Homogeneous Space Construction of Modular Invariant QFT Models with a Chiral $U(1)$ Current *

R. P. Paunov
Department of Physics, University of Sofia

I. T. Todorov
Institute of Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences, Sofia

Abstract

The local field representations of the chiral $U(1)$ conformal current algebra on the circle are classified. A homogeneous space $SO(2n)/SO(n)$ construction of the resulting lowest weight modules is given, which uses level 1 representations of $\widehat{SO}(2n)$ and level 2 representations of the $\widehat{SO}(n)$ gauge Lie algebra. The modular invariant partition function of these models are, essentially, those listed in [Di 1], [Ge 2]. The "squared Ising model", the level 1 $\widehat{A}_1$-theory and the $N = 2$ extended superconformal model (for $c = 1$) appear as special cases.

1 Introduction

We are concerned in this paper with 2-dimensional quantum field theory (QFT) models of the chiral $U(1)$ conformal current algebra. It is another step towards the classification of modular and conformal invariant theories, following recent work in Saclay, Princeton, Nordita etc. -see, e.g., [Ca 1,2], [Di 1], [Ge 1,2], [Ri 1]. It is based on a study of finite temperature conformal QFT [Bu 1] which we proceed to summarize.

The algebra of observables $\mathcal{A}$ is assumed to be generated by the right and left $U(1)$ current algebra. The compact picture right movers' current

$$\mathcal{J}(\vartheta) = \frac{1}{2}(\mathcal{J}^0 + \mathcal{J}^1) = \sum_n J_n e^{-in\vartheta}$$
$$(x^0 - x^1 = 2\tan\frac{\vartheta}{2}) \tag{1.1}$$

commutes with the left current $\overline{\mathcal{J}}(\overline{\vartheta})$ (where $2\tan\frac{\overline{\vartheta}}{2} = x^0 + x^1$). The modes $\widetilde{L}_n$ of the right movers' stress energy tensor

$$\mathcal{T}(\vartheta) = \frac{1}{2}(\mathcal{T}_0^0 + \mathcal{T}_0^1) = \sum_n \widetilde{L}_n e^{-in\vartheta}, \widetilde{L}_n = L_n - \frac{c\delta_n}{24}$$
$$\delta_n \equiv \delta_{n,0} \tag{1.2}$$

*XVI International Colloquium on Group Theoretical Methods in Physics, Varna, Bulgaria 1987

and the current modes J_n satisfy the commutation relations (CR) of the (chiral) $VU(1)$ conformal current algebra, the semidirect product of the canonical Heisenberg algebra

$$[J_n, J_m] = n\delta_{n+m} \tag{1.3}$$

and the *Virasoro algebra* **Vir**

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}n^3\delta_{n+m} \tag{1.4}$$

the mixed CR are determined from the requirement that the current is a primary field of weight 1 with respect to **Vir** [Be 1]:

$$[\tilde{L}_n, \mathcal{J}(\vartheta)] = -i\frac{d}{d\vartheta}(e^{in\vartheta}\mathcal{J}(\vartheta)) \text{ or } [J_m, \tilde{L}_n] = mJ_{m+n}. \tag{1.5}$$

(The term $\frac{c}{24}$ in (1.2) comes from the Schwarz derivative, $\frac{c}{12}\{2\tan\frac{\vartheta}{2}, \vartheta\}$; the cocycle $\frac{c}{12}n^3\delta_{n+m}$ differs from the conventional $SU(1,1)$-invariant choice $\frac{c}{12}n(n^2-1)\delta_{n+m}$ by a (linear in n) coboundary, the transition between the corresponding generators $\tilde{L}_n$ and L_n being displayed in (1.2).

Along with the real (ϑ-) picture, we shall also use the analytic (z-) picture in which the current and the stress energy tensor have the form

$$J(z) = \sum_n J_n z^{-n-1} \tag{1.1.a}$$

$$T(z) = \sum_n L_n z^{-n-2}; \tag{1.2.a}$$

J and T are related to $\mathcal{J}$ and $\mathcal{T}$ by

$$\mathcal{J}(\vartheta) = e^{i\vartheta}J(e^{i\vartheta})$$

$$\mathcal{T}(\vartheta) = e^{2i\vartheta}T(e^{i\vartheta}) + \frac{c}{12}\{e^{i\vartheta}, i\vartheta\}.$$

The Schwarz derivative $\{f(t), t\}$, given by

$$\{f,t\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \text{ (so that } \{e^{i\vartheta}, i\vartheta\} = -\frac{1}{2}),$$

is characterized by the invariance of the quadratic differential $\{f,t\}dt^2$ under fractional linear transformation: if $x = \frac{at+b}{ct+d}$ $(ad - bc \neq 0)$ and $f(t) = F(x(t))$ then

$$\{f,t\}dt^2 = \{F,x\}dx^2.$$

We are concerned in [Bu 1] with a family of local field algebras $\mathcal{F}[g^2] \supset \mathcal{A}$ labelled by integer "charge squares"

$$g^2 = 1, 2, \ldots . \tag{1.6}$$

The right moving part of $\mathcal{F}[g^2]$ is generated by a pair of charged fields $\psi(z, \pm g)$ characterized by the property of being $VU(1)$-primary ([Kn 1] [To 1])

$$[J_n, \psi(z,g)] = gz^n\psi(z,g) \tag{1.7.a}$$

$$[L_n, \psi(z,g)] = z^n\left(z\frac{\partial}{\partial z} + (n+1)\Delta\right)\psi(z,g). \tag{1.7.b}$$

We are dealing with representations of $\mathcal{F}[g^2]$ in a (positive metric) Hilbert space $\mathcal{H}$ with the following properties.

(A) The generators of the conformal current algebra satisfy the hermiticity condition

$$J_n^* = J_{-n} \quad \widetilde{L}_n^* = \widetilde{L}_{-n} \quad (L_n^* = L_{-n}) \tag{1.7}$$

(and a similar relation for the left movers' modes $\overline{J}_n$ and $\overline{L}_n$).

(B) There is a unique vacuum state $\mid 0\rangle \in \mathcal{H}$ ($\langle 0 \mid 0\rangle = 1$) satisfying

$$J_n \mid 0\rangle = 0 \; (= \overline{J_n} \mid 0\rangle) \quad \text{for} \quad n = 0, 1, 2, \dots . \tag{1.8}$$

If we identify the energy with the conformal Hamiltonian

$$H = \widetilde{L}_0 + \widetilde{\overline{L}}_0 \; (= L_0 + \overline{L}_0 - \frac{c}{12}) \tag{1.9}$$

then the vacuum, defined by (1.8), is the lowest energy state in $\mathcal{H}$.

(C) The expectation value of a field variable F in a mixed state of complexified inverse finite temperature ζ,

$$\zeta = \beta + i\gamma, \quad \beta > 0 \tag{1.10}$$

is given by

$$\langle F\rangle_\zeta = tr(e^{-\zeta\widetilde{L}_0 - \overline{\zeta}\widetilde{\overline{L}}_0} F)/Z \tag{1.11}$$

where the *partition function*

$$Z = Z(\tau) = tr \; e^{-\zeta\widetilde{L}_0 - \overline{\zeta}\widetilde{\overline{L}}_0} \tag{1.12}$$

for

$$(q \equiv) \; e^{2\pi i\tau} = e^{-\zeta} \tag{1.13}$$

is invariant under $PSL(2,\mathbf{Z})$-modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbf{Z}) = SL(2,\mathbf{Z})/\mathbf{Z}_2. \tag{1.14}$$

(For a discussion of the meaning of this requirement in various contexts -see [Ca 2], [Se 1], [Ge 3].)

Proposition 1.1 *Condition (B) implies that the stress tensor is expressed in terms of the current by the Sugawara formula, incorporated in the small distance operator product expansion (OPE).*

$$\mathcal{J}(\vartheta_1)\mathcal{J}(\vartheta_2) = -(2\sin\frac{\vartheta_{12} - i\phi}{2})^{-2} + 2(\mathcal{T}(\vartheta) + \frac{1}{24}) + O(\sin^2\frac{\vartheta_{12}}{2}) \tag{1.15}$$

which says, in particular, that the Virasoro central charge is $c = 1$.

Proof. We define the normal product expressions

$$L_n^J = \frac{1}{2}(\sum_{l\geq 1} + \sum_{l\geq -n})J_{-l}J_{l+n} \tag{1.16}$$

which, as a consequence of (1.3), satisfy (1.5). It follows that $l_n = L_n - L_n^J$ commute with J_m and hence with L_m^J, and therefore satisfy the CR of **Vir** with central charge $c[l] = c - 1$ (cf. [Go 1-3]). If $l_{-n} \mid 0\rangle \neq 0$ (for some $n > 0$) then the uniqueness of the vacuum condition (B) would be violated (since $J_m l_{-n} \mid 0\rangle = l_{-n} J_m \mid 0\rangle = 0$ for $m > 0$). Therefore, $l_{-n} \mid 0\rangle = 0 = c-1 (= 2 \| l_{-2} \mid 0\rangle \|^2)$; since the only (hermitian) positive energy representation of **Vir** with a zero central charge is the trivial one [Go 4] we conclude that $l_n = 0$.

We quote without proof the following result of [To 1] [Bu 1].

Proposition 1.2 *The CR (1.7) and the Sugawara formula (1.16) are only compatible among themselves if the conformal weight of ψ is given by*

$$\Delta = \frac{1}{2} g^2 \tag{1.17}$$

and ψ satisfies the differential equation

$$\frac{\partial}{\partial z}\psi(z,g) = g : J(z)\psi(z,g) := g \lim_{\varepsilon\to 0}\Big(J\Big(\sqrt{z^2+\frac{\varepsilon^2}{4}}+\frac{\varepsilon}{2}\Big) - \frac{1}{\varepsilon}\Big)\psi\Big(\sqrt{z^2+\frac{\varepsilon^2}{4}}-\frac{\varepsilon}{2}, g\Big). \tag{1.18}$$

The solution of this equation, normalized by

$$z_{12} g^2 \langle \psi(z_1, g)\psi^*(z_2, g)\rangle = 1, \;\; \psi^*(z,g) = \psi(z,-g), \;\; z_{12} = z_1 - z_2 \tag{1.19}$$

is expressed in terms of an unitary charge shift operator U_g, such that

$$[J_n, U_g] = g\delta_n U_g, \;\; U_g^* = U_{-g} = U_g^{-1} \tag{1.20}$$

and of the current J as follows:

$$\psi(z,g) = e^{ig\phi_{(+)}(z)} U_g z^{gJ_0} e^{ig\phi_{(-)}(z)} \tag{1.22.a}$$

where

$$\begin{aligned} i\phi_{(+)}(z) &= \int_0^z J_{(+)}(\zeta)d\zeta = \sum_{n=1}^{\infty} J_{-n}\frac{z^n}{n} \\ i\phi_{(-)}(z) &= -\int_z^{\infty} J_{(-)}(\zeta)d\zeta = -\sum_{n=1}^{\infty} J_n \frac{z^{-n}}{n} \quad (i\phi'_{(\pm)}(z) = J_{(\pm)}(z)). \end{aligned} \tag{1.22.b}$$

Corollary 1.1 *The fields $\psi(z, \pm g)$ satisfy the OPE [De 1] (see also [Fu 1])*

$$z_{12}^{g^2}\psi(z_1,g)\psi^*(z_2,g) \;=\; :\exp\{-g\int_{z_1}^{z_2} J(z)dz\}: \tag{1.21}$$

$$= \; 1 + g z_{12} J(z) + g^2 z_{12}^2 T(z) + O(z_{12}^2) \tag{1.22}$$

where the normal product is defined with respect to the (free) current modes (and J, T are given by (1.1.a), (1.2.a)).

The cyclic lowest weight (LW) representations of $\mathcal{F}[g^2]$ are realized in a Hilbert space $\mathcal{H}$, characterized by a LW vector $|\nu\rangle$ satisfying

$$J_0 \mid \nu\rangle = g_\nu \mid \nu\rangle,\ J_n \mid \nu\rangle = 0 \quad \text{for} \quad n \geq 1 \tag{1.23}$$

and minimizing L_0:

$$L_0 \geq \frac{1}{2} g_\nu^2 \quad \text{in} \quad \mathcal{H}_\nu (= \mathcal{H}_\nu[g^2]). \tag{1.24}$$

Since $U_{\pm g} \mid g_\nu\rangle = \mid g_\nu \pm g\rangle$ are also vectors in $\mathcal{H}_\nu$ and correspond to L_0 eigenvalue $\frac{1}{2}(g_\nu \pm g)^2$, it follows from (1.24) that

$$g_\nu^2 \leq \frac{1}{4} g^2. \tag{1.25}$$

We demand, following [Bu 1], that the representations of $\mathcal{F}[g^2]$ are at most double valued. For the univalence automorphism

$$\alpha_{2\pi}\psi(z,g) = e^{2\pi i L_0}\psi(z,g)e^{-2\pi i L_0} = e^{ig^2\pi}\psi(e^{2\pi i}z, g) \tag{1.28.a}$$

or, using (1.22),

$$\alpha_{2\pi}\psi(z,g) = \psi(z,g)e^{i\pi(g^2+2J_0 g)} \tag{1.28b}$$

this gives $2gg_\nu \in \mathbf{Z}$; taking into account (1.25) we end up with the following allowed spectrum of LW charges:

$$g_\nu = \frac{\nu}{2g} \quad - g^2 \langle \nu \leq g^2 \tag{1.26}$$

(The value $\nu = -g^2$ is excluded by the convention that the LW vectors of $\mathcal{F}[g^2]$ are annihilated by the zero mode $\psi_0(g)$ of $\varphi(z,g)$ defined by the expansion

$$\psi(z,g)\mathcal{H}_\nu[g] = \sum_{n\in\mathbf{Z}} \psi_{n+\frac{1}{2}\nu-\frac{1}{2}g^2}(g) z^{-n-\frac{\nu}{2}} \mathcal{H}_\nu[g] \tag{1.27}$$

for integer $\frac{1}{2}\nu - \frac{1}{2}g^2$.)

We have demonstrated in [Pa 1] that for each positive integer g^2 there is a finite number of modular invariant 2-dimensional models which give rise to positive energy QFT representations or $\mathcal{F}[g^2]$ with partition functions classified, essentialy, in [Di 1]. Here we shall give another realization of these models following a suggestion by V. Kac.

2 An $SO(2g^2)$ Homogeneous Space, Realization of the LW Representations of $\mathcal{F}[g^2]$

We shall write down a Kac-Moody type construction of the $\mathcal{F}[g^2]$ representation following the pattern, introduced by Goddard, Kent and Olive [Go 1].

Let J_n^a be the current modes for the Kac-Moody algebra $\widehat{dG}$ associated with a simple Lie algebra dG. Denote by VG the semidirect sum of $\widehat{dG}$ with **Vir**. We shall normalize J_n^a in such a way that the structure constants f_{abc} appearing in the CR

$$[J_l^a, J_m^b] = i f_{abc} J_{l+m}^c + \frac{k}{2} l \delta_{ab} \delta_{l+m} \tag{2.1}$$

where $k = 0, 1, 2, \ldots$ is the Kac-Moody central charge or *level*, satisfy

$$f_{ast} f_{bst} (\equiv \sum_{s,t=1}^{dG} f_{ast} f_{bst}) = \tilde{h} \delta_{ab}. \tag{2.2}$$

Here $\tilde{h}$ is the dual Coxeter number of dG (see, e.g., sec.6.1 of [Ka 1] or [Go 3]); in the special case of the unitary and orthogonal groups it takes the values

$$\tilde{h}[SU(n)] = n \text{ (for } n \geq 2), \qquad \tilde{h}[SO(n)] = n - 2 \text{ (for } n \geq 5). \tag{2.3}$$

Under these conditions the counterpart of the Sugawara formula (1.16) takes the form ([Kn 1] [Go 2] [To 1])

$$L_m = \frac{1}{k + \tilde{h}} (\sum_{l \geq 1} + \sum_{l \geq -m}) J^a_{-l} J^a_{m+l} \; ([J^a_m, L_n] = m J^a_{m+n}). \tag{2.4}$$

The Virasoro central charge is then

$$c = 2 \langle L_l \frac{1}{k + \tilde{h}} \vec{J}^2_{-1} \rangle = \frac{2}{k + \tilde{h}} \langle J^a_1 J^a_{-1} \rangle = \frac{k d_G}{k + \tilde{h}} \tag{2.5}$$

(d_G being the dimension of G)

For a simply laced Lie algebra dG (like $D_n = SO(2n)$) the central charge corresponding to level 1 representations is equal to the rank (in our case $g^2 = n$) of G:

$$c[SO(2n); k = 1] = \frac{1}{2n - 1} n(2n - 1) = n. \tag{2.6}$$

It is equal to the level 1 central charge of the semisimple Lie algebra $SO(n) \oplus SO(n) \subset SO(2n)$

$$c[SO(n) \oplus SO(n); k = 1] = 2c[SO(n); k = 1] = \frac{n(n-1)}{n-1} = n. \tag{2.7}$$

The difference between $c[SO(n); k = 1]$ and the central charge of the diagonal $SO(n)$ (corresponding to level 2) is just 1:

$$c = c[SO(2n); k = 1] - c[SO(n); k = 2] = n - (n - 1) = 1. \tag{2.8}$$

We shall now look in more detail into the current algebra corresponding to the homogeneous space

$$SO(2n)/SO(n)_{diag} \tag{2.9}$$

and shall construct, in particular, the $U(1)$ current of Section 1 which commutes with the "gauge subgroup" $SO(n)_{diag}$.

In the conventionally used basis of the rotation group the CR (2.1) assume the form

$$\begin{aligned} [J^{\kappa\lambda}_l, J^{\mu\nu}_m] &= i(\delta^{\kappa\mu} J^{\lambda\nu}_{l+m} - \delta^{\lambda\mu} J^{\kappa\nu}_{l+m} + \delta^{\lambda\nu} J^{\kappa\mu}_{l+m} - \delta^{\kappa\nu} J^{\lambda\mu}_{l+m}) \\ &\quad + lk(\delta^{\kappa\mu}\delta^{\lambda\nu} - \delta^{\kappa\nu}\delta^{\lambda\mu})\delta_{l+m} \\ &\quad \kappa, \lambda, \mu, \nu = 1, 2, \ldots, 2n, \; J^{\mu\nu}_m = -J^{\nu\mu}_m \end{aligned} \tag{2.10}$$

the normalized (according to (2.2)) generators being $\frac{1}{\sqrt{2}}J^{\mu}_{m}$. We shall also need a Cartan-Weyl basis for D_n. To this end we introduce an orthonormal set $\{e_i\}$ in $\mathbf{R}^n$:

$$e_i \cdot e_j = \delta_{ij} \quad i,j = 1,\ldots,n. \tag{2.11}$$

The $2n(n-1)$ roots of D_n are [Go 5]

$$\alpha_{ij} = e_i - e_j \ (i \neq j) \quad \text{and} \quad \pm\beta_{ij} = \pm(e_i + e_j) \ (i < j) \tag{2.12.a}$$

the simple roots being

$$\alpha_i = \alpha_{i\,i+1} = e_i - e_{i+1} \quad i = 1,\ldots,n-1 \quad \alpha_n = e_{n-1} + e_n \tag{2.12.b}$$

(while the positive roots are α_{ij} with $i < j$ and β_{ij}). The current modes H^i_l related to the Cartan basis in D_n (associated with the above ordered roots) will be identified with

$$H^1_l = J^{12}_l \ H^2_l = J^{34}_l, \ldots, H^n_l = J^{2n-1,2n}_l. \tag{2.12}$$

We shall also single out the Weyl type generators $E^{\alpha_{ij}}_l$ which are obtained as multiple commutators of

$$E^{\alpha_j}_l = \frac{1}{2}(J^{2i-1,2j-1}_l + J^{2j,2j+2}_l + i(J^{2j,2j+1}_l - J^{2j-1,2j+1}_l)); \tag{2.14.a}$$

we have, for positive roots

$$[E^{\alpha_{i_1 i_2}}_l, E^{\alpha_{j_1 j_2}}_m] = \delta_{i_2 j_1} E^{\alpha_{i_1 j_2}}_{l+m} - \delta_{i_1 j_2} E^{\alpha_{j_1 i_2}}_{l+m}. \tag{2.14.b}$$

The gauge $SO(n)$ will be defined as the subalgebra generated by

$$^gJ^{ij}_l = -i(E^{\alpha_{ij}}_l - E^{\alpha_{ji}}_l). \tag{2.13}$$

It is easily verified (using (2.14.b)) that $^gJ^{ij}_0$ satisfy the CR of $SO(n)$. Moreover, if $J^{\mu\nu}_m$ span a level 1 representation of $\widehat{D}_n$ then $^gJ^{ij}_l$ give rise to a level 2 representation of $\widehat{SO}(n)$. Furthermore, the $U(1)$-current

$$J(z) = \sum_l J_l z^{-l-1} \quad J_l = \frac{1}{g}\sum_{i=1}^{n} H^i_l \quad g = \sqrt{n} \tag{2.14}$$

commutes with $\widehat{SO}(n)$ and we can write the stress-energy tensor of the constained theory (on the homogeneous space (2.9) as

$$T(z) = T_{\widehat{SO}(2n)}(z) - T_{\widehat{SO}(n)}(z) = \frac{1}{2} : J^2(z) : . \tag{2.15}$$

A straightforward way to exploit the fact that we are only interested in level 1 representations is the use of the Frenkel-Kac vertex operator construction. We shall write, in particular, the current $E^{\vec{\alpha}}(z)$ corresponding to an arbitrary root $\vec{\alpha}$ of D_n in the form ([Fr 1])

$$E^{\vec{\alpha}}(z) = e^{i\vec{\alpha}\cdot\vec{\phi}_{(+)}(z)} U_{\vec{\alpha}} z^{\vec{\alpha}H_0} e^{i\vec{\alpha}\cdot\vec{\phi}_{(-)}(z)} \tag{2.16}$$

where $i\vec{\phi}'_\pm(z) = \vec{H}_\pm(z)$ (cf.(1.22), while the constant unitary operators $U_{\vec{\alpha}}$ satisfy

$$H_n^j U_{\vec{\alpha}} = U_{\vec{\alpha}}(H_n^j + \alpha^j \delta_n) \tag{2.17}$$

$$U_{\vec{\alpha}} U_{\vec{\beta}} = e^{i\pi\vec{\alpha}\cdot\vec{\beta}} U_{\vec{\beta}} U_{\vec{\alpha}}. \tag{2.18}$$

(In other words, $U_{\vec{\alpha}}$ contain the Klein factors necessary to restore the correct CR between current components.) We shall also use the charge fields construction (1.22) with $J(z)$ given by (2.14) and an $SO(n)$ invariant charge shift operator

$$U_g = U_{e_1+\ldots+e_n} \quad (g = \sqrt{n}). \tag{2.19}$$

We now define the physical subspace LW vectors $\mid \phi\rangle$ by the conditions

$$E_0^{\alpha_{ij}} \mid \phi\rangle = 0 \quad \text{for } i < j, \qquad E_m^{\alpha_{ij}} \mid \phi\rangle = 0 \quad \text{for } m \geq 1 \quad (i \neq j) \tag{2.22.a}$$

$$J_m \mid \phi\rangle = 0 \quad \text{for } m \geq 1, \qquad (H_0^i - \lambda^i) \mid \phi\rangle = 0, \quad i = 1, \ldots, g^2 (= n) \tag{2.22.b}$$

$$-\frac{1}{2}g^2 < \sum_{i=1}^{g^2} h^i \geq \frac{1}{2}g^2. \tag{2.22.c}$$

Eq.(2.22.a) guarantees that the matrix elements of the $\widehat{SO}(n)$ gauge current ${}^g J^{ij}(z)$ (see (2.13)) between physical states vanish. Condition (2.22.b) ensures that $\mid \phi\rangle$ is a LW vector for the $U(1)$ current algebra and is an weight vector for D_n (i.e. an eigenvector of H_0^i). Finally, Eq.(2.22.c) is just a translation of the condition (1.26) (that $\mid \phi\rangle$ is a LW state of $\mathcal{F}[g^2]$) in the D_n language.

Eq.(2.22.a) for $\mid \phi\rangle = \mid \lambda^1, \ldots, \lambda^n\rangle$ also reduces to a set of inequalities for λ^j:

$$0 \leq \lambda^i - \lambda^j \leq 1 \quad \text{for} \quad 1 \leq i < j \leq n (= g^2). \tag{2.20}$$

The determinant of the transformation from the *fundamental weights* of D_n

$$\begin{gathered} \lambda_1 = e_1, \quad \lambda_2 = e_1 + e_2, \ldots, \quad \lambda_{n-2} = e_1 + \cdots + e_{n-2} \\ \lambda_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n), \quad \lambda_n = \frac{1}{2}(e_1 + \cdots + e_n) \end{gathered} \tag{2.21}$$

(characterized by $\lambda_i \cdot \alpha_j = \delta_{ij}$ for α_j given by (2.12.b)) to the basic weights of $SO(n) + U(1)$

$$\tilde{\lambda}_1 = \alpha_1, \ldots, \quad \tilde{\lambda}_{n-1} = \alpha_{n-1}, \quad \tilde{\lambda}_n = e_1 + \cdots + e_n \tag{2.22}$$

is

$$\det \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & -1 & 2 & 0 \\ 0 & \cdots & 0 & \cdots & -1 & 0 & 2 \end{pmatrix} = 2n = 2g^2 \tag{2.23}$$

(see, e.g., Sec.5.5.4 of [Go 5]). Consequently the factor $\Lambda/\tilde{\Lambda}$ of the weight lattice $\Lambda = \Lambda(D_n)$ by its sublattice $\tilde{\Lambda}$ generated by (2.22) is a finite abelian group with $2g^2$ elements. It can be represented by the following two sets of weights, satisfying (2.22) and (2.20)

$$e_1, e_1 + e_2, \dots, e_1 + \dots + e_{[\frac{n}{2}]}, 0, -e_n, -e_n - e_{n-1}, \dots, -e_n - \dots - e_{[\frac{n}{2}+2]} \tag{2.27.a}$$

$$\lambda_n, \lambda_{n-1}, \lambda_{n-2} - \lambda_n, \dots, \lambda_1 - \lambda_n. \tag{2.27.b}$$

The set (2.27.a) reproduces the *Neveu-Schwarz charges* $g_{2\nu} = \frac{\nu}{g}$ of Eq.(1.26) for which $\alpha_{2\pi}\psi(z,g) = (-1)^{g^2}\psi(z,g)$ (see Eq.(1.28)). For even g^2 the weights (2.27.b) give rise to the same set of charges; thus for integer spins we only obtain single valued fields by the above construction. For odd g^2 the set (2.27.b) gives rise to the *Ramond sector* charges $g_{2\nu+1}$.

We note that although the D_n series is defined conventionally for $n \geq 4$ and part of our derivation is only legitimate for such n's the final result is also applicable to $n(= g^2) = 2$ and 3 (and even to $n = 1$, if we identify the set (2.27.a) with the zero vector and the set (2.27.b) with $\{1/2e_1\}$).

3 Modular Invariant Partition Functions

The (reducible) $VU(1)$-affine characters of the above described LW representations of $\mathcal{F}[g^2]$,

$$\mathcal{K}_\nu(\tau,\zeta,g^2) = tr_{\mathcal{H}_\nu} q^{\tilde{L}_0} y^{J_0}, \quad q = e^{2\pi i\tau}, y = e^{2\pi i\zeta} \quad (\mathrm{Im}\,\tau > 0) \tag{3.1}$$

are evaluated by means of the Sugawara formula (1.16):

$$\mathcal{K}_\nu(\tau,\zeta,g^2) = \frac{1}{\eta(\tau)}\Theta_{\nu,g^2}(\tau,\zeta,0) \tag{3.2.a}$$

where $\Theta_{\nu,g^2}(\tau,\zeta,u)$ is the *classical* Θ*-function* (see, e.g., [Ka 2] and references therein)

$$\Theta_{\nu,g^2}(\tau,\zeta,u) = e^{2\pi i g^2 u}\sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(ng+\frac{\nu}{2g})^2} y^{(ng+\frac{\nu}{2g})} \tag{3.2.b}$$

and the *Dedekind* η*-function* can be written in either of the two forms (cf. the Euler identity (1.7.4) of [Ka 2])

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = \tilde{\Theta}_{1,3}(\tau,0,0) \tag{3.2.c}$$

$\tilde{\Theta}$ being the *indefinite* Θ*-function*

$$\tilde{\Theta}_{\nu,g^2}(\tau,\zeta,0) = \sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(ng+\frac{\nu}{2g})^2} y^{(ng+\frac{\nu}{2g})} \quad (= \eta(\tau)\tilde{\mathcal{K}}_\nu). \tag{3.2}$$

The indefinite affine characters $\tilde{\mathcal{K}}_\nu$ (defined by (3.2) also appear in the modular transformation law of $\mathcal{K}_\nu$ (for odd $g^2 + \nu^2$):

$$\mathcal{K}_\nu(\tau+1,\zeta,g^2) = \tag{3.3}$$
$$\exp[\frac{i\pi}{4}(\frac{v^2}{g^2} - \frac{1}{3})]\{\frac{1+(-1)^{g^2+\nu}}{2}\mathcal{K}_\nu(\tau,\zeta,g^2) + \frac{1-(1-)^{g^2+\nu}}{2}\tilde{\mathcal{K}}_\nu(\tau,\zeta,g^2)\}.$$

$\tilde{\mathcal{K}}_\nu$ satisfies a similar transformation law (obtained from (3.3) through the interchange $\mathcal{K}_\nu \leftrightarrow \tilde{\mathcal{K}}_\nu$) under τ-translation. In most of what follows we shall be only interested in the reducible Virasoro characters $\overset{(\sim)}{\mathcal{K}}_\nu(\tau, g^2) = \overset{(\sim)}{\mathcal{K}}_\nu(\tau, 0, g^2)$.

The set $\{\mathcal{K}_\nu, \tilde{\mathcal{K}}_\nu \; 1 - g^2 \leq \nu \leq g^2\}$ is also closed under the second generator of the modular group, the involution $\tau \to -1/\tau$. Indeed, using the celebrated Poisson formula and the identity $\eta(-1/\tau) = z\sqrt{-i\tau}\eta(\tau)$, we find

$$\mathcal{K}_\nu(-\frac{1}{\tau}, g^2) = \frac{1}{g^2} \sum_{\substack{-g^2<\mu\leq g^2 \\ \mu \text{ even}}} e^{\pi i \frac{\mu\nu}{2g^2}} \begin{cases} \mathcal{K}_\mu(\tau, g^2) \text{ for even } \nu \\ \tilde{\mathcal{K}}_\mu(\tau, g^2) \text{ for odd } \nu \end{cases} \tag{3.5.a}$$

$$\tilde{\mathcal{K}}_\nu(-\frac{1}{\tau}, g^2) = \frac{1}{g^2} \sum_{\substack{-g^2<\mu\leq g^2 \\ \mu \text{ odd}}} e^{\pi i \frac{\mu\nu}{2g^2}} \begin{cases} \mathcal{K}_\mu(\tau, g^2) \text{ for even } \nu \\ \tilde{\mathcal{K}}_\mu(\tau, g^2) \text{ for odd } \nu \end{cases} \tag{3.5.b}$$

Instead of the characters $\mathcal{K}_\nu$ and $\tilde{\mathcal{K}}_\nu$ we can use their sum and their difference,

$$\mathcal{K}_\nu^{(\pm)} = \frac{1}{2}(\mathcal{K}_\nu(\tau, g^2) \pm \tilde{\mathcal{K}}_\nu(\tau, g^2)) \tag{3.4}$$

which can be expanded into series in powers of q with positive integer coefficients. We are looking for modular invariant *partition functions* of the form

$$Z(\tau, g^2; \{N\}) = \sum_{\substack{\nu\bar{\nu} \\ \epsilon\bar{\epsilon}}} N_{\nu\bar{\nu}}^{\epsilon\bar{\epsilon}} \mathcal{K}_\nu^{(\epsilon)}(\tau, g^2) \overline{\mathcal{K}_{\bar{\nu}}^{(\bar{\epsilon})}(\tau, g^2)} \tag{3.5}$$

where the N's are nonnegative integers, $N_{00}^{++} = 1$(and the bar stands for complex conjugation).

A complete classification of the Neveu-Schwarz (NS) partition functions for even g^2 (and $\nu, \bar{\nu}$) is presented in [Di 1] and [Ge 2]. To every splitting of $\frac{1}{2}g^2$ into a product $p \cdot p'$ of positive integers there correspond a partition function [1]

$$Z_{NS}^{(p,p')}(\tau, g^2) = \frac{1}{2} \sum_{\substack{\mu\in \mathbf{Z}_{2p'} \\ \nu \in \mathbf{Z}_{2p}}} \mathcal{K}_{2(\mu p + \nu p')}(\tau, g^2) \overline{\mathcal{K}_{2(\mu p - \nu p')}(\tau, g^2)}. \tag{3.6}$$

One can use this result to evaluate all modular invariant partition functions for odd g^2 and the partition functions involving the "twisted sector" for even g^2. This is achieved through the following relation between $\mathcal{K}_\nu^{(\pm)}(\tau, g^2)$ and the NS characters for a double charge, $2g$, defining the field algebra:

$$\mathcal{K}_\nu^{(+)}(\tau, g^2) = \mathcal{K}_{2\nu}(\tau, 4g^2), \; \mathcal{K}_\nu^{(-)}(\tau, g^2) = \mathcal{K}_{2\nu \pm 4g^2}(\tau, 4g^2). \tag{3.7}$$

In allowing for both signs of $4g^2$ in the index of $\mathcal{K}$ we are using the periodicity condition

$$\mathcal{K}_{\nu+2n}(\tau, n) = \mathcal{K}_\nu(\tau, n) \;\; (\tilde{\mathcal{K}}_{\nu+2n}(\tau, n) = -\tilde{\mathcal{K}}_\nu(\tau, n)). \tag{3.8}$$

[1] A sum over $\mathbf{Z}_{2p}$ means summation over any $2p$ consecutive integers, e.g. $\sum_{\nu=1-p}^{p}$

We also note the symmetry property

$$\mathcal{K}_\nu^{(\pm)}(\tau,n)=\mathcal{K}_{-\nu}^{(\pm)}(\tau,n)\ \ (\mathcal{K}_n^{(+)}(\tau,n)=\mathcal{K}_n^{(-)}(\tau,n)). \tag{3.9}$$

For $g^2=4$ we thus obtain the partition function for the "complex Ising model", -i.e., for the theory of a free complex Weyl field $\psi^{(*)}$ (of charge ± 1), which mixes together the Ramond and the NS sectors. Both the current and the stress energy tensor can be expressed in terms of $\psi(z)=\psi(z,-1)$ and $\psi^*(z)=\psi(z,1)$ according to the OPE (1.22) for $g=1$. Thus we have two equivalent expressions for L_n: the Sugawara formula (1.16) and

$$L_n=\frac{1}{2}(\frac{1}{2}-\kappa)^2\delta_n+\sum_{l=1}^{\infty}(\varepsilon_n+l-\kappa)[\psi^*_{\kappa-l+[\frac{n}{2}]}\psi_{l-\kappa+[\frac{n+1}{2}]}+\psi_{\kappa-l+[\frac{n}{2}]}\psi^*_{l-\kappa+[\frac{n+1}{2}]}] \tag{3.12.a}$$

where $2\varepsilon_n=[\frac{n+1}{2}]-[\frac{n}{2}]=\frac{1}{2}(1-(-1)^n)$ $\kappa=0$ in the Ramond sector and $\kappa=1/2$ in the NS sector:

$$\kappa\mathcal{H}_\nu[1]=\frac{1-\nu}{2}\mathcal{H}_\nu[1]\text{ for }\nu=0,1 \tag{3.12.b}$$

and we are using the ψ mode expansion (1.27). The equivalence of the two realizations of **Vir** is a consequence of the equality of central charges, $c=1$. The latter is verified by using the infinitesimal conformal law $[\psi_\rho^{(*)},L_n]=(\rho+\frac{n}{2})\psi^{(*)}_{n+\rho}$. We shall now demonstrate that Eq.(3.12) allows to obtain a new expression for $\mathcal{K}_\nu(\tau,\zeta,1)$, thus reproducing a nontrivial *Jacobi triple product identity.*

We observe, first of all, that

$$q^{\tilde{L}_0}y^{J_0}\psi_\rho q^{-\tilde{L}_0}y^{-J_0}=q^{-\rho}y^{-1}\psi_\rho \tag{3.13.a}$$

$$q^{\tilde{L}_0}y^{J_0}\psi^*_\rho q^{-\tilde{L}_0}y^{-J_0}=q^{-\rho}y\psi^*_\rho \tag{3.13.b}$$

This allows to compute the 2-point correlation functions

$$\langle\Pi_\rho^{(\pm)}\rangle_{q,y;\nu}=\frac{1}{\mathcal{K}_\nu}tr_{\mathcal{H}_\nu}(\Pi_\rho^{(\pm)}q^{\tilde{L}_0}y^{J_0})\qquad \nu=0,1 \tag{3.10}$$

where $\mathcal{K}_\nu$ is given by (3.24) and $\Pi_\rho^{(\pm)}$ are the orthogonal projection operators

$$\Pi_\rho^{(+)}=\psi^*_{-\rho}\psi_\rho\quad \Pi_\rho^{(-)}=\psi_{-\rho}\psi^*_\rho\quad (\Pi_\rho^{(\pm)2}=\Pi_\rho^{(\pm)}=\Pi_\rho^{(\pm)*}). \tag{3.11}$$

Indeed, using the KMS condition (cf. [Bu 1])

$$\langle Aq^{\tilde{L}_0}y^{J_0}Bq^{-\tilde{L}_0}y^{-J_0}\rangle_{q,y;\nu}=\langle BA\rangle_{q,y;\nu} \tag{3.12}$$

along with (3.13) and the canonical anticommutation relations for $\psi_\rho^{(*)}$ we find

$$\langle\psi^*_{-\rho}\psi_\rho\rangle_{q,y;\nu}=\frac{yq^\rho}{1+yq^\rho}=\langle\psi_{-\rho}\psi^*_\rho\rangle_{q,y^{-1};\nu}. \tag{3.13}$$

Inserting (3.13) into the expectation value of (3.12) for $n=0$ we obtain

$$\langle\tilde{L}_0\rangle_{q,y;\nu}=q\frac{\partial}{\partial q}\ln\mathcal{K}_\nu(\tau,\zeta,1) \tag{3.14}$$

with

$$
\begin{aligned}
\mathcal{K}_\nu(\tau,\zeta,1) &= q^{\frac{1}{8}(\nu^2-\frac{1}{3})}\prod_{l=1}^{\infty}(1+yq^{l-\kappa_\nu})(1+y^{-1}q^{l-\kappa_\nu}) \\
\kappa_\nu &= \frac{1-\nu}{2}, \qquad \nu = 0,1.
\end{aligned}
\tag{3.15}
$$

The above mentioned Jacobi type identity is obtained by equating (3.2) (for $g = 1$) with (3.15). The (unique) modular invariant partition function for the $g^2 = 1$ model

$$
Z(\tau,1) = \sum_{\nu=0}^{1}\{|\,\mathcal{K}_\nu^{(+)}(\tau,1)\,|^2 + |\,\mathcal{K}_\nu^{(-)}(\tau,1)\,|^2\}
\tag{3.16}
$$

involves the NS ($\nu = 0$) and Ramond ($\nu = 1$) sectors, as stated.

For any even $g^2 = 2M$ there exists a diagonal NS invariant

$$
Z^{diag}(\tau,2M) = \sum_{\nu=1-M}^{M} |\,\mathcal{K}_{2\nu}(\tau,2M)\,|^2 \,.
\tag{3.17}
$$

If we apply Eq.(3.7) to the invariant (3.17) for $M=4$ ($g^2=8$) we obtain a partition function for the $g^2 = 2$ model that includes a twisted sector:

$$
Z_T(\tau,2) = \sum_{\nu=-1}^{2}\sum_{\epsilon=\pm} |\,\mathcal{K}_\nu^{(\epsilon)}(\tau,2)\,|^2 \,.
\tag{3.18}
$$

The minimal charge g_1 and Virasoro LW Δ_1 of (the right moving projection of) the twisted sector are $g_1 = \frac{1}{2\sqrt{2}}$ and $\Delta_1 = \frac{1}{2}{g_1}^2 = \frac{1}{16}$. The NS partition function for this model (given by Eq.(3.17) for $M = 1$) corresponds to the $k = 1$ level of $VSU(2)$ (cf. [To 2] where this model has been analysed by the methods of Sec.2).

For $g^2 = 3$ we obtain the $N = 2$ extended super Virasoro model, corresponding to central chage $c = 1$ (see [Wa 1], [Ra 1] and references to earlier work cited there). There are two modular invariant partition functions in this case, obtained from (3.6) for $g^2 = 12$ (and $p, p' = 2,3$ and $1,6$) with a subsequent application of (3.7). The twisted sector of the $N = 2$, $c = 1$ -model can also be obtained as follows. If we split the free Weyl charged field $\psi(z) = \psi(z,-1)$ into a real and imaginary part,

$$
\psi(z) = \frac{1}{\sqrt{2}}(\psi_1(z) + i\psi_2(z))\ (\psi_a = \psi_a^*\ a = 1,2)
\tag{3.19}
$$

then the current J can be written in the form

$$
J(z) = i\psi_1(z)\psi_2(z)\ (= \frac{i}{2}[\psi_1(z),\psi_2(z)]).
\tag{3.20}
$$

The twisted sector is then obtained by using the NS representation for ψ_1 and the Ramond representation for ψ_2($[\psi_1,\psi_2]_+ = 0$). It turns out to be isomorphic to the twisted sector of the $VSU(2)$ model discussed above. This is a manifestation of a generally valid isomorphism between the twisted sectors of the two models -see [Ri 1].

4 Summary and Discussion

Using the result of Di Francesco, Saleur and Zuber [Di 1] and of Gepner and Qiu [Ge 2] we have classified the modular invariant 2-dimensional QFT models of the chiral $U(1)$ conformal current algebra which involve single or double valued relalizations of the field algebra $\mathcal{F}[g^2]$ generated by a pair of conjugate charged fields $\psi(z, \pm g)$ ($g^2 = 1, 2, \ldots$). The first three of the infinite family of models thus classified are the free Weyl charged field (equivalent to a pair of "coupled" Ising models), the level 1 realization of the $\widehat{A}_1$ ($= \widehat{SU}(2)$) Kac-Moody algebra, and the $c = 1$ level of the $N = 2$ extended super Virasoro algebra.

It turns out that the family of models, considered here, is also distinguished from a purely mathematical point of view. The Θ-functions (3.2.b) exhaust, according to a deep result of Serre and Stark [Se 2], the most general modular forms of weight $1/2$ (corresponding to $c = 1$ - for a review, see [Ka 2]).

The case $g = 0$ also corresponds to a modular invariant partition function

$$Z(\tau, 0) = \frac{1}{\sqrt{2\mathrm{Im}\,\tau}} \mid \eta(\tau) \mid^{-2} \tag{4.1}$$

which can be written as a continuous superposition of $VU(1)$ characters corresponding to charge g (see [Ge 3]):

$$\begin{aligned} Z(\tau, 0) &= \int dg \frac{(q\bar{q})^{\frac{1}{2}g^2}}{\mid \eta(\tau) \mid^2} = \mid \eta(\tau) \mid^{-2} \int dg e^{\pi i(\tau - \bar{\tau})g^2} \\ &= \frac{\mid \eta(\tau) \mid^{-2}}{\sqrt{2\mathrm{Im}\,\tau}}. \end{aligned} \tag{4.2}$$

We would like to thank V. G. Kac for suggesting to us the homogeneous space construction of Sec.2 and both him and J.-B. Zuber for useful comments.

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