

General analysis of LARGE Volume Scenarios with string loop moduli stabilisation

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ABSTRACT: We study the topological conditions for general Calabi-Yaus to get a non-supersymmetric AdS exponentially large volume minimum of the scalar potential in flux compactifications of IIB string theory. We show that negative Euler number and the existence of at least one blow-up mode resolving point-like singularities are necessary and sufficient conditions for moduli stabilisation with exponentially large volumes. We also analyse the general effects of string loop corrections on this scenario. While the combination of α' and nonperturbative corrections are sufficient to stabilise blow-up modes and the overall volume, quantum corrections are needed to stabilise other directions transverse to the overall volume. This allows exponentially large volume minima to be realised for fibration Calabi-Yaus, with the various moduli of the fibration all being stabilised at exponentially large values. String loop corrections may also play a role in stabilising 4-cycles which support chiral matter and cannot enter directly into the non-perturbative superpotential. We illustrate these ideas by studying the scalar potential for various Calabi-Yau three-folds including $K3$ fibrations and briefly discuss the potential phenomenological and cosmological implications of our results.

KEYWORDS: Flux compactifications, Superstring Vacua.

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1. Introduction

In constructing string models that can be compared to experiment, it is essential to understand the vacuum state. It is the vacuum state whose properties will give rise to, for example, low-energy supersymmetry and the pattern of Yukawa couplings. For a long time string compactifications were plagued by the problem of moduli. These were massless

scalars that parameterised the vacuum and whose VEVs entered into the value of physical observables. The existence of moduli leads to fifth forces and also makes the vacuum inherently unpredictable. Perturbative and nonperturbative quantum effects, generically present in $N = 1$ supersymmetric theories, were expected to lead to moduli stabilisation but for a long time no controlled procedure existed.

This situation improved remarkably after the systematic study of flux compactifications and their impact on moduli stabilisation [1] (for reviews see [2, 3]) Fluxes are particularly powerful in type IIB string compactifications, as there the backreaction is well-controlled and gives rise only to warped Calabi-Yau metrics. Furthermore, the combination of non-perturbative effects [4, 5] with perturbative α' corrections [6–8] has allowed a general study of the large volume scalar potential for arbitrary Calabi-Yau compactifications and has led to the discovery of exponentially large volume compactifications [8, 9] with very interesting phenomenological and cosmological implications.

If α' corrections can play a significant role in moduli stabilisation in the phenomenologically relevant regime of large volume and weak coupling, it is natural to wonder whether g_s corrections may also have a significant effect. At first sight this seems unavoidable, as at large volume the corrections to the Kähler potential induced by string loops are parametrically larger than those induced by α' corrections [10]. However, the scalar potential exhibits an extended no-scale structure, and the loop corrections contribute to the scalar potential at a level subleading to their contribution to the Kähler potential and subleading to the α' corrections [10–12]. [13] studied the effect of loop corrections on the $\mathbb{C}P^4_{[1,1,1,6,9]}$ large volume model and found it only gave minor corrections to the moduli stabilisation and sub-sub-leading corrections to the soft term computation. It is then natural to ask whether loop corrections to the scalar potential can give a qualitative, rather than only quantitative, change to moduli stabilisation.

We study this question in this paper and find that the answer is affirmative. The LARGE Volume Scenario (LVS)¹ stabilises the overall volume at an exponentially large value using α' and g_s corrections. Most previous work has focused on ‘Swiss-Cheese’ Calabi-Yaus, where one cycle controls the overall volume (‘size of the cheese’) and the remaining moduli control the volume of blow-up cycles (‘holes in the cheese’). However for Calabi-Yaus with a fibration structure - the torus is the simplest example - multiple moduli enter into the overall volume. For the overall volume to be made large in a homogeneous fashion, several moduli must become large. In these cases, while the existence of at least one blow-up mode is still necessary, loop corrections turn out to be necessary in order to realise the LVS and obtain a stable minimum at exponentially large volume. The loop corrections lift directions transverse to the overall breathing mode and stabilise these.

The fact that the volume is stabilised large is desirable, first to trust the perturbative expansion in the low energy effective field theory and secondly as a powerful tool to generate hierarchies. Moreover we stress that this minimum is found for generic values of W_0 , without the need to fine tune W_0 .

¹The capitalisation of LARGE is a reminder that the volume is exponentially large, and not just large enough to trust the supergravity limit.

We will illustrate our results by applying them to some examples of Calabi-Yau three-folds that are hypersurfaces in complex weighted projected spaces. From this analysis, it turns out that a necessary and sufficient condition for LARGE volume is the presence of blow-up modes resolving point-like singularities. It seems also that the Calabi-Yaus which have a fibration structure cannot present the interesting phenomenological properties of the LVS. However, as we will explain in sections 4 and 5, via the inclusion of the string loop corrections to the Kähler potential, those K3 fibrations can also present an exponentially large volume minimum, provided a blow-up mode exists.

We will also show that the string loop corrections may play a role in addressing the problem stressed in [14]. The authors there argued that the 4-cycle on which the Standard Model lives cannot get non-perturbative corrections since their prefactor is proportional to the VEV of Standard Model fields which, at this stage, is required to vanish. However, due to the constraints of the Standard Model gauge couplings this cycle must still be stabilised at a relatively small size.

This problem may be cured through having at least two blow-up modes and then adding g_s corrections. The loop corrections have the ability to stabilise the Standard Model cycle, while the ‘transverse’ cycle is stabilised non-perturbatively as usual. This possible solution is discussed for the example $\mathbb{C}P_{[1,3,3,3,5]}^4$, studied in detail in section 5.2. We will see that the inclusion of the g_s corrections can freeze τ_{SM} small producing a minimum of the full scalar potential at exponentially large volume.

This paper is organised as follows. In the next section we briefly review the LARGE Volume Scenario and state the general conditions that have to be satisfied in order to have exponentially large volume. The detailed proof of this general result is left to the appendix. Section 3 illustrates our general results for several Calabi-Yaus, including both Swiss cheese models where all Kähler moduli other than the overall volume are blow-ups, and also fibration Calabi-Yaus, such as K3 fibrations. Section 4 reviews the general structure of string loop corrections and section 5 illustrates how these corrections affect the results on the models of section 3. We finish with a short section on conclusions and potential applications of our results.

We shall not discuss obtaining de Sitter vacua in this paper. For a recent analysis of the conditions for de Sitter vacua from supergravity, see [15].

2. The LARGE Volume Scenario

Large volume compactifications of string theories are desirable for several reasons. First, large volume allows massive string states to be consistently integrated out and makes the effective field theory description of the compactification more robust. Second, large volume is desirable for phenomenological reasons. Given the standard relation $M_{\text{planck}}^2 \sim \frac{M_{\text{string}}^2 \mathcal{V}}{g_s}$, it is clear that LARGE volume (or WEAK coupling [16]) is necessary to have the fundamental string scale hierarchically smaller than the Planck scale. In string units a volume $\mathcal{V} \sim 10^4$ is needed for $M_{\text{string}} = M_{\text{GUT}}$ and much larger volumes are needed for an intermediate scale $M_{\text{string}} \sim 10^{11} \text{ GeV}$ ($\mathcal{V} \sim 10^{15}$), as attractive for gravity mediated supersymmetry breaking, or $M_{\text{string}} \sim 1 \text{ TeV}$ ($\mathcal{V} \sim 10^{30}$) in the extreme case of TeV strings.

Explicitly obtaining exponentially large volume in string theory, with all the geometric moduli stabilised, goes much farther than the original large extra dimensions proposals [17] where the volume was simply assumed to be large. In this section we will briefly review how moduli are typically stabilised in IIB string theory and then we will describe the general conditions under which an exponentially large volume Calabi-Yau compactification is obtained.

2.1 Low energy limit for type IIB flux compactifications

To establish notation and conventions we start with a rapid review of type IIB flux compactifications on Calabi-Yau three-folds X [1], dimensionally reduced to 4D $\mathcal{N} = 1$ supergravity. The Kähler potential takes the form:

$$K = -2 \ln \left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}} \right) - \ln (S + \bar{S}) - \ln \left(-i \int_X \Omega \wedge \bar{\Omega} \right). \quad (2.1)$$

In (2.1) we have included the leading α' correction but have not included any string loop corrections. The constant ξ is given by: $\xi = -\frac{\chi(X)\zeta(3)}{2(2\pi)^3}$, where χ is the Euler number of the Calabi-Yau X , $\zeta(3) \approx 1.2$, S is the axio-dilaton, $S = e^{-\varphi} + iC_0$, and Ω is the holomorphic (3,0)-form which implicitly depends on the complex structure moduli U_α , $\alpha = 1, \dots, h_{2,1}(X)$. \mathcal{V} is the internal volume, measured with an Einstein frame metric $g_{\mu\nu,E} = e^{-\varphi/2} g_{\mu\nu,s}$ and in units of $l_s = 2\pi\sqrt{\alpha'}$. It can be expressed in terms of the Kähler form J once this is expanded in a base $\{\hat{D}_i\}$ of $H^{1,1}(X, \mathbb{Z})$ as $J = \sum_{i=1}^{h_{1,1}} t^i \hat{D}_i$ (we focus on orientifold projections such that $h_{1,1}^- = 0 \Rightarrow h_{1,1}^+ = h_{1,1}$):

$$\mathcal{V} = \frac{1}{6} \int_X J \wedge J \wedge J = \frac{1}{6} k_{ijk} t^i t^j t^k. \quad (2.2)$$

Here k_{ijk} are the triple intersection numbers of X and the t^i are 2-cycle volumes.

The fields entering the $\mathcal{N} = 1$ chiral multiplets are the complexified Kähler moduli $T_i = \tau_i + ib_i$ where τ_i is the Einstein frame volume (in units of l_s) of the divisor $D_i \in H_4(X, \mathbb{Z})$, which is the Poincaré dual to \hat{D}_i . Its axionic partner b_i is the component of the RR 4-form C_4 along this cycle: $\int_{D_i} C_4 = b_i$. The 4-cycle volumes τ_i are related to the 2-cycle volumes t^i :

$$\tau_i = \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2} \int_X \hat{D}_i \wedge J \wedge J = \frac{1}{2} k_{ijk} t^j t^k. \quad (2.3)$$

The classical superpotential is generated by background fluxes $G_3 = F_3 + iSH_3$, where F_3 and H_3 are the RR and NSNS 3-form fluxes respectively. This superpotential also receives nonperturbative corrections from either brane instantons ($a_i = 2\pi$) or gaugino condensation ($a_i = 2\pi/N$). The full superpotential is then:

$$W = W_{\text{tree}} + W_{np} = \int_X G_3 \wedge \Omega + \sum_i A_i e^{-a_i T_i}. \quad (2.4)$$

The sum is over cycles generating nonperturbative contributions to W . The A_i correspond to threshold effects and depend on the complex structure moduli and positions of D3-branes. There may additionally be higher instanton effects in (2.4), but these can be neglected so long as each τ_i is stabilised such that $a_i\tau_i \gg 1$.

The $\mathcal{N} = 1$ F-term supergravity scalar potential is given by:

$$V = e^K \left\{ \sum_{i=T,S,U} K_{i\bar{j}}^{-1} D_i W D_{\bar{j}} \bar{W} - 3|W|^2 \right\}, \quad (2.5)$$

where

$$\begin{cases} D_i W = \partial_i W + W \partial_i K, \\ D_{\bar{j}} \bar{W} = \partial_{\bar{j}} \bar{W} + \bar{W} \partial_{\bar{j}} K. \end{cases} \quad (2.6)$$

Classically, the potential for the Kähler moduli is flat due to the no-scale structure:

$$\left(\frac{\partial^2 K_{\text{tree}}}{\partial T_i \partial \bar{T}_j} \right)^{-1} \frac{\partial K_{\text{tree}}}{\partial T_i} \frac{\partial K_{\text{tree}}}{\partial \bar{T}_j} = 3. \quad (2.7)$$

This implies that the dilaton and complex structure moduli are stabilised supersymmetrically at tree level, $D_S W = D_U W = 0$.

The classical flatness of the potential for the Kähler moduli implies that to study Kähler moduli stabilisation we should keep all possible quantum corrections, while for the U and S moduli it is sufficient to stabilise them classically:

$$\begin{cases} K = K_{cs} - 2 \ln \left(\frac{2}{g_s} \right) - 2 \ln \left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}} \right), \\ W = W_0 + \sum_i A_i e^{-a_i T_i}, \end{cases} \quad (2.8)$$

where $K_{cs} = \langle -\ln \left(-i \int_X \Omega \wedge \bar{\Omega} \right) \rangle$ and $W_0 = \langle W_{\text{tree}} \rangle$. Substituting (2.8) in (2.5), we obtain the following potential (defining $\hat{\xi} \equiv \xi/g_s^{3/2}$):

$$V = e^K \left[K_0^{jk} \left(a_j A_j a_k \bar{A}_k e^{-(a_j T_j + a_k \bar{T}_k)} - \left(a_j A_j e^{-a_j T_j} \bar{W} \partial_{\bar{T}_k} K_0 + a_k \bar{A}_k e^{-a_k \bar{T}_k} W \partial_{T_j} K_0 \right) \right) + 3\hat{\xi} \frac{(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2)}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2} |W|^2 \right]. \quad (2.9)$$

The aim of this paper will be a detailed study of the potential (2.9) for various different Calabi-Yaus, incorporating string loop corrections into the form of the Kähler potential. We aim to work out conditions for when the exponentially large volume minimum of [8] is present.

2.2 General analysis for the large volume limit

We now investigate the topological conditions on an arbitrary Calabi-Yau three-fold under which the scalar potential (2.9) admits an AdS non-supersymmetric minimum at exponentially large volume deepening the analysis performed in [8]. We will refer to those constructions as LARGE Volume Scenarios (LVS).

Claim 1. (*LARGE Volume*) Let X be a Calabi-Yau three-fold and let the large volume limit be taken in the following way:

$$\begin{cases} \tau_j \text{ remains small, } \forall j = 1, \dots, N_{\text{small}}, \\ \mathcal{V} \rightarrow \infty \text{ for } \tau_j \rightarrow \infty, \forall j = N_{\text{small}} + 1, \dots, h_{1,1}(X), \end{cases} \quad (2.10)$$

within type IIB $\mathcal{N} = 1$ 4D supergravity where the Kähler potential and the superpotential in Einstein frame take the form:

$$\begin{cases} K = K_{cs} - 2 \ln(\mathcal{V} + \hat{\xi}), \\ W = W_0 + \sum_{j=1}^{N_{\text{small}}} A_j e^{-a_j T_j}. \end{cases} \quad (2.11)$$

Then the scalar potential admits a set H of AdS non-supersymmetric minima at exponentially large volume located at $\mathcal{V} \sim e^{a_j \tau_j} \forall j = 1, \dots, N_{\text{small}}$ if and only if $h_{2,1}(X) > h_{1,1}(X) > 1$, i.e. $\xi > 0$ and τ_j is a local blow-up mode resolving a given point-like singularity $\forall j = 1, \dots, N_{\text{small}}$. In this case

$$\begin{cases} \text{if } h_{1,1}(X) = N_{\text{small}} + 1, H = \{a \text{ point}\}, \\ \text{if } h_{1,1}(X) > N_{\text{small}} + 1, H = \{(h_{1,1}(X) - N_{\text{small}} - 1) \text{ flat directions}\}. \end{cases} \quad (2.12)$$

The proof of the previous Claim is presented in appendix A where we show also that τ_j is the only blow-up mode resolving a point-like singularity if and only if $K_{jj}^{-1} \sim \mathcal{V} \sqrt{\tau_j}$. On the contrary when the same singularity is resolved by several independent blow-ups, say τ_1 and τ_2 , then $K_{11}^{-1} \sim \mathcal{V} h_{1/2}^{(1)}(\tau_1, \tau_2)$ and $K_{22}^{-1} \sim \mathcal{V} h_{1/2}^{(2)}(\tau_1, \tau_2)$ with $h_{1/2}^{(j)}$ homogeneous function of degree 1/2 such that $\frac{\partial^2 h^{(j)}}{\partial \tau_1 \partial \tau_2} \neq 0 \forall j = 1, 2$.

Let us now explain schematically the global picture of LVS for arbitrary Calabi-Yau manifolds according to the LARGE Volume Claim:

1. The Euler number of the Calabi Yau manifold must be negative. More precisely: $h_{12} > h_{11} > 1$. This means that the coefficient $\hat{\xi}$ must be positive in order to guarantee that in a particular direction the potential goes to zero at infinity from below [8]. This is a both sufficient and necessary condition.
2. The Calabi-Yau manifold must have at least one blow-up mode corresponding to a 4-cycle modulus that resolves a point-like singularity. The associated modulus must have an induced non-perturbative superpotential. This is usually guaranteed since these cycles are rigid cycles of arithmetic genus one, which is precisely the condition needed for the existence of non-perturbative superpotentials in the flux-less case [4].
3. This 4-cycle, together with other blow-up modes possibly present, are fixed small by the interplay of non-perturbative and α' corrections, which stabilise also the overall volume mode. Here small means larger than the string scale but not exponentially large unlike the volume.

4. All the other 4-cycles, such as those corresponding to fibrations, cannot be stabilised small even though they may have induced non-perturbative effects. They are sent large making their non-perturbative corrections negligible.
5. At this stage, non blow-up Kähler moduli, except the overall volume mode, remain unfixed giving rise to essentially flat directions.
6. It turns out then that in order to freeze these moduli, it is crucial to study string loop corrections as the leading term in a g_s expansion will be dominant over any potential non-perturbative correction.

Notice that these are conditions to find exponentially large volume minima and our results hold for generic $\mathcal{O}(1)$ values of W_0 . There may exist other minima which do not have exponentially large volume for which our results do not have anything to say. For example, $|W_0| \ll 1$ may give rise to KKLT-like minima.

Summarising, if there are N_{small} blow-up modes and $L = (h_{11} - N_{\text{small}} - 1)$ modes which do not blow-up point-like singularities nor correspond to the overall modulus, then our results state that all the N_{small} can be fixed at values large with respect to the string scale but not exponentially large, the overall volume is exponentially large and the other L Kähler moduli are not fixed by these effects.

In reality, the directions corresponding to the non blow-up modes, if they have non-perturbative effects, will be lifted by these tiny exponential terms, which however we neglect at this level of approximation. The reason is that, as we will see in the next sections, those directions will be lifted by the inclusion of string loop corrections which are always dominant with respect to the non-perturbative ones.

We would also like to stress that the previous general picture shows how we need non-perturbative effects only in the blow-up modes to get an exponentially large volume minimum. As blow-up modes correspond to rigid exceptional divisors, the corresponding non-perturbative corrections will be generally present even in the fluxless case [4]. They can arise from either gaugino condensation of the gauge theory living on the stack of branes wrapping that 4-cycle or from Euclidean D3 brane instantons. On the contrary, it is not clear if all the other cycles can indeed get non-perturbative corrections to W , but this is not necessary to obtain LARGE Volume.

3. Particular examples

Let us illustrate these results in a few explicit examples. At this stage we ignore string loop corrections but as we will show in section 4 and 5, these can in some cases actually be important and change the configuration of the system studied.

3.1 The single-hole Swiss cheese: $\mathbb{C}P^4_{[1,1,1,6,9]}$

The degree 18 hypersurface embedded in the weighted projective space $\mathbb{C}P^4_{[1,1,1,6,9]}$ is the Calabi-Yau on which the LVS was originally realised. The overall volume in terms of 2-cycle

volumes is given by

$$\mathcal{V} = \frac{1}{6} (3t_1^2 t_5 + 18t_1 t_5^2 + 36t_5^3). \quad (3.1)$$

The divisor volumes take the form $\tau_4 = \frac{t_1^2}{2}$, $\tau_5 = \frac{(t_1 + 6t_5)^2}{2}$, from which it is immediate to see that

$$\mathcal{V} = \frac{1}{9\sqrt{2}} \left(\tau_5^{3/2} - \tau_4^{3/2} \right). \quad (3.2)$$

ξ is positive since $h_{1,1} < h_{2,1}$ and the limit (2.10) can be correctly performed with $\tau_5 \rightarrow \infty$ and τ_4 remaining small. Thus $N_{\text{small}} = 1$ and we have to check if this case satisfies the condition of the LARGE Volume Claim which is $K_{44}^{-1} \simeq \mathcal{V} \sqrt{\tau_4}$. This is indeed satisfied as it can be seen either by direct calculation or by noticing that τ_4 is a local blow-up. Omitting numerical factors, the scalar potential takes the form

$$V \simeq \frac{\sqrt{\tau_4} e^{-2a_4 \tau_4}}{\mathcal{V}} - \frac{W_0 \tau_4 e^{-2a_4 \tau_4}}{\mathcal{V}^2} + \frac{\hat{\xi} W_0^2}{\mathcal{V}^3}. \quad (3.3)$$

As the $\mathbb{C}P^4_{[1,1,1,6,9]}$ example is a particular case of the LARGE Volume Claim, we conclude that the scalar potential (3.3) will admit an AdS minimum at exponentially large volume with $(h_{1,1} - N_{\text{small}} - 1) = 0$ flat directions. This is consistent with the original calculation in [8], which shows that the minimum is located at

$$\langle \tau_4 \rangle \simeq (4\hat{\xi})^{2/3}, \quad \langle \mathcal{V} \rangle \simeq \frac{\hat{\xi}^{1/3} W_0}{a_4 A_4} e^{a_4 \langle \tau_4 \rangle}. \quad (3.4)$$

3.2 The multiple-hole Swiss cheese: \mathcal{F}_{11} and $\mathbb{C}P^4_{[1,3,3,3,5]}$

It is straightforward to realise that the LARGE Volume Claim can be used to generalise the previous case by adding several blow-up modes resolving point-like singularities that will be stabilised small. In this case the overall volume looks like

$$\mathcal{V} = \alpha \left(\tau_{\text{big}}^{3/2} - \sum_{i=1}^{N_{\text{small}}} \lambda_i \tau_i^{3/2} \right), \quad (3.5)$$

where α and λ_i are positive model-dependent parameters and the Calabi-Yau manifold presents a typical ‘‘Swiss cheese’’ shape. An explicit example is the Fano three-fold \mathcal{F}_{11} described in [20], which is topologically a \mathbb{Z}_2 quotient of a CY_3 with Hodge numbers $h_{1,1} = 3$, $h_{2,1} = 111$. The total volume of the \mathcal{F}_{11} reads

$$\mathcal{V} = \frac{t_1^2 t_2}{2} + \frac{t_1 t_2^2}{2} + \frac{t_2^3}{6} + \frac{t_1^2 t_3}{2} + 2t_1 t_2 t_3 + t_2^2 t_3 + t_1 t_3^2 + 2t_2 t_3^2 + \frac{2t_3^3}{3}, \quad (3.6)$$

and the 4-cycle moduli are given by

$$\tau_1 = \frac{t_2}{2} (2t_1 + t_2 + 4t_3), \quad \tau_2 = \frac{t_1^2}{2}, \quad \tau_3 = t_3 (t_1 + t_3). \quad (3.7)$$

It is then possible to express \mathcal{V} in terms of the τ -moduli as

$$\mathcal{V} = \frac{1}{3\sqrt{2}} \left(2(\tau_1 + \tau_2 + 2\tau_3)^{3/2} - (\tau_2 + 2\tau_3)^{3/2} - \tau_2^{3/2} \right). \quad (3.8)$$

The resemblance with the general “Swiss cheese” picture (3.5) is now manifest. Two further Calabi-Yau realisations of this Swiss-Cheese structure have been presented in [14]. They are the $h_{1,1} = 3$ degree 15 hypersurface embedded in $\mathbb{C}P^4_{[1,3,3,3,5]}$ and the $h_{1,1} = 5$ degree 30 hypersurface in $\mathbb{C}P^4_{[1,1,3,10,15]}$. All these many moduli “Swiss cheese” Calabi-Yaus will admit a LARGE Volume minimum located at

$$\langle \mathcal{V} \rangle \sim e^{a_i \langle \tau_i \rangle}, \quad \forall i = 1, \dots, N_{\text{small}}, \quad (3.9)$$

with no orthogonal flat directions. Let us briefly review the geometric data of the resolution of the $\mathbb{C}P^4_{[1,3,3,3,5]}$ manifold [14], since it turns out to be an interesting case in which loop corrections may potentially stabilise the Standard Model cycle that does not admit non-perturbative superpotential contributions. In the diagonal basis the total volume becomes

$$\mathcal{V} = \sqrt{\frac{2}{45}} \left(\tau_a^{3/2} - \frac{1}{3} \tau_b^{3/2} - \frac{\sqrt{5}}{3} \tau_c^{3/2} \right). \quad (3.10)$$

A Euclidean D3-brane instanton wraps the rigid 4-cycle $D_{E3} = \frac{1}{3} (D_b + D_c)$, giving a non-perturbative superpotential term $W_{np} = e^{-\frac{2\pi}{3}(\tau_b + \tau_c)}$. There are also two stacks of D7-branes wrapping the rigid four cycles $D_{D7A} = \frac{1}{3} (D_b - 2D_c)$ and $D_{D7B} = D_c$ with line bundles $\mathcal{L}_A = \frac{1}{3} (2D_b + 5D_c)$ and $\mathcal{L}_B = \mathcal{O}$. This choice guarantees that there are no chiral zero modes on the D7-E3 intersections. The “Standard Model” is part of the $U(N_A)$ gauge group on the stack A of D7-branes, with SM matter obtained from the intersections AA' and AB where the prime denotes the orientifold image.

Neglecting the D-term part of the scalar potential we obtain

$$V = \frac{\lambda_1 (\sqrt{5\tau_b} + \sqrt{\tau_c}) e^{-\frac{4\pi}{3}(\tau_b + \tau_c)}}{\mathcal{V}} - \frac{\lambda_2 (\tau_b + \tau_c) e^{-\frac{2\pi}{3}(\tau_b + \tau_c)}}{\mathcal{V}^2} + \frac{\lambda_3}{\mathcal{V}^3}, \quad (3.11)$$

where $\lambda_i > 0, \forall i = 1, 2, 3$ are unimportant numerical factors. Now to make the study of the scalar potential (3.11) simpler, we perform the change of coordinates $\tau_b = 2\tau_{E3} + \tau_{SM}, \tau_c = \tau_{E3} - \tau_{SM}$, bringing (3.11) to the form

$$V = \frac{\lambda_1 \left(\sqrt{5(2\tau_{E3} + \tau_{SM})} + \sqrt{\tau_{E3} - \tau_{SM}} \right) e^{-4\pi\tau_{E3}}}{\mathcal{V}} - \frac{3\lambda_2 \tau_{E3} e^{-2\pi\tau_{E3}}}{\mathcal{V}^2} + \frac{\lambda_3}{\mathcal{V}^3}. \quad (3.12)$$

The scalar potential (3.12) then has a critical point at $\tau_{E3} = 2\tau_{SM}$. However, this is not a minimum of the full scalar potential but is actually a saddle point along τ_{SM} at fixed τ_{E3} and \mathcal{V} . In subsection 5.2 we will show how string loop corrections may give rise to a stable LVS even though no non-perturbative corrections in τ_{SM} are included (see [14] for a discussion of freezing τ_{SM} by including D-terms with (3.12)).

3.3 2-parameter K3 fibration: $\mathbb{C}P^4_{[1,1,2,2,6]}$

Our next example is a K3 fibered Calabi-Yau, the degree 12 hypersurface embedded in $\mathbb{C}P^4_{[1,1,2,2,6]}$. The overall volume in terms of 2-cycle volumes is given by

$$\mathcal{V} = t_1 t_2^2 + \frac{2}{3} t_2^3. \quad (3.13)$$

The 4-cycle volumes take the form $\tau_1 = t_2^2$, $\tau_2 = 2t_2(t_1 + t_2)$, yielding

$$\mathcal{V} = \frac{1}{2}\sqrt{\tau_1} \left(\tau_2 - \frac{2}{3}\tau_1 \right). \quad (3.14)$$

It is possible to invert the relations $\tau_i = \partial\mathcal{V}/\partial t_i$ to produce

$$t_2 = \sqrt{\tau_1}, \quad t_1 = \frac{\tau_2 - 2\tau_1}{2\sqrt{\tau_1}}. \quad (3.15)$$

The Euler characteristic of the Calabi-Yau is negative and the limit (2.10) can be performed only with $\tau_2 \rightarrow \infty$ and keeping τ_1 small. This corresponds to $t_1 \rightarrow \infty$ and t_2 small. In this limit the volume becomes

$$\mathcal{V} = \frac{1}{2}\sqrt{\tau_1}\tau_2 \simeq t_1 t_2^2 \simeq t_1 \tau_1. \quad (3.16)$$

Thus $N_{\text{small}} = 1$ again and we need to check the condition of the LARGE Volume Claim: $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$. However this is clearly not satisfied, as τ_1 is a fibration over the base t_1 .

This is therefore a situation where no exponentially large volume minimum is present, as can be confirmed by the explicit calculation below.

3.3.1 Explicit calculation

Here we verify that the $\mathbb{C}P_{[1,1,2,2,6]}^4$ model does not give a realisation of the LVS. We take the large volume limit in the following way

$$\begin{cases} \tau_1 \text{ small,} \\ \tau_2 \gg 1, \end{cases} \quad (3.17)$$

which, after the axion minimisation ($W_0 > 0$), gives a scalar potential of the form

$$\begin{aligned} V = V_{np} + V_{\alpha'} = & \frac{4}{\mathcal{V}^2} [a_1 A_1^2 \tau_1 (a_1 \tau_1 + 1) e^{-2a_1 \tau_1} - a_1 A_1 \tau_1 W_0 e^{-a_1 \tau_1}] \\ & + \frac{3}{4} \frac{\xi}{\mathcal{V}^3} (W_0^2 + A_1^2 e^{-2a_1 \tau_1} - 2A_1 W_0 e^{-a_1 \tau_1}). \end{aligned} \quad (3.18)$$

We set $A_1 = 1$ and recall that to neglect higher order instanton corrections we need $a_1 \tau_1 \gg 1$. (3.18) becomes

$$V = \frac{4}{\mathcal{V}^2} [(a_1 \tau_1 e^{-a_1 \tau_1} - W_0) a_1 \tau_1 e^{-a_1 \tau_1}] + \frac{3}{4} \frac{\xi}{\mathcal{V}^3} [W_0^2 + (e^{-a_1 \tau_1} - 2W_0) e^{-a_1 \tau_1}]. \quad (3.19)$$

The previous expression (3.19) can be rewritten as

$$\begin{aligned} V = & \frac{e^{-2a_1 \tau_1}}{\mathcal{V}^2} \left(4a_1^2 \tau_1^2 + \frac{3}{4} \frac{\xi}{\mathcal{V}} \right) - \frac{2W_0 e^{-a_1 \tau_1}}{\mathcal{V}^2} \left(2a_1 \tau_1 + \frac{3}{4} \frac{\xi}{\mathcal{V}} \right) + \frac{3}{4} \frac{\xi}{\mathcal{V}^3} W_0^2 \\ \underset{\mathcal{V} \gg 1}{\sim} & \frac{4}{\mathcal{V}^2} [(a_1 \tau_1 e^{-a_1 \tau_1} - W_0) a_1 \tau_1 e^{-a_1 \tau_1}] + \frac{3}{4} \frac{\xi}{\mathcal{V}^3} W_0^2. \end{aligned} \quad (3.20)$$

Assuming a natural value $W_0 \sim \mathcal{O}(1)$, then (3.20) simplifies to

$$V = -\frac{4}{\mathcal{V}^2} W_0 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{3}{4} \frac{\xi}{\mathcal{V}^3} W_0^2. \quad (3.21)$$

Extremising this scalar potential, we get

$$\frac{\partial V}{\partial \tau_1} = \frac{4}{\mathcal{V}^2} W_0 a_1 e^{-a_1 \tau_1} (a_1 \tau_1 - 1) = 0, \quad (3.22)$$

whose only possible solution for $W_0 \neq 0$ is $a_1 \tau_1 = 1$, which is not in the controlled regime of parameter space. However, when $W_0 = 0$, (3.20) gives

$$V = \left(4a_1^2 \tau_1^2 + \frac{3\xi}{4\mathcal{V}} \right) \frac{e^{-2a_1 \tau_1}}{\mathcal{V}^2} \underset{\mathcal{V} \gg 1}{\sim} 4a_1^2 \tau_1^2 \frac{e^{-2a_1 \tau_1}}{\mathcal{V}^2}, \quad (3.23)$$

and the first derivative with respect to τ_1 is

$$\frac{\partial V}{\partial \tau_1} = 8a_1^2 \tau_1 \frac{e^{-2a_1 \tau_1}}{\mathcal{V}^2} (a_1 \tau_1 - 1), \quad (3.24)$$

which also has no minimum. Thus we have shown that for $W_0 \sim \mathcal{O}(1)$, the $\mathbb{C}P^4_{[1,1,2,2,6]}$ model has no exponentially large volume minimum. The last hope is to find a minimum fine tuning $W_0 \ll 1$. In this case taking the derivatives of (3.20), one obtains

$$\frac{\partial V}{\partial \tau_1} = \frac{4}{\mathcal{V}^2} a_1 e^{-2a_1 \tau_1} (a_1 \tau_1 - 1) (W_0 e^{a_1 \tau_1} - 2a_1 \tau_1) = 0, \quad (3.25)$$

whose only possible solution is

$$2a_1 \langle \tau_1 \rangle = W_0 e^{a_1 \langle \tau_1 \rangle}, \quad (3.26)$$

but then fixing τ_1 , the scalar potential (3.20) along the volume direction looks like

$$V \sim \frac{W_0^2}{\mathcal{V}^2} \left(-1 + \frac{3\xi}{4\mathcal{V}} \right) \sim -\frac{W_0^2}{\mathcal{V}^2}. \quad (3.27)$$

The potential (3.27) has no LARGE Volume minimum and so we conclude that the $\mathbb{C}P^4_{[1,1,2,2,6]}$ model does not admit an exponentially large volume minimum for any value of W_0 .

It is still of course possible to fix the moduli using other stabilisation schemes - for example KKL_T. However, in this case there will not be a large hierarchy between the two Kähler moduli, with instead $\tau_1 \lesssim \tau_2$, and the volume can never be exponentially large.

3.4 3-parameter K3 fibration

In the previous sections 3.1, 3.2 and 3.3, we have presented three examples which illustrate two of the three possible situations which the general analysis determines. We now illustrate the case when an exponentially large volume minimum can be found, but with flat directions still present. We will then explain how these can be lifted using string loop corrections.

This example concerns Calabi-Yau three-folds which are single K3 Fibrations with three Kähler moduli. We start off with the following expression for the overall volume in terms of the three moduli

$$\mathcal{V} = \alpha \left[\sqrt{\tau_1} (\tau_2 - \beta \tau_1) - \gamma \tau_3^{3/2} \right], \quad (3.28)$$

where α, β, γ are positive model-dependent constants. While we do not have any explicit realisation of such kind of Calabi-Yau manifold, eq. (3.28) is simply the expression for the $\mathbb{C}P^4_{[1,1,2,2,6]}$ case (3.14), augmented by the inclusion of a blow-up mode τ_3 . We also assume that $h_{2,1}(X) > h_{1,1}(X) = 3$, thus satisfying the other condition of the LARGE Volume Claim. There are then two ways to perform the limit (2.10) without obtaining an internal volume that is formally negative:

1.
$$\begin{cases} \tau_i \rightarrow \infty, \quad \forall i = 1, 2 \text{ with the constraint } \tau_1 < \tau_2, \\ \tau_3 \text{ remains small.} \end{cases} \quad (3.29)$$

This case keeps both cycles associated with the fibration large, while the blow-up cycle remains small. Given that $\tau_1 \rightarrow \infty$, this situation resembles the “Swiss cheese” picture

$$\mathcal{V} = \alpha \left[\underbrace{\sqrt{\tau_1}(\tau_2 - \beta\tau_1)}_{\tau_{\text{big}}^{3/2}} - \gamma\tau_3^{3/2} \right], \quad (3.30)$$

and due to this analogy with the $\mathbb{C}P^4_{[1,1,1,6,9]}$ model, the condition $K_{33}^{-1} \simeq \mathcal{V}\sqrt{\tau_3}$ is verified. Thus we can apply the LARGE Volume Claim which states that the scalar potential will have an AdS exponentially large volume set of minima together with $(h_{1,1} - N_{\text{small}} - 1) = 1$ flat directions. In the following section we shall confirm this with an explicit calculation.

2.
$$\begin{cases} \tau_2 \rightarrow \infty, \\ \tau_1 \text{ and } \tau_3 \text{ remain small.} \end{cases} \quad (3.31)$$

In this case $N_{\text{small}} = 2$ and according to the LARGE Volume Claim there will be an exponentially large volume minimum of the scalar potential if and only if both τ_1 and τ_3 is a blow-up mode. As we show in the next section 3.4.1, $K_{33}^{-1} \sim \mathcal{V}\sqrt{\tau_3}$, as is suggested by the volume form (3.28). However $K_{11}^{-1} \sim \tau_1^2$, as could be guessed from the fact that the overall volume (3.13) in terms of the 2-cycles moduli is linear in t_1 . Hence τ_1 is not a blow-up but a fibration modulus that does not give rise to LVS.

We now confirm these statements with explicit calculations.

3.4.1 Explicit calculation

We focus on the case in which

$$\mathcal{V} = \alpha \left[\sqrt{\tau_1}(\tau_2 - \beta\tau_1) - \gamma\tau_3^{3/2} \right], \quad (3.32)$$

where α, β, γ are positive model-dependent constants and the Kähler potential and the superpotential take the form (defining $\hat{\xi} \equiv \xi g_s^{-3/2}$):

$$K = K_0 + \delta K_{(\alpha')} = -2 \ln \left(\mathcal{V} + \frac{\hat{\xi}}{2} \right), \quad (3.33)$$

$$W = W_0 + A_1 e^{-a_1 T_1} + A_2 e^{-a_2 T_2} + A_3 e^{-a_3 T_3}. \quad (3.34)$$

In the large volume limit the Kähler matrix and its inverse look like:

$$K_{ij}^0 = \frac{1}{4\mathcal{V}^2} \begin{pmatrix} \frac{\mathcal{V}^2}{\tau_1^2} + 2\alpha^2 \beta^2 \tau_1 & \frac{\alpha^2}{\sqrt{\tau_1}} \left(\gamma \tau_3^{3/2} - 2\beta \tau_1^{3/2} \right) & \frac{3\alpha\gamma}{2} \frac{\sqrt{\tau_3}}{\tau_1} \left(2\alpha\beta \tau_1^{3/2} - \mathcal{V} \right) \\ \frac{\alpha^2}{\sqrt{\tau_1}} \left(\gamma \tau_3^{3/2} - 2\beta \tau_1^{3/2} \right) & 2\alpha^2 \tau_1 & -3\alpha^2 \gamma \sqrt{\tau_1} \sqrt{\tau_3} \\ \frac{3\alpha\gamma}{2} \frac{\sqrt{\tau_3}}{\tau_1} \left(2\alpha\beta \tau_1^{3/2} - \mathcal{V} \right) & -3\alpha^2 \gamma \sqrt{\tau_1} \sqrt{\tau_3} & \frac{3\alpha\gamma}{2} \frac{\mathcal{V}}{\sqrt{\tau_3}} \end{pmatrix}, \quad (3.35)$$

and

$$K_0^{ij} = 4 \begin{pmatrix} \tau_1^2 & \beta \tau_1^2 + \gamma \sqrt{\tau_1} \tau_3^{3/2} & \tau_1 \tau_3 \\ \beta \tau_1^2 + \gamma \sqrt{\tau_1} \tau_3^{3/2} & \frac{\mathcal{V}^2}{2\alpha^2 \tau_1} + \beta^2 \tau_1^2 & \tau_3 \left(\frac{\mathcal{V}}{\alpha \sqrt{\tau_1}} + \beta \tau_1 \right) \\ \tau_1 \tau_3 & \tau_3 \left(\frac{\mathcal{V}}{\alpha \sqrt{\tau_1}} + \beta \tau_1 \right) & \frac{2}{3\alpha\gamma} \mathcal{V} \sqrt{\tau_3} \end{pmatrix}. \quad (3.36)$$

Both of the ways outlined above to take the large volume limit have $\tau_2 \gg 1$ and so the superpotential (3.34) can be simplified as follows:

$$W \simeq W_0 + A_1 e^{-a_1 T_1} + A_3 e^{-a_3 T_3}. \quad (3.37)$$

The scalar potential takes its general form (2.9). In the large volume limit, the α' leading contribution to the scalar potential becomes

$$V_{(\alpha')} = 3e^K \hat{\xi} \frac{(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2)}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2} |W|^2 \xrightarrow{\mathcal{V} \rightarrow \infty} \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} |W|^2. \quad (3.38)$$

Adding this to the non-perturbative part, we are left with

$$V = \frac{1}{\mathcal{V}^2} \left[K_0^{ij} a_i A_i a_j \bar{A}_j e^{-(a_i T_i + a_j \bar{T}_j)} + 2a_i A_i \tau_i e^{-a_i T_i} \bar{W} + 2a_j \bar{A}_j \tau_j e^{-a_j \bar{T}_j} W \right] + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} |W|^2. \quad (3.39)$$

We shall focus on the case $W_0 \sim \mathcal{O}(1)$, (since a tuned small value of W_0 cannot give rise to large volume) and for $a_1 \tau_1 \gg 1$, $a_3 \tau_3 \gg 1$, after extremising the axion directions, the scalar potential simplifies to (with $\lambda \equiv 8/(3\alpha\gamma)$, $\nu \equiv 3\hat{\xi}/4$ and $A_1 = A_3 = 1$)

$$V = \frac{\lambda a_3^2}{\mathcal{V}} \sqrt{\tau_3} e^{-2a_3 \tau_3} - \frac{4}{\mathcal{V}^2} W_0 a_1 \tau_1 e^{-a_1 \tau_1} - \frac{4}{\mathcal{V}^2} W_0 a_3 \tau_3 e^{-a_3 \tau_3} + \frac{\nu}{\mathcal{V}^3} W_0^2. \quad (3.40)$$

The scalar potential V depends on \mathcal{V} , τ_1 and τ_3 : $V = V(\mathcal{V}, \tau_1, \tau_3)$, with the dependence on τ_2 implicit in the internal volume \mathcal{V} . The large volume limit can be taken in the two ways (1 and 2) outlined in the previous section 3.4. The difference between these two cases is that in limit 1 $\tau_1 \rightarrow \infty$ whereas in limit 2 τ_1 remains small. Let us now study these two different cases in detail.

1) $\tau_1 \rightarrow \infty \Leftrightarrow \tau_3 \ll \tau_1 < \tau_2$

In this case, the superpotential (3.37) obtains non-perturbative corrections only in τ_3 :

$$W \simeq W_0 + A_3 e^{-a_3 \tau_3}. \quad (3.41)$$

Since the A_1 term is not present in (3.41), we will be unable to stabilise the corresponding Kähler modulus τ_1 , thereby giving rise to an exactly flat direction. In this case the scalar potential (3.40) further reduces to:

$$V = \frac{\lambda a_3^2}{\mathcal{V}} \sqrt{\tau_3} e^{-2a_3 \tau_3} - \frac{4}{\mathcal{V}^2} W_0 a_3 \tau_3 e^{-a_3 \tau_3} + \frac{\nu}{\mathcal{V}^3} W_0^2. \quad (3.42)$$

and V depends only on \mathcal{V} and τ_3 : $V = V(\mathcal{V}, \tau_3)$. The potential (3.42) has the same form as the scalar potential found in section 3.2 of [8] where the $\mathbb{C}P_{[1,1,1,6,9]}^4$ case was first discussed. Following the same reasoning, we look for possible minima of the scalar potential (3.42) by working out the two minimisation conditions:

$$\frac{\partial V}{\partial \mathcal{V}} = 0 \quad \Leftrightarrow \quad (\lambda a_3^2 \sqrt{\tau_3} e^{-2a_3 \tau_3}) \mathcal{V}^2 - (8W_0 a_3 \tau_3 e^{-a_3 \tau_3}) \mathcal{V} + 3\nu W_0^2 = 0, \quad (3.43)$$

$$\frac{\partial V}{\partial \tau_3} = 0 \quad \Leftrightarrow \quad \frac{\lambda a_3}{2\sqrt{\tau_3}} \mathcal{V} e^{-a_3 \tau_3} (1 - 4a_3 \tau_3) + 4W_0 (a_3 \tau_3 - 1) = 0. \quad (3.44)$$

Equation (3.43) admits a solution of the form

$$\frac{\lambda a_3}{4W_0 \sqrt{\tau_3}} \mathcal{V} e^{-a_3 \tau_3} = 1 \pm \sqrt{1 - \frac{3\lambda\nu}{16\tau_3^{3/2}}}, \quad (3.45)$$

whereas in the approximation $a_3 \tau_3 \gg 1$, (3.44) becomes:

$$\frac{\lambda a_3}{2\sqrt{\tau_3}} \mathcal{V} e^{-a_3 \tau_3} = W_0. \quad (3.46)$$

Combining (3.45) and (3.46), we find $\frac{1}{2} = 1 \pm \sqrt{1 - \frac{3\lambda\nu}{16\tau_3^{3/2}}}$, whose solution is given by

$$\langle \tau_3 \rangle = \frac{1}{g_s} \left(\frac{\xi}{2\alpha\gamma} \right)^{2/3} \sim \frac{1}{g_s}. \quad (3.47)$$

On the contrary, from (3.46) we work out

$$\langle \mathcal{V} \rangle = \frac{3(\alpha\gamma)^{2/3} W_0}{4a_3 \sqrt{g_s}} \left(\frac{\xi}{2} \right)^{1/3} e^{\frac{a_3}{g_s} \left(\frac{\xi}{2\alpha\gamma} \right)^{2/3}} \sim \frac{W_0}{a_3 \sqrt{g_s}} e^{\frac{a_3}{g_s}}. \quad (3.48)$$

There is therefore an exponentially large volume minimum. Setting $\alpha = \gamma = 1$, $\xi = 2$, $g_s = 0.1$, $a_3 = \pi$ and $W_0 = 1$, we finally obtain $\langle \tau_3 \rangle = 10$ and $\langle \mathcal{V} \rangle = 3.324 \cdot 10^{13}$. However there is still the presence of an exactly flat direction which can be better appreciated after the following change of coordinates:

$$(\tau_1, \tau_2) \quad \longrightarrow \quad (\mathcal{V}, \Omega) : \quad \begin{cases} \mathcal{V} \simeq \alpha [\sqrt{\tau_1} (\tau_2 - \beta\tau_1)] \\ \Omega = \alpha [\sqrt{\tau_1} (\tau_2 + \beta\tau_1)] \end{cases} \quad (3.49)$$

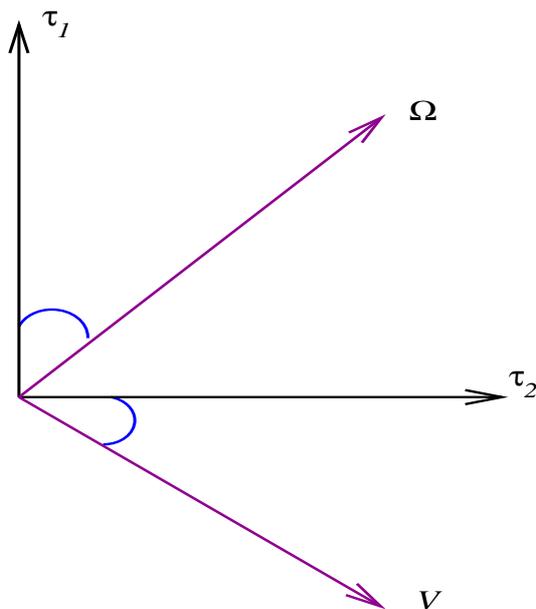


Figure 1: Change of coordinates of the Kähler moduli space.

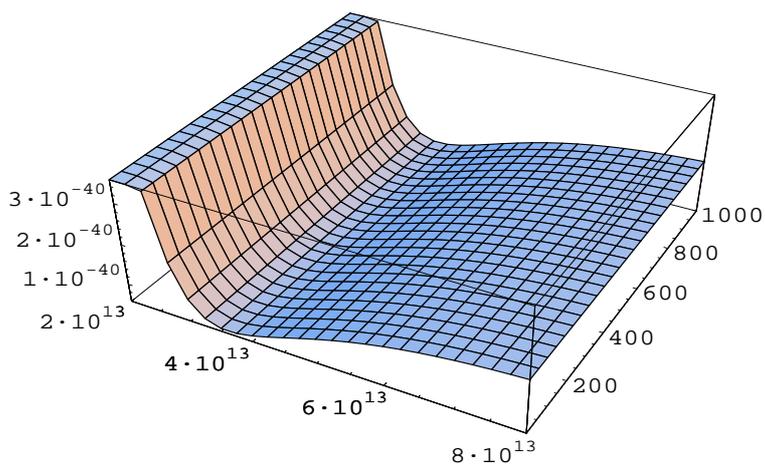


Figure 2: "Sofa" potential with the presence of a flat direction.

From (3.48) and (3.49) we see that the stabilisation of \mathcal{V} and τ_3 does not depend on Ω at all, implying that Ω is a flat direction. We plot below in figure 2 the behaviour of this scalar potential where the flat direction is manifest: τ_3 has been already fixed as $\langle \tau_3 \rangle = 10$, \mathcal{V} is plotted on the x -axis and Ω on the y -axis.

2) τ_1 small

In this case the large volume limit is taken keeping τ_1 small and the scalar potential

takes the general form (3.40). The minimisation equation with respect to τ_1 reads

$$\frac{\partial V}{\partial \tau_1} = 0 \quad \Leftrightarrow \quad \frac{4}{\mathcal{V}^2} W_0 a_1 e^{-a_1 \tau_1} (a_1 \tau_1 - 1) = 0, \quad (3.50)$$

which implies $a_1 \tau_1 = 1$ and so we cannot neglect higher instanton corrections.

There is therefore no trustable minimum for the τ_1 field. We may however think about a situation in which the system still has an exponentially large internal volume, with τ_3 and \mathcal{V} sitting at their minimum $\langle \mathcal{V} \rangle \sim e^{a_3 \langle \tau_3 \rangle}$, while τ_1 plays the rôle of a quintessence field rolling in a region at large $\tau_1 \gg 1$ away from $a_1 \tau_1 = 1$. The quintessence scale would be set by the $e^{-a_1 \tau_1}$ exponent. Setting $a_1 \tau_1 \gg 1$ it is easy to see that this is possible. However, the values of a_1 and τ_1 need to be tuned to get a realistically small mass for τ_1 and even if this is done the fifth force problems of quintessence fields would seem to be unavoidable.

Finally, let us summarize in the table below the results found without string loop corrections to K .

$$\begin{array}{l}
 1) \quad \tau_1 \rightarrow \infty \quad \left\{ \begin{array}{ll} W_0 \text{ small} & \text{No LVS,} \\ W_0 \sim \mathcal{O}(1) & \text{LVS + exactly flat direction } \perp \mathcal{V}, \end{array} \right. \\
 2) \quad \tau_1 \text{ small} \quad \left\{ \begin{array}{l} W_0 \text{ small} \quad \text{No LVS,} \\ W_0 \sim \mathcal{O}(1) \text{ and } \left\{ \begin{array}{ll} a_1 \tau_1 \gtrsim a_3 \langle \tau_3 \rangle & \text{LVS + almost flat direction } \perp \mathcal{V} \text{ (quintessence),} \\ a_1 \tau_1 < a_3 \langle \tau_3 \rangle & \text{No LVS.} \end{array} \right. \end{array} \right.
 \end{array}$$

4. Inclusion of the string loop corrections

One of the purposes of this paper is to study the effect of string loop corrections on moduli stabilisation. On a Calabi-Yau it is of course not possible to explicitly determine the full functional form of loop corrections. Nonetheless, it may still be possible to extract the leading scaling behaviour with the moduli of the loop corrections, even if the detailed form of the prefactors cannot be determined. As the Kähler moduli are not stabilised at tree-level, even this leading scaling behaviour is very significant.

This is the philosophy that was pursued in [13, 12], where the expected parametric behaviour of the string loop corrections to the Kähler potential was studied, even if the detailed prefactors cannot be computed. On the torus, the explicit string computation of [10] does exist. For the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ case the loop corrections take the form

$$\delta K_{(g_s)} = \delta K_{(g_s)}^{KK} + \delta K_{(g_s)}^W, \quad (4.1)$$

where $\delta K_{(g_s)}^{KK}$ comes from the exchange between D7 and D3-branes of closed strings which carry Kaluza-Klein momentum, and reads (for vanishing open string scalars)

$$\delta K_{(g_s)}^{KK} = -\frac{1}{128\pi^4} \sum_{i=1}^3 \frac{\mathcal{E}_i^{KK}(U, \bar{U})}{\text{Re}(S) \tau_i}. \quad (4.2)$$

In the previous expression we assumed the all the three 4-cycles of the torus are wrapped by D7-branes and τ_i denotes the volume of the 4-cycle wrapped by the i -th D7-brane. The other correction $\delta K_{(g_s)}^W$ is interpreted in the closed string channel as due to exchange of winding strings between intersecting stacks of D7-branes. It takes the form

$$\delta K_{(g_s)}^W = -\frac{1}{128\pi^4} \sum_{i=1}^3 \frac{\mathcal{E}_i^W(U, \bar{U})}{\tau_j \tau_k} \Big|_{j \neq k \neq i}, \tag{4.3}$$

where τ_i and τ_j denote the volume of the 4-cycles wrapped by the i -th and the j -th intersecting D7-branes. Note in both cases there is a very complicated dependence of the corrections on the U moduli, encoded in the functions $\mathcal{E}_i(U, \bar{U})$, but a very simple dependence on the T moduli.

These formulae were generalised by [13] for the behaviour of loop corrections on general Calabi-Yau three-folds. Given that these corrections can be interpreted as the tree-level propagation of a closed KK string and a Weyl rescaling is always necessary to convert the string computation to Einstein frame, they proposed

$$\delta K_{(g_s)}^{KK} \sim \sum_{i=1}^{h_{1,1}} \frac{\mathcal{C}_i^{KK}(U, \bar{U}) m_{KK}^{-2}}{\text{Re}(S) \mathcal{V}} \sim \sum_{i=1}^{h_{1,1}} \frac{\mathcal{C}_i^{KK}(U, \bar{U}) (a_{il} t^l)}{\text{Re}(S) \mathcal{V}}, \tag{4.4}$$

where $a_{il} t^l$ is a linear combination of the basis 2-cycle volumes t_l that is transverse to the 4-cycle wrapped by the i -th D7-brane. A similar line of argument for the winding corrections gives

$$\delta K_{(g_s)}^W \sim \sum_i \frac{\mathcal{C}_i^W(U, \bar{U}) m_W^{-2}}{\mathcal{V}} \sim \sum_i \frac{\mathcal{C}_i^W(U, \bar{U})}{(a_{il} t^l) \mathcal{V}}, \tag{4.5}$$

with $a_{il} t^l$ the 2-cycle where the two D7-branes intersect. \mathcal{C}_i^{KK} and \mathcal{C}_i^W are unknown functions of the complex structure moduli; however, as the complex structure moduli are flux-stabilised these reduce to an (unknown) constant. This approach is therefore useful to fix the leading order dependence on Kähler moduli.

In [12] the current authors showed that the proposed form of these correction is consistent with what would be expected based on the form of the Coleman-Weinberg potential in supergravity [22, 23],

$$\delta V_{1-loop} \sim \Lambda^4 \text{STr}(M^0) + 2\Lambda^2 \text{STr}(M^2) + \text{STr}\left(M^4 \ln\left(\frac{M^2}{\Lambda^2}\right)\right), \tag{4.6}$$

where Λ denotes the cut-off scale and $\text{STr}(M^n)$ is the supertrace. The cut-off scale is taken to be the scale at which the higher supersymmetry of the ten-dimensional theory becomes apparent, effectively giving an extended supersymmetry within the loops.

In terms of the corrections to the Kähler potential, the leading contribution of these corrections to the scalar potential is always vanishing, giving an "extended no-scale structure". This result holds in general as long as the corrections are homogeneous functions of degree -2 in the 2-cycle volumes.

In [12] a general formula was also worked out for the first non-vanishing contribution to the effective scalar potential of $\delta K_{(g_s)}$ as given by the conjectures (4.4)–(4.5). It turns out to

be relatively simple and expressible in terms of the tree-level Kähler metric $K_0 = -2 \ln(\mathcal{V})$ and the winding correction to the Kähler potential:

$$\delta V_{(g_s)}^{1-loop} = \sum_{i=1}^{h_{1,1}} \left(\frac{(\mathcal{C}_i^{KK})^2}{\text{Re}(S)^2} K_{ii}^0 - 2\delta K_{(g_s),\tau_i}^W \right) \frac{W_0^2}{\mathcal{V}^2}. \quad (4.7)$$

The low energy interpretation of the 1-loop scalar potential (4.7) comes from matching it with the Coleman-Weinberg potential (4.6). The leading term in both cases vanishes, and so the comparison should involve the leading non-zero terms in both cases. These match precisely for the various cases studied in [12].

This low-energy interpretation can easily be illustrated in the simple one-modulus case where the volume takes the form $\mathcal{V} = \tau^{3/2}$. The formula for the 1-loop scalar potential (4.7) applied to this case produces (dropping the dilaton dependence since S is fixed at tree level)

$$\delta V_{(g_s),1-loop}^{KK} = \left(0 \cdot \frac{-3\mathcal{C}^{KK}}{\mathcal{V}^{8/3}} + \frac{3\alpha_2 (\mathcal{C}^{KK})^2}{\mathcal{V}^{10/3}} - \frac{6\alpha_3 (\mathcal{C}^{KK})^3}{\mathcal{V}^4} + \mathcal{O}\left(\frac{1}{\mathcal{V}^{14/3}}\right) \right) W_0^2. \quad (4.8)$$

To compare with (4.6) we recall that in supergravity the supertrace is proportional to the gravitino mass:

$$STr(M^2) \simeq m_{3/2}^2 = e^K W_0^2 \simeq \frac{1}{\mathcal{V}^2}. \quad (4.9)$$

The cut-off Λ is identified with the compactification scale given by

$$\Lambda = m_{KK} \simeq \frac{M_s}{R} = \frac{M_s}{\tau^{1/4}} = \frac{1}{\tau^{1/4}} \frac{M_P}{\sqrt{\mathcal{V}}} = \frac{M_P}{\mathcal{V}^{2/3}}. \quad (4.10)$$

Therefore in units of the Planck mass, (4.6) scales, in agreement with (4.8), as

$$\delta V_{1-loop} \simeq 0 \cdot \frac{1}{\mathcal{V}^{8/3}} + \frac{1}{\mathcal{V}^{10/3}} + \frac{1}{\mathcal{V}^4}. \quad (4.11)$$

We now use these results for the study of Kähler moduli stabilisation.

4.1 LARGE volume and string loop corrections

The results reviewed in the previous section are very important for Kähler moduli stabilisation. The general picture for LVS which we presented in the previous sections, was neglecting the effect of string loop corrections to the scalar potential. However just looking at the Kähler potential we have seen that, in terms of powers of the Kähler moduli, the leading order α' correction (2.1) scales as $\delta K_{(\alpha')} \sim \frac{1}{\mathcal{V}}$, whereas from (4.4), the scaling behaviour of the Kaluza-Klein loop correction is $\delta K_{(g_s)}^{KK} \sim \frac{\sqrt{\tau}}{\mathcal{V}}$. Naively it seems incorrect to neglect $\delta K_{(g_s)}^{KK}$ while including the effects of $\delta K_{(\alpha')}$. However, as discussed in [10, 13, 12] due to the extended no-scale structure at the level of the scalar potential the α' corrections dominate over the g_s corrections. This allows loop corrections to be neglected compared to α' corrections for the stabilisation of the volume.

However in our general analysis earlier, we saw that for fibration models the inclusion of α' corrections still left almost flat directions corresponding to non blow-up moduli orthogonal to the overall volume. Loop corrections to the scalar potential are much more

important than non-perturbative superpotential corrections, and we realise that they can play a crucial role in stabilising these non blow-up moduli transverse to the overall volume.

Thus we conclude that the extended no-scale structure renders the LVS robust not only because it allows $\delta V_{(g_s)}$ to be neglected when stabilising the volume, but also because it ensures that when $\delta V_{(g_s)}$ is introduced to lift the remaining flat directions, even though it will reintroduce a dependence in V on \mathcal{V} and blow-up moduli, it will not destroy the minimum already found but will give just a small perturbation around it.

The general picture is that all corrections - α' , loop and non-perturbative - play a rôle in a generic Calabi-Yau compactification. We can summarise our general analysis for the LVS as:

1. In order to stabilise all the Kähler moduli at exponentially large volume one needs at least one 4-cycle which is a blow-up mode resolving point-like singularities.
2. This 4-cycle, together with other blow-up modes possibly present, are fixed small by the interplay of non-perturbative and α' corrections, which stabilise also the overall volume mode.
3. The g_s corrections are subleading and so can be safely neglected.
4. All the other 4-cycles, as those corresponding to fibrations, even though they have non-perturbative effects, cannot be stabilised small. Thus they are sent large so making their non-perturbative corrections negligible.
5. These moduli, which are large and transverse to the overall volume, can then be frozen by g_s corrections, which dominate over the (tiny) non-perturbative ones.

In general $\delta V_{(g_s)}$ only lifts the flat directions associated to non blow-up moduli transverse to the overall volume. One could wonder whether they indeed yield a real minimum for such moduli as opposed to a runaway direction. We do not address this problem in general terms here and so in principle this looks like a model-dependent issue. However, as the overall volume is stabilised, the internal moduli space is compact. Therefore these non blow-up moduli cannot run-away to infinity and so we expect that loop corrections will induce a minimum for the potential. In fact, one example in the next section will illustrate this idea explicitly.

5. Moduli stabilisation via string loop corrections

We will now see in detail how the inclusion of string loop corrections can affect the results found in the previous examples which, neglecting g_s corrections, can be summarised as:

1. $CP^4_{[1,1,1,6,9]} \rightarrow$ LVS without flat directions.
2. 3-parameter K3 Fibration with τ_1 “small” and $a_1\tau_1 > a_3\langle\tau_3\rangle \rightarrow$ LVS with an almost flat direction.
3. 3-parameter K3 Fibration with $\tau_3 \ll \tau_1 < \tau_2 \rightarrow$ LVS with one flat direction.

4. $\mathbb{C}P_{[1,3,3,3,5]}^4 \rightarrow$ LVS with a tachyonic direction.
5. $\mathbb{C}P_{[1,1,2,2,6]}^4$ and 3-parameter K3 Fibration with τ_1 "small" and $a_1\tau_1 < a_3\tau_3 \rightarrow$ No LVS.

We shall find that the inclusion of loop corrections modifies the previous picture as follows:

1. $\mathbb{C}P_{[1,1,1,6,9]}^4 \rightarrow$ Not affected by $\delta V_{(g_s)}$.
2. 3-parameter K3 Fibration with τ_1 "small" and $a_1\tau_1 > a_3\tau_3 \rightarrow \delta V_{(g_s)}$ ruins the almost flat direction \implies No LVS.²
3. 3-parameter K3 Fibration with $\tau_3 \ll \tau_1 < \tau_2 \rightarrow \delta V_{(g_s)}$ lifts the flat direction \implies LVS without flat directions.
4. $\mathbb{C}P_{[1,3,3,3,5]}^4 \rightarrow \delta V_{(g_s)}$ stabilises the tachyonic direction \implies LVS without flat directions.
5. $\mathbb{C}P_{[1,1,2,2,6]}^4$ and 3-parameter K3 Fibration with τ_1 "small" and $a_1\tau_1 < a_3\tau_3 \rightarrow$ Not affected by $\delta V_{(g_s)}$ - still no LVS.

The $\mathbb{C}P_{[1,1,2,2,6]}^4$ case can never give large volume due to the fibration 4-cycle τ_1 which is impossible to stabilise small. However in the example of the 3-parameter K3 fibration, LARGE Volume can be achieved by including a third Kähler modulus which is a local blow-up and then sending τ_1 large. We shall use the expression (4.7) for the form of string loop corrections to the scalar potential.

5.1 The single-hole Swiss cheese: $\mathbb{C}P_{[1,1,1,6,9]}^4$

The influence of the g_s corrections in the $\mathbb{C}P_{[1,1,1,6,9]}^4$ case has been studied in detail in [13]. The authors showed that the loop corrections are subleading and so can be neglected, as we claimed above. The loop corrected Kähler potential looks like

$$\begin{aligned}
 K &= K_{\text{tree}} + \delta K_{(\alpha')} + \delta K_{(g_s, \tau_5)}^{KK} + \delta K_{(g_s, \tau_4)}^{KK} \\
 &= -2 \ln \mathcal{V} - \frac{\xi}{\mathcal{V} g_s^{3/2}} + \frac{g_s C_5^{KK} \sqrt{\tau_5}}{\mathcal{V}} + \frac{g_s C_4^{KK} \sqrt{\tau_4}}{\mathcal{V}},
 \end{aligned} \tag{5.1}$$

but due to the "extended no scale structure", we obtain for the scalar potential

$$\begin{aligned}
 V &= V_{np} + V_{(\alpha')} + V_{(g_s, \tau_5)}^{KK} + V_{(g_s, \tau_4)}^{KK} \\
 &= \frac{\lambda_1 \sqrt{\tau_4} e^{-2a_4 \tau_4}}{\mathcal{V}} - \frac{\lambda_2 W_0 \tau_4 e^{-a_4 \tau_4}}{\mathcal{V}^2} + \frac{3\xi W_0^2}{4\mathcal{V}^3 g_s^{3/2}} + \frac{g_s^2 (C_5^{KK})^2}{\mathcal{V}^3 \sqrt{\tau_5}} + \frac{g_s^2 (C_4^{KK})^2}{\mathcal{V}^3 \sqrt{\tau_4}}.
 \end{aligned} \tag{5.2}$$

Without taking the loop corrections into account, we have found a minimum located at $\mathcal{V} \sim e^{a_4 \tau_4} \Leftrightarrow a_4 \tau_4 \sim \ln \mathcal{V}$. Therefore the various terms in (5.2) scale as

$$\begin{aligned}
 V &= V_{np} + V_{(\alpha')} + V_{(g_s, \tau_5)}^{KK} + V_{(g_s, \tau_4)}^{KK} \\
 &\sim \frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}^3} - \frac{\ln \mathcal{V}}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^{10/3}} + \frac{1}{\mathcal{V}^3 \sqrt{\ln \mathcal{V}}},
 \end{aligned} \tag{5.3}$$

²Notice that this case is the same as case 3 below but in a different region of moduli space. The fact that there is no LVS realised here only means that for this model the LARGE Volume is realised as in the conditions of case 3 but not in those for case 2. In particular it requires both fibration moduli to be large.

and it is straightforward to realise that at exponentially large volume the last two terms in (5.3) are suppressed with respect to the first three ones.

5.2 The multiple-hole Swiss cheese: $\mathbb{C}P^4_{[1,3,3,3,5]}$

In section 3.2 we have seen that if the non-perturbative corrections in the SM cycle τ_{SM} are absent, the F-term scalar potential (3.12) for the $\mathbb{C}P^4_{[1,3,3,3,5]}$ Calabi-Yau does not present a LVS with all the Kähler moduli stabilised. Following the same procedure as in [19], we shall now illustrate how the g_s corrections can turn the maximum in the τ_{SM} direction into a minimum without destroying the exponentially large volume minimum $\mathcal{V} \sim \sqrt{\tau_{E3}} e^{2\pi\tau_{E3}}$.

To derive the conjectured scaling behaviour of the loop corrections, we use the formula (4.7) setting $\mathcal{C}_i^{KK} = \text{Re}(S) \forall i$ and $W_0 = 1$. Two stacks of D7-branes wrap the τ_{SM} and τ_c cycle respectively and both will give rise to Kaluza-Klein g_s corrections. From (4.7), we estimate the first kind of corrections by writing the overall volume (3.10) in the $(\tau_a, \tau_{SM}, \tau_c)$ basis and computing the relevant elements of the direct Kähler metric. We find:

$$\mathcal{V} = \sqrt{\frac{2}{45}} \left(\tau_a^{3/2} - \frac{1}{3} (3\tau_{SM} + 2\tau_c)^{3/2} - \frac{\sqrt{5}}{3} \tau_c^{3/2} \right), \quad (5.4)$$

along with

$$\frac{\partial^2 K_{\text{tree}}}{\partial \tau_{SM}^2} \simeq \frac{3}{\sqrt{10}} \frac{1}{\mathcal{V} \sqrt{3\tau_{SM} + 2\tau_c}}, \quad (5.5)$$

and

$$\frac{\partial^2 K_{\text{tree}}}{\partial \tau_c^2} \simeq \frac{2\sqrt{2}}{3\sqrt{5}} \left(\frac{\sqrt{5}}{4\sqrt{\tau_c}} + \frac{1}{\sqrt{3\tau_{SM} + 2\tau_c}} \right) \frac{1}{\mathcal{V}}, \quad (5.6)$$

where in the large volume limit we have approximated the volume as $\mathcal{V} \simeq \sqrt{\frac{2}{45}} \tau_a^{3/2}$. Thus the Kaluza-Klein loop corrections to (3.12) look like

$$\delta V_{(g_s)}^{KK} \simeq \left(\frac{5}{\sqrt{\tau_c}} + \frac{13\sqrt{5}}{\sqrt{3\tau_{SM} + 2\tau_c}} \right) \frac{1}{15\sqrt{2}\mathcal{V}^3}. \quad (5.7)$$

Writing (5.7) back in terms of τ_{SM} and $\tau_{E3} = \tau_c + \tau_{SM}$, we obtain

$$\delta V_{(g_s)}^{KK} \simeq \left(\frac{5}{\sqrt{\tau_{E3} - \tau_{SM}}} + \frac{13\sqrt{5}}{\sqrt{2\tau_{E3} + \tau_{SM}}} \right) \frac{1}{15\sqrt{2}\mathcal{V}^3}. \quad (5.8)$$

Due to the particular simple form of the volume (5.4), it is very sensible to expect that the winding corrections will scale like the Kaluza-Klein ones (5.8). Therefore adding (5.8) to (3.12) we end up with

$$\begin{aligned} V + \delta V_{(g_s)} &= \frac{\lambda_1 \left(\sqrt{5(2\tau_{E3} + \tau_{SM})} + \sqrt{\tau_{E3} - \tau_{SM}} \right) e^{-4\pi\tau_{E3}}}{\mathcal{V}} - \frac{3\lambda_2 \tau_{E3} e^{-2\pi\tau_{E3}}}{\mathcal{V}^2} \\ &+ \frac{\lambda_3}{\mathcal{V}^3} + \left(\frac{\lambda_4}{\sqrt{\tau_{E3} - \tau_{SM}}} + \frac{\lambda_5}{\sqrt{2\tau_{E3} + \tau_{SM}}} \right) \frac{1}{\mathcal{V}^3}. \end{aligned} \quad (5.9)$$

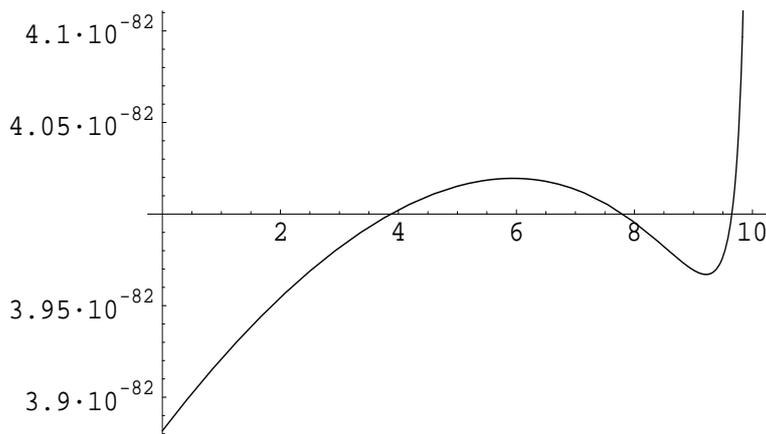


Figure 3: τ_{SM} fixed by string loop corrections. The numerical values used are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 50$, $\lambda_4 = \lambda_5 = 5$ and then we have fixed $\tau_{E3} = 10$ and $\mathcal{V} = \sqrt{10}e^{20\pi}$.

We notice that the string loop corrections are suppressed with respect to the α' ones by a factor of $1/\sqrt{\tau_{E3}}$ and so do not affect the large volume minimum $\mathcal{V} \sim \sqrt{\tau_{E3}}e^{2\pi\tau_{E3}}$ given that we require $\tau_{E3} \gg 1$ to neglect higher order instanton contributions. On the contrary $\delta V_{(g_s)}$ can become important to fix the SM direction when τ_{SM} gets small. In fact, the maximum in that direction is now accompanied by a minimum, as illustrated in figure 3.

Thus we have shown that g_s corrections can indeed freeze the SM direction so giving rise to a LVS without any tachyonic direction. The physics of this stabilisation is simply that if a D7-brane wraps a 4-cycle, then loop corrections induced by the brane will become large as the cycle size collapses. This repels the modulus from collapsing and induces a minimum of the the potential.

This example is illustrative in nature and shows how a cycle, which is required to be small and which does not admit nonperturbative effects, can potentially be stabilised by loop corrections. In a fully realistic model, the D-term contribution to the potential should also be included and the combined F- and D-term potential studied. Usually the D-term will include, besides the Fayet-Iliopoulos term depending on the moduli, also the charged matter fields. Minimising the D-term will generically fix one of the Standard Model singlets to essentially cancel the Fayet-Iliopoulos term. Thus we can foresee a scenario in which the Standard Model cycle is fixed by string loop corrections whereas the D-term fixes not the size of that cycle but instead the VEV of a Standard Model singlet as a function of the moduli. In this way we address the challenge of [14]. The form of the D-term however depends on the model and in particular on the details of the charged matter content and whether or not they acquire VEVs. We therefore do not try and specify this, but note that it will be necessary to include it in a realistic model.

5.3 2-parameter K3 fibration: $\mathbb{C}P^4_{[1,1,2,2,6]}$

One could wonder whether including the string loop corrections in the case of the K3 Fibration with two Kähler moduli treated in section 3.3, could generate an exponentially large volume minimum which was absent when only non-perturbative and the α' corrections are included. In reality, the answer is negative as these further perturbative corrections produce a contribution $V_{(g_s)}^{KK} + V_{(g_s)}^W$ to the scalar potential (3.21), which is subdominant and cannot help to stabilise the moduli. In fact, in the large volume limit (3.17) and for $W_0 \sim \mathcal{O}(1)$, the full corrected scalar potential, now takes the form

$$\begin{aligned}
 V &= V_{np} + V_{(\alpha')} + V_{(g_s, \tau_1)}^{KK} + V_{(g_s, \tau_2)}^{KK} + V_{(g_s, \tau_1)}^W + V_{(g_s, \tau_2)}^W \simeq \\
 &\simeq -\frac{4}{\mathcal{V}^2} W_0 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{3 \xi \text{Re}(S)^{3/2}}{4 \mathcal{V}^3} W_0^2 \\
 &\quad + \frac{W_0^2}{\mathcal{V}^2} \left(\frac{(\mathcal{C}_1^{KK})^2}{\text{Re}(S)^2} \frac{1}{\tau_1^2} + \frac{(\mathcal{C}_2^{KK})^2}{\text{Re}(S)^2} \frac{1}{2} \frac{\tau_1}{\mathcal{V}^2} - 2\mathcal{C}_1^W \frac{\tau_1}{\mathcal{V}^2} - \frac{2\mathcal{C}_2^W}{\mathcal{V} \sqrt{\tau_1}} \right) \\
 &\simeq -\frac{4}{\mathcal{V}^2} W_0 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{3 \xi \text{Re}(S)^{3/2}}{4 \mathcal{V}^3} W_0^2 + \frac{W_0^2}{\mathcal{V}^2} \left(\frac{(\mathcal{C}_1^{KK})^2}{\text{Re}(S)^2} \frac{1}{\tau_1^2} - \frac{2\mathcal{C}_2^W}{\mathcal{V} \sqrt{\tau_1}} \right). \quad (5.10)
 \end{aligned}$$

First of all we have to check that the minimum in the volume is exponentially large. Therefore let us take the derivative

$$\frac{4\mathcal{V}^4}{W_0^2} \frac{\partial V}{\partial \mathcal{V}} = \left(\frac{32}{W_0} a_1 \tau_1 e^{-a_1 \tau_1} - \frac{(\mathcal{C}_1^{KK})^2}{\text{Re}(S)^2} \frac{8}{\tau_1^2} \right) \mathcal{V} + \left(\frac{24\mathcal{C}_2^W}{\sqrt{\tau_1}} - 9\xi \text{Re}(S)^{3/2} \right) = 0, \quad (5.11)$$

whose solution is

$$\langle \mathcal{V} \rangle = \frac{3 \text{Re}(S)^2 \langle \tau_1 \rangle^{3/2} W_0 \left(8\mathcal{C}_2^W - 3\sqrt{\langle \tau_1 \rangle} \xi \text{Re}(S)^{3/2} \right)}{8 \left((\mathcal{C}_1^{KK})^2 W_0 e^{a_1 \langle \tau_1 \rangle} - 4a_1 \text{Re}(S)^2 \langle \tau_1 \rangle^3 \right)} e^{a_1 \langle \tau_1 \rangle}. \quad (5.12)$$

From (5.12) we realise that in order to have an exponentially large volume, we need to fine tune $(\mathcal{C}_1^{KK})^2 \sim e^{-a_1 \tau_1} \ll 1$. We assume that this is possible and so the denominator of (5.12) scales as

$$W_0 - 4a_1 \text{Re}(S)^2 \langle \tau_1 \rangle^3 \simeq -4a_1 \text{Re}(S)^2 \langle \tau_1 \rangle^3, \quad (5.13)$$

given that we are working in a regime where $W_0 \sim \mathcal{O}(1)$, $\text{Re}(S) \simeq 10$ and $a_1 \tau_1 \gg 1$. Finally the VEV of the volume reads

$$\langle \mathcal{V} \rangle \simeq \frac{3}{8} \frac{W_0 \left(8\mathcal{C}_2^W - 3\sqrt{\langle \tau_1 \rangle} \xi \text{Re}(S)^{3/2} \right)}{-4a_1 \langle \tau_1 \rangle^{3/2}} e^{a_1 \langle \tau_1 \rangle}, \quad (5.14)$$

with \mathcal{C}_2^W chosen such that

$$\left(1 - \frac{3\sqrt{\langle \tau_1 \rangle} \xi \text{Re}(S)^{3/2}}{8\mathcal{C}_2^W} \right) < 0, \quad (5.15)$$

to have a positive result. Now we neglect the $(\mathcal{C}_1^{KK})^2$ term in V (5.10) when we perform the derivative with respect to τ_1 and we obtain

$$\frac{\mathcal{V}^2}{W_0} \frac{\partial V}{\partial \tau_1} = 4a_1 e^{-a_1 \langle \tau_1 \rangle} (a_1 \langle \tau_1 \rangle - 1) + \frac{W_0 \mathcal{C}_2^W}{\langle \mathcal{V} \rangle \langle \tau_1 \rangle^{3/2}} = 0. \quad (5.16)$$

Substituting back (5.14), (5.16) becomes

$$a_1 \langle \tau_1 \rangle = 1 + \frac{1}{3 \left(1 - \frac{3\sqrt{\langle \tau_1 \rangle} \xi \text{Re}(S)^{3/2}}{8\mathcal{C}_2^W} \right)}, \quad (5.17)$$

but (5.15) forces us to get $a_1 \langle \tau_1 \rangle < 1$, clearly in disagreement with our starting point when we ignored higher order instanton corrections. Hence we conclude that the inclusion of the string loop corrections does not help to stabilise the moduli at exponentially large volume since they render this attempt even worse.

5.4 3-parameter K3 fibration

The results of the study of the K3 Fibration with three Kähler moduli are summarised in the table at the end of section 3.4.1. We will now try to address the problem left unsolved in that section. Without loop corrections it was possible to find an exponentially large volume in this class of models but there was still a flat direction left, which we named Ω . Let us see now how this direction is lifted. We shall work in the regime $W_0 \sim \mathcal{O}(1)$ where the perturbative corrections are important. We start off wrapping stacks of D7 branes around all the 4-cycle τ_1 , τ_2 and τ_3 . We immediately notice that the Kaluza-Klein loop correction to V in τ_3 takes the form

$$\delta V_{(g_s), \tau_3}^{KK} = \frac{g_s^2 (\mathcal{C}_3^{KK})^2}{\sqrt{\tau_3} \mathcal{V}^3}, \quad (5.18)$$

and so does not depend on Ω and is subdominant to the α' correction. Thus we will confidently neglect it. More precisely, it could modify the exact locus of the minimum but not the main feature of the model, that is the presence of an exponentially large volume. Let us now focus on the region: $\tau_3 \ll \tau_1 < \tau_2$. We recall the form of the scalar potential and the Kähler potential without loop corrections:

$$V = \frac{16a_3^2}{3\mathcal{V}} \sqrt{\tau_3} e^{-2a_3 \tau_3} - \frac{4}{\mathcal{V}^2} a_3 \tau_3 e^{-a_3 \tau_3} + \frac{3}{2g_s^{3/2} \mathcal{V}^3}, \quad (5.19)$$

$$K = K_{\text{tree}} + \delta K_{(\alpha')} \underset{\mathcal{V} \gg 1}{\simeq} -2 \ln \mathcal{V} - \frac{2}{g_s^{3/2} \mathcal{V}}. \quad (5.20)$$

We study now the possible corrections to V coming from τ_1 and τ_2 according to the general 1-loop formula (4.7). We realise that the form of the volume (3.28) implies that in this base of the Kähler cone, the blow-up mode τ_3 has only its triple self-intersection number non-vanishing and so it does not intersect with any other cycle. This is a typical feature of a blow-up mode which resolves a point-like singularity: due to the fact that this exceptional

divisor is a *local* effect, it is always possible to find a suitable basis where it does not intersect with any other divisor. Now we have seen that some string loop corrections come from the exchange of closed winding strings at the intersection of stacks of D7 branes. Hence the topological absence of these intersections, implies an absence of these corrections. At the end, the only relevant loop corrections are

$$\delta V_{(g_s)} = \delta V_{(g_s),\tau_1}^{KK} + \delta V_{(g_s),\tau_2}^{KK} + \delta V_{(g_s),\tau_1\tau_2}^W, \quad (5.21)$$

which look like

$$\begin{aligned} \delta V_{(g_s),\tau_1}^{KK} &= g_s^2 (C_1^{KK})^2 \left(\frac{1}{\tau_1^2} + \frac{2\beta^2}{P} \right) \frac{W_0^2}{\mathcal{V}^2}, \\ \delta V_{(g_s),\tau_2}^{KK} &= g_s^2 (C_2^{KK})^2 \frac{2}{P} \frac{W_0^2}{\mathcal{V}^2}, \\ \delta V_{(g_s),\tau_1\tau_2}^W &= -2C_{12}^W \frac{W_0^2}{\mathcal{V}^3 t_*}, \end{aligned} \quad (5.22)$$

where the 2-cycle t_* is the intersection locus of the two 4-cycles whose volume is given by τ_1 and τ_2 . In order to work out the form of t_* , we need to write down the volume of the K3 Fibration (3.28) in terms of 2-cycle moduli:

$$\mathcal{V} = (\lambda_1 t_1 + \lambda_2 t_2) t_2^2 + \lambda_3 t_3^3. \quad (5.23)$$

Then

$$\begin{cases} \tau_1 = \frac{\partial \mathcal{V}}{\partial t_1} = t_2 (\lambda_1 t_2), \\ \tau_2 = \frac{\partial \mathcal{V}}{\partial t_2} = t_2 (2\lambda_1 t_1 + 3\lambda_2 t_2), \end{cases} \quad (5.24)$$

and so $t_* = t_2 = \sqrt{\frac{\tau_1}{\lambda_1}}$. Therefore the g_s corrections to the scalar potential (5.22) take the general form:

$$\delta V_{(g_s)} = \left(\frac{A}{\tau_1^2} + \frac{B}{\mathcal{V}\sqrt{\tau_1}} + \frac{C\tau_1}{\mathcal{V}^2} \right) \frac{W_0^2}{\mathcal{V}^2}, \quad (5.25)$$

where

$$\begin{cases} A = g_s^2 (C_1^{KK})^2 > 0, \\ B = -2C_{12}^W \sqrt{\lambda_1} \equiv -\frac{C_{12}^W}{\alpha}, \\ C = 2\alpha^2 g_s^2 [(C_1^{KK})^2 \beta^2 + (C_2^{KK})^2] > 0. \end{cases} \quad (5.26)$$

Notice that due to the “extended no-scale structure” which causes the vanishing of the leading Kaluza-Klein correction to V , we know the sign of the coefficients A and C because the parameters are squared (see (4.7)) but we do not have any control over the sign of B . It is now convenient to take advantage of the field redefinition (3.49) and recast the loop corrections (5.22) in terms of \mathcal{V} and Ω . Inverting the relation (3.49), we get

$$\tau_1 = \left(\frac{\Omega - \mathcal{V}}{2\alpha\beta} \right)^{2/3}, \quad \tau_2 = \left(\frac{\beta}{4\alpha^2} \right)^{1/3} \frac{(\Omega + \mathcal{V})}{(\Omega - \mathcal{V})^{1/3}}. \quad (5.27)$$

Substituting these results back in (5.22) we can find the relevant dependence of the scalar potential on Ω :

$$\delta V_{(g_s)} = \frac{d_1 \Omega^2 + d_2 \Omega \mathcal{V} + d_3 \mathcal{V}^2}{(\Omega - \mathcal{V})^{4/3} \mathcal{V}^4}, \quad (5.28)$$

where

$$d_1 = g_s^2 \left(\frac{2\alpha^4}{\beta^2} \right)^{1/3} \left[(C_1^{KK})^2 \beta^2 + (C_2^{KK})^2 \right] W_0^2, \quad (5.29)$$

$$d_2 = - \left(\frac{2}{\alpha^2 \beta^2} \right)^{1/3} \left\{ \beta C_{12}^W + 2\alpha^2 g_s^2 \left[(C_1^{KK})^2 \beta^2 + (C_2^{KK})^2 \right] \right\} W_0^2, \quad (5.30)$$

$$d_3 = \left(\frac{2}{\alpha^2 \beta^2} \right)^{1/3} \left\{ \beta C_{12}^W + \alpha^2 g_s^2 \left[3 (C_1^{KK})^2 \beta^2 + (C_2^{KK})^2 \right] \right\} W_0^2. \quad (5.31)$$

For generic values of d_1 , d_2 and d_3 we expect to lift the flat direction Ω . Consistency requirements imply that any meaningful minimum must lie within the Kähler cone so that no 2-cycle or 4-cycle shrinks to zero and the overall volume is always positive. Let us work out the boundaries of the Kähler moduli space in terms of \mathcal{V} and Ω and then look for a minimum in the Ω direction. Given that we are sending both τ_1 and τ_2 large while keeping τ_3 small we can approximate the volume (3.28)–(5.23) as follows

$$\mathcal{V} \simeq \alpha \sqrt{\tau_1} (\tau_2 - \beta \tau_1) = (\lambda_1 t_1 + \lambda_2 t_2) t_2^2, \quad (5.32)$$

where $\lambda_1 = \frac{1}{4\alpha^2} > 0$ and $\lambda_2 = \frac{\beta}{4\alpha^2} > 0$. Then looking at (5.24) and (5.32) it is clear that when t_1 and t_2 are positive then also $\mathcal{V} > 0$ and $\tau_i > 0 \forall i = 1, 2$. Hence the boundaries of the Kähler cone are where one of the 2-cycle moduli $t_{1,2} \rightarrow 0$. The expression of the 2-cycles in terms of \mathcal{V} and Ω reads

$$t_1 = \left(\frac{2\mathcal{V} - \Omega}{\lambda_1} \right) \left(\frac{\lambda_2}{\Omega - \mathcal{V}} \right)^{2/3}, \quad t_2 = \left(\frac{\Omega - \mathcal{V}}{\lambda_2} \right)^{1/3}, \quad (5.33)$$

and so we realise that the Kähler cone is given by $\mathcal{V} < \Omega < 2\mathcal{V}$. In fact, looking at (5.27) and (5.33) we obtain

$$\left\{ \begin{array}{l} \Omega \rightarrow \mathcal{V}^+ \iff \tau_1 \rightarrow 0 \iff \tau_2 \rightarrow \infty \iff t_1 \rightarrow \infty \iff t_2 \rightarrow 0, \\ \Omega \rightarrow (2\mathcal{V})^- \iff \tau_1 \rightarrow \lambda_1 \left(\frac{\mathcal{V}}{\lambda_2} \right)^{2/3} \iff \tau_2 \rightarrow 3\lambda_2^{1/3} \mathcal{V}^{2/3} \iff t_1 \rightarrow 0 \iff t_2 \rightarrow \left(\frac{\mathcal{V}}{\lambda_2} \right)^{1/3}. \end{array} \right.$$

We look now for possible minima along the Ω direction considering the volume already fixed:

$$\frac{\partial(\delta V_{(g_s)})}{\partial \Omega} = \frac{2d_1 \Omega (\Omega - 3\mathcal{V}) - \mathcal{V} (d_2 (\Omega + 3\mathcal{V}) + 4d_3 \mathcal{V})}{3(\Omega - \mathcal{V})^{7/3} \mathcal{V}^4} = 0. \quad (5.34)$$

Equation (5.34) admits a solution of the form $\langle \Omega \rangle = \kappa \langle \mathcal{V} \rangle$ where

$$\kappa = \frac{6d_1 + d_2 + \sqrt{36d_1^2 + 36d_1 d_2 + d_2^2 + 32d_1 d_3}}{4d_1}. \quad (5.35)$$

A consistent minimum within the walls of the Kähler cone requires a choice of d_1 , d_2 and d_3 such that $1 < \kappa < 2$. In section 3.4.1 we have set the parameters $\alpha = \gamma = 1$, $\xi = 2$, $g_s = 0.1$, $a_3 = \pi$ and $W_0 = 1$, and then obtained $\langle \tau_3 \rangle = 10$ and $\langle \mathcal{V} \rangle \simeq 3.324 \cdot 10^{13}$ from (3.47)–(3.48). We now keep the same choice of parameters and set also $\beta = 1/2$, $C_1^{KK} = 1$, $C_2^{KK} = C_{12}^W = 10$. It follows that $d_1 = 2.005$, $d_2 = -14.01$ and $d_3 = 12.015$

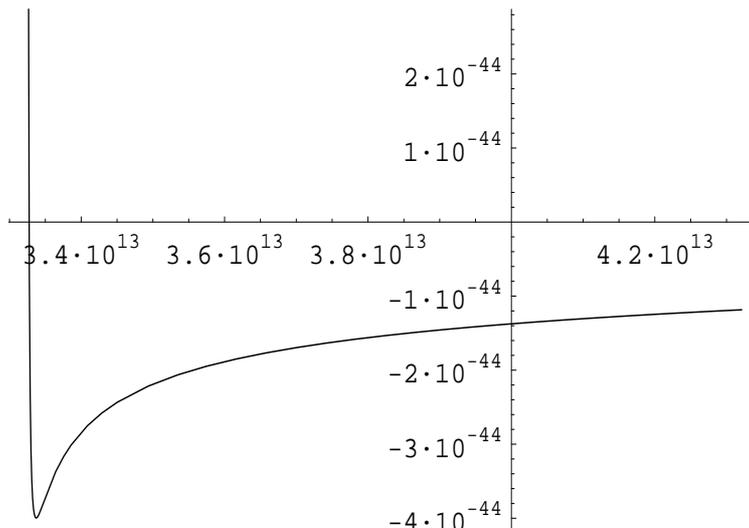


Figure 4: V versus Ω at \mathcal{V} and τ_3 fixed.

which gives $\kappa \simeq 1.004$ correctly in the required regime. Then the minimum for Ω shown in figure 4, is located at $\langle \Omega \rangle = \kappa \cdot 3.324 \cdot 10^{13} \simeq 3.337 \cdot 10^{13}$. We stress that we have stabilised Ω without fine tuning any parameter.

We could also contemplate the case where we do not have D7 branes wrapping one of the 4-cycles τ_1 and τ_2 . In this case there is no correction coming from the exchange of winding strings because we have just one stack of D7 branes with no intersection. Only the Kaluza-Klein corrections would survive.

1. no D7 brane wrapping the τ_1 cycle

In this case the 1-loop correction looks like

$$\delta V_{(g_s)} = \delta V_{(g_s), \tau_2}^{KK} = 2\alpha^2 g_s^2 (C_2^{KK})^2 \frac{W_0^2 \tau_1}{\mathcal{V}^4} = d \frac{(\Omega - \mathcal{V})^{2/3}}{\mathcal{V}^4}, \quad (5.36)$$

where $d = (\alpha^2 \sqrt{2}/\beta)^{2/3} g_s^2 (C_2^{KK})^2 W_0^2$. However (5.36) has no minimum in Ω regardless of the value of d .

2. no D7 brane wrapping the τ_2 cycle

$$\delta V_{(g_s)} = \delta V_{(g_s), \tau_1}^{KK} = g_s^2 (C_1^{KK})^2 \left(\frac{1}{\tau_1^2} + \frac{2(\alpha\beta)^2 \tau_1}{\mathcal{V}^2} \right) \frac{W_0^2}{\mathcal{V}^2} = \frac{\mu_1 \Omega^2 + \mu_2 \Omega \mathcal{V} + \mu_3 \mathcal{V}^2}{(\Omega - \mathcal{V})^{4/3} \mathcal{V}^4}, \quad (5.37)$$

where

$$\mu_1 = \delta, \quad \mu_2 = -2\delta, \quad \mu_3 = 3\delta, \quad \delta = 2^{1/3} (\alpha\beta)^{4/3} g_s^2 (C_1^{KK})^2 W_0^2. \quad (5.38)$$

Since the potential (5.37) has the same form of (5.28), we can follow the same line of reasoning as above and conclude that this case admits a minimum located at

$\langle \Omega \rangle = \kappa \langle \mathcal{V} \rangle$ if $1 < \kappa < 2$, where κ now is given by (5.35) with the replacement $d_i \leftrightarrow \mu_i \forall i = 1, 2, 3$.

We consider now the matrix of second derivatives in the (τ_1, τ_2) space $M_{ij} = V_{ij}$ and using the known expression for the Kähler metric K_{ij} we construct the matrix $K_{ik}^{-1} M_{kj}$. The two eigenvalues of this matrix correspond to the mass-squared of the canonically normalised particles corresponding to the volume modulus and the originally flat direction Ω :

$$m_{\mathcal{V}}^2 \sim 1/\mathcal{V}^3, \quad m_{\Omega}^2 \sim 1/\mathcal{V}^{10/3}.$$

Finally we have looked at minima in which $\tau_2 \gg \tau_1$. This would be an interesting configuration because of the following observation: since $t_2 = \sqrt{\tau_1}$ and $t_1 = (\tau_2 - 2\tau_1)/2\sqrt{\tau_1}$, we can see that $\tau_2 \gg \tau_1$ would imply that $t_1 \gg t_2$ and we would effectively have a very anisotropic compactification with the 2-cycle much bigger than its dual 4-cycle. This could then lead to a realisation of the supersymmetric 2 large extra dimensions scenario [17, 18] in which the extra dimension could be as large as a fraction of a millimetre. This would correspond to looking for solutions $\langle \Omega \rangle = (1 + \epsilon)\langle \mathcal{V} \rangle$ with $\epsilon \rightarrow 0$.

However, we have shown that in the case τ_1 “small”, the inclusion of the string loop corrections ruins the presence of the almost flat direction which we had found before, without being able to produce an actual minimum at exponentially large volume with all moduli stabilised in a reliable region of moduli space.

6. Conclusions and potential applications

This paper has studied the general conditions needed to find exponentially large volume in type IIB compactifications. The necessary and sufficient conditions are simple to state: negative Euler number, more than one Kähler moduli with at least one of them being a blow-up mode resolving a point-like singularity.

We have also uncovered the important rôle played by g_s corrections in moduli stabilisation. This has allowed us to find new classes of LVS with a fibration structure in which not only the volume but the fibre moduli are exponentially large whereas the blow-up modes are stabilised at the usual small values. Therefore in general all of α' , non-perturbative and g_s corrections, may be important to stabilise the different classes of Kähler moduli.

Here we briefly discuss some of the applications. First, our results do not appear to change significantly the standard phenomenology of LVS explored in [26, 27], where we imagine the Standard Model localised on D7 branes wrapping a small 4-cycle. The reason is that the volume modulus is still the main source of supersymmetry breaking leading to an approximate no-scale structure, which can be argued in general terms [28]. As the Standard Model is localised around a blow-up cycle, the effects from other exponentially large moduli will be suppressed. However, it may be interesting to explore the potential implications of hidden sectors localised on those cycles. Also, in the multiple-hole Swiss cheese case where the Standard Model cycle is stabilised by perturbative rather than non-perturbative effects, the general structure of soft terms will not change significantly, again since the main source of supersymmetry breaking is the volume modulus. The only difference could

be the absence of the small hierarchy between the scale of the soft terms M_{soft} and the gravitino mass $m_{3/2}$, since if the SM cycle is not stabilised non-perturbatively, then the suppression of M_{soft} with respect to $m_{3/2}$ by $\ln(M_P/m_{3/2})$ [29] is probably not present, but it would be interesting to study this case in further detail.

A potentially more interesting application is to cosmology. The cosmological implications of LVS have been explored in [30–33] only for Swiss cheese compactifications. Small moduli were found to be good candidates for inflatons as long as $h_{11} > 2$ without the need to fine tune. However a difficulty with this is that loop corrections are expected to modify this result if there is a D7 brane wrapping the inflaton cycle, while if there is no such brane then it is difficult to reheat the Standard Model brane since there is no direct coupling of the inflaton to Standard Model fields.

The volume modulus is not suitable for inflation as $m_{\mathcal{V}} \sim H$ and so it suffers directly from the η problem. However for fibration models such as $\mathbb{C}P^4_{[1,1,2,2,6]}$, there is the transverse field Ω which is stabilised by the loop corrections. As the loop corrections are parametrically weaker than the α' corrections which stabilise the volume, Ω is parametrically lighter than the volume modulus and thus the Hubble scale. In fact $m_{\Omega} \sim \mathcal{V}^{-(3/2+\alpha)}$, with $\alpha = 1/6$. It follows that the slow-roll η parameter is

$$\eta \sim M_P^2 \frac{m_{\Omega}^2}{H^2} \sim \frac{1}{\mathcal{V}^{1/3}} \ll 1. \tag{6.1}$$

Therefore such fibration models seem promising for string theory realisations of modular inflation, as at large volume the mass scale induced by loop corrections is parametrically smaller than the Hubble scale. A detailed study of the potential for large values of the field, away from the minimum, will be required in order to see if this is a viable model of inflation, including the value of density perturbations and the potential for reheating.

The fact that the spectrum of moduli fields includes further candidate light fields, besides the volume modulus, is a new source for the cosmological moduli problem. In the LARGE volume scenario this problem is already present as long as the string scale is smaller than 10^{13} GeV since the volume modulus would be lighter than 10 TeV, and coupling with gravitational strength interactions it would overclose the universe or decay so late to ruin nucleosynthesis. Given a solution to this problem - such as a late period of inflation - the corresponding modulus becomes a dark matter candidate. With an intermediate string scale and TeV supersymmetry, the volume modulus has a mass $m \sim 1\text{MeV}$. The additional light moduli fields are also potential dark matter candidates and have masses $m \sim 10\text{keV}$. Furthermore, they can decay into photons with a clean monochromatic line similar to the volume modulus. A proper analysis of their couplings to photons along the lines of [24] should be made in order to see if this effect could be eventually detected.

It is worth pointing that the multiple hole Swiss cheese example provides an explicit example of Kähler moduli inflation, in which at least three Kähler moduli were needed (but no explicit example was provided in [30]). Also this is a good example to explore the issue about stabilisation of the Standard Model cycle that has to be small (and then a blow-up mode) but without the presence of a non-perturbative superpotential which is not desired if

the corresponding axion is the QCD axion [19] and if D-terms could induce a breaking of the Standard Model group [14]. Our results indicate that it is actually possible to achieve this.

We would finally like to emphasise that this is only a first attempt to investigate the relevance of loop corrections in the LVS and much work remains to be done. In particular, although we have used a well motivated volume dependence of the leading quantum corrections to the Kähler potential, explicit calculations are still lacking. While we believe that given the general importance of loop corrections, it is important to study their effects even with incomplete knowledge of their form, further information about these corrections for general Calabi-Yaus is very desirable.

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A. Proof of the LARGE volume claim

Let us now present a comprehensive argument in favour of the LARGE Volume Claim which establishes the existence of LARGE Volumes in IIB string compactifications.

A.1 Proof for $N_{\text{small}} = 1$

Proof. (LARGE Volume Claim for $N_{\text{small}} = 1$) Let us start from the scalar potential (2.9) which we now rewrite as

$$V = V_{np1} + V_{np2} + V_{(\alpha')}, \tag{A.1}$$

and perform the large volume limit as described in (2.10) with $N_{\text{small}} = 1$ corresponding to τ_1 . In this limit $V_{(\alpha')}$ behaves as

$$V_{(\alpha')} \xrightarrow{\mathcal{V} \rightarrow \infty} + \frac{3\hat{\xi}}{4\mathcal{V}^3} e^{K_{cs}} |W|^2 + \mathcal{O}\left(\frac{1}{\mathcal{V}^4}\right). \tag{A.2}$$

We also point out that

$$e^K \xrightarrow{\mathcal{V} \rightarrow \infty} \frac{e^{K_{cs}}}{\mathcal{V}^2} + \mathcal{O}\left(\frac{1}{\mathcal{V}^3}\right). \tag{A.3}$$

Let us now study V_{np1} which reduces to

$$V_{np1} \xrightarrow{\mathcal{V} \rightarrow \infty} e^K K_{11}^{-1} a_1^2 |A_1|^2 e^{-a_1(T_1 + \bar{T}_1)} = \frac{K_{11}^{-1}}{\mathcal{V}^2} a_1^2 |A_1|^2 e^{-2a_1\tau_1}. \tag{A.4}$$

Switching to the study of V_{np2} , we find that

$$V_{np2} \xrightarrow{\nu \rightarrow \infty} -e^K \sum_{k=1}^{h_{1,1}} K_{1k}^{-1} \left[\left(a_1 A_1 e^{-a_1 \tau_1} e^{-ia_1 b_1} \bar{W} \partial_{T_k} K \right) + \left(a_1 \bar{A}_1 e^{-a_1 \tau_1} e^{+ia_1 b_1} W \partial_{T_k} K \right) \right], \quad (\text{A.5})$$

where we have used the fact that $K_{1k}^{-1} = K_{k1}^{-1}$. Equation (A.5) can be rewritten as

$$\begin{aligned} V_{np2} &\xrightarrow{\nu \rightarrow \infty} -e^K \sum_{k=1}^{h_{1,1}} K_{1k}^{-1} (\partial_{T_k} K) a_1 e^{-a_1 \tau_1} \left[\left(A_1 \bar{W} e^{-ia_1 b_1} \right) + \left(\bar{A}_1 W e^{+ia_1 b_1} \right) \right] \\ &= \left(X_1 e^{+ia_1 b_1} + \bar{X}_1 e^{-ia_1 b_1} \right), \end{aligned} \quad (\text{A.6})$$

where

$$X_1 \equiv -e^K K_{1k}^{-1} (\partial_{T_k} K) a_1 \bar{A}_1 W e^{-a_1 \tau_1}. \quad (\text{A.7})$$

We note that for a general Calabi-Yau, the following relation holds:

$$K_{1k}^{-1} (\partial_{T_k} K) = -2\tau_1, \quad (\text{A.8})$$

and thus the definition (A.7) can be simplified to

$$X_1 \equiv 2e^K a_1 \tau_1 |A_1| e^{-i\vartheta_1} |W| e^{i\vartheta_W} e^{-a_1 \tau_1} = |X_1| e^{i(\vartheta_W - \vartheta_1)}. \quad (\text{A.9})$$

Therefore

$$V_{np2} \xrightarrow{\nu \rightarrow \infty} |X_1| \left(e^{+i(\vartheta_W - \vartheta_1 + a_1 b_1)} + e^{-i(\vartheta_W - \vartheta_1 + a_1 b_1)} \right) = 2 |X_1| \cos(\vartheta_W - \vartheta_1 + a_1 b_1). \quad (\text{A.10})$$

V_{np2} is a scalar function of the axion b_1 whereas ϑ_1 and ϑ_W are to be considered just as parameters. In order to find a minimum for V_{np2} let us set its first derivative to zero:

$$\frac{\partial V_{np2}}{\partial b_1} = -2a_1 |X_1| \sin(\vartheta_W - \vartheta_1 + a_1 b_1) = 0. \quad (\text{A.11})$$

The solution of (A.11) is given by

$$a_1 b_1 = p_1 \pi + \vartheta_1 - \vartheta_W, \quad p_1 \in \mathbb{Z}. \quad (\text{A.12})$$

We have still to check the sign of the second derivative evaluated at b_1 as given in (A.12) and require it to be positive:

$$\frac{\partial^2 V_{np2}}{\partial b_1^2} = -2a_1^2 |X_1| \cos(\vartheta_W - \vartheta_1 + a_1 b_1) > 0 \iff p_1 \in 2\mathbb{Z} + 1. \quad (\text{A.13})$$

Thus we realise that at the minimum

$$V_{np2} = -2 |X_1| = -2 |W| |A_1| a_1 \tau_1 \frac{e^{-a_1 \tau_1}}{\mathcal{V}^2}. \quad (\text{A.14})$$

We notice that the phases of W and A_1 do not enter into V_{np2} once the axion has been properly minimised and so, without loss of generality, we can consider W and $A_1 \in \mathbb{R}^+$ from now on.

We may now study the full potential by combining equations (A.2), (A.4) and (A.14)

$$V \simeq \frac{K_{11}^{-1}}{\mathcal{V}^2} A_1^2 a_1^2 e^{-2a_1 \tau_1} - \frac{W_0}{\mathcal{V}^2} A_1 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2, \quad (\text{A.15})$$

where we have substituted W with its tree-level expectation value W_0 because the non-perturbative corrections are always subleading by a power of \mathcal{V} . Moreover, we have dropped all the factors since they are superfluous for our reasoning.

We would like to emphasize that we know that the first term in (A.15) is indeed positive. In fact it comes from

$$K_{11}^{-1} (\partial_1 W) (\partial_1 \bar{W}), \quad (\text{A.16})$$

and we know that the Kähler matrix is positive definite since it gives rise to the kinetic terms. Moreover, as we have just seen, the second term in (A.15) comes from the axion minimisation as so is definitely negative. Only the sign of $V_{(\alpha')}$ is in principle unknown, but the condition $h_{2,1}(X) > h_{1,1}(X)$ ensures that it is positive. This condition will turn out to be crucial in showing that the volume direction has indeed a minimum at exponentially large volume.

We need now to study the form of K_{11}^{-1} . For a general Calabi-Yau, the inverse Kähler matrix with α' corrections included, reads [25]:

$$K_{ij}^{-1} = -\frac{2}{9} (2\mathcal{V} + \hat{\xi}) k_{ijk} t^k + \frac{4\mathcal{V} - \hat{\xi}}{\mathcal{V} - \hat{\xi}} \tau_i \tau_j, \quad (\text{A.17})$$

which at large volume becomes

$$K_{ij}^{-1} = -\frac{4}{9} \mathcal{V} k_{ijk} t^k + 4\tau_i \tau_j + (\text{terms subleading in } \mathcal{V}). \quad (\text{A.18})$$

Hence we can classify the behaviour of K_{11}^{-1} depending on the volume dependence of the quantity $k_{11j} t^j$ and find 4 different cases:

1. $k_{11j} t^j = 0$ or $k_{11j} t^j \simeq \frac{\tau_1^{1/2+3\alpha/2}}{\mathcal{V}^\alpha}$, $\alpha \geq 1 \implies K_{11}^{-1} \simeq \tau_1^2$,
2. $k_{11j} t^j = \frac{\tau_1^{1/2+3\alpha/2}}{\mathcal{V}^\alpha}$, $0 < \alpha < 1 \implies K_{11}^{-1} \simeq \mathcal{V}^\alpha \tau_1^{2-3\alpha/2}$, $0 < \alpha < 1$,
3. $k_{11j} t^j \simeq \sqrt{\tau_1} \implies K_{11}^{-1} \simeq \mathcal{V} \sqrt{\tau_1}$,
4. $k_{11j} t^j \simeq \mathcal{V}^\alpha \tau_1^{1/2-3\alpha/2}$, $\alpha > 0 \implies K_{11}^{-1} \simeq \mathcal{V}^\alpha \tau_1^{2-3\alpha/2}$, $\alpha > 1$.

One could wonder why we are setting the conditions of the Theorem on the elements of the inverse Kähler matrix and not on the intersection numbers or the form of the overall volume of the Calabi-Yau from which it would be easier to understand their topological meaning. The reason is that it is the inverse Kähler matrix which enters directly with the superpotential into the form of the scalar potential which is the one that determines the physics.

Moreover, the Claim applies if the superpotential has the expression (2.11), but in this case we can still make linear field redefinitions that will not change W , corresponding to proper changes of basis, of the form

$$\begin{cases} \tau_j \longrightarrow \tau'_j = \tau_j, \quad \forall j = 1, \dots, N_{\text{small}}, \\ \tau_j \longrightarrow \tau'_j = \tau_j + g_1(\tau_i), \quad \forall j = N_{\text{small}} + 1, \dots, h_{1,1}(X), \end{cases} \quad (\text{A.19})$$

where $g_1(\tau_i)$, $i = 1, \dots, N_{\text{small}}$, is an homogeneous function of degree 1. This means that the small 4-cycles will stay small and the large ones will just be perturbed by the small ones. We are therefore in the same situation and the physics should not change. We conclude that the inverse Kähler matrix should not change but both the intersection numbers and the form of the volume can indeed vary. In fact, for an arbitrary Calabi-Yau, the elements of the inverse Kähler matrix are given by:

$$K_{ij}^{-1} = -\frac{4}{9} \mathcal{V} k_{ijk} t^k + 4\tau_i \tau_j, \quad (\text{A.20})$$

and so we see that in order to keep the form of K_{ij}^{-1} unaltered, the quantity $(k_{ijk} t^k)$ has not to vary, but the intersection numbers k_{ijk} can indeed change. This is the main reason why we need to put our conditions on the K_{ij}^{-1} .

Let us illustrate this statement in the explicit example of the orientifold of the Calabi-Yau threefold embedded in $\mathbb{C}P^4_{[1,1,1,6,9]}$ whose volume in terms of 2-cycle volumes is given by

$$\mathcal{V} = 6 (t_5^3 + t_4^3). \quad (\text{A.21})$$

The corresponding 4-cycle volumes look like

$$\begin{cases} \tau_4 = \frac{\partial \mathcal{V}}{\partial t_4} = 18t_4^2, \\ \tau_5 = \frac{\partial \mathcal{V}}{\partial t_5} = 18t_5^2, \end{cases} \iff \begin{cases} t_4 = -\frac{\sqrt{\tau_4}}{3\sqrt{2}}, \\ t_5 = +\frac{\sqrt{\tau_5}}{3\sqrt{2}}, \end{cases} \quad (\text{A.22})$$

and the volume in terms of the 4-cycles is

$$\mathcal{V} = \frac{1}{9\sqrt{2}} (\tau_5^{3/2} - \tau_4^{3/2}). \quad (\text{A.23})$$

Finally the superpotential reads

$$W = W_0 + A_4 e^{-a_4 T_4} + A_5 e^{-a_5 T_5}. \quad (\text{A.24})$$

It exists a well defined large volume limit when the 4-cycle τ_4 is kept small and τ_5 is sent to infinity. In this case the superpotential can be approximated as

$$W \simeq W_0 + A_4 e^{-a_4 T_4}. \quad (\text{A.25})$$

We can now perform the following field redefinition

$$\begin{cases} \tau_4 \longrightarrow \tau'_4 = \tau_4, \\ \tau_5 \longrightarrow \tau'_5 = \tau_5 + \tau_4, \end{cases} \quad (\text{A.26})$$

which will not change the form of W (A.25). However now the volume reads

$$\mathcal{V}' = \frac{1}{9\sqrt{2}} \left((\tau'_5 - \tau'_4)^{3/2} - \tau_4'^{3/2} \right) \simeq \frac{1}{9\sqrt{2}} \left(\tau_5'^{3/2} - \tau_4' \sqrt{\tau_5'} - \tau_4'^{3/2} \right), \quad (\text{A.27})$$

which is clearly different from the initial form (A.23). This means that also the intersection numbers are different. However the elements of the inverse Kähler matrix do not change. In particular we are interested in $K_{44}^{-1} \simeq \mathcal{V} \sqrt{\tau_4}$ in this case as τ_4 is the small cycle. Its form stays unchanged since $K_{44}'^{-1} \simeq \mathcal{V}' \sqrt{\tau_4'}$. From (A.20), this implies that

$$\sqrt{\tau_4'} = (k'_{44k} t'^k) = k'_{444} t'_4 + k'_{445} t'_5, \quad (\text{A.28})$$

and one would tend to say that k'_{445} has to be zero but we know from (A.27) that this is definitely not the case. This means that the field redefinition (A.26) will have the corresponding redefinition of the 2-cycle volumes which will produce t'_4 and t'_5 that are both large 2-cycles but such that the combination $(k'_{444} t'_4 + k'_{445} t'_5)$ stays small. This is the reason why the form of the inverse Kähler matrix is left invariant while the intersection numbers do vary. This can be rephrased by saying that if τ_j is a small 4-cycle, in general the corresponding t_j has not to be a small 2-cycle and viceversa. This is clear without the need to perform any field redefinition in the case of the Calabi-Yau K3 fibration described by the degree 12 hypersurface in $\mathbb{C}P^4_{[1,1,2,2,6]}$ whose overall volume in terms of 2-cycle volumes is

$$\mathcal{V} = t_1 t_2^2 + \frac{2}{3} t_2^3, \quad (\text{A.29})$$

giving relations between the 2- and 4-cycle volumes,

$$\begin{aligned} \tau_1 &= t_2^2, & \tau_2 &= 2t_2(t_1 + t_2), \\ t_2 &= \sqrt{\tau_1}, & t_1 &= \frac{\tau_2 - 2\tau_1}{2\sqrt{\tau_1}}, \end{aligned} \quad (\text{A.30})$$

that allow us to write

$$\mathcal{V} = \frac{1}{2} \sqrt{\tau_1} \left(\tau_2 - \frac{2}{3} \tau_1 \right). \quad (\text{A.31})$$

Looking at (A.31) we see that the large volume limit can be performed keeping τ_1 small and taking τ_2 large. Nonetheless, as it is clear from (A.30), t_1 is big whereas t_2 is small. Therefore it is impossible to impose that the quantity $k_{jji} t^i$ does not introduce any volume dependence by requiring that some intersection numbers have to vanish.

Going back to the proof of the LARGE Volume Claim for $N_{\text{small}} = 1$, let us assume that we are in case (3), so that (A.15) becomes

$$V \simeq \frac{\sqrt{\tau_1}}{\mathcal{V}} A_1^2 a_1^2 e^{-2a_1 \tau_1} - \frac{W_0}{\mathcal{V}^2} A_1 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2, \quad (\text{A.32})$$

and when we take the decompactification limit given by

$$\mathcal{V} \rightarrow \infty \quad \text{with} \quad e^{a_1 \tau_1} = \frac{\mathcal{V}}{W_0}, \quad (\text{A.33})$$

all the terms in (A.32) have the same volume dependence

$$V \simeq \frac{W_0^2}{\mathcal{V}^3} \left[(A_1 a_1 - \sqrt{\tau_1}) A_1 a_1 \sqrt{\tau_1} + \hat{\xi} \right]. \quad (\text{A.34})$$

We can finally express the scaling behaviour of (A.34) as

$$V \simeq \frac{W_0^2}{\mathcal{V}^3} \left(C_1 \sqrt{\ln \mathcal{V}} - C_2 \ln \mathcal{V} + \hat{\xi} \right), \quad (\text{A.35})$$

where C_1 and C_2 are positive constants of order 1 for natural values of the parameter $A_1 \simeq 1$. We conclude that at large volume, the dominant term in (A.35) is the second one and the scalar potential approaches zero from below. It is now straightforward to argue that there must exist an exponentially large volume AdS minimum.

In fact, at smaller volumes the dominant term in the potential (A.35) is either the first or the third term, depending on the exact value of the constants. Both are positive as we have explained above. Thus at smaller volumes the potential is positive, and so since it must go to zero at infinity from below, there must exist a local AdS minimum along the direction in Kähler moduli space where the volume changes.

One could argue that if at smaller volumes the dominant term in (A.35) is the first one, then there is no need to require $h_{2,1}(X) > h_{1,1}(X)$. In reality this is wrong, because $\xi < 0$ could still ruin the presence of the large volume minimum. In fact we can rewrite the full scalar potential (A.32) as

$$V = \frac{\lambda}{\mathcal{V}} \sqrt{\tau_1} e^{-2a_1 \tau_1} - \frac{\mu}{\mathcal{V}^2} \tau_1 e^{-a_1 \tau_1} + \frac{\hat{\xi} W_0^2}{\mathcal{V}^3}, \quad (\text{A.36})$$

where λ , μ and ν are positive constants depending on the exact details of this model. We can integrate out τ_1 , so ending up with just a potential for \mathcal{V} . Under the requirement $a_1 \tau_1 \gg 1$, $\partial V / \partial \tau_1 = 0$ gives

$$e^{-a_1 \tau_1} = \frac{\mu \sqrt{\tau_1}}{2 \lambda \mathcal{V}}, \quad (\text{A.37})$$

which substituted back in (A.36) yields

$$V = -\frac{1}{2} \frac{\mu^2 \tau_1^{3/2}}{\lambda \mathcal{V}^3} + \frac{\hat{\xi} W_0^2}{\mathcal{V}^3} \sim \frac{-(\ln \mathcal{V})^{3/2} + \hat{\xi} W_0^2}{\mathcal{V}^3} \quad (\text{A.38})$$

and is straightforward to see that we need $\hat{\xi} > 0$ even though the dominant term at small volumes in (A.36) is the first one.

It remains to show that the scalar potential has also a minimum in the other direction of the moduli space. In order to do that, let us fix the Calabi-Yau volume and see what happens if we vary the small Kähler modulus along that surface. Then as one approaches the walls of the Kähler cone the positive first term in (A.32) dominates since it has the fewest powers of volume in the denominator and the exponential contributions of the modulus that is becoming small cannot be neglected. Thus at large overall volume, we expect the potential to grow in the positive direction towards the walls of the Kähler cone.

On the other hand, when the small Kähler modulus becomes bigger then the dominant term in (A.32) is the positive $V_{(\alpha')}$ due to the exponential suppressions in the other two

terms. Given that the potential is negative along the special direction in the moduli space that we have identified and eventually raises to be positive or to vanish in the other direction, we are sure to have an AdS exponentially large volume minimum.

Since $V \sim \mathcal{O}(1/\mathcal{V}^3)$ at the minimum, while $-3e^K |W|^2 \sim \mathcal{O}(1/\mathcal{V}^2)$, it is clear that this minimum is non-supersymmetric. We can heuristically see why the minimum we are arguing for can be at exponentially large volume. The naive measure of its location is the value of the volume at which the negative term in (A.35) becomes dominant. As this occurs only when $(\ln \mathcal{V})$ is large, we expect to find the vacuum at large values of $(\ln \mathcal{V})$.

In reality the way in which we have taken the limit (A.33), tells us how the volume will scale, even though this can very well not be the correct location of the minimum

$$\mathcal{V} \sim W_0 e^{a_1 \tau_1}. \tag{A.39}$$

Looking at (A.39) we realise that W_0 cannot be too small, otherwise we would get a small volume minimum merging with the KKL_T one and our derivation would not make sense anymore. However W_0 is multiplying an exponential, which means that in order to destroy the large volume minimum W_0 has to be really small.

Furthermore, we stress that there is no need to require $h_{2,1}(X) \gg h_{1,1}(X)$ instead of just $h_{2,1}(X) > h_{1,1}(X)$. In fact, in this proof we have used $\hat{\xi}$ instead of ξ , so obscuring the presence of any factors of g_s but, as it is written explicitly in (2.11), in Einstein frame $\hat{\xi}$ is equivalent to $\xi/g_s^{3/2}$. Therefore if we just have $h_{2,1}(X) > h_{1,1}(X)$ then we can still adjust g_s to make sure that the AdS minimum is indeed at large volume.

We are now able to understand what happens if K_{11}^{-1} is not in case (3). For example, when it is in case (4) then the first term in (A.15) beats all the other ones and along the direction (A.33) the scalar potential either presents a runaway or has no minimum at large volume depending on the exact value of α .

Moreover if K_{11}^{-1} is in case (1) or (2) then the first term in (A.15) is subleading with respect to the other two and at leading order in the volume, the scalar potential looks like

$$V \simeq -\frac{W_0}{\mathcal{V}^2} A_1 a_1 \tau_1 e^{-a_1 \tau_1} + \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \tag{A.40}$$

The minimisation equation for τ_1 , $\partial V/\partial \tau_1 = 0$, admits the only possible solution $a_1 \tau_1 = 1$, that has to be discarded since we need $a_1 \tau_1 \gg 1$ in order to avoid higher instanton corrections.

Finally, let us argue in favour of the last statement of Claim 1. At the end of all our derivation we realised that the small Kähler modulus τ_1 plus a particular combination which is the overall volume are stabilised. Therefore we have in general $(N_{\text{small}} + 1)$ fixed Kähler moduli and is straightforward to see that if we have just one big Kähler modulus then it will be fixed, whereas if we have more than one big Kähler moduli, only one of them will be fixed and the others will give rise to exactly $(h_{1,1}(X) - N_{\text{small}} - 1)$ flat directions. This is because they do not appear in the non-perturbative corrections to the superpotential due to the limit (2.10). This terminates our proof of the LARGE Volume Claim for $N_{\text{small}}=1$.

■

A.2 Proof for $N_{\text{small}} > 1$

Proof. (LARGE Volume Claim for $N_{\text{small}} > 1$) When $N_{\text{small}} > 1$ the situation is more involved due to the presence of cross terms. However $V_{(\alpha')}$ has still the form (A.2). Without loss of generality, we shall focus on the case with $N_{\text{small}} = 2$ Kähler moduli, which we will call τ_1 and τ_2 . V_{np1} generalises to

$$\begin{aligned} V_{np1} &\xrightarrow{\nu \rightarrow \infty} e^K \sum_{j,k=1}^2 K_{jk}^{-1} a_j A_j a_k \bar{A}_k e^{-(a_j T_j + a_k \bar{T}_k)} \\ &= e^K \left\{ \sum_{j=1}^2 K_{jj}^{-1} a_j^2 |A_j|^2 e^{-2a_j \tau_j} + K_{12}^{-1} a_1 A_1 a_2 \bar{A}_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} e^{i(a_2 b_2 - a_1 b_1)} \right\}. \end{aligned} \quad (\text{A.41})$$

In order to consider separately the axion-dependent part of V_{np1} , we write

$$V_{np1} = V_{np1}^{\text{real}} + V_{np1}^{\text{AX}}. \quad (\text{A.42})$$

Switching to the study of V_{np2} , we find that

$$V_{np2} \xrightarrow{\nu \rightarrow \infty} -e^K \sum_{k=1}^{h_{1,1}} \sum_{j=1}^2 K_{jk}^{-1} \left[\left(a_j A_j e^{-a_j \tau_j} e^{-ia_j b_j} \bar{W} \partial_{\bar{T}_k} K \right) + \left(a_j \bar{A}_j e^{-a_j \tau_j} e^{+ia_j b_j} W \partial_{T_k} K \right) \right], \quad (\text{A.43})$$

where we have used the fact that $K_{jk}^{-1} = K_{kj}^{-1}$. Equation (A.43) can be rewritten as

$$\begin{aligned} V_{np2} &\xrightarrow{\nu \rightarrow \infty} -e^K \sum_{k=1}^{h_{1,1}} \sum_{j=1}^2 K_{jk}^{-1} (\partial_{T_k} K) a_j e^{-a_j \tau_j} \left[\left(A_j \bar{W} e^{-ia_j b_j} \right) + \left(\bar{A}_j W e^{+ia_j b_j} \right) \right] \\ &= \sum_{j=1}^2 \left(X_j e^{+ia_j b_j} + \bar{X}_j e^{-ia_j b_j} \right), \end{aligned} \quad (\text{A.44})$$

where

$$X_j \equiv -e^K K_{jk}^{-1} (\partial_{T_k} K) a_j \bar{A}_j W e^{-a_j \tau_j}. \quad (\text{A.45})$$

We note that for a general Calabi-Yau, the following relation holds:

$$K_{jk}^{-1} (\partial_{T_k} K) = -2\tau_j, \quad (\text{A.46})$$

and thus the definition (A.45) can be simplified to

$$X_j \equiv 2e^K a_j \tau_j |A_j| e^{-i\vartheta_j} |W| e^{i\vartheta_W} e^{-a_j \tau_j} = |X_j| e^{i(\vartheta_W - \vartheta_j)}. \quad (\text{A.47})$$

Therefore

$$V_{np2} \xrightarrow{\nu \rightarrow \infty} \sum_{j=1}^2 |X_j| \left(e^{+i(\vartheta_W - \vartheta_j + a_j b_j)} + e^{-i(\vartheta_W - \vartheta_j + a_j b_j)} \right). \quad (\text{A.48})$$

Let us now reconsider V_{np1}^{AX} , which we had set aside for a moment. It can be rewritten as

$$V_{np1}^{\text{AX}} = e^K K_{12}^{-1} a_1 a_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} \left(A_1 \bar{A}_2 e^{i(a_2 b_2 - a_1 b_1)} + A_2 \bar{A}_1 e^{-i(a_2 b_2 - a_1 b_1)} \right), \quad (\text{A.49})$$

	(a)	(b)	(c)	(d)
$\cos \psi_1$	+1	-1	+1	-1
$\cos \psi_2$	+1	-1	-1	+1
$\cos \psi_{12}$	+1	+1	-1	-1

Table 1: Points where the axion potential vanishes.

and finally as

$$V_{np1}^{AX} = Y_{12} e^{i(a_2 b_2 - a_1 b_1)} + \bar{Y}_{12} e^{-i(a_2 b_2 - a_1 b_1)}, \quad (\text{A.50})$$

where

$$Y_{12} \equiv e^K K_{12}^{-1} a_1 a_2 A_1 \bar{A}_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} = |Y_{12}| e^{i(\vartheta_1 - \vartheta_2)}. \quad (\text{A.51})$$

Therefore

$$V_{np1}^{AX} = |Y_{12}| \left(e^{i(\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1)} + e^{-i(\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1)} \right). \quad (\text{A.52})$$

Thus, the full axion-dependent part of the scalar potential V_{AX} looks like

$$V_{AX} = V_{np2} + V_{np1}^{AX} = 2 \sum_{j=1}^2 |X_j| \cos(\vartheta_W - \vartheta_j + a_j b_j) + 2 |Y_{12}| \cos(\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1). \quad (\text{A.53})$$

A.2.1 Axion stabilisation

V_{AX} is a scalar function of the axions b_1 and b_2 whereas ϑ_1 , ϑ_2 and ϑ_W are to be considered just as parameters. In order to find a minimum for V_{AX} let us set its gradient to zero

$$\begin{cases} \partial V_{AX} / \partial b_1 = 0 & \iff |X_1| \sin(\vartheta_W - \vartheta_1 + a_1 b_1) = + |Y_{12}| \sin(\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1), \\ \partial V_{AX} / \partial b_2 = 0 & \iff |X_2| \sin(\vartheta_W - \vartheta_2 + a_2 b_2) = - |Y_{12}| \sin(\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1), \end{cases} \quad (\text{A.54})$$

The solution of (A.54) is given by

$$\begin{cases} \psi_1 \equiv (\vartheta_W - \vartheta_1 + a_1 b_1) = p_1 \pi, & p_1 \in \mathbb{Z}, \\ \psi_2 \equiv (\vartheta_W - \vartheta_2 + a_2 b_2) = p_2 \pi, & p_2 \in \mathbb{Z}, \end{cases} \quad (\text{A.55})$$

and

$$\psi_{12} \equiv (\vartheta_1 - \vartheta_2 + a_2 b_2 - a_1 b_1) = p_{12} \pi, \quad p_{12} \in \mathbb{Z}. \quad (\text{A.56})$$

From (A.55) equation (A.56) requires $p_{12} = p_2 - p_1$. Let us summarise the points where the gradient of the axion potential is zero in table 1.

We notice that the phases of W , A_1 and A_2 will not enter into V_{np2} once the axions have been properly minimised and so, without loss of generality, we can consider W , A_1 and $A_2 \in \mathbb{R}^+$ from now on.

We have still to check the Hessian matrix evaluated at b_1 and b_2 as given in (A.55) and require it to be positive definite. Its diagonal elements are given by

$$\begin{cases} \partial^2 V_{AX} / \partial b_1^2 = -2a_1^2 (|X_1| \cos \psi_1 + |Y_{12}| \cos \psi_{12}), \\ \partial^2 V_{AX} / \partial b_2^2 = -2a_2^2 (|X_2| \cos \psi_2 + |Y_{12}| \cos \psi_{12}), \end{cases} \quad (\text{A.57})$$

whereas the non-diagonal ones read

$$\frac{\partial^2 V_{AX}}{\partial b_2 \partial b_1} = \frac{\partial^2 V_{AX}}{\partial b_1 \partial b_2} = 2a_1 a_2 |Y_{12}| \cos \psi_{12}. \quad (\text{A.58})$$

We can diagonalise the Hessian \mathcal{H} to the identity by decomposing it *a la* Choleski into $\mathcal{H} = \mathcal{U}^T \mathbb{I} \mathcal{U}$, where the elements of the upper triangular matrix \mathcal{U} are given by the following recursive relations:

$$\mathcal{U}_{11}^2 = -2a_1^2 (|X_1| \cos \psi_1 + |Y_{12}| \cos \psi_{12}), \quad (\text{A.59})$$

$$\mathcal{U}_{12} = \frac{2a_1 a_2 |Y_{12}| \cos \psi_{12}}{\mathcal{U}_{11}}, \quad (\text{A.60})$$

$$\mathcal{U}_{22}^2 = -\mathcal{U}_{12}^2 - 2a_2^2 (|X_2| \cos \psi_2 + |Y_{12}| \cos \psi_{12}), \quad (\text{A.61})$$

with $\mathcal{U}_{21} = 0$. Determining if the Hessian is positive definite is equal to checking that \mathcal{U} is a real matrix. Looking at (A.60) we realise that \mathcal{U}_{12} is automatically real if \mathcal{U}_{11} is real. Hence we have to make sure just that both $\mathcal{U}_{11}^2 > 0$ and $\mathcal{U}_{22}^2 > 0$. When we analyse the cases listed in table (1) we realise that:

- (a) can never be a minimum since $\mathcal{U}_{11}^2 < 0$; in reality it turns out to be always a maximum,
- (b) is a minimum only if $|X_1| > |Y_{12}|$ and $|X_1| |X_2| > |Y_{12}| (|X_1| + |X_2|)$,
- (c) is a minimum only if $|Y_{12}| > |X_1|$ and $|X_2| |Y_{12}| > |X_1| (|X_2| + |Y_{12}|)$,
- (d) is a minimum only if $|Y_{12}| |X_1| > |X_2| (|X_1| + |Y_{12}|)$,

where, according to the definitions (A.47) and (A.51), we have

$$\begin{cases} |X_1| = 2 |A_1| a_1 \tau_1 |W| \frac{e^{-a_1 \tau_1}}{\mathcal{V}^2}, \\ |X_2| = 2 |A_2| a_2 \tau_2 |W| \frac{e^{-a_2 \tau_2}}{\mathcal{V}^2}, \\ |Y_{12}| = K_{12}^{-1} |A_1| a_1 |A_2| a_2 \frac{e^{-a_1 \tau_1} e^{-a_2 \tau_2}}{\mathcal{V}^2}. \end{cases} \quad (\text{A.62})$$

In order to study the cases (b), (c) and (d), it is therefore crucial to know the order of magnitude of the two exponentials $e^{-a_1 \tau_1}$ and $e^{-a_2 \tau_2}$ given by their scaling behaviour in the volume. This depends on the direction we are looking at to find the minimum in the large volume limit which can be performed in three different ways:

$$\begin{aligned} I) \mathcal{V} &\sim e^{\gamma a_1 \tau_1} \sim e^{\gamma a_2 \tau_2}, \quad \gamma \in \mathbb{R}^+ \\ II) \mathcal{V} &\sim e^{\beta a_1 \tau_1} \sim e^{\gamma a_2 \tau_2}, \quad \beta < \gamma, \quad \gamma, \beta \in \mathbb{R}^+, \\ III) \mathcal{V} &\sim e^{\beta a_1 \tau_1} \sim e^{\gamma a_2 \tau_2}, \quad \beta > \gamma, \quad \gamma, \beta \in \mathbb{R}^+. \end{aligned} \quad (\text{A.63})$$

Finally we need also to know the form of K_{12}^{-1} . We can classify its behaviour according to the volume dependence of the quantity $k_{12} t^j$ and find 4 different cases:

1. $k_{12} t^j = 0$ or $k_{12} t^j = \frac{f(\tau_1, \tau_2)}{\mathcal{V}^\alpha}$, $\alpha \geq 1 \implies K_{12}^{-1} = \tau_1 \tau_2$;

2. $k_{12j}t^j = \frac{g_\gamma(\tau_1, \tau_2)}{\mathcal{V}^\alpha}$, $0 < \alpha < 1$, g homogeneous function of degree $\gamma = \frac{1+3\alpha}{2} \implies K_{12}^{-1} = \mathcal{V}^\alpha g_{2-3\alpha/2}(\tau_1, \tau_2)$, $0 < \alpha < 1$;
3. $k_{12j}t^j = f_{1/2}(\tau_1, \tau_2)$, f homogeneous function of degree $1/2 \implies K_{12}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$;
4. $k_{12j}t^j = \mathcal{V}^\alpha h_\beta(\tau_1, \tau_2)$, $\alpha > 0$, h homogeneous function of degree $\beta = \frac{1-3\alpha}{2} \implies K_{12}^{-1} = \mathcal{V}^\alpha h_{2-3\alpha/2}(\tau_1, \tau_2)$, $\alpha > 1$.

Let us now focus on the axion minimisation by analysing each of these 4 cases in full detail. For each case we will have to study if the inequalities (b), (c) and (d) admit a solution for any of the three possible ways to take the large volume limit as expressed in (A.63). We will always consider natural values of the parameters $|A_1| \simeq |A_2| \simeq |W| \simeq 1$.

From (I) of (A.63), we can immediately realise that, regardless of the form of K_{12}^{-1} , at large volume $|X_1|$ and $|X_2|$ have the same scaling with the volume and so we can denote both of them as $|X|$. It is then straightforward to see that both the second (c)-condition and the (d)-condition can never be satisfied. In fact they take the form

$$|X| |Y_{12}| > |X| |Y_{12}| + |X|^2, \quad (\text{A.64})$$

which is manifestly an absurd. This implies that neither (c) nor (d) can be a minimum along the direction (I) for any value of K_{12}^{-1} . A further analysis reveals that the points (c) and (d) can never be maxima so since we proved that they cannot be minima, they are forced to be saddle points. We do not present the details of this analysis here since it is not important for our reasoning. On the other hand, the first (b)-condition is automatically satisfied if the second one is true since it reduces to

$$|X|^2 > 2|X| |Y_{12}| \iff |X| > 2|Y_{12}|. \quad (\text{A.65})$$

From (II) of (A.63), we also notice that, regardless of the form of K_{12}^{-1} , at large volume $|X_1| < |X_2|$ since $\beta < \gamma$. It is then straightforward to see that in this situation the (d)-condition can never be satisfied. This implies that (d) is always a saddle point along the direction (II) for any value of K_{12}^{-1} .

Furthermore (III) of (A.63) implies that, regardless of the form of K_{12}^{-1} , at large volume $|X_1| > |X_2|$ as $\beta > \gamma$. Then we immediately see that the second (c)-condition can never be satisfied. Therefore (c) is always a saddle point along the direction (III) for any value of K_{12}^{-1} .

Case (1): $K_{12}^{-1} \simeq \tau_1 \tau_2$

- direction (I)

The volume dependence of the parameters (A.62) is

$$\begin{cases} |X_1| \simeq |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(2+2/\gamma)} (\ln \mathcal{V})^2. \end{cases} \quad (\text{A.66})$$

Looking at (A.66), we realise that at large volume $|X_1| > 2|Y_{12}|$. Therefore the second (b)-condition (A.65) is satisfied and (b) is a minimum of the axion potential.

	(I)	(II)	(III)
(a)	max	max	max
(b)	min	min	min
(c)	saddle	saddle	saddle
(d)	saddle	saddle	saddle

Table 2: Summary of axion stabilisation results for case (1).

- direction (II)

The parameters (A.62) now read

$$\begin{cases} |X_1| \simeq \mathcal{V}^{-(2+1/\beta)} \ln \mathcal{V}, \\ |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(2+1/\beta+1/\gamma)} (\ln \mathcal{V})^2. \end{cases} \quad (\text{A.67})$$

Looking at (A.67), we realise that at large volume $|X_1| > |Y_{12}|$, which implies that (c) is a saddle point. Thus the first (b)-condition is satisfied and the second becomes

$$\frac{(\ln \mathcal{V})^2}{\mathcal{V}^{4+1/\beta+1/\gamma}} > \frac{(\ln \mathcal{V})^2}{\mathcal{V}^{2+1/\beta+1/\gamma}} \left(\frac{\ln \mathcal{V}}{\mathcal{V}^{2+1/\beta}} + \frac{\ln \mathcal{V}}{\mathcal{V}^{2+1/\gamma}} \right) \underset{1/\beta > 1/\gamma}{\simeq} \frac{(\ln \mathcal{V})^3}{\mathcal{V}^{4+1/\beta+2/\gamma}}, \quad (\text{A.68})$$

which is true at large volume for values of $\gamma > \beta > 0$ not extremely big. Thus (b) is a minimum of the axion potential.

- direction (III)

The parameters (A.62) take the same form as (A.67) but now with $\beta > \gamma$. We have still $|X_1| > |Y_{12}|$, which implies that the first (b)-condition is satisfied. The second looks like

$$\frac{(\ln \mathcal{V})^2}{\mathcal{V}^{4+1/\beta+1/\gamma}} > \frac{(\ln \mathcal{V})^2}{\mathcal{V}^{2+1/\beta+1/\gamma}} \left(\frac{\ln \mathcal{V}}{\mathcal{V}^{2+1/\beta}} + \frac{\ln \mathcal{V}}{\mathcal{V}^{2+1/\gamma}} \right) \underset{1/\beta < 1/\gamma}{\simeq} \frac{(\ln \mathcal{V})^3}{\mathcal{V}^{4+2/\beta+1/\gamma}}, \quad (\text{A.69})$$

which is true at large volume. Thus (b) is a minimum of the axion potential. On the contrary the simplified (d)-condition reads

$$\frac{\ln \mathcal{V}}{\mathcal{V}^{1/\beta}} > 1 + \frac{\ln \mathcal{V}}{\mathcal{V}^{1/\gamma}} \quad (\text{A.70})$$

which at large volume is clearly false for values of $\beta > \gamma > 0$ not extremely big. It follows that in this case (d) is a saddle point.

The results found in case (1) are summarised in table 2.

Case (2): $K_{12}^{-1} = \mathcal{V}^\alpha g_{2-3\alpha/2}(\tau_1, \tau_2)$, $0 < \alpha < 1$

- direction (I)

The volume dependence of the parameters (A.62) now reads

$$\begin{cases} |X_1| \simeq |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(2+2/\gamma-\alpha)} (\ln \mathcal{V})^{2-3\alpha/2}. \end{cases} \quad (\text{A.71})$$

Substituting the expressions (A.71) in (A.65) we find

$$1 > 2 \frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\gamma-\alpha}}, \quad (\text{A.72})$$

which at large volume is true if $\alpha < (1/\gamma)$, false if $\alpha < (1/\gamma)$ or $\alpha = (1/\gamma) \leq 2/3$. On the contrary, for $2/3 < \alpha = (1/\gamma) < 1$ the minimum is present. Thus (b) can be a minimum of the axion potential.

- direction (II)

The parameters (A.62) now read

$$\begin{cases} |X_1| \simeq \mathcal{V}^{-(2+1/\beta)} \ln \mathcal{V}, \\ |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(2+1/\beta+1/\gamma-\alpha)} (\ln \mathcal{V})^{2-3\alpha/2}. \end{cases} \quad (\text{A.73})$$

The first (b)-condition becomes

$$1 > \frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\gamma-\alpha}}, \quad (\text{A.74})$$

that is satisfied if either $\alpha < (1/\gamma)$ or $2/3 < \alpha = (1/\gamma) < 1$. Otherwise (A.74) is false unless $\alpha = (1/\gamma) = 2/3$ in which case we cannot conclude anything just looking at the volume dependence. However the second (b)-condition reads

$$1 > (\ln \mathcal{V})^{1-3\alpha/2} \left(\frac{1}{\mathcal{V}^{1/\beta-\alpha}} + \frac{1}{\mathcal{V}^{1/\gamma-\alpha}} \right), \quad (\text{A.75})$$

which is definitely true at large volume if $\alpha < (1/\gamma)$ or $2/3 < \alpha = (1/\gamma) < 1$. On the contrary, in the case $\alpha = (1/\gamma) = 2/3 \Leftrightarrow (1/\beta - 2/3) > 0$, (A.75) becomes

$$1 > 1 + \frac{1}{\mathcal{V}^{1/\beta-2/3}}, \quad (\text{A.76})$$

which is clearly impossible. Thus (b) can be a minimum of the axion potential. We need now just to study the case (c) for which the first inequality is

$$1 < \frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\gamma-\alpha}}, \quad (\text{A.77})$$

that is satisfied if either $(1/\gamma) < \alpha < 1$ or $\alpha = (1/\gamma) < 2/3$. Otherwise (A.77) is false unless $\alpha = (1/\gamma) = 2/3$ in which case we cannot conclude anything just looking at the volume dependence. However the second (c)-condition can be simplified to give

$$\frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\gamma-\alpha}} > 1 + \frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\beta-\alpha}}, \quad (\text{A.78})$$

which is clearly satisfied at large volume if either $(1/\gamma) < \alpha < 1$ or $\alpha = (1/\gamma) < 2/3$. On the contrary when $\alpha = (1/\gamma) = 2/3 \Leftrightarrow (1/\beta - 2/3) > 0$, (A.78) becomes

$$1 > 1 + \frac{1}{\mathcal{V}^{1/\beta-2/3}}, \quad (\text{A.79})$$

which is clearly false.

	(I)	(II)	(III)
(a)	max	max	max
(b)	$\begin{cases} \alpha < 1/\gamma \text{ min,} \\ 2/3 < \alpha = 1/\gamma < 1 \text{ min,} \\ \alpha = 1/\gamma \leq 2/3 \text{ saddle,} \\ \alpha > 1/\gamma \text{ saddle.} \end{cases}$	$\begin{cases} \alpha < 1/\gamma \text{ min,} \\ 2/3 < \alpha = 1/\gamma < 1 \text{ min,} \\ \alpha = 1/\gamma \leq 2/3 \text{ saddle,} \\ \alpha > 1/\gamma \text{ saddle.} \end{cases}$	$\begin{cases} \alpha < 1/\beta \text{ min,} \\ 2/3 < \alpha = 1/\beta < 1 \text{ min,} \\ \alpha = 1/\beta \leq 2/3, \text{ saddle,} \\ \alpha > 1/\beta \text{ saddle.} \end{cases}$
(c)	saddle	$\begin{cases} \alpha < 1/\gamma \text{ saddle,} \\ 2/3 \leq \alpha = 1/\gamma < 1 \text{ saddle,} \\ \alpha = 1/\gamma < 2/3 \text{ min,} \\ 1/\gamma < \alpha < 1 \text{ min.} \end{cases}$	saddle
(d)	saddle	saddle	$\begin{cases} \alpha < 1/\beta \text{ saddle,} \\ 2/3 \leq \alpha = 1/\beta < 1 \text{ saddle,} \\ \alpha = 1/\beta < 2/3, \text{ min,} \\ \alpha > 1/\beta \text{ min.} \end{cases}$

Table 3: Summary of axion stabilisation results for case (2).

- direction (III)

The parameters (A.62) assume the same form as (A.73) but now with $\beta > \gamma$. Following lines of reasoning similar to those used for direction (II), we get the results summarised in table 3 along with all the others for case (2).

Case (3): $K_{12}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$

- direction (I)

The volume dependence of the parameters (A.62) now looks like

$$\begin{cases} |X_1| \simeq |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(1+2/\gamma)} \sqrt{\ln \mathcal{V}}. \end{cases} \quad (\text{A.80})$$

Substituting the expressions (A.80) in (A.65) we find

$$\frac{\ln \mathcal{V}}{\mathcal{V}^{2+1/\gamma}} > 2 \frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}^{1+2/\gamma}}, \quad (\text{A.81})$$

which at large volume is true if $\gamma \leq 1$, false if $\gamma > 1$. Thus (b) can be a minimum of the axion potential.

- direction (II)

The parameters (A.62) now read

$$\begin{cases} |X_1| \simeq \mathcal{V}^{-(2+1/\beta)} \ln \mathcal{V}, \\ |X_2| \simeq \mathcal{V}^{-(2+1/\gamma)} \ln \mathcal{V}, \\ |Y_{12}| \simeq \mathcal{V}^{-(1+1/\beta+1/\gamma)} \sqrt{\ln \mathcal{V}}. \end{cases} \quad (\text{A.82})$$

Looking at (A.82), we realise that the first (b)-condition becomes

$$\frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}} > \frac{1}{\mathcal{V}^{1/\gamma}}, \quad (\text{A.83})$$

	(I)	(II)	(III)
(a)	max	max	max
(b)	$\begin{cases} 0 < \gamma \leq 1 \text{ min,} \\ \gamma > 1 \text{ saddle.} \end{cases}$	$\begin{cases} 0 < \gamma \leq 1 \text{ min,} \\ \gamma > 1 \text{ saddle.} \end{cases}$	$\begin{cases} 0 < \gamma < \beta \leq 1 \text{ min,} \\ \beta > 1 \text{ saddle.} \end{cases}$
(c)	saddle	$\begin{cases} 0 < \gamma \leq 1 \text{ saddle,} \\ \gamma > 1 \text{ min.} \end{cases}$	saddle
(d)	saddle	saddle	$\begin{cases} 0 < \gamma < \beta \leq 1 \text{ saddle,} \\ \beta > 1 \text{ min.} \end{cases}$

Table 4: Summary of axion stabilisation results for case (3).

which is satisfied only if $\gamma \leq 1$. Viceversa the first (c)-condition is satisfied only for $\gamma > 1$. Let us check now the validity of the second (b)-condition which reads

$$\frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}} > \frac{1}{\mathcal{V}^{1/\beta}} + \frac{1}{\mathcal{V}^{1/\gamma}}, \tag{A.84}$$

which, at large volume, is automatically true if $\gamma \leq 1$. The second (c)-condition is also correctly satisfied for $\gamma > 1$ since it reads

$$\frac{1}{\mathcal{V}^{1/\gamma}} > \frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}} + \frac{1}{\mathcal{V}^{1/\beta}}. \tag{A.85}$$

Thus both (b) and (c) can be a minimum of the axion potential.

- direction (III)

The parameters (A.62) assume the same form as (A.82) but now with $\beta > \gamma$. The inequality corresponding to the (d)-condition reads

$$\frac{1}{\mathcal{V}^{1/\beta}} > \frac{1}{\mathcal{V}^{1/\gamma}} + \frac{\sqrt{\ln \mathcal{V}}}{\mathcal{V}}, \tag{A.86}$$

and becomes true if $\beta > 1$. Moreover, the first (b)-condition (A.83) is again satisfied for $\gamma \leq 1$. On the other hand, the second looks like (A.84) and now, at large volume, is true only if $\beta \leq 1$, which implies correctly $\gamma \leq 1$ since in this case $\gamma < \beta$. It follows that both (b) and (d) can be a minimum.

The results found in case (3) are summarised in table 4.

Case (4): $K_{12}^{-1} = \mathcal{V}^\alpha h_{2-3\alpha/2}(\tau_1, \tau_2)$, $\alpha > 1$

- direction (I)

The volume dependence of the parameters (A.62) is given again by (A.71) and (A.65) takes the same form as the inequality (A.72) which at large volume is true if $\alpha < 1/\gamma$, false if $\alpha > 1/\gamma$. The situation $\alpha = 1/\gamma$ is more involved and (A.72) simplifies to

$$1 > 2(\ln \mathcal{V})^{1-3\alpha/2}, \tag{A.87}$$

which gives a positive result if $\alpha > 2/3$. This is definitely true in our case where $\alpha > 1$. Thus (b) can be a minimum of the axion potential.

	(I)	(II)	(III)
(a)	max	max	max
(b)	$\begin{cases} 1 < \alpha \leq 1/\gamma \text{ min,} \\ \alpha > 1/\gamma \text{ saddle.} \end{cases}$	$\begin{cases} 1 < \alpha \leq 1/\gamma \text{ min,} \\ \alpha > 1/\gamma \text{ saddle.} \end{cases}$	$\begin{cases} 1 < \alpha \leq 1/\beta < 1/\gamma \text{ min,} \\ 1/\beta < \alpha \text{ saddle.} \end{cases}$
(c)	saddle	$\begin{cases} 1 < \alpha \leq 1/\gamma \text{ saddle,} \\ \alpha > 1/\gamma \text{ min.} \end{cases}$	saddle
(d)	saddle	saddle	$\begin{cases} \alpha \leq 1/\beta \text{ saddle,} \\ \alpha > 1/\beta \text{ min.} \end{cases}$

Table 5: Summary of axion stabilisation results for case (4).

- direction (II)

The parameters (A.62) now take the same form given in (A.73). It follows then that the first (b)-condition $|X_1| > |Y_{12}|$ looks like (A.74) and is verified for $1/\gamma \geq \alpha$. The second (b)-condition looks like (A.75) which at large volume is correctly true for $1/\gamma \geq \alpha$. Thus (b) is a minimum of the axion potential. On the contrary the first (c)-condition implies $1/\gamma < \alpha$, whereas the second is similar to the inequality (A.78) which is again clearly true for $1/\gamma < \alpha$. It follows that in this case (c) can also be a minimum.

- direction (III)

The parameters (A.62) assume the same form as (A.73) but now with $\beta > \gamma$. The (d)-condition looks like

$$\frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\beta-\alpha}} > 1 + \frac{(\ln \mathcal{V})^{1-3\alpha/2}}{\mathcal{V}^{1/\gamma-\alpha}}, \quad (\text{A.88})$$

and is verified only if $1/\beta < \alpha$. On the other hand, the first (b)-condition is again given by (A.74) and so is still solved for $1/\gamma \geq \alpha$. The second (b)-condition looks like (A.75) but now at large volume it is satisfied for $1/\beta \geq \alpha$. It follows that in this case both (b) and (d) can be a minimum of the axion potential.

The results found in case (4) are summarised in table 5.

A.2.2 Kähler moduli stabilisation

After this long analysis of the axion minimisation, let us now focus again step by step on the four cases according to the different possible values of K_{12}^{-1} . In each case, we shall fix the axions at their possible VEVs and then study the Kähler moduli stabilisation depending on the particular form of K_{11}^{-1} and K_{22}^{-1} .

However, before focusing on each particular case, let us point out some general features. At the axion minimum we will have:

$$\langle V_{AX} \rangle = 2 (\pm |Y_{12}| \pm |X_1| \pm |X_2|), \quad (\text{A.89})$$

where the "±" signs depend on the specific locus of the minimum, that is (a) or (b) or (c), as specified in (1). Now to write (A.89) explicitly, recall (A.62) and get:

$$V_{np2} + V_{np1}^{AX} = \frac{2}{\mathcal{V}^2} \left\{ W \sum_{j=1}^2 (\pm 2a_j \tau_j e^{-a_j \tau_j}) \pm K_{12}^{-1} a_1 a_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} \right\}, \quad (\text{A.90})$$

where we have set $A_1 = A_2 = 1$. We may now study the full potential by combining equations (A.2), (A.41) and (A.90)

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} \pm 2K_{12}^{-1} a_1 a_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] + 4 \frac{W_0}{\mathcal{V}^2} \sum_{j=1}^2 (\pm a_j \tau_j e^{-a_j \tau_j}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2, \quad (\text{A.91})$$

where we have substituted W with its tree-level expectation value W_0 because the non-perturbative corrections are always subleading by a power of \mathcal{V} .

When we take the generic large volume limit $\mathcal{V} \sim e^{\beta a_1 \tau_1} \sim e^{\gamma a_2 \tau_2}$, with β and $\gamma \in \mathbb{R}$ without any particular relation among them to take into account all the possible limits (A.63), (A.91) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} \pm \frac{K_{12}^{-1}}{\mathcal{V}^{2+1/\beta+1/\gamma}} \pm \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} \pm \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.92})$$

Now given that we are already aware of the volume scaling of K_{12}^{-1} , which was our starting point to stabilise the axions, in order to understand what are the leading terms in (A.92), we need to know only the form of K_{jj}^{-1} , $j = 1, 2$. We can classify its behaviour according to the volume dependence of the quantity $k_{jjk} t^k$ and find 4 different cases as we did for K_{12}^{-1} :

1. $k_{jjk} t^k = 0$ or $k_{jjk} t^k = \frac{f(\tau_1, \tau_2)}{\mathcal{V}^\alpha}$, $\alpha \geq 1 \implies K_{jj}^{-1} = \tau_j^2$;
2. $k_{jjk} t^k = \frac{g(\tau_1, \tau_2)}{\mathcal{V}^\alpha}$, $0 < \alpha < 1$, g homogeneous function of degree $\gamma = \frac{1+3\alpha}{2} \implies K_{jj}^{-1} = \mathcal{V}^\alpha g_{2-3\alpha/2}(\tau_1, \tau_2)$, $0 < \alpha < 1$;
3. $k_{jjk} t^k = f_{1/2}(\tau_1, \tau_2)$, f homogeneous function of degree $1/2 \implies K_{jj}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$;
4. $k_{jjk} t^k = \mathcal{V}^\alpha h_\beta(\tau_1, \tau_2)$, $\alpha > 0$, h homogeneous function of degree $\beta = \frac{1-3\alpha}{2} \implies K_{jj}^{-1} = \mathcal{V}^\alpha h_{2-3\alpha/2}(\tau_1, \tau_2)$, $\alpha > 1$.

Before focusing on the Kähler moduli minimisation by analysing all these 4 cases in full detail for each direction (A.63), we stress that we can already show in general that some situations do not lead to any LARGE Volume minimum.

For example, let us assume that the elements of K^{-1} are such that the dominant terms in (A.92) are

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} - \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}, \quad (\text{A.93})$$

with $\beta = \gamma = 1$ and $K_{11}^{-1} = \mathcal{V}f_{1/2}(\tau_1, \tau_2)$. Therefore the potential (A.91) with $W_0 = 1$ looks like

$$V \sim \frac{f_{1/2}(\tau_1, \tau_2)a_1^2 e^{-2a_1\tau_1}}{\mathcal{V}} - \frac{4a_1\tau_1 e^{-a_1\tau_1}}{\mathcal{V}^2} - \frac{4a_2\tau_2 e^{-a_2\tau_2}}{\mathcal{V}^2} + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3}. \quad (\text{A.94})$$

Thus from $\frac{\partial V}{\partial \tau_1} = 0$ we find

$$\mathcal{V} = \frac{4\tau_1}{\left(2a_1 f_{1/2} - \frac{\partial f_{1/2}}{\partial \tau_1}\right)} e^{a_1\tau_1}, \quad (\text{A.95})$$

whereas $\frac{\partial V}{\partial \tau_2} = 0$ gives

$$\mathcal{V} = -4 \left(\frac{a_2}{a_1}\right)^2 \frac{\tau_2}{\frac{\partial f_{1/2}}{\partial \tau_2}} \frac{e^{2a_1\tau_1}}{e^{a_2\tau_2}}. \quad (\text{A.96})$$

Now since we have $\beta = \gamma = 1$, from the form of the large volume limit (I) of (A.63), we infer that the minimum should be located at $a_1\tau_1 \simeq a_2\tau_2$. Making this substitution and combining (A.95) with (A.96), we end up with the following equation

$$\frac{\partial f_{1/2}}{\partial \tau_2} = \frac{a_2}{a_1} \frac{\partial f_{1/2}}{\partial \tau_1} - 2a_2 f_{1/2}. \quad (\text{A.97})$$

Now using the homogeneity property of $f_{1/2}$, that is $\tau_1 \frac{\partial f_{1/2}}{\partial \tau_1} + \tau_2 \frac{\partial f_{1/2}}{\partial \tau_2} = \frac{1}{2} f_{1/2}$, (A.97) takes the form

$$\frac{\partial f_{1/2}}{\partial \tau_2} = a_2 \left(\frac{1}{4a_2\tau_2} - 1\right) f_{1/2}. \quad (\text{A.98})$$

We can solve the previous differential equation getting $f_{1/2} = \tau_2^{1/4} e^{-a_2\tau_2}$, which is not an homogeneous function of degree 1/2. Thus we deduce that this case gives no LVS.

Another case in which we can show explicitly that no LARGE Volume minimum is present, is the one where the dominant terms in (A.92) read

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.99})$$

with $\beta \leq \gamma$. The fact that all the three terms in (A.99) are strictly positive leads us to conclude that there would definitely be no LARGE Volume minimum in the volume direction once we integrate out the small moduli. In fact, (A.99) would take the generic form

$$V \sim \frac{a(\ln \mathcal{V})^b + c}{\mathcal{V}^3}, \quad c > 0, \quad (\text{A.100})$$

which can be easily seen to have a minimum only if $a < 0$ and $b > 0$.

We illustrate now a further case in which it is possible to show explicitly that no LVS is present. The leading terms in (A.92) read

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}}, \quad (\text{A.101})$$

with $\beta < \gamma$ and $\gamma > 1$ to be able to neglect the α' corrections that scale as \mathcal{V}^{-3} . The necessary but not sufficient condition to fix the small Kähler moduli is $K_{11}^{-1} = \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ and $K_{22}^{-1} = \mathcal{V}^\eta f_{2-3\eta/2}(\tau_1, \tau_2)$ with $\delta = 2/\beta - 1/\gamma$ and $\eta = 1/\gamma$. We can now prove that it is never possible to stabilise $a_1 \tau_1 \gg 1$. In fact, the relevant part of the scalar potential (A.91) would read

$$V \simeq \frac{a_1^2 \tau_1^{2-3\delta/2} e^{-2a_1 \tau_1}}{\mathcal{V}^{2-\delta}} + \frac{a_2^2 f_{2-3\eta/2}(\tau_1, \tau_2) e^{-2a_2 \tau_2}}{\mathcal{V}^{2-\eta}} - \frac{4a_2 \tau_2 e^{-a_2 \tau_2}}{\mathcal{V}^2}. \quad (\text{A.102})$$

Now the equation $\frac{\partial V}{\partial \tau_1} = 0$, admits a solution of the form

$$\mathcal{V}^{\eta-\delta} = \frac{2a_1^3 \tau_1^{2-3\delta/2}}{a_2^2 \frac{\partial f_{2-3\eta/2}}{\partial \tau_1}} e^{2(a_2 \tau_2 - a_1 \tau_1)}, \quad (\text{A.103})$$

whereas $\frac{\partial V}{\partial \tau_2} = 0$ gives

$$\mathcal{V}^\eta = \frac{2\tau_2}{a_2 f_{2-3\eta/2}} e^{a_2 \tau_2}. \quad (\text{A.104})$$

The third minimisation equation $\frac{\partial V}{\partial \mathcal{V}} = 0$ looks like

$$(\eta - 2)a_2^2 f_{2-3\eta/2} \mathcal{V}^\eta e^{-2a_2 \tau_2} + (\delta - 2)a_1^2 \tau_1^{2-3\delta/2} \mathcal{V}^\delta e^{-2a_1 \tau_1} + 8a_2 \tau_2 e^{-a_2 \tau_2} = 0, \quad (\text{A.105})$$

and substituting the results (A.104) and (A.103), we obtain

$$2(\eta - 2) + \frac{(\delta - 2)}{a_1 f_{2-3\eta/2}} \frac{\partial f_{2-3\eta/2}}{\partial \tau_1} + 8 = 0. \quad (\text{A.106})$$

Solving the differential equation (A.106), we realise that $f_{2-3\eta/2}$ has an exponential behaviour in τ_1 which is in clear contrast with the requirement that it has to be homogeneous. Following arguments very similar to this one it can be seen that, as in the case with just one small modulus, the presence of the α' corrections is crucial to find a LARGE Volume minimum. In fact if we omit them, either it is impossible to fix the small moduli large enough to ignore higher instanton corrections or, once we integrate them out, we are left with a run-away in the volume direction.

Lastly, we describe the final case in which it is possible to prove the absence of a LARGE Volume vacuum. The leading terms in (A.92) are given by

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}, \quad (\text{A.107})$$

with $\beta < 1$ and the axion minimum along the direction $\mathcal{V} \sim e^{\beta a_1 \tau_1} \sim e^{a_2 \tau_2}$. The necessary but not sufficient condition to fix the small Kähler moduli is $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ with $\delta = 2/\beta - 1$, and $K_{22}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$. The relevant part of the scalar potential (A.91) takes the form

$$V \simeq \frac{a_1^2 \tau_1^{2-3\delta/2} e^{-2a_1 \tau_1}}{\mathcal{V}^{2-\delta}} + \frac{a_2^2 f_{1/2}(\tau_1, \tau_2) e^{-2a_2 \tau_2}}{\mathcal{V}} - \frac{4a_2 \tau_2 e^{-a_2 \tau_2}}{\mathcal{V}^2} + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3}. \quad (\text{A.108})$$

Now the equation $\frac{\partial V}{\partial \tau_1} = 0$, admits a solution of the form

$$a_1^2 \tau_1^{2-3\delta/2} \mathcal{V}^\delta e^{-2a_1 \tau_1} = \frac{a_2^2}{2a_1} \frac{\partial f_{1/2}}{\partial \tau_1} \mathcal{V} e^{-2a_2 \tau_2}, \quad (\text{A.109})$$

whereas $\frac{\partial V}{\partial \tau_2} = 0$ gives

$$a_2 \tau_2 e^{-a_2 \tau_2} = \frac{a_2^2}{2} f_{1/2} \mathcal{V} e^{-2a_2 \tau_2}. \quad (\text{A.110})$$

The third minimisation equation $\frac{\partial V}{\partial \mathcal{V}} = 0$ corresponds to

$$(\delta - 2) a_1^2 \tau_1^{2-3\delta/2} \mathcal{V}^\delta e^{-2a_1 \tau_1} - a_2^2 f_{1/2} \mathcal{V} e^{-2a_2 \tau_2} - 8a_2 \tau_2 e^{-a_2 \tau_2} = \frac{9}{4} \frac{\hat{\xi}}{\mathcal{V}}, \quad (\text{A.111})$$

and substituting the results (A.109) and (A.110), we obtain

$$4a_2^2 \mathcal{V}^2 \left[\frac{(\delta - 2)}{2a_1} \frac{\partial f_{1/2}}{\partial \tau_1} - 5f_{1/2} \right] = 9\hat{\xi} e^{2a_2 \tau_2}. \quad (\text{A.112})$$

Now writing $f_{1/2}(\tau_1, \tau_2) = F\left(\frac{\beta a_1}{a_2}\right) \sqrt{\tau_1}$ for appropriate function F , (A.112) becomes

$$\frac{a_2^2}{a_1 \sqrt{\tau_1}} \mathcal{V}^2 F\left(\frac{\beta a_1}{a_2}\right) (\delta - 2 - 20a_1 \tau_1) = 9\hat{\xi} e^{2a_2 \tau_2}. \quad (\text{A.113})$$

Given that a trustable minimum requires $a_1 \tau_1 \gg 1$, the l.h.s. of (A.113) is negative while the r.h.s. is definitively positive and so this case does not allow us to find any LVS.

The general path that we shall follow to derive the conditions which guarantee that we have enough terms with the correct volume scaling to stabilise all the moduli at exponentially large volume, is the following one. We learnt from the proof of Claim 1 for the case with just one small modulus τ_s , that we need to have two terms in the scalar potential with the same volume scaling that depend on τ_s so that it can be stabilised rather large in order to be able to neglect higher instanton corrections. Then if we integrate out τ_s , we have to be left with at least two terms that depend on the overall volume and have the same volume scaling. Lastly in order to find the exponentially large volume minimum, the leading term at large volume has to be negative. As we have seen before, the same arguments apply here. Thus we shall first work out the conditions to be able to fix both $a_1 \tau_1 \gg 1$ and $a_2 \tau_2 \gg 1$ by having at least two terms in the potential with a dependence on these moduli and the same volume scaling. Then, we shall imagine to integrate out these moduli, and derive the conditions to be left with at least two terms dependent on \mathcal{V} with the leading one which is negative.

Case (1): $K_{12}^{-1} \simeq \tau_1 \tau_2$

The previous analysis tells us that, regardless of the particular direction considered, the axion minimum is always in the case (b). Thus we realise that at the minimum

$$\langle V_{AX} \rangle = 2(|Y_{12}| - |X_1| - |X_2|). \quad (\text{A.114})$$

Now recalling that in case (1) $K_{12}^{-1} \simeq \tau_1 \tau_2$, (A.91) takes the form:

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} + 2\tau_1 \tau_2 a_1 a_2 e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] - 4 \frac{W_0}{\mathcal{V}^2} \sum_{j=1}^2 a_j \tau_j e^{-a_j \tau_j} + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \quad (\text{A.115})$$

We shall now study the behaviour of (A.115) by taking the large volume limit along each direction (A.63) and then considering all the possible forms of K_{jj}^{-1} , $j = 1, 2$. When we take the large volume limit (I) of (A.63), (A.115) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{\tau_1 \tau_2}{\mathcal{V}^{2+2/\gamma}} - \frac{\tau_1}{\mathcal{V}^{2+1/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.116})$$

The third term in (A.116) is subleading with respect to the fourth and the fifth. Thus it can be neglected

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{\tau_1}{\mathcal{V}^{2+1/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.117})$$

We have seen that the presence of the α' corrections is crucial to find the exponentially large volume minimum. Therefore the fact that the third and the fourth terms in (A.117) have to scale as \mathcal{V}^{-3} tells us that $\gamma = 1$.

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^4} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_1}{\mathcal{V}^3} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.118})$$

At this point it is straightforward to realise that if either K_{11}^{-1} or K_{22}^{-1} were in case (4), then we would have a run-away behaviour of the volume direction. Similarly the situation with K_{11}^{-1} and K_{22}^{-1} either in case (1) or (2) is not giving a LARGE Volume minimum since the first two terms in (A.118) should be neglected without then the possibility to stabilise τ_1 and τ_2 large. What happens if either K_{11}^{-1} or K_{22}^{-1} is in case (3) and the other one is either in case (1) or (2)? We do not find any minimum. In fact, let us say that K_{11}^{-1} is in case (3) and K_{22}^{-1} in case (1) or (2): then the second term in (A.118) can be neglected. If we want to have still some hope to stabilise τ_2 large, then K_{11}^{-1} should better depend also on τ_2 : $K_{11}^{-1} \simeq f_{1/2}(\tau_1, \tau_2) \mathcal{V}$. However this case has been studied explicitly to show that it leads to an absurd. Thus only if both K_{11}^{-1} and K_{22}^{-1} is in case (3) we can have a LARGE Volume minimum.

On the contrary, if we took either the large volume limit (II) or (III) in (A.63), (A.115) would scale as

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{\tau_1 \tau_2}{\mathcal{V}^{2+1/\beta+1/\gamma}} - \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.119})$$

Let us focus on the direction (II) where $1/\beta > 1/\gamma$. The third and the fourth term in (A.119) at large volume are subdominant to the fifth and therefore they can be ignored:

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}, \quad (\text{A.120})$$

K_{11}^{-1}	K_{22}^{-1}	(I), (b)	(II), (b)	(III), (b)
1	1	NO	NO	NO
1	2	NO	NO	NO
1	3	NO	NO	NO
1	4	NO	NO	NO
2	1	NO	NO	NO
2	2	NO	NO	NO
2	3	NO	NO	NO
2	4	NO	NO	NO
3	1	NO	NO	NO
3	2	NO	NO	NO
3	3	OK, $\gamma = 1$	NO	NO
3	4	NO	NO	OK, $\beta = 1$, (**)
4	1	NO	NO	NO
4	2	NO	NO	NO
4	3	NO	OK, $\gamma = 1$, (*)	NO
4	4	NO	NO	NO

(*) $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, $K_{11}^{-1} \simeq \mathcal{V}^\alpha \tau_1^{2-3\alpha/2}$ with $\frac{1}{\beta} = \alpha$
(**) $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$, $K_{22}^{-1} \simeq \mathcal{V}^\alpha \tau_2^{2-3\alpha/2}$ with $\frac{1}{\gamma} = \alpha$

Table 6: Summary of Kähler moduli stabilisation results for case (1): $K_{12}^{-1} \simeq \tau_1 \tau_2$.

If $1/\gamma > 1$, then the \mathcal{V}^{-3} term would be the dominant one producing a run-away in the volume direction. Thus we impose $1/\gamma \leq 1$. However we have already showed that the situation with $1/\gamma < 1$ gives no LVS and so we deduce that we need $1/\gamma = 1$. Then we realise that the only possible situation in which we can hope to fix τ_2 large is when either the first or the second term in (A.120) scales as \mathcal{V}^{-3} . Now if the second term involving K_{22}^{-1} were subleading with respect to the fourth term in (A.120), then the first one should scale as \mathcal{V}^{-3} . However at that point, knowing that K_{11}^{-1} will introduce a dependence on τ_1 , we would not be able to stabilise τ_1 large. Hence K_{22}^{-1} has to be in case (3): $K_{22}^{-1} = \mathcal{V}f_{1/2}(\tau_1, \tau_2)$.

Now we have two different situations according to the fact that $f_{1/2}$ indeed depends on both τ_1 and τ_2 or only on τ_2 . The first possibility has already been studied with the final conclusion that it produces no LVS. On the other hand, when K_{22}^{-1} depends only on τ_2 , i.e. $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, we have that the overall volume and τ_2 are both stabilised by the interplay of the second, the third and the fourth term in (A.120). The first term is now subleading and can be used to fix τ_1 if we write $K_{11}^{-1} \simeq \mathcal{V}^\alpha \tau_1^{2-3\alpha/2}$ and then impose $1/\beta = \alpha$ in order to make it scale as the fourth term in (A.119).

We point out that these results apply also to the direction (III) where $1/\gamma > 1/\beta$, if we swap γ with β and τ_1 with τ_2 . Let us finally summarise in table 6 what we have found for this case.

Case (2): $K_{12}^{-1} \simeq \mathcal{V}^\alpha g_{2-3\alpha/2}(\tau_1, \tau_2)$, $0 < \alpha < 1$

This case is more involved than the previous one since, depending on the direction chosen for the large volume limit and the exact value of the parameter α , the axion minimum can not only be in case (b), but also in (c) and (d). Let us start by considering each case in detail:

- axion minimum at (c) along direction (II) for $\alpha = 1/\gamma < 2/3$ or $1/\gamma < \alpha < 1$

We can easily conclude that no LVS is present given that, looking at the general volume scaling of the scalar potential (A.92), we can notice that the fifth term would be dominant with respect to the last one since we have always $1/\gamma < 1$. Thus the α' correction would be negligible at large volume, so producing no LVS.

- axion minimum at (d) along direction (III) for $\alpha = 1/\beta < 2/3$ or $1/\beta < \alpha < 1$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we conclude that no LARGE Volume minimum will be present.

- axion minimum at (b) along direction (II) for $\alpha < 1/\gamma$ or $2/3 < \alpha = 1/\gamma < 1$

First of all we realise that the situation with $2/3 < \alpha = 1/\gamma < 1$ does not give rise to any LVS because the leading order α' correction would be negligible at large volume. We shall therefore focus on the case $\alpha < 1/\gamma$. The scalar potential (A.91) takes the form

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} + 2a_1 a_2 g_{2-3\alpha/2}(\tau_1, \tau_2) \mathcal{V}^\alpha e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] - 4 \frac{W_0}{\mathcal{V}^2} (a_1 \tau_1 e^{-a_1 \tau_1} + a_2 \tau_2 e^{-a_2 \tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \quad (\text{A.121})$$

When we take the large volume limit (II) of (A.63), (A.121) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{g_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+1/\beta+1/\gamma-\alpha}} - \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.122})$$

Setting $1/\gamma = 1$ and recalling that in this direction $1/\beta > 1/\gamma$, the dominant terms in (A.122) become

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.123})$$

Now by noticing that equation (A.123) has the same form of (A.120) if we set $1/\gamma = 1$, we can just repeat the same consideration made before and obtain that K_{22}^{-1} has to be in case (3). Moreover if K_{22}^{-1} depends on both τ_1 and τ_2 , then there is no LARGE Volume minimum, but when K_{22}^{-1} depends only on τ_2 , the first term in (A.123) is negligible at large volume and can be used to fix τ_1 if we make it compete with the fourth term in (A.122) by writing $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ and then imposing $1/\beta = \delta$.

- axion minimum at (b) along direction (III) for $\alpha < 1/\beta$ or $2/3 < \alpha = 1/\beta < 1$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we do not need to discuss this case.

- axion minimum at (b) along direction (I) for $\alpha < 1/\gamma$ or $2/3 < \alpha = 1/\gamma < 1$

In this situation the full scalar potential still looks like (A.121), but the volume scaling behaviour of its terms now reads

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{g_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+2/\gamma-\alpha}} - \frac{\tau_1}{\mathcal{V}^{2+1/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.124})$$

For $2/3 < \alpha = 1/\gamma < 1$ the last term in (A.124) would be subdominant with respect to the fifth one, but we know that its presence is crucial to find a minimum and so we can conclude that this case admits no minimum. On the other hand for $\alpha < 1/\gamma$, setting $1/\gamma = 1$, the dominant terms in (A.124) become

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^4} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_1}{\mathcal{V}^3} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.125})$$

We immediately realise that (A.125) is absolutely similar to (A.118). We can therefore repeat exactly the same analysis and conclude that only if both K_{11}^{-1} and K_{22}^{-1} is in case (3) we can have a LARGE volume minimum. Let us finally summarise in table 7 what we have found for this case.

Case (3): $K_{12}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$

We shall now consider each particular situation according to the different possible locations of the axion minimum:

- axion minimum at (c) along direction (II) for $\gamma > 1$

Thus at the minimum

$$\langle V_{AX} \rangle = 2(|X_1| - |Y_{12}| - |X_2|). \quad (\text{A.126})$$

Therefore the full potential (A.91) looks like

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} - 2a_1 a_2 f_{1/2}(\tau_1, \tau_2) \mathcal{V} e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] + 4 \frac{W_0}{\mathcal{V}^2} (a_1 \tau_1 e^{-a_1 \tau_1} - a_2 \tau_2 e^{-a_2 \tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \quad (\text{A.127})$$

When we take the large volume limit (II) of (A.63), (A.127) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{f_{1/2}(\tau_1, \tau_2)}{\mathcal{V}^{1+1/\beta+1/\gamma}} + \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.128})$$

Given that $1/\gamma < 1$, the leading order α' correction would be subleading in a Large Volume limit. However, we know that its presence is crucial to find the minimum and so we conclude that this case does not present any new Large Volume vacuum.

K_{11}^{-1}	K_{22}^{-1}	(I), (b), $\alpha \leq 1/\gamma$	(II), (b), $\alpha \leq 1/\gamma$	(II), (c), $\alpha \geq 1/\gamma$	(III), (b), $\alpha \leq 1/\beta$	(III), (d), $\alpha \geq 1/\beta$
1	1	NO	NO	NO	NO	NO
1	2	NO	NO	NO	NO	NO
1	3	NO	NO	NO	NO	NO
1	4	NO	NO	NO	NO	NO
2	1	NO	NO	NO	NO	NO
2	2	NO	NO	NO	NO	NO
2	3	NO	NO	NO	NO	NO
2	4	NO	NO	NO	NO	NO
3	1	NO	NO	NO	NO	NO
3	2	NO	NO	NO	NO	NO
3	3	OK, $\gamma = 1$	NO	NO	NO	NO
3	4	NO	NO	NO	OK, $\beta = 1$ (**)	NO
4	1	NO	NO	NO	NO	NO
4	2	NO	NO	NO	NO	NO
4	3	NO	OK, $\gamma = 1$ (*)	NO	NO	NO
4	4	NO	NO	NO	NO	NO

(*) $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ with $\frac{1}{\beta} = \delta$

(**) $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$, $K_{22}^{-1} \simeq \mathcal{V}^\delta \tau_2^{2-3\delta/2}$ with $\frac{1}{\gamma} = \delta$

Table 7: Summary of Kähler moduli stabilisation results for case (2): $K_{12}^{-1} = \mathcal{V}^\alpha g_{2-3\alpha/2}(\tau_1, \tau_2)$, $0 < \alpha < 1$.

- axion minimum at (d) along direction (III) for $\beta > 1$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we conclude that no LARGE Volume minimum is present.

- axion minimum at (b) along direction (I) for $0 < \gamma \leq 1$

Thus at the minimum

$$\langle V_{AX} \rangle = 2(|Y_{12}| - |X_1| - |X_2|), \quad (\text{A.129})$$

and so the full potential (A.91) becomes (setting $W_0 = 1$)

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} + 2a_1 a_2 f_{1/2}(\tau_1, \tau_2) \mathcal{V} e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] - \frac{4}{\mathcal{V}^2} (a_1 \tau_1 e^{-a_1 \tau_1} + a_2 \tau_2 e^{-a_2 \tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3}. \quad (\text{A.130})$$

When we take the large volume limit (I) of (A.63), (A.130) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{f_{1/2}(\tau_1, \tau_2)}{\mathcal{V}^{1+2/\gamma}} - \frac{\tau_1}{\mathcal{V}^{2+1/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.131})$$

In the case where $1/\gamma > 1$, the leading part of (A.131) at Large Volume takes the form

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.132})$$

We have already checked explicitly that this situation does not present any LARGE Volume minimum, therefore we shall focus only on the case $1/\gamma = 1$, for which the volume scaling (A.131) becomes

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^4} + \frac{K_{22}^{-1}}{\mathcal{V}^4} + \frac{f_{1/2}(\tau_1, \tau_2)}{\mathcal{V}^3} - \frac{\tau_1}{\mathcal{V}^3} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.133})$$

We immediately realise that as soon as either K_{11}^{-1} or K_{22}^{-1} is in case (4) then the α' corrections would be subleading in a large volume expansion. Thus we can reject this possibility. Then we are left with three different situations. Firstly when both K_{11}^{-1} and K_{22}^{-1} are subdominant with respect to the last three terms in (A.133) (i.e. each K_{jj}^{-1} with $j = 1, 2$ is either in case (1) or (2)) the scalar potential (A.130) takes the form

$$V = \frac{2}{\mathcal{V}} a_1 a_2 f_{1/2}(\tau_1, \tau_2) e^{-(a_1 \tau_1 + a_2 \tau_2)} - \frac{4}{\mathcal{V}^2} (a_1 \tau_1 e^{-a_1 \tau_1} + a_2 \tau_2 e^{-a_2 \tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3}. \quad (\text{A.134})$$

Since we have to find a minimum such that $a_j \tau_j \gg 1$ for $j = 1, 2$, we can work at leading order in a $\frac{1}{a_1 \tau_1}$ and $\frac{1}{a_2 \tau_2}$ expansion and obtain that $\frac{\partial^2 V}{\partial \tau_1^2} \simeq 0$ due to the presence of just one exponential in τ_1 in both the first and the second term in (A.134). In fact, if we are interested in the dependence of V on just τ_1 , (A.134) can be rewritten as

$$V = c_1 e^{-a_1 \tau_1} \left(a_2 f_{1/2}(\tau_1, \tau_2) e^{-a_2 \tau_2} - \frac{2\tau_1}{\mathcal{V}} \right) + c_2 \equiv c_1 e^{-a_1 \tau_1} g(\tau_1) + c_2, \quad (\text{A.135})$$

where c_1 and c_2 are constants and $g(\tau_1)$ is the sum of two homogeneous functions in τ_1 . Therefore at leading order in a $\frac{1}{a_1 \tau_1}$ expansion, we get:

$$\frac{\partial V}{\partial \tau_1} \simeq -a_1 c_1 e^{-a_1 \tau_1} g(\tau_1) = 0 \Leftrightarrow g(\tau_1) = 0, \quad (\text{A.136})$$

which implies

$$\left. \frac{\partial^2 V}{\partial \tau_1^2} \right|_{\min} \simeq a_1^2 c_1 e^{-a_1 \tau_1} g(\tau_1) \Big|_{\min} = 0. \quad (\text{A.137})$$

Similarly we have $\frac{\partial^2 V}{\partial \tau_2^2} \simeq 0$, whereas

$$\left. \frac{\partial^2 V}{\partial \tau_1 \partial \tau_2} \right|_{\min} \simeq 2a_1^2 a_2^2 f_{1/2} e^{-a_1 \tau_1} \frac{e^{-a_2 \tau_2}}{\mathcal{V}} \Big|_{\min} \equiv c_3 > 0. \quad (\text{A.138})$$

Therefore considering \mathcal{V} constant, the Hessian matrix will look like

$$\mathcal{H} \simeq \begin{pmatrix} 0 & c_3 \\ c_3 & 0 \end{pmatrix} \implies \det \mathcal{H} = -c_3^2 < 0, \quad (\text{A.139})$$

so implying that we can never have a minimum.

Secondly we have to contemplate the possibility that only one of the first two terms in (A.133) is competing with the last three ones while the other is negligible. However even this case does not yield any new LVS due the asymmetry of the dependence of the scalar potential in τ_1 and τ_2 that does not allow us to stabilise the small Kähler moduli large enough. In fact, let us assume for example that K_{11}^{-1} is in case (1) or (2) and hence is negligible, whereas $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$. Consequently we obtain

$$\frac{\partial V}{\partial \tau_1} = 0 \iff \mathcal{V} = \frac{2\tau_1}{a_2 f_{1/2}} e^{a_2 \tau_2}, \quad (\text{A.140})$$

$$\frac{\partial V}{\partial \tau_2} = 0 \iff \mathcal{V} = \frac{2\tau_2}{a_1 f_{1/2} e^{-a_1 \tau_1} + a_2 \sqrt{\tau_2} e^{-a_2 \tau_2}}. \quad (\text{A.141})$$

Now combining (A.140) with (A.141) we find:

$$a_1 \tau_1 f_{1/2} e^{a_2 \tau_2 - a_1 \tau_1} + a_2 \tau_1 \sqrt{\tau_2} = a_2 \tau_2 f_{1/2}, \quad (\text{A.142})$$

which evaluated along the direction $a_1 \tau_1 \simeq a_2 \tau_2$ where the axion minimum is located, becomes

$$a_1 \tau_1 f_{1/2} + a_2 \tau_1 \sqrt{\tau_2} \simeq a_1 \tau_1 f_{1/2} \iff \sqrt{a_1 a_2} \tau_1^{3/2} \simeq 0, \quad (\text{A.143})$$

which is the negative result we mentioned above. Lastly when both K_{11}^{-1} and K_{22}^{-1} is in case (3) all the terms in (A.133) have the same volume scaling. It can be seen that, regardless of the form of $f_{1/2}$, the LARGE Volume minimum is always present.

- axion minimum at (b) along direction (II) for $0 < \gamma \leq 1$

In this situation the full scalar potential still looks like (A.130), but the volume scaling behaviour of its terms now reads

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{f_{1/2}(\tau_1, \tau_2)}{\mathcal{V}^{1+1/\beta+1/\gamma}} - \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.144})$$

Given that along the direction (II) $1/\beta > 1/\gamma$, and the axion minimum is present for $1/\gamma \geq 1$, the dominant terms in (A.144) become

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}, \quad (\text{A.145})$$

where we have already set $\gamma = 1$ because, as we argued before, this is the only possible situation when we can hope to find a LARGE Volume minimum. We notice now that whenever the first two terms in (A.145) are negligible at large volume, then we have only one term in V dependent on τ_2 and so we can never obtain a minimum at $a_2 \tau_2 \gg 1$. This happens if K_{22}^{-1} is either in case (1) or in case (2) and K_{11}^{-1} is in case (1), (2), (3) or even in case (4) if its term in (A.145) still goes like \mathcal{V}^α , $\alpha > 3$. Moreover if K_{22}^{-1} is in case (4) then it would beat the last two terms in (A.145) so giving no LVS.

On the other hand, when only the first term in (A.145) is negligible at large volume, we have the possibility to find a new LVS if K_{22}^{-1} is in case (3) and does not depend

on τ_1 , that is $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$. In fact, at leading order in a large volume expansion the scalar potential looks like the one we studied for the case with just one small modulus and we know that the corresponding LARGE Volume minimum would be present. However we have still to fix τ_1 . If K_{11}^{-1} is in case (1), (2) or (3) then the first term in (A.144) is always subleading with respect to the third and the fourth one and therefore it can be neglected. We can now focus on the third and fourth term in (A.144) which have the same volume scaling. Then combining the solution of $\frac{\partial V}{\partial \tau_1} = 0$ and $\frac{\partial V}{\partial \tau_2} = 0$ we end up with $\tau_1 = \tau_2$ which is correct if we choose $\beta a_1 = a_2$, $\beta < 1$. However we have still to check the sign of $\frac{\partial^2 V}{\partial \tau_1^2}$ which turns out to be positive only if, writing $f_{1/2}(\tau_1, \tau_2) \sim \tau_1^\alpha \tau_2^{1/2-\alpha}$ for arbitrary α , we have $\alpha < 1$.

The situation when $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$ and K_{11}^{-1} is in case (4) needs to be studied more carefully. Writing $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$, $\delta > 1$, if $\delta > \frac{2}{\beta} + 1$ then the first term in (A.145) beats all the other ones so giving no LVS. If $\delta = \frac{2}{\beta} + 1$, then the first term in (A.145) scales as the other ones but we have already shown that this is not an interesting situation. The only way to get a LVS is to impose $\delta \leq \frac{1}{\beta}$ to make the first term in (A.144) scale as the third and fourth term or to render it subdominant with respect to them.

Finally if K^{-1} is in case (4) and K_{22}^{-1} is either in case (1) or (2) then we don't find any new LVS since the second term in (A.145) would be subleading with respect to the other ones so leaving just one term, the first one, which depends on τ_1 .

- axion minimum at (b) along direction (III) for $0 < \gamma < \beta \leq 1$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we do not need to discuss this case. Let us finally summarise in table 8 what we have found for this case.

Case (4): $K_{12}^{-1} \simeq \mathcal{V}^\alpha h_{2-3\alpha/2}(\tau_1, \tau_2)$, $\alpha > 1$

Let us focus on each particular situation according to the different possible positions of the axion minimum:

- axion minimum at (c) along direction (II) for $\alpha > 1/\gamma$

Thus at the minimum

$$\langle V_{AX} \rangle = 2(|X_1| - |Y_{12}| - |X_2|). \tag{A.146}$$

Therefore the full scalar potential (A.91) reads

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j \tau_j} - 2a_1 a_2 h_{2-3\alpha/2}(\tau_1, \tau_2) \mathcal{V}^\alpha e^{-(a_1 \tau_1 + a_2 \tau_2)} \right] + 4 \frac{W_0}{\mathcal{V}^2} (a_1 \tau_1 e^{-a_1 \tau_1} - a_2 \tau_2 e^{-a_2 \tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \tag{A.147}$$

K_{11}^{-1}	K_{22}^{-1}	(I), (b), $0 < \gamma \leq 1$	(II), (b), $0 < \gamma \leq 1$	(II), (c), $\gamma > 1$	(III), (b), $0 < \gamma < \beta \leq 1$	(III), (d), $\gamma \geq 1$
1	1	NO	NO	NO	NO	NO
1	2	NO	NO	NO	NO	NO
1	3	NO	OK, $\beta < \gamma = 1$ (*)	NO	NO	NO
1	4	NO	NO	NO	NO	NO
2	1	NO	NO	NO	NO	NO
2	2	NO	NO	NO	NO	NO
2	3	NO	OK, $\beta < \gamma = 1$ (*)	NO	NO	NO
2	4	NO	NO	NO	NO	NO
3	1	NO	NO	NO	OK, $\gamma < \beta = 1$ (*)	NO
3	2	NO	NO	NO	OK, $\gamma < \beta = 1$ (*)	NO
3	3	OK, $\gamma = 1$	OK, $\beta < \gamma = 1$ (*)	NO	OK, $\gamma < \beta = 1$ (*)	NO
3	4	NO	NO	NO	OK, $\gamma < \beta = 1$ (**)	NO
4	1	NO	NO	NO	NO	NO
4	2	NO	NO	NO	NO	NO
4	3	NO	OK, $\beta < \gamma = 1$ (**)	NO	NO	NO
4	4	NO	NO	NO	NO	NO

(*) $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, $f_{1/2} \sim \tau_1^\alpha \tau_2^{1/2-\alpha}$ with $\alpha < 1$

(**) $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ and $f_{1/2} \sim \tau_1^\alpha \tau_2^{1/2-\alpha}$ with $\alpha < 1$ for $1 < \delta < \frac{1}{\beta}$ and $\forall \alpha$ for $\delta = \frac{1}{\beta}$

(*) $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$, $f_{1/2} \sim \tau_2^\alpha \tau_1^{1/2-\alpha}$ with $\alpha < 1$

(**) $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$, $K_{22}^{-1} \simeq \mathcal{V}^\delta \tau_2^{2-3\delta/2}$ and $f_{1/2} \sim \tau_2^\alpha \tau_1^{1/2-\alpha}$ with $\alpha < 1$ for $1 < \delta < \frac{1}{\gamma}$ and $\forall \alpha$ for $\delta = \frac{1}{\gamma}$

Table 8: Summary of Kähler moduli stabilisation results for case (3): $K_{12}^{-1} = \mathcal{V}f_{1/2}(\tau_1, \tau_2)$.

When we take the large volume limit (II) of (A.63), (A.147) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+1/\beta+1/\gamma-\alpha}} + \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.148})$$

Recalling that in this direction $1/\beta > 1/\gamma$, the dominant terms in (A.148) are

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+1/\beta+1/\gamma-\alpha}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.149})$$

We know that the presence of the last term in (A.149) is crucial to find the LARGE Volume minimum, so in order not to make it subleading with respect to the fourth one, we need to have $1/\gamma \geq 1$. If $1/\gamma = 1$, the new volume scaling looks like

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{3+1/\beta-\alpha}} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}, \quad (\text{A.150})$$

with $1/\beta \geq \alpha$ to keep the last term in (A.150). Now setting $1/\beta = \alpha > 1$, (A.150) reduces to

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2\alpha}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^3} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.151})$$

By studying the expression (A.151), we realise that there are only three possible situations in which the necessary but not sufficient conditions to stabilise $a_j \tau_j \gg 1$, $j = 1, 2$, and not to neglect the leading order α' corrections, can be satisfied. The first one is when K_{22}^{-1} is in case (3) and depends also on τ_1 : $K_{22}^{-1} = \mathcal{V} f_{1/2}(\tau_1, \tau_2)$. In addition, the first term in (A.151) is subleading with respect to the other ones given that K_{11}^{-1} is in case (1) or (2) or (3) or even in case (4) but still being subleading. However we can again show that this case does not lead to any new LVS by noticing that $\partial V / \partial \tau_1 = 0$ combined with $\partial V / \partial \tau_2 = 0$ gives rise to a differential equation for $f_{1/2}$ whose solution is not homogeneous.

The second situations takes place when the second term in (A.151) is subleading with respect to the others. This occurs when K_{22}^{-1} is either in case (1) or (2) and K_{11}^{-1} is in case (4). Moreover if $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ we have to impose $\delta = 2\alpha - 1$. However this case would not work because the minimisation equation $\frac{\partial V}{\partial \tau_2} = 0$ would produce a negative volume:

$$\mathcal{V}^\alpha = -\frac{2\tau_2}{a_1 h_{2-3\alpha/2}} e^{a_1 \tau_1}. \tag{A.152}$$

Finally we have to contemplate the possibility that all the terms in (A.151) have the same volume scaling. This can happen only if $K_{22}^{-1} \simeq \mathcal{V} \sqrt{\tau_2}$ and $K_{22}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ with $\delta = (2\alpha - 1) > 1$. Even this case can be seen to produce no LVS. In fact, it is possible to integrate out the overall volume from one of the usual minimisation equations, so being left with two equations in τ_1 and τ_2 . Then given that we know that we are looking for a minimum located at $\beta a_1 \tau_1 \simeq a_2 \tau_2$, making this substitution, we end up with two equations in just τ_1 which can be seen to disagree.

On the other hand, for $1/\beta > \alpha$, we would be left with

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^4} - \frac{\tau_2}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \tag{A.153}$$

Now by noticing that equation (A.153) has the same form of (A.120) if we set $1/\gamma = 1$, we can just repeat the same consideration made before and obtain that K_{22}^{-1} has to be in case (3). Moreover if K_{22}^{-1} depends on both τ_1 and τ_2 , then we have no LARGE Volume minimum. On the other hand, when K_{22}^{-1} depends only on τ_2 , the first term in (A.153) is now negligible at large volume and can be used to fix τ_1 if we make it compete with the third term in (A.150) by writing $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ and then imposing $1/\beta + \alpha = 1 + \delta$.

On the contrary, if $1/\gamma > 1$, (A.149) takes the form

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+1/\beta+1/\gamma-\alpha}} + \frac{1}{\mathcal{V}^3}. \tag{A.154}$$

Now for $(1/\beta + 1/\gamma - \alpha) > 1$, (A.154) at leading order in a large volume expansion, reduces to

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{1}{\mathcal{V}^3}, \tag{A.155}$$

which has already been proved to produce no LVS. On the other hand, for $(1/\beta + 1/\gamma - \alpha) < 1$, the α' correction would be negligible at large volume, so forcing us to impose $(1/\beta + 1/\gamma - \alpha) = 1$ and obtain:

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} - \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.156})$$

However we can show explicitly that there is no LARGE Volume minimum. In fact, setting $K_{11}^{-1} \simeq \mathcal{V}^\eta \tau_1^{2-3\eta/2}$ with $\eta = \frac{2}{\beta} - 1$, and $K_{22}^{-1} \simeq \mathcal{V}^\delta \tau_2^{2-3\delta/2}$ with $\delta = \frac{2}{\gamma} - 1$, and then substituting the solutions of $\partial V/\partial \tau_1 = 0$ and $\partial V/\partial \tau_2 = 0$ in $\partial V/\partial \mathcal{V} = 0$, one finds that it is never possible to fix the small Kähler moduli large enough to be able to neglect the higher order instanton contributions to W . If $h_{2-3\alpha/2}$ did not depend on both τ_1 and τ_2 but just on one of them, this negative result would not be altered as the term involving $h_{2-3\alpha/2}$ depends always on both the two small moduli via the two exponentials $e^{a_1\tau_1} e^{a_2\tau_2}$ (see (A.147)).

- axion minimum at (d) along direction (III) for $\alpha > 1/\beta$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we do not need to discuss this case.

- axion minimum at (b) along direction (I) for $1 < \alpha \leq 1/\gamma$

Thus at the minimum

$$\langle V_{AX} \rangle = 2(|Y_{12}| - |X_1| - |X_2|), \quad (\text{A.157})$$

and so the full potential (A.91) becomes

$$V \sim \frac{1}{\mathcal{V}^2} \left[\sum_{j=1}^2 K_{jj}^{-1} a_j^2 e^{-2a_j\tau_j} + 2a_1 a_2 h_{2-3\alpha/2}(\tau_1, \tau_2) \mathcal{V}^\alpha e^{-(a_1\tau_1 + a_2\tau_2)} \right] - 4 \frac{W_0}{\mathcal{V}^2} (a_1\tau_1 e^{-a_1\tau_1} + a_2\tau_2 e^{-a_2\tau_2}) + \frac{3}{4} \frac{\hat{\xi}}{\mathcal{V}^3} W_0^2. \quad (\text{A.158})$$

When we take the large volume limit (I) of (A.63), (A.158) has the following volume scaling

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+2/\gamma-\alpha}} - \frac{\tau_1}{\mathcal{V}^{2+1/\gamma}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.159})$$

Due to the fact that $1 < \alpha \leq 1/\gamma$, the third, the fourth and the fifth term in (A.159) are suppressed with respect to the remaining ones by an appropriate power of the volume and so we can neglect them. Thus the leading part of (A.159) takes the form

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.160})$$

We have already checked explicitly that this situation does not present any LARGE Volume minimum.

K_{11}^{-1}	K_{22}^{-1}	(I), (b), $1 < \alpha \leq 1/\gamma$	(II), (b), $1 < \alpha \leq 1/\gamma$	(II), (c), $\alpha > 1/\gamma$	(III), (b), $1 < \alpha < 1/\beta < 1/\gamma$	(III), (d), $\alpha > 1/\beta$
1	1	NO	NO	NO	NO	NO
1	2	NO	NO	NO	NO	NO
1	3	NO	NO	NO	NO	NO
1	4	NO	NO	NO	NO	NO
2	1	NO	NO	NO	NO	NO
2	2	NO	NO	NO	NO	NO
2	3	NO	NO	NO	NO	NO
2	4	NO	NO	NO	NO	NO
3	1	NO	NO	NO	NO	NO
3	2	NO	NO	NO	NO	NO
3	3	NO	NO	NO	NO	NO
3	4	NO	NO	NO	NO	OK, $\beta = 1$ (**)
4	1	NO	NO	NO	NO	NO
4	2	NO	NO	NO	NO	NO
4	3	NO	NO	OK, $\gamma = 1$ (*)	NO	NO
4	4	NO	NO	NO	NO	NO

(*) $K_{22}^{-1} \simeq \mathcal{V}\sqrt{\tau_2}$, $K_{11}^{-1} \simeq \mathcal{V}^\delta \tau_1^{2-3\delta/2}$ with $\frac{1}{\beta} + \alpha = 1 + \delta$, $\frac{1}{\beta} > \alpha$

(**) $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$, $K_{22}^{-1} \simeq \mathcal{V}^\delta \tau_2^{2-3\delta/2}$ with $\frac{1}{\gamma} + \alpha = 1 + \delta$, $\frac{1}{\gamma} > \alpha$

Table 9: Summary of Kähler moduli stabilisation results for case (4): $K_{12}^{-1} = \mathcal{V}^\alpha h_{2-3\alpha/2}(\tau_1, \tau_2)$, $\alpha > 1$.

- axion minimum at (b) along direction (II) for $1 < \alpha \leq 1/\gamma$

In this situation the full scalar potential still looks like (A.158), but the volume scaling behaviour of its terms now reads

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{h_{2-3\alpha/2}(\tau_1, \tau_2)}{\mathcal{V}^{2+1/\beta+1/\gamma-\alpha}} - \frac{\tau_1}{\mathcal{V}^{2+1/\beta}} - \frac{\tau_2}{\mathcal{V}^{2+1/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.161})$$

Given that along the direction (II) $1/\beta > 1/\gamma$, and the axion minimum is present for $1 < \alpha \leq 1/\gamma$, the dominant terms in (A.161) become

$$V \sim \frac{K_{11}^{-1}}{\mathcal{V}^{2+2/\beta}} + \frac{K_{22}^{-1}}{\mathcal{V}^{2+2/\gamma}} + \frac{1}{\mathcal{V}^3}. \quad (\text{A.162})$$

Thus we conclude that this case does not show any LVS, as we have already showed.

- axion minimum at (b) along direction (III) for $1 < \alpha \leq 1/\beta < 1/\gamma$

This situation looks like the previous one if we swap γ with β and τ_1 with τ_2 , therefore we do not need to discuss this case. Let us finally summarise in table 9 what we have found for this case.

Therefore we realise that the positive results represent cases where all the N_{small} small Kähler moduli plus a particular combination, which is the overall volume, are stabilised.

It is then straightforward to see that at this stage there will be $(h_{1,1}(X) - N_{\text{small}} - 1)$ flat directions. This terminates our proof of the LARGE Volume Claim. ■

A.3 General picture

We shall try now to draw some conclusions from the previous LARGE Volume Claim. This can be done by noticing that it is possible to understand the topological meaning of two of the four cases for the form of the elements of the inverse Kähler matrix.

Let us focus on the Kähler modulus τ_1 . From the general expression of the inverse Kähler matrix for an arbitrary Calabi-Yau (A.20), we deduce that in this case, dropping all the coefficients

$$K_{11}^{-1} \simeq -\mathcal{V}k_{11i}t^i + \tau_1^2. \tag{A.163}$$

Case (1) states that $K_{11}^{-1} \simeq \tau_1^2$, therefore the quantity $k_{11i}t^i$ has to vanish. This is definitely true if $k_{11i} = 0, \forall i = 1, \dots, h_{1,1}(X)$, that is if the volume is linear in t_1 , the 2-cycle volume corresponding to τ_1 . This is the definition of a three-fold with a K3 fibration structure over the base t_1 [34]. Thus we realise that Calabi-Yau K3 fibrations correspond to case (1). More precisely if the three-fold is a single fibration only K_{11}^{-1} will be in case (1) but not K_{22}^{-1} . On the contrary, double K3 fibrations will have both K_{11}^{-1} and K_{22}^{-1} in case (1). Thus we have proved that

Claim 2. $K_{11}^{-1} \sim \tau_1^2 \Leftrightarrow \tau_1$ is a K3 fiber over the base t_1 .

One could wonder whether this reasoning is correct being worried about possible field redefinitions since we showed that they can change the intersection numbers. However this argument is indeed correct because, as we have explained above, when one restricts himself to changes of basis which do not alter the form of the superpotential (A.25), the form of the elements of the inverse Kähler matrix do not change as the physics depends only on them and we know that it should not be modified by changes of basis. Therefore it suffices to calculate K_{11}^{-1} in one frame where the geometrical interpretation is clear.

The same procedure can be followed to prove that $K_{11}^{-1} \sim \mathcal{V}\sqrt{\tau_1}$ if and only if τ_1 is a blow-up mode resolving a point-like singularity. The blow-up of a singularity at a point P is obtained by removing the point P and replacing it with a projective space like $\mathbb{C}P^1$. This procedure introduces an extra divisor, called *exceptional*, with the corresponding extra Kähler modulus that is what we call a blow-up mode. An exceptional divisor D_1 is such that it has only its triple self-intersection number non-vanishing [35]

$$D_1 \cdot D_i \cdot D_j \neq 0 \quad \text{only if } i = j = 1. \tag{A.164}$$

Therefore if τ_1 is a blow-up then, in a suitable basis, we can always write the volume as

$$\mathcal{V} = f_{3/2}(\tau_j) - \tau_1^{3/2}, \quad j \neq 1 \tag{A.165}$$

where $f_{3/2}(\tau_j)$ is an homogeneous function of degree $3/2$. It is then clear that these blow-up modes are purely *local* effects, since the change in the volume of the Calabi-Yau as the blow-up cycle is collapsed, only goes as the volume of the cycle, with no dependence on the overall

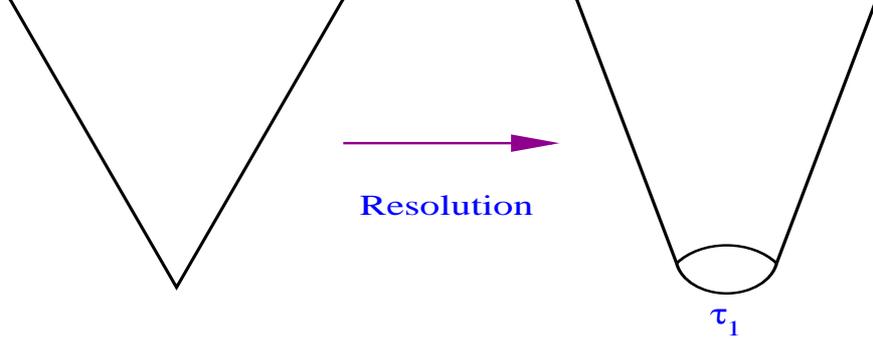


Figure 5: Blow-up cycle τ_1 resolving a point-like singularity.

volume. In fact, a change of τ_1 , $\delta\tau_1$, would generate a change of the total volume of the form

$$\delta\mathcal{V} = \frac{\partial\mathcal{V}}{\partial\tau_1}\delta\tau_1 = -\frac{3}{2}\sqrt{\tau_1}\delta\tau_1. \quad (\text{A.166})$$

Let us approximate the volume (A.165) as $\mathcal{V} \simeq f_{3/2}(\tau_j)$ and calculate the Kähler matrix

$$\frac{\partial K}{\partial\tau_1} \simeq \frac{\sqrt{\tau_1}}{\mathcal{V}} \implies \begin{cases} K_{11} = \frac{\partial^2 K}{\partial\tau_1^2} \simeq \frac{1}{\sqrt{\tau_1}\mathcal{V}}, \\ K_{1j} = \frac{\partial^2 K}{\partial\tau_j\partial\tau_1} \simeq -\frac{\sqrt{\tau_1}}{\mathcal{V}^2} \frac{\partial\mathcal{V}}{\partial\tau_j} \simeq o\left(\frac{1}{\mathcal{V}^{5/3}}\right), \quad j \neq 1. \end{cases} \quad (\text{A.167})$$

It turns out that $\frac{K_{1j}}{K_{11}} \simeq o\left(\frac{1}{\mathcal{V}^{2/3}}\right) \ll 1$, $j \neq 1$ and so we can immediately deduce the leading order term in the "11" element of the inverse Kähler matrix by simply taking the inverse of K_{11} , which gives case (3) or more explicitly $K_{11}^{-1} \simeq \mathcal{V}\sqrt{\tau_1}$. But does $K_{11}^{-1} \sim \mathcal{V}\sqrt{\tau_1}$ imply a form of the volume as $\mathcal{V} = f_{3/2}(\tau_j) - \tau_1^{3/2}$ with $j \neq 1$? We shall prove now that this is indeed the case. Let us focus on $N_{\text{small}} = 1$ without loss of generality. Then

$$K_{ij} \equiv \frac{\partial^2 K}{\partial\tau_i\partial\tau_j} = \frac{2}{\mathcal{V}} \left(\frac{1}{\mathcal{V}} \frac{\partial\mathcal{V}}{\partial\tau_i} \frac{\partial\mathcal{V}}{\partial\tau_j} - \frac{\partial^2\mathcal{V}}{\partial\tau_i\partial\tau_j} \right), \quad \text{for } i, j = 1, 2. \quad (\text{A.168})$$

We can then invert the Kähler matrix and find (by denoting $\frac{\partial\mathcal{V}}{\partial\tau_j} \equiv \mathcal{V}_j$ and similarly for the second derivative)

$$K_{11}^{-1} = \frac{\mathcal{V}}{2} \left[\frac{\frac{1}{\mathcal{V}}\mathcal{V}_2^2 - \mathcal{V}_{22}}{\left(\frac{1}{\mathcal{V}}\mathcal{V}_1^2 - \mathcal{V}_{11}\right)\left(\frac{1}{\mathcal{V}}\mathcal{V}_2^2 - \mathcal{V}_{22}\right) - \left(\frac{1}{\mathcal{V}}\mathcal{V}_1\mathcal{V}_2 - \mathcal{V}_{12}\right)^2} \right]. \quad (\text{A.169})$$

Now if we impose that at leading order $K_{11}^{-1} = c_1\mathcal{V}\sqrt{\tau_1}$ with $c_1 \in \mathbb{R} \setminus \{0\}$, we get that at leading order

$$\left[\frac{\frac{1}{\mathcal{V}}\mathcal{V}_2^2 - \mathcal{V}_{22}}{\left(\frac{1}{\mathcal{V}}\mathcal{V}_1^2 - \mathcal{V}_{11}\right)\left(\frac{1}{\mathcal{V}}\mathcal{V}_2^2 - \mathcal{V}_{22}\right) - \left(\frac{1}{\mathcal{V}}\mathcal{V}_1\mathcal{V}_2 - \mathcal{V}_{12}\right)^2} \right] = 2c_1\sqrt{\tau_1}. \quad (\text{A.170})$$

Now using the homogeneity property of the volume in terms of the 4-cycle moduli, $\tau_1\mathcal{V}_1 + \tau_2\mathcal{V}_2 = \frac{3}{2}\mathcal{V}$, we derive

$$\mathcal{V}_2 = \frac{3\mathcal{V}}{2\tau_2} - \frac{\mathcal{V}_1\tau_1}{\tau_2}, \quad \mathcal{V}_{12} = \frac{\mathcal{V}_1}{2\tau_2} - \frac{\mathcal{V}_{11}\tau_1}{\tau_2}, \quad \mathcal{V}_{22} = \frac{3\mathcal{V} - 4\mathcal{V}_1\tau_1 + 4\mathcal{V}_{11}\tau_1^2}{4\tau_2^2}. \quad (\text{A.171})$$

Now plugging (A.171) back in (A.170), we find that at leading order

$$3\mathcal{V}(1 + 2c_1\sqrt{\tau_1}\mathcal{V}_{11}) - 2\mathcal{V}_1(2\tau_1 + c_1\sqrt{\tau_1}\mathcal{V}_1) = 0. \quad (\text{A.172})$$

We can now write the general form of \mathcal{V}_1 as $\mathcal{V}_1 = c_2\mathcal{V}^\alpha\tau_1^{\frac{(1-3\alpha)}{2}}$ with $c_2 \in \mathbb{R} \setminus \{0\}$ and $\alpha \leq 1$. It follows then that

$$\mathcal{V}_{11} = c_2 \frac{\partial}{\partial \tau_1} \left(\mathcal{V}^\alpha \tau_1^{\frac{(1-3\alpha)}{2}} \right) = \alpha c_2^2 \mathcal{V}^{2\alpha-1} \tau_1^{1-3\alpha} + c_2 \frac{(1-3\alpha)}{2} \mathcal{V}^\alpha \tau_1^{-\frac{(3\alpha+1)}{2}}, \quad (\text{A.173})$$

which for $\alpha < 1$, $\alpha \neq 1/3$, at leading order reduces to

$$\mathcal{V}_{11} = c_2 \frac{(1-3\alpha)}{2} \mathcal{V}^\alpha \tau_1^{-\frac{(3\alpha+1)}{2}}, \quad (\text{A.174})$$

while for $\alpha = 1/3$ reads

$$\mathcal{V}_{11} = \frac{c_2^2}{3} \mathcal{V}^{-1/3}, \quad (\text{A.175})$$

and for $\alpha = 1$ becomes

$$\mathcal{V}_{11} = c_2 \frac{\mathcal{V}}{\tau_1^2} (c_2 - 1). \quad (\text{A.176})$$

Now if $1/2 < \alpha \leq 1$, then (A.172) at leading order looks like

$$3\mathcal{V}_{11} = c_2^2 \mathcal{V}^{2\alpha-1} \tau_1^{(1-3\alpha)}. \quad (\text{A.177})$$

For $1/2 < \alpha < 1$, using (A.174), (A.177) gives

$$3c_2 \frac{(1-3\alpha)}{2} \mathcal{V}^\alpha \tau_1^{-\frac{(3\alpha+1)}{2}} = c_2^2 \mathcal{V}^{2\alpha-1} \tau_1^{(1-3\alpha)}, \quad (\text{A.178})$$

which at leading order reduces to

$$3c_2 \frac{(1-3\alpha)}{2} \mathcal{V}^\alpha \tau_1^{-\frac{(3\alpha+1)}{2}} = 0, \quad (\text{A.179})$$

with the solution $\alpha = 1/3$ that is in contradiction with the fact that we are considering $\alpha > 1/2$. For $\alpha = 1$, using (A.176), (A.177) becomes

$$3(c_2 - 1) = c_2, \quad (\text{A.180})$$

but the solution $c_2 = 3/2 \Rightarrow \tau_1 \mathcal{V}_1 = \frac{3}{2} \mathcal{V}$, using the homogeneity property of the volume, $\tau_1 \mathcal{V}_1 + \tau_2 \mathcal{V}_2 = \frac{3}{2} \mathcal{V}$, would imply $\mathcal{V}_2 = 0$ and so we have to reject it. On the contrary if $\alpha = 1/2$, then (A.172) at leading order takes the form

$$6c_1\sqrt{\tau_1}\mathcal{V}_{11} = (2c_1c_2^2 - 3), \quad (\text{A.181})$$

and by means of (A.174), this expression at leading order becomes

$$c_1c_2\mathcal{V}^{1/2}\tau_1^{-\frac{3}{4}} = 0, \quad (\text{A.182})$$

that clearly admits no possible solution. Finally if $\alpha < 1/2$, then (A.172) at leading order becomes

$$3\mathcal{V}(1 + 2c_1\sqrt{\tau_1}\mathcal{V}_{11}) = 0. \tag{A.183}$$

Due to (A.174), (A.183) for $\alpha \neq 1/3$ reads

$$3\mathcal{V}\left(1 + c_1c_2(1 - 3\alpha)\mathcal{V}^\alpha\tau_1^{-\frac{3\alpha}{2}}\right) = 0, \tag{A.184}$$

whereas using (A.175), (A.183) for $\alpha = 1/3$ takes the form

$$0 = 3\mathcal{V}\left(1 + 2c_1\sqrt{\tau_1}\frac{c_2^2}{3}\mathcal{V}^{-1/3}\right) \simeq 3\mathcal{V}, \tag{A.185}$$

which is impossible to solve. Now focusing on (A.184), if $\alpha < 0$ we do not find any solution, whereas if $0 < \alpha < 1/2$, (A.184) reduces to

$$3c_1c_2(1 - 3\alpha)\mathcal{V}^{\alpha+1}\tau_1^{-\frac{3\alpha}{2}} = 0, \tag{A.186}$$

which is solved by $\alpha = 1/3$ that is in disagreement with the fact that we are considering $\alpha \neq 1/3$. Lastly for $\alpha = 0$, (A.184) looks like

$$3\mathcal{V}(1 + c_1c_2) = 0, \tag{A.187}$$

which admits a solution of the form $c_2 = -\frac{1}{c_1}$. Therefore we have $\mathcal{V}_1 = -\frac{\sqrt{\tau_1}}{c_1}$ and $\mathcal{V}_{11} = -\frac{1}{2c_1\sqrt{\tau_1}}$ that imply an overall volume of the form $\mathcal{V} = \lambda_2\tau_2^{3/2} - \lambda_1\tau_1^{3/2}$. It is easy now to generalise this result for $N_{\text{small}} > 1$ given that we have shown that

Claim 3. $K_{11}^{-1} \sim \mathcal{V}\sqrt{\tau_1} \Leftrightarrow \mathcal{V} = f_{3/2}(\tau_j) - \tau_1^{3/2}$ with $j \neq 1 \Leftrightarrow \tau_1$ is the only blow-up mode resolving a point-like singularity.

We point out also that the stressing that the blow-up has to resolve a point-like singularity is exactly related to the fact that it has to be a purely local effect. In fact, the resolution of a hyperplane or line-like singularity would *not* be a local effect, even though it would still enable us to take a sensible large volume limit by sending τ_b large and keeping τ_1 small. In this case, it is plausible to expect an expression for the overall volume of the form

$$\mathcal{V} = \tau_b^{3/2} - \tau_b\tau_1^{1/2} - \tau_1^{3/2}. \tag{A.188}$$

If we approximate the volume as $\mathcal{V} \simeq \tau_b^{3/2}$, we can see that the change of \mathcal{V} with the increase of the cycle size τ_1 does not depend on powers of τ_1 alone as in (A.166) but it looks like

$$\delta\mathcal{V} = \frac{\partial\mathcal{V}}{\partial\tau_1}\delta\tau_1 \simeq -\frac{1}{2}\frac{\tau_b}{\sqrt{\tau_1}}\delta\tau_1 \simeq -\frac{1}{2}\frac{\mathcal{V}^{2/3}}{\sqrt{\tau_1}}\delta\tau_1. \tag{A.189}$$

Moreover the case (A.188) gives rise to an inverse Kähler metric of the form $K_{11}^{-1} \simeq \mathcal{V}^{1/3}\tau_1^{3/2}$ which does not satisfy the condition of the LARGE Volume Claim exactly because τ_1 is not resolving a point-like singularity.

Let us show now that if we have $N_{\text{small}} = 2$ with one small modulus τ_2 which is a local blow-up mode then the cross term K_{12}^{-1} has to be in case (1): $K_{12}^{-1} \sim \tau_1 \tau_2$. Without loss of generality we can consider just one large modulus τ_3 and so the volume will look like $\mathcal{V} = f_{3/2}(\tau_3, \tau_1) - \tau_2^{3/2}$. The computation of the Kähler metric gives an expression like (A.168) but now for $i, j = 1, 3$ with in addition:

$$K_{22} = \frac{3}{2\mathcal{V}\sqrt{\tau_2}}, \quad K_{2j} = -\frac{3\sqrt{\tau_2}}{\mathcal{V}^2}\mathcal{V}_j, \quad \text{with } j = 1, 3, \quad (\text{A.190})$$

and the "12" element of the inverse Kähler metric in full generality reads

$$K_{12}^{-1} = \frac{\tau_2 \mathcal{V}^2 (\mathcal{V}_1 \mathcal{V}_{13} - \mathcal{V}_1 \mathcal{V}_{33})}{(3\tau_2^{3/2} - \mathcal{V}) (\mathcal{V}_{11} \mathcal{V}_3^2 + \mathcal{V}_{33} \mathcal{V}_1^2 - 2\mathcal{V}_1 \mathcal{V}_3 \mathcal{V}_{13}) - \mathcal{V}^2 (\mathcal{V}_{13}^2 - \mathcal{V}_{11} \mathcal{V}_{33})}. \quad (\text{A.191})$$

Using again the homogeneity property of the volume, we can find the following relations

$$\begin{aligned} \mathcal{V}_3 &= \frac{1}{\tau_3} \left[\frac{3}{2} (\mathcal{V} + \tau_2^{3/2}) - \tau_1 \mathcal{V}_1 \right], & \mathcal{V}_{13} &= \frac{1}{\tau_3} \left[\frac{\mathcal{V}_1}{2} - \tau_1 \mathcal{V}_{11} \right], \\ \mathcal{V}_{33} &= \frac{1}{4\tau_3^2} \left[3 (\mathcal{V} + \tau_2^{3/2}) + 4\tau_1 (\tau_1 \mathcal{V}_{11} - \mathcal{V}_1) \right], \end{aligned}$$

which substituted back in (A.191) give the final result

$$K_{12}^{-1} = \frac{2\mathcal{V}^2 \tau_1 \tau_2}{2\mathcal{V}^2 + 6\mathcal{V}\tau_2^{3/2} - 9\tau_2^3} \simeq \tau_1 \tau_2. \quad (\text{A.192})$$

Similarly one can check the correctness of Claim 3 by finding at leading order $K_{22}^{-1} \sim \mathcal{V}\sqrt{\tau_2}$. Let us summarise this result in the following

Claim 4. *If $N_{\text{small}} = 2$ and $K_{22}^{-1} \sim \mathcal{V}\sqrt{\tau_2} \Rightarrow K_{12}^{-1} \sim \tau_1 \tau_2$.*

We immediately realise that this Claim rules out the possible LARGE Volume minima along the directions (II) and (III) for the case (2), (3) and (4). However following arguments similar to the ones presented to prove Claim 3, one can show that if $K_{22}^{-1} \sim \mathcal{V}\sqrt{\tau_2}$ and so $K_{12}^{-1} \sim \tau_1 \tau_2$, K_{11}^{-1} can never be in case (4). Hence also the new would-be LVS along the directions (II) and (III) for the case (1) have to be rejected because mathematically inconsistent. Claim 4 also implies that the LVS along the direction (I) for case (2) and (3) is viable only if $K_{jj}^{-1} \sim \mathcal{V}h_{1/2}^{(j)}(\tau_1, \tau_2)$ with $\frac{\partial^2 h^{(j)}}{\partial \tau_1 \partial \tau_2} \neq 0 \forall j = 1, 2$. In fact if $\frac{\partial^2 h^{(j)}}{\partial \tau_1 \partial \tau_2}$ were vanishing, then Claim 4 would imply K_{12}^{-1} in case (1) and not (2) or (3).

In reality we understand these two cases better by realising that we can go further in our connection of the topological features of the Calabi-Yau with the elements of K^{-1} . In fact, one could wonder what happens when a singularity is not resolved by just one blow-up cycle but by several independent local blow-ups. A concrete example where this happens, is the resolution of the singularity at the origin of the quotient \mathbb{C}^2/G , where G is a finite subgroup of $SU(2)$ acting linearly on \mathbb{C}^2 . This resolution replaces the singularity by several $\mathbb{C}P^1$'s which correspond to new Kähler moduli whose number is determined by the group G . For example, if $G = \mathbb{Z}_n$, one gets $n - 1$ such $\mathbb{C}P^1$'s which play the rôle of the

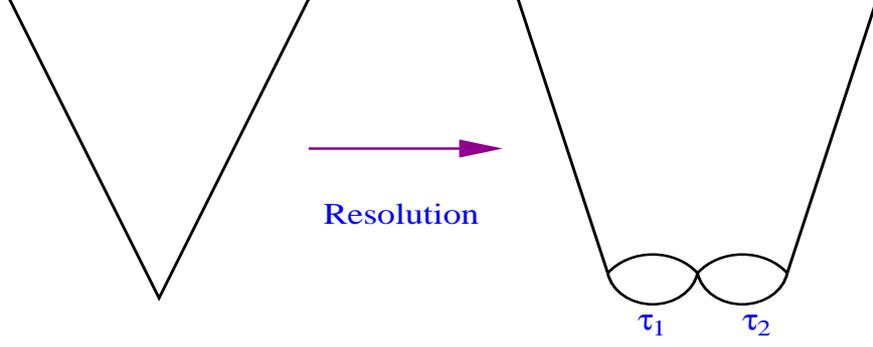


Figure 6: Resolution by two independent blow-ups τ_1 and τ_2 .

simple roots of the Lie algebra $A_{n-1} = su(n)$. After resolving the singularity of \mathbb{C}^2/G , one obtains an example of an ALE space [36].

Focusing on the case $N_{\text{small}} = 2$, in a suitable basis, the overall volume will take the general form

$$\mathcal{V} = \tau_b^{3/2} - g_{3/2}(\tau_1, \tau_2), \quad (\text{A.193})$$

with $g_{3/2}(\tau_1, \tau_2) \neq \tau_1^{3/2} + \tau_2^{3/2}$ since in that special case τ_1 and τ_2 would be blow-up cycles resolving two different point-like singularities. If the total volume is given by (A.193) then the scaling with the volume of the elements of the Kähler metric is (denoting $\partial g_{3/2}/\partial \tau_j \equiv f_j$ and similarly for the second derivative):

$$\begin{aligned} K_{bb} &\sim \frac{1}{\mathcal{V}\sqrt{\tau_b}} + \frac{\tau_b}{\mathcal{V}^2} \sim \frac{1}{\mathcal{V}^{4/3}}, & K_{12} &\sim \frac{g_{12}}{\mathcal{V}^2} + \frac{g_{12}}{\mathcal{V}} \sim \frac{1}{\mathcal{V}}, \\ K_{jb} &\sim \frac{\sqrt{\tau_b}}{\mathcal{V}^2} g_j \sim \frac{1}{\mathcal{V}^{5/3}}, & K_{jj} &\sim \frac{g_j^2}{\mathcal{V}^2} + \frac{g_{jj}}{\mathcal{V}} \sim \frac{1}{\mathcal{V}}, \quad j = 1, 2 \end{aligned}$$

therefore producing

$$K_{ij} \sim \left(\begin{array}{cc|c} \mathcal{V}^{-1} & \mathcal{V}^{-1} & \mathcal{V}^{-5/3} \\ \mathcal{V}^{-1} & \mathcal{V}^{-1} & \mathcal{V}^{-5/3} \\ \hline \mathcal{V}^{-5/3} & \mathcal{V}^{-5/3} & \mathcal{V}^{-4/3} \end{array} \right) \quad (\text{A.194})$$

where we have highlighted with a box the submatrix with the leading powers of the volume. We have just to invert this submatrix to get K_{jj}^{-1} for $j = 1, 2$ and K_{12}^{-1} which turn out to be given by

$$\begin{aligned} K_{11}^{-1} &\sim \mathcal{V} \left(\frac{g_{22}}{2\Delta} \right) = \mathcal{V} h_{1/2}^{(1)}(\tau_1, \tau_2), & K_{22}^{-1} &\sim \mathcal{V} \left(\frac{g_{11}}{2\Delta} \right) = \mathcal{V} h_{1/2}^{(2)}(\tau_1, \tau_2), \\ K_{12}^{-1} &\sim \mathcal{V} \left(\frac{g_{12}}{2\Delta} \right) = \mathcal{V} f_{1/2}(\tau_1, \tau_2), & \text{where } \Delta &\equiv g_{11}g_{22} - g_{12}^2. \end{aligned} \quad (\text{A.195})$$

Following arguments similar to the ones used to prove Claim 3 we can also show that starting from $K_{11}^{-1} \sim \mathcal{V} h_{1/2}^{(1)}(\tau_1, \tau_2)$ with $h^{(1)}$ really dependent on both the small moduli, the form of the volume has to be (A.193). A good intuition for this result is that by setting $\tau_1 = \tau_2$, this is the only way to recover Claim 3. Therefore we have shown that

Claim 5. $K_{11}^{-1} \sim \mathcal{V}h_{1/2}^{(1)}(\tau_1, \tau_2)$ and $K_{22}^{-1} \sim \mathcal{V}h_{1/2}^{(2)}(\tau_1, \tau_2) \Leftrightarrow \mathcal{V} = f_{3/2}(\tau_j) - g_{3/2}(\tau_1, \tau_2)$ with $j \neq 1, 2 \Leftrightarrow \tau_1$ and τ_2 are two independent blow-up modes resolving the same point-like singularity,

along with

Claim 6. If $K_{11}^{-1} \sim \mathcal{V}h_{1/2}^{(1)}(\tau_1, \tau_2)$ and $K_{22}^{-1} \sim \mathcal{V}h_{1/2}^{(2)}(\tau_1, \tau_2) \Rightarrow K_{12}^{-1} \sim \mathcal{V}f_{1/2}(\tau_1, \tau_2)$.

In generality we can conclude that whenever $K_{jj}^{-1} \simeq \mathcal{V}h_{1/2}^{(j)}(\tau_1, \tau_2, \dots, \tau_{N_{\text{small}}})$ then τ_j is a blow-up resolving a point-like singularity. Moreover, if $h_{1/2}^{(j)}$ depends only on τ_j , then τ_j will be the only blow-up cycle resolving the singularity and $K_{ij}^{-1} \sim \tau_i \tau_j \forall i \neq j = 1, \dots, N_{\text{small}}$; on the contrary, if $h_{1/2}^{(j)}$ depends on several 4-cycle moduli, say τ_j and τ_k for $j \neq k$, the singularity is resolved by all those independent 4-cycles with $K_{jk}^{-1} \sim \mathcal{V}f_{1/2}(\tau_j, \tau_k)$ and $K_{il}^{-1} \sim \tau_i \tau_j \forall i \neq l = 1, \dots, N_{\text{small}}$ for $l = j, k$.

These considerations imply that the would-be LVS along direction (I) for case (2) is mathematically inconsistent. Thus for $N_{\text{small}} = 2$ we are left with just two cases that give rise to a LARGE Volume minimum located at $\mathcal{V} \sim e^{a_1 \tau_1} \sim e^{a_2 \tau_2}$:

1. $K_{12}^{-1} \sim \tau_1 \tau_2$, $K_{jj}^{-1} \sim \mathcal{V} \sqrt{\tau_j} \forall j = 1, 2$ where τ_1 and τ_2 are local blow-up modes resolving two different point-like singularities;
2. $K_{12}^{-1} \sim \mathcal{V}f_{1/2}(\tau_1, \tau_2)$, $K_{jj}^{-1} \sim \mathcal{V}h_{1/2}^{(j)}(\tau_1, \tau_2) \forall j = 1, 2$ where τ_1 and τ_2 are two independent blow-up modes resolving the same point-like singularity.

The only difference between these two cases is that the first one works always whereas the second one gives a LVS only if, writing the volume as $\mathcal{V} = f_{3/2}(\tau_j) - g_{3/2}(\tau_1, \tau_2)$ with $j \neq 1, 2$, the homogeneous function $g_{3/2}$ is symmetric in τ_1 and τ_2 . This can be seen easily by comparing the solution of the two minimisation equations $\frac{\partial \mathcal{V}}{\partial \tau_1} = 0$ and $\frac{\partial \mathcal{V}}{\partial \tau_2} = 0$, then substituting the solution we are looking for, that is $a_1 \tau_1 \simeq a_2 \tau_2$, recalling (A.195) and lastly finding that we do not get a contradiction only if $g_{3/2}$ is symmetric.

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