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THE QUANTUM R-MATRIX VIA CANONICAL QUANTIZATION IN THE LIOUVILLE THEORY[1]

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ABSTRACT

Starting from the classical Liouville theory, we study its quantum theory through canonical quantization. We find that if the Poisson bracket relations between two vectors is dominated by the classical r-matrix in the classical case, their quantum analoge is replaced by the exchange relations dominated by the quantum R-matrix. The quantum group structure in the quantum Liouville theory is studied and the central charge of the quantum Liouville theory is also obtained.

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1. Introduction

The Liouville action plays the role of the conformal anomally in the noncritical string theory[1]. As a classical system, the Liouville action has the geometrical meaning related to the uniformization theory of the Riemann surfaces[2]. Much attention on the Liouville theory has been paid in the previous years[2-8]. The integrability of this system can be guaranted by the existence of the Lax pair[5] and hence of the Yang-Baxter equation. The quantum R-matrix can also be obtained from them. In an intriguing paper by Takhtajan and Smirnov [7], it is assumed that if the Poisson bracket relations between two vectors are dominated by the classical r-matrix in the classical case, their quantum analoge is replaced by the exchange relations dominated by the quantum R-matrix. This assumption plays a central role in their work, and they have got a lot of interesting results about the noncritical string theory[8] by using the quantum group structure of the quantum Liouville theory.

However, the quantum group structure of a theory should be obtained via the standard method of canonical quantization. On the other hand, a deep investigation of this problem along this line may also give us much more knowledge about the essence of the quantum group. In this paper, we will mainly prove the assumption made by Smirnov and Takhtajan [8] by using the method of canonical quantization. In section 2, we will briefly review the main results obtained by Gervais and Neveu[4] to give the background knowledges about the classical Liouville theory. We introduce two vectors which relate the energy-momentum tensor of the theory via the Hill operator and prove that their Poisson bracket relations are dominated by the classical r-matrix . In section 3, we first give the definition of these two vectors in the quantum case, and then prove that their exchange relations are dominated by the quantum R-matrix. To reveal its exact meanings, we find that in the semiclassical limit, i.e. when $\hbar \to 0$, these exchange relations is exactly that with the vectors being regarded as operators and their Poisson bracket relations replaced by the commutator divided by the factor $i\hbar$. This is just the theorem due to Dirac[9]. We also claim that the exchange algebra realtions we have obtained is just the quantum analogue of the Poisson bracket relations dominated by the classical r-matrix. In section 4, we study the quantum group structures in the quantum Liouville theory and get the KPZ relations[8]. Finally we end with some concluding remarks.

2. The classical Liouville theory[4]

The Liouville action which plays the role of conformal anomally in the noncritical string theory can be written to be

$$S(\phi) = \frac{1}{\gamma^2}\int d^2x(\frac{1}{2}\partial_\tau\phi\partial_\tau\phi - \frac{1}{2}\partial_\sigma\phi\partial_\sigma\phi - e^{\phi}) \qquad (1)$$

where

$$\gamma^2 = \frac{48\pi}{26-d}$$

and d is the dimension of the matter fields in the string theory. The cosmological constant Λ has been taken to be -1.

The equation of motion reads

$$\partial_\sigma^2\phi - \partial_\tau^2\phi + e^\phi = 0. \tag{2}$$

The Lax pairs to the Liouville equation (2) are

$$\begin{aligned} L &= \tfrac{d^2}{d\sigma^2} + U, \qquad M = \tfrac{d}{d\sigma}, \\ U &= \tfrac{1}{8}e^\phi + \tfrac{1}{16}(\phi_\sigma + \phi_\tau) - \tfrac{1}{4}(\phi_{\sigma\sigma} + \phi_{\sigma\tau}). \end{aligned} \tag{3}$$

It is easy to show that the zero curvature condition

$$\partial_\tau L - \partial_\sigma M + [M, L] = 0 \tag{4}$$

is equivalent to the equation of motion (2).

The classical solution to (2) is well known to be [4]

$$\phi = \ln \frac{8A'(u)B'(v)}{(1 - A(u)B(v))^2}, \tag{5}$$

where $u = \sigma + \tau$ and $v = \sigma - \tau$ and the primes represent derivatives with respect to the variables. If we substitute (5) into (3), we can get

$$U = \frac{1}{2}D[A(\sigma + \tau)], \tag{6}$$

where $D[A]$ denotes the Schwarzian derivative

$$D[A] = \frac{\partial_\sigma^3 A}{\partial_\sigma A} - \frac{3}{2}(\frac{\partial_\sigma^2 A}{\partial_\sigma A})^2. \tag{7}$$

The momentum density conjugate to φ can be shown to be

$$\pi = \frac{1}{\gamma^2}\phi_\tau.$$

Hence we can get the canonical Poisson bracket relation

$$\{\phi(\sigma,\tau), \pi(\sigma',\tau)\} = \delta(\sigma - \sigma').$$

After some discussions and tedious calculations (for details, see ref.[4,5]), we can get the Poisson bracket relations

$$\begin{aligned} \{K(\sigma,\tau), K(\sigma',\tau)\} &= \tfrac{\gamma^2}{8}\delta'(\sigma - \sigma'), \qquad K = -\tfrac{1}{2}\partial_\sigma \ln A_\sigma, \\ \{P(\sigma,\tau), P(\sigma',\tau)\} &= -\tfrac{\gamma^2}{16}\varepsilon(\sigma - \sigma'), \qquad P = -\tfrac{1}{2}\ln A_\sigma. \end{aligned} \tag{8}$$

The Poisson bracket relation between A's can be shown to be [4,5]

$$\{A(\sigma,\tau),A(\sigma',\tau)\} = \frac{\gamma^2}{8}\varepsilon(\sigma-\sigma')(A(\sigma,\tau)-A(\sigma',\tau))^2 + \frac{\gamma^2}{8}(A^2(\sigma,\tau)-A^2(\sigma,\tau)) \quad (9)$$

where $\varepsilon(\sigma-\sigma')$ is the sign function.

Now let us come to see the zero eigenvalue equation to the operator L

$$(-\frac{d^2}{d\sigma^2}+U)\psi_i = 0 \quad (10)$$

whose two linear independent solutions are

$$\psi_1 = \frac{1}{\sqrt{A_\sigma}} = e^{P(\sigma)}, \quad \psi_2 = \frac{A}{\sqrt{A_\sigma}} = \int_{+\infty}^{\sigma} dx e^{-2P(x)} e^{P(\sigma)}. \quad (11)$$

It is noticeable that the definition of $\psi_2(\sigma)$ is similar to that of $\psi_1(\sigma)$ acted by a screening charge in contrast to the conformal field theory[12].

The Poisson bracket relations to these two vectors can be shown from (9) to be

$$\{\psi_i(\sigma) \otimes_, \psi_j(\sigma')\} = -\frac{\gamma^2}{4}[r^+\theta(\sigma-\sigma') + r^-\theta(\sigma'-\sigma)]|_{ij}^{mn}\psi_m(\sigma)\otimes\psi_n(\sigma') \quad (12)$$

where

$$r^+ = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r^- = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (13)$$

which satisfy the classical Yang-Baxter equation for the r-matrix. It is noticeable that U in (6) plays the role of energy-momentum tensor in the Liouville theory [4]. And we have shown that the vectors which relate the energy-momentum tensor through equation (10) have the Poisson bracket relations dominated by the classical r-matrix.

3. Canonical quantization.

Now we are to study the quantum Liouville system through canonical quantization. One of the main tasks in this section is to get the quantum analogue of equation (12). To do this, we should first replace all the classical fields by the relevant operators. The commutation relations between the operators which are free fields in the classical case can be got directly from their Poisson relation according to the Dirac theorem. As to other quantum fields, they should first be defined carefully from the free operators and their commutation relations should be obtained from that of the free fields taking into account the normal ordering etc.

In order to get the quantum analogue of equation (12), we first define the operators ψ_1 and ψ_2 in the quantum case. From equation (11), they can be defined as

$$\hat{\psi}_1(z) =: e^{\hat{P}(z)}:, \qquad \hat{\psi}_2(z) = \int_c dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z)} :, \quad (14)$$

where : : denotes normal ordering and the contour c of the integration is depicted in fig.(1). The dotted line is a cut from z to infinity. $\hat{\psi}_2$ is integrated around the cut. The whole complex plane minus the cut is the analytic region of the function $(x-z)^a$ which appears in the OPE of $: e^{-2\hat{P}(x)} :$ and $: e^{\hat{P}(z)} :$. The coordinate σ in the classical case has now been extended to the complex plane perpendicular to the time coordinate, i.e. the z plane is an equal time plane. From equation (8), we find that $\hat{P}(z)$ is a free quantum field up to a constant. Their commutation relations can directly be obtained from equation (8) according to the Dirac theorem

$$[\hat{P}(z), \hat{P}(z')] = k\varepsilon(|z| - |z'|), \qquad k = -\frac{i\hbar\gamma^2}{16} \tag{15}$$

From the fundmental commutation relation (15), it is easy to get

$$\begin{aligned} &: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} := e^{-k} : e^{\hat{P}(z_2)} :: e^{\hat{P}(z_1)} :, \quad |z_1| < |z_2| \\ &: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} := e^{k} : e^{\hat{P}(z_2)} :: e^{\hat{P}(z_1)} :, \quad |z_1| > |z_2| \end{aligned} \tag{16}$$

In what follows, we will consider only the case when $|z_1| < |z_2|$ without loss of any generality. In this case we get

$$\hat{\psi}_1(z_1)\hat{\psi}_1(z_2) = e^{-k}\hat{\psi}_1(z_2)\hat{\psi}_1(z_1) \tag{17}$$

Direct product of two operators is just put them together, for example

$$\begin{aligned} &\hat{\psi}_2(z_1) \otimes \hat{\psi}_2(z_2) = \int_{c_1} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} : \\ &= \int_{c_2} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} : + \int_{c_3} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} : \end{aligned} \tag{18}$$

Because $|z_1| < |z_2|$, we have set $|z_1|$ in the origin. The original contour c_1 can be divided into $c2$ and $c3$ (see fig.2) where $|z_E - z_1| = |z_2 - z_1|$.

Let us now compute the effect of a braiding on, for instance, $\hat{\psi}_2(z_1) \otimes \hat{\psi}_1(z_2)$. Using (8) and taking account of the deformation of the contour under the action of the braiding, we get

$$\int_{c2} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} := e^{-k} \int_{c4} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_2)} :: e^{\hat{P}(z_1)} : \tag{19}$$

and

$$\int_{c3} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :: e^{\hat{P}(z_2)} := e^{k} \int_{c5} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} : \tag{20}$$

(see fig.3). The phase factors in equation (19) come from the braiding of $: e^{\hat{P}(z_1)} :$ and $: e^{\hat{P}(z_2)} :$. The minus sign in the front of k is due to $|z_1| < |z_2|$ and there is no commutation between $: e^{-2\hat{P}(x)} :$ and $: e^{\hat{P}(z_2)} :$ The phase factor in equation (20) comes from the braiding of $: e^{\hat{P}(z_2)} :$ and $: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} :$ where all the $|x|$ in this region is less than $|z_2|$. The contour $c5$ can be opened to be c_6 and c_7 (see fig.4)

$$\begin{aligned} &\int_{c_5} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} : \\ &= \int_{c_6} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} : + \int_{c_7} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} : \end{aligned} \tag{21}$$

while

$$\int_{C6} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} := -e^{2k} \int_{C8} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_2)} :: e^{\hat{P}(z_1)} : \qquad (22)$$

(see fig.5)From equation (18-22), we conclude that

$$\hat{\psi}_2(z_1)\hat{\psi}_1(z_2) = (e^{-k} - e^{3k})\hat{\psi}_2(z_2)\hat{\psi}_1(z_1) + e^k\hat{\psi}_1(z_2)\hat{\psi}_2(z_1) \qquad (23)$$

Similarly, we can get

$$\hat{\psi}_1(z_1)\hat{\psi}_2(z_2) = e^k\hat{\psi}_2(z_2)\hat{\psi}_1(z_1)\hat{\psi}_2(z_1)\hat{\psi}_2(z_2) = e^{-k}\hat{\psi}_2(z_2)\hat{\psi}_1(z_1) \qquad (24)$$

If we denote

$$e^{1/2}_{1/2} = \hat{\psi}_1, \qquad e^{1/2}_{-1/2} = \hat{\psi}_2$$

Finally, equations (17), (23), (24) can be rewritten in the more compact form as follows

$$e^{j_1}_{m_1}(z_1)e^{j_2}_{m_2}(z_2) = \sum_{m_1',m_2'} R^{j_1 j_2}|^{m_1' m_2'}_{m_1 m_2} e^{j_2}_{m_2'}(z_2)e^{j_1}_{m_1'}(z_1) \qquad (25)$$

where

$$R^{1/2,1/2} = q^{1/4} \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1/2} - q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix} \qquad (26)$$

and $q = e^{4k}$. In the general case, we can get

$$\begin{aligned} & e^{j_1}_{m_1}(z_1)e^{j_2}_{m_2}(z_2) \\ & = \textstyle\sum_{m_1',m_2'}(R^{j_1 j_2}(+)\theta(|z_1| - |z_2|) + R^{j_1 j_2}(-)\theta(|z_2| - |z_1|))|^{m_1' m_2'}_{m_1 m_2} e^{j_2}_{m_2'}(z_2)e^{j_1}_{m_1'}(z_1) \end{aligned} \qquad (27)$$

where $R^{1/2,1/2}(-)$ is shown in equation (26) and $R^{1/2,1/2}(+)$ is the inverse of it. They are nothing but the quantum R-matrix of $SL_q(2)$ which satisfies the Yang-Baxter equation

$$R_{j_1 j_2}R_{j_1 j_3}R_{j_2 j_3} = R_{j_2 j_3}R_{j_1 j_3}R_{j_1 j_2}. \qquad (28)$$

which is defined on $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ and the indices label the spaces on which the R-matrix acts.

So we have found the hidden $SL_q(2)$ symmetry in the quantum Liouville theory through canonical quantization. The vector spaces V^j_z generated by $\hat{\psi}_1$ and $\hat{\psi}_2$ are irreducible representation of a quantum deformation of $SL(2)$. By using the canonical commutation relations of the operators $\hat{\psi}_i$, we get the quantum R-matrix as a map, independent of the coordinates, from $V^{j_1}_{z_1} \otimes V^{j_2}_{z_2}$ to $V^{j_2}_{z_2} \otimes V^{j_1}_{z_1}$

$$R^{j_1 j_2} : V^{j_1}_{z_1} \bigotimes V^{j_2}_{z_2} \rightarrow V^{j_2}_{z_2} \bigotimes V^{j_1}_{z_1} \qquad (29)$$

where $j_1 = j_2 = \frac{1}{2}$.

If we recover the variable z as real one, i.e. $z \to \sigma$ and take the semiclassical limit, $\hbar \to 0$, the exchange relation (27) reduces to

$$[\hat{\psi}_i(\sigma), \hat{\psi}_j(\sigma')] = -\frac{i\hbar\gamma^2}{4}[r^+\theta(\sigma-\sigma') + r^-\theta(\sigma'-\sigma)]|_{ij}^{mn}\hat{\psi}_m(\sigma)\bigotimes\hat{\psi}_n(\sigma'). \tag{30}$$

Corresponding to equation (12), the Poisson bracket is replaced by the commutator and the right of the equation is multiplied by the factor $i\hbar$.

The exchange algebra relation (27) has a very interesting physical meaning. To illustrate this, let us denote p and q as canonical variables, that is they satisfy the following Poisson bracket relation

$$\{p, q\} = 1 \tag{31}$$

In the quantum case, p and q are regarded as operators, we may denote them as $\hat{p}$ and $\hat{q}$ respectively. They satisfy

$$[\hat{p}, \hat{q}] = -i\hbar. \tag{32}$$

Now let us consider two functions $f(p,q)$ and $g(p,q)$. For simplicity, we assume $f(p,q) = p^2$ and $g(p,q) = q^2$. Their Poisson bracket relation can found to be

$$\{f(p,q), g(p,q)\} = 4pq \tag{33}$$

In the quantum case, the operators $\hat{f}(\hat{p},\hat{q})$ and $\hat{g}(\hat{p},\hat{q})$ are defined to be $\hat{f}(\hat{p},\hat{q}) = \hat{p}^2$ and $\hat{g}(\hat{p},\hat{q}) = \hat{q}^2$. Their commutation relation can be obtained to be

$$\begin{aligned}[\hat{f}(\hat{p},\hat{q}), \hat{g}(\hat{p},\hat{q})] &= [\hat{p}^2, \hat{q}^2] \\ &= -2i\hbar(\hat{p}\hat{q} + \hat{q}\hat{p}) = -4i\hbar\hat{p}\hat{q} + 2\hbar^2\end{aligned} \tag{34}$$

If $\hbar \to 0$, these terms propotional to $\hbar^2$ and higher orders can be omitted, the relevant commutation relations of $\hat{f}$ and $\hat{g}$ can also be obtained directly according to the Dirac theorem.

In our case, things are not so easy. We can see from equation (15) that the commutation relation of P is that of $\hbar$ multiplied by a coupling constant of the system. In this case, these terms proportional to $\hbar^2$ and higher orders can't be omitted and hence the Dirac theorem will not valid for the composite fields of the free fields. For the generic quantum fields, their commutation relations must be derived from that of the free fields of this system. In our case, $\hat{\psi}_1$, and $\hat{\psi}_2$ are composite quantum fields. Their commutation relations obtained via the canonical quantization is the quantum analogue of their Poisson bracket relation (12).

Generally, we can prove the following theorem: In the classical case, the chiral components $\psi_i(z)(i = 1, 2, \cdots, 2j+1. j = \frac{N}{2}$ and N is an integer) have the Poisson bracket relation

$$\{\psi_i(\sigma) \overset{\otimes}{,} \psi_j(\sigma')\} = -\pi\alpha[r^+\theta(\sigma-\sigma') + r^-\theta(\sigma'-\sigma)]|_{ij}^{mn}\psi_m(\sigma)\bigotimes\psi_n(\sigma') \tag{35}$$

where r^+ and r^- are the classical r-matrix for the Poisson Lie group $SL(2)$. It can be represented as

$$r^+ = \frac{1}{4}H \otimes H + X^+ \otimes X^-$$
$$(r^{j_1 j_2}(-))|^{m_1 m_2}_{m_1' m_2'} = -(r^{j_2 j_1}(-))|^{m_2 m_1}_{m_2' m_1'} \tag{36}$$

where H, $X+$ and $X-$ are the standard generators of the Lie algebra $SL(2)$. Then we can conclude that in the quantum case, the operators $\hat{\psi}_i(z)$ have the exchange algebra relation

$$e^{j_1}_{m_1}(z_1)e^{j_2}_{m_2}(z_2)$$
$$= \sum_{m_1', m_2'} (R^{j_1 j_2}(+)\theta(|z_1| - |z_2|) + R^{j_1 j_2}(-)\theta(|z_2| - |z_1|))|^{m_1' m_2'}_{m_1 m_2} e^{j_2}_{m_2'}(z_2) e^{j_1}_{m_1'}(z_1) \tag{37}$$

where

$$R^{j_1 j_2}_q(+) == q^{\frac{1}{4}\hat{H} \otimes \hat{H}} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} \hat{e}^n \otimes \hat{f}^n$$

where $q = exp(2\pi i \alpha)$ and

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad \hat{e} = q^{\hat{H}/4}\hat{X}^+, \quad \hat{f} = q^{-\hat{H}/4}\hat{X}^-$$

and $\hat{H}$, $\hat{X}^+$, $\hat{X}^-$ are the generators of $U_q SL(2)$, satisfying the following relations

$$[\hat{H}, \hat{X}^+] = 2\hat{X}^+, [\hat{H}, \hat{X}^-] = -2\hat{X}^-, [X^+, X^-] = [\hat{H}]$$

For the special case ($j_1 = j_2 = \frac{1}{2}$), this theorem reduces to what we have proven in this section via the method of canonical quantization.

4. The quantum group

In this section, we explore the quantum group structure in the quantum Liouville system. In order to do this, we give the explicit representation of this quantum group on the spaces $V_z^{1/2}$ generated by the operators ψ_1 and ψ_2. Because of the relation (22), the creation operator F can be defined by simply acting with $\int dx : e^{-2\hat{P}(x)} :$ on the operator $: e^{\hat{P}(z)} :$ and the contour of the integration is depicted as in fig.6. The identification of $\int dx : e^{-2\hat{P}(x)} :$ with the quantum group generator $F = q^{-H/4}X_-$ comes from the comultiplication rule of $\int dx : e^{-2\hat{P}(x)} :$. The comultiplication can be easily obtained by using contour deformation as was described in [10]. The same arguments lead to

$$\triangle(F) = F \otimes 1 + q^{-H/2} \otimes F, \quad q^{-H/2}(e^{\hat{P}(z)}) = e^{-2k} e^{\hat{P}(z)} \tag{38}$$

where $\triangle$ denotes the coproduct.

At this stage, we have defined the action of the Hopf algebra generated by H and F on the spaces $V_z^{1/2}$. This algebra is isomorphic to a Borel subalgebra of $SL_q(2)$. Moreover, the R-matrix we have obtained in section 3 was the quantum R-matrix of $SL_q(2)$. The quantum group $SL_q(2)$ can be obtained using Drinfeld's quantum double constructure [11] starting from the Hopf algebra generated by H and F.

Once we have the explicit representation of the quantum group $SL_q(2)$ on the spaces $V_z^{1/2}$, we can get the tensor product decomposition rules directly from the operator product expansion of the operators $e_m^{1/2}$. For generic q, the standard Clebsch-Gordan decomposition holds

$$V^{j_1} \bigotimes V^{j_2} = \sum_{|j_1-j_2|}^{j_1+j_2} V^j \tag{39}$$

The Clebsch-Gordan coefficients $K_{j_3}^{j_1 j_2}$ project the representation V^{j_3} out of the product $V^{j_1} \bigotimes V^{j_2}$:

$$K_{j_3}^{j_1 j_2}: \quad V^{j_1} \bigotimes V^{j_2} \rightarrow V^{j_3} \tag{40}$$

The Clebsch-Gordan coefficients $K_{j_3}^{j_1 j_2}$ together with the Yang-Baxter matrix $R^{j_1 j_2}$ which was shown as equation (26) and its inverse satisfies

$$K_j^{j_1 j_2} R^{j_2 j_1}(+) = (-1)^{j_1+j_2-j} q^{(c_j - c_{j_1} - c_{j_2})/2} K_j^{j_2 j_1} \tag{41}$$

where $c_j = j(j+1)$. If we represent the Clebsch-Gordan coefficient as three point vertex (see fig.7), then this equation can be understood graphically as in fig.8. This is analogous to a chiral vertex representing a three point function for three primary fields of spin j_1, j_2, j_3 and equation (41) amounts to the braiding of the fields with spins j_1, j_2 yeilding a factor $(-1)^{j_1+j_2-j} q^{(c_j-c_{j_1}-c_{j_2})/2}$.

According to the same arguments as in [7], we can get that the conformal dimension of ψ_i is

$$\Delta = -\frac{1}{2} + \frac{3\hbar\gamma^2}{16\pi}$$

With conformal dimension obtained, we can easily get the central charge of this system. Because of equation (10) in the classical level, we can assume in the quantum level that the fields ψ_i are degenerated at level two[12]. Hence the central charge of this system can be computed by using the general formula

$$c = \frac{2\,\Delta\,(5 - 8\Delta)}{2\,\Delta + 1} = 13 - 6(\frac{4\pi}{\hbar\gamma^2} + \frac{\hbar\gamma^2}{4\pi}) \tag{42}$$

At this stage, we are easy to reproduce the famous equation in [8] for the balance of charges in 2-d quantum gravity. Namely, consider Liouville theory coupled to the matter fields X^μ, $\mu = 1, \cdots, d$. Since in the conformal gauge, the central charge of the matter fields is d and that of the ghost is -26, so we arrive at the equation

$$c^{tot} = c^{matt} + c^{ghost} + c^{Liou} = d - 13 - 6(\frac{4\pi}{\hbar\gamma^2} + \frac{\hbar\gamma^2}{4\pi}) \tag{43}$$

5. Conclusion

In this paper, we have started from the classical Liouville theory and got its quantum theory via the method of canonical quantization. We find that the Poisson Lie Group of the classical Liouville theory is replaced by the Quantum Group of the quantum theory.

Another point to be noticed is that the quantum theory is considered in the plane perpendicular to the time coordinate. This is different from the case of conformal field theory[12]. Of course, we can get the same results in the complex plane after the Wick rotation($\sigma - \tau \to \sigma - i\tau,\ \sigma + \tau \to \sigma + i\tau$).

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appendix

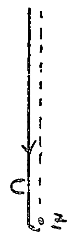

Fig.1 The dotted line is a cut from z to infinity. $\hat{\psi}_2$ is integrated around the cut. The whole complex plane minus the cut is the analytic region of the function $(x-z)^a$ which appears in the OPE of : $e^{-2\hat{P}(x)}$: and : $e^{\hat{P}(z)}$:. The z plane is an equal time plane.

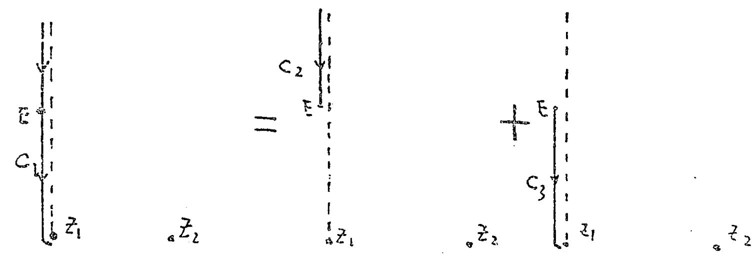

Fig.2 The contour c_1 can be divided into $c2$ and $c3$ where $|z_E - z_1| = |z_2 - z_1|$.

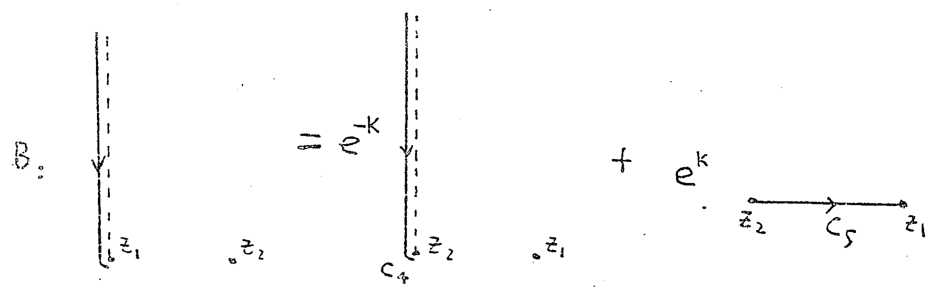

Fig.3 The phase factor e^{-k} comes from the braiding of : $e^{\hat{P}(z_1)}$: and : $e^{\hat{P}(z_2)}$: $(|z_1| < |z_2|)$, and e^k comes from the braiding of : $e^{\hat{P}(z_2)}$: and : $e^{-2\hat{P}(x)}$ :: $e^{\hat{P}(z_1)}$: where all the $|x|$ in this region is less than $|z_2|$.

appendix

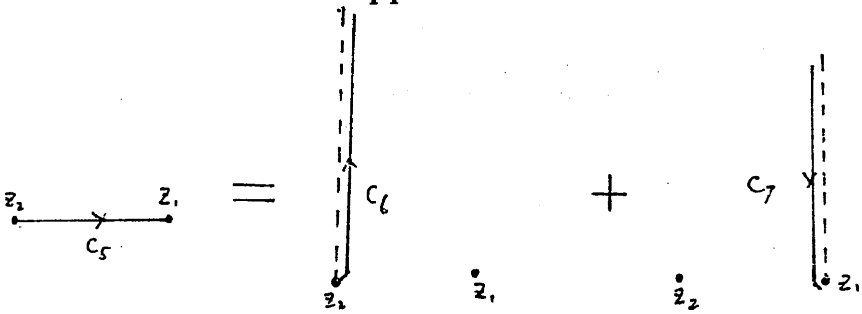

Fig.4 The contour $c5$ can be opened to be c_6 and c_7.

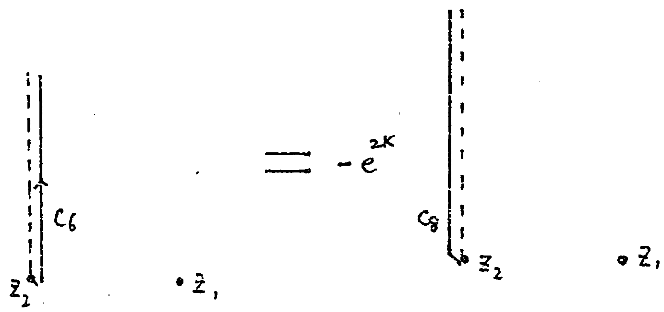

Fig.5 $\int_{c_6} dx : e^{\hat{P}(z_2)} :: e^{-2\hat{P}(x)} :: e^{\hat{P}(z_1)} := -e^{2k} \int_{c_8} dx : e^{-2\hat{P}(x)} :: e^{\hat{P}(z_2)} :: e^{\hat{P}(z_1)} :$

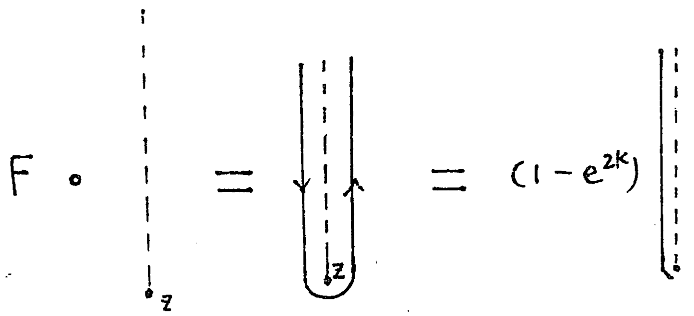

Fig.6 The action of the creation operator F.

appendix

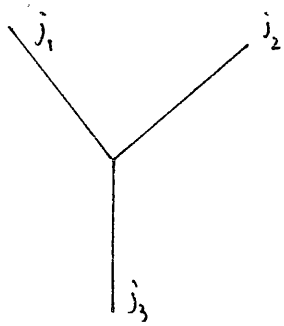

Fig.7 The Clebsch-Gordan coefficient can be represented as three point vertex

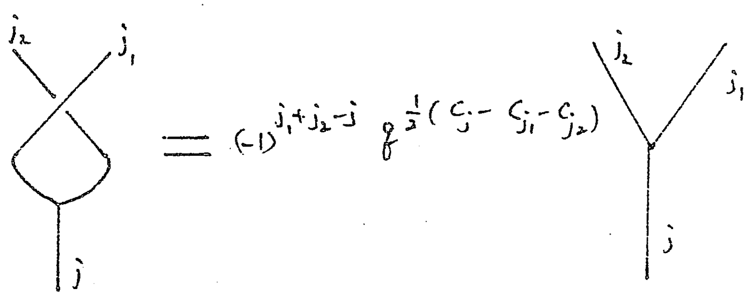

Fig.8 Equation (41) can be understood graphically.