riemann surfaces to the loop spaces we use the twistor approach. Recall that the main idea of this approach is to construct for a given Riemannian manifold N the so called twistor bundle $\pi : Z \to N$, where Z is an almost complex manifold, which has the following property. For any almost holomorphic map $\psi : M \to Z$ of any Riemann surface M into the twistor space Z the projection $\pi \circ \psi : M \to N$ of this map to N is harmonic. One can say that the twistor approach allows to reduce the original "real" problem of description of harmonic maps from Riemann surfaces M to a given Riemannian manifold N to the "complex" problem of description of almost holomorphic maps from M to the almost complex manifold Z. A survey of the general theory of twistor spaces is given in [5] (cf. also [10]), here we are interested in a special case of the twistor construction for the Grassmann manifold $N = G_r(\mathbb{C}^d)$. In this case the role of the twistor bundle $\pi : Z \to N$ is played by the homogeneous flag bundles $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) \to G_r(\mathbb{C}^d)$ so that harmonic maps $\varphi : M \to G_r(\mathbb{C}^d)$ arise as projections of almost holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ with respect to a natural almost complex structure on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$.

In order to use the twistor approach for the description of harmonic maps $\varphi : M \to \Omega G$ into the loop spaces ΩG, we embed the loop space ΩG isometrically into an infinite-dimensional Grassmanian $Gr_{\mathrm{res}}(H)$, associated with a complex Hilbert space H. This Grassmanian is the disjoint union of Grassmanians $G_r(H)$ of virtual dimension r and for each Grassmanian $G_r(H)$ we can construct virtual flag bundles $\mathcal{F}_{\mathbf{r}}(H) \to G_r(H)$ by analogy with the finite-dimensional situation. Using these bundles, we can construct harmonic maps $\varphi : M \to \Omega G$ as projections of almost holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(H)$.

Briefly on the content of the paper. We start by recalling basic properties of harmonic maps of Riemannian manifolds in Sec.2. In Sec.3 we present a general idea of the twistor approach to the study of harmonic maps and in Sec.4 define flag bundles over the Grassmann manifolds. In Sec.5 we remind the Atiyah's correspondence between G-instantons on $\mathbb{R}^4$ and holomorphic spheres in ΩG, based on the Donaldson's description of G-instantons on $\mathbb{R}^4$ as holomorphic $G^{\mathbb{C}}$-bundles over the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ of two projective lines. In Sec.6 we introduce the Hilbert Grassmanian $\mathrm{Gr}_{\mathrm{res}}(H)$ and Grassmanians $G_r(H)$ of virtual dimension r. We construct also an isometric immersion of ΩG into $Gr_{\mathrm{res}}(H)$ by which harmonic maps $\varphi : M \to \Omega G$ can be considered as harmonic maps into Grassmanians $G_r(H)$. The latter harmonic maps may be constructed as projections of almost holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(H)$ to virtual flag manifolds $\mathcal{F}_{\mathbf{r}}(H)$.

2 Harmonic Maps. General Properties

Let $\varphi : (M, g) \to (N, h)$ be a smooth map of a Riemannian manifold M with a Riemannian metric g into a Riemannian manifold N with a Riemannian metric h. We define the *energy* of the map φ as the Dirichlet integral

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi(p)|^2 \mathrm{vol}_g \,. \tag{1}$$

The norm of the differential may be computed in local coordinates as follows. Denote by (x^i) local coordinates at $p \in M$ and by (u^α) local coordinates at $q = \varphi(p) \in N$. Then

$$|d\varphi(p)|^2 = \sum_{i,j} \sum_{\alpha,\beta} g^{ij} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} h_{\alpha\beta}$$

where $\varphi^\alpha = \varphi^\alpha(x)$ are the components of φ, $g = (g_{ij})$ and $h = (h_{\alpha\beta})$ are the metric tensors of M and N respectively, and $g^{-1} = (g^{ij})$ is the inverse matrix of (g_{ij}), vol_g is the volume element of the metric g.

Definition 1 *A smooth map $\varphi : M \to N$ is called* harmonic *if it is extremal for the functional $E(\varphi)$ with respect to all smooth variations of φ with compact supports.*

The Euler–Lagrange equation for the energy functional $E(\varphi)$ is called otherwise the *harmonic map equation.* In the local coordinates (x^i) on M and (u^α) on N, introduced above, it has the following form

$$\Delta_M \varphi^\gamma + \sum_{i,j} g^{ij} \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} = 0 \tag{2}$$

where Δ_M is the standard Laplace–Beltrami operator of M, given by

$$\Delta_M \varphi^\gamma = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x_k} \right\} .$$

Here, ${}^M\Gamma^k_{ij}$ denotes the Christoffel symbol of the Levi-Civita connection ${}^M\nabla$ of M and, respectively, ${}^N\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol of the Levi-Civita connection ${}^N\nabla$ of N.

In the particular case $N = \mathbb{R}^n$ the equation (2) becomes linear and reduces to the Laplace–Beltrami equation $\Delta_M \varphi^\gamma = 0$, $\gamma = 1, \ldots, n$, on the components of the map φ.

A non-trivial nonlinear example of harmonic maps is provided by the so called SO(3)-model, arising in the ferromagnet theory. In this example we consider smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with finite energy $E(\varphi) < \infty$. The finite energy condition implies that such maps should stabilize at infinity, i.e. $\varphi(x) \to \varphi_0$ for $|x| \to \infty$. Therefore, φ extends to a map $\varphi : S^2 = \mathbb{R}^2 \cup \infty \to S^2$ which has a topological invariant, called the *degree* of the map φ:

$$\deg \varphi = \int_{S^2} \varphi^* \mathrm{vol} \ .$$

Here, vol is the normalized volume form on S^2. To have better formulas, it's convenient to introduce complex coordinates. We denote by $z = x_1 + ix_2$ a complex coordinate on $\mathbb{R}^2$ and by w the complex coordinate in the image $S^2 \setminus \{\infty\}$, given by the stereographic projection.

Then the energy of a map $\varphi = w(z)$ in these coordinates will be given by the following formula

$$E(\varphi) = 2 \int\limits_{\mathbb{C}} \frac{|\partial_z w|^2 + |\partial_{\bar{z}}|^2}{(1+|w|^2)^2} \, |dz \wedge d\bar{z}| \ , \tag{3}$$

while the degree of φ is computed, according to

$$\deg \varphi = \frac{1}{2\pi} \int\limits_{\mathbb{C}} \frac{|\partial_z w|^2 - |\partial_{\bar{z}}|^2}{(1+|w|^2)^2} \, |dz \wedge d\bar{z}| \ , \tag{4}$$

Comparing the last two formulas, we obtain the estimate of the energy from below

$$E(\varphi) \geq 4\pi |\deg \varphi| \ .$$

It follows that the minimum of the energy $E(\varphi)$ for a fixed $k = \deg \varphi$ is attained on holomorphic functions $w = \varphi(z)$ for $k \geq 0$, and on antiholomorphic functions $w = \varphi(z)$ for $k \leq 0$.

Fixing the asymptotic value φ_0 by the SO(3)-invariance (we set $\varphi_0 = 1$), one can write down the minima of $E(\varphi)$ for $k \geq 0$ in the form

$$w = \varphi(z) = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j}$$

where a_j, b_j are arbitrary complex numbers. In particular, the space of minima for a fixed k is parametrized by $4k$ real parameters.

If we compare the harmonic map equation with the Yang–Mills duality equations on $\mathbb{R}^4$ then the holomorphic (respectively, antiholomorphic) maps $\varphi : \mathbb{R}^2 \cup \infty \to S^2$ will correspond to the instanton (respectively, anti-instanton) solutions of the duality equations. We shall see later that this correspondence can be established on a more deep level.

It may be shown that in the case of SO(3)-model the energy functional $E(\varphi)$ has no critical points except for the described minima. In other words, there are no other harmonic maps $\varphi : \mathbb{R}^2 \cup \infty \to S^2$ apart from the holomorphic and antiholomorphic ones. We note that holomorphic and antiholomorphic maps yield the local minima of the energy $E(\varphi)$ also for maps between general complex manifolds.

Namely, suppose that our Riemannian manifold (M, g) is provided with a complex (or almost complex) structure ${}^M J$, compatible with the Riemannian metric g, and, likewise, the target manifold (N, h) has a complex (or almost complex) structure ${}^N J$, compatible with the Riemannian metric h.

Definition 2 *A smooth map $\varphi : M \to N$ is called* (almost) holomorphic *iff the tangent map $\varphi_* : TM \to TN$ commutes with (almost) complex structures on M and N, i.e.*

$$\varphi_* \circ {}^M J = {}^N J \circ \varphi_* \,,$$

and it is called (almost)anti-holomorphic *iff φ_* anti-commutes with (almost) complex structures on M and N.*

Generalizing the phenomena, observed for the SO(3)-model, one may prove that for (almost) Kähler manifolds holomorphic and anti-holomorphic maps $\varphi : M \to N$ always realize the minima of the energy functional $E(\varphi)$ but in general there exist another critical points of $E(\varphi)$, i.e. non-minimal harmonic maps.

3 Twistor Approach

To describe harmonic maps $\varphi : M \to N$ from Riemann surfaces into Riemannian manifolds, we are going to use the twistor approach.

Let us recall the informal formulation of the so called *Penrose twistor program* in application to our problem:

Given a Riemannian manifold N, construct a *twistor bundle* $\pi : Z \to N$ with an almost complex *twistor space* Z, establishing a 1–1 correspondence between objects of Riemannian geometry on N and objects of holomorphic geometry on Z.

Thus, the twistor approach yields a method of studying the real geometry of Riemannian manifolds via the complex geometry of their twistor spaces.

The first construction of a twistor space Z of a Riemannian manifold N was given by Atiyah–Hitchin–Singer in [2] who proposed to take for Z the bundle of complex structures on N, compatible with the Riemannian metric h. This bundle has a natural almost complex structure $\mathcal{J}^1$ which we call briefly the *AHS-structure*. (We do not give the precise formulation of the AHS-construction here, referring for it to [2] and also to [5], since we shall give another construction of the twistor bundle in the homogeneous situation below.)

Consider now the twistor space $(Z, \mathcal{J}^1)$ of our manifold (N, h). If the Penrose program applies to our problem then we can expect that harmonic maps $\varphi : M \to N$ from an arbitrary Riemann surface M into our manifold N should arise as projections of almost holomorphic maps $\psi : M \to (Z, \mathcal{J}^1)$. However, it's not completely true. Projections of almost holomorphic maps $\psi : M \to (Z, \mathcal{J}^1)$ do satisfy some second-order differential equations on N but these are the *ultraharmonic equations*, i.e. harmonic equations with a wrong signature. So, if we want to construct harmonic maps by the twistor construction, we should replace the AHS-structure $\mathcal{J}^1$ along some tangent directions in Z by the opposite almost complex structure $-\mathcal{J}^1$. In this way we get an almost complex structure $\mathcal{J}^2$, defined by:

$$\mathcal{J}^2 = \begin{cases} -\mathcal{J}^1 & \text{along vertical } \pi - \text{directions} \,, \\ \mathcal{J}^1 & \text{along horizontal } \pi - \text{directions} \,. \end{cases}$$

This almost complex structure was introduced by Eells–Salamon in [7] and will be called briefly the *ES-structure*.

Now we can give a more formal definition of the twistor bundle:

Definition 3 *A smooth bundle $\pi : Z \to N$ with an almost complex manifold $(Z, \mathcal{J}^2)$ will be called the* twistor bundle *of a Riemannian manifold N iff the projection $\varphi := \pi \circ \psi$ of any almost holomorphic map $\psi : M \to Z$ of any Riemann surface M is a harmonic map $\varphi : M \to N$.*

Note that the almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$ on the twistor space Z are usually non-integrable. More precisely, *the AHS-structure $\mathcal{J}^1$ is integrable $\Leftrightarrow$ N is conformally flat, while the ES-structure $\mathcal{J}^2$ is never integrable.* This result looks dissapointing at the first glance since the non-integrable almost complex structures might be quite bizarre. For example, such a structure may have no holomorphic functions at all. But our advantage is that we are dealing not with holomotphic functions (i.e. holomorphic maps $f : Z \to \mathbb{C}$ from the twistor space Z) but with a dual object — holomorphic maps $\varphi : M \to Z$ from a Riemann surface M into our manifold Z. Such a map $\varphi : M \to Z$ is holomorphic with respect to the almost complex structure $\mathcal{J}^2$ on Z iff it satisfies a $\bar{\partial}_J$-equation on M, i.e. the Cauchy–Riemann equation with respect to the pulled-back almost complex structure $J := \varphi^*(\mathcal{J}^2)$ on M. This structure J is integrable on M (like any almost complex structure on a Riemann surface). So the integrability properties of the almost complex structure $\mathcal{J}^2$ on the twistor space Z are not essential for the problem, we are considering.

4 Harmonic Maps into Grassmann Manifolds

We turn now to some particular class of Riemannian manifolds, namely, to the Grassmann manifolds and give another (and quite explicit) construction of twistor bundles in this case. The role of twistor spaces over the Grassmann manifolds is played by the flag manifolds which we define now.

To define flag manifolds in $\mathbb{C}^d$, we fix a decomposition of d into the sum of natural numbers $d = r_1 + \ldots + r_n$ and denote $\mathbf{r} := (r_1, \ldots, r_n)$.

Definition 4 *A* flag manifold *$\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ of type $\mathbf{r}$ in $\mathbb{C}^d$ consists of collections $\mathcal{E} = (E_1, \ldots, E_n)$ of mutually orthogonal linear subspaces E_i of dimension r_i in $\mathbb{C}^d$ such that $\mathbb{C}^d = E_1 \oplus \ldots \oplus E_n$.*

In particular, for $\mathbf{r} = (r, d - r)$ the flag manifold

$$\mathcal{F}_{(r,d-r)}(\mathbb{C}^d) = \{\mathcal{E} = (E, E^{\perp}) : \dim E = r\} = G_r(\mathbb{C}^d)$$

coincides with the Grassmann manifold of r-dimensional subspaces in $\mathbb{C}^d$.

We have the following homogeneous space representation for the flag manifold

$$\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{U}(d)/\,\mathrm{U}(r_1) \times \ldots \times \mathrm{U}(r_n)\ .$$

There is also another, complex homogeneous space representation

$$\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{GL}(d, \mathbb{C})/\,\mathcal{P}_{\mathbf{r}}$$

where $\mathcal{P}_{\mathbf{r}}$ is the parabolic subgroup of blockwise upper-triangular matrices of type $\mathbf{r}$. These representations imply that $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ has a natural complex structure, which we denote again by $\mathcal{J}^1$, and is a compact Kähler manifold.

We shall construct now flag twistor bundles over the Grassmann manifold. These bundles are parametrized by ordered subsets σ in $\{1,\dots,n\}$. For such a subset σ and $\mathbf{r} := (r_1,\dots,r_n)$ we set $r := \sum_{i\in\sigma} r_i$ and associate with σ the following homogeneous bundle

$$\pi = \pi_\sigma\colon\ \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) = \frac{\mathrm{U}(d)}{\mathrm{U}(r_1)\times\ldots\times\mathrm{U}(r_n)} \longrightarrow \frac{\mathrm{U}(d)}{\mathrm{U}(r)\times\mathrm{U}(d-r)} = G_r(\mathbb{C}^d)$$

by assigning to a flag $\mathcal{E} = (E_1,\dots,E_n)$ the subspace $E := \bigoplus_{i\in\sigma} E_i$.

This homogeneous bundle generates a homogeneous decomposition of the complexified tangent bundle $T^{\mathbb{C}}\mathcal{F}$ into the direct sum of the vertical and horizontal subbundles. Using this decomposition, we can define a $\mathrm{U}(d)$-invariant almost complex structure $\mathcal{J}^2$ on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ from the original complex structure $\mathcal{J}^1$ on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ precisely, as in the previous Section.

Theorem 1 (Burstall–Salamon [4]). *The constructed homogeneous flag bundle π_σ: $(\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d),\mathcal{J}^2) \to G_r(\mathbb{C}^d)$ is a twistor bundle, i.e. for any $\mathcal{J}^2$-holomorphic map $\psi : M \to \mathcal{F}$ its projection $\varphi := \pi_\sigma\circ\psi : M \to G_r(\mathbb{C}^d)$ is a harmonic map.*

Moreover, in the case $M = \mathbb{CP}^1$ Burstall [3] has proved also the converse of this Theorem, namely: <u>any</u> harmonic map $\varphi : \mathbb{CP}^1 \to G_r(\mathbb{C}^d)$ can be obtained as the projection of a $\mathcal{J}^2$-holomorphic map $\psi : \mathbb{CP}^1 \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ with respect to some twistor bundle π_σ : $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) \to G_r(\mathbb{C}^d)$. So the problem of description of harmonic spheres in the Grassmann manifold $G_r(\mathbb{C}^d)$ reduces to the problem of description of $\mathcal{J}^2$-holomorphic spheres in flag manifolds $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$. The latter problem was solved by Wood in [11] (cf. also [4]) and, though the solution, given in [11], is rather technical, the guiding idea is, roughly, the following.

Any smooth map $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ generates at any point $p \in M$ an orthogonal decomposition

$$\mathbb{C}^d = E_1(p)\oplus\ldots\oplus E_n(p)\ ,$$

induced from $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ by ψ. In other words, we have an orthogonal decomposition of the trivial bundle $M\times\mathbb{C}^d$ into the sum of subbundles $E_1,\dots,E_n$. If the original map ψ was $\mathcal{J}^1$-holomorphic, i.e. holomorphic with respect to the canonical complex structure on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$, then the subbundles $E_1,\dots,E_n$ will be holomorphic with respect to the pulled-back complex structure $J_\psi := \psi^*(\mathcal{J}^1)$ on M. We want, starting from $\mathcal{J}^1$-holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$, to convert them into $\mathcal{J}^2$-holomorphic maps. From the definition of the almost complex structure $\mathcal{J}^2$ it's clear that, in order to do it, we should replace some of the holomorphic subbundles E_i by anti-holomorphic subbundles $\bar{E}_i$.

5 Loop Spaces and Yang–Mills Fields

We switch now to the infinite-dimensional target manifolds N, namely, we take for N the loop space ΩG of a compact Lie group G. At the end of this section we shall explain why this example is of special interest for us.

We denote by $LG = C^\infty(S^1, G)$ the loop group of G, i.e. the space of C^∞-smooth maps $S^1 \to G$ where S^1 is identified with the unit circle in $\mathbb{C}$. The *loop space* ΩG of the group G (or the basic loop space) is the homogeneous space of (right conjugacy classes) of the group LG of the form

$$\Omega G = LG/G \tag{5}$$

where the group G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$.

The loop group LG acts on ΩG by left translations. Denote by o the origin in ΩG, given by the class of constant maps: $o := [G]$. The tangent space of ΩG at the origin o is identified with the space $\Omega\mathfrak{g} = L\mathfrak{g}/\mathfrak{g}$.

The loop space ΩG has a natural symplectic structure, invariant under the action of the loop group LG on ΩG. Due to the invariance, it's sufficient to define its restriction to $T_o(\Omega G) = \Omega\mathfrak{g}$. For that we fix an invariant inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$ and consider a 2-form ω on $L\mathfrak{g}$ of the form

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(\theta), \eta'(\theta) > \, d\theta \ , \quad \xi, \eta \in L\mathfrak{g} \ .$$

This formula defines a left-invariant closed 2-form on LG which extends to an invariant symplectic structure on ΩG.

An invariant complex structure on ΩG is provided by the "complex" representation of $\Omega G = LG/G$ as a homogeneous space of the complex loop group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$ where $G^{\mathbb{C}}$ is the complexification of the Lie group G. This representation has the form (cf. [8] and also [9])

$$\Omega G = LG^{\mathbb{C}}/L^+G^{\mathbb{C}} \tag{6}$$

where $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is a subgroup of $LG^{\mathbb{C}}$, consisting of maps $S^1 \to G^{\mathbb{C}}$ which can be smoothly extended to holomorphic maps of the disc $\Delta := \{|z| < 1\} \to G^{\mathbb{C}}$.

The introduced symplectic and complex structures on ΩG are compatible in the sense that $\omega(\mathcal{J}^1\xi, \mathcal{J}^1\eta) = \omega(\xi, \eta)$ for all $\xi, \eta \in T_o(\Omega G)$ and the symmetric form $g^1(\xi, \eta) := \omega(\xi, \mathcal{J}^1\eta)$ on $T_o(\Omega G) \times T_o(\Omega G)$ is positive definite and extends to an invariant Riemannian metric g^1 on ΩG. So, ΩG is a *Kähler Frechet space*, provided with a Kähler metric g^1.

We are going to study harmonic maps from Riemann surfaces M to the loop spaces ΩG. The motivation for such a study comes from a result of Atiyah [1] which asserts that there is a 1–1 correspondence between the moduli space of G-instantons on $\mathbb{R}^4$ and the space of based holomorphic maps $f : \mathbb{CP}^1 \to \Omega G$, sending ∞ to the origin $o \in \Omega G$.

Motivated by this result, we can conjecture that we have also a 1–1 correspondence between the space of based harmonic maps $h : \mathbb{CP}^1 \to \Omega G$ and the moduli space of solutions of full Yang–Mills G-equations on $\mathbb{R}^4$.

6 Harmonic Maps into Loop Spaces

We are going to study harmonic maps into loop spaces by embedding ΩG into an infinite-dimensional Grassmanian $Gr_{textres}(H)$, called otherwise the *Hilbert Grassmanian*. We begin with its definition which mimics the homogeneous space definition of the standard Grassmanian $G_r(\mathbb{C}^d)$.

Let H be a complex Hilbert space, realized as the space $L^2_0(S^1, \mathbb{C})$ of square integrable complex functions on the circle S^1 with zero average over S^1 (or its vector analogue $L^2_0(S^1, \mathbb{C}^d)$).

Suppose that H has a *polarization*, i.e. a decomposition

$$H = H_+ \oplus H_- \tag{7}$$

into the direct orthogonal sum of infinite-dimensional closed subspaces. In the case of $H = L_0^2(S^1, \mathbb{C})$ one can take for such subspaces $H_\pm = \{\gamma \in H : \gamma(z) = \sum_{\pm k>0} \gamma_k z^k\}$. Any bounded linear operator $A \in L(H)$ with respect to the polarization (7) can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Denote by $\mathrm{GL}(H)$ the group of linear bounded operators on H, having a bounded inverse, and introduce the *restricted group* $GL_{\mathrm{res}}(H)$, consisting of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are Hilbert–Schmidt operators (briefly: HS-operators). In other words, the group $GL_{\mathrm{res}}(H)$ consists of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are "small" with respect to the "diagonal" terms a and d. We denote also by $U_{\mathrm{res}}(H)$ the intersection of $GL_{\mathrm{res}}(H)$ with the group $\mathrm{U}(H)$ of unitary operators in H.

As in the finite-dimensional situation, there is a Grassmann manifold $Gr_{\mathrm{res}}(H)$, called the *Hilbert Grassmannian*, related to the group $GL_{\mathrm{res}}(H)$.

Definition 5 *The* Hilbert Grassmanian $Gr_{res}(H)$ *is the set of closed subspaces $W \subset H$ such that the orthogonal projection $pr_+ : W \to H_+$ is a Fredholm operator, and the orthogonal projection $pr_- : W \to H_-$ is a Hilbert–Schmidt operator. Equivalently: a subspace $W \in Gr_{res}(H)$ iff it coincides with the image of a linear operator $w : H_+ \to H$ such that $w_+ := pr_+ \circ w$ is a Fredholm operator, and $w_- := pr_- \circ w$ is a Hilbert–Schmidt operator.*

We have the following homogeneous space representation of $Gr_{\mathrm{res}}(H)$:

$$Gr_{\mathrm{res}}(H) = U_{\mathrm{res}}(H)/\,\mathrm{U}(H_+) \times \mathrm{U}(H_-) .$$

This representation implies that $Gr_{\mathrm{res}}(H)$ is a *Kähler Hilbert manifold*.

The manifold $Gr_{\mathrm{res}}(H)$ has a countable number of connected components, numerated by the index of the Fredholm operator w_+ for a subspace $W \in Gr_{\mathrm{res}}(H)$, coinciding with the image of a linear operator $w : H_+ \to H$. We shall say that a subspace W has the *virtual dimension* d if the index of w_+ is equal to d. Denote by $G_d(H)$ the component of $Gr_{\mathrm{res}}(H)$, consisting of subspaces W of virtual dimension d. Then $Gr_{\mathrm{res}}(H)$ is the disjoint union of its components $G_d(H)$.

Due to this decomposition, the study of harmonic maps of Riemann surfaces into $Gr_{\mathrm{res}}(H)$ is reduced to the study of harmonic maps into the Grassmannians $G_d(H)$ of virtual dimension d which may be carried on along the same lines, as in the case of the Grassmann manifold $G_r(\mathbb{C}^d)$. As in that case, for any decomposition $d = r_1 + \ldots + r_n$ of d into the sum of natural numbers we define the corresponding *virtual flag manifold* $\mathcal{F} = \mathcal{F}_{\mathbf{r}}(H)$ of type $\mathbf{r} = (r_1, \ldots, r_n)$, consisting of collections $\mathcal{W} = (W_1, \ldots, W_n)$ of mutually orthogonal subspaces $W_i \subset H$ of virtual dimension r_i. Next, for any ordered subset $\sigma \subset \{1, \ldots, n\}$ we set $r := \sum_{i\in\sigma} r_i$ and construct a homogeneous flag bundle $\pi\colon \mathcal{F}_{\mathbf{r}}(H) \to G_r(H)$ by assigning to a flag $\mathcal{W} = (W_1, \ldots, W_n)$ the subspace $W = \bigoplus W_i$. We introduce the almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$ on the flag manifold $\mathcal{F}_{\mathbf{r}}(H)$, as in the finite-dimensional situation. We have the following assertion, analogous to the finite-dimensional case.

Theorem 2 *The homogeneous bundle $\pi : (\mathcal{F}_{\mathbf{r}}(H), \mathcal{J}^2) \to G_r(H)$ is a twistor bundle, i.e. for any $\mathcal{J}^2$-holomorphic map $\psi : M \to \mathcal{F}_{\mathbf{r}}(H)$ its projection $\varphi := \pi \circ \psi : M \to G_r(H)$ is a harmonic map.*

Due to this Theorem, one can produce harmonic maps $M \to G_r(H)$ by projecting $\mathcal{J}^2$-holomorphic maps $M \to \mathcal{F}_{\mathbf{r}}(H)$ to $G_r(H)$.

We construct now an isometric immersion of the loop space ΩG into the Hilbert Grassmannian $G_r(H)$, mentioned in the beginning of this Section.

Assume that G is a matrix group, i.e. G is represented as a subgroup of $\mathrm{U}(n)$ for some n. Then we have an isometric embedding $LG \to U_{\mathrm{res}}(H)$, given by the map: $\gamma \in LG = C^\infty(S^1, G) \mapsto M_\gamma \in U_{\mathrm{res}}(H)$ where the multiplication operator M_γ is defined by

$$f \in H = L_0^2(S^1, \mathbb{C}^n) \longmapsto (M_\gamma f)(z) := \gamma(z)f(z) \quad \text{for } z \in S^1 .$$

It's easy to check that $M_\gamma \in U_{\mathrm{res}}(H)$ if γ is smooth ([8]).

The constructed embedding of the loop group LG into $U_{\mathrm{res}}(H)$ induces an isometric immersion $\Omega G \to Gr_{\mathrm{res}}(H)$. So we can consider harmonic maps $M \to \Omega G$ as taking values in $Gr_{\mathrm{res}}(H)$, thus reducing their study to the study of harmonic maps $M \to \mathrm{Gr}_{\mathrm{res}}(H)$, i.e. to the problem, we have addressed above.

7 Conclusion

Suppose that our conjecture on the 1–1 correspondence between the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$ and the space of based harmonic spheres in ΩG, formulated at the end of Sec.5, is true. What kind of description we get for Yang–Mills fields on $\mathbb{R}^4$ from their interpretation as $\mathcal{J}^2$-holomorphic spheres in virtual flag spaces $\mathcal{F}_{\mathbf{r}}(H)$? Since any $\mathcal{J}^2$-holomorphic sphere in $\mathcal{F}_{\mathbf{r}}(H)$ can be obtained from a $\mathcal{J}^1$-holomorphic one by a finite number of replacements (of some holomorphic subbundles by antiholomorphic ones), our construction in terms of Yang–Mills fields will give a procedure of constructing Yang–Mills fields from instantons and anti-instantons by a finite number of reconstructions ("Bäcklund-type" transformations).

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Poincaré Algebra Extension with Tensor Generator

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Abstract

A tensor extension of the Poincaré algebra is proposed for the arbitrary dimensions. Casimir operators of the extension are constructed. A possible supersymmetric generalization of this extension is also found in the dimensions $D = 2, 3, 4$.

1 Introduction

There are many examples for the tensor 'central' extensions of the super-Poincaré algebra (see, for example, [1, 2, 3, 4, 5, 6, 7, 8]). However, there also exists the tensor extension of the Poincaré algebra itself. In the present report we give the example of such an extension with the help of the second rank tensor generator (see also [9, 10]). Such an extension makes common sense, since it is homomorphic to the Poincaré algebra. Moreover, the contraction of the extended algebra leads also to the Poincaré algebra. It is interesting enough that the momentum square Casimir operator for the Poincaré algebra under this extension ceases to be the Casimir operator and it is generalized by adding the term containing linearly the angular momentum[1]. Due to this fact, an irreducible representation of the extended algebra[2] has to contain the fields of the different masses. This extension with non-commuting momenta has also something in common with the ideas of the papers [12, 13, 14] and with the non-commutative geometry idea [15]. It is also shown that for the dimensions $D = 2, 3, 4$ the extended Poincaré algebra allows a supersymmetric generalization.

2 Extension of the Poincaré algebra

The Poincaré algebra for the components of the rotations M_{ab} and translations P_a in D dimensions

$$[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),$$

[1]Note that this reminds the relation for the Regge trajectory, which connects the mass square with the angular momentum.

[2]Concerning the irreducible unitary representations of the extended Poincaré group in $(1 + 1)$ dimensions see, for example, [11].

$$[M_{ab}, P_c] = g_{bc}P_a - g_{ac}P_b,$$

$$[P_a, P_b] = 0 \tag{2.1}$$

can be extended with the help of the tensor 'central' generator Z_{ab} in the following way:

$$[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),$$

$$[M_{ab}, P_c] = g_{bc}P_a - g_{ac}P_b,$$

$$[P_a, P_b] = cZ_{ab},$$

$$[M_{ab}, Z_{cd}] = (g_{ad}Z_{bc} + g_{bc}Z_{ad}) - (c \leftrightarrow d),$$

$$[P_a, Z_{bc}] = 0,$$

$$[Z_{ab}, Z_{cd}] = 0, \tag{2.2}$$

where c is some constant. By taking a set of the generators Z_{ab} as a homomorphism kernel, we obtain that the extended Poincaré algebra (2.2) is homomorphic to the usual Poincaré algebra (2.1). Moreover, in the limit $c \to 0$ the algebra (2.2) goes to the semi-direct sum of the commutative ideal Z_{ab} and Poincaré algebra (2.1).

Casimir operators of the extended Poincaré algebra are

$$Z_{a_1a_2}Z^{a_2a_3}\cdots Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_1}, \quad (k = 1, 2, \ldots); \tag{2.3}$$

$$\begin{aligned} &P^{a_1}Z_{a_1a_2}Z^{a_2a_3} \quad \cdots \quad Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}P_{a_{2k+1}} \\ &+cZ^{aa_1}Z_{a_1a_2} \quad \cdots \quad Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}M_{a_{2k+1}a}, \quad (k = 0, 1, 2, \ldots); \end{aligned} \tag{2.4}$$

$$\epsilon^{a_1a_2\ldots a_{2k-1}a_{2k}}Z_{a_1a_2}\cdots Z_{a_{2k-1}a_{2k}}, \quad 2k = D, \tag{2.5}$$

where $\epsilon^{a_1\ldots a_{2k}}$, $\epsilon^{01\ldots 2k-1} = 1$ is the totally antisymmetric Levi-Civita tensor in the even dimensions $D = 2k$. In particular, there is a Casimir operator generalizing the momentum square

$$P^aP_a + cZ^{ab}M_{ba}, \tag{2.6}$$

which indicates that an irreducible representation of the extended algebra contains the fields having the different masses. Note that for the extended algebra there is no generalization of the Pauli-Lubanski vector of the Poincaré algebra. The expressions (2.3) and (2.4) for the Casimir operators are valid for the extended Poincaré algebra (2.2) in the arbitrary dimensions D, but the expression (2.5) is only true for the even dimensions $D = 2k$.

Note that in the case of the extended two-dimensional Poincaré algebra the Casimir operators (2.3) and (2.4) can be expressed

$$Z_{a_1a_2}Z^{a_2a_3}\cdots Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_1} = 2Z^{2k},$$

$$\begin{aligned} P^{a_1}Z_{a_1a_2}Z^{a_2a_3} \quad \cdots \quad & Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}P_{a_{2k+1}} \\ +cZ^{aa_1}Z_{a_1a_2} \quad \cdots \quad & Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}M_{a_{2k+1}a} = Z^{2k}(P^aP_a + cZ^{ab}M_{ba}) \end{aligned}$$

as degrees of the following generating Casimir operators:

$$Z = \frac{1}{2}\epsilon^{ab}Z_{ab},$$

$$P^aP_a + cZ^{ab}M_{ba},$$

where $\epsilon^{ab} = -\epsilon^{ba}$, $\epsilon^{01} = 1$ is the completely antisymmetric two-dimensional Levi-Civita tensor. In the case of the extended three-dimensional Poincaré algebra these Casimir operators can be expressed

$$Z_{a_1a_2}Z^{a_2a_3}\cdots Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_1} = 2(Z^aZ_a)^k,$$

$$\begin{aligned} P^{a_1}Z_{a_1a_2}Z^{a_2a_3} \quad \cdots \quad & Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}P_{a_{2k+1}} \\ +cZ^{aa_1}Z_{a_1a_2} \quad \cdots \quad & Z_{a_{2k-1}a_{2k}}Z^{a_{2k}a_{2k+1}}M_{a_{2k+1}a} \\ = \quad & (Z^aZ_a)^k(P^aP_a + cZ^{ab}M_{ba}) - (Z^aZ_a)^{k-1}(P^aZ_a)^2 \end{aligned}$$

in terms of the following generating Casimir operators:

$$Z^aZ_a,$$

$$P^aP_a + cZ^{ab}M_{ba},$$

$$P^aZ_a,$$

where

$$Z^a = \frac{1}{2}\epsilon^{abc}Z_{bc}$$

and ϵ^{abc}, $\epsilon^{012} = 1$ is the totally antisymmetric three-dimensional Levi-Civita tensor. In the case of the extended D-dimensional ($D \geq 4$) Poincaré algebra the Casimir operators (2.3) and (2.4) can not be expressed in terms of the finite number of the generating Casimir operators.

Generators of the left shifts, acting on the function $f(y)$ with a group element G,

$$[T(G)f](y) = f(G^{-1}y), \quad y = (x^a, z^{ab})$$

have the form

$$P_a = -\left(\frac{\partial}{\partial x^a} + \frac{c}{2} x^b \frac{\partial}{\partial z^{ab}}\right),$$

$$Z_{ab} = -\frac{\partial}{\partial z^{ab}},$$

$$M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} + z_a{}^c \frac{\partial}{\partial z^{bc}} - z_b{}^c \frac{\partial}{\partial z^{ac}} + S_{ab}, \tag{2.7}$$

where coordinates x^a correspond to the translation generators P_a, coordinates z^{ab} correspond to the generators Z_{ab} and S_{ab} is a spin operator.

On the other hand, generators of the right shifts

$$[T(G)f](y) = f(yG)$$

have the form

$$D_a \stackrel{\text{def}}{=} P_a{}^r = \frac{\partial}{\partial x^a} - \frac{c}{2} x^b \frac{\partial}{\partial z^{ab}},$$

$$Z_{ab}{}^r = -Z_{ab} = \frac{\partial}{\partial z^{ab}}. \tag{2.8}$$

By taking into account (2.7) and (2.8), the Casimir operators (2.4) can be rewritten with the help of the generators D_a in the following way:

$$\begin{aligned} &D^{a_1} Z_{a_1 a_2} Z^{a_2 a_3} \quad \cdots \quad Z_{a_{2k-1} a_{2k}} Z^{a_{2k} a_{2k+1}} D_{a_{2k+1}} \\ &+ c Z^{a a_1} Z_{a_1 a_2} \quad \cdots \quad Z_{a_{2k-1} a_{2k}} Z^{a_{2k} a_{2k+1}} S_{a_{2k+1} a}, \quad (k = 0, 1, 2, \ldots). \end{aligned}$$

Note that the algebra

$$[M_{ab}, M_{cd}] = (g_{ad} M_{bc} + g_{bc} M_{ad}) - (c \leftrightarrow d),$$

$$[M_{ab}, P_c] = g_{bc} P_a - g_{ac} P_b,$$

$$[M_{ab}, D_c] = g_{bc} D_a - g_{ac} D_b,$$

$$[P_a, P_b] = c Z_{ab},$$

$$[D_a, D_b] = -c Z_{ab},$$

$$[P_a, D_b] = 0,$$

$$[M_{ab}, Z_{cd}] = (g_{ad}Z_{bc} + g_{bc}Z_{ad}) - (c \leftrightarrow d),$$

$$[P_a, Z_{bc}] = 0,$$

$$[D_a, Z_{bc}] = 0,$$

$$[Z_{ab}, Z_{cd}] = 0, \tag{2.9}$$

formed by the generators M_{ab}, P_a, D_a and Z_{ab}, has as Casimir operators the operators (2.3) and the following operators:

$$(P-D)^{a_1} Z_{a_1a_2} Z^{a_2a_3} \quad \cdots \quad Z_{a_{2k-1}a_{2k}} Z^{a_{2k}a_{2k+1}} (P+D)_{a_{2k+1}}$$

$$+cZ^{aa_1} Z_{a_1a_2} \quad \cdots \quad Z_{a_{2k-1}a_{2k}} Z^{a_{2k}a_{2k+1}} M_{a_{2k+1}a}, \quad (k = 0, 1, 2, \ldots). \tag{2.10}$$

The algebra (2.9) can be considered as another extension with the help of the vector generator $\frac{1}{2}(P+D)_a$ and tensor generator Z_{ab} of the Poincaré algebra formed by the generators M_{ab} and $\frac{1}{2}(P-D)_a$. By using the expressions (2.7) and (2.8), the Casimir operators (2.10) can be represented in the form

$$cZ^{aa_1} Z_{a_1a_2} \cdots Z_{a_{2k-1}a_{2k}} Z^{a_{2k}a_{2k+1}} S_{a_{2k+1}a}, \quad (k = 0, 1, 2, \ldots).$$

3 Supersymmetric generalization

In the dimensions $D = 2, 3, 4$ the extended Poincaré algebra (2.2) admits the following supersymmetric generalization:

$$\{Q_\alpha, Q_\beta\} = -d(\sigma^{ab}C)_{\alpha\beta} Z_{ab},$$

$$[M_{ab}, Q_\alpha] = -(\sigma_{ab}Q)_\alpha,$$

$$[P_a, Q_\alpha] = 0,$$

$$[Z_{ab}, Q_\alpha] = 0 \tag{3.1}$$

with the help of the super-translation generators

$$Q_\alpha = -\left[\frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{d}{2}(\sigma^{ab}\theta)_\alpha \frac{\partial}{\partial z^{ab}}\right],$$

where $\theta = C\bar{\theta}$ is a Majorana Grassmann spinor, C is a charge conjugation matrix, d is some constant and $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$.

The rotation generators acquire the terms depending on the Grassmann variables θ_α

$$M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} + z_a{}^c \frac{\partial}{\partial z^{bc}} - z_b{}^c \frac{\partial}{\partial z^{ac}} - (\sigma_{ab}\theta)_\alpha \frac{\partial}{\partial\theta_\alpha} + S_{ab},$$

whereas the expressions (2.7) for the translations P_a and tensor generator Z_{ab} remain unchanged.

The validity of the Jacobi identities

$$[P_a, \{Q_\alpha, Q_\beta\}] = \{Q_\alpha, [P_a, Q_\beta]\} + \{Q_\beta, [P_a, Q_\alpha]\}$$

and

$$[M_{ab}, \{Q_\alpha, Q_\beta\}] = \{Q_\alpha, [M_{ab}, Q_\beta]\} + \{Q_\beta, [M_{ab}, Q_\alpha]\}$$

for the supersymmetric generalization of the extended Poincaré algebra (2.2) verified for the dimensions $D = 2, 3, 4$ with the use of the symmetry properties of the matrices C and $\gamma_a C$ and the relations (A.1)–(A.3) of the Appendix.

One of the generating Casimir operator in the dimensions $D = 2, 3$ is generalized into the following form:

$$P^a P_a + cZ^{ab} M_{ba} - \frac{c}{2d} Q_\alpha (C^{-1})^{\alpha\beta} Q_\beta, \tag{3.2}$$

while the form of the rest generating Casimir operators in these dimensions are not changed. Note that in the case $D = 3$ there is also the following Casimir operator:

$$Z^a Q_\alpha (C^{-1}\gamma_a)^{\alpha\beta} Q_\beta.$$

One of the simplest Casimir operator (2.6) in $D = 4$ is also generalized into the form (3.2). The supersymmetric generalization of the more complicated Casimir operators in the four-dimensional case has the following structure:

$$\begin{aligned} P^a Z_{ab} Z^{bc} P_c \;&+\; cZ^{ab} Z_{bc} Z^{cd} M_{da} + \frac{2c}{5d} Q_\alpha (C^{-1}\sigma^{ab} Z_{ab} \sigma^{cd} Z_{cd})^{\alpha\beta} Q_\beta \\ &+\; \frac{c}{2d} Z^{ab} Z_{ab} Q_\alpha (C^{-1})^{\alpha\beta} Q_\beta, \end{aligned}$$

$$\begin{aligned} P^a Z_{ab} Z^{bc} Z_{cd} Z^{de} P_e \;&+\; cZ^{ab} Z_{bc} Z^{cd} Z_{de} Z^{ef} M_{fa} \\ &+\; \frac{2c}{5d} Q_\alpha \left[C^{-1}\sigma^{ab} Z_{ab} \sigma^{cd} \left(Z_{ce} Z^{ef} Z_{fd} + \frac{3}{10} Z^{gh} Z_{hg} Z_{cd} \right) \right]^{\alpha\beta} Q_\beta \\ &-\; \frac{c}{20d} \left[7 Z_{ab} Z^{bc} Z_{cd} Z^{da} + 3 (Z^{ef} Z_{fe})^2 \right] Q_\alpha (C^{-1})^{\alpha\beta} Q_\beta. \end{aligned}$$

An algorithm for the construction of the supersymmetric generalization of the Casimir operators (2.4) is obvious and based on the use of the following commutation relations:

$$\left[\frac{1}{2d} Q_\alpha (C^{-1})^{\alpha\beta} Q_\beta, Q_\gamma \right] = Z^{ab} (\sigma_{ab} Q)_\gamma,$$

$$\begin{aligned} \left[\frac{2}{5d} Q_\alpha (C^{-1}\sigma^{ab} Z_{ab} \sigma^{cd} \tilde{Z}_{cd})^{\alpha\beta} Q_\beta, Q_\gamma \right] = \Big(Z^{ab} Z_{bc} \tilde{Z}^{cd} \;&+\; \frac{7}{10} Z_{bc} \tilde{Z}^{cb} Z^{ad} \\ &+\; \frac{3}{10} Z_{bc} Z^{cb} \tilde{Z}^{ad} \Big) (\sigma_{ad} Q)_\gamma, \end{aligned}$$

where

$$\tilde{Z}^{ab} = Z^{aa_1} Z_{a_1 a_2} \cdots Z_{a_{2k-1} a_{2k}} Z^{a_{2k} b}, \quad (k = 0, 1, \ldots).$$

4 Conclusion

Thus, in the present report we proposed the extension of the Poincaré algebra with the help of the second rank tensor generator. Casimir operators for the extended algebra are constructed. The form of the Casimir operators indicate that an irreducible representation of the extended algebra contains the fields with the different masses. A consideration is performed for the arbitrary dimensions D. A possible supersymmetric generalization of the extended Poincaré algebra is also given for the particular cases with the dimensions $D = 2, 3, 4$.

It would be interesting to find the spectra of the Casimir operators and to construct the models based on the extended Poincaré algebra. The work in these directions is in progress.

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A Appendix

As a real (Majorana) representation for the two-dimensional γ-matrices and charge conjugation matrix C we adopt

$$\gamma^0 = C = -C^T = -i\sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma_5 = \frac{1}{2}\epsilon^{ab}\gamma_a\gamma_b = \sigma_3;$$

$$\{\gamma_a, \gamma_b\} = 2g_{ab}, \quad g_{11} = -g_{00} = 1, \quad C^{-1}\gamma_a C = -{\gamma_a}^T,$$

where σ_i are Pauli matrices. The matrices γ_a satisfy the relations

$$\gamma_a\gamma_5 = \epsilon_{ab}\gamma^b, \quad \gamma_a\gamma_b = g_{ab} - \epsilon_{ab}\gamma_5. \tag{A.1}$$

For the Majorana three-dimensional γ-matrices and charge conjugation matrix C we take

$$\gamma^0 = C = -C^T = -i\sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3;$$

$$\{\gamma_a, \gamma_b\} = 2g_{ab}, \quad g_{11} = g_{22} = -g_{00} = 1, \quad C^{-1}\gamma_a C = -{\gamma_a}^T.$$

The matrices γ_a obey the relations

$$\gamma_a\gamma_b = g_{ab} - \epsilon_{abc}\gamma^c. \tag{A.2}$$

At last, the real four-dimensional γ-matrices and matrix C are

$$\gamma^0 = C = -C^T = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix},$$

$$\gamma^2 = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = - \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$\{\gamma_a, \gamma_b\} = 2g_{ab}, \quad g_{11} = g_{22} = g_{33} = -g_{00} = 1, \quad C^{-1}\gamma_a C = -{\gamma_a}^T, \quad \gamma_5 = \frac{1}{4}\epsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d.$$

The matrices γ_a and σ_{ab} meet the relations

$$\gamma_a\sigma_{bc} = \frac{1}{2}\epsilon_{abcd}\gamma^d\gamma_5 + \frac{1}{2}(\gamma_c g_{ab} - \gamma_b g_{ac}), \quad \sigma_{ab}\gamma_c = \frac{1}{2}\epsilon_{abcd}\gamma^d\gamma_5 + \frac{1}{2}(\gamma_a g_{bc} - \gamma_b g_{ac}),$$

$$\sigma_{ab}\sigma_{cd} = \frac{1}{4}(g_{ad}g_{bc} - g_{ac}g_{bd} - \epsilon_{abcd}\gamma_5) + \frac{1}{2}(\sigma_{ad}g_{bc} + \sigma_{bc}g_{ad} - \sigma_{ac}g_{bd} - \sigma_{bd}g_{ac}). \qquad \text{(A.3)}$$

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Construction of TYM via Field Redefinitions

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Abstract

By constructing a nilpotent extended BRST operator $\bar{s}$ that involves the N=2 global supersymmetry transformations of one chirality, we find exact field redefinitions that allows to construct the Topological Yang Mills Theory from the ordinary Euclidean N=2 Super Yang Mills theory in flat space. We also show that the given field redefinitions yield the Baulieu-Singer formulation of Topological Yang Mills theory when after an instanton inspired truncation of the theory is used.

1 Introduction

Topological Yang-Mills (TYM) theory was first constructed by Witten [1] in 1988 as the twisted version of Euclidean N=2 Super Yang-Mills (SYM) theory in order to study the topological invariants of four-manifolds. Soon after [1], it was shown by Baulieu and Singer that TYM can be fully obtained as a pure gauge fixing term (i.e. as an exact BRST term) [2].

Moreover, N=2 SYM and TYM are also intertwined together when physical calculations are considered. For instance, the instanton calculations of N=2 SYM by using semi-classical approximation [3] and the ones of TYM, where no approximation is needed to perform these calculations [4] give the same result. Since some position independent correlators exist in supersymmetric gauge theories [5], one can interpret this result that a subset of correlators of N=2 SYM coincide with a subset of the observables in TYM [4]. The non-renormalition theorems of N=2 SYM can also be proved by using twisted version [6]. As a consequence, one may conclude that twisting can be thought as a variable redefinition in flat space [7].

Therefore, two questions are in order: First of all, is it also possible to write the action of N=2 SYM as an exact term like the twisted version of the theory and second, if twisting can be thought as really a variable redefinition in flat space, is it possible to find these field redefinitions explicitly?

The answers to both of the above questions are found to be affirmative [8] and the strategy to find these answers is to use the BRST formalism (also called BV or field-antifield formalism [9, 10]) that is extended to include global supersymmetry (SUSY). [11, 12, 13, 14, 15].

2 Extended BRST transformations and N=2 SYM action as an exact term

The off-shell Euclidean N=2 SYM action [16] is given as[1],

$$
\begin{aligned}
I &= Tr\int d^4x(\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{8}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}-\lambda^i \not{D}\bar{\lambda}_i+MD_\mu D_\mu N- \\
&\quad -\frac{i\sqrt{2}}{2}(\lambda_i[\lambda^i,N]+\bar{\lambda}^i[\bar{\lambda}_i,M])-\frac{1}{2}[M,N]^2+\frac{1}{2}\vec{D}.\vec{D})
\end{aligned} \tag{1}
$$

where the (anti-hermitian) gauge field A_μ and the scalar fields M , N are singlets, the Weyl spinors $\lambda_{i\alpha}$ $\bar{\lambda}^i_{\dot{\alpha}}$ are doublets and the auxiliary field $\vec{D}$ is a triplet under the $SU(2)_R$ symmetry group.

Since, the action is translation, gauge and N=2 SUSY invariant one can define an extended BRST symmetry [13]:

$$
s = s_0 - i\xi^i Q_i - i\bar{\xi}_i\bar{Q}^i - i\eta^\mu\partial_\mu \tag{2}
$$

where s_0 is the ordinary BRST transformations, Q_i , $\bar{Q}^i$ are chiral and antichiral parts of N=2 SUSY transformations and $\xi^{i\alpha}$, $\bar{\xi}_{i\dot{\alpha}}$ and η_μ are the constant *commuting* chiral, antichiral SUSY ghosts and constant imaginary anticommuting translation ghost respectively. By choosing s-transformations of the ghosts suitably, the extended BRST operator s becomes nilpotent [13] and one can construct a cohomology problem.

On the other hand, from the definition of s it is still possible to derive another nilpotent operator by using a suitable filtration of global ghosts [17]. We choose this filtration to be

$$
\mathcal{N} = \bar{\xi}_{i\dot{\alpha}}\frac{\delta}{\delta\bar{\xi}_{i\dot{\alpha}}} + \eta_\mu\frac{\delta}{\delta\eta_\mu} \quad ; \quad s = \sum s^{(n)} \quad , \quad [\mathcal{N}, s^{(n)}] = ns^{(n)}, \tag{3}
$$

so that the zeroth order in the above expansion is an operator that includes ordinary BRST and chiral SUSY on the space of the fields of the N=2 vector multiplet $M, N, A_\mu, \lambda_i, \bar{\lambda}^i, \vec{D}$

$$
\bar{s} := s^{(0)} = s_0 - i\xi^i Q_i \quad , \quad \bar{s}^2 = 0. \tag{4}
$$

The cohomology of s is isomorphic to a subset of the cohomology of the filtered operator $\bar{s}$ [17]. The $\bar{s}$ transformation of the fields are given as,

$$
\begin{aligned}
\bar{s}A_\mu &= D_\mu c - \xi_i e_\mu \bar{\lambda}^i \quad , \quad \bar{s}M = -[c,M] + i\sqrt{2}\xi^i\lambda_i \quad , \quad \bar{s}N = -[c,N] \\
\bar{s}\lambda_i &= -\{c,\lambda_i\} - e_{\mu\nu}\xi_i F_{\mu\nu} + \xi_i[M,N] + \vec{\tau}_i^{\,j}\xi_j.\vec{D} \quad , \quad \bar{s}\bar{\lambda}^i = -\{c,\bar{\lambda}^i\} + i\sqrt{2}\bar{e}_\mu\xi^i D_\mu N \\
\bar{s}\vec{D} &= -[c,\vec{D}] + \vec{\tau}_i^{\,j}(\xi_j e_\mu D_\mu\bar{\lambda}^i + i\sqrt{2}\xi^i[\lambda_j,N]) \\
\bar{s}c &= -\frac{1}{2}\{c,c\} + i\sqrt{2}\xi_i\xi^i N \quad , \quad \bar{s}\eta_\mu = \bar{s}\xi_i = \bar{s}\bar{\xi}_i = 0
\end{aligned} \tag{5}
$$

Since, the actions of SYM theories can be represented as chiral (or antichiral) multiple supervariations of lower dimensional gauge invariant field polynomials [18], it is straightforward to assume that the action can also be written as an $\bar{s}$ exact term of a gauge invariant field polynomial which is independent of Fadeev-Popov ghost fields[2],

$$
I = \bar{s}\Psi. \tag{6}
$$

[1]Our conventions are explained in Ref.[8] in detail.

[2]In other words, we assume that the action can be chosen to be a trivial element of equivariant cohomology of $\bar{s}$. See for instance Ref.s[6] and the references therein.

It is clear that Ψ, the so called gauge fermion in BV formalism, has negative ghost number, $Gh(\Psi) = -1$. However, since no fields with negative ghost number has been introduced and since we have chosen the gauge fermion to be free of Fadeev-Popov ghosts, the only way to assign a negative ghost number to Ψ is to choose Ψ to depend on the negative powers of the global SUSY ghosts:

$$\Psi = \frac{1}{\xi_k \xi^k} \xi^i \int d^4x \psi_i \tag{7}$$

where ψ_i^α is a dimension 7/2 fermion that is made from the fields of the N=2 vector multiplet. The most general such gauge fermion that is covariant in its Lorentz, spinor and $SU(2)_R$ indices is easy to find:

$$\Psi_E = \frac{1}{\xi_k \xi^k} Tr \int d^4x (\frac{1}{2}\xi^i \lambda_i [M, N] - \frac{1}{2}\xi^i \vec{\tau}_i^j \lambda_j . \vec{D} - \frac{1}{2}\xi^i e_{\mu\nu} \lambda_i F_{\mu\nu} - \frac{i\sqrt{2}}{2} M \xi^i e_\mu D_\mu \bar{\lambda}_i). \tag{8}$$

The coefficients of the terms in Ψ are fixed in order that the $\bar{s}$ variation of Ψ is free of chiral ghosts:

$$\begin{aligned} I_E &= \bar{s}\Psi_E \\ &= Tr \int d^4x (\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{8}\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} - \lambda^i \not{D} \bar{\lambda}_i + M D_\mu D_\mu N \\ &\qquad - \frac{i\sqrt{2}}{2}(\lambda_i [\lambda^i, N] + \bar{\lambda}^i [\bar{\lambda}_i, M]) - \frac{1}{2}[M, N]^2 + \frac{1}{2}\vec{D}.\vec{D}) \end{aligned} \tag{9}$$

This is exactly the N=2 supersymmetric Euclidean action, that is constructed by Zumino [19], up to the topological term $\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}$ and the auxiliary term $\frac{1}{2}\vec{D}.\vec{D}$.

Here, we should remark that, the action belongs to the trivial cohomology of $\bar{s}$ and therefore to that of the complete operator s, if and only if the functional space where s is defined is the polynomials of the fields that are not necessarily analytic in the constant ghosts.

3 TYM as a variable redefinition

After twisting physical nature of some fields are interpreted differently, i.e. some fields become ghosts while some others become anti-ghosts [1]. In order to derive the topological fields that have the correct dimensions and ghost numbers via field redefinitions the aforementioned non-analyticity argument can be used. Since the SUSY ghosts ξ_i have ghost number one and dimension 1/2, by studying the structure of the gauge fermion Ψ_E as given in (8), the only consistent field redefinitions that assign the correct dimensionality and ghost number to the topological fields[8] are found to be

$$A_\mu = A_\mu \quad , \quad \psi_\mu = -\xi_i e_\mu \bar{\lambda}^i \quad , \quad \Phi = i\sqrt{2}\xi_i \xi^i N \quad , \quad \bar{\Phi} = \frac{i}{\sqrt{2}\xi_i \xi^i} M \tag{10}$$

$$\eta = \frac{1}{\xi_k \xi^k} \xi_i \lambda^i \quad , \quad \mathcal{X}_{\mu\nu} = \frac{-2}{\xi_k \xi^k} \xi^i e_{\mu\nu} \lambda_i \quad , \quad B_{\mu\nu} = \frac{-2}{\xi_k \xi^k} \xi^i e_{\mu\nu} \vec{\tau}_i^j \xi_j . \vec{D} \tag{11}$$

It is straightforward to show that when the above variable redefinitions are inserted in the transformations[3] (5),

$$\bar{s}A_\mu = D_\mu c + \Psi_\mu \quad , \quad \bar{s}\psi_\mu = -\{c, \Psi_\mu\} - D_\mu \Phi \quad , \quad \bar{s}\Phi = -[c, \Phi] \quad , \quad \bar{s}c = -\frac{1}{2}\{c, c\} + \Phi$$

$$\bar{s}\bar{\Phi} = -[c, \bar{\Phi}] + \eta \quad , \quad \bar{s}\eta = -\{c, \eta\} + [\Phi, \bar{\Phi}] \quad , \quad \bar{s}\mathcal{X}_{\mu\nu} = -[c, \mathcal{X}_{\mu\nu}] + F^+_{\mu\nu} + \mathcal{B}_{\mu\nu}$$

$$\bar{s}B_{\mu\nu} = -[c, B_{\mu\nu}] + [\Phi, \mathcal{X}_{\mu\nu}] - (D_\mu\psi_\nu - D_\nu\psi_\mu)^+ \tag{12}$$

one can exactly extract the scalar supersymmetry transformations δ introduced by Witten [1] if one decomposes $\bar{s}$ on the fields (A_μ, Φ, $\bar{\Phi}$, ψ_μ, η, $\mathcal{X}_{\mu\nu}$) as $\bar{s} = s_o + \delta$ [4].

Similarly, the corresponding action that can also be found by these field redefinitions

$$\begin{aligned} I_{top} &= \bar{s}\Psi_{top} = \bar{s}Tr\int d^4x(-\frac{1}{2}\eta[\Phi, \bar{\Phi}] + \frac{1}{8}\mathcal{X}_{\mu\nu}F^+_{\mu\nu} - \frac{1}{8}\mathcal{X}_{\mu\nu}B_{\mu\nu} + \bar{\Phi}D_\mu\psi_\mu) & (13) \\ &= Tr\int d^4x(\frac{1}{8}F^+_{\mu\nu}F^+_{\mu\nu} + \eta D_\mu\psi_\mu - \frac{1}{4}\mathcal{X}_{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ - \bar{\Phi}D^2\Phi - \\ &\quad -\frac{1}{2}\Phi\{\eta, \eta\} - \frac{1}{8}\Phi\{\mathcal{X}_{\mu\nu}, \mathcal{X}_{\mu\nu}\} + \bar{\Phi}\{\psi_\mu, \psi_\mu\} - \frac{1}{2}[\Phi, \bar{\Phi}]^2 - \frac{1}{8}B_{\mu\nu}B_{\mu\nu}). & (14) \end{aligned}$$

is exactly the Topological Yang Mills action [1] with an auxiliary field term . We remark that the inclusion of the auxiliary field is crucial in order to write the action as an exact term[5].

In other words, TYM theory in flat Euclidean space can be obtained directly as variable redefinitions from the ordinary N=2 SYM theory [8]. As it is obvious from the above definitions of the topological fields, the ghost numbers and the dimensions that are assigned to the fields in the twisting procedure by hand, appears here naturally due to the composite structure of the topological fields in terms of global ghosts ξ_i and the original fields i.e. with respect to the power of ξ_i's in the definitions.

4 Baulieu-Singer formulation of TYM

Aiming to incorporate the instantons into supersymmetric theories Zumino have constructed a supersymmetric field theory directly in the Euclidean space [19]. It is then observed by Zumino that when one imposes for instance an anti self-dual field strength, i.e. $F^+_{\mu\nu} = 0$ with the restrictions $M = \lambda_i = 0$ the equations of motion from (1) reduce to a simple form [19],

$$F^+_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} = 0\,,\; D^2N = \frac{i\sqrt{2}}{2}\{\bar{\lambda}^i, \bar{\lambda}_i\}\,,\; e_\mu D_\mu\bar{\lambda}^i = 0. \tag{15}$$

that are invariant under the corresponding truncated SUSY.

[3]Here, $F^+_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}$ is the self-dual part of the field strength $F_{\mu\nu}$.

[4]Note that this scalar SUSY generator can also be written as a composite generator, $\delta = -i\xi^i Q_i$ where Q_i are the chiral SUSY generators.

[5]The reason why the action could not be written as an exact term in the original paper [1] is that the twisted theory was obtained from the on-shell SYM. Note that, since Ψ_{top} is gauge invariant, we have $I_{top} = \bar{s}\Psi_{top} = \delta\Psi_{top}$.

The equations (15) are also the saddle point equations in the context of constraint instanton method [3]. On the other hand, similar equations are obtained in Baulieu-Singer formulation of TYM without any approximation [4]. Since, both of the approaches to the instanton calculations give the same result [4] and Wittens TYM [1] can be obtained by using simple field redefinitions (10,11), it is natural to look for another analogy between the above instanton inspired truncation of Euclidean N=2 SYM theory and the Baulieu-Singer approach to TYM.

Indeed, when the above instanton inspired truncation is used to define another nilpotent operator $\tilde{s}$,

$$\tilde{s} = \bar{s}|_{F^+_{\mu\nu}=\not{D}\bar{\lambda}^i=M=\lambda_i=0} \;,\; \tilde{s}^2 = 0 \tag{16}$$

such that

$$\tilde{s}A_\mu = D_\mu c - \xi_i e_\mu \bar{\lambda}^i \,,\; \tilde{s}\bar{\lambda}^i = -\{c,\bar{\lambda}^i\} + i\sqrt{2}\bar{e}_\mu \xi^i D_\mu N \tag{17}$$

$$\tilde{s}N = -[c,N]\,,\; \tilde{s}c = -\frac{1}{2}\{c,c\} + i\sqrt{2}\xi_i\xi^i N \tag{18}$$

and[6]

$$\tilde{s}M = i\sqrt{2}\xi^i\lambda_i\,,\; \tilde{s}\lambda_i = \vec{\tau}_i^{\,j}\xi_j.\vec{D}\,,\; \tilde{s}\vec{D} = 0 \tag{19}$$

$\tilde{s}$-transformations are found to be exactly that of Baulieu-Singer [2] after performing the field redefinition given in (10,11) [8].

On the other hand the gauge fermion that is compatible with the restrictions of Zumino [19] has to be chosen slightly different then the one given for Euclidean case (8),

$$\Psi_{inst.} = \frac{1}{\xi_k\xi^k} Tr \int d^4x \big(-\frac{\alpha}{2}\xi^i\vec{\tau}_i^{\,j}\lambda_j.\vec{D} - \frac{1}{2}\xi^i e_{\mu\nu}\lambda_i F^+_{\mu\nu} + \frac{i\sqrt{2}}{2}\xi^i e_\mu \bar{\lambda}_i D_\mu M\big) \tag{20}$$

so that the corresponding action is

$$\begin{aligned} I^{(\alpha)}_{inst.} &= \tilde{s}\Psi_{inst.} \qquad (21)\\ &= Tr\int d^4x\big(-\frac{\alpha}{8}B_{\mu\nu}B_{\mu\nu} + \frac{1}{4}B_{\mu\nu}F^+_{\mu\nu} - \lambda^i e_\mu D_\mu \bar{\lambda}_i + M(D_\mu D_\mu N - \frac{i\sqrt{2}}{2}\{\bar{\lambda}^i,\bar{\lambda}_i\}) + \\ &+ \frac{1}{\xi_k\xi^k}\big(-\frac{1}{2}\xi^i e_{\mu\nu}\lambda_i[c,F^+_{\mu\nu}] + \frac{i\sqrt{2}}{2}M\{c,\xi^i e_\mu D_\mu\bar{\lambda}_i\}\big)\,\big) + \frac{1}{\xi_k\xi^k}Tr\int d^4x\partial_\mu\big(\tilde{s}\frac{i\sqrt{2}}{2}M\xi^i e_\mu\bar{\lambda}_i\big) \end{aligned}$$

where we have used the definition of $B_{\mu\nu}$ in order to have notational simplification.

First of all, the gauge fermion Ψ_{inst} (20) and the above action I_{inst} are exactly the ones given in Baulieu-Singer approach [2] up to ordinary gauge fixing. However, if the above relations are considered on their own, to be able to derive the instanton equations (15) from the action functional without having any dependence on the constant ghosts, the coefficient of $Tr\xi^i\lambda_i[M,N]$ in the Euclidean Ψ_E has to be chosen to vanish whereas the coefficient α of $Tr\tilde{s}\xi^i\vec{\tau}_i^{\,j}\lambda_j.\vec{D}$ can be left arbitrary. Therefore, the gauge fermion Ψ_{inst} is the only consistent choice up to total derivatives that gives the right action to derive the exact instanton equations, when the truncated transformations (17-19) are used.

[6] The reason why we do not set $\lambda_i = \vec{D} = 0$ in Eq.(20) is that the pairs $(M,\xi^i\lambda_i)$ and $(\xi^i\vec{\tau}_i^{\,j}\lambda_j,\vec{D})$ behaves like the trivial pairs (BRST doublets) It is known that the cohomology of an operator does not depend on inclusion of such trivial pairs (see for instance [10, 17]).

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Gravitational and axial anomalies for generalized Euclidean Taub-NUT metrics

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Abstract

Investigating the Dirac equation in curved backgrounds we point out the role of the Killing-Yano tensors in the construction of the Dirac-type operators. The gravitational and axial anomalies are studied for generalized Euclidean Taub-NUT metrics which admit hidden symmetries analogous to the Runge-Lenz vector of the Kepler-type problem. Using the Atiyah-Patodi-Singer index theorem for manifolds with boundaries, it is shown that the these metrics make no contribution to the axial anomaly.

1 Introduction

In the case of gravitational interaction, a consistent perturbative quantization is not available, even if there are no fermions. It is of crucial importance in the construction of any quantum theory for gravitation to understand the problem of anomalies which can affect the conservation laws.

From the symmetry viewpoint, the following two generalization of the Killing vector equation have become of interest in physics:

1. A symmetric tensor field $K_{\mu_1 \dots \mu_r}$ is called a Stäckel-Killing (S-K) tensor of valence r if and only if
$$K_{(\mu_1 \dots \mu_r ; \lambda)} = 0. \tag{1}$$
The usual Killing vectors correspond to valence $r = 1$ while the hidden symmetries are encapsulated in S-K tensors of valence $r > 1$.

2. A tensor $f_{\mu_1 \dots \mu_r}$ is called a Killing-Yano (K-Y) tensor of valence r if it is totally anti-symmetric and it satisfies the equation
$$f_{\mu_1 \dots (\mu_r ; \lambda)} = 0. \tag{2}$$
The K-Y tensors play an important role in models for relativistic spin-$\frac{1}{2}$ particles having in mind that they produce first-order differential operators of the Dirac-type which anticommute with the standard Dirac one.

The family of Taub-NUT metrics with their plentiful symmetries provides an excellent background to investigate the classical and quantum conserved quantities on curved spaces. The Taub-Newman-Unti-Tamburino (Taub-NUT) metrics were found by Taub [1] and extended by Newman-Unti-Tamburino [2]. The Euclidean Taub-NUT metric has lately attracted much attention in physics. Hawking [3] has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analog of the Yang-Mills instanton. This metric is the space part of the line element of the celebrated Kaluza-Klein monopole of Gross and Perry and Sorkin. On the other hand, in the long distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [4]. The Taub-NUT family of metrics is also involved in many other modern studies in physics like strings, membranes, etc.

In the Taub-NUT geometry there are four K-Y tensors. Three of these are complex structures realizing the quaternion algebra and the Taub-NUT manifold is hyper-Kähler [5]. In addition to these three vector-like K-Y tensors, there is a scalar one which has a non-vanishing field strength and which exists by virtue of the metric being type D.

For the geodesic motions in the Taub-NUT space, the conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, and its components are S-K tensors which can be expressed as symmetrized products of K-Y tensors [5, 6].

In Section 2, considering the Dirac equation in curved spaces, we point out the role of the K-Y tensors in the construction of new Dirac-type operators [7]. The Dirac-type operators constructed with the aid of covariantly constant K-Y tensors are equivalent with the standard Dirac operator. The non-covariantly constant K-Y tensors generates non-standard Dirac operators which are not equivalent to the standard Dirac operator and they are associated with the hidden symmetries of the space [8].

In the next Section we consider the generalization of the Euclidean Taub-NUT space as it was done by Iwai and Katayama [9, 10, 11]. The extended Euclidean Taub-NUT metrics still admits a Kepler-type symmetry as the standard Taub-NUT metric.

In Section 4 we investigate the gravitational anomalies pointing out the role of the K-Y tensors to prevent the appearance of quantum anomalies for scalar fields.

The importance of anomalous Ward identities in particle physics is widely appreciated. The anomalous divergence of the axial vector current in a background gravitational field was large discussed in the literature and directly related with the index theorem. In even-dimensional spaces one can define the index of a Dirac operator as the difference in the number of linearly independent zero modes with eigenvalue $+1$ and -1 under γ_5. The index is useful as a tool to investigate topological properties of the space, as well as in computing anomalies in quantum field theory.

In Section 5 we compute the index of the Dirac operator for the generalized Taub-NUT metric with the APS boundary condition and we find that the extended Taub-NUT metric does not contribute to the axial anomaly at least for not too large deformations of the standard Taub-NUT metric.

The last Section contains some concluding remarks.

2 Dirac equation on a curved background

In what follows we shall consider the Dirac operator on a curved background which has the form

$$D_s = \gamma^\mu \hat{\nabla}_\mu . \tag{3}$$

In this expression the Dirac matrices γ_μ are defined in local coordinates by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \tag{4}$$

and $\hat{\nabla}_\mu$ denotes the canonical covariant derivative for spinors. The essential properties of this covariant derivative are summarized in the following equations

$$\begin{aligned} &\hat{\nabla}_\mu \gamma^\mu = 0, \\ &\hat{\nabla}_{[\rho} \hat{\nabla}_{\mu]} = \frac{1}{4} R_{\alpha\beta\rho\mu} \gamma^\alpha \gamma^\beta \end{aligned} \tag{5}$$

where $R_{\alpha\beta\rho\mu}$ denotes the components of the Riemann curvature tensor.

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with Killing vector R_μ there is an appropriate operator [7]:

$$X_k = -i(R^\mu \hat{\nabla}_\mu - \frac{1}{4}\gamma^\mu \gamma^\nu R_{\mu;\nu}) \tag{6}$$

which commutes with the *standard* Dirac operator (3).

Moreover each K-Y tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form

$$D_f = -i\gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\gamma^\nu \gamma^\rho f_{\mu\nu;\rho}) \tag{7}$$

which anticommutes with the standard Dirac operator D_s.

3 Euclidean Taub-NUT metrics

Let us consider the Taub-NUT space and the chart with Cartesian coordinates $x^\mu (\mu, \nu = 1, 2, 3, 4)$ having the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{16m^2}(dx^4 + A_i dx^i)^2 \tag{8}$$

where $\vec{x}$ denotes the three-vector $\vec{x} = (r, \theta, \varphi)$, $(d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ and $\vec{A}$ is the gauge field of a monopole

$$\mathrm{div} \vec{A} = 0\,, \quad \vec{B} = \mathrm{rot}\, \vec{A} = 4m \frac{\vec{x}}{r^3} . \tag{9}$$

The real number m is the parameter of the theory which enter in the form of the functions

$$f(r) = g^{-1}(r) = V^{-1}(r) = \frac{4m + r}{r} \tag{10}$$

and the so called NUT singularity is absent if x^4 is periodic with period $16\pi m$. Sometimes it is convenient to make the coordinate transformation $4m(\chi + \varphi) = -x^4$ with $0 \le \chi < 4\pi$.

In the standard Taub-NUT geometry there are four Killing vectors $R_A^\mu,\ A=1,2,3,4$ and there are known to exist four K-Y tensors of valence 2. The first three K-Y tensors are covariantly constant

$$\begin{aligned} f_i &= 8m(d\chi+\cos\theta d\varphi)\wedge dx_i-\epsilon_{ijk}(1+\frac{4m}{r})dx_j\wedge dx_k, \\ D_\mu f^\nu_{i\lambda} &= 0\ , \qquad i,j,k=1,2,3. \end{aligned} \tag{11}$$

The fourth K-Y tensor is

$$f_Y=8m(d\chi+\cos\theta d\varphi)\wedge dr+4r(r+2m)(1+\frac{r}{4m})\sin\theta d\theta\wedge d\varphi \tag{12}$$

having a non-vanishing covariant derivative

$$f_{Y\, r\theta;\varphi}=2(1+\frac{r}{4m})r\sin\theta. \tag{13}$$

In Taub-NUT space there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem [5, 12]

$$\vec{K}=\frac{1}{2}\vec{K}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu=\vec{p}\times\vec{j}+\left(\frac{q^2}{4m}-4mE\right)\frac{\vec{r}}{r} \tag{14}$$

where $\vec{p}$ is the mechanical momentum, $\vec{j}$ is the angular momentum, q is the so called *relative electric charge*

$$q=g(r)(\dot{\chi}+\cos\theta\dot{\varphi})\,, \tag{15}$$

and the conserved energy is

$$E=\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{16}$$

The components $K_{i\mu\nu}$ involved with the Runge-Lenz vector (14) are S-K tensors and can be expressed as symmetrized products of the K-Y tensors f_i, f_Y and Killing vectors R_A^μ [6, 13]

$$K_{i\mu\nu}-\frac{1}{8m}(R_{4\mu}R_{i\nu}+R_{4\nu}R_{i\mu})=m\left(f_{Y\mu\lambda}{f_i}^\lambda{}_\nu+f_{Y\nu\lambda}{f_i}^\lambda{}_\mu\right). \tag{17}$$

Iwai and Katayama [9, 10, 11] generalized the Taub-NUT metric so that it still admit a Kepler-type symmetry. It was proved that the extensions of the Taub-NUT metric do not admit K-Y tensors, even if they possess S-K tensors [14, 15]. The only exception is the original Taub-NUT metric which possesses four K-Y tensors of valence two.

The extended Taub-NUT metric, denoted by ds_K^2, is defined on $\mathbb{R}^4-\{0\}$ by

$$ds_K^2=f(r)(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2)+g(r)(d\chi+\cos\theta\,d\varphi)^2 \tag{18}$$

where $r>0$ is the radial coordinate, the angle variables (θ,φ,χ) parametrize the unit sphere S^3 with $0\le\theta<\pi, 0\le\varphi<2\pi, 0\le\chi<4\pi$, and $f(r)$ and $g(r)$ are functions given, with constants a,b,c,d by

$$f(r)=\frac{a+br}{r}\quad,\quad g(r)=\frac{ar+br^2}{1+cr+dr^2}\,. \tag{19}$$

If one takes the constants

$$c=\frac{2b}{a},\quad d=\frac{b^2}{a^2} \tag{20}$$

with $4m = \frac{a}{b}$, the extended Taub-NUT metric becomes the original Euclidean Taub-NUT metric up to a constant factor. In the original Kaluza-Klein context the Taub-NUT parameter m is positive.

Spaces with a metric of the above form have an isometry group $SU(2) \times U(1)$ with four Killing vectors. The remarkable result of Iwai and Katayama is that the extended Taub-NUT space (18) still admits a conserved vector, quadratic in 4-velocities, analogous to the Runge-Lenz vector of the following form

$$\vec{K} = \vec{p} \times \vec{j} + \kappa \frac{\vec{r}}{r} . \tag{21}$$

The constant κ involved in the Runge-Lenz vector (21) is

$$\kappa = -a\,E + \frac{1}{2} c\, q^2 \tag{22}$$

where the conserved energy E is

$$E = \frac{\vec{p}^{\,2}}{2f(r)} + \frac{q^2}{2g(r)} . \tag{23}$$

4 Gravitational anomalies

For the classical motions, a S-K tensor $K_{\mu\nu}$ generate a quadratic constant of motion

$$K = K_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu . \tag{24}$$

In the case of the geodesic motion of classical scalar particles, the fact that $K_{\mu\nu}$ is a S-K tensor satisfying (1), assures the conservation of (24).

Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (24) is [16, 17]:

$$\mathcal{K} = D_\mu K^{\mu\nu} D_\nu \tag{25}$$

where D_μ is the covariant differential operator on the manifold with the metric $g_{\mu\nu}$. Working out the commutator of (25) with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu \tag{26}$$

we get

$$\begin{aligned}[D_\mu D^\mu, \mathcal{K}] = \quad & 2K^{\mu\nu;\lambda} D_{(\mu} D_\nu D_{\lambda)} + 3K^{(\mu\nu;\lambda)}{}_{;\lambda} D_{(\mu} D_{\nu)} \\ & + \{\frac{1}{2} g_{\lambda\sigma} (K_{(\lambda\sigma;\mu);\nu} - K_{(\lambda\sigma;\nu);\mu}) - \frac{4}{3} K_\lambda{}^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu \end{aligned} \tag{27}$$

Concerning the hidden symmetry of the quantized system, the above commutator does not vanishes on the strength of (1). Taking into account (1) we get:

$$[\mathcal{H}, \mathcal{K}] = -\frac{4}{3} \{K_\lambda{}^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu \tag{28}$$

which means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [18]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

It is obvious that for a Ricci-flat manifold this quantum anomaly is absent. However, a more interesting situation is represented by the manifolds in which the S-K tensor $K_{\mu\nu}$ can be written as a product of K-Y tensors $f_{\mu\nu}$ [7].

The integrability condition for any solution of (2), written for K-Y tensors of valence $r = 2$, is

$$R_{\mu\nu[\sigma}{}^{\tau} f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^{\tau} f_{\nu]\tau} = 0\,. \tag{29}$$

Now contracting this integrability condition on the Riemann tensor for any solution of (2) we get

$$f^{\rho}{}_{(\mu} R_{\nu)\rho} = 0. \tag{30}$$

Let us suppose that there exist a *square* of the S-K tensor $K_{\mu\nu}$ of the form of a K-Y tensor $f_{\mu\nu}$ [7]:

$$K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^{\rho}. \tag{31}$$

In case this should happen, the S-K equation (1) is automatically satisfied and the integrability condition (30) becomes

$$K^{\rho}{}_{[\mu} R_{\nu]\rho} = 0. \tag{32}$$

It is interesting to observe that in this last equation an antisymmetrization rather than symmetrization is involved this time as compared to (30). But this relation implies the vanishing of the commutator (28) for S-K tensors which admit a decomposition in terms of K-Y tensors.

Using the S-K tensor components of the Runge-Lenz vector (21) we can proceed to the evaluation of the quantum gravitational anomaly for the generalized Taub-NUT metrics. A direct evaluation [20] shows that the commutator (28) does not vanish.

In conclusion the operators constructed from symmetric S-K tensors are in general a source of gravitational anomalies for scalar fields. However, when the S-K tensor is of the form (31), then the anomaly disappears owing to the existence of the K-Y tensors.

5 Index formulas and axial anomalies

Atiyah, Patodi and Singer (APS) [19] discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their index formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. Endless trouble is caused in this theory by the condition that the metric and the operator be of "product type" near the boundary.

In our previous paper [20] we computed the index of the Dirac operator on annular domains and on disk, with the non-local APS boundary condition. For the generalized Taub-NUT metrics, we found that the index is a number-theoretic quantity which depends on the metrics. In particular, our formula shows that the index vanishes on balls of sufficient large radius, but can be non-zero for some values of the parameters c, d and of the radius.

Theorem 1 *If $c > -\frac{\sqrt{15d}}{2}$ then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof: We delegate the proof to Ref. [20]. ■

The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this would amount to a continuous change in the metric. The absence of axial anomalies is due to the fact there exists an underlying structure that does not depend on the metric. However for larger deformations of the metric there could appear discontinuities in the boundary conditions and therefore the index could present jumps. Our formula for the index involves a computable number-theoretic quantity depending on the parameters of the metric.

We also examined the Dirac operator on the complete Euclidean space with respect to this metric, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

We mentioned in [20] some open problems in connection with unbounded domains. The paper [21] brings new results in this direction. First we showed that the Dirac operator on $\mathbb{R}^4$ with respect to the standard Taub-NUT metric does not have L^2 harmonic spinors. This follows rather easily from the Lichnerowicz formula, since the standard Taub-NUT metric has vanishing scalar curvature. In particular, the index vanishes.

Theorem 2 *For the standard Taub-NUT metric on $\mathbb{R}^4$ the Dirac operator does not have L^2 solutions.*

Proof: Recall that the standard Taub-NUT metric is hyper-Kähler, hence its scalar curvature κ vanishes.

By the Lichnerowicz formula,

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4} = \nabla^*\nabla.$$

Let $\phi \in L^2$ be a solution of D in the sense of distributions. Then, again in distributions, $\nabla^*\nabla\phi = 0$. The operator $\nabla^*\nabla$ is essentially self-adjoint with domain $\mathcal{C}_c^\infty(\mathbb{R}^4, \Sigma_4)$, which implies that its kernel equals the kernel of ∇. Hence $\nabla\phi = 0$. Now a parallel spinor has constant pointwise norm, hence it cannot be in L^2 unless it is 0, because the volume of the metric ${ds_K}^2$ is infinite. Therefore $\phi = 0$. ■

6 Concluding remarks

There is a relationship between the absence of anomalies and the existence of the K-Y tensors. For scalar fields, the decomposition (31) of S-K tensors in terms of K-Y tensors guarantees the absence of gravitational anomalies. Otherwise operators constructed from symmetric tensors are in general a source of anomalies proportional to the Ricci tensors.

However for the axial anomaly the role of K-Y tensors is not so obvious. The topological aspects are more important and the absence of K-Y tensors does not imply the appearance of anomalies.

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Twistors and Aspects of Integrability of self-dual SYM Theory

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Abstract

With the help of the Penrose-Ward transform, which relates certain holomorphic vector bundles over the supertwistor space to the equations of motion of self-dual SYM theory in four dimensions, we construct hidden infinite-dimensional symmetries of the theory. We also present a new and shorter proof (cf. hep-th/0412163) of the relation between certain deformation algebras and hidden symmetry algebras.

1 Introduction and conclusions

By analyzing the linearized [1] and full [2] field equations and by virtue of the Penrose-Ward transform [3], it was shown that there is a one-to-one correspondence between the moduli space of holomorphic Chern-Simons theory on supertwistor space and of self-dual $\mathcal{N} = 4$ SYM theory in four dimensions.[1] This correspondence has then been used for a twistorial construction of hidden infinite-dimensional symmetry algebras in the self-dual truncation of SYM theory [7]. Therein, the results known for the purely bosonic self-dual YM equations (see, e.g., Refs. [8]–[13]) have been generalized to the supersymmetric setting. Here, we shall briefly report on those results thereby also presenting a new and shorter proof of the relation between certain deformation algebras on the twistor side and symmetry algebras on the gauge theory side. For the sake of clarity, the discussion presented below is given in the complex setting only but, of course, it is also possible to implement real structures (see, e.g., [2, 7] for details).

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2 Preliminaries

2.1. Supertwistor space

The starting point of our discussion is the complex projective supertwistor space $\mathbb{C}P^{3|\mathcal{N}}$ with homogeneous coordinates $[z^\alpha, \lambda_{\dot\alpha}, \eta_i]$ obeying the equivalence relation

$$(z^\alpha, \lambda_{\dot\alpha}, \eta_i) \;\sim\; (tz^\alpha, t\lambda_{\dot\alpha}, t\eta_i) \tag{2.1}$$

[1] For reviews of twistor theory, we refer to [4, 5, 6, 12].

for any $t \in \mathbb{C}^*$. Here, the spinorial indices $\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots$ run from 1 to 2 and the R-symmetry indices $i, j, \ldots$ from 1 to $\mathcal{N}$. In the following, we are interested in the open subset $\mathcal{P}^{3|\mathcal{N}} := \mathbb{C}P^{3|\mathcal{N}} \setminus \mathbb{C}P^{1|\mathcal{N}}$ defined by $\lambda_{\dot{\alpha}} \neq 0$. This space can be covered by two patches, say $\mathcal{U}_+$ and $\mathcal{U}_-$, for which $\lambda_{\dot{1}} \neq 0$ and $\lambda_{\dot{2}} \neq 0$, respectively. On those patches we have the coordinates

$$\begin{array}{llllll} z_+^\alpha := \frac{z^\alpha}{\lambda_{\dot{1}}}, & z_+^3 := \frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}} =: \lambda_+ & \text{and} & \eta_i^+ := \frac{\eta_i}{\lambda_{\dot{1}}} & \text{on} & \mathcal{U}_+, \\ z_-^\alpha := \frac{z^\alpha}{\lambda_{\dot{2}}}, & z_-^3 := \frac{\lambda_{\dot{1}}}{\lambda_{\dot{2}}} =: \lambda_- & \text{and} & \eta_i^- := \frac{\eta_i}{\lambda_{\dot{2}}} & \text{on} & \mathcal{U}_-, \end{array} \tag{2.2}$$

which are related by

$$z_+^\alpha = \frac{1}{z_-^3} z_-^\alpha, \qquad z_+^3 = \frac{1}{z_-^3} \qquad \text{and} \qquad \eta_i^+ = \frac{1}{z_-^3}\eta_i^- \tag{2.3}$$

on $\mathcal{U}_+ \cap \mathcal{U}_-$. This in particular shows that $\mathcal{P}^{3|\mathcal{N}}$, which we simply call supertwistor space, is a holomorphic fibration over the Riemann sphere $\mathbb{C}P^1$,

$$\mathcal{P}^{3|\mathcal{N}} = \mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^{\mathcal{N}} \rightarrow \mathbb{C}P^1. \tag{2.4}$$

From this definition it is clear that global holomorphic sections of the fibration (2.4) are degree one polynomials. In a given trivialization, they are locally of the form

$$z_\pm^\alpha = x^{\alpha\dot{\alpha}} \lambda_{\dot{\alpha}}^\pm \qquad \text{and} \qquad \eta_i^\pm = \eta_i^{\dot{\alpha}} \lambda_{\dot{\alpha}}^\pm \tag{2.5}$$

and parametrized by the moduli $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}) \in \mathbb{C}^{4|2\mathcal{N}}$. Here, we also introduced the common abbreviations

$$(\lambda_{\dot{\alpha}}^+) := \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \qquad \text{and} \qquad (\lambda_{\dot{\alpha}}^-) := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}. \tag{2.6}$$

Therefore, $\mathcal{P}^{3|\mathcal{N}}$ naturally fits into the following double fibration

$$\begin{tikzcd} & \mathcal{F}^{5|2\mathcal{N}} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}} \end{tikzcd} \tag{2.7}$$

where $\mathcal{F}^{5|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1$ is called the correspondence space. The (holomorphic) projections are given according to

$$\begin{array}{lll} \pi_1 : (x^{\alpha\dot{\alpha}}, \lambda_\pm, \eta_i^{\dot{\alpha}}) & \mapsto & (x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}), \\ \pi_2 : (x^{\alpha\dot{\alpha}}, \lambda_\pm, \eta_i^{\dot{\alpha}}) & \mapsto & (z_\pm^\alpha = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}^\pm, z_\pm^3 = \lambda_\pm, \eta_i^\pm = \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}^\pm). \end{array} \tag{2.8}$$

Next let us take a closer look at the relations (2.5). Fixing a point $(z_\pm^\alpha, \lambda_\pm, \eta_i^\pm)$ in supertwistor space and solving (2.5) for $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}})$, one determines an isotropic (null) plane $\mathbb{C}^{2|\mathcal{N}}$ in $\mathbb{C}^{4|2\mathcal{N}}$. On the other hand, a fixed point $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}) \in \mathbb{C}^{4|2\mathcal{N}}$ gives a holomorphic embedding of the Riemann sphere into supertwistor space. Thus, we have

$$\begin{array}{llll} \text{(i)} & \text{a point } p \in \mathcal{P}^{3|\mathcal{N}} & \longleftrightarrow & \text{an isotropic plane } \mathbb{C}_p^{2|\mathcal{N}} \hookrightarrow \mathbb{C}^{4|2\mathcal{N}}, \\ \text{(ii)} & \mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}} & \longleftrightarrow & \text{a point } (x,\eta) \in \mathbb{C}^{4|2\mathcal{N}}. \end{array}$$

2.2. Holomorphy and self-dual SYM theory in the twistor approach

In order to study super gauge theory, some additional data on the manifolds appearing in the double fibration (2.7) is required. Let us consider a rank n holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ which is characterized by the transition function $f = \{f_{+-}\}$ and its pull-back $\pi_2^*\mathcal{E}$ to the supermanifold $\mathcal{F}^{5|2\mathcal{N}}$. For notational reasons, we denote the pulled-back transition function by the same letter f. By definition of a pull-back, the transition function f is constant along the fibers of $\pi_2 : \mathcal{F}^{5|2\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}$. Therefore, it is annihilated by the vector fields

$$D_\alpha^\pm \ := \ \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha} \qquad \text{and} \qquad D_\pm^i \ := \ \lambda_\pm^{\dot\alpha}\partial_{\dot\alpha}^i, \tag{2.9}$$

where $\partial_{\alpha\dot\alpha} = \partial/\partial x^{\alpha\dot\alpha}$ and $\partial_{\dot\alpha}^i = \partial/\partial\eta_i^{\dot\alpha}$. Spinorial indices are raised and lowered via the ϵ-tensors, $\epsilon^{12} = \epsilon^{\dot1\dot2} = -\epsilon_{12} = -\epsilon_{\dot1\dot2} = 1$, together with the normalizations $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}$. Let $\bar\partial_{\mathcal{P}}$ and $\bar\partial_{\mathcal{F}}$ be the anti-holomorphic parts of the exterior derivatives on the supertwistor space and the correspondence space, respectively. Then we have $\pi_2^*\bar\partial_{\mathcal{P}} = \bar\partial_{\mathcal{F}} \circ \pi_2^*$, and hence, the transition function of $\pi_2^*\mathcal{E}$ is also annihilated by $\bar\partial_{\mathcal{F}}$.

Next we want to assume that the bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ is holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. This condition implies that there exist some smooth $GL(n,\mathbb{C})$-valued functions $\psi = \{\psi_\pm\}$, which define a trivialization of $\pi_2^*\mathcal{E}$, such that $f = \{f_{+-}\}$ can be decomposed as

$$f_{+-} \ = \ \psi_+^{-1}\psi_- \tag{2.10}$$

and

$$\bar\partial_{\mathcal{F}}\psi_\pm \ = \ 0. \tag{2.11}$$

In particular, this formula implies that the $\psi_\pm$ depend holomorphically on $\lambda_\pm$. Applying the vector fields (2.9) to (2.10), we realize by virtue of an extension of Liouville's theorem that the expressions

$$\psi_+ D_\alpha^+ \psi_+^{-1} \ = \ \psi_- D_\alpha^+ \psi_-^{-1} \qquad \text{and} \qquad \psi_+ D_+^i \psi_+^{-1} \ = \ \psi_- D_+^i \psi_-^{-1} \tag{2.12}$$

must be at most linear in $\lambda_\pm$. Therefore, we may introduce a Lie-algebra valued one-form $\mathcal{A}$ such that

$$\begin{aligned} D_\alpha^\pm \lrcorner \mathcal{A} \ &:= \ \mathcal{A}_\alpha^\pm \ := \ \lambda_\pm^{\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} \ = \ \psi_\pm D_\alpha^\pm \psi_\pm^{-1}, \\ D_\pm^i \lrcorner \mathcal{A} \ &:= \ \mathcal{A}_\pm^i \ := \ \lambda_\pm^{\dot\alpha}\mathcal{A}_{\dot\alpha}^i \ = \ \psi_\pm D_\pm^i \psi_\pm^{-1}, \end{aligned} \tag{2.13}$$

and hence

$$\begin{aligned} \lambda_\pm^{\dot\alpha}(\partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha})\psi_\pm \ &= \ 0, \\ \lambda_\pm^{\dot\alpha}(\partial_{\dot\alpha}^i + \mathcal{A}_{\dot\alpha}^i)\psi_\pm \ &= \ 0 \end{aligned} \tag{2.14}$$

and $\bar\partial_{\mathcal{F}}\psi_\pm = 0$. The compatibility conditions for the linear system (2.14) read as

$$[\nabla_{\alpha(\dot\alpha}, \nabla_{\beta\dot\beta)}] \ = \ 0, \quad [\nabla^i_{(\dot\alpha}, \nabla_{\alpha\dot\beta)}] \ = \ 0 \quad \text{and} \quad \{\nabla^i_{(\dot\alpha}, \nabla^j_{\dot\beta)}\} \ = \ 0, \tag{2.15}$$

where we have introduced

$$\nabla_{\alpha\dot\alpha} \ := \ \partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \nabla_{\dot\alpha}^i \ := \ \partial_{\dot\alpha}^i + \mathcal{A}_{\dot\alpha}^i. \tag{2.16}$$

Eqs. (2.15) have been known for quite some time and it has been shown that they are equivalent to the equations of motion of $\mathcal{N}$-extended self-dual SYM theory [14, 15] on four-dimensional space-time. Note that Eqs. (2.14) imply that the gauge potentials $\mathcal{A}_{\alpha\dot\alpha}$ and

$\mathcal{A}^i_{\dot\alpha}$ do not change when we perform transformations of the form $\psi_\pm \mapsto \psi_\pm h_\pm$, where the $h = \{h_\pm\}$ are annihilated by the vector fields (2.9) and $\bar\partial_{\mathcal{F}}$. Under such transformations the transition function $f = \{f_{+-}\}$ of $\pi_2^*\mathcal{E}$ transform into a transition function $h_+^{-1} f_{+-} h_-$ of a bundle which is said to be equivalent to $\pi_2^*\mathcal{E}$. On the other hand, gauge transformations of the gauge potentials are induced by transformations of the form $\psi_\pm \mapsto g^{-1}\psi_\pm$ for some smooth λ-independent $GL(n,\mathbb{C})$-valued g. Under such transformations the transition function f is unchanged. In fact, we have

Theorem 1 *There is a one-to-one correspondence between equivalence classes of holomorphic vector bundles over the supertwistor space which are holomorphically trivial when restricted to any* $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ *and gauge equivalence classes of solutions to the equations of motion of* $\mathcal{N}$*-extended self-dual SYM theory in four dimensions. In fact, Eqs. (2.13) give the Penrose-Ward transform, i.e., the relation between fields on supertwistor space and fields on space-time.*

3 Hidden symmetries

3.1. Infinitesimal deformations

In order to study solutions to the linearized equations of motion (i.e., symmetries), one considers small perturbations of the transition functions $f = \{f_{+-}\}$ of a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and its pull-back $\pi_2^*\mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$, respectively. Note that any infinitesimal perturbation of f is allowed, as small enough perturbations will, by Kodaira's theorem on deformation theory, preserve its trivializability properties on the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. This follows directly from $H^1(\mathbb{C}P^1, \mathcal{O}) = 0$. Thus, we find

$$f_{+-} + \delta f_{+-} \;=\; (\psi_+ + \delta\psi_+)^{-1}(\psi_- + \delta\psi_-) \tag{3.17}$$

for the deformed transition function of $\pi_2^*\mathcal{E}$. Upon introducing the Lie-algebra valued function

$$\phi_{+-} \;:=\; \psi_+(\delta f_{+-})\psi_-^{-1}, \tag{3.18}$$

and linearizing Eq. (3.17), we have to find the splitting

$$\phi_{+-} \;=\; \phi_+ - \phi_-. \tag{3.19}$$

Here, the Lie-algebra valued functions $\phi_\pm$ can be extended to holomorphic functions in $\lambda_\pm$ on the respective patches, and which eventually yield

$$\delta\psi_\pm \;=\; -\phi_\pm\psi_\pm. \tag{3.20}$$

Moreover, we point out that finding such $\phi_\pm$ from ϕ_{+-} means to solve the infinitesimal variant of the Riemann-Hilbert problem. Obviously, the splitting (3.19) and hence solutions to the Riemann-Hilbert problem are not unique, as we certainly have the freedom to consider new $\tilde\phi_\pm$ shifted by some function function ω which is globally defined, i.e., $\tilde\phi_\pm = \phi_\pm + \omega$. In fact, such shifts eventually correspond to infinitesimal gauge transformations.

Infinitesimal variations of the linear system (2.14) yield

$$\delta\mathcal{A}^\pm_\alpha \;=\; \lambda^{\dot\alpha}_\pm\delta\mathcal{A}_{\alpha\dot\alpha} \;=\; \lambda^{\dot\alpha}_\pm\nabla_{\alpha\dot\alpha}\phi_\pm \qquad \text{and} \qquad \delta\mathcal{A}^i_\pm \;=\; \lambda^{\dot\alpha}_\pm\delta\mathcal{A}^i_{\dot\alpha} \;=\; \lambda^{\dot\alpha}_\pm\nabla^i_{\dot\alpha}\phi_\pm, \tag{3.21}$$

where the covariant derivatives have been introduced in (2.16). Note that they act adjointly in these equations. The λ-expansion of Eqs. (3.21) eventually gives the infinitesimal transformation $\delta\mathcal{A}_{\alpha\dot{\alpha}}$ and $\delta\mathcal{A}^i_{\dot{\alpha}}$, which satisfy by construction the linearized equations of motion. Note that the equivalence relations as defined at the end of Sec. 2.2 have an infinitesimal counterpart. Therefore, we altogether have

Corollary 1 *There is a one-to-one correspondence between equivalence classes of deformations of the transition functions of holomorphic vector bundles over the supertwistor space which are holomorphically trivial when restricted to any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and equivalence classes of symmetries of $\mathcal{N}$-extended self-dual SYM theory in four dimensions.*

3.2. Hidden symmetry algebras

Suppose we are given some indexed set $\{\delta_a\}$ of infinitesimal deformations $\delta_a f_{+-}$ of the transition function of our holomorphic vector bundle $\pi_2^*\mathcal{E}$. Suppose further that the δ_as satisfy a deformation algebra of the form

$$[\delta_a, \delta_b\} f_{+-} \; = \; f_{ab}{}^c \delta_c f_{+-}, \tag{3.22}$$

where the $f_{ab}{}^c$s are generically structure functions and $[\cdot,\cdot\}$ denotes the graded commutator. Let us assume that the $f_{ab}{}^c$s are constant. Above we have seen that any such deformation $\delta_a f_{+-}$ yields a symmetry of $\mathcal{N}$-extended self-dual SYM theory. So, given such an algebra, what is the corresponding symmetry algebra on the gauge theory side? To answer this question, we consider

$$[\delta_1, \delta_2] \; = \; (-)^{p_a p_b} \varepsilon^a \varrho^b [\delta_a, \delta_b\}, \tag{3.23}$$

where ε^a and ϱ^b are the infinitesimal parameters of the transformations δ_1 and δ_2, respectively, and p_a denotes the Graßmann parity of the transformation δ_a. Explicitly, we may write

$$[\delta_1, \delta_2]\mathcal{A}^\pm_\alpha \; = \; \delta_1(\mathcal{A}^\pm_\alpha + \delta_2\mathcal{A}^\pm_\alpha) - \delta_1\mathcal{A}^\pm_\alpha - \delta_2(\mathcal{A}^\pm_\alpha + \delta_1\mathcal{A}^\pm_\alpha) + \delta_2\mathcal{A}^\pm_\alpha \tag{3.24}$$

and similarly for $\mathcal{A}^i_\pm$; cf. also (3.21). Then one easily checks that

$$[\delta_1, \delta_2]\mathcal{A}^\pm_\alpha \; = \; \lambda^{\dot{\alpha}}_\pm \nabla_{\alpha\dot{\alpha}} \Sigma^\pm_{12}, \qquad \text{with} \qquad \Sigma^\pm_{12} \; := \; \delta_1\phi^2_\pm - \delta_2\phi^1_\pm + [\phi^1_\pm, \phi^2_\pm]. \tag{3.25}$$

Note that we use the notation $\phi^1_\pm = \varepsilon^a \phi_{\pm a}$ and similarly for $\phi^2_\pm$. Next one considers the commutator

$$[\delta_1, \delta_2] f_{+-} \; = \; \delta_1(f_{+-} + \delta_2 f_{+-}) - \delta_1 f_{+-} - \delta_2(f_{+-} + \delta_1 f_{+-}) + \delta_2 f_{+-}. \tag{3.26}$$

Using the definition (3.18) and the resulting splittings (3.19) for the deformations $\delta_{1,2} f_{+-}$, one can show after some tedious but straightforward algebraic manipulations that the commutator (3.26) is given by

$$[\delta_1, \delta_2] f_{+-} \; = \; \psi_+^{-1}(\Sigma^+_{12} - \Sigma^-_{12})\psi_-, \tag{3.27}$$

where $\Sigma^\pm_{12}$ has been introduced (3.25). By hypothesis (3.22), it must also be equal to

$$[\delta_1, \delta_2] f_{+-} \; = \; \delta_3 f_{+-}, \qquad \text{with} \qquad \delta_3 \; = \; (-)^{p_a p_b} \varepsilon^a \varrho^b f_{ab}{}^c \delta_c, \tag{3.28}$$

i.e.,

$$[\delta_1, \delta_2] f_{+-} \;=\; \psi_+^{-1}(\phi_+^3 - \phi_-^3)\psi_-, \tag{3.29}$$

where $\phi_\pm^3 = (-)^{p_a p_b}\varepsilon^a \varrho^b f_{ab}{}^c \phi_{\pm c}$. By comparing this equation with the result (3.27), we therefore conclude

$$\Sigma_{12}^{\pm} \;=\; \phi_\pm^3 + \omega^3 \;=\; (-)^{p_a p_b}\varepsilon^a \varrho^b (f_{ab}{}^c \phi_{\pm c} + \omega_{ab}), \tag{3.30}$$

since the $f_{ab}{}^c$s are assumed to be constant. Here, ω^3 (respectively, ω_{ab}) is some function independent of $\lambda_\pm$ and therefore representing an infinitesimal gauge transformation. Combining this result with Eq. (3.25), we get the following

Theorem 2 *Suppose we are given a deformation algebra of the form (3.22) with constant $f_{ab}{}^c$. Then the corresponding symmetry algebra on the gauge theory side has exactly the same form modulo possible gauge transformations.*

It should be stressed that this theorem does, however, not give the explicit form of the gauge parameter ω_{ab}. In order to compute it, one has to perform the splitting procedure explicitly; see [7] for details.

3.3. Examples: Affine extensions of gauge type and superconformal symmetries

So far, we have been quite general. Let us now exemplify our discussion. Let X_a be some generator of the gauge algebra $\mathfrak{gl}(n, \mathbb{C})$ and consider

$$\delta_a^m f_{+-} \;:=\; \lambda_+^m [X_a, f_{+-}], \qquad \text{for} \qquad m \in \mathbb{Z}. \tag{3.31}$$

For $m = 0$, the transformations of the components of the gauge potential are given by

$$\delta_a^0 \mathcal{A}_{\alpha\dot\alpha} \;=\; [X_a, \mathcal{A}_{\alpha\dot\alpha}] \qquad \text{and} \qquad \delta_a^0 \mathcal{A}_{\dot\alpha}^i \;=\; [X_a, \mathcal{A}_{\dot\alpha}^i]. \tag{3.32}$$

Thus, they represent a gauge type transformation with constant gauge parameter (a global symmetry transformation). The corresponding deformation algebra is easily computed to be

$$[\delta_a^m, \delta_b^n] f_{+-} \;=\; f_{ab}{}^c \delta_c^{m+n} f_{+-}, \tag{3.33}$$

where the $f_{ab}{}^c$s are the structure constants of $\mathfrak{gl}(n, \mathbb{C})$, i.e., we get a centerless Kac-Moody algebra. By virtue of our above theorem, we will get the same algebra (modulo gauge transformations) on the gauge theory side (for explicit calculations see also [7]). Note that such deformations can be used for the explicit construction of solutions to the field equations [16].

Another example is concerned with affine extensions of superconformal symmetries. In [7], it was shown that the generators of the superconformal algebra when viewed as vector fields have to be pulled back to the correspondence space (and hence to the supertwistor space) in a very particular fashion. Their pull-backs are explicitly given by

$$\begin{aligned}
\widetilde{P}_{\alpha\dot\alpha} &= P_{\alpha\dot\alpha}, \qquad \widetilde{Q}_{i\alpha} = Q_{i\alpha}, \qquad \widetilde{Q}^i_{\dot\alpha} = Q^i_{\dot\alpha},\\
\widetilde{D} &= D,\\
\widetilde{K}^{\alpha\dot\alpha} &= K^{\alpha\dot\alpha} + x^{\alpha\dot\beta} Z_{\dot\beta}{}^{\dot\alpha}, \qquad \widetilde{K}^{i\alpha} = K^{i\alpha}, \qquad \widetilde{K}_i^{\dot\alpha} = K_i^{\dot\alpha} + \eta_i^{\dot\beta} Z_{\dot\beta}{}^{\dot\alpha},\\
\widetilde{J}_{\alpha\beta} &= J_{\alpha\beta}, \qquad \widetilde{J}_{\dot\alpha\dot\beta} = J_{\dot\alpha\dot\beta} - \tfrac{1}{2} Z_{\dot\alpha\dot\beta},\\
\widetilde{T}_i^j &= T_i^j, \qquad \widetilde{A} = A,
\end{aligned} \tag{3.34}$$

where the untilded quantities, commonly denoted by N_a in the sequel, are the usual vector field expressions for the superconformal generators on four-dimensional superspace-time and

$$Z_{\dot\alpha\dot\beta} \;:=\; \lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\partial_{\lambda_\pm} + \hat\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta}\partial_{\bar\lambda_\pm}. \tag{3.35}$$

Here, $(\hat\lambda^+_{\dot\alpha}) := {}^t(-\bar\lambda_+, 1)$ and $\hat\lambda^+_{\dot\alpha} = \bar\lambda_-^{-1}\hat\lambda^-_{\dot\alpha}$. Let us define the following (holomorphic) action on the transition function:

$$\delta^m_a f_{+-} \;:=\; \lambda^m_+ \widetilde N_a f_{+-}, \qquad \text{for} \qquad m\in\mathbb{Z}, \tag{3.36}$$

where $\widetilde N_a$ represents any of the generators given above. Note that the $\bar\lambda$-derivative drops out as f_{+-} is holomorphic in $\lambda_\pm$. For $m=0$, we find

$$\delta^0_a\mathcal{A}_{\alpha\dot\alpha} \;=\; \mathcal{L}_{N_a}\mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \delta^0_a\mathcal{A}^i_{\dot\alpha} \;=\; \mathcal{L}_{N_a}\mathcal{A}^i_{\dot\alpha}, \tag{3.37}$$

where $\mathcal{L}_{N_a}$ is the Lie derivative along N_a. Furthermore, one straightforwardly deduces

$$[\delta^m_a, \delta^n_b\} f_{+-} \;=\; (f_{ab}{}^c + n g_a \delta^c_a - (-)^{p_a} m g_b \delta^c_a)\delta^{m+n}_c f_{+-}, \tag{3.38}$$

what represents a centerless Kac-Moody-Virasoro type algebra. Here, the $f_{ab}{}^c$s are the structure constants of the superconformal algebra. The g_as are abbreviations for $\lambda_+^{-1}\widetilde N_a^{\lambda_+}$, where $\widetilde N_a^{\lambda_+}$ represents the ∂_{λ_+}-component of $\widetilde N_a$. Hence, this time we obtain structure functions rather than structure constants and therefore we have to restrict our discussion to a certain subalgebra of the superconformal algebra in order to apply the above theorem. The most naive way of doing this is simply by dropping the special conformal generators $\widetilde K^{\alpha\dot\alpha}$ and $\widetilde K^{\dot\alpha}_i$ and the rotation generators $J_{\dot\alpha\dot\beta}$. Then one eventually obtains honest structure constants and can therefore use the theorem. However, in [7] we have seen that one need not to exclude the rotation generators $J_{\dot\alpha\dot\beta}$, since the structure functions for the maximal subalgebra of the superconformal algebra which does not contain $\widetilde K^{\alpha\dot\alpha}$ and $\widetilde K^{\dot\alpha}_i$ are only dependent on $\lambda_\pm$. By inspecting the formulas (3.21), we see that such λ-dependent functions do not spoil the generic form of the transformations on the gauge theory side, i.e., the corresponding symmetry algebra still closes. This is not the case when $\widetilde K^{\alpha\dot\alpha}$ and $\widetilde K^{\dot\alpha}_i$ are included as the structure functions also depend on $x^{\alpha\dot\alpha}$ and $\eta^{\dot\alpha}_i$, respectively. For more details see [7].

Finally, let us stress that the existence of such such algebras originates from the fact that the full group of continuous transformations acting on the space of holomorphic vector bundles over the supertwistor space is a semi-direct product of the group of local holomorphic automorphisms (i.e., complex structure preserving maps) of the supertwistor space and of the group of one-cochains for a certain covering of the supertwistor space with values in the sheaf of holomorphic maps of the supertwistor space into the gauge group. This can be shown by following and generalizing the lines presented in the case of the purely bosonic self-dual YM equations [12, 13].

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Twistor spaces, mirror symmetry and self-dual Kähler manifolds

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Abstract

We present the evidence for two conjectures related to the twistor string. The first conjecture states that two super-Calabi Yaus – the supertwistor space and the superambitwistor space – form a mirror pair. The second conjecture is that the B-model on the twistor space can be seen as describing a 4-dimensional gravitational theory, whose partition function should involve a sum over "space-time foams" related to D1 branes in the topological string.

1 Introduction

Witten's reformulation of perturbative $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory as a topological string theory with the supertwistor space as target [1] has inspired a large number of works that have extended the original insight and fulfilled one of the initial expectations, namely that of simplifying the calculations of YM amplitudes by exploiting the holomorphic properties they exhibit in twistor space (see [2] for a review of these results and a fuller list of references). This achievement was in keeping with the general philosophy of the twistor program, which has at its starting point the observation that complicated field equations in Minkowski space can be recast in the form of simpler cohomology problems via the so-called Penrose transform ([3] and [4] for a general presentation).

By contrast, some puzzles that were pointed out by Witten from the beginning are still unsolved. Probably the most important open question is that of the coupling (or decoupling) of the closed string modes. While they can be ignored at tree level, they inevitably contribute in loops and one has to find a way of disentangling them from field theory, or else of making sense of field theory coupled to conformal supergravity. In other directions there has been some progress, *e.g.* in extending the correspondence to situations with lower supersymmetry, and in setting up the correspondence using the ambitwistor construction, which avoids breaking the parity symmetry between self-dual and anti-self-dual solutions.

One remarkable aspect of Witten's proposal is the fact that the complete 4-dim gauge theory is described in terms of a topological string; in their usual applications to Calabi-Yau compactifications, topological strings can describe only the F-terms of the corresponding field theory. However the Calabi-Yau appearing here as the twistor space is not part of some internal geometry, but it contains the space-time as well. Moreover, it is a supermanifold. So even though formally the twistor topological string is not different from its more usual incarnations, it encodes the physical information in a rather different way. This begs the question of how much of the common lore concerning these models can be directly applied to this situation. The most striking result of the investigations of topological strings is the existence of mirror symmetry: Calabi-Yau spaces come in pairs, and two spaces in a mirror pair must satisfy some conditions, *e.g.* relations between their Hodge numbers. It is not entirely obvious whether this should extend to the case of super-CY, and how. One problem is that a complete cohomology theory for supermanifolds is still lacking (for discussions on these issues see [5, 6, 7]). Nevertheless, this question can be addressed at a formal level, applying the usual techniques of gauged linear sigma model. In section 2 we discuss these issues and present results obtained with S.P. Kumar [8].

In section 3 we look at the closed string sector, and try to interpret the D1 branes as gravitational sources. This work was done with S. Hartnoll [9]. It can be argued that the backreaction of the branes on the geometry is related, via the twistor correspondence, to a transition between different gravitational instantons in 4 dimensions. This raises the possibility of having a sector of 4-dim gravity that can be described by a gas of D1-branes; in spirit this would be similar to the quantum foam description of the gravity sector of the A-model topological string on CY 3-folds [10]. Whether such a description can be useful in this case and whether may lead to some exact results remains to be seen.

2 Mirror symmetry

Let us now briefly review the twistor and ambitwistor constructions and the arguments in favor of the existence of mirror symmetry put forward in [11, 12] and then in [8].

The (bosonic) twistor space $\mathcal{Z} = \mathbb{CP}^3$ is a $\mathbb{CP}^1$ fibration over S^4. The fibre over a point p parametrizes almost-complex structures on T_pS^4. There is a 1-1 correspondence (Penrose-Ward) between rank r holomorphic vector bundles on $\mathcal{Z}$, trivial when restricted to the fibres, and $SU(r)$ anti-self dual connections on S^4 (or suitable open sets on both sides). This is, in essence, because an ASD gauge field has a field strength of type $(0, 2)$ for any choice of complex structure compatible with the orientation. Topologically, the bundle on $\mathcal{Z}$ is the pull-back of the gauge bundle on S^4, and the holomorphic structure is given by a $(0, 1)$ connection $\mathcal{A}$ satisfying $\bar{\partial}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$. The latter are the field content and the equations of motion of holomorphic Chern-Simons theory, which in turn is the effective space-time theory of the open string sector of the topological B-model on a Calabi-Yau. The action is

$$S = \int_{\mathcal{Z}} \Omega \wedge (\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \,. \tag{1}$$

Here Ω is a $(3, 0)$ form. $\mathcal{Z}$ is not a Calabi-Yau, but the problem is circumvented by adding fermionic coordinates. The total space is then taken to be $\mathbb{CP}^{3|4}$, which can also be seen as a fermionic bundle $\Pi(\mathcal{O}(1) \otimes \mathbb{C}^4)$ over the bosonic twistor space. This is a super-CY, since it has a nowhere vanishing holomorphic volume form, $\Omega = \epsilon_{\mu\nu\rho\sigma}\epsilon_{ABCD}\, z^\mu dz^\nu dz^\rho dz^\sigma d\psi^A d\psi^B d\psi^C d\psi^D$. The connection $\mathcal{A}$ becomes a superfield and its component expansion

$$\mathcal{A} = d\bar{z}\left(A + \psi^I \chi_I + \psi^I \psi^J \phi_{IJ} + \psi^I \psi^J \psi^K \epsilon_{IJKL} \tilde{\chi}^L + \psi^4 G\right) \tag{2}$$

yields the $\mathcal{N} = 4$ SYM multiplet in a helicity basis. The Chern-Simons action reproduces the action for self-dual YM fields, and the complete action is recovered by including the contributions of D1 branes wrapped on rational curves in $\mathcal{Z}$. The simplest case is when the curves have degree 1; they are specified by equations

$$\omega_{\dot{\alpha}} = x_{\alpha\dot{\alpha}} \lambda^\alpha\,, \quad \psi^I = \theta^I_\alpha \lambda^\alpha\,. \tag{3}$$

These equations admit a dual interpretation: a point (x, θ) in superspace defines a fibre of the twistor space, and viceversa a point $(\lambda, \omega) \in \mathcal{Z}$ defines a self-dual plane in (complexified) Minkowski space[1]. It may be surprising that twistors, in principle well apt to describe self-dual field configurations, may turn out to be the most useful way of describing the complete perturbative sector of Yang-Mills. In fact, the breaking of parity invariance is somewhat unsatisfactory, and can be avoided by using the ambitwistor construction [13].

The idea underlying the ambitwistor construction is the fact that the classical equations of motion of $\mathcal{N} = 4$ SYM follow from the condition of integrability of gauge fields on supersymmetric null lines. A null line is the intersection of a self-dual and an anti-self-dual plane, and so is defined by two points in two copies of $\mathbb{CP}^3$ with a condition to ensure that the intersection of the corresponding planes is not empty. The suitably supersymmetrized space turns out to be a quadric in $\mathbb{CP}^{3|3} \times \widetilde{\mathbb{CP}}^{3|3}$:

$$\mathcal{N} = \{\omega_{\dot{\alpha}} \tilde{\omega}^{\dot{\alpha}} - \lambda^\alpha \tilde{\lambda}_\alpha + \psi^A \tilde{\psi}_A = 0\}\,. \tag{4}$$

This is again a super-CY, and one could envision finding a topological string that lives on this space and reproduces the full YM amplitudes. This seems to be a very non-trivial problem (see attempts at its solution in [14, 15]).

As it has been argued in [11] the S-duality of physical string theory descends to the topological sector. This implies the existence of an A-model realization of $\mathcal{N} = 4$ Yang-Mills, where the D1-instantons would be replaced by worldsheet instantons (and the D5-branes by NS5-branes). Given such an A-model description, one would expect the mirror B-model to realize the perturbative Yang-Mills amplitudes classically, without the help of instantons. The natural candidate for such a mirror is then the ambitwistor space.

Since the quadric $\mathcal{N}$ is a hypersurface in a toric variety, it is possible to use the techniques explained in [16, 17] in order to find the mirror manifold. The homogeneous coordinates of the toric variety correspond to chiral fields Φ_i in a $(2, 2)$-susy $U(1)$ gauged linear sigma model, and mirror symmetry amounts to performing a T-duality. A degree d hypersurface is described in the sigma model via a superpotential $W = \; P\, G(\Phi_i)$, where P is a field of charge $-d$. The vacuum equations set $P = 0, G = 0$. We also have to use the results of [6], where it is shown that A-model observables on a hypersurface $M \subset V$ can

[1] We ignore here all the issues concerning the signature of spacetime.

be computed on a supermanifold that is a fermionic bundle over V. One gets it by simply replacing the auxiliary field P with a fermionic superfield Ψ of charge d. In trading M for the bundle one is throwing away all the information about the superpotential, except its degree; this is not a problem since the parameters of W are complex structure deformations to which the A-model is insensitive.

The T-duality action on the fermions is similar to the one on bosons: the phase of Ψ dualizes into a bosonic twisted chiral multiplet Y, such that $Y + \bar{Y} = \bar{\Psi}\Psi$ while the imaginary part is periodic. However one needs something more, namely two additional fermions η, χ to preserve the virtual (bosonic minus fermionic) dimension of the space. There must also be a superpotential $W = -q\ \Sigma(Y - \eta\chi) + e^{-Y}$, to account for all the massive excitations. It is now straightforward to apply these results to the hypersurface (4) (see [8]). Let us note that the quadric has two Kähler moduli t_1, t_2 inherited from the embedding, corresponding to the sizes of the two $\mathbb{CP}^3$. The computation yields the following partition function for the Landau-Ginzburg B-model:

$$Z = e^{t_1} \int \prod_{a=1,2} [dx_a du_a dv_a d\eta_a d\chi_a] \prod_a \delta(u_a v_a + \eta_a \chi_a - x_a + 1)\, \delta(e^{t_1}(e^{-t_2} - 1) x_1 x_2 - 1)\,. \quad (5)$$

The δ-functions inside the integral contain the information on the geometry of the mirror manifold. The first thing to note is that it has dimension $(3|4)$ as expected. In the limit $t_2 \to 0$ the partition function (5) can be interpreted as an integral in an affine patch of $\mathbb{CP}^{3|4}$, thus confirming the conjecture that the latter and the quadric are mirror partners. However for generic t_2 the equations we find describe some deformation of the space. Strictly speaking $\mathbb{CP}^3$ is a rigid manifold and does not have complex deformations, and the same probably holds for its supersymmetric version. It is likely that one can make sense of the deformations only in an affine patch. It would be interesting to understand better these deformations, as they could give more insight into the nature of the corresponding A-model deformations.

It is worthwhile to notice in this respect that the above derivation of the mirror manifold is somewhat formal. One would like to substantiate the conjecture with the computation of observables on both sides. This would require first of all a proper definition of the observables, which has been problematic especially on the A-model side. The S-duality conjecture described above suggests that there should also be a B-model description in ambitwistor space, but, as already observed, at present this has not yet been found.

3 A spacetime quantum foam

While most works on the twistor string have focused on the open string sector, it is also interesting to consider the closed string sector, corresponding to conformal supergravity [18]. Here the correspondence rests upon a generalization of Penrose's construction that associates a complex 3-fold $\mathcal{Z}$ to any 4-dimensional *conformally self-dual* (*i.e.* with a self-dual Weyl tensor C_{abcd}) Riemannian manifold $\mathcal{M}$ [19]. Specializing to the case $\mathcal{M} = S^4$ one recovers the twistor space $\mathbb{CP}^3$ we have used so far. One consequence is that deformations of the complex structure of $\mathcal{Z}$ correspond to perturbations of the metric of $\mathcal{M}$ that preserve self-duality. We will argue that the effect of adding D1 branes in $\mathcal{Z}$ is equivalent to considering blow-ups of the 4-dim space $\mathcal{M}$, and that this may lead to a quantum foam description of (a sector of) quantum gravity on $\mathcal{M}$.

Manifolds with self-dual Weyl tensor can be thought of as gravitational instantons, as they minimize the action $\int \mathrm{d}vol\, C_{abcd}C^{abcd}$ in a topological sector given by the value of the Hirzebruch signature $\tau = b_2^+ - b_2^-$. In the B-model realization, D1 branes act as sources for the holomorphic form of the CY, whose periods parametrize the complex moduli. Therefore adding D1 branes should be equivalent to consider complex deformations once the backreaction is taken into account [20]. In a sense this means generalizing the twistor correspondence to non-perturbative gravitational fluctuations. It is natural to conjecture that the number of branes should correspond to the Hirzebruch signature. One can increase the value of τ by taking the connected sum $\mathcal{M}\#\overline{\mathbb{CP}^2}$; it is known that the resulting manifold is again conformally self-dual. We have then the picture that a gas of gravitational instanton can be described as a gas of D-branes in the B-model.

In order to understand what our goal is, it is helpful to compare to the S-dual description: the effective action for the A-model is a theory of Kähler gravity in 6 dimensions, but (at least on toric manifolds) the strong coupling regime is more usefully described as a "quantum foam" made up of successive blow-ups of the manifold at the corners of its toric base [10]. The twistor string offers the possibility of a similar result for a version of 4-dimensional gravity that is dominated by instantons in some regime. We have explored this issue in [9].

Although we conjecture that the picture presented above should be quite general, we focus on a particular class of (asymptotically flat) self-dual Kähler manifolds, which are blow-ups of $\mathbb{C}^2$ at a finite number of points. Their twistor spaces are explicitly known, they have been constructed by Lebrun [21], and they have the special property of being bimeromorphic to projective spaces, which makes them amenable to investigation with algebraic-geometric techniques. In the following we outline the construction of the twistor spaces and we show how this class of examples verifies our conjecture, in the sense that the data of the Kähler geometry are captured by the periods of the holomorphic form on $\mathcal{Z}$, in turn related to the D1-brance charge. We add some details about the supersymmetric version of the twistor spaces, which is necessary, just as in the flat case, in order to have a well-defined topological string.

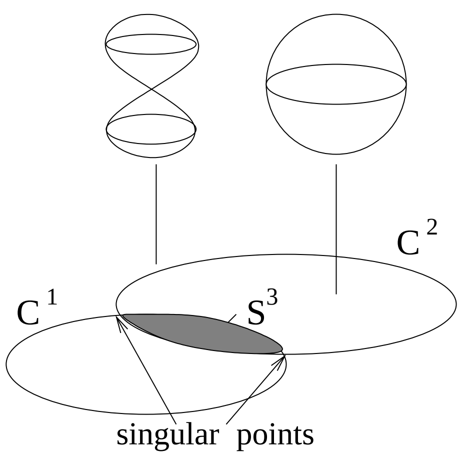
We start from the 4-fold $\mathcal{B}$ (a projective bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$) obtained as a quotient of $\mathbb{C}^7$ by the following identifications

$$\begin{aligned} [z_0, z_1] \sim &\quad \lambda[z_0, z_1]\,, \\ [\zeta_0, \zeta_1] \sim &\quad \mu[\zeta_0, \zeta_1]\,, \\ [x, y, t] \sim &\quad \nu[\lambda^{n-1}\mu\, x, \lambda\mu^{n-1}\, y, t]\,. \end{aligned} \tag{6}$$

The equation

$$F \equiv xy - t^2 \prod_{j=1}^{n} P^j = 0\,. \tag{7}$$

defines a singular hypersurface $\widetilde{Z} \subset \mathcal{B}$. The P^j are n polynomials of the form $P^i = a^i_{mn}\zeta_m z_n$, and their zeroes are n curves $\{C^i\}_{i=1}^n$ in the base. We assume that the curves are nondegenerate and generic, implying that they all mutually intersect at precisely two points. Away from the curves the manifold $\widetilde{Z}$ is a $\mathbb{CP}^1$ fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Over the curves the fibration degenerates to two spheres joined at a point, as illustrated in the figure. The degeneration is regular except for the points where two curves intersect. Each pair of curves intersect at two points, so there will be a total of $n(n-1)$ singular points.



The singularities are conifold points, and they can be smoothed by taking a small resolution. The resulting manifold $\mathcal{Z}$ is smooth and is a twistor space[2], since it has a foliation by $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and it has an antiholomorphic involution under which the fibres are invariant. One can show that $\mathcal{Z}$ is the twistor space of a blow-up of $\mathbb{C}^2$ at n points, and that $b_2(\mathcal{Z}) = n + 1$. The most important property of $\mathcal{Z}$ for our purposes is that it contains a divisor D that is a section of the twistor fibration, *i.e.* it intersects every fibre in one point. It can be written as $\{z = c\}$ in terms of a local coordinate $z = z_0/z_1$. The involution takes it to another divisor $\bar{D} = \{\zeta = \bar{c}\}$. For twistor spaces of Kähler manifolds it is known that the line bundle corresponding to the divisor $D \cup \bar{D}$ is $K_{\mathcal{Z}}^{-1/2}$. This means that if s is a section of this line bundle, we can use it to write a meromorphic 3-form globally defined on $\mathcal{Z}$. Explicitly we take $s = (z_0 - cz_1)(\zeta_0 - \bar{c}\zeta_1)$, and

$$\Omega_{\mathcal{Z}} = \frac{dz\, d\zeta\, dt}{t\, s^2}\,. \tag{8}$$

This is in fact the Penrose transform of the Kähler form ω of the 4-manifold $\mathcal{M}$. The Penrose transform involves a contour integral on the twistor fibres. The Kähler moduli of $\mathcal{M}$ are given by the integrals of ω on a basis of 2-cycles $\Sigma_i \subset \mathcal{M}$. If we take the corresponding 3-cycles $L_i \subset \mathcal{Z}$ that include the chosen contours in the fibres, we obtain

$$\int_\Sigma \omega = \int_L \Omega_{\mathcal{Z}}\,. \tag{9}$$

This is the anticipated relation that allows us to describe the Kähler geometry of $\mathcal{M}$ in terms of period integrals that are computed by the topological B-model on $\mathcal{Z}$. We stress again that this relation is not limited to the class of examples we study, rather it holds for every self-dual Kähler 4-manifold.

Two observations are in order at this point. First, we wanted to understand the complex deformations of $\mathcal{Z}$ as coming from D1 brane charge, as given by

$$\int \Omega = g_s N\,. \tag{10}$$

But the number N is an integer, so we might expect that the Kähler moduli of $\mathcal{M}$ will also turn out to be quantized, exactly as it happens for the CY quantum foam of [10].

[2]This is true only after some additional blow-downs, we skip the details here. Note also that a deformation of the conifold singularity, as opposed to the resolution, would not give a twistor space.

It would be interesting to make this conjecture more precise. Second, the 3-form we found is meromorphic. This could not be otherwise since $\mathcal{Z}$ is not Calabi-Yau, but we need a holomorphic 3-form to define the B-model. We already know the solution from the flat case: it is necessary to add fermionic coordinates. There is a general recipe for producing a super-CY out of any twistor space, but for Lebrun's manifolds we can use a simpler approach. First we extend the projective bundle $\mathcal{B}$ adding a rank 4 fermionic vector bundle E, with coordinates η_i, and define the super-twistor space $\mathcal{Z}_S$ by a suitable modification of the hypersurface equation. The ansatz

$$F \to \mathcal{F} = F + \frac{F}{ts^2}\eta^1\eta^2\eta^3\eta^4 \tag{11}$$

is well-defined provided the fermions are assigned the right scaling properties under (6). Then we can define a holomorphic top-form on $\mathcal{Z}_S$ using Poincare's residue map. It is given by

$$\Omega_{\mathcal{Z}_S} = \frac{\Omega_{\mathcal{B}}}{d\mathcal{F}}\,. \tag{12}$$

where $\Omega_{\mathcal{B}}$ is the unique form on $\mathcal{B}$ with a pole along $\mathcal{F} = 0$. The form (12) is well-defined and holomorphic, and it reduces to (8) if the fermions are integrated out. This construction involved some arbitrary choices and it would be interesting to understand to what extent these choices are fixed. Note that a similar holomorphic superform has been suggested in [15] for the ambitwistor space.

Finally, we observe that an interesting question is whether the mirror symmetry between twistor and ambitwistor space extends beyond the flat case. Obviously it is necessary to find appropriate supersymmetric generalizations so that one has Calabi-Yaus on both sides, and for the ambitwistor space, at least to the author's knowledge, the general prescription is not known.

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