MYSTERY OF THE DUALITY ROTATION*

Maciej Przanowski
Institute of Physics, Technical University of Łódź
Wólczańska 219, 93-005 Łódź, Poland

Duality rotation group in the linear or nonlinear electrodynamics is considered. Conservation law generated by the symmetry under the duality rotation is found from the Noether theorem.

Then, using the Noether theorem, we get the infinite number of conservation laws in the Maxwell electrodynamics. They involve the mysterious conservation laws of Lipkin.

*This modest work is dedicated to Professor
Jerzy F. Plebański, my Teacher and Friend,
who showed me the dual world.

I. INTRODUCTION

Heaviside (1893) was perhaps the first who observed that the source-free Maxwell equations were invariant under the transformation: $\vec{E} \to \vec{H}$, $\vec{H} \to -\vec{E}$, where $\vec{E}$ and $\vec{H}$ stand for the vectors of electric and magnetic fields, respectively. Then, Larmor (1928) and Rainich (1925) extended this transformation to the one-parameter *duality rotation group*

$$\vec{E}' = \vec{E}\cos\varphi + \vec{H}\sin\varphi, \; \vec{H}' = -\vec{E}\sin\varphi + \vec{H}\cos\varphi, \; \varphi \epsilon \Re. \quad (1.1)$$

In terms of the electromagnetic tensor f_{ij}, $i, j = 1, \ldots, 4$, the duality rotation group is defined by

$$f'_{ij} = f_{ij}\cos\varphi + i * f_{ij}\sin\varphi, \quad \varphi\epsilon\Re, \quad (1.2)$$

where $*f_{ij} = -\frac{i}{2}\sqrt{-g} \; \epsilon_{ijkl}f^{kl}$, and it has been shown (Rainich (1925), Misner and Wheeler (1957), Ibragimov (1967), Plebański and Przanowski (to appear)) that the group (1.2) appears to be the maximal group of transformations of the form

$$x'^i = x^i, \; g'_{ij} = g_{ij}, \; f'_{ij} = f'_{ij} \; (x^l, \; f_{mk}) \quad (1.3)$$

leaving the Einstein-Maxwell equations invariant.

Mutatis mutandis, this holds true for some models of nonlinear electrodynamics (Salazar, García and Plebański (1987,1989), Plebański and Przanowski (to appear)). But, as it has been shown by Deser and Teitelboim (1976), the duality rotation is not well defined in non-abelian field theories. Ferrara et al (1977) have found some generalized duality transformations in the extended 0(2) and 0(3) supergravity theories.

The present work is devoted to the various aspects of the duality rotation in linear or nonlinear electrodynamics. We consider conservation laws generated by the symmetry under the duality rotation group and by its generalization (Secs. 3 and 4). Of course the fundamental problem is whether or not the symmetry under the duality rotation is broken when the sources are present. If not, as we expect, then it seems that the Dirac quantization condition for the electric and magnetic charges should be revised (Schwinger (1975)).

II. DUALITLY ROTATION IN ELECTRODYNAMICS

We consider a vacuum electromagnetic field in a space-time (M_4, g_{ij}), $i, j, = 1, 2, 3, 4$. The electromagnetic field is described by the potential A_i and the antisymetric tensor $p_{ij} = -p_{ji}$. Lagrangian for the electromagnetic field is taken to be

$$L = -\frac{1}{2}p^{ij}f_{ij} + K(P,Q) \tag{2.1}$$

$$f_{ij} := \partial_i A_j - \partial_j A_i, \quad P := \frac{1}{4}p_{ij}p^{ij},$$

$$Q := \frac{1}{4}p_{ij} * p^{ij}, \quad *p_{ij} := -\frac{i}{2}\sqrt{-g}\ \epsilon_{ijkl}p^{kl};$$

$K(P,Q)$ is called the *structural function.*

For the action of the gravitational and electromagnetic field one has

$$S = \int d^4x\sqrt{-g}\cdot\left[\frac{1}{16\pi}\cdot(R+2\Lambda)+L\right] \tag{2.2}$$

where R stands for the curvature scalar and Λ is the cosmological constant.

Then the variation of S with respect to g_{ij}, A_i and p_{ij} yields the following set of equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} + \Lambda g_{ij} \quad (Einstein \;\; equations) \tag{2.3}$$

$$df = 0, \quad d * p = 0 \quad (Maxwell\ equations) \tag{2.4}$$

$$f_{ij} = \frac{\partial K}{\partial P} p_{ij} + \frac{\partial K}{\partial Q} * p_{ij} \quad (Material\ equations) \tag{2.5}$$

with $T_{ij} = p_i^k f_{jk} + L g_{ij}, \;\; f := \frac{1}{2} f_{ij} dx^i \wedge dx^j, \;\; *p := \frac{1}{2} * p_{ij} dx^i \wedge dx^j$

(Born and Infeld (1934), Plebański (1970), Białynicki-Birula and Białynicka-Birula (1975), Salazar et al (1987)).

It is convenient to deal with a (3+1)- decomposition of M_4, $M_4 = M_3 \times M_1$, dim $M_3 = 3$, dim $M_1 = 1$.

We define the following 3-objects (Landau and Lifschitz (1973), Møller (1972), Deser and Teitelboim (1976))

$$D^\mu = \sqrt{-g}\, p^{4\mu}, \quad H_\mu = \frac{1}{2}\sqrt{-g}\; \epsilon_{\mu\nu\sigma}\, p^{\nu\sigma} \tag{2.6a}$$

$$E_\mu = -f_{4\mu}, \quad B^\mu = \frac{1}{2}\, \epsilon^{\mu\nu\sigma} f_{\nu\sigma}; \tag{2.6b}$$

Greek indices run through $1, 2, 3$.

Then the Maxwell equations (2.4) take the form

$$\partial_4 B^\mu + \epsilon^{\mu\nu\sigma} \partial_\nu E_\sigma = 0, \quad \partial_\mu B^\mu = 0,$$

$$\partial_4 D^\mu - \epsilon^{\mu\nu\sigma} \partial_\nu H_\sigma = 0; \quad \partial_\mu D^\mu = 0 \tag{2.7}$$

The material equations read

$$E_\mu = -\frac{\partial \mathcal{K}}{\partial D^\mu}, \quad B^\mu = \frac{\partial \mathcal{K}}{\partial H_\mu}, \tag{2.8}$$

where $\mathcal{K} := \sqrt{-g}\, K$.

Now from the second set of equations (2.8) we find $H_\mu = H_\mu(g_{ij}, D^\nu, B^\sigma)$; and then we define $M = M(g_{ij}, D^\nu, B^\sigma)$ to be

$$M := B^\mu H_\mu - \mathcal{K}, \tag{2.9}$$

$$E_\mu = \frac{\partial M}{\partial D^\mu}, \quad H_\mu = \frac{\partial M}{\partial B^\mu} \tag{2.10}$$

(the Legendre transformation).

Finally, for the Maxwell equations we get

$$\partial_4 B^\mu + \epsilon^{\mu\nu\sigma} \partial_\nu \frac{\partial M}{\partial D^\sigma} = 0, \quad \partial_\mu B^\mu = 0,$$

$$\partial_4 D^\mu - \epsilon^{\mu\nu\sigma} \partial_\nu \frac{\partial M}{\partial B^\sigma} = 0, \ \partial_\mu D^\mu = 0 \tag{2.11}$$

It is natural to assume that the electrodynamics considered corresponds to the linear Maxwell electrodynamics for weak fields i.e.

$$K = P + 0 \ \ (P^2, Q^2) \tag{2.12}$$

Moreover, as $K(P,Q)$ is a scalar function

$$K(P,Q) = K(P,-Q). \tag{2.13}$$

Then one shows that the dominant energy condition is satisfied iff

$$\frac{\partial K}{\partial P} > 0 \quad and \quad P\frac{\partial K}{\partial P} + Q\frac{\partial K}{\partial Q} - K \geq 0 \tag{2.14}$$

(Plebański (1970)).

Now we state the following problem:

Find the maximal (connected) local group of transformations G of the form

$$x'^i = x^i, g'_{ij} = g_{ij}, \vec{D}' = \vec{D}'(x^i, \vec{D}, \vec{B}; \tau_1, \ldots, \tau_r),$$

$$\vec{B}' = \vec{B}' \ (x^i, \vec{D}, \vec{B}; \ \tau_1, \ldots, \tau_r), \ \tau_1, \ldots, \tau_r \ \epsilon \Re \tag{2.15}$$

leaving the Einstein-Maxwell equations, (2.3) and (2.11), invariant for every decomposition $M_4 = M_3 \times M_1$ and every coordinate system.

Under the assumption (2.12) one finds the solution of the problem to be

Theorem 2.1

G is of the form

$$x'^i = x^i, \quad g'_{ij} = g_{ij},$$

$$\vec{D}' = \vec{D}' \cos\varphi + \vec{B}\sin\varphi, \ \vec{B}' = -\vec{D}\sin\varphi + \vec{B}\cos\varphi, \varphi \ \epsilon \ \Re \tag{2.16}$$

It is admitted iff

$$[(\frac{\partial K}{\partial P})^2 + (\frac{\partial K}{\partial Q})^2 - 1] \ .Q + 2\frac{\partial K}{\partial P} \ \frac{\partial K}{\partial Q} \ P = 0, \tag{2.17}$$

or equivalently, iff

$$f \wedge f = p \wedge p. \tag{2.18}$$

[Plebański and Przanowski (to appear). See also Salazar et al (1987), Ibragimov (1967), Ovsiannikov (1982), Białynicki-Birula (1983)].

The group G is called the *duality rotation group* (DR-group).

Using the trick given by Salazar, García and Plebański (1987) we can find the general solution of (2.18) ($\Longleftrightarrow$ (2.17)).

This trick consist in choosing the Lorentzian coordinate system at some point $q \epsilon M_4$ so that

$$\vec{D} = (0,0,D) \quad and \quad \vec{H} = (0,0,H) \quad at \quad q \tag{2.19}$$

and consequentetly

$$\vec{E} = (0,0,E) \quad and \quad \vec{B} = (0,0,B) \quad at \quad q. \tag{2.20}$$

Then

$$P = \frac{1}{2} \cdot (H^2 - D^2), \quad Q = iDH. \tag{2.21}$$

Equation (2.18) reads

$$BE = DH \Longleftrightarrow B\frac{\partial M}{\partial D} - D\frac{\partial M}{\partial B} = 0 \tag{2.22}$$

and the latter equation has the general solution

$$M = M \ (\frac{1}{2} \cdot (D^2 + B^2)), \tag{2.23}$$

where $M(\frac{1}{2}\cdot(D^2+B^2))$ is an arbitrary function of the variable $\frac{1}{2}\cdot(D^2+B^2)$.

From the relation $H = \frac{\partial M}{\partial B}$ we get $B = B(D, H)$; and then using the relation

$$K = H \cdot B(D,H) - M\Big(\frac{1}{2} \cdot (D^2 + (B(D,H))^2\Big) \qquad (2.24)$$

and (2.21) one finds the general solution $K = K\ (P,Q)$ of (2.17)

To ensure (2.12) we assume that

$$M(0) = 0 \qquad and \qquad \partial_y M(0) = 1, \qquad (2.25)$$

$$y := \frac{1}{2}\ (D^2 + B^2).$$

Moreover, the dominant energy condition holds iff

$$M \geq y.\partial_y M > 0 \quad for \quad y > 0. \qquad (2.26)$$

Example:

$$M = b^2.[\ \sqrt{1 + b^{-2}.(D^2 + B^2)}\ \ - 1], \quad 0 \neq b \epsilon \Re \qquad (2.27)$$

Then

$$K = b^2 - \sqrt{(b^2 + D^2).(b^2 - H^2)}\ \ = b^2 - \sqrt{b^4 - 2b^2 P + Q^2}. \qquad (2.28)$$

This K generates the Born-Infield nonlinear electrodynamics.

Finally one can easily show that in terms of the 2-forms f and p the duality rotation takes the form

$$f' \ = \ f \cos\varphi + i * p \sin\varphi,$$

$$p' = p\cos\varphi + i * f\sin\varphi. \tag{2.29}$$

III. CONSERVATION LAW

From the preceding section one infers that an electrodynamics admits the duality rotation grup iff the condition (2.18) holds.

Now as

$$p \wedge p = *p \wedge *p, \tag{3.1}$$

(2.18) is equivalent to

$$f \wedge f = *p \wedge *p. \tag{3.2}$$

From the second set of the Maxwell equations (2.4) it follows that (at least locally)

$$*p = -i\, da, \tag{3.3}$$

where a is some 1-form. Them the Chern-Simons formula gives

$$f \wedge f = d(A \wedge dA) \quad and \quad * p \wedge *p = -d(a \wedge da). \tag{3.4}$$

Substituting (3.4) into (3.2) one gets

$$d(A \wedge dA + a \wedge da) = 0 \tag{3.5}$$

or the following conservation law

$$-i * d(A \wedge dA + a \wedge da) = 0 \tag{3.6}$$

[In the present work we define the Hodge star $*$ to be (Plebański (1974), Plebański and Przanowski (1989))

$$\sigma \wedge *\omega \;=\; -i\; exp\,[\frac{i\pi}{2}\, r.(4-r)].(\sigma|\omega).\vartheta \tag{3.7}$$

$$\sigma = \frac{1}{r!}\,\sigma_{i_1 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r}, \qquad \omega \;=\; \frac{1}{r!}\,\omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

$(\sigma|\omega) := \frac{1}{r!}\sigma_{i_1 \ldots i_r}\omega^{i_1 \cdots i_r}$, ϑ is the volume 4-form.

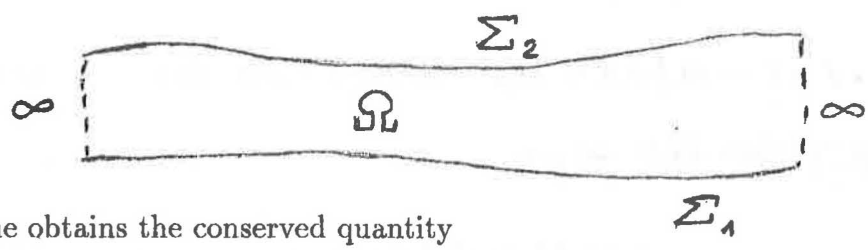
Thus

$$** = 1, \tag{3.8a}$$

$$\sigma \wedge *\omega \;=\; \omega \wedge *\sigma = (-1)^{r.(4-r)} * \sigma \wedge \omega, \tag{3.8b}$$

$$*1 = -i\vartheta, \quad *\vartheta = i]. \tag{3.8c}$$

Integrating (3.5) or (3.6) over a 4-dimensional domain Ω



one obtains the conserved quantity

$$C := -\frac{1}{2}\int_{\Sigma_1} d^3x \epsilon^{\mu\nu\rho}.(A_\mu \partial_\gamma A_\varrho + a_\mu \partial_\nu a_\rho)$$

$$= -\frac{1}{2}\int_{\Sigma_2} d^3x \epsilon^{\mu\nu\rho}.(A_\mu \partial_\nu A_\rho + a_\mu \partial_\nu a_\varrho) \tag{3.9}$$

In the Minkowski space-time, neglecting some 3-divergences, we can write

$$C = \frac{1}{2}\int_{R^3} d^3x \, [-\vec{A}\cdot(\nabla\times\vec{A}) + \vec{D}\cdot(\nabla^{-2}\nabla\times\vec{D})]$$

$$= \frac{1}{2}\int_{R^3} d^3x[\vec{B}\cdot(\nabla^{-2}\nabla\times\vec{B}) + \vec{D}\cdot(\nabla^{-2}\nabla\times\vec{D})] \tag{3.10}$$

[For the Maxwell electrodynamics the constant C has been found by Deser and Teitelboim (1976). In nonlinear electrodynamics it has been given by Białynicki-Birula (1983)].

We intend to show that the conservation law (3.6) is generated by DR-group according to the famous Noether construction.

To this end we write down the general duality rotation in terms of the Lagrange variables A_i and p_{ij}

$$A_i' = A_i \cos\varphi + a_i \sin\varphi,$$

$$p_{ij}' = p_{ij}\cos\varphi + i * f_{ij}\sin\varphi, \quad \varphi \epsilon \Re \tag{3.11}$$

$$f_{ij} = \partial_i A_j - \partial_j A_i, \quad i * p_{ij} = \partial_i a_j - \partial_j a_i.$$

[Notice that the first formula of (3.11) should be considered *mod* $\partial_i h$; h is an arbitrary function].

The infinitesimal operador X of (3.11) reads

$$X = a_i \frac{\partial}{\partial A_i} + i * f_{ij}\frac{\partial}{\partial p_{ij}}. \tag{3.12}$$

As a_i are nonlocal "function" of p_{ij} our infinitesimal operator X is not the standard one. [The standard infinitesimal operator appears to be the field on a relevant jet bundle (Trautman (1972), Kuperschmidt (1980), Ibragimov (1985)]. Nevertheless, in our case one can also find the Noether - Ibraginov identity and consequently the Noether constuction can be applied (Noether (1918), Trautman (1972), Kuperschmidt (1980), Ibragimov (1985), Olver (1986)).

First, we present X in a concise form

$$X = \eta^m \frac{\partial}{\partial u^m} \tag{3.14}$$

where $(u^1, \cdots, u^{16}) = (A_1, \cdots, p_{43}), (\eta^1, \cdots, \eta^{16}) = (a_1, \cdots, i * f_{43})$.

Then the Noether-Ibragimov identity reads

$$prX = d_i N^i + \eta^m \frac{\delta}{\delta u^m}, \tag{3.14}$$

where d_i stands for the total derivative with respect to the variable x^i and

$$prX = \eta^m \frac{\partial}{\partial u^m} + \sum_{s \geq 1} d_{i_1} \cdots d_{i_s} (\eta^m) \cdot \frac{\partial}{\partial u^m_{i_1 \cdots i_s}} \tag{3.15a}$$

(the Lie-Bäcklund operator);

$$N^i = \eta^m . [\frac{\partial}{\partial u^m_i} + \sum_{s \geq 1} (-1)^s d_{j_1} \cdots d_{j_s} \frac{\partial}{\partial u^m_{ij_1 \cdots j_s}}]$$

$$+ \sum_{r \geq 1} d_{k_1} \cdots d_{k_r} (\eta^m) . [\frac{\partial}{\partial u^m_{ik_1 \cdots k_r}}$$

$$+ \sum_{s \geq 1} (-1)^s d_{j_1 \cdots d_{j_s}} \frac{\partial}{\partial u^m_{ik_1 \cdots k_r j_1 \cdots j_s}}] \tag{3.15b}$$

(the Noether-Ibragimov operator);

$$\frac{\delta}{\delta u^m} = \frac{\partial}{\partial u^m} + \sum_{s\geq 1}(-1)^s \, d_{i_1}...d_{i_s} \frac{\partial}{\partial u^m_{i_1\cdots i_s}} \tag{3.15c}$$

(the Euler-Lagrange operator);

$$u^m_{ij_1\cdots j_s} := \partial_i\partial_{j_1}\cdots\partial_{j_s}\, u^m, \cdots etc. \tag{3.15d}$$

From (3.14) one gets

$$prX(\mathcal{L}) = d_i N^i(\mathcal{L}) + \eta^m \frac{\delta\mathcal{L}}{\delta u^m} \tag{3.16}$$

$$\mathcal{L} := \sqrt{-g}\, L$$

Assuming that u^m satisfy the field equations (2.4) and (2.5), i.e.,

$$\frac{\delta\mathcal{L}}{\delta u^m} = 0 \tag{3.17}$$

we have

$$prX(\mathcal{L}) = d_i N^i(\mathcal{L}) \tag{3.18}$$

Then performing simple calculations, using also Theorem 2.1 one arrives at the formula

$$-i\sqrt{-g} \, * \, d(A\wedge dA + a\wedge da) = 0 \tag{3.19}$$

Concluding, we have shown that the conservation law (3.6) is in fact generated by the DR-group according to the Noether theorem.

IV. NEW CONSERVATION LAWS IN THE MAXWELL ELECTRODYNAMICS

In this section we deal with the Maxwell electrodynamics in the Minskowski space-time.

Here one has

$$(g_{ij}) = diag(1,1,1,-1),\ p = f,\ \ L = -\frac{1}{2}p^{ij} \cdot (\partial_i A_j - \partial_j A_i) + \frac{1}{4}p^{ij}p_{ij},$$

$$\vec{D} = \vec{E}, \quad \vec{B} = \vec{H}. \tag{4.1}$$

To simplyfy the considerations we use the gauge

$$A_4 = 0, \quad \nabla \cdot \vec{A} = 0; \tag{4.2}$$

and we choose $\vec{E}$ and $\vec{A}$ to be the field variables (the *first-order formalism*). Then the Maxwell equations read

$$\partial_t \vec{E} + \nabla^2 \vec{A} = 0, \quad \partial_t \vec{A} + \vec{E} = 0, \quad (t \equiv x^4) \tag{4.3a}$$

$$\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{A} = 0 \tag{4.3b}$$

Equations (4.3 a) can be obtained from the Euler-Lagrange equations for the Lagrangian

$$L_1 = -\{\vec{E} \cdot \partial_t \vec{A} + \frac{1}{2} \cdot [\vec{E}^2 + (\partial_\mu A_\nu)^2]\}. \tag{4.4}$$

As it has been shown by Fushchich and Nikitin (1983) (see also Przanowski and Maciołek-Niedźwiecki (1992)) it is very convenient to employ here the Fourier representation.

Let $\vec{\mathcal{E}}$ and $\vec{\mathcal{A}}$ denote the Fourier transforms of $\vec{E}$ and $\vec{A}$, respectively, i.e.

$$\vec{\mathcal{E}}(\vec{k},t) = (2\pi)^{-\frac{3}{2}} \int d^3x \vec{E}(\vec{x},t) \cdot e^{-i\vec{k}\cdot\vec{x}}$$

$$\vec{\mathcal{A}}(\vec{k},t) = (2\pi)^{-\frac{3}{2}} \int d^3x \vec{A}(\vec{x},t) \cdot e^{-i\vec{k}\cdot\vec{x}} \tag{4.5}$$

As $\vec{E}$ and $\vec{A}$ are real vectors

$$\vec{\mathcal{E}}^*(\vec{k},t) = \vec{\mathcal{E}}(-\vec{k},t), \quad \vec{\mathcal{A}}^*(\vec{k},t) = \vec{\mathcal{A}}(-\vec{k},t); \tag{4.6}$$

(the star "*" stands for the complex conjugation)

The Maxwell equations in the Fourier representation takes the form

$$\partial_t\vec{\mathcal{E}} - k^2 \cdot \vec{\mathcal{A}} = 0, \quad \partial_t\vec{\mathcal{A}} + \vec{\mathcal{E}} = 0 \tag{4.7a}$$

$$\vec{k}\cdot\vec{\mathcal{E}} = 0, \quad \vec{k}\cdot\vec{\mathcal{A}} = 0 \tag{4.7b}$$

It is an easy matter to show that Eqs. (4.7 a) and their complex conjugate can be obtained from the following Lagrangian

$$\mathcal{L}_1 = -\frac{1}{2} \cdot \{\vec{\mathcal{E}} \cdot \partial_t\vec{\mathcal{A}}^* + \vec{\mathcal{E}}^* \cdot \partial_t\vec{\mathcal{A}} + \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^* + k^2 \cdot \vec{\mathcal{A}} \cdot \vec{\mathcal{A}}^*\} \tag{4.8}$$

Notice that by the Parseval-Plancheler formula,

$$\int d^3x dt L_1 = \int d^3k dt \mathcal{L}_1 \tag{4.9}$$

Now as in terms of $\vec{E}$ and $\vec{A}$ the general duality rotation reads (Deser and Teitelboim (1976), Przanowski and Maciołek-Niedźwiecki (1992)).

$$\vec{E}' = \vec{E} \cdot \cos\varphi + \nabla \times \vec{A} \cdot \sin\varphi,$$

$$\vec{A}' = \nabla^{-2}\nabla \times \vec{E} \cdot \sin\varphi + A \cdot \cos\varphi, \varphi \epsilon \Re, \tag{4.10}$$

(compare also with (3.11)), in terms of $\vec{\mathcal{E}}$ and $\vec{\mathcal{A}}$ one has

$$\vec{\mathcal{E}}' = \vec{\mathcal{E}} \cdot \cos\varphi + i\vec{k} \times \vec{\mathcal{A}} \cdot \sin\varphi,$$

$$\vec{\mathcal{A}}' = -\frac{i}{k^2}\vec{k} \times \vec{\mathcal{E}} \cdot \sin + \mathcal{A} \cdot \cos\varphi, \quad \varphi \epsilon \Re \tag{4.11}$$

and we arrive at the conclusion that in terms of the Fourier transform $\vec{\mathcal{E}}$ and $\vec{\mathcal{A}}$ the DR-group appears to be the *group of point transformations.*

Then the Noether - Ibragimov identity leads to the conservation law

$$d_t\{\frac{1}{2} \cdot [i\vec{k} \cdot (\vec{\mathcal{A}} \times \vec{\mathcal{A}}^*) + ik^{-2}\vec{k} \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{E}}^*)]\} = 0 \tag{4.12}$$

(Fushchich and Nikitin (1983), Przanowski and Maciołek-Niedźwiecki (1992)).

By the Parseval-Plancheler formula one has

$$\frac{1}{2} \int d^3k \cdot [i\vec{k} \cdot (\vec{\mathcal{A}} \times \vec{\mathcal{A}}^*) + ik^{-2}\vec{k} \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{E}}^*)]$$

$$= \frac{1}{2} \int d^3x \, [-\vec{A} \cdot (\nabla \times \vec{A}) + \vec{E} \cdot (\nabla^{-2}\nabla \times \vec{E})] = C \tag{4.13}$$

Note that if we use the *second order formalism,* i.e. we deal with A_i as the field variables, then in the Fourier representation the infinitesimal operator of the DR-group reads (see (4.11) and (4.7 a))

$$\tilde{X} = ik^{-2} \cdot (\vec{k} \times \partial_t \vec{\mathcal{A}}) \frac{\partial}{\partial \vec{\mathcal{A}}}. \tag{4.14}$$

To find the DR-group as expressed in terms of $\vec{A}$ one should solve the Lie equations

$$\frac{\partial}{\partial\varphi}\frac{\partial^s\vec{A}'}{\partial t^s} = ik^{-2}\cdot(\vec{k}\times\frac{\partial^{s+1}\vec{A}'}{\partial t^{s+1}}); \quad \frac{\partial^s\vec{A}'}{\partial t^s}|_{\varphi=0} = \frac{\partial^s\vec{A}}{\partial t^s} \tag{4.15}$$

$$s = 0, 1, \cdots.$$

The solutions of the set (4.15) constitutes a *general Lie-Bäcklund one-parameter transformation group* as a *formal one-parameter group* (Ibragimov (1985), Przanowski and Maciołek-Niedźwiecki (1992)).

Now we intend to consider some simple generalization of (4.11).

To this end we assume the infinitesimal operator to be of the form

$$\tilde{Y} = \xi_\mu \frac{\partial}{\partial\mathcal{E}_\mu} + \zeta_\mu \frac{\partial}{\partial\mathcal{A}_\mu},$$

$$\xi_\mu = a_{\mu\nu}\mathcal{E}_\nu + b_{\mu\nu}A_\nu, \quad \zeta_\mu = c_{\mu\nu}\mathcal{E}_\nu + d_{\mu\nu}\mathcal{A}_\nu, \tag{4.16}$$

where $a_{\mu\nu}$, $b_{\mu\nu}$, $c_{\mu\nu}$ and $d_{\mu\nu}$ are functions of $\vec{k}$. Then one shows that the Maxwell equations (4.7 a, b) are invariant under the one-parameter group of transformations generated by $\tilde{Y}$ iff

$$b_{\mu\nu} = -k^2 c_{\mu\nu} - \omega_\mu k_\nu, \quad d_{\mu\nu} = a_{\mu\nu} + \lambda_\mu k_\nu \tag{4.17}$$

and

$$a_{\mu\nu}\cdot k_\mu = \alpha\cdot k_\nu, \qquad c_{\mu\nu}\cdot k_\mu = \beta\cdot k_\nu \tag{4.18}$$

where $\omega_\mu, \lambda_\mu, \alpha$ and β are functions of $\vec{k}$. It is evident that by the constraint equations (4.7 b) we can put

$$b_{\mu\nu} = -k^2 c_{\mu\nu}, \quad d_{\mu\nu} = a_{\mu\nu} \tag{4.19}$$

Denoting the matrices $(a_{\mu\nu})$ and $(c_{\mu\nu})$ by $\hat{a}$ and $\hat{c}$, respectively, we denote also the relevant infinitesimal operator (4.16) by $\tilde{Y}_{\hat{a}\hat{c}}$. Then one finds the commutators $[\tilde{Y}_{\hat{a}\hat{c}}, \tilde{Y}_{\hat{a}'\hat{c}'}]$ to be $[\tilde{Y}_{\hat{a}\hat{c}}, \; \tilde{Y}_{\hat{a}'\hat{c}'}] = \tilde{Y}_{\hat{a}''\hat{c}''}$

$$\hat{a}'' = [\hat{a}', \hat{a}] - k^2 \cdot [\hat{c}', \hat{c}], \quad \hat{c}'' = [\hat{a}', \hat{c}] + [\hat{c}', \hat{a}]. \tag{4.20}$$

Therefore the operators $\tilde{Y}_{\hat{a}\hat{c}}$ constitute an *infinite - dimensional Lie algebra* of some symmetry group of the point transfomations for the Maxwell equations (4.7a,b)

Then the straightforward calculations show that the Noether-Ibragimov identity taken for $\tilde{Y}_{\hat{a}\hat{c}} + (\tilde{Y}_{\hat{a}\hat{c}})^*$ and for $\mathcal{L}_1$ given by (4.8) leads to the following conservation laws

$$d_t\{\frac{1}{2} \cdot [\frac{1}{2} \cdot (c_{\mu\nu} + c^*_{\nu\mu}) \cdot (k^2 \mathcal{A}_\mu \mathcal{A}^*_\nu + \mathcal{E}_\mu \mathcal{E}^*_\nu)]\} \; = \; 0 \tag{4.21}$$

$$d_t\{\frac{1}{2} \cdot [\frac{1}{2} \cdot (a_{\mu\nu} - a^*_{\nu\mu}) \cdot (\mathcal{A}_\mu \mathcal{E}^*_\nu - \mathcal{E}_\mu \mathcal{A}^*_\nu)\} \; = \; 0 \tag{4.22}$$

(Przanowski, Rajca and Tosiek (to appear)).

One can easily check that the conservation lews (4.21) and (4.22) hold for *arbitrary* $c_{\mu\nu}$ and $a_{\mu\nu}$ and not only for ones satisfying the conditions (4.18). It means that (4.21) and (4.22) are, in fact, the consequences of the symmetry of the set of equations (4.7a) only.

It may seem that (4.21) and (4.22) are defined for our specific gauge $\vec{k} \cdot \vec{A} \; = \; 0$ and they are not the gauge invariant relations.

However, it is not so. Using the formula

$$\mathcal{A}_\mu = ik^{-2} \epsilon_{\mu\varrho\sigma} k_\rho \mathcal{H}_\sigma \tag{4.23}$$

where $\mathcal{H}_\sigma$ stands for the Fourier transformation of H_σ, one can write (4.21) and (4.22) in the gauge invariant form

$$d_t\{\frac{1}{2}\cdot[\frac{1}{2}\cdot(c_{\mu\nu}+c^*_{\nu\mu})\cdot(\epsilon_{\mu\rho\sigma}\epsilon_{\nu\gamma\delta}k^{-2}k_\rho k_\gamma \mathcal{H}_\sigma \mathcal{H}^*_\delta+\mathcal{E}_\mu\mathcal{E}^*_\nu)]\} \;=\; 0 \qquad (4.24)$$

$$d_t\{\frac{1}{2}\cdot[\frac{1}{2}\cdot(a_{\mu\nu}-a^*_{\nu\mu})ik^{-2}k_\rho\cdot(\epsilon_{\mu\rho\sigma}\mathcal{H}_\sigma\mathcal{E}^*_\nu+\epsilon_{\nu\rho\sigma}\mathcal{E}_\mu\mathcal{H}^*_\sigma)]\} \qquad (4.25)$$

Examples:

(i) $a_{\mu\nu} = 0, \quad c_{\mu\nu} = ih(\vec{k})k^{-2}k_\kappa\epsilon_{\kappa\mu\nu}$

Then by (4.24) we get

$$\frac{1}{2}\int d^3x\cdot[\vec{H}\cdot(h(-i\nabla)\nabla^{-2}\nabla\times\vec{H})+\vec{E}\cdot(h(-i\nabla)\nabla^{-2}\nabla\times\vec{E})] = \; const. \; (4.26)$$

for $h(\vec{k}) = 1$ the constant (4.26) is exactly C given by (3.10), For $h(\vec{k}) = k^2$ ones has

$$\frac{1}{2}\int d^3x\cdot[\vec{H}\cdot(\nabla\times\vec{H})+\vec{E}\cdot(\nabla\times\vec{E})] = \quad const. \qquad (4.27)$$

(compare with Deser and Teitelboim (1976)).

(ii). $a_{\mu\nu} = +k^2\epsilon_{\gamma\mu\nu}, \quad c_{\mu\nu} = 0$

Here, (4.25) yields

$$-\frac{1}{2}\int d^3x\cdot(\vec{H}\times\partial_t\vec{H}+\vec{E}\times\partial_t\vec{E})_\gamma = \quad const. \qquad (4.28)$$

(iii). $a_{\mu\nu} = 0, \quad c_{\mu\nu} = -ik_\gamma\epsilon_{\kappa\mu\nu}$

Then from (4.24) one gets

$$-\frac{1}{2}\int d^3x\cdot(\vec{H}\times\partial_\gamma\vec{H}+\vec{E}\times\partial_\gamma\vec{E})_\kappa = \quad const. \tag{4.29}$$

Now we can write the constant (4.27), (4.28) and (4.29) in a compact form

$$Z_{jk}=\frac{i}{2}\cdot\int d^3x(*f^{4l}\partial_k f_{lj}-f^{4l}\partial_k * f_{lj}) \tag{4.30}$$

and this, *mutatis mutandis*, corresponds to the "zilch" of Lipkin (1964). (see also Morgan (1964), Kibble (1965), Fradkin (1965), O'Connell and Tompkins (1965), Deser and Nicolai (1981)).

It is supposed that the conservation laws (4.24) and (4.25) (eventually combined with the conservation law for the energy-momentum tensor) involve all conservation laws given by Morgan (1964). Of course our formulas (4.24) and (4.25) yield the infinite number of nonlocal conserved quantities.

It is of some interest to write down the infinitesimal operator $\tilde{Y}_{\hat{a}\hat{c}}$ in the " coordinate representation". Denoting this operator by $Y_{\hat{a}\hat{c}}$ we get

$$Y_{\hat{a}\hat{c}}=\left[a_{\mu\nu}(-i\nabla)E_\nu+c_{\mu\nu}(-i\nabla)\nabla^2A_\nu\right]\frac{\partial}{\partial E_\mu}$$
$$+[c_{\mu\nu}(-i\nabla)\;E_\nu+a_{\mu\nu}(-i\nabla)A_\nu]\frac{\partial}{\partial A_\mu} \tag{4.31}$$

Observe that if $Y_{\hat{a}\hat{c}}$ is a local operator (for example, it is the case for the Lipkin conservation law) then it gives rise to a (general) Lie-Bäcklund transformation group.

Consider now the *zilch* in a nonlinear electrodynamics. By analogy to the Maxwell case (see (4.30)) we put

$${}'Z^i_{jk}=\frac{i}{2}\;(*f^{il}\partial_k f_{lj}-p^{il}\partial_k * p_{lj}) \tag{4.32}$$

(compare with Lipkin (1964), Morgan (1964), Kibble (1965)). Note that $'Z^i_{jk}$ is invariant under the duality rotation group given by (2.29).

Then using the Maxwell equations (2.4) one gets

$$\partial'_i Z^i_{jk} = -\frac{i}{4}(*f^{il}\partial_j\partial_k f_{il} - p^{il}\partial_j\partial_k * p_{il}) \qquad (4.33)$$

Thus, in contrary to the Maxwell electrodynamics, we don't have the conservation low for $'Z^i_{jk}$ within a nonlinear electrodynamics, even when this electrodynamics admits the duality rotation.

[In the latter case, by (2.18), we can write $'Z^i_{jk}$ in the following form

$$'Z^i_{jk} = \frac{i}{2}\cdot(*f^{il}\partial_k f_{lj} + p_{jl}\partial_k p^{li} \qquad (4.34)$$

where $A\partial_k B := \frac{1}{2}\cdot[A\cdot\partial_k B - (\partial_k A)\cdot B]$

(compare with Kibble (1965), Deser and Nicolai (1981)).

We end this section with some remarks on the duality rotation and the zilch in non-abelian field theories. It has been shown by Deser and Teitelboim (1976) that the duality rotation is not well defined for the non-abelian field. Then from the work by Deser and Nicolai (1981) we acknowledge that the non-abelian zilch fails to be conserved. This is also the case in the Einstein gravitation.

REFERENCES

Białynicki-Birula, I., and Białynicka-Birula, Z. (1975). *Quantum Electrodynamics*, Pergamon Press, Oxford.

Białynicki-Birula, I. (1983), *Nonlinear Electyrodynamics: Variations on a Theme by Born and Infeld*, in *Quantum Theory of Particle and Fields*, Birthay volume dedicated to Jan Łopuszański, ed. B. Jancewicz and Jerzy Lukierski, Wold Scientific, Singapore (p. 31-48.)

Born, M., and Infeld, L. (1934). Proc. Roy. Soc. *A*144, 425.

Deser. S., and Teitelboim, C. (1976). Phys. Rev. *D*13, 1592.

Deser. S., and Nicolai, H. (1981). Phys. Lett. 98*B*, 45

Ferrara, S., Scherk, Y., and Zumino, R. (1977). Nucl. Phys. *B*121, 393.

Fradkin, D.M. (1965). J. Math. Phys. 6, 879.

Fushchich, V.I., and Nikitin, A. G. (1983). Sov. J. Part. Nucl. 14, 1.

Heaviside, O. (1893). Trans. R. Soc. *A*183, 423.

Ibragimov, N.H. (1967). *Gruppouye Svoistva Nekotoryh Differencialnyh Uravnenii,* Nauka, Novosybirsk.

Ibragimov, N.H. (1985), *Transformation groups Applied to Mathematical Physics.* Riedel, Boston.

Kibble,, T.W.B. (1965). J. Math. Pys. 6, 1022.

Kuperschmidt, B.A. (1980). *Geometry of Jet Bundles and the Struture of Langrangian and Hamiltonian Formalisms*, in Lecture Notes in Mathematics 775, Springer-Verlag, Berlin (p. 162-218).

Landau, L.D., and Lifschitz, E.M. (1973). *Teoria Polia,* Nauka, Moskva (in Russian).

Larmor, I. (1928). *Collected Papers*, London.

Lipkin, D.M. (1964). J. Math. Phys. 5, 696.

Misner, C.W., and Wheeler, J. A. (1957). Ann. Phys. 2, 525.

Morgan, T.A. (1964). J. Math. Phys. 5, 1659.

Møller, C. (1972). *The Theory of Relativity*, Clarendon Press, Oxford.

Noether, E. (1918). *Invariante Variationprobleme*, Nach. König. Gesell. Wissen. Göting. Math. Phys. Kl. 235.

O'Connel, R.F., and Tompkins, D.R. (1965). J. Math. Phys. 6, 1952.

Olver, P. J. (1986). *Applications of Lie groups to Differential Equations*, Springer-Verlag, New York.

Ovsiannikov, L.V. (1982). *Group Analysis of Differential Equations*, Academic, New York.

Plebański, J. F. (1970). *Lectures on Non-Linear Electrodynamics*, Monograph of the Niels Bohr Institute Nordita, Copenhagen.

Plebański, J. F. (1974). *Spinors, Tetrads and Forms*, unpublished Monograph of CINVESTAV, Mexico City.

Plebański, J. F., and Przanowski, M. (1988). Acta Phys. Pol. *B19*, 805

Plebański, J. F. and Przanowski, M. *Duality Transformations in Electrodynamics*, submitted to Int. J. Theor. Phys.

Przanowski, M., and Maciołek-Niedźwiecki, A. (1992). J. Math. Phys. 33, 3978.

Przanowski, M., Rajca, B., and Tosiek, Y. *On Some Conservation Laws in Electrodynamics,* submitted for publication.

Rainich, G.Y. (1925). Trans. Am. Math. Soc. 27, 106

Salazar, H.I., García, D. A., and Plebański, J. F. (1987) J. Math. Phys.

28, 2171.

Salazar, H.I., García, D.A., and Plebański, J. F. (1989). J. Math. Phys. 30, 2689.

Schwinger, J. (1975). Phys. Rev. D12, 3105.

Trautman, A. (1972). *Invariance of Lagrangian Systems, in Gravitation*, ed. by L.O' Raifeartaigh, Oxford.