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Noncommutative Instantons Revisited

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Abstract. We discuss the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of $U(N)$ instantons in noncommutative (NC) space and prove the one-to-one correspondence between moduli spaces of the noncommutative instantons and the ADHM data, together with an origin of the instanton number for $U(1)$. We also give a derivation of the ADHM construction from the viewpoint of the Nahm transformation of instantons on four-torus.

1. Introduction

Instantons are finite-action solutions of anti-self-dual (ASD) Yang-Mills equation and become exact solutions of classical Yang-Mills theories. They can reveal non-perturbative aspects of the quantum theories. Actually, the path-integrations, formulating the quantum theories, could reduce to finite-dimensional integrations over the instanton moduli spaces. The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [2] is a powerful method to obtain the instantons. Furthermore, via the construction, the moduli space is mapped to the set of quadruple matrices. The aforementioned integration, being thereby an integration over the matrices, becomes tractable [10]. To evaluate the integration, the use of noncommutative instantons [34] is relevant so that a localization formula can be applied to the integration [33, 30]. In the procedures, various formulas and relations of the ADHM construction are required. Hence it is worthwhile to elucidate the noncommutative ADHM construction together with the group actions and to present all the ingredients in the construction explicitly.

In our paper [17], we discuss the ADHM construction of $U(N)$ instantons in noncommutative spaces and prove one-to-one correspondence between moduli spaces of the noncommutative instantons and the ADHM data. We also argue group actions (especially torus actions) on noncommutative instantons. The present article is a brief report of it with some co-supplement materials.

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2. Noncommutative Field Theory

Noncommutative (NC) space is a space which coordinate ring is noncommutative. Let x^μ be the spacial coordinates. The noncommutativity is expressed by the following commutation relations.

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a real antisymmetric tensor. When $\theta^{\mu\nu}$ vanishes identically, the coordinate ring is commutative and the underlying space reduces to a commutative one. The commutation relations (1), like the canonical commutation relations in quantum mechanics, lead to "space-space uncertainty relation." Singularities in commutative space could resolve in noncommutative space thereby. This is one of the prominent features of field theories on noncommutative space, NC field theories, for short, and yields various new physical objects such as $U(1)$ instantons.

In four-dimensional Euclidean space, the antisymmetric tensor takes the canonical form as

$$\theta^{\mu\nu} = \left(\begin{array}{cc|cc} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{array}\right), \tag{2}$$

where the real numbers θ_1, θ_2 are called *noncommutative parameters*. The spacial coordinates x^μ has a Fock representation. To see this, introduce complex coordinates z_1, z_2 by $z_1 = x^2 + ix^1$ and $z_2 = x^4 + ix^3$. By using the complex coordinates, the noncommutativity is expressed as

$$[z_1, \bar{z}_1] = 2\theta_1,\ [z_2, \bar{z}_2] = 2\theta_2. \tag{3}$$

This shows that each pair of the complex coordinate and its complex conjugates gives annihilation and creation operators of a harmonic oscillator. In this article, we set $\theta_{j=1,2} \leq 0$. $a_j = (1/\sqrt{-2\theta_j})\bar{z}_j$ and $a_j^\dagger = (1/\sqrt{-2\theta_j})z_j$ are the annihilation and creation operators of harmonic oscillators.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a Fock space spanned by the occupation number basis $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$ $(n_1, n_2 = 0, 1, \dots)$. Field variables on the NC space can be represented as an infinite matrices of the form

$$\hat{f}(\hat{x}) = \sum_{m_1,m_2,n_1,n_2=0}^{\infty} f_{m_1,m_2,n_1,n_2} |m_1, m_2\rangle\langle n_1, n_2|. \tag{4}$$

In the above, the hat symbol is used to emphasize that these are operators. We remark that, as seen in subsection 4.4, to take the inverse of these operators is a delicate issue because the operator zero-modes, if they exist, need to be accounted.

For more details on noncommutative field theories, see e.g. [6, 11, 17, 19, 23, 39, 40].

3. Nahm Transformation and Origin of the ADHM Duality

The ADHM construction is a descendant of the twistor theory founded by R. Penrose. Twistor theoretical idea was first applied by R. Ward to instantons. It converts the self-duality of gauge fields on S^4 to the holomorphy of complex vector bundles on $\mathbb{C}P_3$ (see e.g. [44]), where the construction reduces to an algebro-geometric one, and finally Atiyah, Drinfeld, Hitchin and Manin obtained an algebraic method to generate *all* instanton solutions on S^4 [2]. (Instantons on S^4 are equivalent to those on $\mathbb{R}^4$ by the conformal invariance and Uhlenbeck's theorem.) The construction is based on one-to-one correspondence between the moduli space of instantons and that of the ADHM data. The correspondence is thereafter called the *ADHM duality*.

It was also applied to the case of the BPS monopoles by W. Nahm, which is called the *ADHMN construction* or the *Nahm construction*. Schenk [38] and Braam and van Baal [5], extracting the Fourier transform, known as the Nahm transformation, from the construction and applying to instantons on four-torus, obtained the duality between instantons on four-torus and its dual torus, which is also known as the *Nahm duality*. Nahm transformations of explicit ASD gauge fields are performed in e.g. [1, 16, 20, 43]. For surveys of the Nahm transformation, see e.g. [4, 9, 22].

The ADHM duality is also understood as a version of the Nahm duality. The ADHM duality in this line was first outlined in [7] and made rigorously in [25] with a generalization to ALE spaces. (A beautiful treatment can be also found in [9].) Noncommutative version of the ADHM duality is discussed in [17, 27, 32]. We start this section by giving a brief review on the Nahm transformation and then argue the ADHM/Nahm duality by taking certain limits.

3.1. Poincaré Line Bundle

The Poincaré line bundle plays a central role in the Nahm duality.

The Poincaré line bundle $\mathcal{P}$ is a $U(1)$ line bundle over the product space $T^4 \times \widehat{T}^4$, where T^4 is a four-torus and $\widehat{T}^4$ is the dual four-torus. The four torus is realized as $T^4 = \mathbb{R}^4/\Lambda$, where Λ is a maximal lattice in $\mathbb{R}^4$. The dual four-torus is $\widehat{T}^4 = \mathbb{R}^{4*}/2\pi\Lambda^*$, where $\mathbb{R}^{4*}$ is the dual vector space and Λ^* is the dual lattice. We use x^μ and ξ_μ as the coordinates of $\mathbb{R}^4$ and $\mathbb{R}^{4*}$ with the dual pairing $\xi \cdot x = \xi_\mu x^\mu$. The line bundle $\mathcal{P}$ is equipped with a $U(1)$ potential of the form

$$\omega(x,\xi) = i\xi_\mu dx^\mu. \tag{5}$$

Note that, taking another pair (x',ξ'), where $x' \in x + \Lambda$ and $\xi' \in \xi + 2\pi\Lambda^*$, amounts to $U(1)$ gauge transformation. Actually, it is apparent to see that, for $x' = x+\lambda$ ($\lambda \in \Lambda$) and $\xi' = \xi+2\pi\mu$ ($\mu \in \Lambda^*$), two gauge potentials $\omega(x',\xi')$ and $\omega(x,\xi)$ are gauge equivalent.

$$\omega(x+\lambda, \xi+2\pi\mu) = \omega(x,\xi) + g^{-1}(x,\xi)dg(x,\xi), \tag{6}$$

where $g(x,\xi) = e^{2\pi i\mu\cdot x}$ is single-valued on $T^4 \times \widehat{T}^4$.

The field strength $\Omega = d\omega$ becomes constant.

$$\Omega(x,\xi) = id\xi_\mu \wedge dx^\mu. \tag{7}$$

The following form of the gauge potential is also used conveniently.

$$\omega'(x,\xi) = -ix^\mu d\xi_\mu. \tag{8}$$

The above form can be obtained from (5) by the gauge transformation $\exp(-i\xi\cdot x)$ on $\mathbb{R}^4 \times \mathbb{R}^{4*}$.

$$\omega'(x,\xi) = \omega(x,\xi) + e^{i\xi\cdot x}de^{-i\xi\cdot x}. \tag{9}$$

Let π, $\hat{\pi}$ be natural projections from $T^4 \times \widehat{T}^4$ onto T^4 and $\widehat{T}^4$. We summarize on the Poincaré line bundle as follows.

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d] & \\
 & T^4 \times \widehat{T}^4 \arrow[dl, "\pi"'] \arrow[dr, "\hat{\pi}"] & \\
T^4 & & \widehat{T}^4
\end{tikzcd}$$

3.2. Nahm Transformation

Let $E \to T^4$ be a $U(N)$ vector bundle with the 2nd Chern number $c_2(E) = k$ and A be a $U(N)$ gauge potential on E. The tensor product bundle $\pi^* E \otimes \mathcal{P}$ is the $U(N)$ vector bundle over $T^4 \times \widehat{T}^4$ equipped with the gauge potential $A \otimes 1_{[1]} + 1_{[N]} \otimes \omega$. Taking the slice along $T^4 \times \{\xi\}$, we regard $\pi^* E \otimes \mathcal{P}|_{T^4 \times \{\xi\}} \to T^4 \times \{\xi\}$ as a $U(N)$ vector bundle over T^4. In particular, the gauge potential on $\pi^* E \otimes \mathcal{P}|_{T^4 \times \{\xi\}}$ is given by $A_\xi = A \otimes 1_{[1]} + 1_{[N]} \otimes i\xi_\mu dx^\mu$. The field strength F_ξ of A_ξ equals to F of A.

We introduce the Dirac operator. Let $S^\pm \to T^4$ be a spinor bundle on T^4. The Dirac operator acting on the sections $\Gamma(T^4, S^\pm \otimes \pi^* E \otimes \mathcal{P}|_{T^4 \times \{\xi\}})$ is given by

$$\begin{aligned} \mathcal{D}[A_\xi] &= e_\mu \otimes (\partial_\mu + A_\mu + i\xi_\mu), \\ \bar{\mathcal{D}}[A_\xi] &= \bar{e}_\mu \otimes (\partial_\mu + A_\mu + i\xi_\mu), \end{aligned} \tag{10}$$

where $e_\mu = (-i\sigma_i, 1)$, $\bar{e}_\mu = (i\sigma_i, 1)$. These are the *Weyl operators* rather than the Dirac operators, however, we use the word the "Dirac operator" for simplicity.

The dual vector bundle $\widehat{E} \to \widehat{T}^4$ can be constructed by using the Dirac zero modes. The fiber $\widehat{E}_\xi$ is identified with $\mathrm{Ker}\, \bar{\mathcal{D}}[A_\xi]$. Since the Atiyah-Singer family index theorem implies $\dim \mathrm{Ker}\, \bar{\mathcal{D}}[A_\xi] = k$, the rank of $\widehat{E}$ is k. Actually, $\widehat{E}$ is a $U(k)$ vector bundle equiped with the dual $U(k)$ gauge potential $\widehat{A}$. For the description of the dual gauge potential, we regard $\widehat{E}$ as a sub-bundle of $\widehat{H}$, where $\widehat{H} \to \widehat{T}^4$ is an infinite-dimensional trivial vector bundle with the fiber $\widehat{H}_\xi = L^2(T^4, S^+ \otimes \pi^* E \otimes \mathcal{P}|_{T^4 \times \{\xi\}})$. (See Fig. 1.) By using the projection $P : \widehat{H} \to \widehat{E}$, diffrential $\widehat{d}$ on $\Gamma(\widehat{T}^4, \widehat{H})$ induces the covariant derivative $\widehat{D} = \widehat{d} + \widehat{A}$ as

$$\widehat{D} = P\widehat{d} : \Gamma(\widehat{T}^4, \widehat{E}) \to \Gamma(\widehat{T}^4, \Lambda^1 \otimes \widehat{E}). \tag{11}$$

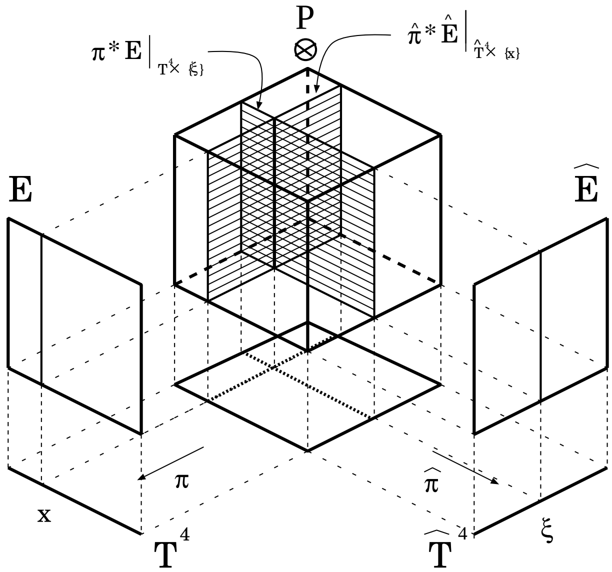

Figure 1. The stage of the Nahm transformation

The $U(k)$ gauge potential $\widehat{A}$ is expressed by using the Dirac zero modes as

$$\widehat{A}_\mu^{pq}(\xi) = \int_{T^4} d^4x \; \psi_\xi^{p\dagger}(x) \frac{\partial}{\partial \xi^\mu} \psi_\xi^q(x) \tag{12}$$

where ψ_ξ^p $(p = 1, 2, \ldots, k)$ are the normalized Dirac zero-modes.

$$\bar{\mathcal{D}}[A_\xi]\psi_\xi^p(x) = 0, \quad \int_{T^4} d^4x\ \psi_\xi^{p\dagger}(x)\psi_\xi^q(x) = \delta^{pq}. \tag{13}$$

The 2nd Chern number of $\widehat{E}$ turns out to be N [5, 38]. Thus we obtain the $U(k)$ vector bundle $\widehat{E} \to \widehat{T}^4$ with $c_2(\widehat{E}) = N$ and the $U(k)$ gauge potential $\widehat{A}$ on $\widehat{E}$. This gives the Nahm transformation $\mathcal{N} : (E, A) \mapsto (\widehat{E}, \widehat{A})$.

$$
\begin{array}{ccccc}
 & & \pi^*E\otimes\mathcal{P} & & \\
 & & \downarrow & & \\
E & & T^4\times\widehat{T}^4 & & \widehat{E}\\
\downarrow & \overset{\pi}{\swarrow} & & \overset{\hat{\pi}}{\searrow} & \downarrow\\
T^4 & & & & \widehat{T}^4
\end{array}
$$

The inverse transformation is carried out by following the same procedure. Let $\widehat{E} \to \widehat{T}^4$ be a $U(k)$ vector bundle with $c_2(\widehat{E}) = N$ and $\widehat{A}$ be a $U(k)$ gauge potential on $\widehat{E}$. We regard the slice of the tensor product bundle $\widehat{\pi}^*\widehat{E} \otimes \mathcal{P}$ along $\{x\} \times \widehat{T}^4$ as a $U(k)$ vector bundle over $\widehat{T}^4$ equipped with the gauge potential $\widehat{A}_x = \widehat{A} \otimes 1_{[1]} - 1_{[k]} \otimes ix^\mu d\xi_\mu$. The field strength $\widehat{F}_x$ of $\widehat{A}_x$ equals to $\widehat{F}$ of $\widehat{A}$. The Dirac operator acting on the sections $\Gamma(\widehat{T}^4, \widehat{S}^\pm \otimes \pi^*\widehat{E} \otimes \mathcal{P}|_{\{x\}\times\widehat{T}^4})$ takes the form

$$\begin{aligned}
\widehat{\mathcal{D}}[\widehat{A}_x] &= e_\mu \otimes (\widehat{\partial}^\mu + \widehat{A}^\mu - ix^\mu),\\
\widehat{\bar{\mathcal{D}}}[\widehat{A}_x] &= \bar{e}_\mu \otimes (\widehat{\partial}^\mu + \widehat{A}^\mu - ix^\mu).
\end{aligned} \tag{14}$$

By using the zero modes of $\widehat{\bar{\mathcal{D}}}[\widehat{A}_x]$, consisting of N normalizable ones, we obtain a $U(N)$ vector bundle $E \to T^4$ with $c_2(E) = k$ and a $U(N)$ gauge potential A on E. This gives the inverse Nahm transformation $\widehat{\mathcal{N}} : (\widehat{E}, \widehat{A}) \mapsto (E, A)$.

$$
\begin{array}{ccccc}
 & & \hat{\pi}^*\widehat{E}\otimes\mathcal{P} & & \\
 & & \downarrow & & \\
E & & T^4\times\widehat{T}^4 & & \widehat{E}\\
\downarrow & \overset{\pi}{\swarrow} & & \overset{\hat{\pi}}{\searrow} & \downarrow\\
T^4 & & & & \widehat{T}^4
\end{array}
$$

It can be shown $\mathcal{N}\widehat{\mathcal{N}} = \widehat{\mathcal{N}}\mathcal{N} = id.$ [5, 38]. Thus, eventually, $\widehat{\mathcal{N}}$ turns out to be the inverse of $\mathcal{N}$. The Nahm duality is summarized in the following dictionary.

$$
\begin{array}{ccc}
G = U(N) & & \widehat{G} = U(k)\\
k\text{-instanton on } T^4 & \overset{1\,:\,1}{\longleftrightarrow} & N\text{-instanton on } \widehat{T}^4\\
(\text{coord. } x^\mu) & & (\text{coord. } \xi^\mu)\\
 & \text{massless Dirac eq.} & \\
 & \bar{\mathcal{D}}\psi = 0 & \\
\text{instanton} \ :\ A_{\mu[N]} & \overset{k \text{ solutions: } \psi(\xi,x)}{\longrightarrow} & \widehat{A}_{\mu[k]} = \displaystyle\int_{T^4} d^4x\ \psi^\dagger \frac{\partial}{\partial \xi^\mu}\psi\\
 & & \\
 & \text{massless Dirac eq.} & \\
 & \widehat{\bar{\mathcal{D}}}v = 0 & \\
A_{\mu[N]} = \displaystyle\int_{\widehat{T}^4} d^4\xi\ v^\dagger \frac{\partial}{\partial x^\mu} v & \overset{N \text{ solutions: } v(x,\xi)}{\longleftarrow} & \text{instanton} \ :\ \widehat{A}_{\mu[k]}
\end{array}
$$

3.3. A Derivation of the ADHM construction from the Nahm transformation

We can understand the duality in the ADHM/Nahm construction in some limit of the Nahm duality.

- Taking all four radii of the four-torus infinity $\Rightarrow$ ADHM construction of instantons
  Then the radii of the dual torus become zero. Hence the dual torus shrinks into one point and derivatives become meaningless because derivatives measure difference between two points. As the result, all derivatives in the dual ASD equation and the dual massless Dirac equation drop out and the differential equations becomes matrix equations. This naively yields the ADHM duality: one-to-one correspondence between the moduli space of the ASD equations on $\mathbb{R}^4$ (=infinite-size torus) and the moduli space of the matrix equations. For more detailed discussion, see [43].
- Taking three radii infinity and the other radius zero $\Rightarrow$ Nahm construction of monopoles

4. ADHM Construction of (Noncommutative) Instantons

In this section, we discuss the ADHM construction of $U(N)$ k-instantons on commutative and noncommutative Euclidean $\mathbb{R}^4$, especially focusing on the case of $(N,k) = (2,1)$, that is, $U(2)$ 1-instantons. The construction reads as follows.

Firstly, we take the ADHM data for $U(N)$ k-instantons. They are quadruple complex matrices $\big(B_1, B_2, I, J\big)$ which satsifies the noncommutative ADHM equation

$$\mu_{\mathbb{R}} \equiv [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta, \quad \mu_{\mathbb{C}} \equiv [B_1, B_2] + IJ = 0, \tag{15}$$

where $B_{1,2}$ are $k \times k$ matrices, and I and J are $k \times N$ and $N \times k$ matrices. $\zeta \equiv -[z_1, \bar{z}_1] - [z_2, \bar{z}_2] = -2(\theta_1 + \theta_2)$ in the first equation of (15) is a non-negative constant originated in the noncommutativity (cf. section 2). When the antisymmetric tensor $\theta^{\mu\nu}$ is ASD, the constant vanishes and the noncommutative ADHM equation (15) coincides with the commutative one.

Secondly, with the ADHM data $\big(B_1, B_2, I, J\big)$, we associate "0-dimensional Dirac equation"

$$\nabla^\dagger V = 0, \quad \nabla(x) = \begin{pmatrix} I^\dagger & J \\ \bar{z}_2 - B_2^\dagger & -(z_1 - B_1) \\ \bar{z}_1 - B_1^\dagger & z_2 - B_2 \end{pmatrix}. \tag{16}$$

The solution V for equation (16) is a $(N + 2k) \times N$ matrix. In the noncommutative case, we have to take care about the existence of operator zero-modes of V.

Finally, normalize V so that

$$V^\dagger V = 1_N. \tag{17}$$

Then, by using the above V, we get the corresponding instanton solution as (cf. (12))

$$A_\mu = V^\dagger \partial_\mu V. \tag{18}$$

Actually, due to the construction, it satisfies the (noncommutative) ASD Yang-Mills equation

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \; F_{z_1 z_2} = 0. \tag{19}$$

The instanton number is computed to be $-k$.

For reviews on noncommutative instantons, see [14, 15, 24, 26, 31, 37].

4.1. Comments on Instanton Moduli Spaces
Geometry of the instanton moduli space depends on the constant ζ. [28, 29]. For the case of $(N, k) = (2, 1)$, see Fig. 2.

- When $\zeta = 0$, the moduli space contains the small instanton singularities. This is also the case of the instantons on commutative $\mathbb{R}^4$.
- When $\zeta > 0$, the small instanton singularities disapper and replaced by a new class of smooth instantons, $U(1)$ instantons.

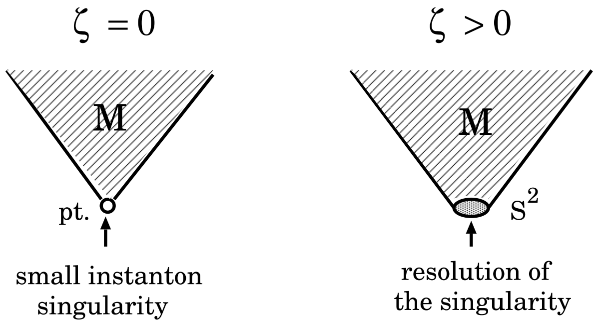

Figure 2. $(N, k) = (2, 1)$ Instanton Moduli Space

4.2. ADHM Construction of Instantons
Here we present the ADHM construction of the Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton solution [3], which is the $SU(2)$ 1-instanton on commutative Euclidean $\mathbb{R}^4$.

In the case of $k = 1$, the commutators in the ADHM equation (15) are dropped out and the equation becomes quite simple. $B_{1,2}$ are chosen as arbitrary complex numbers $\alpha_{1,2}$, and then I, J in the equation ($\zeta = 0$) are easily solved by using a real number ρ. Thus we have

$$B_1 = \alpha_1,\ B_2 = \alpha_2,\ I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}, \quad \alpha_{1,2} \in \mathbb{C},\ \rho \in \mathbb{R}. \tag{20}$$

The normalized solution of the 0-dimensional Dirac equation (16) takes the form

$$V = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \bar{e}_\mu (x_\mu - b_\mu) \\ \begin{matrix} -\rho & 0 \\ 0 & -\rho \end{matrix} \end{pmatrix}, \quad \phi = |x - b|^2 + \rho^2, \tag{21}$$

where b_μ are the real and imaginary parts of $\alpha_{1,2}$ as $\alpha_1 = b_2 + ib_1,\ \alpha_2 = b_4 + ib_3$. The normalization factor ϕ is determined by (17).

Finally the instanton solution is constructed as follows

$$A_\mu = V^\dagger \partial_\mu V = \frac{i(x - b)^\nu \eta_{\mu\nu}}{(x - b)^2 + \rho^2}, \tag{22}$$

where $\eta_{\mu\nu} = \eta^i_{\mu\nu} \sigma_i$ denotes the contraction of 't Hooft's ASD eta symbol by the Pauli matrices. The field strength is computed to

$$F_{\mu\nu} = \frac{2i\rho^2}{(|z - \alpha|^2 + \rho^2)^2} \eta_{\mu\nu}. \tag{23}$$

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This is exactly the BPST instanton solution [3]. Distribution of $\mathrm{Tr} F\tilde{F}$ is depicted at the upper right-hand side in Fig. 3. The moduli space is a five-dimensional space, where each point describes the positions b^μ and the size ρ of the instanton.

In the zero-size limit $\rho \to 0$, the field strength $F_{\mu\nu}$ concentrates at $x^\mu = b^\mu$ and becomes singular. This gives a singularity in the (complete) instanton moduli space, called the *small instanton singularity*. (See Fig. 3.) On noncommutative spaces, the small instanton singularity is resolved and a new class of instantons appears.

4.3. ADHM Construction of Noncommutative BPST Instantons

Noncommutative BPST solution is a noncommutative version of the BPST instanton and is obtained also by applying the ADHM procedure. A solution of the noncommutative ADHM equation is easily found:

$$B_{1,2} = 0, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}. \tag{24}$$

By comparing with the previous solution, I is now deformed by the noncommutativity of the coordinates. This particularly implies that the size of instanton is larger than that of the commutative one. In fact, even in the $\rho \to 0$ limit, the configuration keeps to be smooth. This configuration is essentially the same as a $U(1)$ instanton solution which will be constructed more in detail in the next subsection.

BPST instantons on commutative and noncommutative spaces are summarized as follows.

BPST instanton		NC BPST instanton
$\mu_\mathbb{R} = 0, \ \mu_\mathbb{C} = 0$	ADHM equation	$\mu_\mathbb{R} = \zeta, \ \mu_\mathbb{C} = 0$
$B_{1,2} = \alpha_{1,2}$, $I = (\rho, 0), J^t = (0, \rho)$	ADHM data	$B_{1,2} = \alpha_{1,2}$, $I = (\sqrt{\rho^2 + \zeta}, 0), J^t = (0, \rho)$
$\mathbb{R}^4 \times$ orbifold $\mathbb{C}^2/\mathbb{Z}_2$ (singular)	moduli space	$\mathbb{R}^4 \times$ Eguchi-Hanson $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ (regular)
$F_{\mu\nu} \to$ (singular)	zero-size limit	$F_{\mu\nu} \to$ (regular)

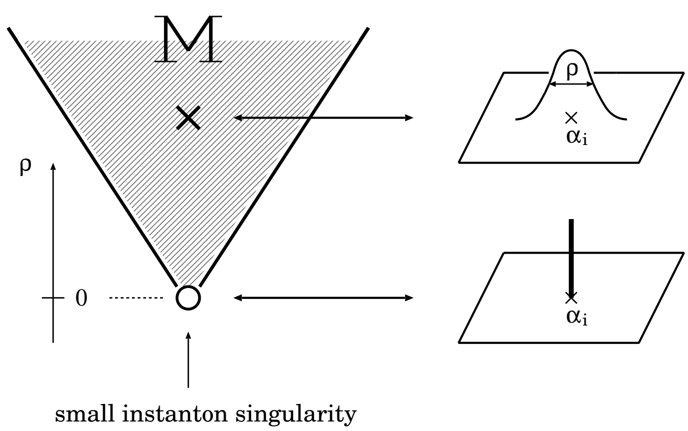

Figure 3. Instanton moduli space and the configuration of the BPST instanton

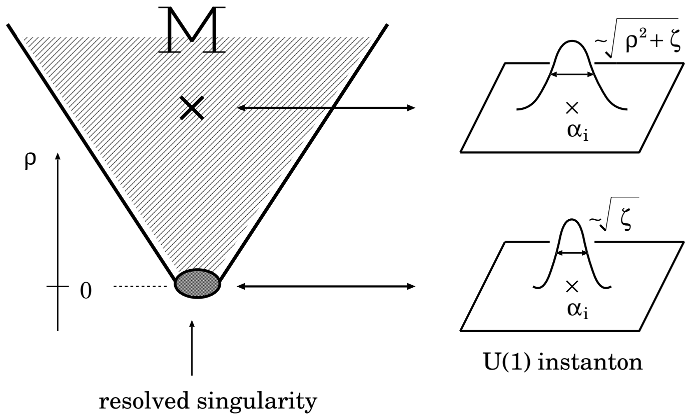

Figure 4. Instanton moduli space and the configutration of the NC BPST instanton

4.4. ADHM Construction of Noncommutative $U(1)$ 1-instanton

We put $U(1)$ 1-instanton at the origin. Solution of the noncommutative ADHM equation (15) relevant to describe such $U(1)$ 1-instanton, is

$$B_{1,2} = 0, I = \sqrt{\zeta}, J = 0. \tag{25}$$

The 0-dimensional Dirac equation (16) reads

$$\hat{\nabla}^\dagger \hat{V} = \begin{pmatrix} \sqrt{\zeta} & \hat{z}_2 & \hat{z}_1 \\ 0 & -\hat{\bar{z}}_1 & \hat{\bar{z}}_2 \end{pmatrix} \hat{V} = 0. \tag{26}$$

Apart from the normalization, the following form of $\hat{V}$ gives a naive solution of (26).

$$\hat{V}_1 = \begin{pmatrix} \hat{z}_1\hat{\bar{z}}_1 + \hat{z}_2\hat{\bar{z}}_2 \\ -\sqrt{\zeta}\hat{\bar{z}}_2 \\ -\sqrt{\zeta}\hat{\bar{z}}_1 \end{pmatrix}. \quad \hat{\nabla}^\dagger \hat{V}_1 = 0. \tag{27}$$

$\hat{V}_1$ is an operator acting on the Fock space $\mathcal{H}$. As described in section 2, the coordinates $\hat{\bar{z}}_1$ and $\hat{\bar{z}}_2$ are the annihilation operators. Hence $\hat{V}_1$ has an operator zero-mode $|0,0\rangle$ and there exists no inverse of $\hat{V}_1^\dagger \hat{V}_1$ on $\mathcal{H}$. This means that $\hat{V}_1$ cannot be normalized on the whole of the Fock space in the operator sense.

K. Furuuchi [12] shows that $\hat{V}_1$ yields a smooth ASD instanton solution on the subspace $\mathcal{H}_1 \equiv \mathcal{H} - \mathbb{C}|0,0\rangle$. By using the shift operators, he also discussed that the solution can be converted to $\hat{V}$ which satisfies the normalization condition (17) on $\mathcal{H}$ [13]:

$$\hat{V} = \hat{V}_1\hat{\beta}_1\hat{U}_1^\dagger, \quad \hat{V}^\dagger\hat{V} = 1, \tag{28}$$

where

$$\hat{\beta}_1 = (1-\hat{P}_1)(\hat{V}_1^\dagger\hat{V}_1)^{-\frac{1}{2}}(1-\hat{P}_1) = \sum_{(n_1,n_2)\neq(0,0)} \frac{1}{\sqrt{(n_1+n_2)(n_1+n_2+\zeta)}}|n_1,n_2\rangle\langle n_1,n_2|, \tag{29}$$

$$\hat{U}_1\hat{U}_1^\dagger = 1, \quad \hat{U}_1^\dagger\hat{U}_1 = 1 - |0,0\rangle\langle 0,0|. \tag{30}$$

The shift operator $\hat{U}_1$ converts operators on $\mathcal{H}_1$ into those on $\mathcal{H}$. The problem of operator zero-modes can be solved in this manner to give the correct operators on $\mathcal{H}$.

Finally, the smooth ASD instanton solution on $\mathcal{H}$ is obtained by further taking the ADHM procedure. In particular, the instanton number is computed to be -1. More discussion is seen in our paper [18].

4.5. Origin of Instanton Numbers

We apply Furuuchi's observation and several properties on noncommutative field theory to all the other ingredients of the ADHM construction and prove the existence of them in the operator sense. As a result, we obtain the following formula (a noncommutative version of the *Corrigan-Goddard-Osborn-Templeton or Osborn formula* [8, 35])

$$\int d^4x \ \mathrm{Tr}_N(F_{\mu\nu}F^{\mu\nu}) = -\int d^4x \partial^2\partial_\mu(\mathrm{Tr}_k f^{-1}\partial^\mu f), \ f := (\nabla^\dagger\nabla)^{-1}, \qquad (31)$$

which is crucial to show that any ADHM data labeled by k yields k-instanton solutions even for the $U(1)$ case. This directly shows an origin of the instanton number in the context of the ADHM construction, which has been discussed by several authors for explicit instanton solutions e.g. [14, 21, 36, 41, 42].

Detailed proofs and other results are seen in our paper [17].

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