Phase transitions and linear response in strongly coupled systems

A holographic approach

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Declaration

This thesis is based on research that was carried out along with my supervisor, Jerome Gauntlett, and several collaborators between October 2013 and October 2016. Chapters 2, 4, 5, 6 and 7 are based on the following papers (referred to as [1–5] in the bibliography):

- E. Banks, A. Donos, and J. P. Gauntlett, "Thermoelectric DC conductivities and Stokes flows on black hole horizons," *JHEP* **10** (2015) 103, `arXiv:1507.00234 [hep-th]`
- E. Banks, A. Donos, J. P. Gauntlett, T. Griffin, and L. Melgar, "Holographic thermal DC response in the hydrodynamic limit," *Class. Quant. Grav.* **34** no. 4, (2017) 045001, `arXiv:1609.08912 [hep-th]`
- E. Banks, A. Donos, J. P. Gauntlett, T. Griffin, and L. Melgar, "Thermal backflow in CFTs," *Phys. Rev.* **D95** no. 2, (2017) 025022, `arXiv:1610.00392 [hep-th]`
- E. Banks and J. P. Gauntlett, "A new phase for the anisotropic N=4 super Yang-Mills plasma," *JHEP* **09** (2015) 126, `arXiv:1506.07176 [hep-th]`
- E. Banks, "Phase transitions of an anisotropic N=4 super Yang-Mills plasma via holography," *JHEP* **07** (2016) 085, `arXiv:1604.03552 [hep-th]`

In addition, chapter 3 is based on unpublished work arising from discussions with Jerome Gauntlett. The numerical plots in [3] were produced by Aristomenis Donos, whilst all other plots were produced by myself, unless stated otherwise. To the best of my knowledge, all the material in this thesis that is not my own work has been properly acknowledged.

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Abstract

Understanding strongly coupled systems is an important area of theoretical physics, and has wide ranging applications from quantum chromodynamics to condensed matter physics. This thesis uses holographic methods to understand two particular aspects of strongly coupled systems - linear response and phase transitions.

Firstly, we consider a general class of electrical black holes in Einstein-Maxwell-scalar theory, that are holographically dual to conformal field theories at finite charge density and explicitly break translational invariance. By considering the linearised perturbations of these background black holes, we show that the DC thermoelectric conductivity of these systems can be determined by solving a set of linearised Navier-Stokes equations on the event horizon of the dual black hole. We demonstrate how to apply this framework in practice with several examples.

Next, we consider this framework in the hydrodynamic limit, for the simpler case of Einstein gravity. We show that the full stress-energy response, rather than just the thermal conductivity, can be determined in this limit, and compare the results with the fluid/gravity correspondence. We then consider more general hydrodynamics, and demonstrate that periodically deformed field theories exhibit thermal backflow when a DC thermal source is applied

Finally, we study black hole solutions of type IIB supergravity that describe N=4 supersymmetric Yang-Mills plasma with an anisotropic spatial deformation. We show that, by preserving additional scalar modes from the consistent truncation of IIB supergravity on the five-sphere, these black holes have low temperature instabilities. We construct new thermodynamically preferred black hole solutions, and show that the phase transition between these black hole solution has unusual critical exponents that is not captured by the normal Landau-Ginzburg exponents. We consider various extensions to this, such as introducing a chemical potential, and construct a more complete phase diagram for the theory.

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Introduction

Chapter 1

Introduction and thesis outline

Strongly coupled systems describe some of the most interesting and important physical phenomena, with examples ranging from quantum chromodynamics (QCD) and the theory of quarks, to topics in condensed matter physics such as high temperature superconductors and in questions about the early stages of the universe and the big bang. At the same time, the lack of mathematical techniques and a well understood framework to describe these systems, in particular a lack of a perturbative expansion, make strongly coupled systems some of the most poorly understood and difficult problems to tackle. The purpose of this thesis is to present novel techniques and findings to further our understanding of strongly coupled systems, especially with regards to their phase transitions and linear response.

Since its introduction in 1997 by Maldacena, the AdS/CFT correspondence[1] has provided one possible framework to understand strongly coupled systems [6]. The correspondence establishes a relationship between a certain class of quantum field theories - namely conformal field theories (CFTs) that have a conformal symmetry, and gravitational theories in an anti-de Sitter ("AdS") spacetime. More precisely, the correspondence states that a d dimensional CFT lives on the boundary of a dual string theory or M-theory realised on AdS_{d+1}. The power of the correspondence lies in the fact that it is a strong/weak duality - a strongly coupled CFT is dual to a weakly coupled gravitational theory and vice-versa. Specifically, there is a one to one map between strongly coupled CFTs in the large N limit (when we only consider planar Feynman diagrams) in d dimensions, and weakly curved classical supergravity in $d+1$ dimensions. The supergravity problem is often substantially easier to solve, and has led to numerous studies focused on condensed matter and QCD applications (see for example, [7–9], and references therein).

In this thesis, we will utilise the AdS/CFT correspondence to demonstrate a number of results, ultimately motivated by condensed matter physics. The rest of this introduction will therefore be devoted to explaining the important background physics required, and some of the previous work that has been done in the field. We will first present the

[1]We will use holography, AdS/CFT and gauge/gravity duality interchangeably throughout the thesis, and we will be referring to the more general correspondence between gauge fields and gravitational theories than in Maldacena's original work.

framework and physics behind holography, before describing the holographic dictionary. Then we will present some applications of the correspondence, and discuss linear response, phase transitions and the fluid/gravity correspondence. Finally, we will give a brief outline of the rest of the thesis.

1 Gauge/gravity duality

We will now present a brief introduction to the underlying physics of the AdS/CFT correspondence. For a complete review on the correspondence, please refer to [10].

1.1 Conformal field theories

The principle of symmetry is one of major importance across all areas of physics. Important examples in relativistic quantum field theories are the Lorentz and Poincaré symmetries, which describe the symmetries due to rotations and boosts (as well as translations in the case of Poincaré) in Minkowski space. The conformal group is a natural extension to the Poincaré group, and describes relativistic systems that also possess scale invariance[2], meaning that the physics looks the same at all length scales. Under conformal transformations, lengths can change, but angles are locally invariant and the casual structure remains.

To be more precise, under conformal transformations the metric scales by $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$, where Ω is a positive arbitrary scale factor. It is possible to show that the set of conformal transformations contains Poincaré transformations, as well as the dilatation and special conformal transformations, defined by (in Minkowski spacetime)

$$\begin{aligned} x^\mu &\to \lambda x^\mu\,, \\ x^\mu &\to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}\,, \end{aligned} \tag{1.1}$$

respectively.

Let us now turn to the infinitesimal conformal transformation. The algebra generating these transformations contains the generators of the Lorentz transformations, $M_{\mu\nu}$, spacetime translations, P_μ, the dilatations, D, and special conformal transformations, K_μ, with the generators obeying the following commutation relations

$$\begin{gathered} [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\,, \qquad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)\,, \\ [M_{\nu\mu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}\,, \qquad [M_{\mu\nu}, D] = 0\,, \qquad [D, K_\mu] = iK_\mu\,, \\ [D, P_\mu] = -iP_\mu\,, \qquad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D\,, \end{gathered} \tag{1.2}$$

[2]Note that a theory can be scale invariant but not conformal, as discussed in [11].

It is easy to see that this algebra is isomorphic to $SO(2,d)$, by making the following redefinitions [12]

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu}\,, \\ J_{-1\mu} &= \frac{1}{2}(P_\mu - K_\mu)\,, \\ J_{0\mu} &= \frac{1}{2}(P_\mu + K_\mu)\,, \\ J_{-10} &= D\,, \end{aligned} \tag{1.3}$$

which leads to the algebra

$$[J_{mn}, J_{pq}] = i(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp})\,, \tag{1.4}$$

where η is the metric diag(-1,-1,1,....,1).

Since mass can be rescaled under a conformal transformations, they are no longer good quantum numbers to describe the observables in our CFT. Instead, CFTs have a different set of quantum numbers. Any field, $\phi(x)$ that transforms covariantly under the irreducible representations of the conformal algebra has a fixed scaling dimension, Δ, that is defined by the transformation under $x \to \lambda x$

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x)\,. \tag{1.5}$$

This Δ captures the transformation property of ϕ of dilatations with $[D,\phi] = i(x_\mu\partial^\mu + \Delta)\phi$. Along with the quantum numbers from the Lorentz representation, these quantum numbers can label the observables in a CFT.

It is interesting to note that the condition of unitarity on the fields imposes a lower bound on the allowed value of Δ for any field in a CFT. For example, in d dimensions, the scaling dimension of a scalar field must satisfy

$$\Delta \geq \frac{d-2}{2}\,. \tag{1.6}$$

For a discussion of this result in detail, we refer the reader to [13, 14]. We will return to the unitary bound in the context of AdS/CFT later.

It is important to stress that conformal invariance at the level of a classical theory does not necessarily imply conformal invariance of a quantum theory. Indeed, there is a natural component of quantum theories that in general will break conformal invariance - renormalisation. In general a quantum theory will have a renormalisation scale, μ, which mean that the dimensionless couplings, g, run

$$\mu\frac{\partial}{\partial\mu}g = \beta(g)\,. \tag{1.7}$$

Therefore in general, quantum theories will only be conformal at certain points where $\beta(g) = 0$, such as UV or IR fixed points. In addition, some theories, such as $\mathcal{N} = 4$ super Yang-Mills have $\beta(g) = 0$ for all values of g, and hence are conformal everywhere [15].

1.2 AdS spacetimes

On the other side of the AdS/CFT correspondence, we have AdS space. Whilst AdS spacetimes were first studied in the 1970s, it was only with the advent of AdS/CFT [6, 16, 17] that their full usefulness became apparent. AdS_{d+1} spacetime is the maximally symmetric solution of the Einstein equations with a negative cosmological constant, Λ,

$$R_{\mu\nu} = \frac{2\Lambda}{d-1} g_{\mu\nu} \,. \tag{1.8}$$

AdS_{d+1} space is a Minkowskian generalisation of a hyperboloid, defined in $d+2$ dimensions by a solution to the equation

$$y_0^2 + y_{d+1}^2 - \sum_{i=1}^{d} y_i^2 = L^2 \,, \tag{1.9}$$

with line element $ds^2 = -dy_0^2 - dy_{d+1}^2 + \sum_{i=1}^{d} dy_i^2$. Here, L is the AdS radius, which will often be set to one for convenience. There is a clear $SO(2, d)$ isometry in the spacetime, the same as the conformal group in $d + 1$ dimensions.

An alternative parameterisation of the hyperboloid, labelled by $(t, r, \vec{x})$ is given by

$$y_0 = \frac{1}{2r}(1 + r^2(L^2 + \vec{x}^2 - t^2)) \tag{1.10}$$

$$y_{d+1} = Lrt \tag{1.11}$$

$$y_{1,\ldots,d-1} = Lrx_{1,\ldots,d-1} \tag{1.12}$$

$$y_d = \frac{1}{2r}(1 - r^2(L^2 - \vec{x}^2 + t^2)) \,. \tag{1.13}$$

This is the Poincaré patch, with metric

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i}^{d-1} dx_i^2\right) \,. \tag{1.14}$$

As we shall discuss later, this particular geometry is very important in the context of holography. In this coordinate system, the constant r slices are isomorphic to d dimensional Minkowski spacetime, scaled by a factor of r, hence the name. The boundaries of the spacetime are the conformal boundary at $r = \infty$ and a Poincaré killing horizon at $r = 0$. This coordinate system covers half of the AdS hyperboloid, as shown in figure 1.

Often, calculations on the Poincaré patch can be made easier by defining $z = L^2/r$ as

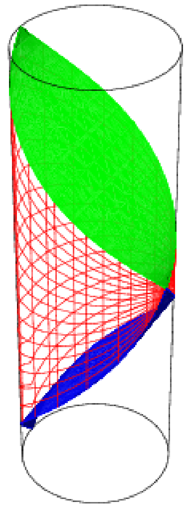


Figure 1: Cartoon showing Poincaré patch in global AdS space. The past horizon of the patch is shown in blue, the future horizon in green and the AdS boundary is in red. The patch covers half of the global AdS space. Figure taken from [18].

a rescaled coordinate. Now the conformal boundary is located at $z = 0$, whilst the killing horizon is at $z = \infty$. In this coordinate system, the metric is given by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + \sum_i dx_i^2\right) = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) . \tag{1.15}$$

One remarkable fact is that AdS spacetimes are stable in the presence of scalar fields with tachyonic masses, provided that the absolute value of the mass is sufficiently small. This is known as the Breitenlohner-Freedman (BF) bound [19], and in $d + 1$ dimensions is given by

$$m^2 \geq -\frac{d^2}{4} . \tag{1.16}$$

As we shall see later, this result has many consequences for holographic superconductors.

1.3 AdS/CFT and the holographic principle

Having laid the groundwork, we are now in a position to discuss AdS/CFT. Whilst there are many examples [10], here we will discuss the original and most prominent example, which relates $\mathcal{N} = 4$ super Yang-Mills (SYM) theories to IIB superstring theory on $AdS_5 \times S^5$. The exact form of the correspondence is as follows:

$\mathcal{N} = 4$ super Yang-Mills theory with gauge group SU(N) and coupling constant g_{YM} is *dual* to type IIB string theory with string length l_s and coupling constant g_s on $AdS_5 \times \mathrm{S}^5$, with radius of curvature L.

The free parameters in the two theories are related to each other by

$$g_{YM}^2 = 2\pi g_s\,, \qquad (L/l_s)^4 = 2N g_{YM}^2\,. \tag{1.17}$$

What is the meaning of the word *dual* in the above description? The statement of the correspondence is that the two theories are identical, and describe exactly the same physics. If this conjecture holds, then all of the physics on one side of the duality will exactly map to the physics on the other side. This is incredibly powerful - on the one hand, we have a theory of quantum gravity, whilst on the other we have a gauge theory in flat space, without any hint of gravity. The correspondence states that these two theories are describing the same underlying physics.

From our previous discussion, we can see the isometry group of AdS space is exactly the same as the conformal group of the SYM theory, SO(2,4) in both. The correspondence goes much deeper than that, and actually says that the theories are the same at the level of partition functions. We will discuss this idea further in the next section.

A useful picture to geometrically visualise AdS/CFT is with the Poincaré patch. We start with the CFT, which is in Minkowski space on the conformal boundary of our bulk theory. As we move along the radial coordinate, we are still on Minkowski space since we are in the Poincaré patch. However, our metric is rescaled by a factor of r. The different radial slice corresponds to the boundary theory at a different wavelengths or energies, and so the radial coordinate has a very particular meaning - it is the geometrical realisation of the renormalisation group (RG) flow of the CFT.

String theory is currently best understood in the perturbative regime, so it is helpful to analyse the correspondence in the limit of weak coupling, so taking $g_s \ll 1$, whilst keeping L/l_s constant. At leading order, this string theory then reduces to classical string theory, meaning that we only include tree level diagrams in string perturbation theory, rather than the full perturbative expansion that would include higher genus diagrams. In the dual description, the equivalent statement is that we are taking $g_{YM} \ll 1$ whilst keeping $g_{YM}^2 N$ fixed. This is known as the 't Hooft limit, and is the planar limit of the gauge theory (when only Feynman diagrams that can be constructed on a plane survive) [20].

For what follows we will be interested in strongly coupled theories. In that case, we can set the 't Hooft parameter, $g_{YM}^2 N \to \infty$. From the duality above, that corresponds to taking $l_s/L \to 0$. Here, the string length is much smaller than the radius of curvature, and so we are in the point particle limit of string theory - supergravity. Our strongly coupled field theory in the large N limit is dual to a weakly coupled classical supergravity theory. It is this form of the correspondence that we will be concerned with in this thesis. Following our earlier discussion, we are interested in theories that are CFTs at fixed points (having say, a UV fixed point), rather than everywhere in the RG flow. This corresponds to geometries in the bulk gravitational theory that are asymptotically AdS, rather than AdS everywhere. We have therefore reduced our strongly coupled QFT problem to something that is much simpler, solving a weakly coupled supergravity problem in a geometry that is asymptotically AdS.

The original AdS/CFT correspondence of Maldacena related two particular theories - $\mathcal{N} = 4$ SYM in four dimensions with gauge group $SU(N)$, and type IIB superstring theory defined on $AdS_5 \times S^5$. Since then, many examples in holography have been found, although a general proof of AdS/CFT is an outstanding problem. For example, ABJM theory is holographically dual to $AdS_4 \times S^7$ in M theory [21], whilst it is possible to construct conformal field theories that are dual to $AdS_5 \times X$, where X is some suitably chosen manifold rather than being restricted to S^5 (see, for example [10]). In addition, the constraint that the field theory is conformal can be relaxed. For example, non relativistic spacetimes are dual to Liftshitz spacetimes (geometries where space and time scale differently) [22], whilst relevant or marginal operators can break conformal invariance away from a particular UV fixed point of our theory. We will return to this later.

1.4 The holographic dictionary

Now that we have established the connection between quantum field theories (QFTs) and their dual gravitational description, we will make precise the map between two theories - this is the so-called "holographic dictionary".

The single most important quantity in any quantum field theory is the partition function. It tells you everything you could possibly want to know about a system - obtaining a partition function means that you have completely solved the theory. Schematically, for theories with a Lagrangian description, the partition function of a QFT can be represented as

$$Z_{QFT}[\phi_0] = \int \mathcal{D}A \exp\left(i(S_{QFT} + \int \phi_0\, \mathcal{O}(A))\right) , \tag{1.18}$$

which is a path integral over the fields, A. $\mathcal{O}(A)$ is an operator of theory, expressed as a function of A, with a corresponding source, ϕ_0.

In high energy physics, ϕ_0 is often simply used as a mathematical trick allowing us to calculate the correlation functions of the corresponding operator. However, in condensed matter physics, ϕ_0 has a physical meaning as some source, such as an electric source or thermal gradient. The idea of holography is to turn the source in the boundary theory into a dynamical field in the bulk gravitational theory that satisfies its own equations of motion, and can couple to the other fields in the gravitational theory.

We now explain the precise definition of *duality* in holography. First, we introduce a field in the bulk that is dual to our particular observable, and impose suitable boundary conditions at the conformal boundary. The two theories are then dual if the partition function of the QFT is equivalent to a partition function on the bulk. We will discuss what we mean by suitable boundary conditions later in this section, but roughly speaking, we can take $\phi(r, \vec{x}) \to \phi_0(\vec{x})$ as $r \to \infty$. Mathematically, this idea is expressed in the GKPW formula[3] [16, 17]

$$Z_{QFT}[\phi_0] = Z_{string}[\phi(r, \vec{x})|_{r\to\infty} = \phi_0(\vec{x})] \,. \tag{1.19}$$

At this point we impose the conditions that we have a strongly coupled CFT and are in the perturbative regime of string theory. From the argument in the previous section, we now have a weakly coupled classical supergravity, and we can approximate its partition function by its dominant saddle point. We now have

$$Z_{QFT}[\phi_0] \approx e^{iS_{bulk}}|_{\phi\to\phi_0} \,, \tag{1.20}$$

where the right hand side is the on-shell bulk action, and $\phi \to \phi_0$ is understood to be at the boundary. The right hand side is something that modern physics has a good handle of - our difficult, strongly coupled field theory has been reduced to something more manageable.

A toy example: scalar fields

Whilst string theory has an infinite spectrum of different fields (in towers of increasing mass states), we are typically only interested in the lowest mass states in the dual field theory. The most important fields are the metric and $U(1)$ gauge field. As we shall discuss later, these are dual to the stress energy tensor and an operator with a *global* $U(1)$ symmetry. Physically, sources of these dual operators correspond to deforming our CFT through matter deformations (such as putting the system on a lattice) and allowing an electric current source in our system.

Another important field, which serves as an excellent model to demonstrate the dictionary in practice, is the scalar field, which is dual to a scalar operator in the field theory. To see the dictionary in action, let's consider a simple toy model of a massive bulk scalar field, described by the action

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x\sqrt{-g} \left(R + \frac{d(d-1)}{L^2} - \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right) \,. \tag{1.21}$$

Solutions of ϕ with non trivial radial dependence will, in general, deform the bulk geometry away from AdS spacetime. This corresponds to deforming our QFT away from a CFT by some operator in our dual field theory. However, for theories of interest, we

[3] Note that, in general, the field theory and bulk theory will both have divergences that have to be appropriately renormalised - we will return to this later.

want our field theory to approach some fixed point in the UV, and so we impose that our boundary condition is asymptotically AdS (which we will chose to be $z = 0$). Near the conformal boundary, the scalar equation of motion is therefore given by

$$z^2\partial_z^2\phi - (d-1)z\partial_z\phi = m^2L^2\phi\,, \tag{1.22}$$

where we have assumed that ϕ only has radial dependence.

One can show that the asymptotic expansion of ϕ is given by

$$\phi = z^{d-\Delta}\phi_{(0)} + z^{\Delta}\phi_{(1)} + ... \qquad \text{as}\, z \to 0\,. \tag{1.23}$$

We see that our scalar field is fixed by two boundary conditions (as expected for a second order differential equation), as well as another parameter Δ, which satisfies

$$\Delta(\Delta - d) = (Lm)^2\,. \tag{1.24}$$

It can be shown [16] that this Δ corresponds to the scaling dimension of the operator in the CFT, as discussed in section 1.1. The CFT unitary bound (1.6) implies the condition that $\Delta \geq (d-2)/2$, whilst demanding that Δ is real (a unitary bound in the bulk) gives us the condition that

$$m^2 \geq -\frac{d^2}{4}\,. \tag{1.25}$$

This is exactly the BF bound from (1.16) that was discussed earlier.

When $d/2 < \Delta \leq d$, $\phi_{(0)}$ and $\phi_{(1)}$ correspond to the source and expectation value (VEV) of the scalar operator in the dual CFT. Simlarly, when $(d-2)/2 \leq \Delta < d/2$, there is an alternative quantisation in which $\phi_{(0)}$ and $\phi_{(1)}$ correspond to the VEV and source of the field instead. Whilst both theories have the same bulk behaviour, the CFTs they correspond to are fundamentally different. Depending on the value of Δ, we impose one of these two conditions on the asymptotic fall off of the dual bulk field. We now have a complete prescription to start using the holographic dictionary. For a given source and VEV, we fix the asymptotic fall off of the scalar fields through (1.23), solve in the bulk, and use (1.20) to construct the partition function of the CFT.

The scalar operator will be a relevant or marginal deformation of the theory if $\Delta \leq d$. In this case, this implies that the leading term in (1.23) is either constant or zero on the boundary - this is the exact requirement that we need to have an asymptotically AdS spacetime. We therefore see that relevant operators can be turned on in the field theory without destroying the UV fixed point, as one would expect. On the other hand, if $\Delta > d$, we have an irrelevant deformation which will blow up on the conformal boundary, and take us outside the understood realm of AdS/CFT.

Note that if $\Delta = d/2$, then the expansion (1.23) no longer holds. Instead, have an asymptotic expansion of the form

$$\phi = z^{d/2}\phi_{(0)} + z^{d/2}\log(z)\phi_{(1)} + ... \qquad \text{as}\, z \to 0\,. \tag{1.26}$$

In this case there is only one possible quantisation [23], with the source corresponding to $\phi_{(0)}$ and $\phi_{(1)}$ is the VEV.

Finite temperature and charge density

So far we have been concerned with theories at zero temperature, in which our bulk theory is AdS. In the previous section, we introduced relevant operators to deform our theory away from AdS in the bulk, which allowed us to have some RG flow away from a fixed point in the UV in the dual field theory. An obvious question is therefore if there is some deformation that we can make to introduce some form a temperature scale into our system.

More precisely, we want to introduce some sort of scale into our theory, whilst preserving translational and rotational symmetries. If we take the simplest case of pure Einstein gravity action with negative cosmological constant $\Lambda = -\frac{d(d-1)}{2}$, we find the following solution to the equations of motion that preserves the appropriate symmetries

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + \sum_i dx_i^2\right)\,, \tag{1.27}$$

where

$$f(z) = 1 - \left(\frac{z}{z_h}\right)^d\,. \tag{1.28}$$

This is the AdS-Schwarschild black hole, and there is now a scale in the theory, the radius of the black hole, z_h. The presence of temperature in our theory therefore leads to a planar black hole, with Hawking temperature [24]

$$T = \frac{d}{4\pi z_h}\,. \tag{1.29}$$

To calculate the temperature of the dual field theory, it is helpful to recall the calculation of the Hawking temperature by analytic continuation. In that calculation, we analytically continue the metric to a Euclidean signature, and demand regularity of the spacetime at the horizon. This changes our time coordinate, t, to a Euclidean coordinate, τ, which is periodic with period θ. The temperature of the black hole is then given by $T = 1/\theta$.

Returning to AdS/CFT, we can determine the background metric of the boundary theory by reading off the asymptotic expansion of the metric

$$g_{\mu\nu} = \frac{L^2}{z^2}g^0_{\mu\nu} + ... \qquad \text{as} \quad z \to 0\,, \tag{1.30}$$

and we can identify $g^0_{\mu\nu}$ as the non-dynamical boundary metric. At this point, notice that the τ coordinate will be the same in both the bulk metric and the boundary metric, and will therefore have the same periodicity. Thus, in the Weyl frame of 1.30, we conclude that the temperature of our boundary theory is given by the Hawking temperature of the black hole.

This is a remarkable conclusion. Information about the temperature of the CFT (which lives on the boundary of our black hole), is stored on the event horizon[4]. In fact, it turns out other important physical observables of the dual CFT, such as shear viscosity [25] can also be determined solely by information stored on the event horizon in particular cases. This idea will form an important part of our later work when we discuss the DC thermoelectric conductivity of holographic systems. Specifically, we will show that, for a general class of holographic systems, the DC thermoelectric conductivity of a class of holographic systems can be determined by solving the Navier-Stokes equations for an auxiliary fluid on the event horizon of the dual black hole.

In the AdS-Schwarschild solution, the only scale in our theory is temperature. Therefore we can always rescale our temperature so that non zero temperatures are equivalent. For the systems we are interested in, we require additional scales. A popular choice of a second scale is a chemical potential, since many physically relevant condensed matter systems are charged. We can introduce this through a global $U(1)$ symmetry in our quantum theory, with a corresponding $U(1)$ gauge field in the bulk.

Our action is

$$S = \int d^{d+1}x \left(R + \frac{d(d-1)}{L^2} - \frac{1}{4}F^2 \right) , \tag{1.31}$$

where $F = dA$ is the field strength of the $U(1)$ gauge field, A. A solution to this is the AdS-Reissner-Nördstron ("AdS-RN") black hole,

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + \sum_i dx_i^2 \right) \\ A &= \mu \left(1 - \left(\frac{z}{z_h} \right)^{d-2} \right) dt \, , \end{aligned} \tag{1.32}$$

where

$$\begin{aligned} f(z) &= 1 - (1 + \frac{\mu^2 z_h^2}{\gamma^2}) \left(\frac{z}{z_h} \right)^d + \frac{\mu^2 z_h^2}{\gamma^2} \left(\frac{z}{z_h} \right)^{2(d-1)} \\ \gamma &= \frac{(d-1)L^2}{d-2} \, . \end{aligned} \tag{1.33}$$

[4]Note that this is up to a normalisation of a Killing vector, that we must fix using boundary data.

In this case the temperature is given by

$$T = \frac{1}{4\pi z_h}\left(d - \frac{(d-2)z_h^2\mu^2}{\gamma^2}\right), \tag{1.34}$$

while μ corresponds to the chemical potential. We see that the temperature depends on both z_h and μ. Unlike with (1.29), if we try to rescale our theory to remove z_h dependence, we are left with a scale set by μ. Thus non-zero temperature are no longer equivalent, and the dimensionless ratio T/μ can be continuously taken to zero. It is this ratio that is the physically measured observable quantity.

There are some particularly interesting properties of our theory that emerge at low temperatures. First, let us consider the *extremal* black hole. This occurs when the theory is at zero temperature, and implies that $z_h^2\mu^2/\gamma^2 = d/(d-2)$. Near the event horizon, the geometry of the AdS-RN in $d+1$ dimensions approaches $AdS_2 \times \mathbb{R}^{d-1}$, with metric of the form

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2) + d\Omega_{d-1}, \tag{1.35}$$

where Ω_{d-1} is a metric on $\mathbb{R}^{d-1}$. But we know that an AdS_2 geometry must be holographically dual to a one dimensional conformal field theory. The IR dynamics of the dual field theory are therefore governed by a one-dimensional CFT. Thus, a d dimensional CFT in the UV is broken by finite chemical potential, an example of *emergent* quantum criticality, emerging in the IR [26]. At finite temperatures, in the limit where temperature tends to zero, the near horizon geometry of an AdS-RN black hole also approaches $AdS_2 \times \mathbb{R}^{d-1}$, and we will later see that the emergence of this AdS_2 geometry can lead to holographic phase transitions.

It is also possible to show that extremal AdS-RN black holes have finite entropy density, violating the third law of black hole thermodynamics. This suggests that the geometry is unstable, and it is possible that this finite entropy means that AdS-RN are never true zero temperature ground states of gravitational theories (see e.g the discussion in [7]), or it could indicate new physics. In particular, there are holographic black hole solutions that have an AdS_2 geometry that are supersymmetric [27], although there is evidence that extremal black holes can be unstable [28–31].

1.5 Holographic renormalisation

In field theory calculations, we are often interested in determining the partition function of a particular theory. If we know the partition function, then we can calculate all possible correlation functions and have in effect solved our theory. In any QFT, however, there is a problem. The action will have divergences, and one must renormalise the theory by adding suitable counterterms in order to render the action finite. There is an analogous procedure in holography - holographic renormalisation. For a detailed introduction, a good review is [32].

One therefore needs to add appropriate counterterms to the bulk action in order to render it finite. In addition, the standard Gibbons-Hawking term is needed to make the variational problem well defined since we have a boundary in our gravitational theory. To determine the necessary counterterms, we first determine the asymptotic expansion of the action, and add minimal numbers of counterterms to the action to ensure the action is finite. Alternatively, one can use the Hamilton-Jacobi formalism to determine the appropriate counterterms [33].

Once the action has been regularised, the partition function can be calculated through $Z = e^{-I_{OS}}$, where I_{OS} is the regularised Euclidean continuation of the action, evaluated on-shell. From the partition function, important thermodynamic quantities can be determined. For example, the free energy is given by $W = -T \log(Z)$, whilst the other thermodynamic quantities such as entropy can be calculated in the standard way.

In addition to thermodynamic quantities, holographic renormalisation allows correlation functions to be determined. For example, the stress energy tensor can be calculated by the functional derivative of the partition function with respect to the induced metric on the boundary.

2 Applications of holography

The previous section focused on holography in a general context, free from any particular application beyond understanding strongly coupled QFTs. In this section we present some of the more recent advances in applied holography, especially relating to condensed matter (dubbed "AdS/CMT"). We will focus on the three broad themes, which later chapters will then extend; linear response and DC conductivity, holographic superconductivity and quantum phase transitions, and the fluid/gravity correspondence and the hydrodynamic regime.

Before we discuss some recent research in these themes, we will briefly mention the two alternative approaches to modeling systems using holography. In the "top-down" approach, one starts with a particular consistent truncation of some string theory reduced down to an appropriate number of dimensions. A consistent truncation means that the solution to the lower dimensional theory is also a solution to the higher dimensional theory. More precisely, if we split the fields from our full theory into "light" and "heavy fields", then a theory is consistent if it is consistent to set all heavy fields to zero in the equations of motion.

Whilst top down models have the advantage that the holographic dictionary is well defined and understood, guaranteeing the existence of a dual field theory, there are several challenges that make them computationally difficult. Firstly, in general there is not a sharp separation of energy scales in supergravity, meaning that it is unclear *a-priori* which fields to truncate. This means that whilst you may have a model based on some consistent truncation, it is unclear how the results will change when fewer fields are truncated. Secondly, realising consistent truncations in practice is very difficult (see, for example [34]). An alternative approach is to therefore perform a "bottom-up" calculation, where one starts with a minimal phenomenological theory of gravity to describe a particular phenomenon of interest. Whilst this is mathematically much simpler, there are questions as to whether a particular model has a string theory embedding, and thus a well defined holographic dual. This thesis will present results using both methods.

2.1 Linear response of holographic systems

The subject of linear response is a hugely important one in theoretical and experimental physics. Many interesting and novel phases exhibit characteristic responses to thermal and electrical perturbations, some of which are poorly understood. An interesting example is the AC electrical conductivity as a function of frequency for different phases, as shown in figure 2. A conventional metal exhibits a sharp Drude peak as $\omega \to 0$, indicating a small amount of momentum relaxation[5], whilst insulating phases have a vanishing DC ($\omega = 0$) conductivity. In addition to these characteristic DC conductivities, we can place bounds can be placed on certain transport quantities, such as the Mott-Ioffe-Regel (MIR) resistivity bound [35, 36], which states that the resistivity of a material is bounded above by a value linearly proportional to the temperature, provided that these materials have suitably weak coupling and hence a quasi-particle description.

Whilst the above classes of material are well understood, there are several novel phases of matter that are poorly understood and appear to be strongly coupled. For example, "bad metals" that appear in certain superconducting phases [38] violate the MIR bound and have no Drude peak. Similarly, the phase diagram of cuprate superconductors include the so-called "strange-metals", which have unusual scaling laws with respect to temperature and conductivity, such as linear in temperature resistivity. There is evidence that cuprates are not described by quasi-particles, as they violate the MIR bound [39] and have broad optical conductivity peaks at the order of temperature (see e.g [36, 39–42]), implying strong coupling. Holography is therefore a potential tool to understand the linear response of these systems.

To understand the linear response of a system, we want to calculate the retarded Green's function, which linearly relates a source to a corresponding expectation value

$$\delta\langle\mathcal{O}_A\rangle(\omega, k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)\delta\phi_B(\omega, k)\,. \tag{1.36}$$

We know that the VEV[6] and the source of the particular operator can be read off by

[5] Historically this is understood in terms of quasi-particle scattering off electrons or impurities.

[6] Note that although we refer to VEVs, our theories are normally not in the vacuum, and so our VEV

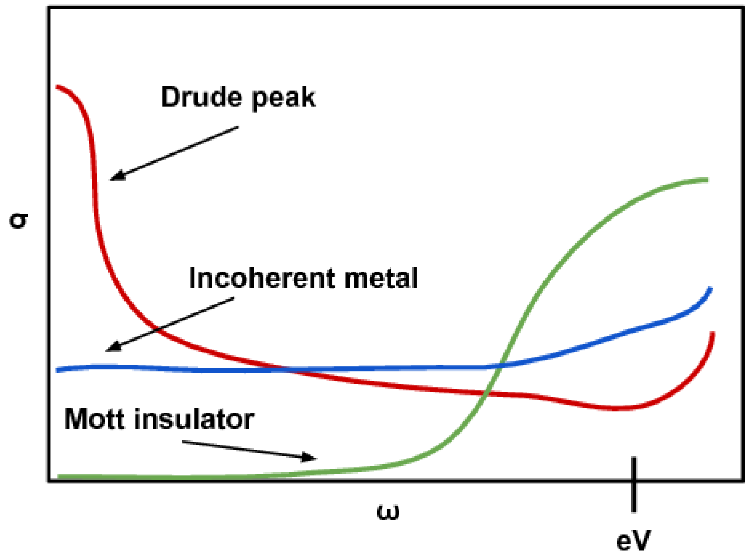


Figure 2: Cartoon showing AC conductivity for different condensed matter phases that have been experimentally realised. Convential metals (red line) show a characteristic Drude peak, indicating some momentum relaxation. Incoherent metals (blue line), including bad and strange metals have no sharp Drude peak, whilst insulators (green line) have no conductivity as $\omega \to 0$. Figure taken from [37].

considering the asymptotic fall off of the dual bulk field. Thus, to determine the linear response in holography, we apply the following prescription:

- Introduce a linearised source term to perturb the background black hole (e.g a gauge field term $A_t = \delta A e^{i\omega t}$ for the electric response, where ω is the frequency of the perturbation).

- Impose appropriate boundary conditions at the horizon. For dissipation we require ingoing boundary conditions on the event horizon.

- Solve the resulting linear equations of motion, and read off the resulting VEV. From this calculate the Green's function.

The earliest AC conductivity calculations found a delta function response in the DC limit [43]. This was unsurprising. The simplest background system one can consider is the Reissner-Nördstrom-AdS black hole. This system is transitionally invariant and therefore has no mechanism to dissipate momentum - it is describes a perfect metal. Therefore, the lack of momentum dissipation will naturally lead to an infinite response. To model real systems, we need to add some form of momentum dissipation.

A natural framework to study such dissipation is given by holographic lattices[7]. These are black holes with asymptotic behaviour that corresponds to the introduction of spatially

is really just an expectation value.

[7]Another approach is to use massive gravity and various interesting results have been obtained e.g. [44–49]. However, massive gravity is currently not well defined in a holographic setting, so at best the equations of motion should be treated as some approximation to an underlying theory.

dependent sources to the dual field theory. In general, this will involve solving PDEs [50–56], which is computationally very intense. However, there are several examples of sources that, whilst still breaking translational invariance, are homogeneous, leading to ODEs. Such examples include Q-lattices [57, 58] which exploit a global symmetry in the bulk leading to a problem involving ODEs, and massless "axions" to obtain sources linear in the spatial coordinates [4, 5, 59–65]

In what follows, we will be particularly interested in the DC conductivity (both thermal and electrical) of strongly coupled systems. The DC conductivity can be calculated by utilising the fact that the conductivities are related to retarded Greens functions via $i\omega\sigma = G^R_{\mathcal{O}\mathcal{O}}$, and taking appropriate limits as $\omega \to 0$ (This is the Kubo formula).

It is a natural question to ask if there are ways to directly determine the DC conductivity of holographic systems, rather than as an AC limit. It was shown in [55, 66], building on the earlier work of [67], that the DC conductivity of certain classes of black holes can be determined directly from information on the black hole horizon, rather than the above prescription, by including a source term that is linearly, rather than exponentially, time dependent. This is another example where the event horizon of the black hole captures dissipation, reminiscent of the membrane paradigm [68], and the holographic bound on the shear viscosity of a strongly coupled system [25, 69]. As we shall discuss in the subsequent chapters, there is in fact a general prescription to calculate the DC conductivity of holographic systems directly, without turning to any sort of AC limit, and it is deeply related to the idea that physical properties of a CFT are stored on the event horizon of the dual black hole.

2.2 Holographic superconductors

Much of the richness of condensed matter physics lies in the dynamics involved in the onset of an ordered phase below some critical temperature. One of the main aims of AdS/CMT has been to shed light on high temperature (high-T_c) superconductivity. Superconductors are a class of materials that exhibit infinite DC conductivity and expel a magnetic field at temperatures below some critical temperature, T_c. An explanation for this phenomenon was first provided by Bardeen, Cooper and Scheiffer in 1957, with the introduction of BCS theory [70]. In this theory, pairs of nearly free electrons can interact with phonons and bind into pairs of bosons (Cooper pairs). At the critical temperature, these bosons condense, and the system undergoes a second order phase transition to the new superconducting phase.

It was once believed that the highest temperature for T_c was around 30K [71]. Recently, high-T_c superconductors, with T_c up to 160K have been discovered [72]. Although these materials do seem to form Cooper pairs, the mechanism for their formation is poorly understood. In particular, the coupling mechanism is strongly, rather than weakly coupling. Gauge/gravity duality therefore provides a potential tool to better understand this phenomena.

The first steps in using holography is to therefore produce a toy model that can explain a simple superconductor in terms of a phase transition. Let us consider a minimal theory that includes a $U(1)$ gauge field, the graviton and a complex scalar field in $d+1$ dimensions, with a Lagrangian given by

$$\mathcal{L} = (R + d(d-1)) - \frac{1}{4}F^2 - |\nabla\phi - iqA\phi|^2 - m^2\phi^2\,, \tag{1.37}$$

where ϕ is the complex scalar field, A is the $U(1)$ gauge field, dual to a global $U(1)$ operator in the field theory, with field strength F and R is the Ricci scalar. We have set various scalings such as the AdS radius and string length to unity for simplicity, whilst m and q are parameters that we can tune for our particular theory.

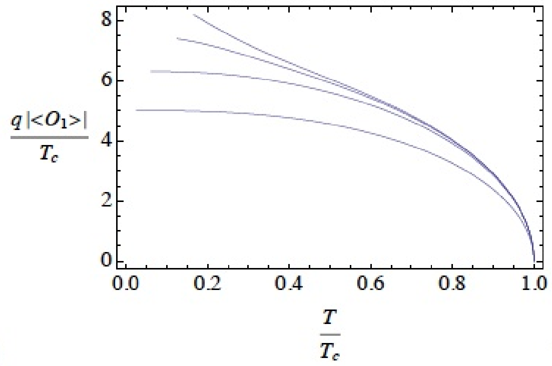
When $\phi = 0$, we can choose our background solution to be the Reissner-Nördstron-AdS, with a scale set by the chemical potential, μ. This is the normal phase of the solution.

Next, we want to know whether this phase is unstable to perturbations of the charged scalar field. We therefore look for a zero-mode of the black hole solution. We introduce a perturbation $\delta\phi$, and consider the linearised equations of motion for $\delta\phi$. At the onset of instability, there will be a solution to this equation (combined with the background equations of motion) that satisfies ingoing boundary conditions on the horizon, with no source term but a non zero expectation value for ϕ. Physically, we are saying that the system undergoes spontaneous symmetry breaking at this point, and the temperature of this solution corresponds to the critical temperature, T_c, of the phase transition.

To understand the mechanism for the presence of a zero mode, let's work momentarily in the specific case of $d = 3$. As we go to low temperatures, the horizon geometry approaches AdS_2, as opposed to AdS_4 on the boundary. Thus, recalling from (1.16), a tachyonic mass that is stable on the boundary may be unstable as we move into the bulk geometry, and can drive the instability. We will return to this idea later on in part III. As well as this mechanism, there is another way to see how the instability arises. For the case of a complex scalar field, the covariant derivative gives an additional contribution to the effective negative mass squared of the scalar. Again, this can cause the scalar field to violate the BF bound, rendering the geometry unstable.

Now that we have our critical point for our phase transition, the next question is what does our theory look like below this point. As we cool down below T_c, the previous background solution is unstable. We therefore look for hairy black hole solutions, which have non vanishing ϕ, with the metric ansatz

$$ds^2 = \frac{1}{z^2}\left(-f(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right) \tag{1.38}$$



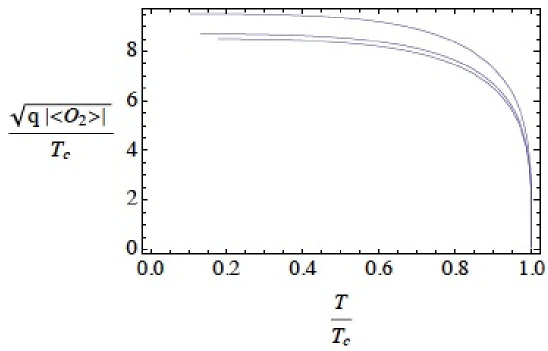


Figure 3: The condensate as a function of the temperature for the case when scalar scaling dimension $\Delta = 1$ (left) and $\Delta = 2$ (right). In curve (a), from bottom to top, q = 1, 3, 6, 12. In curve (b), from top to bottom, q = 3, 6, 12. A characteristic $(T - T_c)^{1/2}$ scaling is seen near the critical point, consistent with Ginzburg-Landau mean field theory with quadratic exponents. Figure taken from [43].

and matter content

$$A = A_t(z)dt\,, \qquad \phi = \phi(z)\,. \tag{1.39}$$

These black holes are then constructed by solving the resulting equations of motion. To determine if this new branch of solution is actually realised in nature, the free energy between the two solutions for a given temperature must be compared, and the branch of solution with a lower free energy is the thermodynamically preferred solution. The numerical calculations were first performed in [43], and the new hairy black hole was found to be the stable phase.

The behaviour of physical quantities near continuous phase transitions is described by critical exponents, which classify phase transitions into universality classes. The holographic phase transition we have just discussed is the simplest and most common phase transition that is seen in holographic calculations. That is, a second order phase transition with critical exponents that are found in the standard Ginzburg-Landau mean field theory with quadratic exponents [73]. In particular, the plot of $\langle\mathcal{O}\rangle$ versus temperature has a characteristic form of $(T - T_c)^{1/2}$ as shown in 3.

Whilst this particular universality class is the most widely seen, in general, different critical exponents (and hence different universality classes of phase transitions) have been constructed in holographic settings [4,74,75]. Note that different critical exponents do not mean that the theory is a not mean-field theory, but simply means that the theory belongs to a different universality class, with a different Ginburg-Landau expansion. Indeed, there are many examples in the literature that have non standard critical exponents but are completely homogeneous, with only radial dependence in the bulk. These examples must be mean field theories, but simply with phase transitions that belong to different universality classes.

The above example describes the earliest example of a bottom-up holographic superconductor, where the order parameter is a scalar field. This is known as an s wave superconductor. Since then, there has been considerable progress in describing p-wave superconductors (where the order parameter is a vector field [76–78], or a two-form [79–81]), and d-wave superconductors (where the order parameter is tensor field [82]). In addition, examples of phase transitions in top-down settings have been constructed (see, for example [83–86]). In a top-down setting, a neutral, rather than charged, scalar field is often a natural field to drive instabilities. Top-down models pose many additional challenges, especially in determining the true ground states of a particular theory, as to determine with any certainty the true ground state would require solving high dimensional equations with a large matter content.

In part III of this thesis, we will explore some new findings regarding holographic phase transitions for a particular holographic lattice model, and demonstrate some of the subtleties described above.

2.3 The hydrodynamic approximation and fluid/gravity correspondence

When considering problems in holography, many calculations can be made simpler by taking a hydrodynamic approximation (see e.g [87]). Before discussing hydrodynamics in the context of holography, we will quickly highlight some of the key concepts of general hydrodynamics. The idea of hydrodynamics is to take small fluctuations about thermal equilibrium when the wavelength, λ_{wave}, is much larger than the mean free path, l_{mfp}. The hydrodynamic limit can therefore be viewed as an effective field theory, where the high frequency and large wavenumber degrees of freedom have been integrated out.

Perturbations vary slowly on the scale of λ_{wave}, so the system is locally in equilibrium, and described by thermodynamic variables that are functions of spacetime coordinates. The behaviour of hydrodynamics is determined by the conservation of currents, such as the stress energy tensor and gauge current, and their constitutive relations, which express conserved quantities in terms of fluid variables. In what follows, we will be interested in the *relativistic hydrodynamic limit*, and the conservation of the stress tensor is expressed as

$$\nabla_a T^{ab} = 0\,, \tag{1.40}$$

where T is the stress-energy tensor, and ∇_a is the covariant derivative for the background metric of the fluid. In the case of charged fluids, the associated charged currents will be similarly conserved.

Since we are considering fluctuations away from some global equilibrium state, we need to start with the thermal equilibrium state itself. The idea is to move away from this equilibrium state so that we have local, rather than global, equilibrium in a controlled manner. Since we are in the long wavelength limit, we can achieve this by making the appropriate fields functions of the boundary coordinates, and consider a derivative expansion of the system. The coefficients in this expansion are the transport coefficients of our theory.

More precisely, let's start with an ideal and isotropic fluid in $d+1$ dimensions, which has no mechanism for energy dissipation and is invariant under the rotations of the d spatial dimensions. The stress-energy tensor takes the form

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + pP^{\mu\nu}\,, \tag{1.41}$$

where ϵ is the energy density of the fluid, p is the pressure, u is the fluid velocity that satisfies $u_\mu u^\mu = -1$, and P is the projection operator onto the spatial coordinates, given by

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\,. \tag{1.42}$$

Now consider the derivative expansion. We write the stress-energy tensor as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + pP^{\mu\nu} - \sigma^{\mu\nu} + \mathcal{O}(\partial^2)\,, \tag{1.43}$$

where σ is first order in the derivatives of T and u. There is an arbitrariness is choosing u, so we have to make a choice for our fluid velocity (a choice of *frame*). One choice is the Landau frame

$$u_\mu \sigma^{\mu\nu} = 0\,. \tag{1.44}$$

Once a particular frame has been chosen, we impose (1.40) and one further condition, namely that entropy current, $s^\mu = su^\mu$, increases locally over time

$$\nabla_\mu s^\mu \geq 0\,. \tag{1.45}$$

These two constraints then fix our derivative expansion. This procedure can be generalised to include charges in a straightforward manner.

Now let's turn to AdS/CFT. (1.40) is a dynamical equation solely involving the stress tensor. But AdS/CFT asserts that the behaviour of any field theory can be solely described by the Einstein equations in the dual bulk theory. Thus there should be a correspondence between gravitational theories in $d+1$ dimensions and d dimensional hydrodynamics in an appropriate high temperature and long distance limit (the long wavelength limit). A mechanism to realise this correspondence order by order in an appropriate derivative expansion was first demonstrated in [88], and is the fluid/gravity correspondence. For a good review of topic, we refer the reader to [89, 90].

Here we will briefly sketch the main arguments for this result. We start with our equilibrium solution, the AdS-Schwarschild black hole in $d+1$ dimensions. This is a one-parameter family of solutions (the parameter corresponding to the temperature of the black hole). However, by boosting the solution along the spatial directions, we can find a d parameter family of solutions. These parameters are exactly the parameters in d dimensional relativistic hydrodynamics, namely the temperature and fluid velocity. In order to make regularity at horizon manifest, it is helpful to write the metric in ingoing Eddington-Finkelstein coordinates. We therefore have

$$ds^2 = -2u_\mu dx^\mu dz - r^2 f(bz) u_\mu u_\nu dx^\mu dx^\nu + z^2 P_{\mu\nu} dx^\mu dx^\nu \tag{1.46}$$

where b is related to the Hawking temperature, T, by

$$b = \frac{d}{4\pi T}\,, \tag{1.47}$$

and the velocities, β are related to field, u, with corresponding projector, P, by

$$u^t = \frac{1}{\sqrt{1-\beta^2}}\,, \qquad u^i = \frac{\beta^i}{\sqrt{1-\beta^2}}\,. \tag{1.48}$$

Now we perturb this away from global equilibrium in order to introduce dissipation. We can achieve local equilibrium by making b and β slow varying functions of the boundary coordinates, and construct a derivative expansion of our metric, by adding additional perturbative terms to the metric (1.46) and solving the Einstein equations order by order in the derivatives. The choice of Eddington-Finkelstein coordinates ensures that the black hole remains regular, and gives a good visualisation of the approximation process. As shown in figure 4, the boundary domains that are in local equilibrium extend in tubes along radial null geodesics into the bulk spacetime. These tubes are then patched together, giving a solution to Einstein's equations order by order in the derivative expansion.

The original fluid/gravity correspondence has been extended to include charged fluids [92,93], whilst [94] introduced a scalar field. In [95,96], fluid/gravity was used to determine DC conductivity of systems with momentum relaxation.

3 Outline of thesis

The first part of this thesis is concerned with the linear response of holographic systems. In chapter 2, we consider a general class of electrical black holes of Einstein-Maxwell-scalar theory that are holographically dual to conformal field theories at finite charge density and which break translation invariance explicitly. By considering linearised perturbations of these background black holes, we show that the DC thermoelectric conductivity for current fluxes of a holographic lattice can be determined by solving a set of linearised Navier-Stokes equations on the event horizon of the dual black hole. We then demonstrate how to apply this framework in practice, with several specific examples, including one-

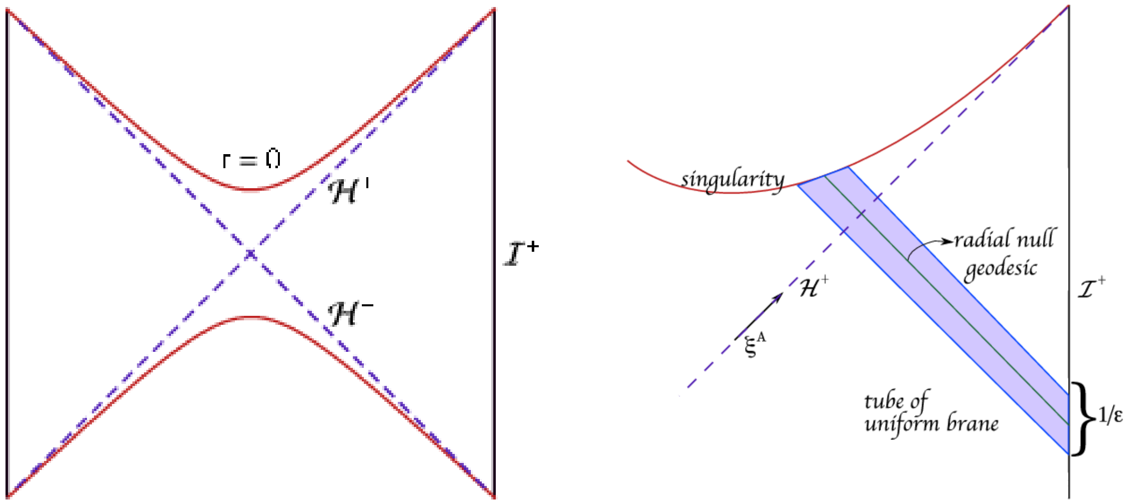


Figure 4: Penrose diagram of the uniform black brane (left) and the causal structure of the spacetimes dual to fluid mechanics, illustrating the tube structure (right). The dashed line in the second figure denotes the future event horizon, while the shaded tube indicates the region of spacetime where the solution is well approximated by a tube of the uniform black brane. These tubes are then patched together to give an approximate solution to Einstein's equations. Figure taken from [91].

dimensional and perturbative lattices.

Having introduced a robust and general framework, chapter 3 considers more general forms of DC linear response for a specific class of holographic lattices that contains a gauge field and three scalar fields. Specifically, we determine the full DC linear response matrix due to perturbations of one of the scalar fields.

Next, we consider the hydrodynamic limit of the results from chapter 2. We show that, in the hydrodynamic limit, local thermoelectric currents, rather than current fluxes, can be determined by solving the Navier-Stokes equations on the bulk horizon. We then show that the full stress energy response due to the DC source can be obtained in this limit, and compare our results with previous fluid/gravity calculations.

Finally, we utilise some of the results in chapter 4 in chapter 5 when we discuss hydrodynamic backflow. We study CFTs in flat spacetime in the hydrodynamic limit, that have been deformed by spatially dependent and periodic local temperature variations or strains, and show that specific deformations lead to thermal backflow. In particular, we show that when a DC thermal source is applied, in certain setups, the periodic strains can lead to thermal currents that locally flow in the opposite direction to the source.

In the second part of the thesis we turn to phase transitions, and study black hole solutions of type IIB supergravity that describe the N=4 supersymmetric Yang-Mills plasma with an anisotropic spatial deformation, which were constructed in [60, 61]. The zero temperature limit of these black holes approach a Lifshitz-like scaling solution in the infrared. In chapter 6, we show that these black holes become unstable at low temperature and construct a new class of black hole solutions which are thermodynamically preferred.

The phase transition is third order, and has unusual critical exponents for the phase transition, which differ from the regular quadratic Ginzburg-Landau exponents.

In chapter 7, we extend our analysis to include a $U(1)$ gauge field in the bulk, and also consider more scalar fields in our top-down model. We show that the results of the previous chapter generalise in the presence of a chemical potential, and construct new black hole solutions. We then demonstrate that the model of the previous chapter may in fact be unstable at low temperatures, whilst the charged case appears to be stable. This highlights some of the challenges in top-down holographic constructions.

Finally, we will discuss our conclusions in part IV of the thesis, followed by a bibliography and appendix.

Part II

Linear response

Chapter 2

Thermoelectric DC conductivities and Stokes flows on black hole horizons

1 Introduction

The holographic correspondence provides a powerful framework for obtaining precise results about strongly coupled systems using weakly coupled gravitational descriptions. A key cornerstone is that the dual description of the field theory at finite temperature is provided by a gravitational spacetime, and is often a black hole. It is a remarkable fact that various properties of the thermal system are captured by the properties of the gravitational solutions at the black hole horizon. For example, the entropy of the thermal system at equilibrium is given by the Bekenstein-Hawking forumla, $S = A/4G$, where A is the area of the black hole event horizon. Another well-known example is provided by the shear viscosity, η, which, for certain classes of black hole solutions, is given by $\eta = 4\pi s$, where s is the entropy density [25,69]. In this chapter, which expands upon and generalises the results presented in [97], we explain how the DC thermoelectric conductivity can be obtained by solving equations for a non-relativistic fluid on the black hole horizon. Moreover, we will see that, unlike η, our result for the DC conductivity holds in a very general context, being applicable to arbitrary static black holes for which the DC conductivity is finite. The extension to stationary black holes is presented in [98].

We first recall that the DC thermal conductivity, κ, is a very natural observable to study in holography. Indeed, in the regime of linear response it determines the heat current, or momentum flow, that is produced after applying a constant external thermal gradient. Although this naively appears to be a low-energy quantity it is in fact sensitive to the UV physics. For example, if the system is translationally invariant, and hence conserves momentum, then κ is infinite. More precisely, there is a delta function in the AC conductivity at $\omega = 0$. To obtain a finite κ it is necessary to introduce a mechanism for momentum to dissipate.

For systems with a $U(1)$ symmetry, another natural observable to consider is the DC electric conductivity. Momentum dissipation is also required in order to obtain a finite result when the charge density is non-vanishing. More generally, for these systems there is a mixing of electric and heat currents and one should consider the matrix of thermoelectric conductivities

$$\begin{pmatrix} \bar{J} \\ \bar{Q} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \bar{\alpha} T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \end{pmatrix}, \tag{2.1}$$

where $\bar{J}$ and $\bar{Q}$ are the total electric and heat current flux densities (which we define precisely later) and E and ∇T are constant applied electric field and thermal gradients, respectively.

As we discussed in the introduction, holographic lattices provide a natural framework[1] for studying momentum dissipation. These are black hole solutions whose asymptotic behaviour at the holographic boundary corresponds to the addition of spatially dependent sources to the dual field theory. Various holographic lattices with sources that depend periodically on just one of the non-compact spatial directions have been constructed by solving PDEs in two variables [50–56]. Constructing lattices that depend periodically on additional spatial dimensions generically requires solving PDEs in more variables, which becomes increasingly challenging at the technical level. An important class of exceptions are provided by Q-lattices [57, 58] which exploit a global symmetry in the bulk leading to a problem involving ODEs. Other constructions involving ODEs use massless "axions" to obtain sources linear in the spatial coordinates [59–64], or use the metric or matter fields to obtain helical sources when $D \geq 5$ [37, 101–103]. A particularly interesting application of holographic lattices is that they can lead to novel incoherent metal ground states [37, 58, 104], insulating ground states [37, 57, 58] and transitions between them [37, 58]. Connections between holographic lattices and superconductivity have also been explored in [103, 105, 106].

For the special case of translationally invariant black holes at zero chemical potential, while the thermal conductivity is infinite the electric conductivity is finite and can be expressed in closed form in terms of the behaviour of the solution at the black hole horizon [67]. For the above holographic lattices involving ODEs, formulae for the thermoelectric DC conductivity, also expressed in terms of the black hole solutions at the black hole horizon, were obtained in [58, 66, 101, 102]. These results were then extended to a class of one-dimensional holographic lattices in [55], although the details were more involved. Given these results it is natural to anticipate that similar results can be obtained

[1] Another approach is to use massive gravity and various interesting results have been obtained e.g. [44–49]. The DC conductivity has also been examined in the context of translationally invariant probe branes e.g. [99, 100]. In these constructions a finite DC conductivity can arise because the delta function is suppressed by $1/N$, where N is the number of branes providing the background geometry.

for all holographic lattices. Here we show that for a broad class of holographic lattices, depending on all spatial directions, in general one cannot obtain such explicit formulae for the DC conductivity. However, it is possible to obtain the DC conductivity after solving a set of fluid equations on the black hole horizon. These equations are a generalisation of the forced Stokes equations for a charged fluid on the curved black hole horizon, with additional viscous terms arising from bulk scalar fields. Recall that the Stokes equations are a time-independent and linearised limit of the Navier-Stokes equations for an incompressible fluid arising at low Reynolds numbers (see, for example [107]). We will show that the previous results on the DC conductivity can all be obtained as special cases where the Stokes equations can be solved explicitly in closed form.

The fact that the fluid equations which arise are linear and time-independent is not too surprising since we are calculating in the regime of linear response and we are calculating the DC conductivity. Similarly, the forcing terms are very natural since they arise from the applied sources for the electric and heat currents. On the other hand only a subset of the linearised perturbation appears in the equations on the horizon and it is remarkable that the equations form a closed system.

We emphasise that unlike in the relativistic fluid-gravity correspondence [88], and the associated non-relativistic limit [108, 109], we do not take any hydrodynamic limit in obtaining our fluid equations. In the presence of spatially dependent sources a natural hydrodynamic limit would arise for temperatures much bigger than all other scales, including those of the lattice. By contrast our results are valid for all temperatures. Our results differ but are also reminiscent of the "membrane paradigm" [68] and the more recent work[2] which relates solutions of the non-linear Navier-Stokes equations on hypersurfaces in Minkowski space to obtain black hole solutions [110] (see also [111–113]). We expect that the time-dependent and non-linear versions of our equations will play a role in studying momentum dissipation for holographic lattices, possibly after taking a hydrodynamic limit and this will be reported on elsewhere. Discussions of momentum dissipation, conductivities and hydrodynamics can be found in [95, 114–116] and some recent low-frequency conductivity results are presented in [117].

The plan of the rest of the chapter is as follows. In section 2 we introduce the holographic model and the class of electrically charged black hole solutions that we shall be considering. In section 3 we analyse the linearised perturbations, containing sources for the electric and heat currents, which are associated with the DC conductivity. We will show how the Stokes equations can be obtained by expanding the Hamiltonian, momentum and Gauss law constraints, associated with a radial Hamiltonian decomposition, at the black hole horizon. The fluid equations determine electric and heat currents at the horizon in terms of sources for the electric and heat currents. In turn these can be used to

[2]Ref. [110] also contains a discussion and references to some of the earlier work on fluids and black hole horizons.

obtain suitably defined constant electric and heat current fluxes which are independent of the radial direction and hence give rise to the DC conductivity. If the deformed CFT is living on Σ_d then the DC conductivity is a $b_1(\Sigma_d)\times b_1(\Sigma_d)$ matrix, where $b_1(\Sigma_d)$ is the first Betti number of Σ_d. We emphasise that we provide a precise procedure for calculating the DC conductivity of the boundary theory (the spectral weight of a two-point function) by solving an auxiliary set of fluid equations on the black hole horizon.

In section 4 we analyse some examples. We first generalise our results to an arbitrary number of scalar fields which allows us to reconsider Q-lattices. For the Q-lattices and also for general one-dimensional lattices, we show that the fluid equations can be explicitly solved and we can obtain formulae for the DC conductivity explicitly in terms of the black hole solution at the horizon. We also examine holographic lattices that can be obtained as perturbative expansions about translationally invariant solutions, including the AdS-RN black brane. We show that the leading order DC conductivity can also be found in closed form. We briefly conclude in section 5 where we put some of the main results in a more general setting of general static black hole spacetimes. The chapter contains four appendices, including a discussion of the Hamiltonian decomposition of the equations of motion with respect to the radial coordinate in appendix A.1.

2 The background black holes

We will consider theories in D spacetime dimensions which couple the metric to a gauge-field, A, and a single scalar field, ϕ. The extension of our analysis to additional scalar fields is straightforward as we will discuss later. We focus on $D \geq 4$. The action is given by

$$S = \frac{1}{16\pi G_N}\int d^D x\sqrt{-g}\left(R - V(\phi) - \frac{Z(\phi)}{4}F^2 - \frac{1}{2}\left(\partial\phi\right)^2\right). \tag{2.2}$$

The equations of motion are given by

$$\begin{aligned} R_{\mu\nu} - \frac{V}{D-2}g_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}Z(\phi)\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(D-2)}g_{\mu\nu}F^2\right) &= 0\,, \\ \nabla_\mu\left[Z(\phi)F^{\mu\nu}\right] &= 0\,, \\ \nabla^2\phi - V'(\phi) - \frac{1}{4}Z'(\phi)F^2 &= 0\,. \end{aligned} \tag{2.3}$$

The only restrictions that we will make on the functions $V(\phi)$, $Z(\phi)$ is that $V(0) = -(D-1)(D-2)$, $V'(0) = 0$ and $Z(0) > 0$. This ensures that a unit radius AdS_D solves the equations of motion with $\phi = 0$ and this is dual to a CFT with a stress tensor, dual to the metric, a global $U(1)$ current, dual to A, and an additional operator dual to ϕ. When $Z'(0) = 0$, the D-dimensional electrically charged AdS-Reissner-Nordström black hole also solves the equations of motion and describes the CFT at constant charge density. Note that we have set the AdS radius to unity, for convenience.

We will focus on a general class of electrically charged static black holes with metric and gauge-field given by

$$\begin{aligned} ds^2 &= -UG\,dt^2 + \frac{F}{U}\,dr^2 + ds^2(\Sigma_d)\,, \\ A &= a_t\,dt\,, \end{aligned} \tag{2.4}$$

where $ds^2(\Sigma_d) \equiv g_{ij}(r,x)dx^i dx^j$ is a metric on a ($d \equiv D-2$)-dimensional manifold, Σ_d, at fixed r. In addition, $U = U(r)$, while G, F, a_t and ϕ are all functions of (r, x^i)[3]. In section 5 we will discuss our main results in the context of a more general class of static black hole solutions.

Asymptotically, as $r \to \infty$, the solutions are taken to approach AdS_D with

$$\begin{aligned} &U \to r^2, \qquad F \to 1, \qquad G \to \bar{G}(x), \quad g_{ij}(r,x) \to r^2 \bar{g}_{ij}(x), \\ &a_t(r,x) \to \mu(x), \qquad \phi(r,x) \to r^{\Delta-d-1}\bar{\phi}(x)\,. \end{aligned} \tag{2.5}$$

The spatial dependence of the boundary metric given by $\bar{G}(x)$, $\bar{g}_{ij}(x)$ corresponds to providing a source for the stress tensor in the dual CFT living on $\mathbb{R} \times \Sigma_d$. Similarly, $\mu(x)$ is a spatially dependent chemical potential for the global abelian symmetry and $\bar{\phi}(x)$ gives rise to a spatially dependent source for the associated dual operator, which we have assumed has scaling dimension Δ.

A particularly interesting class of black holes is associated with adding sources to CFTs in flat Minkowski space, $\mathbb{R}^{1,d}$. In this case Σ_d is topologically $\mathbb{R}^d$. Periodic lattices, which have been a focus of study, are obtained by taking the functions $\bar{G}(x)$, $\bar{g}_{ij}(x)$, $\mu(x)$ and $\bar{\phi}(x)$ to be periodic functions on $\mathbb{R}^d$. If we denote the period in each of the spatial directions be L_i then for this class of black holes we can, in effect, take Σ_d to parametrise a torus with periods $x^i \sim x^i + L_i$.

The black hole horizon, which has topology Σ_d, is assumed to be located at $r = 0$. By considering the Kruskal coordinate $v = t + \frac{\ln r}{4\pi T} + \ldots$ we deduce that the near horizon

[3]Note we could set $G = 1$ through a change of coordinate. However, this approach was chosen to be useful for numerical calculation. In addition, we could also include rx^i terms in our metric through a coordinate redefintion. Again, we choose this approach for the simplicity of presentation.

expansions are given by

$$
\begin{aligned}
U(r) &= r\left(4\pi T + U^{(1)} r + \dots\right) , \\
a_t(r,x) &= r\left(a_t^{(0)} G^{(0)}(x) + a_t^{(1)}(x)\, r + \dots\right) , \\
G(r,x) &= G^{(0)}(x) + G^{(1)}(x)\, r + \dots , \\
F(r,x) &= F^{(0)}(x) + F^{(1)}(x)\, r + \dots , \\
g_{ij} &= g_{ij}^{(0)} + g_{ij}^{(1)} r + \dots , \\
\phi &= \phi^{(0)}(x) + r\phi^{(1)}(x) + \dots ,
\end{aligned}
\tag{2.6}
$$

with

$$
G^{(0)}(x) = F^{(0)}(x) \; . \tag{2.7}
$$

We have added the extra factor of $G^{(0)}$ in the leading expression for a_t for convenience. For later use, we observe that boundary electric charge density at the horizon is simply

$$
\rho_H = \frac{1}{16\pi G_N}\sqrt{-g}Z(\phi)F^{tr}|_H = \frac{1}{16\pi G_N}\sqrt{-g_0}a_t^{(0)}Z^{(0)} , \tag{2.8}
$$

where $Z^{(0)} \equiv Z(\phi^{(0)})$. For the averaged holographic charge density, ρ, we have

$$
\rho \equiv \frac{1}{\mathrm{vol}_d 16\pi G_N}\int d^d x(\sqrt{-g}Z(\phi)F^{tr})|_\infty = \frac{1}{\mathrm{vol}_d}\int d^d x \rho_H , \tag{2.9}
$$

where $\mathrm{vol}_d \equiv \int d^d x\sqrt{-\bar{g}}$ is the volume of the spatial metric at the AdS boundary. This result follows from the fact that the gauge-equations of motion implies $\partial_r(\sqrt{-g}ZF^{tr}) + \partial_i(\sqrt{-g}ZF^{ti}) = 0$ and for the case of non-compact Σ_d we have assumed any boundary terms vanish.

This class of black holes includes almost all of the holographic lattices that have been constructed to date as special cases. For example, periodic lattices with modulated chemical potential with non-vanishing zero mode were studied in [50, 53, 55], while the case of vanishing zero mode was studied in [52]. Periodic lattices with a single real scalar field have been studied in [51, 56]. These examples have spatially inhomogeneous sources in one direction only. By contrast, the Q-lattice construction using two (or more) scalar fields [57, 58, 66, 104, 118] the non-periodic "axionic" lattices studied in [59–64, 66, 119] are homogeneous and the sources can be in any number of the spatial directions. Other homogeneous constructions using $D = 5$ helical lattices have been studied in [101, 102] (an additional gauge field is needed to be included to cover the examples of [37]). Metric deformations in one spatial dimension were studied in [54]. Holographic lattices in the presence of magnetic fields have been studied in [120]; the generalisation of these results to include magnetic fields is discussed in [98].

3 Perturbing the black holes

We want to study the holographic, linear response of the black holes after applying suitable one-form sources (E, ζ) on Σ_d for the electric and heat currents, respectively. Generalising [55, 58, 66] we incorporate the sources by the addition of terms that are linear in time. Specifically, we consider the following linear perturbation[4]

$$\begin{aligned}\delta\left(ds^2\right) &= \delta g_{\mu\nu}dx^\mu dx^\nu - 2tM\zeta_i dt dx^i\,,\\ \delta A &= \delta a_\mu dx^\mu - tE_i dx^i + tN\zeta_i dx^i\,,\\ \delta\phi\,. \end{aligned} \tag{2.10}$$

Here, $\delta g_{\mu\nu}$,δa_μ, $\delta\phi$ are all functions of (r, x^i), while $E_i = E_i(x)$, $\zeta_i = \zeta_i(x)$. We demand that E, ζ are closed one-forms on Σ_d:

$$d(E_i dx^i) = d(\zeta_i dx^i) = 0\,. \tag{2.11}$$

This means that the one forms E, ζ can uniquely be written as the sum of a harmonic form plus an exact form on Σ_d. Later we will see later that the harmonic piece is the important part of the source. In the case that Σ_d is $\mathbb{R}^d$ or a torus, for example, we could take an independent basis of sources to be the d one-forms $E_i dx^i$ (no sum on i) and the d one-forms $\zeta_i dx^i$ (no sum on i) with constant E_i, ζ_i.

The functions M, N in (2.10) depend on (r, x^i) and we will fix them in terms of the background black hole solution via

$$M = GU\,, \qquad N = a_t\,. \tag{2.12}$$

This is assumed, in addition to (2.11), in order to solve the time dependence of the equations of motion at linear order. In general, this ansatz for the perturbation contains some residual gauge symmetry, which, for our purposes, will not need to be fixed.

At the AdS_D boundary, as $r \to \infty$, we will demand that the fall-off of $\delta g_{\mu\nu}$,δa_μ, $\delta\phi$ is such that the only applied sources are parametrised by (E, ζ). The asymptotic fall-offs of $\delta g_{\mu\nu}$,δa_μ, $\delta\phi$ are associated with currents and other expectation values that are produced by the sources. The resulting currents will be our primary interest here.

[4]Throughout this chapter, we will only be concerned with linear perturbations of this metric, and will not consider backreaction. While one may be tempted to think that we could make the time coordinate arbitrarily large, and therefore induce non-linear effects, we will later see that the time dependence will completely drop out of the linear equations of motion without imposing any condition on t, and so at the level of linear response there is no backreaction provided that the source terms, ζ and E, are sufficiently small. Furthermore, as we will later show in the discussion in section 4, the time dependence can be locally removed from the metric through a coordinate transformation.

At the black hole horizon, as $r \to 0$, regularity implies that we must have

$$\begin{aligned}
&\delta g_{tt} = U\left(r\right)\,\left(\delta g^{(0)}_{tt}\left(x\right) + \mathcal{O}(r)\right), \quad \delta g_{rr} = \frac{1}{U}\,\left(\delta g^{(0)}_{rr}\left(x\right) + \mathcal{O}(r)\right), \\
&\delta g_{ij} = \delta g^{(0)}_{ij}\left(x\right) + \mathcal{O}(r), \quad \delta g_{tr} = \delta g^{(0)}_{tr}\left(x\right) + \mathcal{O}(r)\,, \\
&\delta g_{ti} = \delta g^{(0)}_{ti}\left(x\right) - M\,\zeta_i\,\frac{\ln r}{4\pi T} + \mathcal{O}(r), \quad \delta g_{ri} = \frac{1}{U}\,\left(\delta g^{(0)}_{ri}\left(x\right) + \mathcal{O}(r)\right)\,, \\
&\delta a_t = \delta a^{(0)}_t\left(x\right) + \mathcal{O}(r), \quad \delta a_r = \frac{1}{U}\,\left(\delta a^{(0)}_r\left(x\right) + \mathcal{O}(r)\right)\,, \\
&\delta a_i = \frac{\ln r}{4\pi T}(-E_i + N\zeta_i) + \mathcal{O}(r^0)\,,
\end{aligned} \tag{2.13}$$

with the following constraints on the leading functions of x:

$$\delta g^{(0)}_{tt} + \delta g^{(0)}_{rr} - 2\,\delta g^{(0)}_{rt} = 0, \qquad \delta g^{(0)}_{ri} = \delta g^{(0)}_{ti}, \qquad \delta a^{(0)}_r = \delta a^{(0)}_t\,. \tag{2.14}$$

It is worth emphasising that the logarithm terms that appear in (2.13) are a direct consequence of the applied sources (E, ζ). For the scalar field we have $\delta\phi = \delta\phi^{(0)}(x) + \mathcal{O}(r)$.

In the Kruskal coordinates, at leading order in the expansion in r we have

$$\begin{aligned}
\delta(ds^2) \sim\ & 4\pi T r \delta g^{(0)}_{tt} dv^2 + 2dvdr(-\delta g^{(0)}_{tt} + \delta g^{(0)}_{tr}) + 2\delta g^{(0)}_{ti} dvdx^i + \delta g^{(0)}_{ij} dx^i dx^j\,, \\
& + 2(vG^{(0)}\zeta_i - \delta g^{(1)}_{ti} + \delta g^{(1)}_{ri})drdx^i + \frac{1}{4\pi T}\left(\delta g^{(1)}_{tt} + \delta g^{(1)}_{rr} - 2\,\delta g^{(1)}_{rt}\right) dr^2\,, \\
\delta A \sim\ & \delta a^{(0)}_t dv + \left(\delta a^{(0)}_i - vE_i\right) dx^i + \frac{1}{4\pi T}(\delta a^{(1)}_r - \delta a^{(1)}_t)dr\,.
\end{aligned} \tag{2.15}$$

Note that to obtain the leading order pieces in the perturbed field strength one should calculate the field strength first and then take the limit $r \to 0$.

3.1 Electric current

We define the bulk electric current density via

$$J^i = \frac{1}{16\pi G_N}\sqrt{-g}Z(\phi)F^{ir}\,. \tag{2.16}$$

When evaluated at the AdS_D boundary we obtain $J^i|_\infty$ which is the electric current density of the dual field theory as explained in appendix A.3.

In the background geometry we have $J^i = 0$. At linearised order for the perturbed black holes we have

$$J^i = \frac{1}{16\pi G_N}\frac{\sqrt{g_d}g^{ij}_d}{(FG)^{1/2}}GUZ(\phi)\left(\partial_j a_t \frac{\delta g_{rt}}{GU} - \partial_r a_t(\frac{\delta g_{jt}}{GU} - \frac{tM\zeta_j}{GU}) + \partial_j \delta a_r - (\partial_r \delta a_j + t\partial_r N\zeta_j)\right), \tag{2.17}$$

and we see that the time-dependence drops out because of (2.12).

The gauge equations of motion given in (2.3) can be written in the form

$$\begin{aligned} \partial_i J^i &= 0\,, \\ \partial_r J^i &= \frac{1}{16\pi G_N}\partial_j\left(\sqrt{-g}Z(\phi)F^{ji}\right)\,, \end{aligned} \tag{2.18}$$

as well as

$$\partial_r\left(\sqrt{-g}Z(\phi)F^{rt}\right) = -\partial_i\left(\sqrt{-g}Z(\phi)F^{it}\right)\,. \tag{2.19}$$

For later use we also note that the perturbation satisfies, at linearised order, the condition

$$d(i_k * Z(\phi)F) = 0\,, \tag{2.20}$$

where $k = \partial_t$. Indeed, this easily follows by writing the components of the $d-1$ form as

$$\begin{aligned} i_k * Z(\phi)F =& (-1)^{d-2}\Bigg[\tfrac{1}{(d-1)!}\epsilon(i_i \dots i_{d-1}j)J^j dx^{i_1}\wedge\dots\wedge dx^{i_{d-1}} \\ &+ \tfrac{1}{2(d-2)!}\epsilon(i_1\dots i_{d-2}jk)\sqrt{-g}Z(\phi)F^{jk}dx^{i_1}\wedge\dots dx^{i_{d-2}}\wedge dr\Bigg]\,, \end{aligned} \tag{2.21}$$

where ϵ is the alternating symbol with $\epsilon(1\dots d) = 1$ and then using (2.18). Note that in the special case when $\zeta = 0$ we have that $k = \partial_t$ is a Killing vector with $\mathcal{L}_k F = \mathcal{L}_k\phi = 0$. It is then very easy to establish (2.20). When $\zeta \neq 0$, k is no longer a Killing vector and furthermore $\mathcal{L}_k F \neq 0$. Nevertheless, at linearised order, we still have (2.20) as we have just shown.

3.2 Heat current

We now define the bulk heat current. To do so we want to identify equations of motion involving the metric perturbation that have a similar structure to the gauge equations of motion. We do this using[5] the vector $k \equiv \partial_t$. The procedure is slightly subtle when $\zeta \neq 0$ since in this case ∂_t is no longer a Killing vector. We proceed as follows. Consider a general vector k which satisfies

$$\nabla_\mu k^\mu = 0, \qquad \nabla_\mu\nabla^{(\mu}k^{\nu)} = \alpha k^\nu\,, \tag{2.22}$$

[5] Heuristically, one can view this as a Kaluza-Klein reduction on the time direction. Alternatively, the analysis is inspired by derivations of the first law of black hole mechanics e.g. [121].

for some function α. Note, in particular that a Killing vector satisfies these conditions with $\alpha = 0$. The conditions (2.22) imply that

$$\nabla_\mu \left(\nabla^{[\mu} k^{\nu]}\right) = (-R^\nu{}_\sigma + \alpha \delta^\nu_\sigma) k^\sigma \,. \tag{2.23}$$

We next write $\varphi = i_k A$ and $i_k F = d\theta + \psi$, with ψ a one-form and θ a globally defined function. In the special case that $\mathcal{L}_k F = 0$ we have $d\psi = 0$. We now define[6] the two-form G:

$$G^{\mu\nu} = -2\nabla^{[\mu} k^{\nu]} - \frac{2Z(\phi)}{D-2} k^{[\mu} F^{\nu]\sigma} A_\sigma - \frac{1}{(D-2)} \left[(3-D)\,\theta + \varphi\right] Z(\phi)\, F^{\mu\nu} \,. \tag{2.24}$$

If we assume that $\mathcal{L}_k \phi = 0$, using the equations of motion (2.3), we can deduce that

$$\nabla_\mu G^{\mu\nu} = \left(\alpha + \frac{2V}{D-2}\right) k^\nu - \frac{3-D}{D-2} Z(\phi) F^{\nu\rho} \psi_\rho - \frac{1}{D-2} Z(\phi) A_\rho L_k (F^{\nu\rho}) \,. \tag{2.25}$$

For our setup, with $k = \partial_t$ and working at linearised order, we can choose $\theta = -\varphi$ and we have

$$\begin{aligned} \alpha &= -\nabla^{(d)}_i (g^{ij}_{(d)} \zeta_j) - \frac{1}{2} \partial_i \log(G^3 F) g^{ij}_{(d)} \zeta_j \,, \\ \varphi &= -\theta = a_t + \delta a_t \,, \qquad \psi = -E_i\, dx^i + a_t\, \zeta_i\, dx^i \,. \end{aligned} \tag{2.26}$$

We now define the bulk heat current density via

$$Q^i = \frac{1}{16\pi G_N} \sqrt{-g} G^{ir} \,. \tag{2.27}$$

When evaluated at the AdS_D boundary, we show in appendix A.3 that $Q^i|_\infty$ is the time-independent part of the heat current density of the dual CFT:

$$\bar{G}^{1/2} \sqrt{\bar{g}_d} (\bar{G} t^{ti} - \mu j^i) = Q^i|_\infty - t \bar{G}^{3/2} \sqrt{\bar{g}_d} t^{ij} \zeta_j \,. \tag{2.28}$$

Here t^{ti}, j^i are the expectation values of the holographic stress tensor and current vector, with e.g. $J^i|_\infty = \bar{G}^{1/2} \sqrt{\bar{g}_d} j^i$ The precise combination that appears on the left hand side in (2.28) is the operator that is sourced by $-t\zeta_i$ and is, by definition, what we call the heat current density. Notice that in the case when the holographic lattice has no spatially dependent sources for the metric this reduces to the standard expression $t^{ti} - \mu j^i$. The time dependent piece on the right hand side is associated with the static susceptibility for the heat current two point function (see appendix C of [66]).

In the background geometry we have $Q^i = 0$. At linearised order for the perturbed

[6]This definition slightly differs from the definition used in [55, 66]. The expression here has the advantage that it is globally defined.

black holes we have

$$Q^i = \frac{1}{16\pi G_N}\frac{G^{3/2}U^2}{F^{1/2}}\sqrt{g_d}g_d^{ij}\left(\partial_r\left(\frac{\delta g_{jt}}{GU}\right) - \partial_j\left(\frac{\delta g_{rt}}{GU}\right)\right) - a_t J^i \,. \tag{2.29}$$

$$\begin{aligned}\partial_i Q^i &= 0\,,\\ \partial_r Q^i &= \frac{1}{16\pi G_N}\partial_j\left(\sqrt{-g}G^{ji}\right)\,,\end{aligned} \tag{2.30}$$

as well as

$$\partial_r\left(\sqrt{-g}G^{rt}\right) + \partial_i\left(\sqrt{-g}G^{it}\right) = \sqrt{-g}\left(\left(\alpha + \frac{2V}{D-2}\right) - \frac{3-D}{D-2}Z(\phi)F^{tj}\psi_j\right)\,. \tag{2.31}$$

Note that (2.31) includes an equation for the background as well as the linearised perturbation. We also record here that

$$\sqrt{-g}G^{ij} = -(GF)^{1/2}\sqrt{g_d}g_d^{ik}g_d^{jl}\left((UG)\partial_k\left(\frac{\delta g_{lt}}{GU}\right) + Z(\phi)a_t\left(\partial_k a_t\left(\frac{\delta g_{lt}}{GU}\right) + \partial_k\delta a_l\right) \quad -k\leftrightarrow l\right), \tag{2.32}$$

and we note that the time dependence drops out because of the conditions (2.11) and (2.12).

Finally, following the discussion for the electric currents, with $k = \partial_t$ we conclude that (2.30) implies

$$d(i_k * G) = 0\,. \tag{2.33}$$

3.3 Currents at the horizon

We now obtain expressions for the electric and the heat current densities by expanding at the black hole horizon. We find:

$$\begin{aligned} J^i_{(0)} &\equiv J^i\big|_H = \frac{1}{16\pi G_N}Z(\phi^{(0)})\sqrt{g_{(0)}}g^{ij}_{(0)}\left(\left(\partial_j\delta a_t^{(0)} + E_j\right) - a_t^{(0)}\delta g_{jt}^{(0)}\right)\,,\\ Q^i_{(0)} &\equiv Q^i\big|_H = -\frac{1}{4G_N}T\sqrt{g_{(0)}}g^{ij}_{(0)}\delta g^{(0)}_{jt}\,. \end{aligned} \tag{2.34}$$

From the first equations in (2.18) and (2.30) we immediately obtain

$$\partial_i J^i_{(0)} = 0\,, \qquad \partial_i Q^i_{(0)} = 0\,, \tag{2.35}$$

which give two equations for a subset of the perturbations at the horizon. We can obtain a closed system of equations, which are the generalised Stokes equations, by considering the second equation of (2.30). We explain how this can be achieved in appendix A.4. The same system of equations can also be obtained, in a more illuminating manner, by evaluating the Hamiltonian, momentum and Gauss law constraints on the black hole horizon, as we now discuss.

3.4 Constraints at the horizon

We carry out a Hamiltonian decomposition of the equations of motion using a radial decomposition in appendix A.1. The momentum constraints and the Gauss-law constraints, can be written in the form $H^\nu = C = 0$ where

$$\begin{aligned} H^\nu =& -2\sqrt{-h}\, D_\mu \left((-h)^{-1/2}\pi^{\mu\nu}\right) + h^{\nu\sigma} f_{\sigma\rho}\pi^\rho - h^{\nu\sigma} a_\sigma \partial_\rho \pi^\rho + h^{\nu\sigma}\partial_\sigma\phi\pi_\phi\,, \\ C =& \partial_\mu \pi^\mu\,, \end{aligned} \tag{2.36}$$

with

$$\begin{aligned} \pi^{\mu\nu} &= -\sqrt{-h}\,\left(K^{\mu\nu} - K\,h^{\mu\nu}\right)\,, \\ \pi^\mu &= \sqrt{-h} Z F^{\mu\rho} n_\rho\,, \\ \pi_\phi &= -\sqrt{-h} n^\nu \partial_\nu \phi\,. \end{aligned} \tag{2.37}$$

Here n is the unit norm normal vector, $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric, $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}$ is the extrinsic curvature and $K = g^{\mu\nu}K_{\mu\nu}$. In addition D_μ is the Levi-Civita connection with respect to $h_{\mu\nu}$, $b_\mu = h_\mu{}^\nu A_\nu$ and $f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$.

We want to analyse these constraints for the perturbed metric on a surface of constant r, near the horizon, and then take the limit $r \to 0$. In local coordinates we have $n = N dr$ where N is the lapse function. We immediately notice that

$$\pi^\mu = \sqrt{-h} Z F^{\mu r} n_r = 16\pi G_N J^\mu\,, \tag{2.38}$$

and hence the Gauss-Law constraint $C = 0$ is simply $\partial_\mu J^\mu = 0$. Evaluated at the horizon we obtain $\partial_i J^i_{(0)} = 0$ as in (2.35). Turning to the momentum constraint, which we discuss further in appendix A.2, we find that evaluating $H_t = 0$ as an expansion at the horizon gives $\partial_i Q^i_{(0)} = 0$ as in (2.35). Next, $H_i = 0$ evaluated at the horizon gives the extra equation mentioned above which, combined with $\partial_i J^i_{(0)} = \partial_i Q^i_{(0)} = 0$ gives the Stokes system of equations which we summarise in the next subsection. Finally, we note that the leading order term of the Hamiltonian constraint, which is explicitly given in (A.9), also gives the condition $\partial_i Q^i_{(0)} = 0$.

3.5 Generalised Stokes equations at the horizon

We can now summarise the closed system of equations that we have shown a subset of the linearised perturbations must satisfy at the black hole horizon. The black hole horizon is as in (2.4), (2.6) and (2.7). The perturbation at the horizon is given as in (2.13),(2.14) and it is illuminating to now introduce the following notation:

$$v_i \equiv -\delta g_{it}^{(0)}, \qquad w \equiv \delta a_t^{(0)}, \qquad p \equiv -4\pi T \frac{\delta g_{rt}^{(0)}}{G^{(0)}} - \delta g_{it}^{(0)} g_{(0)}^{ij} \nabla_j \ln G^{(0)} . \tag{2.39}$$

These $d+2$ unknowns satisfy the following $d+2$ linear system of partial differential equations:

$$\nabla_i v^i = 0 , \tag{2.40}$$

$$\nabla_i (Z^{(0)} \nabla^i w) + v^i \nabla_i \left(Z^{(0)} a_t^{(0)} \right) = -\nabla_i (Z^{(0)} E^i) , \tag{2.41}$$

$$-2\, \nabla^i \nabla_{(i} v_{j)} - Z^{(0)} a_t^{(0)} \nabla_j w + \nabla_j \phi^{(0)} \nabla_i \phi^{(0)} v^i + \nabla_j p = 4\pi T \, \zeta_j + Z^{(0)} a_t^{(0)} E_j , \tag{2.42}$$

where the covariant derivatives ∇ in this subsection are with respect to the metric, $g_{ij}^{(0)}$, on the black hole horizon Σ_d, and all indices are being raised and lowered with this metric. The first two equations are simply $\partial_i Q_{(0)}^i = \partial_i J_{(0)}^i = 0$, where $Q_{(0)}^i, J_{(0)}^i$ are the heat current and electric current densities at the horizon, respectively:

$$\begin{aligned} Q_{(0)}^i &= \frac{1}{4G_N} T \sqrt{g_{(0)}} v^j , \\ J_{(0)}^i &= \frac{1}{16\pi G_N} \sqrt{g_{(0)}} g_{(0)}^{ij} Z^{(0)} \left(\partial_j w + a_t^{(0)} v_j + E_j \right) . \end{aligned} \tag{2.43}$$

It is helpful to note that in the third equation we can also write

$$2\, \nabla^i \nabla_{(i} v_{j)} = \nabla^2 v_j + R_{ji} v^i . \tag{2.44}$$

We emphasise that by evaluating the constraints at the horizon we obtain a system of equations for a *subset* of the linear perturbation, namely, $\delta g_{it}^{(0)}, \delta a_t^{(0)}, \delta g_{rt}^{(0)}$, and we obtain a closed system for this set. Furthermore, the equations we have obtained are a generalisation of the forced Stokes equations for a charged fluid on the curved black hole horizon. Indeed in the special case of electrically neutral black hole horizons with $a_t^{(0)} = w = E = 0$ and in addition constant $\phi^{(0)}$, the equations are simply the Stokes equations with fluid velocity v_i, pressure p and forcing term given by the closed one-form $4\pi T \zeta$. The curvature of the horizon gives rise to an extra viscosity term as in (2.44). In the general case we have a charged fluid with scalar potential w and an additional forcing term given by the closed one-form E. It is also interesting to note that the scalar field is giving viscosity terms of the form $\nabla^j \phi^{(0)} \nabla^i \phi^{(0)} v_i$. We will see in section 4 how these extra terms play a direct role in determining the DC conductivity. We emphasise that we have not taken any hydrodynamical limit in obtaining these equations.

We now establish a number of interesting properties of this set of equations. Firstly, by taking the divergence of (2.42) and using (2.40), (2.41) we obtain the "pressure Poisson equation"

$$\nabla^2 p = \nabla_j \left(2R^j{}_k v^k + Z^{(0)} a^{(0)} (\nabla^j w + E^j) + 4\pi T \zeta^j - \left(\nabla^j \phi^{(0)} \nabla^k \phi^{(0)}\right) v_k \right). \tag{2.45}$$

For a compact horizon and given background data, the pressure term is uniquely specified by v_j, w, E_j, ζ_j.

Second, we multiply (2.42) by v_j from the left, and then integrate over the horizon, and use (2.40),(2.41) to obtain

$$\begin{aligned} \int d^d x \sqrt{g_0} \left[2\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + Z^{(0)} \left(\nabla w + E\right)^2 + v_i \left(\nabla^i \phi^{(0)} \nabla^j \phi^{(0)}\right) v_j \right] \\ = \int d^d x \left(Q^i_{(0)} \zeta_i + J^i_{(0)} E_i \right) \end{aligned} \tag{2.46}$$

In the case of non-compact horizons, we have assumed that possible boundary terms vanish. Observe that the left hand side is a manifestly positive quantity and this is related to the positivity of the thermoelectric conductivities, which we discuss later.

Third, we consider the issue of uniqueness of the equations (2.40)-(2.42). If we have two solutions then the difference of the functions, which we again write as (v_i, w, p), will satisfy the same equations but with vanishing forcing terms, $\zeta_j = E_j = 0$. From (2.46) we immediately conclude that

$$\nabla^{(i} v^{j)} = 0, \quad \nabla^i w = 0, \quad v^i \nabla_i \phi^{(0)} = 0\,. \tag{2.47}$$

We also have $\mathcal{L}_v a_t^{(0)} = 0$ from (2.41) and $\nabla_i p = 0$ from (2.42). We conclude that the solution space of equation (2.42) is unique up to Killing vectors of the horizon metric, with p, w constant. We then have $\delta g_{rt} = (4\pi T)^{-1} \mathcal{L}_v G^{(0)}$ plus a constant. This result agrees with the intuition that one should be able to boost along the orbits of Killing vectors to obtain a solution with momentum at the horizon.

Fourth, we observe that when (E, ζ) are exact forms, $(E, \zeta) = (de, dz)$ with e, z globally defined functions on Σ_d, we can solve the equations (2.40)-(2.42) by taking $w = -e$ and $p = 4\pi T z$, plus possible constants, and $v^i = 0$. We observe that this solution gives no contribution to the current densities (2.43) at the horizon. We will see later that this solution gives no contribution to suitably averaged currents at the AdS boundary and hence no contribution to the DC thermoelectric conductivity *i.e.* the DC conductivity is determined by the harmonic part of E and ζ. A basis for the non-trivial part of these sources is thus given by a basis for the first cohomology group of Σ_d.

Fifth, we point out that the fluid equations that we have obtained can be obtained by varying the following functional:

$$L = \int d^d x \sqrt{g_0} \Bigg[-\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} - \frac{1}{2}(v^i \nabla_i \phi)^2 + p(\nabla_i v^i) + \frac{1}{2} Z^{(0)} \left(a^{(0)} v + \nabla w \right)^2 - \frac{1}{2} Z^{(0)} a^{(0)2} v^2 + 4\pi T \zeta_i v^i + Z^{(0)} E_i \left(a^{(0)} v^i + \nabla^i w \right) + \frac{1}{2} Z^{(0)} E_i E^i \Bigg] , \quad (2.48)$$

and we remind the reader that the covariant derivative ∇ in this subsection is with respect to the metric g^0_{ij}. Varying with respect to the pressure, which is a Lagrange multiplier, gives the incompressibility condition. Varying with respect v and w then gives the remaining Stokes equations. It is also interesting to note that if we vary with respect to E_i and ζ_i then we get the currents at the horizon $J^i_{(0)}$ and $Q^i_{(0)}$, respectively. On shell we therefore can deduce, for example, that

$$\frac{\delta J^i_{(0)}}{\delta \zeta_j} = \frac{\delta Q^j_{(0)}}{\delta E_j} . \quad (2.49)$$

This is a kind of Onsager reciprocal relation for the currents at the horizon. After considering the current fluxes, to be described in the next subsection, we obtain Onsager relations for the DC conductivities.

Finally, we comment on the fact that, *locally*, the sources can be eliminated from the Stokes equations (2.40)-(2.42). Indeed since the sources E, ζ are closed, locally we can write $E = de$, $\zeta = dz$ and after defining $\tilde{w} = w + e$, $\tilde{p} = p - 4\pi T z$ we have

$$\nabla_i v^i = 0 , \quad (2.50)$$

$$\nabla_i (Z^{(0)} \nabla^i \tilde{w}) + v^i \nabla_i \left(Z^{(0)} a^{(0)}_t \right) = 0 , \quad (2.51)$$

$$-2 \nabla^i \nabla_{(i} v_{j)} - Z^{(0)} a^{(0)}_t \nabla_j \tilde{w} + \nabla_j \phi^{(0)} \nabla_i \phi^{(0)} v^i + \nabla_j \tilde{p} = 0 . \quad (2.52)$$

It is important to emphasise that now $\tilde{w}$ and $\tilde{p}$ are not globally defined functions on the black hole horizon. For example, if the horizon was a torus with $x^i = x^i + L_i$, and the source $E = c dx^1$ for some constant c, then $\tilde{w}$ would satisfy the twisted boundary condition $\tilde{w}(x^1 + L_1) = \tilde{w} + cL_1$. Note that this would give extra contributions to (2.46). It is therefore most natural to work with the formulation with sources, as in [98].

It is worth noting, however, that the sources can also be removed, locally, from the full linearised perturbation. Indeed, suppose we carry out the gauge transformation $A \to A = d(te) + B$ and in addition change the time coordinate via $t = \tilde{t}(1 - z) + C$, where B, C are functions independent of the time coordinate. We then find that at linearised order we obtain the same perturbed ansatz with vanishing sources and

$$\begin{aligned}&\delta g_{\tilde{t}\tilde{t}} = \delta g_{tt} + 2UGz, \quad \delta g_{\tilde{t}r} = \delta g_{tt} - UG\partial_r C \quad \delta g_{\tilde{t}i} = \delta g_{tt} - UG\partial_i C\\ &\delta a_{\tilde{t}} = \delta a_t + e - a_t z, \quad \delta a_r = \delta a_r + \partial_r B, \quad \delta a_i = \delta a_i + \partial_i B\end{aligned} \tag{2.53}$$

We choose B, C to vanish suitably fast at the AdS boundary and at the horizon we choose $B = \ln r/(4\pi T)e + \dots$ and $C = -\ln r/(4\pi T)z + \dots$. Evaluating at the horizon we see that in the new coordinates and gauge we have induced $w \to \tilde{w}$, $p \to \tilde{p}$ and $\delta g_{tt}^{(0)} \to \delta g_{tt}^{(0)} + 2G^{(0)}z$.

3.6 The DC thermoelectric conductivity

For a given set of sources (E, ζ) we can solve the Stokes equations (2.40)-(2.42) at the black hole horizon and hence obtain expressions for the electric and heat current densities at the black hole horizon. From this data we would like to deduce something about the current densities at the holographic boundary as a function of (E, ζ). The radial dependence of the current densities are given by (2.18) and (2.30). For some simple special classes of black hole solutions the current densities J^i, Q^i are independent of the radial coordinate. This occurs for the Q-lattices and the holographic lattices that depend on just one of the spatial dimensions, for example. However, for general classes of black holes J^i, Q^i will depend on the radial direction.

On the other hand, remarkably, we can always define "current flux densities" $\bar{J}^i, \bar{Q}^i$ which are independent of r and hence we can obtain the associated DC conductivity. To consider a concrete example, we assume that we are in $D = 4$ with a periodic holographic lattice and $\Sigma_d = \mathbb{R}^2$ or a two-torus. In particular, the lattice deformations $\bar{G}(x), \bar{g}_{ij}(x)$, $\mu(x)$ and $\bar{\phi}(x)$ in (2.5) are all periodic functions of the spatial coordinates x^i with period L_i. We define the following current flux densities

$$\bar{J}^1 \equiv \frac{1}{L_2}\int J^1 dx^2\,, \qquad \bar{J}^2 \equiv \frac{1}{L_1}\int J^2 dx^1\,. \tag{2.54}$$

where $\bar{J}^1$ and $\bar{J}^2$ is the current flux density through the x_2 and x_1 planes, respectively. We define $\bar{Q}^i$ in a similar way. We can then immediately deduce from (2.18), (2.30) that $\partial_r \bar{J}^i = \partial_r \bar{Q}^i = 0$, which is simply Stokes theorem in the bulk. Notice that $\bar{J}^i, \bar{Q}^i$ are also independent of the x^i coordinates and hence they are just constants. Similarly in $D = 5$ with a periodic lattice on $\mathbb{R}^{1,3}$ we would define the following constant current flux densities

$$\bar{J}^1 \equiv \frac{1}{L_2 L_3}\int dx^2 dx^3 J^1\,, \quad \bar{J}^2 \equiv -\frac{1}{L_3 L_1}\int dx^1 dx^3 J^2\,, \quad \bar{J}^3 \equiv \frac{1}{L_1 L_2}\int dx^1 dx^2 J^3\,, \tag{2.55}$$

The current flux densities of the boundary theory are, by definition, $\bar{J}^i, \bar{Q}^i$ evaluated at the AdS boundary. In order to evaluate the DC conductivity matrix we want to relate these to constant sources $\bar{E}_i, \bar{\zeta}_i$ at the AdS boundary via

$$\begin{pmatrix} \bar{J}^i \\ \bar{Q}^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} \bar{E}_j \\ \bar{\zeta}_j \end{pmatrix} . \tag{2.56}$$

We have just shown that $\bar{J}^i, \bar{Q}^i$ are constant and so their value at the AdS boundary is the same as at the black hole horizon, and that in turn these are fixed by the closed forms E, ζ evaluated at the horizon by solving the Stokes equations (2.40)-(2.42). Continuing with the case that $\Sigma_d = \mathbb{R}^d$ or a torus, we can take an independent basis of sources to be the d one-forms $\bar{E}_i dx^i$ (no sum on i) and the d one-forms $\bar{\zeta}_i dx^i$ (no sum on i) with constant $\bar{E}_i, \bar{\zeta}_i$ and this defines the DC conductivity matrix (2.56).

Note that an equivalent way to characterise the constant sources $\bar{E}_i, \bar{\zeta}_i$ at the AdS boundary is to write a general closed form source on $\Sigma_d = \mathbb{R}^d$ or a torus, as $E = \bar{E}_i dx^i + de$, where e is a periodic function and then extract $\bar{E}_i$ by integrating E over an appropriate basis of one-cycles. It is worth emphasising that in the paragraph following (2.47) we noted that it is only the harmonic part of the sources that contribute to the currents at the horizon and hence this procedure gives the same DC conductivity.

After substituting (2.56) in (2.46), we can now explicitly see that the positivity of the left hand side of equation (2.46) implies that the thermoelectric DC conductivity is a positive definite matrix. Continuing on from the discussion following (2.48) we can deduce that the thermoelectric matrix is symmetric.

We will discuss the DC conductivity matrix when Σ_d is not $\mathbb{R}^d$ or a d-dimensional torus in the next subsection.

We have presented a procedure for calculating the DC conductivity of the boundary theory in terms of a calculation at the black hole horizon. One might wonder if the calculation could also be done at any constant radial hypersurface. In fact this cannot be done since evaluating the constraint equations on a constant r hypersurface with $r \not\to 0$ will not lead to a closed system of equations for a subset of the linear perturbation and hence we cannot obtain the current fluxes.

3.7 Perspective using forms

We have focussed on a particular class of black holes given in (2.4), with a single black hole horizon, and with perturbation given in (2.10) with (2.11), (2.12). We have also focussed on black holes for which Σ_d is topologically either $\mathbb{R}^d$ or a d-dimensional torus. In this section we briefly discuss the DC conductivity calculation using the language of forms, which illuminates some global issues as well as revealing generalisations for Σ_d with other topologies.

One key point is that the two-forms F and G for the perturbed metric satisfy, at linearised order, the following closure conditions (see (2.20), (2.33)):

$$d(i_k * Z(\phi)F) = 0\,, \qquad d(i_k * G) = 0\,, \tag{2.57}$$

where $k = \partial_t$. These conditions are valid without k being a Killing vector, but instead satisfying the weaker conditions in (2.22).

We now observe that for any two-forms satisfying (2.57) we can define a natural set of current fluxes. Specifically, at the deformed AdS_D boundary and at fixed t, we let C_a, $a = 1, \ldots, b_{d-1}(\Sigma_d)$, be a basis of $d-1$ closed cycles on Σ_d, where $b_{d-1}(\Sigma_d)$ is the $(d-1)$ Betti number of Σ_d. We can then define the current fluxes through these cycles via

$$\bar{J}^a \equiv -\int_{C_a} i_k * Z(\phi)F\,, \qquad \bar{Q}^a \equiv -\int_{C_a} i_k * G\,. \tag{2.58}$$

We now consider a d-dimensional surface S in the bulk spacetime which has boundary C_a at the AdS boundary and possibly another boundary at a black hole horizon. Then since the integrand in (2.58) is closed, we deduce that these current fluxes are also equal to their values at the black hole horizon. If the cycle C_a is contractible in the bulk, then the current flux would necessarily have to be zero.

For the special cases of periodic lattices in $D = 4$ and 5 spacetime dimensions, for which Σ_d is topologically $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, using (2.21) we immediately see that the definition (2.58) agrees with the definitions[7] given in (2.54), (2.55) after choosing an obvious basis of one and two cycles, respectively. For these cases, the number of current fluxes is the same as the dimension of Σ_d. However, this is not the case for more general Σ_d. In $D = 4$, for example, we can envisage black holes in which Σ_2 is a Riemann surface with genus $g > 1$, and it is possible to define $2g$ current fluxes. There are many more possibilities for solutions in $D = 5$. We also note that when Σ_d is a sphere, which is relevant for solutions associated with deformations of global AdS, these current fluxes are all trivial since $b_{d-1}(S^d) = 0$.

The above comments were based on general two-forms satisfying (2.57). A second key point in our derivation of the DC conductivity is that the two-forms were constructed with specific source terms parametrised by the one-forms E, ζ. For the class of solutions that we considered we assumed there was a single black hole horizon with the same topology Σ_d as the spatial boundary of the deformed AdS space. In order to satisfy (2.57) it was necessary to take the one forms E, ζ to be closed one-forms on Σ_d and independent of the radial coordinate. Corresponding to the basis of $(d-1)$-cycles, C_a, we can define a basis of harmonic one-forms, ϕ^a, on Σ_d by Poincaré duality. We can then write $E = \bar{E}_a\phi^a + de$ and $\zeta = \bar{\zeta}_a\phi^a + dz$ with constant $\bar{E}_a, \bar{\zeta}_a$. Recalling the discussion in the paragraph following (2.47), in solving the Stokes equations at the horizon in order to obtain the currents at the

[7]Note that in (2.54), (2.55) we have divided by suitable L_i in order to obtain current flux *densities*.

horizon only the harmonic part of the sources, $\bar{E}_a\phi^a$, $\bar{\zeta}_a\phi^a$ are important. We therefore can relate the $b_{d-1}(\Sigma_d)$ independent constant source terms to the $b_{d-1}(\Sigma_d)$ current fluxes at the horizon, after solving the Stokes equations, and hence to the $b_{d-1}(\Sigma_d)$ current fluxes at the AdS_D boundary. This procedure gives rise to thermoelectric conductivity matrices σ^{ab}, α^{ab}, $\bar{\alpha}^{ab}$ and $\bar{\kappa}^{ab}$, all of which are $b_1(\Sigma_d)\times b_1(\Sigma_d)$ matrices, where we used the fact that $b_{d-1}(\Sigma_d)=b_1(\Sigma_d)$.

One can ask if this aspect of the formalism can be adopted to more general classes of black holes in which there are multiple black hole horizons (an example of such a solution, but without spatially dependent sources, is given in [122]). This would require identifying suitable source terms E,ζ that depend on the radial direction while still maintaining the condition (2.57). We return to this point in section 5.

Finally, recalling (2.46) we note that we can write

$$\begin{aligned}\int d^dx\left(Q^i_{(0)}\zeta_i+J^i_{(0)}E_i\right)&=-\int_{\Sigma_d}(i_k*Z(\phi)F)\wedge E+(i_k*G)\wedge\zeta\,,\\&=-\int_{C^{(E)}}(i_k*Z(\phi)F)-\int_{C^{(\zeta)}}(i_k*G)\,,\end{aligned}\tag{2.59}$$

where in the first line we are integrating over any surface at constant r and t. In the second line $C^{(E)}$ and $C^{(\zeta)}$ are $(d-1)$ cycles, unique up to homology, that are Poincaré dual to the closed one-forms E and ζ. By definition the right hand side is thus the sum of the current fluxes $\bar{J}^{(E)}+\bar{Q}^{(\zeta)}$, through the cycles $C^{(E)}$ and $C^{(\zeta)}$, repsectively. The positivity of the left hand-side, which we obtain from (2.46), is associated with the positivity of the thermoelectric DC conductivity.

4 Examples

In this section we examine some special examples of holographic lattices for which we can solve the fluid equations on the horizon and hence obtain expressions for the DC conductivity in terms of the behaviour of the black hole solutions at the horizon. We first discuss how extra scalar fields manifest themselves in extra terms in the fluid equations at the horizon and then use this to study a general class of Q-lattices. We next analyse general holographic lattices that depend on just one spatial dimension. Finally we examine holographic lattices that can be obtained as a perturbative expansion about the AdS-RN black brane.

4.1 Extra scalars and Q-lattices

For simplicity we derived the Stokes equations for the model given in (2.42) which involved a single scalar field. However, the generalisation to extra scalar fields, which can parametrise a non-trivial target space manifold, is straightforward. Specifically, if we replace (2.2) with several scalars, ϕ^I, with the functions V, Z depending on all of the scalars and the kinetic-energy terms generalised via

$$-\frac{1}{2}\partial\phi^2 \to -\frac{1}{2}\mathcal{G}_{IJ}(\phi)\partial\phi^I\partial\phi^J\,, \tag{2.60}$$

then this leads to the Stokes equations as before, with the only change in (2.42) given by

$$-\nabla_j\phi^{(0)}\nabla_i\phi^{(0)}v^i \to -\mathcal{G}_{IJ}(\phi^{(0)})\nabla_j\phi^{I(0)}\nabla_i\phi^{J(0)}v^i\,. \tag{2.61}$$

With this result in hand we can now obtain previous results for the DC conductivities for Q-lattices [58, 66]. They key feature of the Q-lattice is that it exploits a global symmetry in the bulk to construct the black hole solutions. In the present context we assume that the model admits n global shift symmetries of the scalars:

$$\phi^{I_\alpha} \to \phi^{I_\alpha} + \epsilon^{I_\alpha}\,, \tag{2.62}$$

with $\alpha = 1, \ldots, n$. For example, if we had a single complex scalar field with a global $U(1)$ symmetry, then the associated shift symmetry of this type is obtained by parametrising the scalar manifold locally with the modulus and phase of the complex scalar field. This gives rise to a periodic lattice. Another example, is a massless "axion" field with only derivative couplings.

The spatial coordinates x^i are taken to parametrise $\mathbb{R}^d$ or possibly a torus. The black hole solutions are then constructed based on an ansatz in which the scalars associated with these shift symmetries take the form

$$\phi^{I_\alpha} = \mathcal{C}^{I_\alpha}{}_j\, x^j\,, \tag{2.63}$$

everywhere in bulk with $\mathcal{C}$ a constant n by d matrix. For simplicity of presentation we assume that all spatial coordinates are involved and hence the DC conductivity in all spatial directions is finite. The metric, the gauge-field and the remaining scalar fields will depend on the radial direction but will be independent of the spatial coordinates x^i. The metric on the black hole horizon is flat and in addition, $Z^{(0)}$, $G^{(0)}$ and $a_t^{(0)}$ are all constant.

After these remarks, the fluid equations (2.40)-(2.42) are solved with v^i, p and w all constant on the horizon. The fluid velocity is given by

$$v^i = 4\pi T\,\left(\mathcal{D}^{-1}\right)^{ij}\,\left(\zeta_j + \frac{\rho}{Ts}\,E_j\right)\,, \tag{2.64}$$

with constant E_i, ζ_i and we have defined the $d \times d$ matrix:

$$\mathcal{D}_{ij} = G_{I_{\alpha_1} I_{\alpha_2}} \mathcal{C}^{I_{\alpha_1}}{}_i \mathcal{C}^{I_{\alpha_2}}{}_j \,. \tag{2.65}$$

Furthermore, the averaged charge density, ρ, defined in (2.9), and the entropy density, s, are given by

$$\rho = \rho_H = \frac{1}{16\pi G_N} \sqrt{g_{(0)}} Z^{(0)} a_t^{(0)}, \qquad s = \frac{1}{4G_N} \sqrt{g_{(0)}} \,. \tag{2.66}$$

The current densities J^i, Q^i are independent of the radius and are given by their horizon values:

$$\begin{aligned} J^i = & \left(\frac{s\, Z^{(0)}}{4\pi} g_{(0)}^{ij} + \frac{4\pi\rho^2}{s} \left(\mathcal{D}^{-1}\right)^{ij} \right) E_j + 4\pi T \rho \left(\mathcal{D}^{-1}\right)^{ij} \zeta_j \,, \\ Q^i = & 4\pi T s \left(\mathcal{D}^{-1}\right)^{ij} \left(\zeta_j + \frac{\rho}{Ts} E_j \right) \,. \end{aligned} \tag{2.67}$$

The DC conductivities are thus given by

$$\begin{aligned} \sigma^{ij} = & \frac{s\, Z^{(0)}}{4\pi} g_{(0)}^{ij} + \frac{4\pi\rho^2}{s} \left(\mathcal{D}^{-1}\right)^{ij} \,, \\ \alpha^{ij} = & \bar{\alpha}^{ij} = 4\pi\rho \left(\mathcal{D}^{-1}\right)^{ij} \,, \\ \bar{\kappa}^{ij} = & 4\pi T s \left(\mathcal{D}^{-1}\right)^{ij} \,. \end{aligned} \tag{2.68}$$

Note that the conductivity when $Q = 0$, $\sigma_{Q=0} \equiv \sigma - T\alpha\bar{\kappa}^{-1}\bar{\alpha}$, is given by

$$\sigma_{Q=0}^{ij} = \frac{s\, Z^{(0)}}{4\pi} g_{(0)}^{ij} \,. \tag{2.69}$$

A final point worth emphasising is that the origin of the matrix $\mathcal{D}$ appearing in the final expressions arises from the extra terms involving the scalars in the Stokes equations, underscoring the significance of the latter.

4.2 One-dimensional lattices

We now consider a class of black hole solutions with metrics on the horizon that break translations in just one of the spatial directions. As special sub-cases we will recover the results for the inhomogeneous lattices with varying chemical potential studied in [55] as well as the helical lattices studied in [101]. Recently formulae for the DC conductivity for a scalar lattice were obtained in [56] in terms the behaviour of the solution at the black hole horizon as well as sub-leading terms. We improve upon those results by providing a new formula that depends just on the solution at the horizon.

We assume that the horizon geometry depends on the spatial coordinate x and is independent of the remaining $d-1$ spatial coordinates which, for definiteness and without loss of generality, we take to parametrise a torus. The moduli of this torus can depend on x. For simplicity we restrict our considerations to metrics on the black hole horizon of the form

$$ds^2_d = g^{(0)}_{ij}\, dx^i dx^j = \gamma(x)\, dx^2 + ds^2_{d-1}\,(x)\ , \tag{2.70}$$

where $ds^2_{d-1} = g_{ab} dx^a dx^b$ is a flat metric on the torus. We now solve the relevant system of equations (2.40)-(2.42). The incompressibility condition (2.40) is solved by taking the non-vanishing components of v^i to be

$$v^x = (\gamma\, g_{d-1})^{-1/2}\, v_0\,, \tag{2.71}$$

with g_{d-1} the determinant of the $d-1$ dimensional metric on the torus and v_0 a constant. The non-trivial component of the current density is $J^x_{(0)}$, which must be a constant, and we have

$$16\pi G_N \frac{\gamma^{1/2}}{g^{1/2}_{d-1} Z^{(0)}} J^x_{(0)} = \partial_x w + \frac{\gamma^{1/2} a^{(0)}_t}{g^{1/2}_{d-1}}\, v_0 + E_x\,. \tag{2.72}$$

With a little effort we can now write the Stokes equation (2.42) in the form

$$2\, v_0\, \partial_x \left(\gamma^{-1/2}\, \partial_x g^{-1/2}_{d-1}\right) - Y\, v_0 + 16\pi G_N \frac{\gamma^{1/2} a^{(0)}_t}{g^{1/2}_{d-1}} J^x_{(0)} - \partial_x p = -4\pi T\, \zeta_x\,, \tag{2.73}$$

where we have defined

$$\begin{aligned} Y \equiv\ & \frac{1}{2(\gamma g_{d-1})^{1/2}}\ \left[(\partial_x \ln g_{d-1})^2 + \partial_x g_{ab}\, \partial_x g_{cd}\, g^{ac}\, g^{bd}\right] \\ & + \frac{1}{(\gamma\, g_{d-1})^{1/2}}\ \left(\partial_x \phi^{(0)}\right)^2 + \frac{\gamma^{1/2} Z^{(0)} a^{(0)2}_t}{g^{1/2}_{d-1}}\,, \end{aligned} \tag{2.74}$$

with $g_{ab}(x)$ the metric components for ds^2_{d-1}. Equations (2.72) and (2.73) may now be used to fix the functions w and p. Since these are periodic functions, we must have that the expressions for $\partial_x w$ and $\partial_x p$ have no zero modes on the torus and this imposes constraints on $J^x_{(0)}$ and v_0. A simple way to establish these constraints is take an average of the two equations. In fact doing this completely fixes $J^x_{(0)}$ and v_0 in terms of the sources. Indeed, if we define

$$\int \quad \leftrightarrow \quad \frac{1}{L_1} \int dx^1\,, \tag{2.75}$$

where L_1 is the period of the lattice, we obtain

$$J^x_{(0)} = \frac{1}{16\pi G_N X}\left(E_x \int Y + 4\pi T\,\zeta_x \int \frac{\gamma^{1/2} a_t^{(0)}}{g_{d-1}^{1/2}}\right),$$
$$v_0 = \frac{1}{X}\left(4\pi T\,\zeta_x \int \frac{\gamma^{1/2}}{g_{d-1}^{1/2} Z^{(0)}} + E_x \int \frac{\gamma^{1/2} a_t^{(0)}}{g_{d-1}^{1/2}}\right), \tag{2.76}$$

where

$$X \equiv \left(\int \frac{\gamma^{1/2}}{g_{d-1}^{1/2} Z^{(0)}}\right)\left(\int Y\right) - \left(\int \frac{\gamma^{1/2} a_t^{(0)}}{g_{d-1}^{1/2}}\right)^2. \tag{2.77}$$

The expression for the heat current at the horizon is simply $Q^x_{(0)} = T\,v_0/4G_N$. Now for these one-dimensional lattices the electric current and heat current densities are independent of the radial direction and so we have deduced their values at the AdS boundary. Thus we can immediately extract the thermoelectric conductivities in the x direction and we find

$$\sigma = \frac{1}{X 16\pi G_N}\int Y, \quad \alpha = \bar{\alpha} = \frac{1}{4 X G_N}\int \frac{\gamma^{1/2} a_t^{(0)}}{g_{d-1}^{1/2}}, \quad \bar{\kappa} = \frac{\pi T}{X G_N}\int \frac{\gamma^{1/2}}{g_{d-1}^{1/2} Z^{(0)}}. \tag{2.78}$$

Observe that the final result for the conductivity is invariant under reparametrisations of the x coordinate, as it should be.

We can also write the electrical conductivity in the form

$$\sigma = \sigma_{Q=0} + \frac{1}{16\pi G_N X}\left(\int \frac{\gamma^{1/2} a_t^{(0)}}{g_{d-1}^{1/2}}\right)^2 \left(\int \frac{\gamma^{1/2}}{g_{d-1}^{1/2} Z^{(0)}}\right)^{-1} \tag{2.79}$$

where $\sigma_{Q=0} \equiv \sigma - T\alpha\bar{\kappa}^{-1}\bar{\alpha}$ is the conductivity when $Q = 0$ (as opposed to $\zeta = 0$) and is given by

$$\sigma_{Q=0} = \frac{1}{16\pi G_N}\left(\int \frac{\gamma^{1/2}}{g_{d-1}^{1/2} Z^{(0)}}\right)^{-1}. \tag{2.80}$$

Notice that the second term in (2.79) vanishes if $a_t^{(0)} = 0$.

We finish by indicating how to obtain some previous results. For the one-dimensional lattice of $D = 4$ Einstein-Maxwell theory given in [55] we simply need to use the translation given by

$$\gamma = \Sigma(x)\, e^{B(x)}, \quad ds_1^2 = \Sigma(x)\, e^{-B(x)}\, dy^2, \quad G^{(0)} = H_{tt}^{(0)}(x), \tag{2.81}$$

in order to obtain the expressions for $\sigma, \alpha, \bar{\alpha}, \bar{\kappa}$ given in [55]. Similarly for the helical lattice of pure $D = 5$ gravity studied in [101] we should use

$$\gamma = h_+^2, \quad ds_2^2 = r_+^2 \left(e^{2\alpha_+}\,\omega_2^2 + e^{-2\alpha_+}\,\omega_3^2\right) , \tag{2.82}$$

where two of the three left-invariant one-forms for Bianchi VII_0 are given by

$$\omega_2 = \cos kx\, dy - \sin kx\, dz, \quad \omega_3 = \sin kx\, dy + \cos kx\, dz\,. \tag{2.83}$$

It is straightforward to show that $Y = 4k^2 \sinh^2 2\alpha_+/(r_+^2 h_+)$ and hence recover the formula for κ in the x direction given in [101]. Finally charged helical lattices in D=5 gravity with two gauge-fields coupled to a scalar were studied in [102]. Setting the second gauge-field to zero, we can compare our results by setting

$$\gamma = C_1, \quad ds_2^2 = C_2\omega_2^2 + C_3\omega_3^2\,. \tag{2.84}$$

A short calculation shows that in the x direction we have

$$\begin{aligned}
\sigma &= \frac{1}{16\pi G_N}\frac{C_2^{1/2}C_3^{1/2}Z_0}{C_1^{1/2}} + \frac{C_2C_3}{k^2(C_2-C_3)^2}\frac{4\pi\rho^2}{s}\,,\\
\alpha &= \bar{\alpha} = \frac{C_2C_3}{k^2(C_2-C_3)^2}4\pi\rho\,, \qquad \bar{\kappa} = \frac{C_2C_3}{k^2(C_2-C_3)^2}4\pi sT\,,
\end{aligned} \tag{2.85}$$

where $s = (C_1C_2C_3)^{1/2}/(4G_N)$ and $\rho = (C_1C_2C_3)^{1/2}a_t^{(0)}Z^{(0)}/(16\pi G_N)$ (see (2.9)). The expression for σ agrees with [102] and the expressions for $\alpha, \bar{\alpha}$ and $\bar{\kappa}$ are new.

4.3 Perturbative lattices

We now consider the case of a periodic lattice that is constructed as a perturbative expansion about the electrically charged AdS-RN black brane solution with a flat horizon. As we have noted before since everything is periodic in the spatial directions, in effect, we can take Σ_d to be a torus. If λ is the perturbative parameter, then at the black hole horizon we will assume that we can write

$$\begin{aligned}
g_{(0)ij} &= g\,\delta_{ij} + \lambda\, h_{ij}^{(1)} + \cdots\,, & G^{(0)} &= f_{(0)} + \lambda\, f_{(1)} + \cdots\,,\\
Z^{(0)}a_t^{(0)} &= a + \lambda\, a_{(1)} + \cdots\,, & \phi^{(0)} &= \psi_{(0)} + \lambda\, \psi_{(1)} + \cdots\,,\\
Z^{(0)} &= z_{(0)} + \lambda\, z_{(1)} + \cdots\,,
\end{aligned} \tag{2.86}$$

with a, $z_{(0)}$, $\psi_{(0)}$, $f_{(0)}$ and g being constant and the sub-leading terms are periodic functions of, generically, all of the spatial coordinates x^i. Note that the entropy density and the electric current density on the horizon are given by

$$s = \frac{g^{d/2}}{4G_N}, \qquad \rho_H = \frac{1}{16\pi G_N} a g^{d/2}\,. \tag{2.87}$$

For the Ricci tensor and Christoffel symbols we have the expansions

$$\begin{aligned} R_{(0)ij} &= \lambda\, R^{(1)}_{ij} + \lambda^2\, R^{(2)}_{ij} + \cdots\,, \\ \Gamma^i_{jk} &= \lambda \Gamma^{(1)i}_{jk} + \lambda^2 \Gamma^{(2)i}_{jk} + \cdots\,. \end{aligned} \tag{2.88}$$

It turns out that we can solve equations (2.40)-(2.42) perturbatively in λ using the following expansion:

$$\begin{aligned} v^i &= \frac{1}{\lambda^2}\, v^i_{(0)} + \frac{1}{\lambda}\, v^i_{(1)} + v^i_{(2)} + \cdots\,, \qquad w = \frac{1}{\lambda}\, w_{(1)} + w_{(2)} + \cdots\,, \\ p &= \frac{1}{\lambda}\, p_{(1)} + p_{(2)} + \cdots\,. \end{aligned} \tag{2.89}$$

Expanding (2.40)-(2.42) in λ, we find at leading order that

$$\partial_i\, v^i_{(0)} = 0\,, \qquad \Box\, v^i_{(0)} = 0\,, \tag{2.90}$$

where $\Box = \delta^{ij}\partial_i\partial_j$ and we deduce that $v^i_{(0)}$ are just constant on the torus. We proved earlier that for the full non-linear problem, in the absence of horizon Killing vectors, there is a unique solution to the Stokes equations. Therefore, it must be the case that these integration constants are fixed at higher orders in the perturbative expansion, and this will be confirmed shortly.

At next order in the expansion we find

$$\begin{aligned} \partial_i v^i_{(1)} + \frac{g^{-1}}{2}\, \partial_j h^{(1)}\, v^j_{(0)} &= 0\,, \\ g^{-1} z_{(0)}\, \Box w_{(1)} + v^i_{(0)}\, \partial_i\, a^{(1)} &= 0\,, \\ \Box v^i_{(1)} + \partial^k (\Gamma^{(1)})^i_{ks} v^s_{(0)} + R^{(1)i}{}_j\, v^j_{(0)} + a\, \partial^i w_{(1)} - \partial^i p_{(1)} &= 0\,, \end{aligned} \tag{2.91}$$

where

$$h^{(1)} = h^{(1)k}{}_k\,, \qquad (\Gamma^{(1)})^i_{ks} = \frac{g^{-1}}{2}\left(\partial_s h^{(1)i}{}_k + \partial_k h^{(1)i}{}_s - \partial^i h^{(1)}{}_{ks}\right)\,, \tag{2.92}$$

and all indices are raised and lowered with δ. By considering the pressure Poisson equation (2.45) and using $\nabla_j R^j{}_i = \frac{1}{2}\nabla_i R$, we deduce that

$$-a\, w_{(1)} + p_{(1)} = (\Box^{-1}\partial_j R^{(1)})\, v^j_{(0)}\,, \tag{2.93}$$

with $w_{(1)} = -g z^{-1}_{(0)}\, (\Box^{-1}\partial_j a^{(1)})\, v^j_{(0)}$ from (2.91). Combining these results we can obtain an expression for $v^i_{(1)}$ in terms of $v^i_{(0)}$:

$$v^i_{(1)} = N^i_{(1)j}\, v^j_{(0)}\,, \tag{2.94}$$

with

$$N^i_{(1)j} = -\Box^{-1}\left(\partial^k(\Gamma^{(1)})^i_{kj} + R^{(1)i}{}_j - \partial^i\left(\Box^{-1}\partial_j R^{(1)}\right)\right). \tag{2.95}$$

The function $\Box^{-1}f$ is defined up to a constant on a torus. The associated constant for $v^i_{(1)}$ will be fixed at third order in the perturbative expansion and it does not affect the leading order DC result.

We now integrate equation (2.42) to find

$$\begin{aligned}\int g^{1/2}_{(0)}\,\nabla^{(k}v^{l)}\partial_j g_{(0)kl} - \int g^{1/2}_{(0)}\,Z^{(0)}a^{(0)}_t\,\partial_j w + \int g^{1/2}_{(0)}\,\partial_j p + \int g^{1/2}_{(0)}\,\partial_i\phi^{(0)}\partial_j\phi^{(0)}v^j \\ = \left(\int g^{1/2}_{(0)}\right)4\pi T\,\zeta_j + \left(\int g^{1/2}_{(0)}Z^{(0)}a^{(0)}_t\right)E_j\,,\end{aligned} \tag{2.96}$$

where we have taken E_i, ζ_i to be constants and $\int$ is defined to be the average over a period:

$$\int \quad\leftrightarrow\quad \frac{1}{L_1\ldots L_d}\int dx^1\ldots dx^d\,. \tag{2.97}$$

We next expand equation (2.96) with respect to λ and keep the λ^0 pieces. Using (2.93), (2.94), after some work we find that the left hand side can be expressed in terms of $v^i_{(0)}$. Indeed we deduce that

$$\lambda^{-2}L_{ji}v^i_0 = 4\pi T\,\zeta_j + a\,E_j\,, \tag{2.98}$$

where L is a matrix that only depends on the background data given by

$$\begin{aligned}L_{ji} =& \lambda^2 g^{-1}\int\left(\frac{g^{-1}}{2}\partial_j h^{(1)}_{kl}\partial_i h^{(1)kl} + \partial_j h^{(1)}_{kl}\,\partial^k N^l{}_i + \frac{1}{2}h^{(1)}\,\partial_j(\Box^{-1}\partial_i R^{(1)})\right) \\ &+ \lambda^2\,g z^{-1}_{(0)}\int a_{(1)}\,\partial_j\left(\Box^{-1}\partial_i a_{(1)}\right) + \lambda^2\int\partial_i\psi_{(0)}\partial_j\psi_{(0)}\,,\end{aligned} \tag{2.99}$$

with N as given by (2.95). Notice, in particular, that the integration constants associated with $\Box^{-1}$ drop out since the relevant terms are all covered by an extra spatial derivative. Thus at leading order we have

$$v^i \approx (L^{-1})^{ij}\left(4\pi T\,\zeta_j + a\,E_j\right)\,, \tag{2.100}$$

and

$$J^i|_H \approx \rho_H v^i\,, \qquad Q^i|_H \approx T s v^i\,. \tag{2.101}$$

Recalling the definition of the radially independent current flux densities given in (2.54),(2.55), we finally obtain the holographic current flux densities in terms of E, ζ:

$$\bar{J}^i \approx \rho v^i\,, \qquad \bar{Q}^i \approx T s v^i\,, \tag{2.102}$$

where we used (2.9). Thus we can determine the leading order behaviour of the conductivities:

$$\bar{\kappa} = L^{-1} 4\pi s T\,, \qquad \alpha = \bar{\alpha} = L^{-1} 4\pi \rho\,, \qquad \sigma = L^{-1} \frac{4\pi\rho^2}{s}\,. \tag{2.103}$$

We observe that $\bar{\kappa}(\sigma T)^{-1} = s^2/\rho^2$, which corresponds to a kind of Wiedemann-Franz law. In addition we note that the thermal conductivity at zero current flow, $\kappa \equiv \bar{\kappa} - T\bar{\alpha}\sigma^{-1}\alpha$, appears at a higher order in the expansion: $\kappa \sim \lambda^0$. It is interesting to compare these results to the discussion in [123]. We similarly find that $\sigma_{Q=0}$ also appears at a higher order in the expansion.

Perturbative one-dimensional lattices

We conclude this subsection by discussing the special case of one-dimensional perturbative lattices and hence make contact with the results of section 4.2. We first notice that for an arbitrary periodic function of a single coordinate x we have

$$\Box^{-1}\partial_x F = c + \int_0^x dx\, F(x) - x \int F\,, \tag{2.104}$$

where c is an arbitrary constant and in the last term $\int$ refers to the average integral over a period in the x direction, as given in (2.75). Hence

$$\int F\,\partial_x \left(\Box^{-1}\partial_x F\right) = \int F^2 - \left(\int F\right)^2\,. \tag{2.105}$$

Recalling (2.70) we next assume that

$$h^{(1)}_{ij}\, dx^i dx^j = \delta\gamma\, dx^2 + \delta g_{ab}\, dx^a dx^b\,, \tag{2.106}$$

where $\delta\gamma$ and δg_{ab} only depend on x. The only non-zero matrix element of the matrix L is given by

$$\begin{aligned}\lambda^{-2} L_{xx} =& g^{-2}\int\left(\frac{1}{2}\,\partial_x\delta g_{ab}\,\partial_x\delta g^{ab}+\frac{1}{2}\,\partial_x\delta g^a{}_a\,\partial_x\delta g^b{}_b\right)+\int(\partial_x\psi)^2\\ &+g z_{(0)}^{-1}\left(\int a_{(1)}^2-\left(\int a_{(1)}\right)^2\right),\end{aligned}\tag{2.107}$$

where indices are raised with δ^{ab}. In obtaining the above result we used that

$$R_{xx}=-\frac{1}{2g}\,\partial_x^2\,\delta g^a{}_a,\quad R_{ab}=-\frac{1}{2g}\,\partial_x^2\,\delta g_{ab}\,.\tag{2.108}$$

In the notation of section 4.2 we have

$$\gamma=g+\lambda\,\delta\gamma,\quad g_{ab}=g\,\delta_{ab}+\lambda\,\delta g_{ab}\,,\tag{2.109}$$

and the rest of the functions are expanded exactly as in (2.86). After this identification we find that X as defined in (2.77) takes the form

$$X=\frac{1}{g^d\,z_{(0)}}\,g\,L_{xx}\,.\tag{2.110}$$

It is then straightforward to see that the leading order expansion in the lattice strength of the DC conductivities give in (2.78) agree with the perturbative results given in (2.103) when restricted to lattices that depend on one spatial coordinate only.

5 Discussion

The main results that we have obtained in previous sections apply in a more general setting as we now discuss. Specifically, we consider the following ansatz for a general class of static solutions

$$ds^2=g_{tt}dt^2+ds^2(M_{D-1}),\qquad A=a_t dt,\tag{2.111}$$

where g_{tt}, a_t, ϕ and the metric $ds^2(M_{D-1})$ are all independent of time and are just functions of the coordinates x^a on $M_{(D-1)}$.

The spacetime may have various types of asymptotic boundaries, but our primary interest is when there is an AdS boundary[8]. In this case we can introduce a local radial coordinate and then impose the same boundary conditions as in (2.4),(2.5), corresponding to the CFT living on Σ_d deformed with various spatially dependent sources. The spacetime

[8] We can also consider other holographic boundary conditions, for example, asymptotically Lifshitz, or even asymptotically flat boundary conditions.

may have one or possibly more black hole horizons (examples have been discussed in [122]). Near each black hole horizon we can again introduce a local radial coordinate and then demand that the metric has the behaviour that we gave in (2.6). Note that we do not assume that the topology of the black hole horizons are all the same, nor do we assume that that have the same topology as Σ_d.

We now consider the following linear perturbation

$$\begin{aligned}\delta\left(ds^2\right) &= \delta g_{\mu\nu}dx^\mu dx^\nu + 2tg_{tt}\zeta_a dtdx^a\,,\\ \delta A &= \delta a_\mu dx^\mu - tE_a dx^a + ta_t\zeta_a dx^a\,,\\ &\delta\phi\,,\end{aligned} \tag{2.112}$$

where the one-forms E, ζ are now defined on M_{D-1} (not just on Σ_d as before) and are still taken to be closed. In addition $\delta g_{\mu\nu}$, δa_μ and $\delta\phi$ are all independent of t. It is an interesting fact that at linearised order in the perturbation (and independent of any boundary conditions) we still have the key results

$$di_k * Z(\phi)F = 0, \qquad di_k * G = 0\,, \tag{2.113}$$

where $k = \partial_t$. Note that k still satisfies the conditions (2.22), and we have defined G as before in (2.24).

We now consider the boundary conditions on the perturbation. At the holographic boundary E, ζ approach closed one-forms E^0, ζ^0 on Σ_d and we impose that these are the only sources deforming the CFT. Similarly, for each black hole horizon the perturbation behaves in local coordinates as in (2.13), with E, ζ again approaching closed one-forms on each horizon. Using the local coordinates at each horizon we can now impose the Hamiltonian, momentum and Gauss-law constraints exactly as described in section 3.4 and obtain a set of generalised Stokes equations on each horizon. By solving these equations we can thus obtain currents on each horizon. Note that the precise source terms that appear in the Stokes equations on each horizon follows after imposing that E, ζ are closed one-forms in the bulk and that they approach E^0, ζ^0 at the AdS boundary.

At the AdS boundary we can define the current fluxes through each $d-1$ cycle $\mathcal{C}_a$ on Σ_d as in (2.58). Now consider any orientable d-dimensional manifold in the bulk with boundary $\mathcal{C}_a$ at the AdS boundary and a $d-1$ cycle at the black hole horizon (which might be disconnected). Then using (2.113) and Stokes's theorem, we deduce that the current fluxes are equal to the fluxes on the black hole horizon. In turn these fluxes can be obtained by solving the Stokes fluid equations on the black hole horizon, which only depend on the cohomology class of the sources at the horizon, which in turn only depend on the cohomology class of E^0, ζ^0 on Σ_d at the AdS boundary. Thus by expanding the fluxes E^0, ζ^0 in a basis of harmonic one-forms that are Poincaré dual to the $\mathcal{C}_a$ we have a procedure for obtaining the DC conductivities. Note that in all these cases, one still needs to know the behaviour of the background black hole on the horizon, which may be a non-trivial task. However, with this method, the problem of subsequently calculating the DC response has been significantly simplified.

There are a number of interesting directions to pursue. In our analysis we assumed that the black holes have vanishing magnetic field. Following the original work that this chapter was based on, this condition was relaxed in [98], generalising the results obtained in [120, 124] for Q-lattices.

More generally, we have shown the DC conductivity of the boundary theory can be obtained by solving the generalised Stokes equations at the black hole horizon. In a narrow sense the fluid equations are simply an auxiliary set of equations to solve this holographic problem. However, the innate physical character of the equations (with their novel viscous terms) suggest that their may be a deeper significance. As a first step, it would be interesting to determine whether the full time-dependent and non-linear generalised Navier-Stokes equations at the black hole horizon can also be used to obtain exact holographic information for the dual CFT. It is natural to expect that the time-dependent equations will be useful in extracting the small frequency behaviour of the AC conductivity. A related point is to use our results to develop a systematic hydrodynamic framework in the presence of holographic lattices. Finally it would also be very interesting to obtain some explicit lattice black hole solutions that depend on more than one of the spatial dimensions and analyse the fluid equations.

Chapter 3

Generalised DC linear response of scalar fields

1 Introduction

In the previous chapter, we introduced a general framework to calculate DC thermoelectric conductivities for a class of black holes that have broken translational invariance - holographic lattices. We showed that, universally, the DC thermoelectric conductivity matrix can be obtained for holographic lattices by solving a set of linearised Navier-Stokes equations on the black hole horizon. However, whilst there is now a framework in place to determine DC thermoelectric conductivities directly, in general we are left with the difficult problem of solving the Stokes equations. It is a remarkable fact that in many interesting and non-trivial holographic lattice systems, the associated Stokes equations can be analytically solved, and so the associated DC thermoelectric conductivity can obtained directly.

Historically, the thermoelectric DC conductivity for Q-lattices and one-dimensional lattices were found before the general framework was discovered [55, 66]. In both of these cases, the Navier-Stokes equations can be solved directly, leading to exact solutions. In the case of the Q-lattices and one-dimensional homogeneous lattices, the current itself, rather than the current flux, is radially conserved, so the DC conductivity can simply be read off in terms of horizon quantities.

In addition to a metric and gauge field, the general holographic lattices we considered in the last chapter contain scalar fields. A natural extension is to therefore ask what is the response to linear in time perturbations of these scalar fields. In this case, our DC response matrix is of the form:

$$\begin{pmatrix} J \\ Q \\ \langle O_\psi \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T & \beta \\ \bar{\alpha} T & \bar{\kappa} T & \gamma \\ \bar{\beta} & \bar{\gamma} & \bar{\rho} \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \\ \psi_c \end{pmatrix} , \tag{3.1}$$

where ψ is a scalar field dual to some operator in the CFT with vacuum expectation $\langle O_\psi \rangle$, and ψ_c is the source of this operator.

It is the aim of this chapter to build on this idea, and determine the response of the class of CFTs studied in [66] to linear in time perturbations of scalar fields. The models considered in [66] contain both Q-lattices and linear axion models, and it is the linear axionic field, χ_1, that we will perturb. In the background black hole $\chi_1 = k_1 x_1$, where k_1 is a constant and x_1 is a spatial direction, and is dual to a marginal operator with zero expectation value.

The rest of the chapter proceeds as follows. In the next section, we describe the background black hole solutions that we will study. In section 3, we determine the response of χ_1 to a heat and electric current. We will then turn on perturbations of χ_1, and show that that the DC response can be written in terms of horizon data. In the case of holographic Q-lattices, we will discuss the physical interpretation of these perturbations. Next, we turn to a more general case, and show that perturbations of the scalar field lead to a more general Stokes equations, provided the scalar field does not couple to the gauge field and has no potential. Finally we will discuss our findings in more detail in section 5.

2 Holographic models

We will focus on holographic models in $D = 4$ spacetime dimensions which are dual to $d = 3$ CFTs with a global $U(1)$ symmetry. The $D = 4$ fields include a metric and a gauge field, which are dual to the stress tensor and the $U(1)$ current of the CFT, respectively. We will also include a real scalar field, ϕ, and two real "axion" fields, χ_i, which are dual to additional scalar operators in the CFT. The action is given by

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \left[(\partial\phi)^2 + \Phi_1(\phi)(\partial\chi_1)^2 + \Phi_2(\phi)(\partial\chi_2)^2 \right] - V(\phi) - \frac{Z(\phi)}{4} F^2 \right], \tag{3.2}$$

which involves four functions, Φ_i, V and Z, of the real scalar field ϕ where demand $\Phi_i, Z \geq 0$, and we have set $16\pi G = 1$. We assume the model admits a unit radius AdS_4 vacuum with $\phi = 0$ (in particular $V(0) = -6$) and for convenience we set $Z(0) = 1$. The action is invariant under the global symmetries corresponding to shifts of the axion fields.

This model captures two types of black hole solution we are interested in. It firstly includes holographic Q-lattices, where the fields χ_i are periodic. These models arise when $\Phi_i \sim \phi^2$ near $\phi = 0$. For example, for a single axion (i.e. setting $\chi_2 = 0$), we could consider $\Phi_1 = \phi^2$ and then ϕ, χ_1 are the norm and phase of a complex scalar field. Furthermore we would choose the mass of this complex field, by choosing V, so that the complex field is dual to a relevant operator with dimension $\Delta < 3$. A deformation of the CFT by this complex operator with χ_1 linear in a spatial direction would necessarily comprise a periodic deformation and hence what we call a holographic lattice. Indeed decomposing the complex field into two real fields, reveals that the construction has two real periodic lattices in the same spatial direction with a phase shift of $\pi/2$. This was precisely the construction of the anisotropic Q-lattices in [57]. Similarly the models with two χ_i can arise from two complex scalar fields with a Z_2 symmetry that equates their norms; these constructions lead to isotropic Q-lattices as considered in [58].

Our model also includes other types of black hole solutions where the χ_i are, instead, massless fields, and are dual to marginal operators with $\Delta = 3$. These models arise when $\Phi_i(0) \neq 0$. For example, the case when $\Phi_i = 1$ has been considered in [62]. Another case is for a single axion (i.e. setting $\chi_2 = 0$) and $\Phi_1 = e^{2\phi}$, corresponding to the axion and dilaton of string theory after performing a dimensional reduction of type IIB supergravity on a five-dimensional Einstein space, and anisotropic black holes have been studied in [60,61,63]. In these cases, the linear axions do not give a periodic deformation of the CFT. Nevertheless, like the Q-lattices, they do incorporate momentum dissipation and have finite DC conductivities.

2.1 The black hole backgrounds

Rather than consider the general ansatz (2.2), the solutions that we shall consider here all lie within the ansatz

$$\begin{aligned} &ds^2 = -U\,dt^2 + U^{-1}\,dr^2 + e^{2V_1}dx_1^2 + e^{2V_2}dx_2^2, \\ &A = a\,dt, \qquad \chi_1 = k_1\,x_1, \qquad \chi_2 = k_2\,x_2\,, \end{aligned} \tag{3.3}$$

where U, V_i, a and ϕ are only functions of r. In general the black hole solutions are anisotropic, with $V_1 \neq V_2$, but isotropic solutions with $V_1 = V_2$ are possible when we can choose $k_1^2\Phi_1(\phi) = k_2^2\Phi_2(\phi)$.

We will assume that there is a single, regular event horizon at $r = r_+$ and the functions have the following expansions

$$\begin{aligned} &U \sim 4\pi T(r - r_+) + \dots, \qquad &V_i \sim V_{i+} + \dots, \\ &a \sim a_+(r - r_+) + \dots, \qquad &\phi \sim \phi_+ + \dots\,, \end{aligned} \tag{3.4}$$

where T is the temperature of the black hole. In what follows, we will use ingoing Eddington-Finklestein coordinates (v, r) where $v = t + \frac{1}{4\pi T}\ln(r - r_+)$.

As $r \to \infty$, the location of the AdS_4 boundary, we assume that there is an asymptotic expansion of the form

$$\begin{aligned} U &\sim r^2 + \dots, & e^{2V_i} &\sim r^2 + \dots\,, \\ a &\sim \mu - q r^{-1} + \dots, & \phi &\sim \lambda r^{\Delta-3} + \dots\,. \end{aligned} \tag{3.5}$$

For the case of the Q-lattice, as discussed above, we would demand that $\Delta < 3$ and λ denotes the strength of the Q-lattice deformation (assuming a standard quantisation for the scalar). For the Q-lattice black holes the axions are periodic, $\chi_i = \chi_i + 2\pi$, and these UV boundary conditions explicitly break the translation symmetry in a periodic manner. The UV data specifying these black holes is given by T/μ, k_1/μ, k_2/μ and $\lambda/\mu^{3-\Delta}$. For the case of massless linear axions, as discussed above, ϕ can also be massless or absent and the axions are not periodic.

Our starting point is once again to obtain a general expression for the electric charge of the black holes in terms of horizon data. The current density $J^a = (J^t, J^x, J^y)$ in the dual field theory has the form

$$J^a = \frac{1}{16\pi G_N}\sqrt{-g}Z(\phi)F^{ar}\,, \tag{3.6}$$

where the right hand side is evaluated at the AdS boundary $r \to \infty$. We find that the only non-zero component of the equation of motion for the gauge-field is in the t-direction and can be written $\sqrt{-g}\nabla_\mu(Z(\phi)F^{\mu t}) = \partial_r(\sqrt{-g}Z(\phi)F^{rt}) = 0$. Thus we can write

$$q \equiv J^t = \frac{1}{16\pi G_N} e^{V_1+V_2} Z(\phi) a'\,, \tag{3.7}$$

where q is the charge of the black hole and the right hand side can be evaluated on any radial slice, including at the event horizon, $r = r_+$. In general, this charge q depends on the UV data of the Q-lattices including the temperature of the dual field theory.

3 Scalar field DC response

We want to find the response of the axion fields to the heat and electric currents in our model. As we discussed in the previous section, there are two different types of black hole solution which the model supports, solutions where the axions are massless fields dual to marginal operators with conformal dimension $\Delta = 3$, and Q lattices, where the axion and scalar fields can be considered as a combination of complex fields. We will consider both of these cases.

To start, let's consider perturbations which have a source for the scalar, χ_1. Since in general the source of a field will give a response, we can represent this linear response through a 3x3 matrix that represents a generalised DC response matrix

$$\begin{pmatrix} J \\ Q \\ \langle O_{\chi_1} \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T & \beta \\ \bar{\alpha} T & \bar{\kappa} T & \gamma \\ \bar{\beta} & \bar{\gamma} & \bar{\rho} \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \\ \chi_c \end{pmatrix} . \tag{3.8}$$

To calculate this matrix, consider the following linearised perturbation about the black hole background

$$\begin{aligned} A_{x_1} &= (-E + \zeta a)t + \delta a_{x_1}(r) \,, \\ g_{tx_1} &= -\zeta U t + \delta g_{tx_1}(r) \,, \\ g_{rx_1} &= \delta g_{rx_1}(r) \,, \\ \chi_1 &= k_1 x_1 + \chi_c t + \delta \chi_1(r) \,. \end{aligned} \tag{3.9}$$

In addition to sources for the heat and electric current, E and ζ, here we have also introduced a linear in time source for the scalar field χ_1, in an analogous way to the previous chapter where we turned on linear in time sources for the gauge field and metric. The top left 2x2 matrix can be determined using the results of the previous chapter, so here we will focus on the final row and column, and determine their entries for the model that we are considering.

3.1 Determining β and γ

First, consider the equation of motion for χ_1. At linearised order, we find

$$\partial_r \left(\Phi_1(\phi) e^{V_1+V_2} U (\delta \chi_1' - k_1 e^{-2V_1} \delta g_{rx_1}) \right) = 0 \,. \tag{3.10}$$

We see that the quantity O_χ, defined by

$$O_\chi = \frac{1}{16\pi G_N} \Phi_1(\phi) e^{V_1+V_2} U (\delta \chi_1' - k_1 e^{-2V_1} \delta g_{rx_1}) \,, \tag{3.11}$$

is radially conserved. We will see that this quantity corresponds to the current for χ_1, $\langle \mathcal{O}_{\chi_1} \rangle$, when evaluated on the boundary.

Similarly, the gauge equation of motion implies that $\partial_r J = 0$ with

$$J = -\frac{1}{16\pi G_N} e^{V_2 - V_1} Z(\phi) U \delta a_{x_1}' - q e^{-2V_1} \delta g_{tx_1} \,, \tag{3.12}$$

where the right-hand side can be evaluated at any value of r, including at the black hole horizon at $r = r_+$.

Finally, consider the linearised Einstein equations. If we define

$$Q \equiv \frac{1}{16\pi G_N} e^{-V_1+V_2} U^2 \left(U^{-1}\delta g_{tx}\right)' - aJ\,, \tag{3.13}$$

we find, remarkably, that $\partial_r Q = 0$. We identify Q is the heat current, using similar analysis to that found in appendix A.3.

We also find we can algebraically solve for δg_{rx_1} giving

$$\delta g_{rx_1} = -\frac{(E-\zeta a)q e^{V_1-V_2}}{k_1^2 \Phi_1(\phi) U} - \frac{e^{4V_1}(e^{-2V_1}\zeta U)'}{k_1^2 U \Phi_1} + \frac{e^{2V_1}\delta\chi_1'}{k_1}\,. \tag{3.14}$$

Suppose that the only non zero source is χ_c. Then regularity at the black hole event horizon implies that

$$\delta\chi_1(r) \sim \frac{\chi_c}{4\pi T}\ln(r-r_+) + \dots\,, \tag{3.15}$$

which implies that δg_{rx_1} will be diverging at the horizon. This can be remedied by demanding

$$\delta g_{tx_1} \sim \left.\frac{e^{2V_1}\chi_c}{k_1}\right|_{r=r_+} + \dots\,. \tag{3.16}$$

We now want to calculate our conserved currents. Evaluating J at the black hole horizon using (3.12), we now deduce that

$$J = -\frac{q\chi_c}{k_1}\,. \tag{3.17}$$

Similarly, evaluating Q at the black hole horizon we deduce that

$$Q = -\frac{sT}{k_1}\chi_c\,. \tag{3.18}$$

We are now able to fully calculate the first two rows of our DC response matrix. We see that

$$\begin{aligned}\beta &= \frac{\partial}{\partial\chi_c}J = -\frac{q}{k_1}\,,\\ \gamma &= \frac{\partial}{\partial\chi_c}Q = -\frac{sT}{k_1}\,.\end{aligned}$$

3.2 Determining $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\rho}$

Now we have determined the response of the heat and electric current to the scalar source, we wish to determine the response of the scalar current to all three sources. To do this, we first demonstrate that the conserved quantity, (3.11), is equivalent to the scalar current of the dual field theory.

For any holographic theory, in order to render the action finite, suitable boundary terms must be added to the action, in a process known as holographic renormalization. For our background theories, for $\Delta < 3$, we must add the following counterterms to the action

$$\frac{1}{16\pi G_N}\int d^3x\sqrt{-\gamma}\left(2K - 4 - \frac{3-\Delta}{2}\phi^2 + ...\right) \,, \tag{3.19}$$

where the ... indicates additional counterterms which are not needed in this analysis. Similarly, the counterterms when $\Delta = 3$ are

$$\frac{1}{16\pi G_N}\int d^3x\sqrt{-\gamma}\left(2K - 4 + \frac{1}{2}\Phi_1(\phi)\partial_i\chi_1\partial^i\chi_1 + \frac{1}{2}\Phi_2(\phi)\partial_i\chi_2\partial^i\chi_2 + ...\right) \,, \tag{3.20}$$

which comes from the fact that we no longer have the restriction that Φ_1 and its derivatives vanish asymptotically.

Let us consider the expectation of χ_1. In general, we have on-shell that

$$\delta S = \int d^3x\sqrt{-\gamma}\,\bar{\mathcal{O}}_\chi\delta\chi\,. \tag{3.21}$$

It will be convenient to define $r^3\bar{\mathcal{O}}_\chi = \langle\mathcal{O}_\chi\rangle$, where $\langle\mathcal{O}_\chi\rangle$ is the VEV of the scalar field, since $\bar{\mathcal{O}}_\chi \sim r^{-3}$ as $r \to \infty$. Varying the on-shell action and the counterterms, we find that, for $\Delta < 3$, we have

$$\langle\mathcal{O}_{\chi_1}\rangle = -\frac{1}{16\pi G_N}r^3\bar{\Phi}_1 U^{1/2}\left(\delta\chi' - k_1 e^{-2V_1}\delta g_{rx_1}\right) \,, \tag{3.22}$$

where $\bar{\Phi}_1$ is the leading order term in the expansion of Φ_1 on the boundary, which is equal to $1/2\Phi_1''(0)\lambda^2 r^{2\Delta-6}$.

Expanding (3.11) near the AdS boundary, we find that

$$O_\chi = \frac{1}{16\pi G_N}r^4\bar{\Phi}_1\left(\delta\bar{\chi}' - k_1 e^{-2V_1}\delta g_{rx_1}\right) \,, \tag{3.23}$$

and we see that $O_\chi = -\langle\mathcal{O}_{\chi_1}\rangle$. Thus, we are able to identify the conserved current with the scalar current. A similar calculation is needed in the case that $\Delta = 3$. In this case, an additional piece from the counterterm is used, which cancels a diverging term in the expansion of $\delta\chi$.

Now we evaluate the conserved current on the horizon. Substituting (3.14) into (3.11), we have

$$O_\chi = E\frac{q}{k_1} + \frac{\zeta}{k_1}\left(e^{3V_1+V_2}(e^{-2V_1}U)' - aq\right) \,. \tag{3.24}$$

Evaluating this quantity on the horizon, we have

$$O_\chi = E\frac{q}{k_1} + \zeta\frac{sT}{k_1}\,. \tag{3.25}$$

We therefore conclude that

$$\begin{aligned}
\bar{\beta} &= \frac{\partial}{\partial E}\langle\mathcal{O}_\chi\rangle = -\frac{q}{k_1} \\
\bar{\gamma} &= \frac{\partial}{\partial\zeta}\langle\mathcal{O}_\chi\rangle = -\frac{sT}{k_1} \\
\bar{\rho} &= \frac{\partial}{\partial\chi_c}\langle\mathcal{O}_\chi\rangle = 0.
\end{aligned} \tag{3.26}$$

As expected, this implies a symmetric response matrix. We have now fully calculated the DC response matrix (3.8). Note that this result can also be determined in this case directly from the Ward identity, as highlighted in appendix B.1.

In summary, we see that perturbations of the scalar field will induce thermoelectric currents in our theory, but will not alter the scalar field current. To understand this, consider the equations of motion for our theory. In many ways, the new perturbation that we have introduced is similar to a linear axion of the form kx, with the difference being that the axion is in the time direction, rather than a spatial direction. In particular, it is easy to see that a scalar field of the form $\chi_c t$ will in fact solve the equations of motion of our theory in the absence of boundary conditions. However, when we apply regularity of the event horizon, we see that we now need perturbations of the tx component of the metric and the scalar field in order to ensure the scalar field is well behaved on the horizon. It is these perturbations that sources the thermoelectric currents. However, these perturbations do not source the scalar current due to a cancellation between the metric and scalar perturbations at linear order.

3.3 Physical interpretation for holographic Q-lattices

While it is simple to physically interpret the source term for χ as a linear in time source for a massless scalar field in the case when $\Delta = 3$, the situation where $\Delta < 3$ is less clear. In this section we will try to shed some light on the physical interpretation of the above results for Q-lattices

For simplicity, in this section we will consider the case of the simplest Q-lattice, where $\chi_2 = 0$ and $\Phi_1 = \phi^2$. In this case, we can treat the ϕ and $\chi = \chi_1$ as the norm and phase of a single complex scalar field, Ψ. However, we can also write the single complex field as two real scalar fields, ϕ_1 and ϕ_2, corresponding to the real and imaginary parts of Ψ

respectively.

$$\begin{aligned}\phi_1 &= \phi(r)\cos(\chi)\,, \\ \phi_2 &= \phi(r)\sin(\chi).\end{aligned} \tag{3.27}$$

Consider the perturbations from the previous sections. We can write this as

$$\begin{aligned}\phi_1 &= \phi(r)\cos(k_1 x_1) - \phi(r)\delta\chi(r,t)\sin(k_1 x_1)\,, \\ \phi_2 &= \phi(r)\sin(k_1 x_1) + \phi(r)\delta\chi(r,t)\cos(k_1 x_1)\ .\end{aligned} \tag{3.28}$$

The perturbation has therefore introduced an extra mode for each of the scalar fields, in an orthogonal direction. If we now look at the asymptotic expansion of the two fields, we find that the scalar field expectation values are given by:

$$\begin{aligned}\langle \mathcal{O}_{\phi_1} \rangle &= (2\Delta - 3)\left(\phi_c \cos(k_1 x_1) + \left(\frac{Eq + \zeta s T}{(2\Delta - 3)\lambda k_1} - \phi_c \chi_c t\right)\sin(k_1 x_1)\right)\ , \\ \langle \mathcal{O}_{\phi_2} \rangle &= (2\Delta - 3)\left(\phi_c \sin(k_1 x_1) - \left(\frac{Eq + \zeta s T}{(2\Delta - 3)\lambda k_1} - \phi_c \chi_c t\right)\cos(k_1 x_1)\right)\ ,\end{aligned} \tag{3.29}$$

where ϕ_c is the expectation value of the background scalar field $\phi(r)$, and Δ is the conformal dimension of the field.

We see that each of the real scalar fields has acquired an additional constant term in its vacuum expectation, at a phase difference of $\pi/2$ to the initial VEV. There is also a contribution to the expectation that is time dependent. Recall that, in general, a time dependent source will give rise to a time dependent and time independent part of the VEV - these lead to the susceptibility and conductivity respectively. We therefore find the susceptibility due to switching on a time dependent source for the phase is given by

$$\begin{aligned}\tilde{G}_{\chi\phi_1} &= -(2\Delta - 3)\phi_c \chi_c \sin(k_1 x_1)\,, \\ \tilde{G}_{\chi\phi_2} &= (2\Delta - 3)\phi_c \chi_c \cos(k_1 x_1)\ .\end{aligned} \tag{3.30}$$

Note that if we convert back to our previous case of one real scalar, and two axion fields, we find this time dependent part of the scalar expectation vanishes at the linear level, and so is consistent with our previous results. We therefore see that for Q-lattices, the perturbation of the axion field leads to a time dependent response from two real scalar fields, at a phase difference of $\pi/2$ to each of the scalar fields.

4 Generalised DC response

Motivated by the results from the previous section, we now turn a more general setup. We will consider the theory from the previous chapter, with a scalar field with no potential, and no coupling between the gauge field and the scalar field. The action is given by

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left(R - \frac{Z}{4} F^2 - \frac{1}{2} (\partial \phi)^2 \right) , \tag{3.31}$$

where the constant Z is independent of ϕ.

We will consider the same static black hole ansatz as given in (2.4), and will use the same notation throughout. It will be helpful to write the scalar field as

$$\phi(r,x) = k_i x^i + \bar{\phi}(r,x) , \tag{3.32}$$

where k_i are constants, whilst $\bar{\phi}(r,x)$ is periodic in x^i.

We now proceed in a very similar manner to the method outlined in section 3 of chapter 2. The perturbations in the metric and the gauge field are the same as before. However, we consider a linear in time perturbation of the scalar field, giving a scalar field perturbation

$$\delta\phi = \chi_c t + \delta\phi(r,x) . \tag{3.33}$$

By changing to ingoing Eddington-Finkelstein coordinates, we can see that regularity of the solution implies the behaviour of the scalar perturbation on the horizon is given by

$$\delta\phi(r,x) = \chi_c \frac{\ln r}{4\pi T} + \delta\phi^0(x) + \mathcal{O}(r) , \tag{3.34}$$

where the near horizon behaviour of $\bar{\phi}(r,x)$ is $\bar{\phi}^0(x) + \mathcal{O}(r)$.

Now recall that in the radial Hamiltonian decomposition (see Appendix A.1), the momentum conjugate to ϕ is given by

$$\pi_\phi = -\sqrt{-h} n^\mu \partial \mu \phi , \tag{3.35}$$

where n is the normal vector and h is the metric induced on the radial slice. On the horizon, this momentum evaluates to

$$- \sqrt{g}(v^i(k_i + \partial_i \bar{\phi}^0) + \chi_c). \tag{3.36}$$

which gives an additional contribution to the momentum constraint from the previous chapter. Thus, we have the modified Navier-Stokes equations

$$\nabla_i v^i = 0\,, \tag{3.37}$$

$$\nabla_i(Z^{(0)}\nabla^i w) + v^i\,\nabla_i\left(Z^{(0)}a_t^{(0)}\right) = -\nabla_i(Z^{(0)}E^i)\,, \tag{3.38}$$

$$\begin{aligned}-2\,\nabla^i\nabla_{(i}v_{j)} - Z^{(0)}a_t^{(0)}\nabla_j w + \nabla_j\,p + Z^{(0)}a_t^{(0)}E_j &= 4\pi T\,\zeta_j \\ &+ (k_j + \nabla_j\phi^{(0)})((\nabla_i\phi^{(0)} + k_i)v^i + \chi_c)\,.\end{aligned} \tag{3.39}$$

If the one point function of the scalar field, $\langle\mathcal{O}_\phi\rangle$, is taken as $\pi_\phi/(16\pi G_N)$, then the equation of motion for the massless scalar field, $\nabla^2\phi = 0$, implies that $\langle\mathcal{O}_\phi\rangle$ is radially conserved. We therefore find that we have the conserved current

$$\delta\langle\mathcal{O}_\phi\rangle = \int_H \delta\pi_\phi = -\frac{k_i\bar{Q}^i}{4\pi T} - \frac{s}{4\pi}\chi_c\,, \tag{3.40}$$

where $\bar{Q}^i$ is the heat current flux density, with the heat current density given by (2.43).

Setting $\chi_c = 0$, we can immediately calculate the final row of the generalised flux response matrix, given by (3.1). We find that

$$\begin{aligned}\bar{\beta}^i &= -\frac{1}{4\pi}k_j\bar{\alpha}^{ji}\,, \\ \bar{\gamma} &= -\frac{1}{4\pi}k_j\bar{\kappa}^{ji}\,,\end{aligned} \tag{3.41}$$

where we are interested in conserved fluxes here, rather than the conserved currents themselves. In order to determine the final row of the DC matrix, we set $E = \zeta = 0$. If we look at the Stokes equations, we see that we can absorb the term $\nabla_j\phi^{(0)}\chi_c$ into the pressure, p, leaving the term $k_j\chi_c$ as the only term that depends on χ_c. But this is simply equivalent to solving the Stokes equations with $E = \chi_c = 0$ and $4\pi T\zeta_i = -k_i\chi_c$. Hence, we deduce that

$$\begin{aligned}\beta^i &= -\frac{1}{4\pi}\alpha^{ij}k_j\,, \\ \gamma &= -\frac{1}{4\pi}\bar{\kappa}^{ij}k_j\,, \\ \bar{\rho} &= -\frac{k_i\bar{\kappa}^{ij}k_j}{(4\pi)^2 T} - \frac{s}{4\pi}\,.\end{aligned} \tag{3.42}$$

In the case of the Q-lattice, we have $\bar{\kappa} = 4\pi Ts/k^2$, and it is then easy to verify that these results are consistent with the previous section. We now see that the fact that the scalar perturbations do not induce a response in the scalar field is a result of the exact cancellation between two terms.

5 Discussion

We have extended the previous work of [66], by finding the full 3x3 DC conductivity matrix for a class of CFTs dual to a black hole containing a gauge field, two axion fields and a scalar field, in the case when we turn on time dependent sources for electric current and heat current in the x_1 direction, and an axionic field that is linearly dependent on the x_1 coordinate, χ_1. The DC conductivity was first determined directly, rather than as a solution to the Stokes equations. In addition, we also determined the general response for scalar fields in the case that the scalar field is massless and does not couple to gauge field.

Interestingly, we see that for the Q lattice, the axionic perturbations source a thermoelectric current, but do not induce a current in the scalar field itself, leading to a term that is zero in the DC response matrix. In the case of Q-lattices, since the axion can be thought of as a phase of two scalar fields, the perturbation is essentially a perturbation in the phase of the field. This perturbation induces a susceptibility term in the fields which is time dependent, but results in no time independent conductivity piece. In particular, for the scalar fields the susceptibility is at a phase difference of $\pi/2$ to the original scalar fields.

After initially determining the DC conductivity for a Q-lattice, we then showed that for a general system, then provided the scalar field has no potential and does not couple to the gauge field, there is an additional contribution to the Navier-Stokes equations. Interestingly, these extra terms can be absorbed into the definition of the pressure when calculating the DC response, and it would be interesting to understand exactly the physical relevance of this absorption.

Whilst our results for the scalar field are somewhat general, they do require the scalar field to have no potential term and no coupling to the gauge field. To understand why this is necessary, it is helpful to consider the equations of motion for the scalar field in the previous chapter, given by (2.3). If there is a $V(\phi)$ or $Z(\phi)$ term, then at linear order the equations of motion are no longer time independent, and so the analysis breaks down. It would be interesting to understand if there are any cases when this is not the case, in which case a more general Navier-Stokes equations could be constructed.

In the calculations in the first part of this chapter, the Stokes equations could be solved exactly. We have already seen several examples when the Stokes equations can be solved, including perturbative lattices and homogeneous and inhomogeneous lattices. In general, however, one has to solve the Stokes equations, which will result in PDEs. One topic of further study would be to attempt to understand why the above results can be solved exactly, and if there are more classes of black hole solution that have exact DC response.

Chapter 4

Holographic thermal DC response in the hydrodynamic limit

1 Introduction

In this chapter, we continue using holographic techniques to study the response of a strongly coupled system to an applied DC thermal or electric source. As we have previously discussed, in order to get finite results, one needs, in general, a mechanism to dissipate momentum. Within holography, the most natural way to achieve this is via the framework of holographic lattices [37, 50, 52, 57, 62, 125]. These are stationary black hole geometries dual to CFTs in thermal equilibrium that have been deformed by marginal or relevant operators that explicitly break the translation invariance of the CFT. Several different kinds of holographic lattices have now been studied and they give rise to a wide variety of interesting phenomena including Drude physics [50, 55, 125], novel metallic ground states [58, 104], metal-insulator transitions [37, 57] and anomalous temperature scaling of the Hall angle [124].

In chapter 2, we showed that the DC thermoelectric conductivity matrix can be obtained for holographic lattices, universally, by solving a system of linearised Navier-Stokes equations for an auxiliary incompressible fluid on the black hole horizon. In a nutshell, this works as follows. By solving the fluid equations, which we also refer to as Stokes equations, one obtains local thermal and electric currents on the black hole horizon, at the level of linear response, as functions of the applied DC sources. Furthermore, the total thermal and electric current fluxes at the horizon are conserved in the radial direction and hence one also obtains the total thermal current fluxes of the dual field theory and thus the DC conductivity matrix. The results of this, and [1, 97, 98], have recently been used to obtain interesting bounds on DC conductivities [126, 127].

We now elaborate a little further on several important aspects of the results of the earlier chapter and the papers [1, 97, 98]. Firstly, although not essential, it is helpful to introduce the DC sources via perturbations that are linear in time. This enables one to directly parametrise the thermal and electric DC sources using globally defined one-forms, rather than locally defined functions. This is helpful in constructing a globally defined bulk perturbation, and hence extracting the DC response. Such linear in time sources for a DC electric field and thermal gradient were first discussed in [58] and [66], respectively.

The second aspect we want to highlight is the result of [97] that the total thermal and electric current fluxes at the horizon are equal to the current fluxes at infinity. This was first observed for the special case of electric currents in the early paper [67], extending [128], but in a very simplified setting. In particular, the black holes of [67] did not incorporate momentum dissipation and the finite electric DC conductivity arose because the background black holes were electrically neutral, giving rise to constant electric currents. In fact, the thermal DC conductivity, which was not discussed in [67], is infinite for these black holes. For general holographic lattices, both the electric and thermal currents have non-trivial spatial dependence and it is only the total current fluxes that are conserved in moving from the horizon to the AdS boundary. This general result, articulated in [97], came in a series of stages, starting with studies of electric currents [58] and then for the more subtle case of thermal currents [66], both for Q-lattice examples, followed by electric and thermal currents for inhomogeneous holographic lattices, with momentum dissipation in one spatial dimension [55].

The non-renormalisation of the total current fluxes is widely referred to as a kind of membrane paradigm, following [67, 128]. However, this is somewhat of a misnomer. A membrane paradigm, in this context, should determine the currents, or perhaps just the current fluxes, on the black hole horizon and hence the conductivity of the black hole horizon. However, for a general holographic lattice, *a priori*, it could have been the case that in order to obtain the current fluxes on the horizon, one would need to solve the full bulk equations of motion for the perturbation. If this was the case then one would not have been able to obtain the current fluxes of the dual field theory just from the horizon data and the notion of any kind of local membrane would be irrelevant. Happily this turns out not to be the case.

Indeed the third aspect we want to highlight, and logically distinct from the previous one, is that one can determine the currents at the horizon by solving a closed set of fluid equations, the Stokes equations, on the horizon [97]. The Stokes equations, which only involve a subset of the perturbation, can be obtained by considering a radial Hamiltonian decomposition of the bulk equations of motion and then imposing the constraint equations on a surface of constant radial coordinate, in the limit as one approaches the horizon. The fact that one has a closed set of equations on the horizon and that they arise from a set of constraint equations makes the result of [97] a version of the membrane paradigm, if one finds that terminology helpful. We also emphasise that this membrane paradigm is universal and makes no assumption about taking any hydrodynamic limit. As we demonstrated in the previous chapter, solving the hydrostatic problem on the event horizon gives rise to exact statements about correlation functions in relativistic CFTs with momentum dissipation, without taking any hydrodynamic limit.

The two key stepping stones that led to the general result about Stokes equations were the results obtained for Q-lattice examples [58, 66] followed by the results for the more involved example of an inhomogeneous holographic lattice with momentum dissipation in just one spatial direction [55]. In hindsight, the results of [55, 58, 66], which were originally obtained by brute force, were possible because they are two cases in which the Stokes equations can be solved analytically. The Stokes equations also emphasise the fundamental role that is played by thermal currents in studying DC response. Indeed, even if one wants to obtain just a purely electric DC response in the field theory, in general one still must solve for both the thermal and electric currents at the horizon, in order to extract this information.

For the special classes of one-dimensional holographic lattices (i.e. there is only momentum dissipation in one spatial direction) and also for Q-lattices [57], the DC conductivity can be explicitly solved in terms of the horizon data [1, 97, 98]. For more general classes of lattices, there are two limiting situations where we can make some universal statements. The first limiting situation is associated with what have been called "perturbative lattices" and was discussed in section 4.3 of chapter 2, as well as in [1, 97]. The second, and in general distinct, limiting situation occurs in a long-wavelength, hydrodynamic limit and will be the main focus of this chapter.

We first briefly discuss the perturbative lattices, which are associated with weak momentum dissipation, in order to contrast with the hydrodynamic limit. By definition, perturbative lattices can be constructed perturbatively about a translationally invariant black hole solution using a small amplitude deformation. For example, imagine starting with the AdS-Schwarzschild black brane solution or, if one wants to be at finite charge density, the AdS-Reissner-Nordström solution. These solutions are dual to translationally invariant CFTs with no momentum dissipation and hence have infinite DC thermal conductivity. We then deform the dual field theory by marginal or relevant operators that depend on the spatial coordinates of the CFT. It is assumed that the strength or amplitude of the deformation is fixed by a small dimensionless UV parameter λ. On the other hand we make no restriction on how the UV deformation depends on the spatial coordinates, and so we allow deformations with arbitrary wave numbers k. Generically, for small λ, the UV deformation will not change the leading order IR of the black hole geometry. As such the horizon geometry can be expanded perturbatively in λ about the flat horizon.

As we have seen, one can then solve the Stokes equations on the horizon perturbatively in λ to obtain expressions for the currents and hence the DC conductivity. At leading order in the perturbative expansion the currents at the horizon are of order λ^{-2}, which shows the DC response is parametrically large as expected for weak momentum dissipation. The leading order currents at the horizon are constant and by considering the full bulk perturbation one can further show that at leading order in λ the local currents of the dual CFT are also constant.

The final expressions for the currents depend on the corrections to the horizon data at order λ^1 . While these are not explicitly known in terms of the UV data, one can still obtain some additional general results for the DC conductivity. For example, for the case of perturbative lattices which have non-vanishing charge density at leading order[1] one should expand about the AdS-Reissner-Nordström (AdS-RN) solution. The conductivities are then all proportional to the same matrix, of order λ^{-2}, and as shown in [1, 97], extending [66], this leads to a generalised Wiedemann-Franz law[2], $\bar{\kappa}/(\sigma T) = s^2/\rho^2$, as well as a simple expression for the Seeback coefficient (thermopower) equal to $\alpha/\sigma = s/\rho$.

We now turn to the hydrodynamic limit of holographic lattices. We will study this in the context of holographic lattices in Einstein gravity without matter fields in arbitrary spacetime dimensions $D \geq 4$. As such our analysis is applicable to all CFTs in $D-1$ spacetime dimensions which have a classical gravity dual. The deformed CFTs can either be viewed as arising from deformations associated with the stress-energy tensor operator or, equivalently, by placing the CFT on a curved geometry.

If we let k be the largest wave number associated with the spatial deformations in the holographic lattice then we consider the hydrodynamic limit[3] $\epsilon = k/T << 1$. In general this is distinct from the perturbative lattice. This can be seen from the following very explicit example. Consider a CFT at finite T on a conformally flat spatial metric $h_{ij} = \Phi(x)\delta_{ij}$ with $\Phi(x) = \lambda \cos kx$. The perturbative lattice takes $\lambda << 1$ whereas the hydrodynamic limit takes $k/T << 1$. Note, however, that in both limits $\lambda \to 0$ and $\epsilon \to 0$ we end up with the AdS-Schwarzschild black brane.

A key simplification which we show occurs in the hydrodynamic limit is that the black hole horizon geometry can be simply expressed explicitly in terms of the UV data. Thus, to obtain the DC linear response we solve the Stokes equations on a geometry that is explicitly known. Furthermore, we will show that at leading order in the perturbative expansion in ϵ, the local thermal currents at the horizon are conserved in moving to the boundary. In other words, by solving the Stokes equations on the horizon one obtains the leading order local currents that are produced by the DC source, at the level of linear response. The fact that there can be non-trivial local currents in the hydrodynamic limit at leading order should be contrasted with the constant currents in the perturbative lattices. Conversely, similar to the perturbative lattices, the thermal conductivity diverges like ϵ^{-2} at fixed T as $\epsilon \to 0$ and we will see that the limit is again associated with weak momentum dissipation.

[1]If the charge density vanishes at leading order then one should expand about the AdS-Schwarzschild solution. In this case the conductivity σ will be of order λ^0.

[2]This result was also be obtained via memory matrix techniques in [123], building on [125].

[3]In the bulk we will take the limit $k/r_H << 1$, where r_H is the radial location of the black hole. We can thus also consider the hydrodynamic limit as $k/s^{1/(D-2)} << 1$ where s is the entropy density.

It is also interesting to develop the perturbative hydrodynamic expansion in more detail, and we do this to determine the corrections of the thermal currents to the DC linear response that are sub-leading in ϵ. In addition we will also show that it is possible to solve for the full radial dependence of the leading order bulk, linearised perturbations after applying a DC thermal source. This allows us to extract expressions not only for the local thermal currents but also for the full linearised stress tensor of the boundary theory as a function of the applied DC thermal source. As one might expect, the stress tensor takes the form of constitutive relations for a forced fluid and after imposing the Ward identities one obtains, at leading order in the ϵ perturbative expansion, the non-relativistic Stokes equations. The calculation also indicates how the sub-leading corrections to the stress tensor, perturbative in ϵ, can be obtained in terms of solutions to the Stokes equations on the horizon geometry, which is also expanded in ϵ. As we shall see, our results also reveal a subtle interplay between sub-leading corrections in the perturbative expansion, solutions of the Stokes equations and regularity of the bulk perturbation at the horizon.

It is natural to ask how our results relate to work on fluid-gravity [88] which relates perturbative solutions of gravity equations to hydrodynamic equations. In particular, CFTs on curved manifolds have been studied in the context of theories of pure gravity from a fluid-gravity perspective in [129]. Using a Weyl covariant formalism a hydrodynamic expansion ansatz for the bulk metric was presented. The ansatz gives rise to a boundary stress tensor which, when the Ward identities are imposed, leads to perturbative solutions of the bulk Einstein equations. One might expect our perturbative expansion in ϵ, driven by the linearised DC source is a special case of the expansion in [88]. Happily, this is found to be the case.

Before concluding this introduction we note that there have been various other studies of hydrodynamics in the context of momentum dissipation in holography, including [46,54, 95,96,115,117,130]. We highlight that a fluid gravity expansion was developed in [95,96] for the special sub-class of holographic lattices with spatial translations broken by bulk massless axion fields that depend linearly on the spatial coordinates. We also note that general results on hydrodynamic transport for quantum critical points have been presented in, for example, [114,116,131,132]. In contrast to the approach of [116] we will see that within holography in the hydrodynamic limit we need to solve covariantised Navier-Stokes equations with constant viscosity.

2 Holographic lattices in the hydrodynamic limit

We will consider a much simpler gravitational theory than what was studied in the previous chapters, and focus theories of pure gravity in D bulk spacetime dimensions, with action given by

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g}\,[R + (D-1)(D-2)]\ . \tag{4.1}$$

We have set $16\pi G = 1$ for simplicity. We have also chosen the cosmological constant so that a unit radius AdS_D spacetime solves the equations of motion. The holographic lattice solutions which we study are static black holes that lie within the ansatz:

$$ds^2 = -U\,G\,dt^2 + \frac{F}{U}\,dr^2 + g_{ij}dx^i dx^j\ . \tag{4.2}$$

Here G, F and the $d = (D-2)$-dimensional spatial metric, g_{ij}, can depend on both the radial coordinate r and the spatial coordinates of the dual field theory, x^i. The function U is a function of r only, and is also included for convenience. Thus,

$$G = G(r,x), \quad F = F(r,x), \quad g_{ij} = g_{ij}(r,x), \quad U = U(r)\,. \tag{4.3}$$

We assume that the solutions have a single black hole Killing horizon located at $r = r_H$. It will be helpful to introduce another radial coordinate $\rho = r/r_H$, so that the horizon is located at $\rho = 1$, and choose

$$U = r_H^2\,\rho^2\,u(\rho)\,, \qquad u(\rho) = 1 - \rho^{1-D}\,. \tag{4.4}$$

The temperature of the black hole is given by $4\pi T = (D-1)\,r_H$ and we note that regularity of the metric at the horizon imposes the condition $F|_{\rho=1} = G|_{\rho=1}$.

The AdS boundary is located at $r \to \infty$, where we demand that

$$G \to h_{tt}(x)\,, \qquad g_{ij} \to r_H^2\,\rho^2 h_{ij}(x)\,, \tag{4.5}$$

corresponding to studying the dual CFT on the curved background with metric $ds^2 = -h_{tt}dt^2 + h_{ij}dx^i dx^j$. Equivalently, we are considering the CFT with deformations of the stress tensor with sources parametrised by h_{tt} and h_{ij}. We will focus on cases in which the spatial sections, parametrised by the x^i are non-compact with planar topology. Furthermore, we assume the deformations are periodically modulated in the spatial directions, with associated wave numbers in the x^i direction to be an integer multiple of a minimum $2\pi/L^i$. Effectively, this means that we can take the x^i to parametrise a torus with $x^i \sim x^i + L^i$.

To obtain these holographic lattice solutions one needs to solve non-linear PDEs subject to these boundary conditions. Some explicit examples have appeared in [54,101] with the construction in [101] exploiting a residual symmetry leading to solving a system of ODEs. In this chapter we will be interested in studying the hydrodynamic limit of these solutions. More precisely if we suppose k is the largest wave number associated with the spatial deformations then we are interested in studying the limit $\epsilon \equiv k/T \ll 1$. Equivalently, for these black holes we can consider the limits $k/r_H \ll 1$ or $k/s^{1/(D-2)}$ where s is the entropy density. In fact, when solving the bulk equations of motion the latter are more natural.

By directly solving Einstein equations we find that the leading order solution is given by

$$ds^2 = \frac{(1+\rho\partial_\rho \ln w)^2}{\rho^2\, u(w\rho)} d\rho^2 + r_H^2 \rho^2 \frac{w^2}{h_{tt}(x)} \left[-u(w\rho) h_{tt}(x) dt^2 + h_{ij}(x) dx^i dx^j\right] , \tag{4.6}$$

with $w(\rho, x)$ an arbitrary function satisfying

$$\begin{aligned} &w(\rho, x) \to 1, \quad \rho \to 1\,, \\ &w(\rho, x) \to [h_{tt}(x)]^{1/2}, \quad \rho \to \infty\,. \end{aligned} \tag{4.7}$$

This solution solves the Einstein equations in the limit of ignoring spatial derivatives. It should be viewed as the leading term in an asymptotic expansion in ϵ, where have neglected corrections of order ϵ^2 as well as corrections that are non-perturbative in ϵ.

For the remainder of the chapter we will focus on the case with $w = 1$ and hence consider the leading order solution given by

$$ds^2 = \frac{d\rho^2}{\rho^2\, u(\rho)} + r_H^2 \rho^2 \left[-u(\rho) dt^2 + h_{ij}(x) dx^i dx^j\right] . \tag{4.8}$$

This corresponds to setting $h_{tt} = 1$ and hence studying the dual CFT on the metric $ds^2 = -dt^2 + h_{ij}(x)dx^i dx^j$. In fact this is almost without loss of any generality. Indeed we can incorporate a non-vanishing h_{tt} by simply performing a Weyl transformation of this boundary metric. Since we are considering CFTs the physics will be the same with the exception that in the case of odd D, *i.e.* when the CFT is in even spacetime dimensions, for certain metrics we will need to take into account the conformal anomaly, which will, in any case, be a sub-leading effect in ϵ.

Notice that the leading order solution (4.8) is just the standard AdS_D black brane solution but with the flat metric on spatial sections, δ_{ij}, replaced with the metric h_{ij}, which parametrises the UV deformations of the dual CFT. A corollary, which will be important in the following, is that the metric on the black hole horizon is given by $r_H^2 h_{ij}(x^i)dx^i dx^j$, at leading order in the ϵ expansion. This accords with the intuition that at high temperatures the black hole horizon is approaching the AdS boundary.

It is straightforward to show that (4.8) solves the Einstein equations, to leading order, after using the radial coordinate ρ and taking the limit $k/r_H \to 0$. A simple way to derive (4.6) from (4.8) is to simply rescale the radial coordinate, $\rho \to w\rho$. If w was a constant this would give a constant rescaling of the boundary metric. Taking w to be a function of (ρ, x) satisfying (4.7) leads to a Weyl transformation of the boundary metric at leading order in the hydrodynamic limit. In particular this transformation introduces a $d\rho dx^i$ cross term of order ϵ, but after suitably shifting the x^i coordinates one obtains the solution (4.6) dropping just ϵ^2 corrections (or the corrections that are polynomial in ϵ). In general, from (4.6) we see that the metric on the black hole horizon in the hydrodynamic limit is given by

$$ds_H^2 = r_H^2 \frac{h_{ij}}{h_{tt}} dx^i dx^j \,. \tag{4.9}$$

In particular, we emphasise that it is invariant under Weyl transformations of the boundary metric $h_{\mu\nu} \to e^\gamma h_{\mu\nu}$.

3 Thermal currents from a DC source

The linear response that arises from the application of a DC thermal gradient can be calculated within holography by studying a suitable linearised perturbation of the gravitational background. Having gone through the technical details in chapter 2, here we will summarise the key ideas, applicable to all holographic lattices, before turning to the long wavelength limit. In particular, we show how the local thermal currents can be obtained at leading order in the hydrodynamic limit.

3.1 Review of general DC response via Stokes equations

As first discussed in [66] it is convenient to introduce the DC source by considering the following linearised perturbation about the general background holographic lattice geometry given in (4.2) via:

$$\delta ds^2 = -2UGt\zeta_i dt dx^i + \delta g_{\mu\nu} dx^\mu dx^\nu \,, \tag{4.10}$$

where $x^\mu = (t, r, x^i)$. The key piece here is the linear in time source that is parametrised by the one-form $\zeta \equiv \zeta_i(x) dx^i$, which just depends on the spatial coordinates[4] and is closed $d\zeta = 0$. It is important to emphasise that ζ is a globally defined one-form on the boundary spacetime and can easily be shown to parametrise a thermal gradient. Indeed if we write $\zeta = d\phi(x)$, for some locally defined function $\phi(x)$, after making the coordinate transformation $t \to t(1-\phi)$ the linear perturbation appears in the perturbed metric as

$$ds^2 + \delta ds^2 = -(1-\phi)^2 UG dt^2 + \frac{F}{U} dr^2 + g_{ij} dx^i dx^j + \delta g_{\mu\nu} dx^\mu dx^\nu \,, \tag{4.11}$$

[4]Note that if we write (x) the argument will always refer to the spatial field theory coordinates.

and locally, we identify $\phi(x) = -\ln T(x)$ so that $\zeta = -T^{-1}dT$. One might consider, for example, $\phi(x) = \bar{\zeta}_i x^i$, with $\bar{\zeta}_i$ constant, and then we have $T^{-1}\partial_i T = -\bar{\zeta}_i$.

The remaining part of the perturbation, $\delta g_{\mu\nu}(r,x)$, is whatever is required in order to get a consistent set of equations. The perturbation components $\delta g_{\mu\nu}$ are taken to be globally defined functions of the spatial coordinates[5]. They are chosen to suitably fall off fast enough at the AdS boundary to ensure that the only source being applied is parametrised by ζ. The boundary conditions of $\delta g_{\mu\nu}$ at the black hole horizon are chosen so that the full perturbation given in (4.10) is regular. It is worth commenting that a helpful aspect of working with the globally defined one-form and time coordinate as in (4.10) is that it makes it clear that $\delta g_{\mu\nu}(r,x)$ should be globally defined functions. Indeed, *a priori*, it is not clear what conditions one should impose on the spatial dependence of the perturbations $\delta g_{\mu\nu}$ in (4.11) given the presence of the locally defined function ϕ.

We now introduce the *bulk* thermal current density, $Q^i(r,x)$, depending on both the radial direction and the spatial directions of the field theory, defined by

$$Q^i(r,x) \equiv \frac{1}{16\pi G_N}\frac{G^{3/2}U^2}{F^{1/2}}\sqrt{g_d}\,g^{ij}\left(\partial_r\left(\frac{\delta g_{jt}}{GU}\right) - \partial_j\left(\frac{\delta g_{rt}}{GU}\right)\right), \tag{4.12}$$

where $\sqrt{g_d}$ refers to the volume element of the d-dimensional spatial metric $g_{ij}(r,x)$ in (4.2). By evaluating this at the AdS boundary we obtain

$$Q^i_{QFT}(x) \equiv \lim_{r\to\infty} Q^i(r,x)\,, \tag{4.13}$$

where $Q^i_{QFT}(x)$ is the *local* thermal current density of the dual quantum field theory given by the stress-tensor, $Q^i_{QFT}(x) = -\sqrt{h}T^i{}_t(x)$, which is induced by the DC thermal gradient. We also define the local currents at the black hole horizon via

$$Q^i_{BH}(x) \equiv \lim_{r\to r_H} Q^i(r,x)\,. \tag{4.14}$$

In chapter 2, we demonstrated that the bulk thermal currents satisfy a differential equation which can be integrated in the radial direction leading to the following important relation:

$$Q^i_{QFT} = Q^i_{BH} - \frac{1}{16\pi G_N}\partial_j\int_{r_H}^{\infty} dr\left((GF)^{1/2}\sqrt{g_d}\,g_d^{ik}g_d^{jl}\left((UG)\partial_k\left(\frac{\delta g_{lt}}{GU}\right) - k\leftrightarrow l\right)\right). \tag{4.15}$$

Notice that the second term on the right hand-side of this expression is a magnetisation current[6] and is trivially conserved.

[5]Note that in this paper we are considering solutions in which there is a single black hole horizon and in which the spatial coordinates continue from the boundary to the horizon. More general solutions are discussed in [1, 97, 98].

[6]A magnetisation current is a current that can be expressed in the form $J^i = \partial_j M^{ij}$, where M^{ij} is an

Next, in a radial Hamiltonian formulation of the equations of motion, by evaluating the constraints at the black hole horizon, one can show that a *subset* of the perturbation is governed by a system of forced linearised Navier-Stokes equations, also called Stokes equations, on the horizon:

$$-2\,\nabla^{Hi}\nabla^{H}{}_{(i}v_{j)} = 4\pi T\zeta_j - \partial_j p\,, \qquad \nabla^H_i v^i = 0\,, \tag{4.16}$$

where

$$v_i = -\delta g_{it}|_H, \qquad p = -\big(\delta g_{rt}\frac{4\pi T}{G} + \delta g_{it}g^{ij}\nabla_j \ln G\big)|_H\,, \tag{4.17}$$

and here ∇^H is the Levi-Civita connection associated with spatial metric on the horizon $g_{ij}|_H$. Furthermore, the local thermal currents on the horizon are given by

$$Q^i_{BH} = \frac{T}{4G_N}\sqrt{g_d}|_H g^{ij}_H v_j\,. \tag{4.18}$$

It should be emphasised that, in general, the fluid on the horizon is an auxiliary fluid and only indirectly related to observables in the boundary CFT. If we solve the Stokes equations on the horizon geometry, we obtain the local thermal currents $Q^i_{BH}(x)$ from (4.18). We can then obtain the physical local thermal currents $Q^i_{QFT}(x)$ from (4.15), but only if we have solved the full radial dependence of the perturbation in the bulk.

However, still quite generally, if we have solved the Stokes equations we can easily obtain the total current flux $\bar{Q}^i_{QFT}$ and hence the DC thermal conductivity matrix defined via $\bar{Q}^i_{QFT} = T\kappa^{ij}\bar{\zeta}_j$. To see this we define the total current flux, or equivalently, the zero mode of the current, via

$$\bar{Q}^i_{QFT} \equiv \int Q^i_{QFT}\,, \tag{4.19}$$

where[7] $\int \equiv (\Pi_i L_i)^{-1}\int \Pi_i dx^i$ refers to an average integral over a period in the spatial directions. Defining $\bar{Q}^i_{BH}$ in a similar way, we can immediately deduce from (4.15) that $\bar{Q}^i_{QFT} = \bar{Q}^i_{BH}$.

3.2 DC thermal current response for $k/T \ll 1$

We now consider the universal results for holographic lattices that we just summarised in the context of the hydrodynamic limit. We will consider a boundary metric given by

antisymmetric magnetisation density. Black holes with magnetisation currents arise at finite charge density in the context of phases in which translations and time reversal invariance are broken spontaneously - for an example see [133].

[7]This way of defining the zero modes is widely used. One could also define them by averaging with $\int \Pi_i dx^i \sqrt{h}$.

$ds^2 = -dt^2 + h_{ij}(x)dx^i dx^j$. There are two key simplifications.

The first is that when $\epsilon = k/T \ll 1$ the black hole horizon metric can be expressed explicitly in terms of the physical UV deformation of the dual CFT. Indeed, from the leading order form of the solution given in (4.8) we deduce that the black hole horizon metric is given by $r_H^2 h_{ij} dx^i dx^j$, where $h_{ij}(x)$ is the UV deformation of the dual CFT. Thus, we know the explicit geometry for which we need to solve the Stokes equations (4.16) in order to obtain the local currents on the black hole horizon $Q^i_{BH}(x)$.

The second simplification is that the second term on the right hand side of (4.15) will be suppressed by order ϵ^2 and hence, at leading order, we deduce that the *local* currents at the horizon are the same as those in the dual field theory:

$$Q^i_{QFT}(x) = Q^i_{BH}(x) + \mathcal{O}(\epsilon^2)\,. \tag{4.20}$$

Putting these two facts together we draw the following important conclusion. Consider a holographic lattice describing a CFT on a curved manifold with arbitrary spatial metric $h_{ij}(x)$. After applying a DC thermal source, parametrised by the closed one-form ζ, at leading order in ϵ we can obtain the local thermal currents $Q^i_{QFT}(x)$ of the dual field theory by solving the Stokes equations (4.16) using the metric $r_H^2 h_{ij} dx^i dx^j$ where $r_H = 4\pi T/(D-1)$. In the next section we will show how to obtain the full stress tensor of the fluid at leading order in ϵ.

It is worth highlighting that in the hydrodynamic limit we are therefore solving the covariantised Navier-Stokes equations with metric $r_H^2 h_{ij} dx^i dx^j$ and with constant viscosity given by $\eta = r_H^{D-2}$. This should be contrasted with the speculation in section 4 of [116] that within holography one should consider spatially dependent viscosity and a conformally rescaled metric on the horizon. In fact it is also worth recalling here a point made earlier that a Weyl transformation of the boundary metric $h_{\mu\nu}$ leads to the same black hole horizon metric, as we see from (4.9).

We conclude this section by deriving the scaling behaviour of the thermal conductivity in the hydrodynamic limit. We can remove all dimensionful quantities from the Stokes equations by scaling $\hat{x} = kx$, $\hat{p} = kp$, $\hat{\zeta} = T\zeta$ and $\hat{v}^i = k^2 h^{ij} v_j$. Indeed the Stokes equations (4.16) can then be written in terms of the hatted variables as well as the UV metric deformation h_{ij}. We thus deduce that $\hat{v}^i$ is related to $\hat{\zeta}_i$ via a dimensionless matrix. From (4.18), and recalling that the metric on the horizon is $r_H^2 h_{ij} dx^i dx^j$, we deduce that the heat current scales as scaling $Q^i_{QFT} \propto T^{D-1}/k^2 \sqrt{h}\hat{v}^i$ and thus we deduce that in the hydrodynamic limit we have

$$\kappa \propto \frac{s}{T\epsilon^2}\,, \tag{4.21}$$

where $s \propto T^{D-2}$ is the entropy density. Clearly at fixed T, this is parametrically large as $\epsilon \to 0$ as one expects for weak momentum dissipation.

4 The full perturbation for the thermal DC source

In this section we examine the full perturbation that is induced by the thermal DC source for a holographic lattice in the hydrodynamic limit $\epsilon = k/T \ll 1$. This will allow us to obtain not only the local heat currents that are produced but also the full local stress tensor of the dual field theory. In addition, this analysis will display in more detail the structure of the perturbative expansion in ϵ.

The strategy is to first solve the full radial dependence of the linearised perturbation about the background geometry that is associated with a DC thermal source parametrised by the closed one-form ζ. At leading order in ϵ we obtain a local stress tensor that takes the form of constitutive relations for a fluid. Imposing the boundary Ward identities, or equivalently the constraint equations with respect to a radial Hamiltonian decomposition, then implies that the stress tensor satisfies the Stokes equations given in (4.16). The details of how the perturbation scales with ϵ is somewhat subtle as we shall see.

4.1 The thermal gradient source on the boundary

We begin by considering a static boundary metric of the form $ds^2 = -dt^2 + h_{ij}(x)dx^i dx^j$, where the spatial metric $h_{ij}(x)$, which depends periodically on the spatial coordinates x^i, parametrises the holographic lattice. As above, the thermal gradient source can be introduced by writing $\zeta = d\phi(x)$, for some locally defined function $\phi(x)$, and then considering the perturbed boundary metric:

$$ds^2 = -(1-\phi)^2 dt^2 + h_{ij}dx^i dx^j \,. \tag{4.22}$$

We identify $\phi(x) = -\ln T(x)$ so that $\zeta = -T^{-1}dT$. If we make the coordinate transformation $t \to t + t\phi(x)$ we obtain

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j - 2t\zeta_i dx^i dt \,, \tag{4.23}$$

and we see the 'linear in time source'. Note that the perturbed metric is now expressed in terms of the globally defined one-form ζ and not the locally defined function, $\phi(x)$.

In carrying out the calculations presented below, we found it convenient to work with a Weyl transformed version of this metric, with Weyl factor given by $(1+p/(4\pi T))^2$ where $p(x)$ is a globally defined (periodic) function. At linearised order, we therefore consider

$$ds^2 = (1 + \frac{2p}{4\pi T})\left[-dt^2 + h_{ij}dx^i dx^j\right] - 2t\zeta_i dx^i dt \,. \tag{4.24}$$

If we now employ the additional coordinate transformation $t \to t(1-p/(4\pi T))$ we obtain

$$ds^2 = -dt^2 + (1+\frac{2p}{4\pi T})h_{ij}dx^i dx^j - 2t[\zeta_i - (4\pi T)^{-1}\partial_i p]dx^i dt\,. \tag{4.25}$$

An appealing feature of using these coordinates is that the thermal gradient source is now appearing in the combination $\zeta_i - (4\pi T)^{-1}\partial_i p$, as in the Stokes equations (4.16).

We will employ one further coordinate change by taking $x^i \to x^i - th^{ij}\xi_j$, where $\xi_j = \xi_j(x)$, to finally write the perturbed metric in the form

$$\begin{aligned} ds^2 &= -dt^2 + h_{ij}dx^i dx^j - 2t\left(\zeta_i - (4\pi T)^{-1}\partial_i p\right)dx^i dt \\ &\quad - 2\left(\xi_j\, dt + t\,\nabla_{(i}\xi_{j)}\, dx^i\right)\, dx^j + \frac{2p}{4\pi T}h_{ij}dx^i dx^j\,. \end{aligned} \tag{4.26}$$

Despite obscuring the fact that the perturbation corresponds to a simple thermal gradient, we found this way of writing things helpful in extending the perturbation into the bulk as we discuss next.

4.2 The linearised perturbation: solving the radial equations

Writing the source on the boundary in the form given in (4.26) is rather non-intuitive. However, it has two virtues when we extend it into the bulk. The first is that it allows us to work in a radial gauge with $\delta g_{rt} = \delta g_{ri} = 0$. The second is that it helps to obtain a perturbation that is regular at the black hole horizon.

Recall from section 2 that the bulk background solution, at leading order in ϵ, with boundary metric $ds^2 = dt^2 + h_{ij}dx^i dx^j$, is given by

$$ds^2 = \frac{d\rho^2}{\rho^2\, u(\rho)} + r_H^2\rho^2\left[-u(\rho)dt^2 + h_{ij}(x^i)dx^i dx^j\right]\,. \tag{4.27}$$

Furthermore, this has corrections at order ϵ^2. We begin by considering the following bulk linearised perturbation that is induced by the applied source $\zeta_i(x)dx^i$:

$$\delta ds^2 = -2\, r_H^2\rho^2\, u\, t\,[\zeta_i - (4\pi T)^{-1}\partial_i p]\, dx^i\, dt \tag{4.28}$$

$$- 2r_H^2\rho^2\,(\xi_j\, dt + t\,\nabla_{(i}\xi_{j)}\, dx^i)\, dx^j \tag{4.29}$$

$$+ r_H^2\rho^2 2\frac{p}{4\pi T}h_{ij}dx^i dx^j \tag{4.30}$$

$$- 2(\rho^{3-D}V_j + \dots)\, dx^j\, dt \tag{4.31}$$

$$+ \rho^2\,\frac{\ln u}{4\pi T}\, s_{ij}\, dx^i\, dx^j\,. \tag{4.32}$$

Here p, ξ_i, V_i, and s_{ij} are all periodic functions of the boundary spatial coordinates x^i. In this section we will raise and lower indices using the UV boundary metric h_{ij} and ∇ is the associated Levi-Civita connection. We also assume that $h^{ij}s_{ij} = 0$.

We claim that each of the five lines separately solves the radial equations of motion arising in the Einstein equations at leading order in ϵ, with corrections to each line of order ϵ^2. However, an important subtlety, which we will carefully discuss in more detail, is that to ensure the perturbation is well defined at the black hole horizon when we consider these perturbations altogether, we will have to determine the order in the ϵ expansion at which the various terms in (4.28)-(4.32) first appear.

We now discuss the various terms in more detail. The first three lines (4.28)-(4.30) are the radial extensions of the boundary source terms that we discussed in (4.26). The first line (4.28) is almost the standard linear in time perturbation of the metric that we saw in (4.10). By shifting the time coordinate one can then easily show that this solves the Einstein equations, with corrections of order ϵ^2. Notice that $\zeta_i - (4\pi T)^{-1}\partial_i p$ appears rather than just ζ_i as in (4.10). This change, which is associated with our source terms given in (4.26), allows us to work in a radial gauge.

The second line (4.29) can locally be generated from the background solution (4.27) via the coordinate transformation

$$x^i \to x^i - t\, h^{ij}\xi_j \,, \tag{4.33}$$

and it therefore satisfies, trivially, the equations of motion, with corrections of order ϵ^2. The third line (4.30) can also be generated from the background solution (4.27). Indeed the spatial metric in (4.27) was arbitrary and so the perturbation (4.30) is simply obtained by taking $h_{ij} \to (1 + 2\frac{p}{4\pi T})h_{ij}$.

Having explained how the source terms given in (4.26) extend into the bulk, we now discuss the last two lines (4.31), (4.32), which are needed in order to obtain a consistent perturbed metric. Neither of these lines are associated with source terms in the boundary theory; instead they are associated with the response to the DC thermal gradient. Both (4.31) and (4.32) are solutions of Einstein's equations at leading order in ϵ. We will discuss (4.31) in detail in appendix C.1 since, as we discuss below, sub-leading corrections to this perturbation will play an important role when we analyse regularity of the perturbation at the black hole event horizon. We have explicitly added the dots in (4.31) to emphasise this point.

For the line (4.32) one requires the condition $h^{ij}s_{ij} = 0$ and, ignoring spatial derivatives of s_{ij}, one finds the given radial dependence after solving the ij component of the Einstein equations. Indeed if we consider the ansatz $\delta ds^2 = A(\rho)s_{ij}(x)$ then we find that the function $A(\rho)$ satisfies

$$\partial_\rho[\rho^D u \partial_\rho(\rho^{-2}A)] = 0 \,, \tag{4.34}$$

and we have chosen the solution $A \propto \rho^2 \ln u$ in order that it has no source at infinity. Corrections are again of order ϵ^2.

Let us now examine the behaviour of the perturbed metric at at the AdS boundary and at the black hole horizon. It is clear by construction that as we approach the AdS boundary at $\rho \to \infty$, the perturbation (4.28)-(4.32) gives rise to precisely the source terms in (4.26), associated with a thermal gradient parametrised by the closed one-form ζ.

To examine the issue of regularity on the horizon, located at $\rho = 1$, we employ the Eddington-Finklestein ingoing coordinate $v \sim r_H(t + \ln u/(4\pi T))$. We find that we should impose

$$\xi_j = -r_H^{-2} V_j , \quad s_{ij} = 2\, \nabla_{(i} V_{j)} , \tag{4.35}$$

and, after recalling that we imposed $h^{ij} s_{ij} = 0$, the latter implies the incompressibility condition $\nabla_i V^i = 0$. After making these identifications we see that the metric is almost regular at the horizon, but there is a remaining singular term of the form

$$-2r_H \frac{\ln u}{4\pi T} (\zeta_i - (4\pi T)^{-1} \partial_i p)\, dx^i d\rho , \tag{4.36}$$

as $\rho \to 1$.

Before returning to this crucial issue we see that writing the boundary metric in the form (4.26), which arose from a boundary coordinate transformation, implies that additional modes had to be activated in the bulk, namely (4.31) and (4.32) with (4.35), in order to get a regular perturbation at the horizon. A closely related fact is that the boundary coordinate transformation leading to (4.26) can be extended smoothly into the bulk. Specifically, consider the bulk coordinate transformations

$$\begin{aligned} t &\to t\,(1 + (4\pi T)^{-1} p) + g(\rho, x^i) , \\ x^i &\to x^i + t\, h^{ij} \xi_j + g^i(\rho, x^i) . \end{aligned} \tag{4.37}$$

acting on the perturbed metric (4.27) combined with (4.28)-(4.32) If g and g^i vanish fast enough as $\rho \to \infty$ the conformal boundary will be given in the form (4.24). Furthermore, the coordinate transformations are smooth as we approach the horizon, provided that we choose $g(\rho, x^i) \to p\, \frac{\ln u}{(4\pi T)^2}$ and $g^i(\rho, x^i) \to h^{ij} \xi_j \frac{\ln u}{4\pi T}$ as $\rho \to 1$.

We now return to the remaining divergence (4.36) at the horizon. To cancel this it is necessary to consider sub-leading terms in the expansion in ϵ, both for the background and for the perturbation. As we discuss in more detail in appendix C.1, there is a delicate cancelation between the sub-leading terms in the ϵ expansion of (4.31) (denoted by dots) and (4.36). We find that regularity of the perturbation at the horizon implies that the leading order pieces of the perturbation have a dependence on ϵ given by

$$\begin{aligned} V_i(x) &= \epsilon^{-2} v_i^{(0)}(x)\,, \\ p(x) &= \epsilon^{-1} p^{(0)}(x)\,, \end{aligned} \tag{4.38}$$

with $v_i^{(0)}$ and $p^{(0)}$ the same order as $4\pi T\zeta_i$. We emphasise that this behaviour is fixed by regularity at the horizon.

Furthermore, the first line (4.28) and the fourth line (4.31) of the perturbation read

$$\begin{aligned} \delta ds^2 =& -2\, r_H^2 \rho^2\, u\, t\, (\zeta_i - (4\pi T)^{-1} \epsilon^{-1} \partial_i p^{(0)})\, dx^i\, dt\,, \\ & - 2\epsilon^{-2} \rho^{3-D} (v_i^{(0)}(x) + \epsilon^2 V_i^{(2)}(\rho, x))\, dx^j\, dt\,, \end{aligned} \tag{4.39}$$

where an explicit expressions for $V_i^{(2)}(\rho, x)$is given in appendix C.1 (see (C.27)). Note that we have only explicitly written the one sub-leading term, $V_i^{(2)}$, which is relevant to the discussion here. As we approach the horizon the sub-leading term behaves as

$$V_i^{(2)}(\rho, x) \to \frac{r_H^2}{(4\pi T)^2} u \log(\rho - 1) \left(-\frac{2}{(\epsilon r_H)^2}\, \nabla^i \nabla_{(i} v_{j)}^{(0)} \right)\,, \qquad \rho \to 1\,. \tag{4.40}$$

We now see that the singular term (4.36) in the combined perturbation, arising from the first line in (4.39), is cancelled by the sub-leading term in the second line of (4.39) providing that $v_i^{(0)}$ satisfies the Stokes equations,

$$-\frac{2}{(\epsilon r_H)^2}\, \nabla^i \nabla_{(i} v_{j)}^{(0)} = 4\pi T \zeta_j - \epsilon^{-1} \partial_j p^{(0)}\,. \tag{4.41}$$

It is satisfying to see that these equations are none other than the leading order expansion in ϵ of (4.16), where we should recall that the black hole horizon metric is given, at leading order in ϵ, by $(r_H)^2 h_{ij} dx^i dx^j$. We thus conclude that there is a subtle interplay between sub-leading terms in the expansion, regularity of the perturbation at the horizon and the Stokes equations.

4.3 The boundary stress tensor

We now calculate the holographic stress tensor for the dual field theory at leading order in the ϵ expansion. We also present the sub-leading contributions to the local heat current, based on the calculations carried out in appendix C.1.

We will express the result for the stress tensor in the coordinates and Weyl frame for the boundary metric given by

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j - 2t\zeta_i dx^i dt\,. \tag{4.42}$$

To achieve this we first carry out the coordinate transformations[8]

$$\begin{aligned} t &\to t\,(1+(4\pi T)^{-1}p)\,,\\ x^i &\to x^i + t\,h^{ij}\xi_j\,. \end{aligned} \tag{4.43}$$

This leads to a perturbed metric which asymptotes to the boundary metric in (4.24) with an overall Weyl factor given by $(1+\frac{2p}{4\pi T})$. We can move to Fefferman-Graham type coordinates, which are a slight modification of those for AdS-Schwarzschild, and also eliminate the boundary Weyl factor, via

$$\begin{aligned} \rho &= \frac{(1+\frac{1}{4}[r_H z(1+(4\pi T)^{-1}p)]^{D-1})^{\frac{2}{D-1}}}{r_H z(1+(4\pi T)^{-1}p)}\,,\\ &\approx \frac{1}{r_H z(1+(4\pi T)^{-1}p)}\left(1+\frac{1}{2(D-1)}[r_H z(1+(4\pi T)^{-1}p)]^{D-1}\right)\,, \qquad \rho\to\infty\,. \end{aligned} \tag{4.44}$$

In these coordinates we have $g_{zz}=1/z^2$ and the AdS boundary is now located at $z=0$. Note that to the order in ϵ we are working with, we can drop spatial derivatives of p. The boundary conformal metric is now given by (4.42) and by determining the terms with factors $z^{D-3}/(D-1)$ a consideration of [134] then implies that the stress tensor has components,

$$\begin{aligned} 16\pi G_N T_{tt} &= (D-2)r_H^{D-1} + (D-2)r_H^{D-2}p\,,\\ 16\pi G_N T_{ij} &= r_H^{D-1}h_{ij} + r_H^{D-2}\left(-r_H^{-2}2\nabla_{(i}V_{j)} + p h_{ij}\right)\,,\\ 16\pi G_N T_{it} &= r_H^{D-1}\left[(D-2)\zeta_i t - (D-1)r_H^{-2}V_i\right]\,. \end{aligned} \tag{4.45}$$

Recall from our discussion above that $V_i=\epsilon^{-2}v_i^{(0)}(x)$, $p=\epsilon^{-1}p^{(0)}(x)$ and we recall that $r_H=4\pi T/(D-1)$. Clearly the perturbed stress tensor takes the form of constitutive relations for a fluid with source parametrised by ζ_i. We should also note that we have already imposed the incompressibility condition $\nabla_i V^i=0$.

When the constraint equations, in a radial Hamiltonian decomposition of Einstein equations, are imposed at the AdS boundary they imply that the stress tensor is traceless, $T^\mu{}_\mu=0$, and also satisfies the Ward identity $\nabla_\mu T^{\mu\nu}=0$, both with respect to the deformed boundary metric (4.42). The first of these is already satisfied due to the incompressibility condition

$$\nabla_i V^i = 0\,, \tag{4.46}$$

[8]Note that since we are only interested in the boundary expansion here we don't need to consider the g and g^i terms in (4.37).

and the second gives rise to the linearised Navier-Stokes equations:

$$-\frac{2}{(r_H)^2}\nabla^i\nabla_{(i}V_{j)} = 4\pi T\zeta_j - \partial_j p\,. \tag{4.47}$$

exactly as in (4.41).

To summarise, at leading order in the ϵ expansion, after applying a DC thermal gradient source parametrised by the closed one-form ζ, we can obtain the full, local stress tensor response, given in (4.45), after solving the Stokes equations (4.47) on the curved manifold with metric h_{ij}. Notice, in particular, from the expression for the stress tensor given in (4.45) we find that the local heat current of the dual field theory is given by

$$\begin{aligned} Q^i_{QFT} &= -\sqrt{h}T^i{}_t\,,\\ &= \frac{1}{4G_N}T r_H^{D-4}\sqrt{h}h^{ij}V_j\,,\\ &= \frac{1}{4G_N}T r_H^{D-4}\epsilon^{-2}\sqrt{h}h^{ij}v_j^{(0)}\,, \end{aligned} \tag{4.48}$$

where the index was raised using the inverse of the perturbed metric (4.42). This is just the local heat current on the horizon given in (4.18) and hence we have demonstrated that to leading order $Q^i_{QFT}(x) = Q^i_{BH}(x)$ as stated in (4.20).

In the previous sub-section we discussed how certain sub-leading terms in the ϵ expansion are required to cancel divergences at the horizon. In appendix C.1 we show that these lead to the following sub-leading contribution to the heat currents:

$$Q^i_{QFT} = Q^i_{BH} + \frac{1}{16\pi G_N}r_H^{D-3}\left(\epsilon\, r_H\right)^{-2}\sqrt{h}\nabla_k\nabla^{[k}v^{i]}_{(0)}\,. \tag{4.49}$$

Observe that the right hand side is order ϵ^0 because of the two spatial derivatives and hence is lower order. Also notice that the right hand side of (4.49) is a total derivative (a magnetisation current, in fact) and hence this result is clearly consistent with the universal result of [97] that the total heat current flux of the field theory is always the same as the total heat current flux on the boundary, $\bar{Q}^i_{BH} = \bar{Q}^i_{QFT}$.

5 Comparison with fluid-gravity approach

In this section we would like to compare our linearised expansion of the response to a DC source with the fluid-gravity approach discussed in [129].

We begin by recalling that our regular solution is given by

$$\begin{aligned} ds^2 &= r_H^2\,\rho^2\left(-u\,dt^2+h_{ij}\,dx^i dx^j\right)+\frac{d\rho^2}{\rho^2 u}-2r_H^2\rho^2 u\,t\,\left(\zeta_i-(4\pi T)^{-1}\,\partial_i p\right)\,dx^i dt \\ &-2\rho^{3-D}\left(V_j+V_j^{(2)}\right)\,dx^j dt+2\,\rho^2\frac{\ln u}{4\pi T}\,\nabla_{(i}V_{j)}\,dx^i dx^j \\ &+2\rho^2\,\left(V_j dt+t\,\nabla_{(i}V_{j)}\,dx^i\right)\,dx^j\,, \end{aligned} \tag{4.50}$$

where we have explicitly added the one sub-leading term $V_j^{(2)}(\rho,x)$ which we have seen is required to obtain a regular solution at the black hole horizon at leading order in the expansion. Recall that we have imposed $\nabla_i V^i=0$.

We next carry out the coordinate transformations

$$\begin{aligned} &t\to v-\,R, \qquad R=-\int_\rho^\infty\frac{d\rho}{r_H\rho^2 u}\,, \\ &x^i\to x^i-r_H^{-2}\,v\,V^i\,, \end{aligned} \tag{4.51}$$

to obtain the metric

$$\begin{aligned} ds^2 =& r_H^2\rho^2\,\left(-dv^2-2v\,\left(\zeta_j-(4\pi T)^{-1}\,\partial_j p\right)dx^j dv+h_{ij}\,dx^i dx^j\right) \\ &-2r_H d\rho\,\left[-dv+\left(r_H^{-2}V_j-v\,\left(\zeta_j-(4\pi T)^{-1}\,\partial_j p\right)\right)\,dx^j\right] \\ &+\rho^{-D+3}\,r_H^2 dv\,\left[dv-2\left(r_H^{-2}V_j-v\,\left(\zeta_j-(4\pi T)^{-1}\,\partial_j p\right)\right)\,dx^j\right] \\ &+2r_H\rho^2 F(\rho)\left(r_H^{-2}\nabla_{(i}V_{j)}\right)\,dx^i dx^j \\ &+2r_H^2\rho^2 uR\,\left[\left(\zeta_j-(4\pi T)^{-1}\,\partial_j p\right)-\frac{\rho^{1-D}}{r_H^2 uR}V_j^{(2)}\right]dx^j\left(dv-\frac{d\rho}{r_H\rho^2 u}\right)\,, \end{aligned} \tag{4.52}$$

where

$$F(\rho)\equiv r_H\left(\frac{\ln u}{4\pi T}-R\right)\,. \tag{4.53}$$

We note that this function F is the same function defined below equation (4.1) of [129].

We now perform another coordinate transformation

$$x^i\to x^i-h^{ij}\int_\rho^\infty d\rho\left[\frac{R}{r_H\rho^2}\left(\zeta_j-(4\pi T)^{-1}\,\partial_j p\right)-\frac{\rho^{-1-D}}{r_H^3 u}V_j^{(2)}\right]\,, \tag{4.54}$$

to finally write our metric in the form

$$\begin{aligned} ds^2 =& r_H^2\rho^2\left(-dv^2 - 2v\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right)dx^j dv + h_{ij}\,dx^i dx^j\right) \\ &- 2r_H\,d\rho\left[-dv + \left(r_H^{-2}V_j - v\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right)\right)dx^j\right] \\ &+ \rho^{-D+3}\,r_H^2 dv\left[dv - 2\left(r_H^{-2}V_j - v\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right)\right)dx^j\right] \\ &+ 2r_H\rho^2 F(\rho)\left(r_H^{-2}\nabla_{(i}V_{j)}\right)dx^i dx^j \\ &+ 2r_H^2\rho^2 uR\left[\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right) - \frac{\rho^{1-D}}{r_H^2 uR}V_j^{(2)}\right]dx^j dv\,, \end{aligned} \tag{4.55}$$

where here we have dropped covariant derivatives of $\zeta, V^{(2)}$ and second derivatives of p as these would be even higher order corrections. Notice that as we approach the horizon at $\rho = 1$ there is a cancellation in the last line after using (4.40) as well as going on-shell by imposing the Stokes equations (4.41).

We now consider the formalism of [129]. The basic idea is construct an expansion for the D-dimensional metric using a boundary $(D-1)$-vector u^μ combined with a radial coordinate r. An expansion for the boundary stress tensor, $T^{\mu\nu}$, is constructed in terms of u^μ and the boundary metric, $h_{\mu\nu}$, and then it is shown that the Ward identity $\nabla_\mu T^{\mu\nu} = 0$ implies that the bulk Einstein equations are solved order by order in the expansion. An elegant feature of the analysis in [129] is the use of a Weyl covariant formalism.

Prompted by (4.55), let us consider the boundary metric to be given by

$$h_{\mu\nu}dx^\mu dx^\nu = -dv^2 - 2v\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right)dx^j dv + h_{ij}\,dx^i dx^j\,, \tag{4.56}$$

and we note that we should identify $h_{\mu\nu}$ with $g_{\mu\nu}$ in the notation of [129]. With some foresight we choose the $(D-1)$-dimensional boundary fluid vector u^μ to have components

$$u_v = -1\,, \qquad u_j = -v\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right) + \delta u_j\,. \tag{4.57}$$

It is now straightforward to calculate the components of the Weyl gauge-field A_μ, defined in eq. (2.2) of [129], and we find, to the order we are working to,

$$A_v = \frac{1}{D-2}\nabla_i\delta u^i\,, \qquad A_j = -\left(\zeta_j - (4\pi T)^{-1}\partial_j p\right)\,. \tag{4.58}$$

We can calculate the Weyl covariant derivative of u^μ, as defined in eq. (2.3) of [129] and after writing $\mathcal{D}_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu}$, we obtain the the symmetric shear strain tensor $\sigma_{\mu\nu}$ and the antisymmetric vorticity tensor $\omega_{\mu\nu}$. The components of $\sigma_{\mu\nu}$ are given by

$$\sigma_{vv} = 0, \qquad \sigma_{vi} = 0, \qquad \sigma_{ij} = \nabla_{(i}\delta u_{j)} - \frac{h_{ij}}{D-2}\nabla_k\delta u^k\,, \tag{4.59}$$

where the covariant derivative on δu, here and below, is with respect to h_{ij}. The only non-vanishing component of the vorticity tensor is given by $\omega_{ij} = \nabla_{[i}\delta u_{j]}$, but it does not contribute to the hydrodynamics at the order we are working with.

We can now calculate the stress tensor as given eq. (4.6) of [129]. Again to the order we are working with we have

$$T_{\mu\nu} = \frac{1}{16\pi G_N} p(h_{\mu\nu} + (D-1)u_\mu u_\nu) - 2\eta\sigma_{\mu\nu}\,, \tag{4.60}$$

where $p = r_H^{D-1}$ and constant[9] shear viscosity $\eta = r_H^{D-2}$. Explicitly we obtain the components

$$\begin{aligned} 16\pi G_N T_{vv} &= r_H^{D-1}(D-2)\,, \\ 16\pi G_N T_{vi} &= r_H^{D-1}[(D-2)v\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right) - (D-1)\delta u_i]\,, \\ 16\pi G_N T_{ij} &= r_H^{D-1} h_{ij} - 2r_H^{D-2}\left(\nabla_{(i}\delta u_{j)} - \frac{h_{ij}}{D-2}\nabla_k \delta u^k\right)\,. \end{aligned} \tag{4.61}$$

The Ward identity $\nabla_\mu T^{\mu\nu} = 0$ now gives, at leading order, both the incompressibility condition and the linearised Navier-Stokes equations:

$$\nabla_i \delta u^i = 0, \qquad -2\,\nabla^i\nabla_{(i}\,\delta u_{\,j)} = 4\pi T\zeta_j - \partial_j p\,. \tag{4.62}$$

Given this data, we can now obtain the bulk 'fluid-gravity' metric given by eq. (4.1) of [129]. To the order we are considering, and after writing

$$\delta u_i = r_H^{-2} V_i\,, \tag{4.63}$$

we find that it takes the form

$$\begin{aligned} ds^2 =& r_H^2\rho^2\,\left(-dv^2 - 2v\,\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right) dx^j dv + h_{ij}\,dx^i dx^j\right) \\ & - 2r_H\,d\rho\,\left[-dv + \left(r_H^{-2}V_j - v\,\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right)\right)\,dx^j\right] \\ & + \rho^{-D+3} r_H^2 dv\,\left[dv - 2\left(r_H^{-2}V_j - v\,\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right)\right)\,dx^j\right] \\ & + 2r_H\rho^2\,F(\rho) r_H^{-2}\,\nabla_{(i}V_{j)}\,dx^i dx^j \\ & - 2r_H\rho\,\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right) dx^j dv. \end{aligned} \tag{4.64}$$

By comparing (4.64) with (4.55) we find precise agreements in the first four lines. The difference in the last line will be accounted for, on shell, by contributions coming from

[9] The formalism of [129] is Weyl covariant. In particular, we can make a Weyl transformation of the boundary metric (4.56) to introduce a non-vanishing h_{tt}. Under this Weyl transformation the stress tensor (4.61) transforms with a Weyl factor and it may seem that the shear viscosity becomes spatially dependent. However, we will still be led to exactly the same Navier-Stokes equations with constant shear viscosity. This is consistent with the earlier discussion of the Weyl invariance of the black hole horizon metric given in (4.9).

higher order terms[10] in the expansion.

To close this section we present an equivalent way of describing the DC linear response from the fluid-gravity perspective. Consider, carrying out the simple coordinate transformation on the boundary metric (4.56), $v \to (1-\psi)v$ with ψ a locally defined function satisfying

$$d\psi = \left(\zeta_i - (4\pi T)^{-1}\,\partial_i p\right)\,dx^i\,. \tag{4.65}$$

In the new coordinates the boundary metric takes the form

$$h_{\mu\nu}dx^\mu dx^\nu = -(1-2\psi)dv^2 + h_{ij}\,dx^i dx^j\,, \tag{4.66}$$

which is another way to introduce the DC thermal gradient. The fluid velocity is then given by the more intuitive expressions

$$u_v = -(1-\psi)\,, \qquad u_j = \delta u_j\,, \tag{4.67}$$

while the stress tensor takes the form

$$\begin{aligned}
16\pi G_N T_{vv} &= r_H^{D-1}(D-2)(1-2\psi),\\
16\pi G_N T_{vi} &= -r_H^{D-1}(D-1)\delta u_i\,,\\
16\pi G_N T_{ij} &= r_H^{D-1}h_{ij} - 2r_H^{D-2}\left(\nabla_{(i}\delta u_{j)} - \frac{h_{ij}}{D-2}\nabla_i\delta u^i\right)\,.
\end{aligned} \tag{4.68}$$

Imposing the Ward identity $\nabla_\mu T^{\mu\nu} = 0$ again gives the incompressibility condition and the linearised Navier-Stokes equations (4.62). This fluid can then be used to construct the bulk metric, in the hydrodynamic limit, using the formulae in [129]. It should be noted, however, that in these coordinates the bulk metric is constructed from the local function ψ and the regularity of the solution is not immediately transparent.

It is interesting to point out that if we take the boundary metric (4.66), but allow ψ to also depend on time, we can still take the fluid velocity as in (4.67) and the stress tensor is still given as in (4.68). The Ward identities still imply the incompressibility condition $\nabla_i\delta u^i = 0$ but now we obtain the linearised Navier-Stokes equation

$$4\pi T\,\partial_t\delta u_i - 2\nabla^j\nabla_{(j}\delta u_{i)} + \partial_i p = 4\pi T\zeta_i\,. \tag{4.69}$$

By employing the scaling discussed at the end of section 3.2, we deduce that this equation can be used to consider time-dependence that is associated with frequencies $\omega \sim \epsilon k \sim \epsilon^2 T$.

[10]For example, there will be contributions coming from the last term in the third line of eq. (4.1) of [129], proportional to $u_{(\mu}P_{\nu)}^{\lambda}\mathcal{D}_\alpha\sigma^\alpha{}_\lambda$ that are proportional to $\nabla^k\nabla_{(k}V_{i)}dvdx^i$ and hence proportional to $\left(\zeta_j - (4\pi T)^{-1}\,\partial_j p\right)dx^j dv$ on-shell.

This indicates that the poles of the current-current correlator will be order ϵ^2 for fixed T, on the negative imaginary axis in the complex ω plane and hence, combining with (4.21) we infer that as $\epsilon \to 0$ we have weak momentum dissipation and an associated Drude peak.

6 Discussion

In this chapter, we have discussed the DC response of holographic lattices in theories of pure gravity, in a hydrodynamic limit. We have shown that by solving the linearised, covariantised Navier-Stokes equations for an incompressible fluid one can extract out the local heat currents of the dual field theory as well as determining the leading order correction. In addition we also determined the full local stress tensor response at leading order. For simplicity we only considered static holographic lattice black hole geometries. However, our results can be extended to the stationary case, corresponding to having local momentum current deformations in the dual CFT, using the results of [98]. In particular, by solving the generalised Navier-Stokes equations of [98] with specific Coriolis terms, one can extract the local transport heat currents. The effects of the Coriolis term could give a sharp diagnostic to determine the presence of such magnetisation currents in real systems.

We have focussed on holographic lattices that are periodic in non-compact spatial dimensions. In particular, in the limit that $\epsilon \to 0$ we obtain the AdS-Schwarzschild black brane geometry. Our analysis can easily be adapted to lattices that are associated with the AdS-Schwarzschild black holes with hyperbolic horizons. An interesting feature is that by taking suitable quotients of the hyperbolic space one can study DC thermal conductivities on higher genus Riemann surfaces. By contrast one cannot use AdS-Schwarzschild black holes with spherical horizons in the same way because of the absence of one-cycles to set up a DC source (recall that the DC source was parametrised by a closed one-form ζ).

For holographic lattices we have emphasised that the hydrodynamic limit is, in general, distinct from the perturbative lattices analysed in chapter 2, and [1, 97]. The hydrodynamic limit in this chapter corresponds to an expansion in k/T (or $k/s^{1/(D-2)}$), where k is the largest wave number of the UV deformation, while for the perturbative lattice one expands in a dimensionless parameter λ associated with the amplitude of the UV deformation. Despite the differences in the limits, there are some similarities. In this chapter, by demanding regularity of the perturbation at the black hole horizon we saw that at leading order $V_i = \epsilon^{-2} v_i^{(0)}$ and $p = \epsilon^{-1} p^{(0)}$, while in [1, 97] there was an analogous expansion with ϵ replaced with λ. In both cases the DC conductivity is parametrically large and there is weak momentum dissipation. However, in the case of perturbative lattices the leading order solution to the Stokes equations is spatially homogeneous (i.e. constant) on the torus, while in the hydrodynamic limit, the leading order solution generically has a non-trivial local structure.

We have also discussed how our results are related to the fluid-gravity approach and in particular the general results of [129] who examined CFTs on boundary manifolds with arbitrary metrics. We find that the results are the same, as one would expect. This therefore provides a good sanity check for the results of the previous chapters in the hydrodynamic limit.

We saw that using the fluid-gravity formalism one can rather easily obtain the result that the local heat current of the dual field theory can be obtained by solving Stokes equations. However, it should be pointed out that this approach obscures the fact that the boundary heat current is equal to the heat current at the horizon, as we have shown to be the case in section 3. Moreover, we note that extracting the universal result for holographic lattices of [1, 97, 98] concerning Navier-Stokes equations on the black hole horizon, as an exact statement in holography, is highly non-trivial in the fluid-gravity expansion. In essence this is because the Ward identities are imposed at the AdS boundary in the fluid-gravity approach, while to get the Navier-Stokes equations the constraints should be imposed on the black hole horizon. In effect, to obtain the result of [1, 97, 98], one would need to sum up an infinite expansion from the fluid-gravity point of view.

We also note that there is not an existing general fluid-gravity formalism that can be deployed for studying the hydrodynamic limit of holographic lattices in more general theories of gravity coupled to various matter fields. By contrast it is rather clear how some results of this paper can be generalised. For example, suppose we have a theory of gravity coupled to a massless scalar field with no potential in the bulk. Such a scalar is dual to an exactly marginal operator in the dual CFT. If we consider holographic lattices with UV metric deformations, $h_{ij}(x)$, as well as UV scalar deformations, $\phi(x)$, then the metric on the horizon will be $r_H^2 h_{ij}(x)$, as we discussed in this paper, while the scalar field on the horizon will be given by the UV function $\phi(x)$. We can now obtain the local heat currents on the horizon, and hence for the dual field theory in the hydrodynamic limit, by solving the generalised Navier-Stokes equations on the horizon with the scalar viscous terms derived in [1]:

$$-2\,\nabla^i\nabla_{(i}v_{j)} + (v^i\nabla_i\phi)\nabla_j\phi = 4\pi T\zeta_j - \partial_j p\,, \qquad \nabla_i v^i = 0\,, \tag{4.70}$$

and metric $r_H^2 h_{ij}$.

Consider now a theory of gravity coupled to a scalar field that is dual to a relevant operator with dimension $\Delta < D-1$. At the AdS boundary the scalar field will behave as $\phi \sim \phi_s\bar{\phi}(x)r^{\Delta-D+1}$ where ϕ_s is a dimensionful source amplitude and $\bar{\phi}(x)$ is a dimensionless function. By taking the hydrodynamic limit we again find a horizon metric $r_H^2 h_{ij}$. By analysing the radial behaviour of the scalar (as in section 3.3 of [66]), we can determine that at the black hole horizon the scalar field will be given by $c\phi_s T^{\Delta-D+1}\bar{\phi}(x)$, where c is a numerical constant. We can then study the DC response using (4.70).

It is interesting to point out that if we consider the high temperature limit with both $\epsilon \ll 1$ and $\lambda \equiv \phi_s T^{\Delta-D+1}\bar{\phi}(x) \ll 1$, then providing that there is also momentum dissipation arising from a spatially dependent metric, it is clear from (4.70) that the scalar field will not play a role at leading order. However, for holographic lattices in which the UV boundary metric deformations are trivial, $h_{ij} = \delta_{ij}$, but the relevant scalar field deformations are non-trivial, the origin of the momentum dissipation comes purely from the scalar field and hence to obtain the leading order heat currents one will need to solve the Navier-Stokes equations with the scalar viscous terms. In the high temperature limit the scalar viscous terms are small, so we are now in the domain of perturbative lattices associated with weak momentum dissipation, and we can obtain the leading DC conductivity in terms of the scalar field on the horizon using the results of [1, 97].

Finally, we can also consider the addition of gauge fields. In this case we can take the hydrodynamic limit of holographic lattices at finite charge density, by just demanding $k/s^{1/(D-2)} \ll 1$ while holding fixed $\mu/s^{1/(D-2)}$. One should then study the charged Navier-Stokes equations of [1, 97].

Chapter 5

Thermal backflow in CFTs

1 Introduction

A wide variety of strongly correlated states of matter are expected to display collective behaviour described by viscous hydrodynamics. This occurs on time scales when the momentum preserving self interactions of the strongly coupled matter dominate over momentum dissipating processes such as the scattering with phonons. For some further discussion, including some experimental realisations in graphene and other materials, we refer to [131, 132, 135–145]. It is has recently been emphasised that, for matter at finite charge density, a directly verifiable macroscopic signature of viscous flows is provided by the phenomenon of electric current backflow [140, 141]. That is, for suitable set-ups the application of an external electric field leads to a fluid flow that produces an electric current which flows, locally, in the opposite direction to the applied field.

Here we want to discuss thermal backflow. In this case a local heat current flows in the same direction to that of an applied external temperature gradient and in principle can occur in the absence of charge carriers. While electric backflow can be caused both by viscous effects and by spatially modulated regions of charge density ("charge puddles"), thermal backflow would be caused purely by viscous effects of the fluid. For matter at finite charge density, both are special cases of the more general phenomenon of thermoelectric current backflow.

In this chapter we initiate a study of thermoelectric current backflow for relativistic quantum field theories, focussing on conformal field theories (CFTs). More specifically, we will investigate the possibility of thermal backflow by applying an external DC thermal gradient to CFTs at finite temperature and vanishing charge density. We then calculate the local currents that are produced at the level of linear response by solving leading order viscous hydrodynamic equations. We are interested in studying this phenomenon for infinite systems. Thus, in order to get a finite DC response, we will need a set-up in which the total momentum is a not a conserved quantity, or phrased differently, momentum dissipates in the bulk of the CFT. This should be contrasted with other setups where a finite DC response arises because one imposes no-slip or other momentum dissipating boundary conditions on the electronic fluid in a finite volume, as in some of the discussion in [140, 141], for example. A natural way to achieve this is to consider CFTs in Minkowski spacetime that are then deformed by marginal or relevant operators that explicitly break the translation invariance of the CFT. Interestingly, this is precisely the set-up that has received much attention in the AdS/CFT correspondence via the construction of black holes called "holographic lattices" [37, 50, 52, 57, 62, 125].

Here we will focus on the universal class of deformations that arise from placing the CFT on a curved geometry with spacetime metric $g_{\mu\nu}(x)$. We assume that the metric is time independent, *i.e.* it has a timelike Killing vector ∂_t, corresponding to a CFT in local thermal equilibrium. The metric $g_{\mu\nu}(x)$ can also be viewed as parametrising spatially dependent sources for the stress tensor of the CFT. These deformations include applying strains, thermal gradients as well as sources for local rotations to a CFT in flat spacetime, for example. In thinking of potential applications to real materials we can envisage applying such deformations to a plasma that has arisen from some underlying collective behaviour. For example, we note that there has been extensive work on studying the behaviour of strained graphene, *e.g.* [146–148] and it is also worth highlighting the exceptional thermal conductivity properties of graphene [149].

We will study the linear response of the deformed CFTs at vanishing charge density after applying an external thermal gradient source, possibly time dependent, in the hydrodynamic limit, $\epsilon = k/T << 1$, where k is the largest wavenumber associated with the deformations. For the special case of CFTs with holographic duals, we showed in previous chapters that there is a universal connection between thermal DC conductivity and Navier-Stokes equations on black hole horizons for holographic lattices. Furthermore, we have just seen that in the hydrodynamic limit, the local heat current that is produced by a thermal source can be obtained by solving a system of linearised, forced Navier-Stokes equations on a curved manifold fixed by the metric $g_{\mu\nu}$. In this chapter we will show that this result is much more general, applying also to general CFTs without holographic duals. We will also show how it also arises for non-conformally invariant relativistic field theories.

To illustrate thermal backflow for a DC source we will study static metrics with spatial sections that are conformally flat, with the conformal factor a periodic function of the spatial coordinates. This corresponds to applying an isotropic periodic strain to the CFT. After applying a Weyl transformation it also corresponds to deforming by a spatially modulated energy distribution, or equivalently a spatially modulated local temperature variation. For suitably chosen conformal factors, by solving the time-independent Navier-Stokes equations numerically, we are able to find explicit examples that do indeed exhibit thermal backflow for this setup. We emphasise that this thermal backflow arises at the level of the linear response to the application of an external DC thermal gradient, and is thus associated with specific two point functions of the stress tensor in the strained CFT. Moreover, the backflow is due to the spatial inhomogeneities of the metric on which the CFT lives and this should be contrasted with fluid backflow in ordinary fluids at the level of linear response, that is caused by momentum dissipating processes at boundaries. The key point in this chapter is not that fluids exhibit backflow (which has been demonstrated by considering no-slip or other momentum dissipating boundary conditions - see e.g [150]), but rather that inhomogeneities in the spatial metric can lead to backflow at the level of linear response.

We will focus on CFTs in the chapter because their hydrodynamic description depends on fewer parameters. However, much of our analysis can be straightforwardly generalised to arbitrary relativistic quantum field theories and we present some details in appendix D.1. It is interesting that for static metric backgrounds with the time-like Killing vector having constant norm, we also find that the response to a thermal source, possibly time dependent, is again governed by linearised Navier-Stokes equations. For non-constant norm, we obtain more general equations.

2 Thermal transport for CFTs in the hydrodynamic limit

We consider general CFTs on curved manifolds in $d \geq 2$ spacetime dimensions with metric $g_{\mu\nu}$. Using the general results of [151] (see also [152]), we will derive the leading order viscous hydrodynamic equations relevant for studying thermal transport after applying an external thermal gradient source, possibly time dependent, at the level of linear response.

For a general CFT we must impose the Ward identities

$$D_\mu T^{\mu\nu} = 0\,, \qquad T^\mu{}_\mu = 0\,. \tag{5.1}$$

When d is even we have set the conformal anomaly to zero as it will be higher order in the derivative expansion than we wish to consider. In order to obtain a closed set of hydrodynamical equations we need constitutive relations for the stress tensor. We let T denote the local temperature and introduce the fluid velocity u^μ, satisfying $u^\mu u_\mu = -1$. Both T and u^μ can depend on all of the spacetime coordinates, x^μ. Including the leading order viscous terms we have

$$T_{\mu\nu} = P(g_{\mu\nu} + d\, u_\mu u_\nu) - 2\eta\sigma_{\mu\nu}\,, \tag{5.2}$$

where the shear tensor is given by

$$\sigma_{\mu\nu} = D_{(\mu}u_{\nu)} + u_{(\mu}u^\rho D_\rho u_{\nu)} - (g_{\mu\nu} + u_\mu u_\nu)\frac{D_\rho u^\rho}{d-1}\,. \tag{5.3}$$

Conformal invariance fixes the equation of state to be $P = c_0\, T^d$ and the viscosity to be $\eta = c_1\, T^{d-1}$, where c_0 and c_1 are dimensionless numbers fixed by the CFT[1].

Notice that the equations are covariant under Weyl transformations, in which the metric and fluid velocity transform as $g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$, $u_\mu \to e^{\omega} u_\mu$, where ω is an arbitrary function of spacetime coordinates, while the scalars T, P, η transform as $T \to e^{-\omega} T$, $P \to e^{-d\,\omega} P$ and $\eta \to e^{(-d+1)\omega}\eta$. We also notice that $u^\mu T_{\mu\nu} = -(d-1)P u_\nu = -\varepsilon\, u_\nu$, where ε is the energy density and we also have $\varepsilon + P = sT$.

Introducing a time coordinate via $x^\mu = (t, x^i)$, then the heat current density, or equivalently, momentum current density, of the CFT is given by the components

$$Q^i = -\sqrt{-g}\, T^i{}_t\,. \tag{5.4}$$

Notice that Q^i is invariant under Weyl transformations. Also, in stationary spacetimes, for which ∂_t is a Killing vector, we deduce that this current is conserved $\partial_i Q^i = 0$.

To simplify the presentation, we now consider the background metric to be static with line element given by $ds^2 = -g_{tt} dt^2 + g_{ij} dx^i dx^j$, and $\partial_t g_{tt} = \partial_t g_{ij} = 0$. This corresponds to studying the CFT in thermal equilibrium, with g_{tt} and g_{ij} parametrising sources for the stress tensor components, T^{tt} and T^{ij}, respectively. It will be convenient to set $g_{tt} = 1$ and consider the background metric

$$ds^2 = -dt^2 + g_{ij}(x^k) dx^i dx^j\,, \tag{5.5}$$

since a non-vanishing g_{tt} can be reinstated by simply performing a Weyl transformation. We next consider the spatial metric $ds^2 = g_{ij}(x^k) dx^i dx^j$ as a harmonic expansion about some fiducial metric. If k is the largest wave-number in this expansion, then the hydrodynamic limit has $\epsilon = k/T << 1$. A concrete example, and one we will focus on, is to take the fiducial metric to be flat space and consider g_{ij} to be periodic in the spatial directions. In this case, focussing on a fundamental domain, g_{ij} also defines a curved metric on a torus.

We now consider perturbing the CFT by an external thermal gradient source parametrised by a closed one form, $\zeta = \zeta_\mu dx^\mu$. To study the linear response of the CFT to this source, similar to chapter 2, we consider the following linearised perturbation about the equilibrium configuration. For the metric we take[2]

$$ds^2 = -(1-2\phi)\, dt^2 + g_{ij}(x) dx^i dx^j\,, \tag{5.6}$$

where $\zeta_\mu = \partial_\mu \phi$. We now highlight an important aspect of the choice of ζ and ϕ. To

[1]In holography we have $c_0 = \frac{4\pi}{d} c_1$.

[2]Employing the coordinate transformation $t = (1+\phi)\bar{t}$ implies that the linearised perturbed metric is given by $ds^2 = -d\bar{t}^2 + g_{ij}(x) dx^i dx^j - 2\bar{t}\zeta_\mu dx^\mu d\bar{t}$, which has been used in related contexts [97].

illustrate, we focus on the planar case with $g_{ij}(x)$ periodic in the spatial directions. In this case we can write $\zeta = \bar{\zeta}_i(t)dx^i + dz(t,x)$, or $\phi = \bar{\zeta}_i(t)x^i + z(t,x)$, where $z(t,x)$ are periodic functions of the x^i. The $\bar{\zeta}_i$ parametrise the thermal source of most interest. For example, for the DC case, the choice $\phi(x) = z(x)$ would just correspond to considering the CFT on a deformed metric still in thermal equilibrium (we return to this at the end of the section). On the other hand $\phi = \bar{\zeta}_i x^i$, with constant $\bar{\zeta}_i$ corresponds to a constant external thermal gradient source, of strength $\bar{\zeta}_i$, in the x^i direction[3].

We consider the perturbed fluid velocity to be

$$u_t = -(1-\phi), \qquad u_j = \delta u_j\,. \tag{5.7}$$

We vary the local temperature via $T = T_0 + \delta T$, where T_0 is the equilibrium temperature of the CFT. Note that ϕ, δu_i and δT all depend on (t, x^i); in the planar case they are taken to be periodic functions of the x^i. If ω is a characteristic frequency then we should demand that $\omega/T_0 << 1$ in addition to $k/T_0 << 1$, in order to stay in the hydrodynamic limit.

After substituting into (5.2) we find that the stress tensor takes the form

$$\begin{aligned}
T_{tt} &= c_0\,(d-1)\,T_0^d(1-2\phi) + c_0 d\,(d-1)\,T_0^{d-1}\,\delta T,\\
T_{ti} &= -c_0 d\,T_0^d\,\delta u_i\,,\\
T_{ij} &= c_0 T_0^d\,g_{ij} + c_0\,d\,T_0^{d-1}\,\delta T\,g_{ij} - 2c_1 T_0^{d-1}\left(\nabla_{(i}\delta u_{j)} - \frac{g_{ij}}{d-1}\nabla_k\delta u^k\right)\,,
\end{aligned} \tag{5.8}$$

where here, and below, the covariant derivative ∇ is now with respect to g_{ij}. The Ward identities (5.1) then give the following linearised, forced Navier-Stokes equations for δu_i and δT:

$$\begin{aligned}
T_0\,\partial_t\delta u_i - 2\frac{c_1}{d\,c_0}\left(\nabla^j\nabla_{(j}\delta u_{i)} - \frac{1}{d-1}\,\nabla_i\,\nabla_j\delta u^j\right) + \nabla_i\delta T = T_0\zeta_i\,,\\
(d-1)T_0^{-1}\partial_t\delta T + \nabla_i\delta u^i = 0\,.
\end{aligned} \tag{5.9}$$

Furthermore, the heat current (5.4) now reads

$$Q^i = c_0 d T_0^d\sqrt{g}\,\delta u^i = T_0 s_0\sqrt{g}\,\delta u^i\,. \tag{5.10}$$

The system of equations (5.9) is the key result of this section. Observe that they only depend on the one-parameter of the CFT, $c_1/(dc_0)$, which is just η_0/s_0. We also note that ζ_t does not enter these equations. When we set all time derivatives to zero, which is

[3] Note that $\phi(x) = z(x)$ is globally defined and bounded both on the plane and on the torus (i.e. associated with a fundamental domain of the background). On the other hand $\phi = \bar{\zeta}_i x^i$ is globally defined on the plane, but not bounded, and is not a well defined function on the torus. Furthermore, the one-forms $dz(x)$ and $\bar{\zeta}_i dx^i$ are cohomologically trivial and non-trivial on the torus, respectively.

appropriate for studying thermal DC response, we have an incompressible fluid $\nabla_i \delta u^i = 0$. We will refer to the time-independent equations as Stokes equations.

We conclude this section with a few general comments. We first make some observations about conserved currents for general relativistic field theories satisfying the Ward identity $D_\mu T^{\mu\nu} = 0$ on curved manifolds, setting to zero the thermal sources (i.e. $\phi = 0$). Contracting with an arbitrary vector k^μ, we obtain

$$D_\mu \left(T^\mu{}_\nu k^\nu\right) = \tfrac{1}{2}\mathcal{L}_k g_{\mu\nu} T^{\mu\nu} \,, \tag{5.11}$$

where $\mathcal{L}$ is the Lie derivative. We immediately see that if k is a Killing vector then $T^\mu{}_\nu k^\nu$ is a conserved current. For a CFT this is also true if k is a conformal Killing vector, satisfying $\mathcal{L}_k g_{\mu\nu} \propto g_{\mu\nu}$. Thus, in order to have momentum dissipation in the spatial directions, we should only consider background metrics without (conformal) Killing vectors, apart from ∂_t. Equivalently, for a CFT, the metric should not be related by a Weyl transformation to a metric with additional Killing vectors. If we let $k = \partial_i$ and assume that it is not a (conformal) Killing vector, then there is no conserved momentum in the x^i direction. In this case, if we consider perturbing around thermal equilibrium, (5.11) might be viewed as saying that momentum is being dissipated by the non-vanishing of $\partial_i g_{\mu\nu} \delta T^{\mu\nu}$. This can be contrasted with the work of [114] who, instead, modify the Ward identities in order to achieve momentum dissipation.

We now consider a stationary metric $g_{\mu\nu}$ and assume that k^μ is a Killing vector (or conformal Killing vector if we have a CFT), in addition to ∂_t. After considering a DC perturbation (5.6), with all time derivatives vanishing, from the Ward identity we deduce that

$$\frac{1}{\sqrt{g}}\partial_i \left(\sqrt{g} T^i{}_\mu k^\mu\right) = -(k^i \zeta_i) T^t{}_t \,. \tag{5.12}$$

After integrating over the spatial directions, the left hand side vanishes[4] and hence so does the right hand side. Thus, we have deduced, just from the Ward identity (i.e. independent of the constitutive relations), that if there are any (conformal) Killing vectors over and above ∂_t, then the DC response is not well defined in the direction $k^i \zeta_i$. More physically, there will be a delta function at zero frequency in the AC response.

In studying DC response for background metrics as in (5.5), we are thus only interested in spatial metrics $g_{ij} dx^i dx^j$ without Killing vectors. The solutions to the Stokes equations (*i.e.* (5.9) with $\partial_t = 0$) are then unique [1,97] up to an undetermined constant, the zero-mode of δT. Physically, this zero mode can be fixed by demanding that when $\zeta_i = \delta u_i = 0$ the full stress tensor of the CFT is not modified. In any event, this zero mode does not affect the local heat current response given in (5.10).

[4]With appropriate boundary conditions imposed for non-compact spaces.

The final comment relates to the closed one form source ζ in the DC context. For the periodic, planar case we again write $\zeta = \bar{\zeta}_i dx^i + dz(x)$, where $\bar{\zeta}_i dx^i$, with constant $\bar{\zeta}_i$, parametrise the DC thermal source of most interest, and $z(x)$ is an arbitrary periodic function which can be dealt with exactly. Indeed as noted in [97], and section 3.5 in chapter 2, if $\zeta = dz(x)$, associated with $\phi = z(x)$, we can solve the Stokes equations with $\delta u_i = 0$ and $\delta T = T_0 z$, giving rise to a simple response to the full stress tensor with no heat flow. Note that we cannot take the solution $\delta u_i = 0$ and $\delta T = T_0\phi$ when $\phi(x) = \bar{\zeta}_i x^i$ since we have demanded that δu_i and δT are periodic functions[5].

3 Thermal backflow

We now consider specific background static metric deformations of the form (5.5), parametrised by $g_{ij}(x^k)$, that lead to thermal backflows driven by external DC thermal gradients, in the hydrodynamic limit. We will assume that we have a planar spatial topology with g_{ij} a periodic function of the spatial coordinates. For a given g_{ij} we want to numerically solve the Stokes equations (*i.e.* (5.9) with $\partial_t = 0$), effectively on a torus, and then obtain the local heat current density, $Q^i(x)$, at leading order in k/T, using (5.10).

For simplicity we will assume that the deformation is periodic in each of the spatial directions with the same period, $L \equiv 2\pi/k$, with $\epsilon = k/T << 1$. For the numerics we eliminate the dimensionful quantity L by defining new coordinates via $x^i = L\hat{x}^i$ with the $\hat{x}^i$ having unit period. It is convenient to introduce dimensionless variables via[6]

$$v_i = \delta u_i, \qquad p = \frac{d\, c_0}{c_1}\, L\delta T, \qquad \hat{\zeta}_i = \frac{d\, c_0}{c_1}\, L^2 T_0\, \zeta_i\,. \tag{5.13}$$

Then in the hatted coordinates the linear Stokes equations coming from (5.9) take the dimensionless form

$$-2\,\nabla^i \nabla_{(i} v_{j)} = \hat{\zeta}_j - \partial_j p\,, \qquad \nabla_i v^i = 0\,, \tag{5.14}$$

where here we are raising indices with respect to the metric g_{ij} and ∇ is the associated covariant derivative. In the new variables it is natural to define the heat current density

$$\hat{Q}^i \equiv \sqrt{g} g^{ij} v_j = \frac{1}{c_0 d T_0^d} Q^i\,. \tag{5.15}$$

Writing $Q^i = T_0 \kappa^{ij} \zeta_j$, where κ is the thermal conductivity matrix, then we have $\hat{Q}^i = (c_1/c_0 d)(T_0\kappa^{ij}/s_0)\epsilon^2 \zeta_j$, where s_0 is the entropy density. This displays the fact that $(T_0\kappa^{ij}/s_0)$

[5] Note that an alternative approach would have been to allow non-periodic perturbations δT instead of non-periodic functions $\phi(x)$.

[6] Note that p, here should not be confused with the pressure, P, of the background CFT appearing in (5.2).

is of order ϵ^{-2} as we pointed out in the last chapter.

To illustrate examples of backflow, we now restrict to CFTs with metric deformations given by[7]

$$g_{ij} = \Phi\delta_{ij}, \qquad \Phi > 0\,. \tag{5.16}$$

By solving the Stokes equations (5.14) numerically, we find that various choices of Φ lead to thermal backflow. To be specific we discuss the special case of CFTs in two spatial dimensions and set $d = 3$. We present some results for the specific choice

$$\Phi = \alpha + \frac{\beta}{N}\sum_{a,b=-N}^{N} e^{i\,2\pi\,(a\,(\hat{x}-1/2)+b\,(\hat{y}-1/2))}\,. \tag{5.17}$$

Moreover, we restrict to the specific case of $N = 2$ and consider varying α and β. We have plotted Φ for the specific case of $\alpha = 0.98$ and $\beta = 0.3$ in figure 5. We apply a

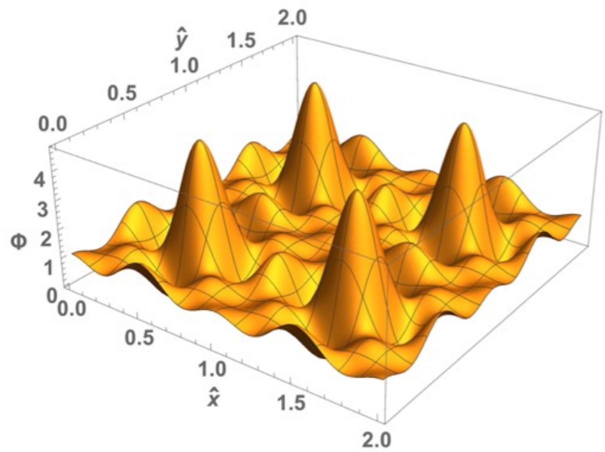


Figure 5: A plot of the function Φ which determines the static metric deformation of the CFT with $g_{tt} = 1$ and $g_{ij} = \Phi\delta_{ij}$. Note that we have plotted twice the period in both spatial directions. This specific choice of Φ is as in (5.17) with $\alpha = 0.98$ and $\beta = 0.3$ and gives rise to thermal backflow as shown in figure 7.

constant DC thermal gradient just in the $\hat{x}$ direction with $\hat{\zeta} = d\hat{x}$. For various choices of α, β we then numerically solve the Stokes equations (5.14), as described in appendix D.2, to extract $\hat{Q}^i(\hat{x})$ and $p(\hat{x})$.

For small values of $1-\alpha$ and β, we are not only in the hydrodynamic limit, we are also in the perturbative limit that is associated with small amplitudes that we considered in section 4.3 in chapter 2. In this limit, at leading order in a perturbative expansion in the amplitude of the metric deformation around flat spacetime, the solutions to the Stokes equations are homogenous, *i.e.* constant [1, 97]. In figure 6 we have plotted the

[7]Note that for this choice of metric, (5.11) with $k = \partial_i$ gives $\nabla_\mu \delta T^\mu{}_i = -\frac{1}{2}(\partial_i \ln\Phi)\delta T^t{}_t$, revealing the origin of momentum non-conservation in this setting.

solutions to the Stokes equations for $\alpha = 1$ and $\beta = 6.6 \times 10^{-4}$. As expected we find nearly homogeneous flows. There are various ways of quantifying this: for example the approximate range of components of the current are $\hat{Q}^1 \in (4565, 4595)$ and $\hat{Q}^2 \in (-8.498, 8.498)$. The background colour in figure 6 depicts the norm of the vector field. The maximum value of the norm (red) has components $(4595, < 10^{-6})$ while the minimum (purple) has components[8] $(4565, < 10^{-6})$. To compare with the perturbative lattice analysis of [1, 97] we let the perturbative parameter, λ, be equal to the difference between the maximum and the minimum values of Φ within one period and we find $\lambda = 0.01$. From the above data we see that, roughly, $\hat{Q}^1$ scales like $\lambda^{-2}/2$ while p scales like $2\,\lambda^{-1}$. In any event, there is no thermal backflow for these lattices.

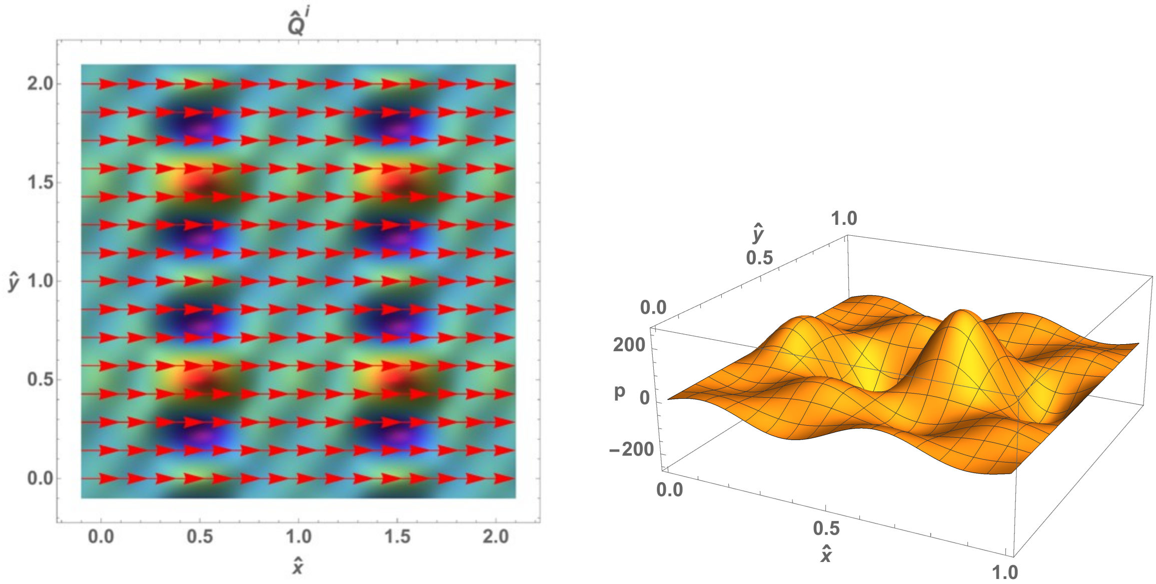


Figure 6: Plot of $\hat{Q}^i$ and p corresponding to the metric deformation Φ as in (5.17) with $\alpha = 1$ and $\beta = 6.6 \times 10^{-4}$ and thermal gradient just in the $\hat{x}$ direction given by $\hat{\zeta} = d\hat{x}$. The left plot displays the vector heat current density $\hat{Q}^i$, for twice the period in both spatial directions. The background colour depicts the norm of the vector, $(\hat{Q}^i \hat{Q}^j \delta_{ij})^{1/2}$, and we note that it is nearly uniform with the maximum and minimum varying by about 1%. The right plot shows p for a single period in the spatial directions. The variation in p is uniform enough to lead to a roughly homogenous response.

By increasing the overall amplitude, by varying α, β, we find that solving the Stokes equations gives rise to sharper peaks in p, which are associated with larger internal fluid forces. We find that for amplitude fixed by $\alpha = 0.98$ and $\beta = 0.3$, that thermal backflow does indeed occur as shown in figure 7. In particular, we see that there is a distinct region of thermal backflow with $\hat{y} \sim 0.5$ and $0.8 < \hat{x} < 1.2$

Finally, it is worth revisiting the original assumptions concerning our hydrodynamic expansion with $\epsilon << 1$. Recall that throughout this paper we have been assuming the constitutive relation given in (5.2). This will receive corrections at higher order in ϵ and

[8]The second component, in both cases, converges to zero within our numerical accuracy.

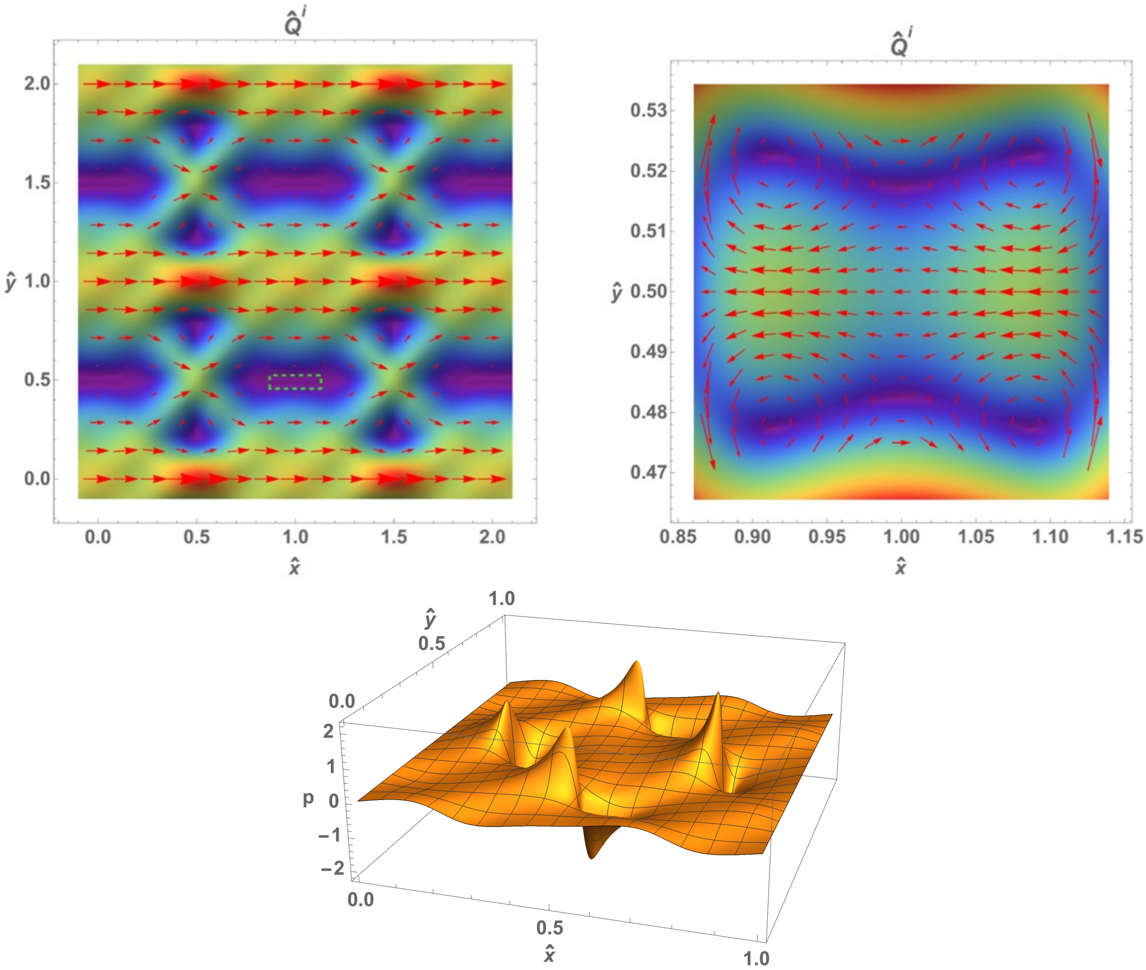


Figure 7: Thermal backflow corresponding to the metric deformation in figure 5, with Φ as in (5.17) with $\alpha = 0.98$, $\beta = 0.3$, and thermal gradient just in the $\hat{x}$ direction given by $\hat{\zeta} = d\hat{x}$. The upper plots display the vector heat current density $\hat{Q}^i$, with the background colour emphasising the norm of the vector $(\hat{Q}^i\hat{Q}^j\delta_{ij})^{1/2}$. The upper left plot shows $\hat{Q}^i$ for twice the period in both spatial directions. Thermal backflow occurs in the elongated purple regions: the upper right plot is an enlargement of the green dashed rectangle. The bottom plot displays p for a single period in the spatial directions.

will include terms involving the curvature of the background metric. For the specific example with $\alpha = 0.98$ and $\beta = 0.3$ we can estimate that the next order curvature contributions will be of the order ϵ^2 times $\Phi^{-1}\nabla^2 \ln \Phi$. Since the latter has spikes of the order 10^5, in order to ensure that these terms are indeed sub-leading we should impose not just $\epsilon << 1$ but the stricter bound $\epsilon << 10^{-3}$. It would be interesting to determine by how much this can be weakened for other examples exhibiting backflow.

4 Discussion

By solving a system of Stokes equations we have shown that thermal backflow driven by an applied external DC thermal source is possible for CFTs in the leading order viscous hydrodynamic limit. We explicitly demonstrated this for CFTs defined on static spacetime metrics with a conformally flat spatial metric, with the conformal factor depending periodically on the spatial coordinates. We did not have to make any assumption concerning the strength of the viscosity η in (5.2); we only demanded that it is non-zero. The thermal backflow occurs at the level of linear response, and is associated with specific two point functions of the stress tensor in the CFT. The thermal backflow solutions are steady state solutions to the linearised equations. If one was interested in going beyond linear response, then one would have to take into account Joule heating and there would not be such steady state solutions. It would be interesting to understand the time scale for when the linearised approximation breaks down.

We have discussed in section 2 how thermal transport properties of CFTs are invariant under Weyl transformations. This means, for example, that since backflow occurs if suitable isotropic strains are applied to a CFT, associated with a conformally flat metric $\Phi dx^i dx^i$, then we should also see exactly the same backflow by applying a periodic local temperature profile parametrised by Φ^{-1} with a flat spatial metric $dx^i dx^i$. Thus, if one were able to experimentally engineer such isotropic strains and local temperature profiles for some strongly coupled matter and one found the same thermal response, this would provide a sharp diagnostic that the matter was described by a conformal field theory in the hydrodynamic limit. Perhaps it is possible to investigate this with graphene, which is known to be described as a relativistic fluid at the Dirac point.

For general CFTs it is straightforward to generalise our analysis from static to stationary metrics. This corresponds to allowing for deformations of the CFT which have sources for local rotations in thermal equilibrium as discussed in [98]. The linear response to applying a thermal source can then be examined in the leading order viscous hydrodynamic limit by studying Navier-Stokes equations that contain Coriolis terms which are determined by the non-vanishing vorticity tensor of the background fluid in thermal equilibrium. In the case of DC thermal sources the relevant time independent Stokes equations were given in [98]. In general it is necessary to focus on the transport currents, which are obtained by subtracting off certain magnetisation currents that depend on the applied thermal source [98, 114, 120, 153].

In this chapter we have discussed the DC response of general CFTs in the leading order viscous hydrodynamic limit, by solving a system of Stokes equations. For the special class of CFTs that have holographic duals we can also study DC response for deformed CFTs far from the hydrodynamic limit, by analysing suitable black hole solutions. Over the course of this thesis, we have seen that, somewhat remarkably, the total thermoelectric current fluxes, and hence the thermoelectric DC conductivities, can be obtained by solving the same system of Stokes equations for an auxiliary fluid on the horizon of the black holes [1, 97, 98]. We also explained the connection between this result and the hydrodynamic limit. Another interesting direction would be to use holography to examine what happens to the backflow as a function of $\epsilon = k/T$.

We can also generalise the analysis in this paper to CFTs that have additional conserved currents. From the work on our previous holographic calculations in chapter 4 we can conclude that we will need to solve the Stokes equations presented in chapter 2. There is a range of possibilities to examine, including the role of charge puddles and magnetic fields, and we aim to report on some of this soon.

We have also presented the equations needed to be solved to examine the thermal response for a general relativistic quantum field theory in appendix D.1. For the special case of DC response, for background spacetimes in which the norm of the timelike Killing vector is constant, the relevant equations are, up to constants, the same Stokes equations that need to be solved for the case of CFTs. In particular, the examples of thermal backflow that we showed in section 3 are applicable to a much more general class of quantum field theories. When the norm of the Killing vector is not constant, the equations that need to be solved are given in (D.9),(D.10) and it would be interesting to explore them in more detail.

Part III

Phase transitions

Chapter 6

A new phase for the anisotropic N=4 super Yang-Mills plasma

1 Introduction

Having studied linear response in some detail, we now turn our attentions to a different topic, the phase transitions of strongly coupled systems that have been deformed by spatially dependent sources. Such systems can be studied by constructing novel black hole solutions using the AdS/CFT correspondence (e.g. [37, 50–55, 57–64, 66, 101, 102, 104, 106, 115, 118, 120, 125, 154]). These studies are interesting for a number of reasons - as well as finite DC response, the spatially dependent sources provide a useful tool to search for novel holographic ground states which can appear in the far IR; insulators, coherent metals and incoherent metals have been realised in this way, as well as transitions between them [37, 57, 58, 104, 118]. A more specific motivation derives from the properties of the quark gluon plasma observed in heavy ion collisions. In particular, the plasma appears to have regimes where it is described by a strongly coupled and spatially anisotropic fluid [155, 156].

An interesting framework for analysing spatial anisotropy in N=4 super Yang-Mills (SYM) theory was initiated in [59] and then further developed in [60, 61]. Specifically, black hole solutions of type IIB supergravity were constructed in [60, 61] that asymptotically approach AdS_5 at the UV boundary with the type IIB axion having a linear dependence on one of the three spatial coordinates. The linear axion source is associated with a distribution of D7-branes that intersect D3-branes in two of the spatial directions and is smeared in the third. At low temperatures these black holes approach a $T = 0$ solution, constructed in [59], which becomes a Lifshitz-like scaling solution in the far IR.

It is natural to interpret this scaling solution as the $T = 0$ ground state of the anisotropically deformed N=4 SYM theory. Here, however, we will show that the black holes of [60, 61] are unstable at low temperatures and there is a phase transition which spontaneously breaks the global $SO(6)$ symmetry down to $SO(4) \times SO(2)$. The origin of this instability was already noticed in [59]. In particular, by analysing the Kaluza-Klein spectrum of the five-sphere it was found that there are scalar modes, transforming in the $\mathbf{20}'$ of $SO(6)$, which saturate the BF bound in AdS_5 background but violate an analogous bound in the Lifshitz-like background. This suggests that the Lifshitz-like scaling solution is unstable. It is natural to suspect that such an instability is also present for the $T = 0$ solution that interpolates between AdS_5 in the UV and the Lifshitz-like solution in the IR. By continuity one then expects that the finite temperature black hole solutions should become unstable at some finite temperature.

Here we will show that these expectations are realised. We find that the black holes become unstable at some critical temperature T_c with a new branch of black hole solutions appearing. Above the critical temperature, these solutions seem to be a non-physical branch of "exotic hairy black holes" [86, 157]. Specifically, it seems that the black holes at temperatures $T \geq T_c$ and are never thermodynamically preferred. At temperatures $T \leq T_c$ the new solution is thermodynamically preferred. The phase transition is continuous and, somewhat surprisingly, third order. Furthermore, we calculate the critical exponents of the phase transition finding $(\alpha, \beta, \gamma, \delta) = (-1, 1, 1, 2)$ rather than the standard mean-field values $(\alpha, \beta, \gamma, \delta) = (0, 1/2, 1, 3)$ associated with most holographic phase transitions. The critical exponents we find have been previously realised in bottom-up models of holographic superconductors with a Lagrangian containing cubic terms in the modulus of the complex scalar field [74, 75, 158, 159]. While such terms are a bit unnatural for a complex scalar field they are natural for a neutral scalar field, provided the potential does not have any discrete symmetry. The critical exponents that we find can, in a certain sense, be realised by a Landau-Ginzburg (LG) model with a scalar order parameter and cubic term in the free energy. While such cubic terms in LG models are associated with first order phase transitions, our holographic transition appears to be continuous and third order.

We construct our new solutions using a consistent KK truncation of type IIB supergravity on S^5 that keeps the $D = 5$ metric coupled to the axion and dilaton, as in [60, 61], and in addition keeps an extra single neutral scalar field. Any solution of the $D = 5$ theory gives rise to an exact solution of type IIB supergravity. The potential for the neutral scalar field in the $D = 5$ theory does not have any discrete symmetry and this is associated with the non-standard values of the critical exponents that we just discussed. The fact that the phase transition spontaneously breaks $SO(6)$ to $SO(4) \times SO(2)$ is only apparent after uplifting to type IIB.

We construct the new branch of black hole solutions down to low temperatures and elucidate the $T = 0$ behaviour. Similar to [37] and unlike many holographic studies, the IR part of the geometry at $T = 0$ does not approach a scaling solution of the equations of motion, but instead approaches the leading terms of an expansion, which eventually approaches AdS_5 in the UV. The leading terms of this IR expansion are similar to the hyperscaling violation solutions [160–162] but with anisotropic scaling in the spatial direction rather than the time direction. We calculate the thermal conductivity of the black holes, essentially importing the results of [66]. We show that the scaling behaviour implies that at low temperatures the system is a thermal insulator with $\kappa \sim T^{10/3}$.

The remainder of the chapter is organised as follows. In section 2, we present the $D = 5$ top-down model and show how it is arises from a KK reduction of type IIB supergravity on S^5. The construction of the black holes and a study of their properties is contained in section 3. We conclude in section 4 and we have two appendices. Appendix E contains some technical results concerning a Smarr formula, and discusses how the critical exponents that we find can be extracted, in a certain sense, from a Landau-Ginzburg type analysis.

2 The top-down model

We will consider a $D = 5$ gravity theory coupled to three scalar fields, the axion and dilaton, ϕ and χ, as in [60, 61], and an additional scalar X. The bulk action is given by:

$$S =\frac{1}{16\pi G_N}\int d^5x\sqrt{-g}\left(R-3X^{-2}(\partial X)^2+4(X^2+2X^{-1})-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}e^{2\phi}(\partial\chi)^2\right), \tag{6.1}$$

where for simplicity of presentation we have set the AdS radius to unity. The corresponding equations of motion are given by:

$$\begin{aligned}
\nabla^2\phi &= e^{2\phi}(\partial\chi)^2\,,\\
\nabla_\mu\left(e^{2\phi}\nabla^\mu\chi\right) &= 0\,,\\
\nabla_\mu\left(X^{-1}\nabla^\mu X\right) &= -\frac{4}{3}(X^2-X^{-1})\,,\\
R_{\mu\nu} &= 3X^{-2}\partial_\mu X\partial_\nu X-\frac{4}{3}(X^2+2X^{-1})g_{\mu\nu}+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi\,.
\end{aligned} \tag{6.2}$$

This top-down model arises as a consistent truncation of the Kaluza-Klein (KK) reduction of type IIB supergravity on a five-sphere. That is, any solution to the equations of motion (6.2) gives rise to an exact solution of type IIB supergravity with $D = 10$ metric and self-dual five-form given by:

$$\begin{aligned}
ds^2_{10} &= \bar\Delta^{1/2}ds^2_5+X\bar\Delta^{1/2}d\xi^2+X^2\bar\Delta^{-1/2}\sin^2\xi d\tau^2+\bar\Delta^{-1/2}X^{-1}\cos^2\xi d\Omega_3\,,\\
F_{(5)} &= 2U\mathrm{vol}_5+3\sin\xi\cos\xi X^{-1}*_5 dX\wedge d\xi\\
&\quad+\bar\Delta^{-2}\sin\xi\cos^3\xi(2Ud\xi-3\sin\xi\cos\xi X^{-2}dX)\wedge d\tau\wedge\mathrm{vol}_3\,,
\end{aligned} \tag{6.3}$$

where ds^2_5 and vol_5 are the $D = 5$ metric and volume form, respectively, $d\Omega_3$ and vol_3 are the metric and volume form on a round three-sphere, respectively, and

$$\bar{\Delta} = X^{-2}\sin^2\xi + X\cos^2\xi, \qquad U = X^2\cos^2\xi + X^{-1}\sin^2\xi + X^{-1}\,. \tag{6.4}$$

The $D = 10$ dilaton and axion are the same as the $D = 5$ scalar fields ϕ and χ, respectively, and the $D = 10$ three-forms are both zero. When $X \neq 0$ this class of $D = 10$ metric and five-form has the $SO(6)$ symmetry of the round five-sphere reduced to $SO(4)\times SO(2)$, with the first factor acting on the round S^3 and the second acting on the circle parametrised by τ.

That this is a consistent truncation can be established using the results of [163]. Indeed it was shown in [163] that there is a consistent KK truncation of type IIB supergravity on a five-sphere to Romans $D = 5$ $SU(2)\times U(1)$ gauged supergravity, whose bosonic fields consist of a $D = 5$ metric, a scalar X, $SU(2)\times U(1)$ gauge-fields and two two-forms. This truncation can simply be extended to include the $D = 10$ axion and dilaton and we can then truncate away the gauge-fields and the two-form to obtain our model.

Notice that the unit radius AdS_5 vacuum solution to the equations of motion (6.2) has $\chi = 0$, $X = 1$ with constant ϕ, and uplifts to the standard $AdS_5 \times S^5$ solution of type IIB. Around this vacuum solution, perturbations of the fields ϕ, χ are massless and are associated with marginal operators in $N = 4$ SYM theory with scaling dimension $\Delta = 4$. Perturbations of X have $m^2 = -4$, which saturates the BF bound, and is associated with an operator $\mathcal{O}_\psi$ with dimension $\Delta = 2$. This operator is part of a multiplet, transforming in the $\mathbf{20'}$ of $SO(6)$ which is dual to operators in N=4 SYM constructed from the six adjoint scalar fields ϕ^I of the form $Tr(\phi^I\phi^J) - trace$. When $\mathcal{O}_\psi$ acquires an expectation value spontaneously, as it will in our solutions, it breaks the $SO(6)$ global R-symmetry down to $SO(4)\times SO(2)$.

Notice that setting $X = 1$, which is a further consistent truncation, we have $\bar{\Delta} = 1$, $U = 2$, from (6.4), and we recover the $D = 5$ model that was studied in [60, 61]. In particular the metric on the five-sphere in (6.3) becomes the round metric. Thus, our model extends the top-down model studied in [60, 61] to include one extra scalar field, X, which saturates the BF bound. It is sometimes convenient to consider a canonically normalised scalar field, ψ, instead of X, defined by

$$X \equiv e^{-\psi/\sqrt{6}}\,, \tag{6.5}$$

in terms of which the action reads

$$S = \frac{1}{16\pi G_N}\int d^5x\sqrt{-g}\left(R - \frac{1}{2}(\partial\psi)^2 + 4(e^{-2\psi/\sqrt{6}} + 2e^{\psi/\sqrt{6}}) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right)\,. \tag{6.6}$$

2.1 Brief review of previous work

The anisotropic solutions constructed in [59–61], with the axion linear in one of the spatial coordinates, all lie within the ansatz

$$\begin{aligned} ds^2 &= \frac{e^{-\frac{1}{2}\phi}}{u^2}\left(-\mathcal{F}\mathcal{B}dt^2 + \frac{du^2}{\mathcal{F}} + dx^2 + dy^2 + \mathcal{H}dz^2\right), \\ \chi &= az, \qquad \phi = \phi(u), \end{aligned} \tag{6.7}$$

with trivial X-field, $X = 1$ (i.e. $\psi = 0$ in (6.5)). The functions $\mathcal{F}, \mathcal{B}$ are functions of the radial coordinate u, and the function $\mathcal{H}$ is taken to be $\mathcal{H} = e^{-\phi}$ which, remarkably, can be imposed consistent with the equations of motion.

The black hole solutions constructed numerically in [60,61] approach in the UV, located at $u \to 0$, a unit radius AdS_5 with a linear axion deformation. As $T \to 0$ the black hole solutions approach a $T = 0$ domain wall solution whose IR limit approaches a fixed point solution with

$$\mathcal{F} = \frac{49}{36}\left(\frac{12}{11}\right)^{\frac{6}{5}} u^{\frac{2}{7}}, \qquad \mathcal{B} = \mathcal{F}^{-1}, \qquad e^{\phi} = \left(\frac{11}{12}\right)^{\frac{2}{5}} u^{-\frac{4}{7}}, \tag{6.8}$$

which was first found in [59]. After switching to a new radial coordinate $u = r^{-7/6}$, this solution can be written in the form

$$\begin{aligned} ds^2 &= L^2\left(\frac{dr^2}{r^2} + r^2(-d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2) + r^{4/3}d\bar{z}^2\right), \\ \chi &= \bar{a}\bar{z}, \qquad e^{\phi} = L^{4/5}r^{2/3}, \end{aligned} \tag{6.9}$$

with $X = 1$, where the bars denote quantities that have been rescaled, and $L^2 = 11/12$. This metric is manifestly invariant under the anisotropic Lifshitz-like scaling $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, r) \to (\lambda\bar{t}, \lambda\bar{x}, \lambda\bar{y}, \lambda^{2/3}\bar{z}, \lambda^{-1}r)$.

Following [59] (and in an analagous way to the derivation of (1.16)), we can study the properties of a massive scalar field, satisfying $\nabla^2\sigma = m^2\sigma$, in the background (6.9). By considering solutions of the form $r^{\Delta_\pm}$ and demanding that $\Delta_\pm$ are real, we deduce that $m^2 \geq -11/3$. Since we would like to identify $\Delta_\pm$ as scaling dimensions in a putative field theory dual to these Lifshitz-like solutions, this suggests that the anisotropic solution (6.9) will be unstable under perturbations by any massive field with $m^2 < -11/3$. If the Lifshitz solution is unstable we expect that the $T = 0$ domain wall solution itself will be unstable and hence, by continuity, that the finite temperature black hole solutions will be unstable up to some critical temperature T_c.

Thus, since the scalar field X in our model has $m^2 = -4$, we anticipate that the black hole solutions of [60, 61] will still describe the high temperature phase of the system but will become unstable at T_c leading to a phase transition. The critical temperature can be found by establishing the existence of a suitable zero-mode in the linearised fluctuations of the X-field about the numerically constructed black holes of [60, 61]. We carried out this analysis but we will omit the details. Instead we will focus on the construction of the new branch of fully back reacted black holes that emerge at $T = T_c$ and examine some of the physical properties of the new low-temperature phase.

3 Construction of new anisotropic black holes

3.1 Ansatz and equations of motion

We extend the ansatz of [59–61] by allowing for a non-trivial X-field and consider

$$\begin{aligned} ds^2 &= \frac{e^{-\frac{1}{2}\phi}}{u^2}\left(-\mathcal{F}\mathcal{B}dt^2 + dx^2 + dy^2 + \mathcal{H}dz^2 + \frac{du^2}{\mathcal{F}}\right), \\ \chi &= az, \qquad \phi = \phi(u), \qquad X = X(u), \end{aligned} \tag{6.10}$$

where $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{H}$ are functions of u. The function $\mathcal{H}$ is associated with the anisotropy in the z direction that is sourced by the axion field. By combining the equation of motion for the dilaton with the Einstein equations, as in [59–61], we find that it is possible to choose the function $\mathcal{H}$ to be related to the dilaton via

$$\mathcal{H} = e^{-\phi}, \tag{6.11}$$

and we will do so in the sequel[1].

We now discuss the resulting equations of motion for this ansatz, following the approach of [61]. The equation for the axion χ in (6.2) is trivially satisfied. The X equation of motion implies that

$$\begin{aligned} &12u^2\mathcal{F}XX'' + 3\Big(-5u^2X\mathcal{F}\phi' + 2u^2X\mathcal{F}\frac{\mathcal{B}'}{\mathcal{B}} + 4u^2X\mathcal{F}' - 12uX\mathcal{F}\Big)X' \\ &- 12u^2\mathcal{F}\left(X'\right)^2 + 16e^{-\phi/2}X(X^3-1) = 0, \end{aligned} \tag{6.12}$$

while the ϕ equation of motion gives

$$4u\mathcal{F}\phi'' + 2u\mathcal{F}\frac{\mathcal{B}'}{\mathcal{B}}\phi' + 4u\mathcal{F}'\phi' - 5u\mathcal{F}\left(\phi'\right)^2 - 12\mathcal{F}\phi' - 4a^2ue^{3\phi} = 0. \tag{6.13}$$

There are also four independent components of the Einstein equations arising from (6.2). By taking a suitable combination of one of these equations with the ϕ equation of motion, in order to eliminate $\mathcal{B}'$ terms, we can arrive at an equation which can be algebraically

[1]Our preliminary investigations into relaxing this condition did not reveal any other solutions of physical interest.

solved for $\mathcal{F}$:

$$\mathcal{F} = \frac{e^{-\frac{1}{2}\phi}}{4(\phi' + u\phi'')}\left(e^{\frac{7}{2}\phi}a^2(4u + u^2\phi') + 16\phi'\right) + \frac{4e^{-\frac{1}{2}\phi}(1-X)^2(2+X)\phi'}{3X(\phi' + u\phi'')}\,. \tag{6.14}$$

Next, by taking a suitable combination of two of the remaining three Einstein equations, in order to eliminate $\mathcal{F}''$ terms, we arrive at an equation that we can solve for $\mathcal{B}'/\mathcal{B}$:

$$\frac{\mathcal{B}'}{\mathcal{B}} = \frac{(24\phi' - 9u\phi'^2 + 20u\phi'')}{24 + 10u\phi'} - \frac{24uX'^2}{X^2\,(12 + 5u\phi')}\,, \tag{6.15}$$

and we observe that only the combination $\mathcal{B}'/\mathcal{B}$ appears in (6.12) and (6.13). It is now possible to show that (6.12), (6.13), (6.14) and (6.15) imply that all of the Einstein equations are solved. To see this we can use (6.13) to solve for $\mathcal{F}'$ in terms of ϕ and X and their derivatives, after using (6.14) and (6.15). Furthermore, comparing this equation with the expression for $\mathcal{F}'$ that can be obtained by differentiating (6.14), we obtain a third order equation for ϕ, which can be used instead of (6.13). One can then check that the remaining Einstein equations are satisfied. Observe that if we set $X = 1$ we recover the equations of motion given in [61].

In summary, the equations of motion are equivalent to (6.12)-(6.15) and are, effectively, second order in X, third order in ϕ and first order in $\mathcal{B}$, with $\mathcal{F}$ algebraically specified by ϕ, X and their first and second derivatives. Thus, a solution is specified by six integration constants.

We note that the ansatz and hence the equations of motion are invariant under the following two scaling symmetries

$$\begin{aligned} &u \to \lambda u, \quad (t,x,y,z) \to \lambda(t,x,y,z), \quad a \to \lambda^{-1}a; \\ &t \to \lambda t, \quad \mathcal{B} \to \lambda^{-1/2}\mathcal{B}; \end{aligned} \tag{6.16}$$

where λ is a constant.

3.2 The UV and IR expansions

We now discuss the boundary conditions that we will impose on (6.12)-(6.15). In the UV, as $u \to 0$, we demand that the asymptotic behaviour is given by

$$\begin{aligned}
\phi &= -\frac{a^2u^2}{4} + u^4\frac{(121a^4 + 1152\mathcal{B}_4 + 2304(X_2)^2)}{4032} - u^4 \log u \frac{a^4}{6} + \dots\,, \\
\mathcal{F} &= 1 + \frac{11a^2u^2}{24} + u^4\mathcal{F}_4 + u^4 \log u \frac{7a^4}{12} + \dots\,, \\
\mathcal{B} &= 1 - \frac{11a^2u^2}{24} + u^4\mathcal{B}_4 - u^4 \log u \frac{7a^4}{12} + \dots\,, \\
X &= 1 + u^2X_2 - u^4\frac{5a^2X_2}{24} + \dots\,.
\end{aligned} \tag{6.17}$$

The solutions are asymptotically approaching AdS_5 with an anisotropic deformation of the axion field in the z-direction with strength a. This UV expansion is specified by four parameters, $\mathcal{F}_4, \mathcal{B}_4, X_2$, whose physical interpretation will be discussed below, and a. It is important to observe that we have set a possible $u^2 \log u$ term in the expansion of X to zero, as this would correspond to sourcing the operator dual to X which we don't want[2]. We next note that the second scaling symmetry in (6.16) has been used to set the leading term in $\mathcal{B}$ to unity. We also observe that associated with the first scaling symmetry in (6.16) the UV expansion is preserved under the transformations $u \to \lambda u$ and

$$\begin{aligned}
\mathcal{B}_4 &\to \lambda^{-4}\mathcal{B}_4 + \frac{7}{12}a^4\lambda^{-4}\log\lambda\,, \\
\mathcal{F}_4 &\to \lambda^{-4}\mathcal{F}_4 - \frac{7}{12}a^4\lambda^{-4}\log\lambda\,, \\
X_2 &\to \lambda^{-2}X_2\,.
\end{aligned} \tag{6.18}$$

The presence of the log terms is associated with the fact that the linear axion deformation gives rise to a non-vanishing conformal anomaly, as discussed in [60, 61].

In the IR, we will assume that we have a regular black hole horizon located at $u = u_h$.

[2]In section 3.5, when we calculate critical exponents of the phase transition, we will briefly consider black holes with such a source for X.

We therefore will demand that as $u \to u_h$ we have

$$
\begin{aligned}
\phi &= \phi_h - \frac{12 u_h X_h a^2 e^{\frac{7\phi_h}{2}}}{32 + 3a^2 e^{\frac{7\phi_h}{2}} u_h^2 X_h + 16 X_h^3} (u - u_h) + \ldots\,, \\
\mathcal{F} &= \mathcal{F}_h\,(u - u_h) + \ldots\,, \\
\mathcal{B} &= \mathcal{B}_h + \frac{2\mathcal{B}_h (45 a^4 e^{7\phi_h} u_h^4 X_h^2 - 256 (X_h^3 - 1)^2 - 96 a^2 e^{\frac{7\phi_h}{2}} u_h^2 X_h (2 + X_h^3))}{u_h \left(32 + 3a^2 e^{\frac{7\phi_h}{2}} u_h^2 X_h + 16 X_h^3\right)^2} (u - u_h) + \ldots\,, \\
X &= X_h + \frac{16 X_h (X_h^3 - 1)}{u_h \left(32 + 3a^2 e^{\frac{7\phi_h}{2}} u_h^2 X_h + 16 X_h^3\right)} (u - u_h) + \ldots\,.
\end{aligned} \tag{6.19}
$$

This IR expansion is specified by four parameters, $\phi_h, \mathcal{B}_h, X_h$ and u_h, with $\mathcal{F}_h$ fixed via

$$
\mathcal{F}_h \equiv \mathcal{F}'(u_h) = -\frac{e^{-\frac{\phi_h}{2}} \left(32 + 3a^2 e^{\frac{7\phi_h}{2}} u_h^2 X_h + 16 X_h^3\right)}{12 u_h X_h}\,. \tag{6.20}
$$

We have noted that the equations of motion are specified by six integration constants. We have eight parameters appearing in the asymptotic expansion minus one for the remaining scaling symmetry in (6.16). We thus expect to find a one-parameter family of solutions which can be parametrised by the quantity T/a. We note that the presence of the conformal anomaly introduces an additional dynamical scale which we hold fixed to be unity throughout our discussion.

3.3 Stress tensor and thermodynamics

To calculate the free-energy and the stress tensor, we need to supplement the bulk action with boundary counter terms (e.g. [32]). We write

$$
S_{total} = S_{bulk} + S_{ct}\,, \tag{6.21}
$$

where S_{bulk} is the bulk action given in (6.1) (or (6.6)) and, for the configurations of interest, we can take [164, 165]

$$
S_{ct} = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-\gamma} \left(2K - 6 + \frac{1}{4} e^{2\phi} \partial_i \chi \partial^i \chi - \psi^2 \left(1 + \frac{1}{2\log u}\right)\right) + \log u \int \mathrm{d}^4x \sqrt{-\gamma}\mathcal{A} \tag{6.22}
$$

where $\partial^i = \gamma^{ij}\partial_j$ and $\mathcal{A}$ is the conformal anomaly in the axion-dilaton-gravity system given by [165]

$$
\mathcal{A} = \frac{1}{96\pi G_N} e^{4\phi} |\partial\chi|^4\,. \tag{6.23}
$$

Note that here we have expressed the X scalar field in terms of the canonically normalised scalar ψ defined by $X = e^{-\frac{\psi}{\sqrt{6}}}$, which we will continue to use throughout this section. We note that the $1/\log u$ term is only relevant for solutions where the X-field is sourced, which are only briefly discussed in section 3.5.

The expectation value of the stress energy tensor is obtained by taking the functional derivative of the total action with respect to the boundary metric [166, 167]

$$
\begin{aligned}
16\pi G_N T^{ij} = \lim_{u\to 0}\Bigg(& -2K^{ij} + \gamma^{ij}\left(2K - 6 + \frac{1}{4}e^{2\phi}\partial_i\chi\partial^i\chi - \psi^2(1+\frac{1}{2\log u})\right) \\
& - \frac{1}{2}e^{2\phi}\partial^i\chi\partial^j\chi + \log u\left(\mathcal{A}\gamma^{ij} - \frac{2}{3}e^{4\phi}\partial^i\chi\partial^j\chi(\partial\chi)^2\right)\Bigg), \qquad (6.24)
\end{aligned}
$$

where K_{ij} is the extrinsic curvature of a $u =$ constant hypersurface, and $\partial^i = \gamma^{ij}\partial_j$. Using the boundary expansion of the fields in the previous section, we find the expectation value of the stress-energy tensor has the following non-vanishing components:

$$
\begin{aligned}
T^{tt} &= \frac{1}{16\pi G_N}\left(-3\mathcal{F}_4 - \frac{23}{7}\mathcal{B}_4 + \frac{2945}{4032}a^4 - \frac{4}{7}(X_2)^2\right), \\
T^{xx} = T^{yy} &= \frac{1}{16\pi G_N}\left(-\mathcal{F}_4 - \frac{5}{7}\mathcal{B}_4 + \frac{443}{4032}a^4 + \frac{4}{7}(X_2)^2\right), \\
T^{zz} &= \frac{1}{16\pi G_N}\left(-\mathcal{F}_4 - \frac{13}{7}\mathcal{B}_4 + \frac{2731}{4032}a^4 - \frac{12}{7}(X_2)^2\right). \qquad (6.25)
\end{aligned}
$$

This result is consistent with [61] when $X_2 \to 0$.

Similarly, we can calculate the one-point functions of the theory in order to find the vacuum expectation of the fields. For the scalar fields we find that expectation values and sources are given by

$$
\begin{aligned}
\langle\mathcal{O}_\chi\rangle &= 0, & \chi_{(0)} &= az\,, \\
\langle\mathcal{O}_\phi\rangle &= \frac{1}{16\pi G_N}\left(-\frac{143}{252}a^4 + \frac{8}{7}(\mathcal{B}_4 + 2X_2^2)\right), & \phi_{(0)} &= 0\,, \\
\langle\mathcal{O}_\psi\rangle &= -\frac{\sqrt{6}X_2}{16\pi G_N}, & \psi_{(0)} &= 0\,. \qquad (6.26)
\end{aligned}
$$

We can now easily check that the Ward identities for the theory are satisfied. Firstly, diffeomorphism invariance gives us the conservation of the stress-energy tensor

$$
\nabla^i T_{ij} + \langle\mathcal{O}_\phi\rangle\nabla_j\phi_{(0)} + \langle\mathcal{O}_\chi\rangle\nabla_j\chi_{(0)} + \langle\mathcal{O}_\psi\rangle\nabla_j\psi_{(0)} = 0\,, \qquad (6.27)
$$

which in our case is simply $\nabla^i T_{ij} = 0$, and is trivially satisfied. Similarly, the invariance of the theory under Weyl transformations leads to the conformal Ward anomaly

$$T_i^i = -(4 - \Delta_\psi)\psi_{(0)}\langle\mathcal{O}_\psi\rangle + \mathcal{A}\,, \tag{6.28}$$

where $\mathcal{A}$ is the conformal anomaly. From (6.25) we have $T_i^i = a^4/6$ and hence

$$\mathcal{A} = \frac{a^4}{96G_N}\,, \tag{6.29}$$

in agreement with a direct calculation of (6.23).

By analytically continuing the time coordinate via $t = -i\tau$ and demanding regularity of the metric at $u = u_h$, we find that the Hawking temperature of the black holes is given by

$$T = \frac{\mathcal{B}_h^{1/2}|\mathcal{F}_h|}{4\pi}\,. \tag{6.30}$$

The entropy density of the black holes, s, can be obtained from the area of the black hole horizon, giving

$$s = \frac{e^{-\frac{5}{4}\phi_h}}{4G_N u_h^3}. \tag{6.31}$$

We can calculate the free-energy density, w, by calculating the total on-shell Euclidean action via $w\text{vol}_3 = TI_{total}|_{os}$. In fact, using the results of [168] we can immediately obtain

$$w = E - Ts\,, \tag{6.32}$$

where $E = T^{tt}$ as well as the Smarr formula

$$E - Ts = -T^{xx}\,. \tag{6.33}$$

As we explain in the appendix, these results can also be obtained by explicitly writing the bulk action as a total derivative in two different ways.

Finally, we note that we can determine how various quantities transform under the scaling given in (6.18). For example, the free-energy transforms as

$$w \to \lambda^{-4}w - \mathcal{A}\lambda^{-4}\log\lambda\,. \tag{6.34}$$

One can check that the Smarr formula is invariant under (6.18).

3.4 Numerical construction of the black hole solutions

We construct the black hole solutions by numerically solving the ODEs (6.12)-(6.15), subject to the boundary conditions given in (6.17),(6.19). Recall that for a fixed dynamical

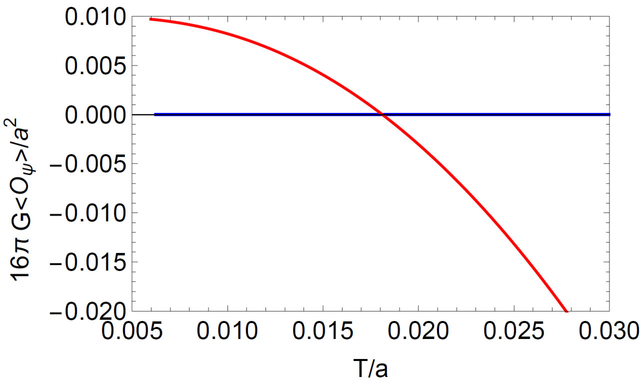


Figure 8: Plot showing the expectation value of the operator $\mathcal{O}_\psi$ dual to the scalar field ψ (recall $X = e^{-\psi/\sqrt{6}}$) for the black hole solutions. The blue line corresponds to the solution of [61], while the red lines indicate the new branch of solution, that exists for $T \leq T_c$ and $T \geq T_c$.

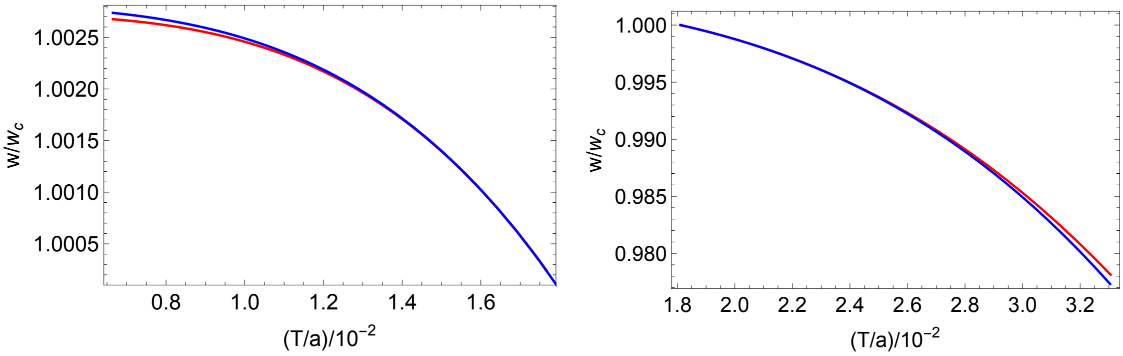


Figure 9: Plot showing the free energy of the black hole solutions, relative to the free energy at the critical temperature, w_c. The red line is the new branch of solution, while the blue line is the solution of [61]. The left panel shows that the branch of black holes that exist for $T \leq T_c$ has lower free energy, and hence is thermodynamically preferred. The right panel shows the black holes with $T \geq T_c$ that are not preferred.

scale the black hole solutions can be parametrised by T/a. In practise we set $a = 1$ and use a numerical shooting method in which we shoot from both near the black hole horizon and the holographic boundary and then match in the middle.

We find that a new branch of black hole solutions appears at the critical temperature $T_c/a \sim 1.8 \times 10^{-2}$, and there are black hole solutions that exist for both $T \leq T_c$ and $T \geq T_c$, as illustrated in figures 8 and 9.

The solutions with $T \leq T_c$ are the physically relevant solutions. In particular, we see from figure 9 that for $T \leq T_c$, where the black hole solutions of [60, 61] are unstable, this new branch of solutions has lower free energy. We thus conclude that there is a phase transition which, moreover, is a continuous phase transition. We emphasise that these hairy black holes are associated with a spontaneous phase transition since the boundary conditions we imposed for the field X corresponded to the dual operator $\mathcal{O}_\psi$ acquiring an expectation value with no source. From the point of view of the $D = 5$ model this new phase does not appear to break any more symmetries than the background black holes. In particular, one can see from that potential in (6.6) does not have, for example, a $\mathbb{Z}_2$

symmetry. However, after uplifting to type IIB, following an earlier discussion we know that when the ψ-field acquires an expectation value then the $SO(6)$ global symmetry is spontaneously broken to $SO(4) \times SO(2)$.

The black hole solutions with $T \geq T_c$ appear to be "exotic hairy black holes". In particular, they only seem[3] to exist for $T \geq T_c$, in contrast to black holes associated with a first order transition which start existing for $T \geq T_c$ and then turn around at some maximum temperature before continuing down to lower temperatures. We also observe from figure 9 that these black holes have higher free energy than the black holes of [60,61] and hence are not thermodynamically preferred. We note that such exotic hairy black holes have appeared in other holographic constructions, both bottom-up [157] and top-down [86].

3.5 Critical Exponents

Having shown that there is a continuous phase transition at T_c, we now investigate the critical exponents of the transition. Somewhat surprisingly, we find that the phase transition does not have the same critical exponents as the majority of holographic phase transitions.

The simplest critical exponent to calculate is β, which is defined by

$$\langle \mathcal{O}_\psi \rangle \sim (T_c - T)^\beta \,. \tag{6.35}$$

For our phase transition, from our numerics we find that $\beta = 1$, differing from the standard value $\beta = 1/2$. There are several other important critical exponents for a phase transition[4]. For example, the behaviour of the specific heat for $T < T_c$ defines the exponent α via

$$C \sim (T_c - T)^{-\alpha} \,. \tag{6.36}$$

This can be read off from the behaviour of the difference between the free energies of the two phases via $\Delta w \sim (T_c - T)^{2-\alpha}$. We find that in our transition $\alpha = -1$ in contrast to the standard value of $\alpha = 0$. Since the free energy is continuous in both the first and second derivatives, and has a discontinuity in the third derivative, we conclude that this is a third order phase transition. The remaining critical exponents are fixed by α, β using scaling relations. For example we have

$$\gamma = 2 - \alpha - 2\beta^{-1}, \qquad \delta = (2-\alpha)\beta^{-1} - 1 \,, \tag{6.37}$$

[3]We have checked that this is true up to $T/a \sim 3$.

[4]For a discussion in the context of holography see [169].

where γ, δ are defined by

$$\frac{\partial \langle \mathcal{O}_\psi \rangle}{\partial \psi} \sim (T_c - T)^\gamma, \qquad \psi \sim \langle \mathcal{O}_\psi \rangle^\delta \,. \tag{6.38}$$

These results can be established using the renormalization group, as discussed in chapter 13 of [170]. For our black holes we obtain $\gamma = 1, \delta = 2$ in contrast to the standard results of $\gamma = 1, \delta = 3$.

As a check that the scaling relations are indeed satisfied, we carried out a direct calculation of the exponent δ. To do this we constructed a more general class of black hole solutions with a source for the operator dual to ψ. This required changing the boundary conditions in (6.17) to allow for terms of the form $u^2 \log u$. Having done this (which requires some effort), it is possible to see how the behaviour of $\langle \mathcal{O}_\psi \rangle$ needs to be changed as one switches on the source, while keeping the temperature fixed to be at the value $T = T_c$. Carrying out this procedure we found $\delta = 2$ in agreement with above. To summarise, the critical exponents for the new phase of black holes are[5]

$$(\alpha, \beta, \gamma, \delta) = (-1, 1, 1, 2)\,, \tag{6.39}$$

in contrast to the standard values $(\alpha, \beta, \gamma, \delta) = (0, 1/2, 1, 3)$. Recall that the standard values arise from a Landau-Ginzburg model with quadratic and quartic terms in the free energy. In appendix E, we discuss how the exponents (6.39) are associated with a free energy for a scalar order parameter with quadratic and cubic terms.

Some bottom-up holographic superconducting phase transitions have been studied in the probe approximation [74, 158, 159] and with back-reaction [75], which also exhibit non-standard critical exponents. The specific critical exponents that we have found for our new black holes have also been found in models with a potential which contained terms that are cubic in the modulus of a complex scalar field. Although such couplings are rather unnatural for a charged scalar field, and it is difficult to see how they would arise from a top-down setting, we find that cubic terms in the potential for the neutral scalar field ψ in our model are responsible for the non-mean field behaviour.

To see this we first note that if we expand our Lagrangian for the scalar field ψ around $\psi = 0$ we have

$$\mathcal{L}_\psi = -\frac{1}{2}(\partial\psi)^2 + 12 + 2\psi^2 - \frac{1}{2}\sqrt{\frac{2}{3}}\psi^3 + \frac{1}{12}\psi^4 + \mathcal{O}(\psi^5), \tag{6.40}$$

In particular, the absence of a $\mathbb{Z}_2$ symmetry $\psi \to -\psi$ allows for the cubic term. We can

[5]It is interesting to contrast the parameters that we have here to that of the standard Ising model [171]. In our model, T/a corresponds to the temperature of the Ising model, while $\langle \mathcal{O}_\psi \rangle$ corresponds to the magnetisation, and ψ to the magnetic field.

contrast this model with a bottom up model in which the cubic term is absent:

$$\mathcal{L}_\psi = -\frac{1}{2}(\partial\psi)^2 + 12 + 2\psi^2 + \frac{1}{12}\psi^4 + \mathcal{O}(\psi^5), \tag{6.41}$$

with the rest of the Lagrangian unchanged. We have constructed the back-reacted black holes for this model and we find that the critical exponents for the phase transition now take the standard values. Furthermore, we have checked that varying the quartic terms does not change this result.

3.6 Zero temperature scaling solution

To investigate the low temperature behaviour of black hole solutions it is often illuminating to examine the low temperature behaviour of Ts'/s, since if it approaches a constant it indicates an emergent scaling behaviour which one can then try to identify. For the black hole solutions constructed in [61] with $X = 0$ one finds that s scales as $s \sim T^{\frac{8}{3}}$ and this is exactly the scaling behaviour that is associated with the Lifshitz-like anisotropic geometry found by [59] that appears at $T = 0$ in the far IR. In fact this scaling behaviour is approximately present at the critical temperature phase transition as we see from figure 10.

For our new black hole solutions with $X \neq 0$ we also see from figure 10 that at very low temperatures $s \sim T^{\frac{11}{3}}$. We therefore look for the existence of a scaling solution to the equations of motion of the form

$$e^{\phi(u)} = e^{\phi_0} u^{\phi_c}, \;\; \mathcal{F}(u) = \mathcal{F}_0 u^{\mathcal{F}_c}, \;\; \mathcal{B}(u) = \mathcal{B}_0 u^{\mathcal{B}_c}, \;\; X(u) = X_0 u^{X_c}\,. \tag{6.42}$$

However, by analysing the resulting algebraic equations one concludes that when $X \neq 0$ such solutions do not exist. Instead, we have found that the equations of motion admit the following expansion as $u \to \infty$:

$$\begin{aligned}
e^{\phi(u)} &= \frac{\phi_0}{(au)^{4/9}} - \frac{15232}{10935(au)^{16/9}\phi_0^{19/2}} + \dots\,, \\
\mathcal{F}(u) &= \frac{81}{112}\phi_0^3(au)^{2/3} + \frac{28}{45\phi_0^{15/2}(au)^{2/3}} + \dots\,, \\
\mathcal{B}(u) &= \frac{\mathcal{B}_0}{(au)^{2/3}} - \frac{3136\mathcal{B}_0}{3645(au)^2\phi_0^{21/2}} + \dots\,, \\
X(u) &= \frac{4}{3(au)^{4/9}\phi_0^{7/2}} - \frac{28672}{32805(au)^{16/9}\phi_0^{14}} + \dots\,,
\end{aligned} \tag{6.43}$$

where $\mathcal{B}_0, \phi_0$ are constant. Furthermore, we have checked that the new black hole solutions start to approach this behaviour at low temperatures. Moreover, we can also show that this behaviour is associated with the observed scaling, $s \sim T^{\frac{11}{3}}$.

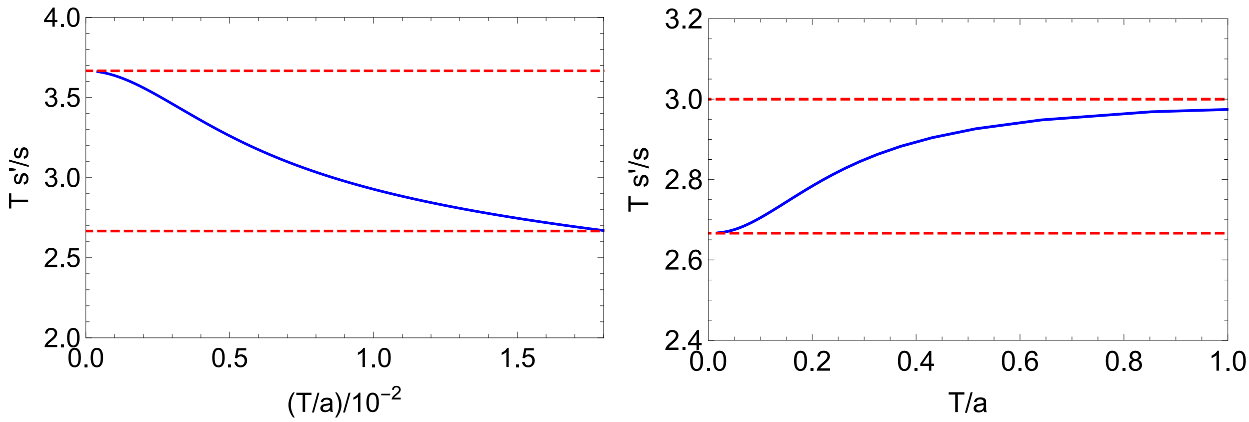


Figure 10: Plots showing the temperature dependence of entropy, both along the high temperature branch of solution (right), and for the new low temperature branch of solution (left). The dotted lines are 11/3 and 8/3 on the left plot, and 3 and 8/3 for the right plot. We see that at very high temperatures s exhibits the standard scaling of the $N = 4$ SYM plasma, before cooling towards a Lifshitz geometry. After the phase transition, the system moves away from the Lifshitz geometry, and towards the new scaling solution.

To see this we first observe that the above expansion can be generalised to finite temperatures, with the leading order expansion of $\mathcal{F}$ replaced with

$$\mathcal{F}(u) = \frac{81}{112}\phi_0^3(au)^{2/3}\left(1-\left(\frac{u}{u_h}\right)^{28/9}\right)+\dots, \tag{6.44}$$

where u_h is the horizon radius. By combining (6.31) and the above finite temperature solution, the entropy is given by

$$s = \frac{1}{4G_N}a^3\phi_0^{-5/4}(au_h)^{-22/9}+\dots \tag{6.45}$$

while the Hawking temperature is given by

$$T = \frac{9a\sqrt{\mathcal{B}_0}\phi_0^3}{16\pi}(au_h)^{-2/3}+\dots\,. \tag{6.46}$$

Combining these two expressions we find, as claimed:

$$16\pi G_N s = \left(\frac{4}{3}\right)^{1/3}\frac{65536\pi^{14/3}a^3}{2187\phi_0^{49/4}\mathcal{B}_0^{11/6}}\left(\frac{T}{a}\right)^{11/3}+\dots\,. \tag{6.47}$$

It is interesting to point out that the leading behaviour of the $T = 0$ solution given in (6.43) can be recast in the following form, after making the coordinate transformation

$u = c\rho^{3/2}$, for some constant c:

$$ds^2 \sim \rho^{\frac{-2(3-\theta)}{3}} \left(d\rho^2 - d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + \rho^{-2(z-1)} d\bar{z}^2\right) ,$$
$$e^{\phi} \sim \rho^{-2/3}, \qquad X \sim \rho^{-2/3} , \tag{6.48}$$

with $\theta = -1$, $z = 2/3$ and the bars denote that we have rescaled the coordinates. This is similar to the hyper-scaling solutions with Lifshitz exponent z and a hyper-scaling violation exponent θ [160–162], but here the Lifshitz exponent is associated with a spatial direction and not a time direction. Under the scaling $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \rho) \to (\lambda\bar{t}, \lambda\bar{x}, \lambda\bar{y}, \lambda^{2/3}\bar{z}, \lambda\rho)$ we find that metric transforms as $ds \to \lambda^{\frac{\theta}{3}} ds$. It is curious that the exponent $z = 2/3$ is the same value as for the unstable Lifshitz-like ground state with $X = 1$ given in (6.9).

3.7 Thermal conductivity

Having established the low temperature behaviour of the black holes, it is of interest to derive the DC thermal conductivity in the z-direction[6], κ. Earlier, we demonstrated in general that the DC conductivity can be calculated by solving the appropriate Stokes equations on the black hole horizon. However, since our lattice is a one dimensional homogeneous lattice, the Stokes equations can be solved exactly. For simplicity, we therefore follow the approach of [66], which showed how κ can be directly solved in terms of black hole horizon data.

To make contact with [66] and our earlier chapters, it is convenient to write the black hole solutions in a slightly different form

$$ds^2 = -U dt^2 + \frac{dr^2}{U} + e^{V_1}(dx^2 + dy^2) + e^{V_3} dz^2 ,$$
$$\chi = az, \qquad \phi = \phi(r), \qquad X = X(r) \tag{6.49}$$

where U, V_1 and V_3 are functions of r. We assume that as $r \to \infty$, the functions have the following asymptotic form

$$U \sim r^2 + ..., \quad e^{2V_i} \sim r^2 + ..., \tag{6.50}$$

and $\phi \to 0 + o(u^2)$, $X \to 1 + o(u^2)$. We now consider a small linearised perturbation

[6] Since the solutions are still translationally invariant in the x and y directions, the thermal conductivity in these directions is infinite.

about this class of black hole solutions that includes a piece that is linear in time:

$$\begin{aligned} g_{tz} &= t\delta f_2(r) + \delta g_{tx_1}(r)\,, \\ g_{rz} &= e^{2V_e}\delta h_{rz}(r)\,, \\ \chi_1 &= az + \delta\chi_1(r)\,. \end{aligned} \tag{6.51}$$

A key point is that this perturbation does not source the X-field. As a result the calculation of κ is virtually unchanged from the derivation given in [66]. Rather than repeat the steps, we just quote the final result:

$$16\pi G_N \kappa = \left[\frac{4\pi s T}{a^2 e^{2\phi}}\right]_{r=r_h}\,. \tag{6.52}$$

We showed in the previous section that the black holes with $X \neq 0$ have $s \sim T^{11/3}$ at low temperatures. From (6.43) we can also determine the low temperature scaling behaviour of the dilaton to be $(e^{2\phi})_{r=r_h} \sim T^{4/3}$. Hence the low temperature scaling of the thermal conductivity is given by $\kappa \sim T^{10/3}$. We see that the the black hole solution is dual to a ground state that is thermally insulating in the direction of the linear axion. It is also interesting to contrast this result with the result for the (unstable) Lifshitz ground state, where $\kappa \sim T^{7/3}$ [66].

4 Discussion

We have shown that the anisotropically deformed $N = 4$ Yang-Mills plasma studied in [60, 61] has low temperature instabilities. The plasma undergoes a third-order phase transition, spontaneously breaking the global $SO(6)$ symmetry down to $SO(4) \times SO(2)$. We showed that critical exponents of the phase transition are given by $(\alpha, \beta, \gamma, \delta) = (-1, 1, 1, 2)$ in contrast to the standard mean field theory values usually seen in holography. These critical values can be associated with a cubic Landau-Ginzburg free energy for a scalar order parameter, as discussed in appendix E. However, such a free energy is unstable. In addition, stabilising the free energy with higher powers of the order parameter leads to a first order phase transition. By contrast, in our holographic model the transition appears to be continuous, in fact third order, provided that the branch does not turn around at some temperature $T > T_c$ and then go down to lower energies. Thus, our model underscores the difficulty in making a precise identification of the properties of the phase transition just using a Landau-Ginzburg mean field approach. Perhaps a Landau-Ginzburg model with more fields might give a better description. It would be interesting to further clarify this point.

It would also be interesting to know which critical exponents can be realised in string/M-theory constructions. In addition to the exponents that we found here, it was shown that for a class of top-down R-charged black holes the critical exponents are given by $(\alpha, \beta, \gamma, \delta) = (1/2, 1/2, 1/2, 2)$ [172–174]. It is not clear if the bottom up constructions discussed in [74, 75, 158, 159], which had more general exponents, can be embedded into a top-down setting. Following [169], it would also be of interest to explicitly calculate the dynamic critical exponents for our transition as well as others.

We analysed the $T = 0$ limiting behaviour of the black holes describing the new low temperature phase. We showed that in the far IR there is an emergent leading order behaviour that is similar to the hyperscaling violation geometries but with spatial anisotropic scaling. This scaling behaviour implies that the thermal conductivity scales with temperature as $\kappa \sim T^{10/3}$ at low temperatures, revealing that the ground state is a thermal insulator.

It is not difficult to show that the $D = 5$ model with metric, axion and dilaton (i.e. when $X = 1$) arises as a consistent truncation on an arbitrary five-dimensional Sasaki-Einstein (SE) manifold, not just the five-sphere. Therefore, the original black hole solutions of [59–61] also describe the high temperature phase of the whole class of dual $N = 1$ SCFT plasmas with an anisotropic deformation. For a given SE space, if there are no BF saturating modes in the spectrum then the black holes will not suffer the instabilities that we have described in this paper, and the Lifshitz ground state constructed in [59] may be the true ground state of the system. On the other hand if there are BF saturating modes then the black holes will become unstable at some critical temperature. For a general SE manifold it is unlikely that there is a consistent truncation maintaining just one extra scalar field as we have studied in this paper for the case of the five-sphere. This would mean that the corresponding black hole solutions describing the low temperature phase would need to be constructed directly in ten spacetime dimensions. Although this is likely to be a challenging task, it may be tractable to study the solutions near the phase transition and it would be particularly interesting to determine the critical exponents.

Finally, the black hole solutions were constructed using a consistent KK truncation that keeps a single scalar field X with $m^2 = -4$. This scalar field is part of a multiplet of twenty scalars that transform in the $\mathbf{20'}$ of $SO(6)$. All of these scalars become unstable at the critical temperature T_c and it would be very interesting to investigate the full class of black hole solutions that emerges at T_c, which will generically break all of the $SO(6)$ symmetry, and then follow them down to low temperatures. Although challenging, this could be investigated using the consistent truncation of [175] that keeps twenty scalars parametrised by a symmetric, unimodular six by six matrix T_{ij}. As a first step one could analyse the truncation that keeps five scalar fields, parametrised by the diagonal subset [176], or even simpler, the truncations that keeps just two scalar fields [177]. We will explore this further in the next chapter.

Chapter 7

Further phases of the anisotropic N=4 super Yang-Mills plasma

1 Introduction

In this chapter, we continue our analysis of the top-down framework to study spatially anisotropic systems in $N = 4$ super Yang-Mills ("SYM") theory that was developed in [59], and further analysed in various works [60, 61, 63]. In [60, 61], black hole solutions of type IIB supergravity that approach AdS_5 in the UV were constructed, with a linearly dependent type IIB axion. The solutions were made anisotropic by making the axion a function of only one of the three spatial coordinates. At low temperatures, these black hole solutions approach a $T = 0$ solution from [59], which has a Liftshitz-like scaling in the IR.

In the previous chapter, we showed that this Liftshitz-like scaling solution is actually unstable at sufficiently low temperatures, and below some critical temperature, the black hole undergoes a phase transition, which spontaneously breaks the global $SO(6)$ symmtery down to $SO(4) \times SO(2)$. An explanation for this instability in the zero temperature limit was first presented in [59]. There, it was shown that the Kaluza-Klein (KK) spectrum of the five-sphere contains scalar modes, which transform in the $\mathbf{20'}$ of $SO(6)$, that saturate the BF bound in an AdS_5 background, but violate a similar bound in the Liftshitz scaling solution. In the last chapter, it was show that the presence of a scalar field from the KK mode at finite temperatures leads to a phase transition, and a new thermodynamically favoured branch of black hole solutions was constructed. These black holes have interesting properties, such as unusual mean field critical exponents, and undergo a third order phase transition. Whilst the instability we found was due to a single scalar field, this scalar field is only one from the multiplet of twenty, all of which become unstable at the same critical temperature, and it is an interesting question to ask whether the other scalar fields will add to the phase diagram, and what is the true low temperature ground state of the theory.

The work in [60, 61] was generalised in [63] to include a finite $U(1)$ gauge field dual to a global $U(1)$ electric field. Whilst most of the physics generalised as expected, it was claimed that the theory has a further global phase transition with the addition of the gauge field. The authors argued that there was an instability in the theory below some critical point, as there are two black hole solutions at the same temperature with different black hole radii, leading to a Hawking-Page transition. It is unclear how the $U(1)$ chemical potential would affect the results of the previous chapter and [4].

In this chapter, we will address both of these questions. We will consider various consistent truncations from type IIB supergravity, first studied in [175], which preserve multiple scalar fields from the $\mathbf{20}'$ multiplet, as well as $U(1)$ gauge fields. The presence of additional scalar fields leads to further branches of black hole solution. However, these solutions form at temperatures above the critical temperature of the phase transition and have a higher free energy - an example of "retrograde condensation". Furthermore, by constructing the static normalisable modes of the theory, we will demonstrate that below some critical temperature the solution we constructed in chapter 5 is actually unstable.

The addition of a finite $U(1)$ chemical potential generalises the results of the last chapter, and has many of the same features. In particular, the properties of the phase transition, such as the critical exponents and its order, remain unchanged when a chemical potential is switched on. However, for a sufficiently large chemical potential, there is no evidence of retrograde condensation, and the solution appears to be stable all the way down to zero temperatures. It is important to note, however, that we do not observe the Hawking-Page transition that was observed in [63].

We can then ask questions about the low temperature behaviour of the stable black holes that we have constructed. In particular, by analysisng the DC thermoelectric conductivity of the black holes, using the techniques from earlier in the thesis [1, 97], we find that at zero temperature, the DC thermoelectric conductivity matrix diverges. This is a particular effect of the top-down model we have chosen. With the consistent truncation that we have used, at $T = 0$ the gauge coupling in the Lagrangian diverges, and hence the electrical conductivity is infinite.

The rest of the chapter proceeds as follows. In section 2, we present the model we will use, discuss the previous work which will be the basis for this study, and introduce our ansatz for the black holes. The black holes are numerically constructed in section 3, and in this section we also dicuss the DC thermoelectric response of the $U(1)$ charged black holes. In the final section we present some conclusions and possible topics of further study.

2 The model

Our starting point is type IIB supergravity. On the field theory side, we are interested in field theories in $3 + 1$ dimensions, and so we will Kaluza-Klein (KK) reduce the full supergravity theory to $D = 5$ using a consistent truncation. By this, we mean that any solution to the equations of motion in the $D = 5$ theory must also be an exact solution to the full ten dimensional theory. In general, obtaining a consistent KK reduction is highly non-trivial in the case of fully backreacted supergravity theories.

A possible starting point is the reduction first obtained in [175]. This truncation preserves 15 $SO(6)$ gauge fields, the type IIB axion and dilaton, and 20 scalar fields that transform in the $\mathbf{20'}$ of $SO(6)$, which can be parameterised by a single unimodular symmetric tensor, T_{ij}. All twenty scalar fields have mass $m^2 = -4$[1], which saturates the BF bound in five dimensions, and are dual to operators $\mathcal{O}_\psi$ that have dimension $\Delta = 2$. It is possible to further truncate this theory to all the theories that have been previously studied in [4, 60, 61, 63].

Given the computational difficulty in numerically solving the field equations containing this many fields, it will be helpful to further truncate this theory to get something which is more numerically manageable. There are two cases that we will consider - the "neutral" case, where five of the twenty scalar fields are non zero in the consistent truncation with all the gauge fields set to zero, and the "charged" case, which preserves three $U(1)$ gauge fields as well as two scalar fields.

We stress that our use of neutral and charged here simply refers to whether or not the consistent truncation leads to non-zero gauge fields. Unlike in the case of holographic superconductors (e.g [178]), the phase transitions here will be driven by neutral scalar fields.

2.1 Neutral case

We will consider a $D = 5$ gravitational theory coupled to five scalar fields, the dilaton and axion, with a Lagrangian given by

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi G_N}\left(R - \frac{1}{2}(\partial\vec{\Psi})^2 - V - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right), \tag{7.1}$$

where the potential, V, is given by

$$V = -\frac{1}{2}\left((\sum_{i=1}^{6} X_i)^2 - 2\sum_{i=1}^{6} X_i^2\right), \tag{7.2}$$

$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$ and

$$X_i = e^{-\frac{1}{2}\vec{b_i}.\vec{\Psi}}. \tag{7.3}$$

[1] Throughout this chapter we have set the AdS radius, $l = 1$

The $\vec{b_i}$ satisfy

$$\vec{b_i}.\vec{b_j} = 8\delta_{ij} - \frac{4}{3}\,, \qquad \sum_{i=1}^{6} b_i = 1\,, \qquad \sum_{i=1}^{6}(\vec{u}.\vec{b_i})\vec{b_i} = 8\vec{u}\,, \tag{7.4}$$

and one convinient choice is [179]

$$\begin{aligned}
\vec{b_1} &= \left(2, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{15}}\right)\,, \quad & \vec{b_2} &= \left(-2, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{15}}\right)\,,\\
\vec{b_3} &= \left(0, -\frac{4}{\sqrt{3}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{15}}\right)\,, \quad & \vec{b_4} &= \left(0, 0, -\sqrt{6}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{15}}\right)\,,\\
\vec{b_5} &= \left(0, 0, 0, -\frac{8}{\sqrt{10}}, \frac{2}{\sqrt{15}}\right)\,, \quad & \vec{b_6} &= \left(0, 0, 0, 0, 0, -\frac{10}{\sqrt{15}}\right)\,.
\end{aligned} \tag{7.5}$$

This truncation can be obtained from the truncation discussed in the previous section by setting the gauge fields to zero and diagonalising T_{ij} [177, 180],

$$T_{ij} = \mathrm{diag}(X_1, X_2, X_3, X_4, X_5, X_6)\,, \qquad \prod_{i=1}^{6} X_i = 1\,, \tag{7.6}$$

and can be further truncated by setting pairs of the elements of (7.6) to be equal to each other. For example, we can set the X_i to be pairwise equal, e.g

$$X_1 = X_2 \equiv Y_1\,, \qquad X_3 = X_4 \equiv Y_2\,, \qquad X_5 = X_6 \equiv Y_3\,, \tag{7.7}$$

so that

$$T_{ij} = \mathrm{diag}\left(Y_1, Y_1, Y_2, Y_2, Y_3, Y_3\right)\,, \tag{7.8}$$

which leaves us with two remaining scalar fields (since we still have the condition that T_{ij} has unit determinant). The constistency of this truncation was first deduced in [176]. One way to obtain this truncation is to set

$$\vec{\Psi} = \left(0, \sqrt{\frac{2}{3}}\psi_1, \sqrt{\frac{1}{3}}\psi_1, \sqrt{\frac{3}{5}}\psi_2, \sqrt{\frac{2}{5}}\psi_2\right)\,, \tag{7.9}$$

which explicitly demonstrates that there are two scalar fields remaining.

Furthermore, if we set

$$\vec{\Psi} = \left(0, 0, 0, \sqrt{\frac{3}{5}}\psi, \sqrt{\frac{2}{5}}\psi\right)\,, \tag{7.10}$$

our matrix is now

$$T_{ij} = \mathrm{diag}\left(X, X, X, X, X^{-2}, X^{-2}\right)\,, \tag{7.11}$$

where $X = e^{-\psi/\sqrt{6}}$, and we have the consistent truncation considered from the last chapter [4]. Finally, we can set $X = 1$ to recover the Einstein-dilaton-axion theory that was studied by Mateos and Trancanelli [60, 61].

Note that in order to truncate the five scalar fields to two, we could have equally chosen $\vec{\Psi}$ so that (7.8) was instead $(X_1, X_2, X_3, X_1, X_2, X_3)$, with

$$\vec{\Psi} = \left(\psi_1, \psi_2, -\sqrt{\frac{3}{8}}\psi_1 - \sqrt{\frac{1}{8}}\psi_2, \sqrt{\frac{5}{8}}\psi_1 - \frac{1}{2}\sqrt{\frac{3}{10}}\psi_2, \frac{2}{\sqrt{5}}\psi_2\right). \tag{7.12}$$

In either case, if both scalar fields acquire a different expectation value, then on uplifting to the full type IIB solution, the global R-symmetry will be broken from $SO(6)$ to $SO(2)^3$. Whilst these two examples would appear to be two different solutions in $D = 5$, they are simply part of the moduli space of $D = 10$ IIB solutions, and are physically equivalent.

2.2 Charged case

We now turn our attention to theories that include gauge fields. We will consider the truncation that preserves two of the scalar fields, three $U(1)$ gauge fields of the maximal $U(1)^3$ subgroup of $SO(6)$, as well as the axion and dilaton, ϕ and χ, which was first obtained in [177]. The Lagrangian for this theory is

$$\begin{aligned}\frac{\sqrt{-g}}{16\pi G_N}\mathcal{L} =& R - \frac{1}{2}(\partial\psi_1)^2 - \frac{1}{2}(\partial\psi_2)^2 + 4\sum_i Y_i^{-1} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \\ & - \frac{1}{4}\sum_i Y_i^{-2}(F^i)_{\mu\nu}(F^i)^{\mu\nu} + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}F^1_{\mu\nu}F^2_{\rho\sigma}A^3_\lambda\,,\end{aligned} \tag{7.13}$$

where the Y_i and ψ_i are the same as in the action described by (7.8), and can be parameterised by

$$Y_i = e^{-\frac{1}{2}\vec{a}_i.\vec{\psi}} \tag{7.14}$$

with

$$\vec{a}_1 = (\frac{2}{\sqrt{6}}, \sqrt{2}), \qquad \vec{a}_2 = (\frac{2}{\sqrt{6}}, -\sqrt{2}), \qquad \vec{a}_3 = (-\frac{4}{\sqrt{6}}, 0)\,. \tag{7.15}$$

Although it is not possible to consistently set the gauge fields to zero in the truncation that preserves twenty scalar fields, here we can consistently set the gauge fields to zero to get the consistent truncation which is described by the neutral Lagrangian and (7.8).

If we turn the dilatonic fields, ψ_1 and ψ_2, into a single scalar field (the simplest way to do this is to set $\psi_2 = 0$ and let $F^1 = F^2 = F/\sqrt{2}$), our theory reduces to the truncation studied previously [4] with two additional $U(1)$ gauge fields[2]. Finally, we note that setting

[2]This is Romans $D = 5$ $SU(1) \times U(1)$ gauged supergravity [163] with an extra dilaton and axion.

both ψ_1 and ψ_2 to zero and $F^1 = F^2 = F^3 = F/\sqrt{3}$, we now have Einstein-Maxwell-axion-dilaton gravity, which is the model studied in [63].

2.3 Equations of motion

We now extend our earlier analysis and the work of [4, 59–61, 63], and consider an ansatz of the form

$$
\begin{aligned}
ds^2 &= \frac{e^{-\frac{1}{2}\phi}}{u^2}\left(-\mathcal{F}\mathcal{B}dt^2 + \frac{du^2}{\mathcal{F}} + dx^2 + dy^2 + \mathcal{H}dz^2\right), \\
\chi &= az, \qquad \phi = \phi(u), \qquad \psi_i = \psi_i(u), \qquad A^j = b_j(u)dt = b_j dt\,,
\end{aligned} \tag{7.16}
$$

where $i = 5$ and $j = 0$ for the netural case, and $i = 2$ and $j = 3$ for the charged case. The UV boundary of our theory is located at $u \to 0$, whilst there is a black hole horizon at u_h. The metric and axion ansatz are the same as in previous works, whilst all other fields only depend on the radial coordinate, u, which ensures the ansatz takes the same form as previous work.

The form of the ansatz ensures that the equations of motions will be ODEs, rather than the more technically challenging PDEs. In addition, the form of the gauge potential mean that the Chern-Simons term in (7.13) vanishes. The rest of this section proceeds in a similar fashion to section 3 of the last chapter [4]. We therefore only highlight a few key details here, and the refer the reader to the previous chapter for full details. Furthermore, whilst this section will refer to the case where the gauge fields are non-zero, the results translate in a straightforward manner if the gauge fields are truncated out.

Substituting this ansatz into the equations of motion, the equation of motion for the axion is trivially satisfied, whilst there are second order equations of motion for the dilaton, gauge and scalar fields in the consistent truncation. In addition, there are four independent components of the Einstein equations, which can be written as equations for $\mathcal{F}', \mathcal{F}'', \mathcal{B}', \mathcal{B}''$. Through appropriate linear combinations of these equations, combined with the equations of motion for the scalar and gauge fields, these four equations reduce to first order equations in $\mathcal{F}$ and $\mathcal{B}$.

The equations of motion are therefore second order in the dilaton, scalar and gauge fields, and first order in $\mathcal{F}$ and $\mathcal{B}$. We can therefore specify the equation of motion by $4 + 2(n_s + n_g)$ integration constants, where n_s and n_g are the number of scalar and gauge fields respectively in the consistent truncation.

At this stage, it is also helpful to note that the ansatz and hence the equations of motion are invariant under the following two scaling symmetries

$$
\begin{aligned}
&u \to \lambda u, \quad (t, x, y, z) \to \lambda(t, x, y, z), \quad a \to \lambda^{-1}a, \quad b_i \to \lambda^{-1}b_i; \\
&t \to \lambda t, \quad \mathcal{B} \to \lambda^{-1/2}\mathcal{B}, \quad b_i \to \lambda^{-1}b_i;
\end{aligned} \tag{7.17}
$$

where λ is a constant.

2.4 Boundary conditions

We will now discuss the boundary conditions for our theory, and hence derive the expansions for the functions near the boundary $u \to 0$, the UV, and near the black hole horizon at u_h, the IR.

First, we consider the UV Expansion. We require that our solution asymptotically approaches AdS_5 with an axionic field that is deformed by strength a in the z direction. In order to have the correct falloff, we require $\phi \to 0$, and $\psi_i \to 0$. Furthermore, we require that the three gauge fields tend to constant values at the boundary, corresponding to switching on a chemical potential. In order to make a connection with [63], we will take the three gauge fields to have the same chemical potential, μ.

By imposing these boundary conditions on the solutions, and solving the equations of motion order by order, we see that the solution has an asymptotic expansion:

$$\begin{aligned}
\phi &= -\frac{a^2u^2}{4} + \dots\,, \\
\mathcal{F} &= 1 + \frac{11a^2u^2}{24} + u^4\mathcal{F}_4 + u^4\log u\frac{7a^4}{12} + \dots\,, \\
\mathcal{B} &= 1 - \frac{11a^2u^2}{24} + u^4\mathcal{B}_4 - u^4\log u\frac{7a^4}{12} + \dots\,, \\
\psi_i &= \langle\psi\rangle_i u^2 + \dots\,, \\
b_i &= \mu + \frac{1}{2}\rho_i u^2 + \dots\,,
\end{aligned} \tag{7.18}$$

where $\langle\psi\rangle_i$ corresponds to the VEV of the operator dual to ψ_i and ρ_i is the electric current for the gauge field A_i[3]. We have set terms proportional to $u^2\log u$ to zero in the expansions of ψ_i, which mean that there is no source term for the operator dual to ψ_i, and we have used the second scaling symmetry from (7.17) to set $\mathcal{B}$ to 1. We can see that the expansion is determined by $4 + n_s + n_g$ terms. The log terms in the expansion indicate that there is a conformal anomaly in our theory, which introduces an additional dynamical scale that is part of the general freedom in the choice of renormalization scheme. However, for the present purposes it will suffice to hold this scale to be fixed to unity throughout.

We now consider the IR expansion, and demand that the black hole has a regular event horizon at u_h, which requires the gauge fields and $\mathcal{F}$ vanish on the horizon. We find that the leading order expansion about u_h for the fields is given by

$$\begin{aligned}
&\mathcal{F} = -\frac{4\pi T}{\sqrt{\mathcal{B}_h}}(u-u_h) + \dots\,, \qquad \mathcal{B} = \mathcal{B}_h + \dots\,, \qquad \phi = \phi_h + \dots\,, \\
&\psi_i = \psi_{ih} + \dots\,, \qquad b_i = a_{ih}(u-u_h) + \dots,
\end{aligned} \tag{7.19}$$

[3]This is up to a scaling by a factor og $16\pi G$.

where T is the Hawking temperature of the black hole, which can be expressed in terms of u_h and the other free parameters in the expanion. There are therefore $3 + n_s + n_g$ parameters in the IR expansion of the fields.

Combining these two results, we see that our system is determined by $7 + n_s + n_g$ constraints, which, after applying the remaining symmetry from (7.17) gives us $6+2(n_s+n_g)$ constants of integration. Recalling that the order of our equations is $4 + 2(n_s + n_g)$, the black holes are specified by a two parameter family of solutions. Throughout the remainder of the paper, we will use the grand canonical ensemble to describe the system and so these parameters will be T/a and μ/a. In the "neutral case" we have a one parameter family of solutions specified by T/a.

3 Numerical construction of the black holes

We can now solve the equations of motion numerically in order to construct black hole solutions. As discussed previously, there are two cases that we will consider - the "neutral" case where we have set the chemical potential to zero, and "charged" case, where we source all gauge fields with the same, constant chemical potential, μ. Since our boundary conditions ensure that there is no source to the scalar fields, ψ_i, any new branch of solution will correspond to a phase transition by spontaneous symmetry breaking, since the dual operator to the field, $\mathcal{O}_\psi$, will have a finite expectation value but no source.

For both cases, our numerical method has been to set the anisotropic strength, a, to 1, and use a shooting method to solve the equations. This means that we numerically integrate the solution from both the black hole horizon and the UV boundary, and match at some midpoint between the two boundaries. The Smarr relation

$$E - Ts + \frac{1}{16\pi G_N}\sum_i \rho_i \mu = -T^{xx} \,, \tag{7.20}$$

where $E = T^{tt}$, $T^{\mu\nu}$ is the stress energy tensor of our black hole and s is the entropy density given by

$$s = \frac{e^{-\frac{5}{4}\phi_h}}{4G_N u_h^3} \,, \tag{7.21}$$

provides a useful check of the numerics. This relation can be verified using the method outlined in Appendix E.1.

Since there are several branches of black hole solution that have been numerically constructed, in what follows we will refer to the black hole solutions from [60, 61] as the "Mateos-Trancanelli" solution, the black hole from [63] as the "Cheng-Ge-Sin" (CGS) solution, and the new branch of black hole solution constructed in the last chapter as the "1 scalar" solution.

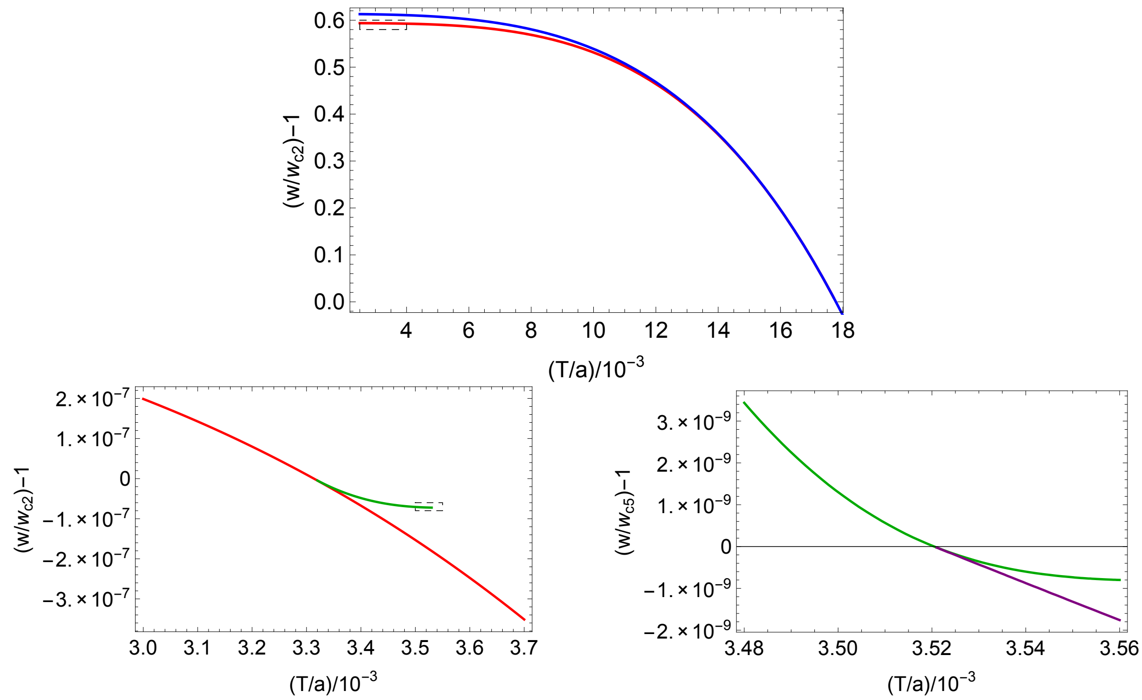


Figure 11: Plot showing the free energy of the black hole solutions in the neutral case, scaled in relation to the free energy at the critical temperature of the particular phase transition. The dashed boxes in the top plot shows the region that has been magnified in the bottom left plot, whilst the dashed box in the bottom left chart has been magnified in the bottom right plot. The blue line is the Mateos-Trancanelli solution, the red line is the 1 scalar solution from [4], whilst the green and purple lines indicate new branches of black hole solution, that only form at a temperature above the critical temperature of the phase transition. Furthermore, these solutions have a higher free energy than the 1 scalar solution and are not thermodynamically preferred, but instead correspond to retrograde condensation.

3.1 Neutral case

Recall that it was previously shown that the Mateos-Trancanelli solution (denoted by the blue line in figure 11) is unstable below a critical temperature, $T_{c1}/a \sim 1.8 \times 10^{-2}$, and a new branch of black hole solution is formed which is thermodynamically preferred (denoted by the red line). This instability is driven by the condensation of a single scalar field that transforms in the $\mathbf{20'}$ of $SO(6)$ and has a mass $m^2 = -4$. Here, we extend this analysis by including five of the scalar fields from the multiplet, all of which all have $m^2 = -4$, and find further branches of black hole solution.

The additional scalar fields in the Lagrangian (7.1) lead to further branches of black hole solution, which are show in figure 11. At a critical temperature $T_{c2}/a \sim 3.3 \times 10^{-3}$, a new branch of solution (the green line) appears from the 1 scalar solution, whilst further along this branch there is another phase transition that occurs at T_{c5}/a and leads to a further branch of solutions (the purple line). The green branch of solution has two independent scalar fields (and so is also a solution to the consistent truncation described by (7.8)), whilst the purple branch of solution has three independent scalar fields (and so is a solution to equations of motion with two pairs of X_i in 7.6 set equal). On uplifting to

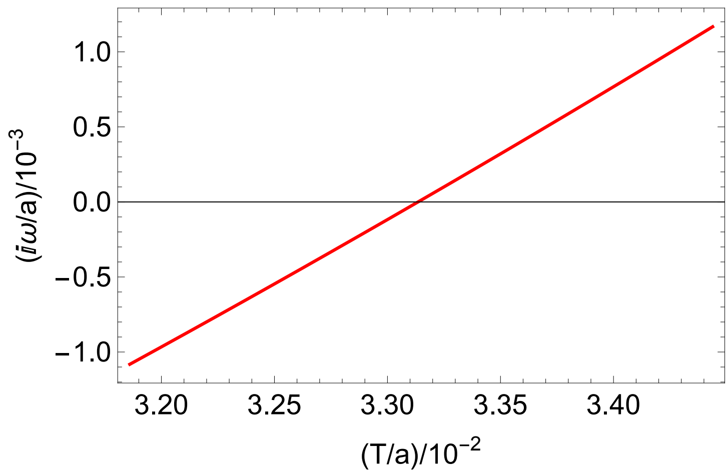


Figure 12: Frequency of normalisable mode versus temperature, for the 1 scalar branch of solution. Below the critical temperature, T_{c2}/a, any perturbation of the form $e^{-i\omega t}$ will grow exponentially in time, and will cause an instability. The quasinormal mode frequency appears to be purely imaginary and hence is a purely decaying mode.

the full $D = 10$ theory we see that the global R-symmetry of the solution is first broken from $SO(4) \times SO(2)$ to $U(1)^3$ and then to $U(1)^2$ along the branches.

Although our Lagrangian contains five scalar fields, it appears that in our solutions there are only a maximum of three fields that are actually independent. This can be seen by analysing the linear perturbation of the five scalar fields around the green branch of solutions. It is only possible to get consistent equations of motion when one of the scalar fields is perturbed in this background, and so only one additional scalar field will condense at T_{c5}/a.

Unlike in our earlier analysis [4], when the Mateos-Trancanelli solution undergoes a phase transtiion to the 1 scalar solution, the branches of solution constructed here only exist at temperature $T \geq T_{c2}/a$, and are not thermodynamically perferred. Furthermore, we find no evidence that the branch will turn back to lower temperatures (which would indicate a first order transition). This is an example of retrograde condensation and has been observed in top-down models for holographic superconductors [85, 86].

In [86], black brane solutions were constructed that also displayed this retrograde condensation. In this example, the black hole solution itself was unstable below the critical point, with a nakedly singular solution at $T = 0$. It is an interesting question to therefore ask if the geometry here is unstable below T_{c2}/a. To do this, we introduce a perturbation of the form $e^{-i\omega t}\delta\psi_2$ and impose infalling boundary condition. We then ask what value of ω do we get a normalisable fall-off, ie at what value of ω can we get a VEV for the scalar field without a source. A plot of $i\omega$ for values of T/a is shown in figure 12. We find that for $T > T_{c2}$, $i\omega$ is positive, and so any perturbation of the scalar field will decay over time. However, below T_{c2}, $i\omega$ changes sign, which means that a perturbation of the scalar field will grow exponentially with time. We therefore conclude that the theory becomes unstable below the critical point, T_{c2}/a.

3.2 Charged case

When we add a single, finite $U(1)$ chemical potential, we observe interesting results which highlight the competing effects that geometry and charge have on phase transitions in top-down holographic models. Our starting point is the CGS solution from [63], which is the charged analogue of the Mateos Trancanelli solution. In their paper, the authors found that the CGS solution has some minimum temperature and the black hole undergoes a Hawking-Page style phase transition below this point. Here, we do not find this, but instead find that the black hole approaches an extremal black hole in the low temperature limit, which we discuss in further detail in Appendix F.1. However, as we now explain, the CGS solution is unstable below a critical temperature, T_{c1}/a

The phase transition observed in [4] is also present when the chemical potential is turned on - in this case, the CGS solution becomes unstable below T_{c1}/a (which now depends on μ/a), and the plasma undergoes a third order phase transition (with the same critical exponents as [4]) to a new branch of thermodynamically preferred black hole solutions. As before, there is also an unphysical branch of solutions that forms above the critical temperature, with a higher free energy than the background solution, corresponding to an exotic hairy black hole. The free energy for these solutions with $\mu/a = 1$ is shown in the left hand plot of figure 13, where the blue line is the CGS solution and the red line is the new branch.

As one would expect from the discussion in (2.2), the multiple gauge fields in our theory can now be explicitly seen in this new branch of solution, due to the breaking of the symmetry in the $D = 10$ supergravity theory. In left plot of figure 13, we plot the electric charge density of the solutions against temperature. The blue line is the charge density for the CGS solution, whilst the two red lines are the two different charge densities in this new branch of solution below the critical temperature.

We find that the critical temperature of this phase transition, T_{c1}/a, increases as we increase the chemical potential, and so the phase transition found in [4] is at the lowest critical temperature in our class of black holes. We have checked the relationship between critical temperature and chemical potential up to $\mu/a \sim 1.5$, and have no reason to suspect that this will change at higher chemical potentials. The chemical potential dependence of the critical temperature for this phase transition is shown in the red plot of figure 14.

However, for the condensation of a second scalar field (analagous to the second plot in figure 11), there is a different story. Although this phase transition is seen for small μ/a, when $\mu/a \geq \mu_c/a \sim 0.015$, this lower temperature phase transition is no longer seen. The green plot in figure 14 shows the effect of increasing μ/a has on this critical temperature for the phase transition, T_{c2}/a. The critical temperature decreases until at $\mu = \mu_c$ the

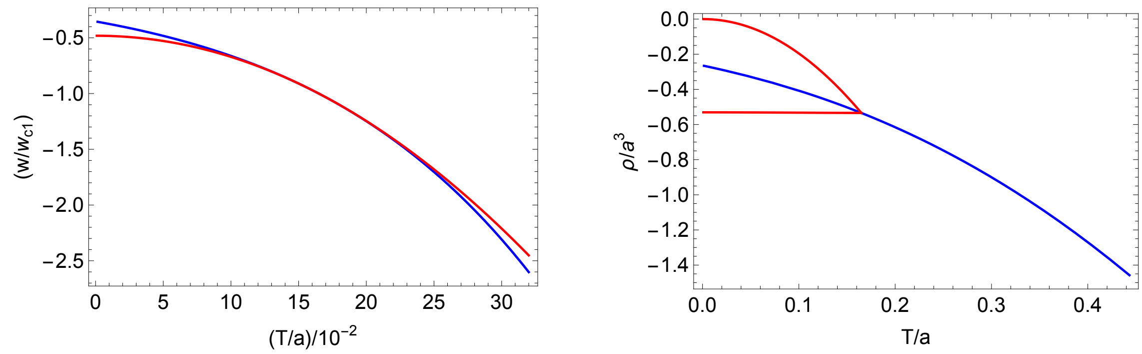


Figure 13: Plot showing the charged branches of black hole solutions when $\mu/a = 1$. The blue line is the CGS soution from [63], whilst the red line is the branch of solution analagous to that found in [4]. The left plot shows the free energies of the solutions, and there is a branch of solutions that forms at temperatures above and below the phase transition. However, only the lower temperature new solution is thermodynamically preferred. The right hand plot shows the electric charge density when the chemical potential is turned on, for the CGS solution and the new branch of solution. The solution has only been shown below the critical temperature for the ease of presentation. Below the critical temperature, the new branch of solution has two different electric charge densities due to the symmetry breaking.

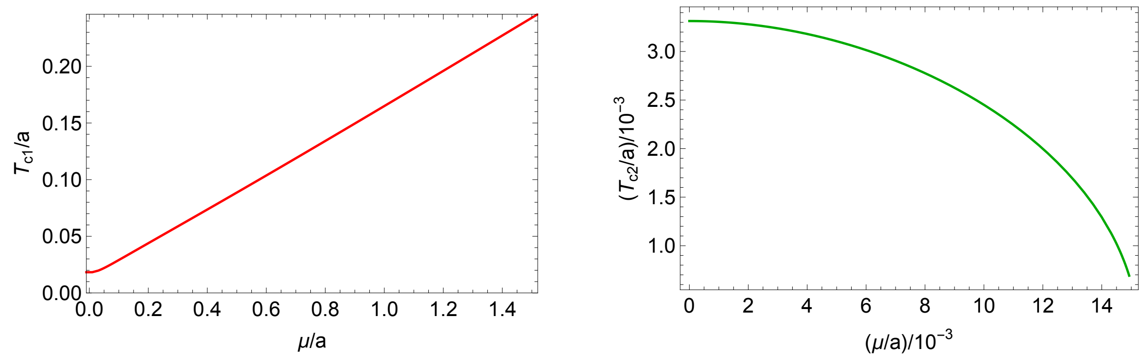


Figure 14: Plot showing critical temperatures, T_{c1}/a (left) and T_{c2}/a (right) as a function of chemical potential, μ/a. The phase transition seen in [4] occurs for all values of chemical potential. However, the instability that discussed in section 3.1 is only seen for $\mu/a < 0.015$. The symmetry of the gauge field ensures these results are symmetric if we take $\mu \to -\mu$.

critical temperature is zero, and the phase transition no longer occurs. Interestingly, this means that although our black holes appear to be unstable to scalar field perturbations at low temperatures in the neutral case, the solutions can be "saved" by turning on a sufficiently large chemical potential.

3.3 Charged solution thermoelectric DC conductivity

We now change tack, and study some of the properties of the new charged branch of solutions[4]. As the solution is cooled down to low temperatures, thermodynamic quantities such as entropy begin to show temperature dependent scaling. This indicates that, unlike

[4]For all the calculations in this section, unless otherwise stated, we have set $\mu/a = 0.02$, to ensure that the solution is stable at low temperatures, as per the discussion in the previous section.

in the CGS solution, the zero temperature black hole solution is not extremal and instead the black hole reaches zero temperature as $u_h \to \infty$.

Since the translational invariance of the theory has been broken in the z direction, it is interesting to determine the DC transport coefficients of the phase in this direction. Since there are two distinct gauge fields, we anticipate that there should be two different electric conductivities. This leads to a 3×3 thermoelectric conductivity matrix in the form

$$\begin{pmatrix} J_1 \\ J_2 \\ Q \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \alpha_1 T \\ \sigma_{21} & \sigma_{22} & \alpha_2 T \\ \bar{\alpha}_1 T & \bar{\alpha}_2 T & \kappa \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ -(\nabla T)/T \end{pmatrix}, \tag{7.22}$$

where the J's are the electric current densities from the two gauge fields, the Q is the heat current density, while E_i and ∇T are the applied electric fields and thermal gradients. The symmetries of our ansatz imply that $\sigma_{12} = \sigma_{21}$, and $\bar{\alpha}_i = \alpha_i$.

In order to determine this DC thermoelectric conductivity matrix, we adopt the method from chapter 2 [55, 66] and consider a linearised perturbation of the form

$$\begin{aligned} A_z^a &= -\delta f_1^a(u)t + \delta a_z^a(u)\,, \\ g_{tz} &= t\delta f_2(r) + \delta g_t z(u)\,, \\ g_{uz} &= \delta g_{uz}(u)\,, \\ \delta\phi &= \delta\phi(u)\,, \\ \delta\chi &= \delta\chi(u)\,, \end{aligned} \tag{7.23}$$

where a is a label for the gauge fields, and δf_1 and δf_2 are related to the sources for the electric and heat current respectively. The matrix (7.22) can then be determined by evaluating the heat and electric current on an appropriate radial hypersurface. The details of this calculation in a general setting, which extends the earlier results [1, 97], are described in appendix F.2.

We have calculated the thermoelectric conductivity matrix for the low temperature charged phase, as shown in figure 15. Determining the DC thermoelectric conductivities allows us to compare the effects of adding a chemical potential to the thermal conductivity, $\bar{\kappa}$. For our theory, $\bar{\kappa}$ is given by

$$16\pi G_N \bar{\kappa} = \frac{16\pi^2}{a^2 u_h^3 e^{13\phi_h/4}} T\,. \tag{7.24}$$

Our numerical calculations reveal that the thermal conductivity scales with temperature as $\kappa \sim T^c$, where c is a constant with value ~ 2 in the case where $\mu = 1$, as shown in figure 16. We therefore see that the ground state is a thermal insulator. We have checked for

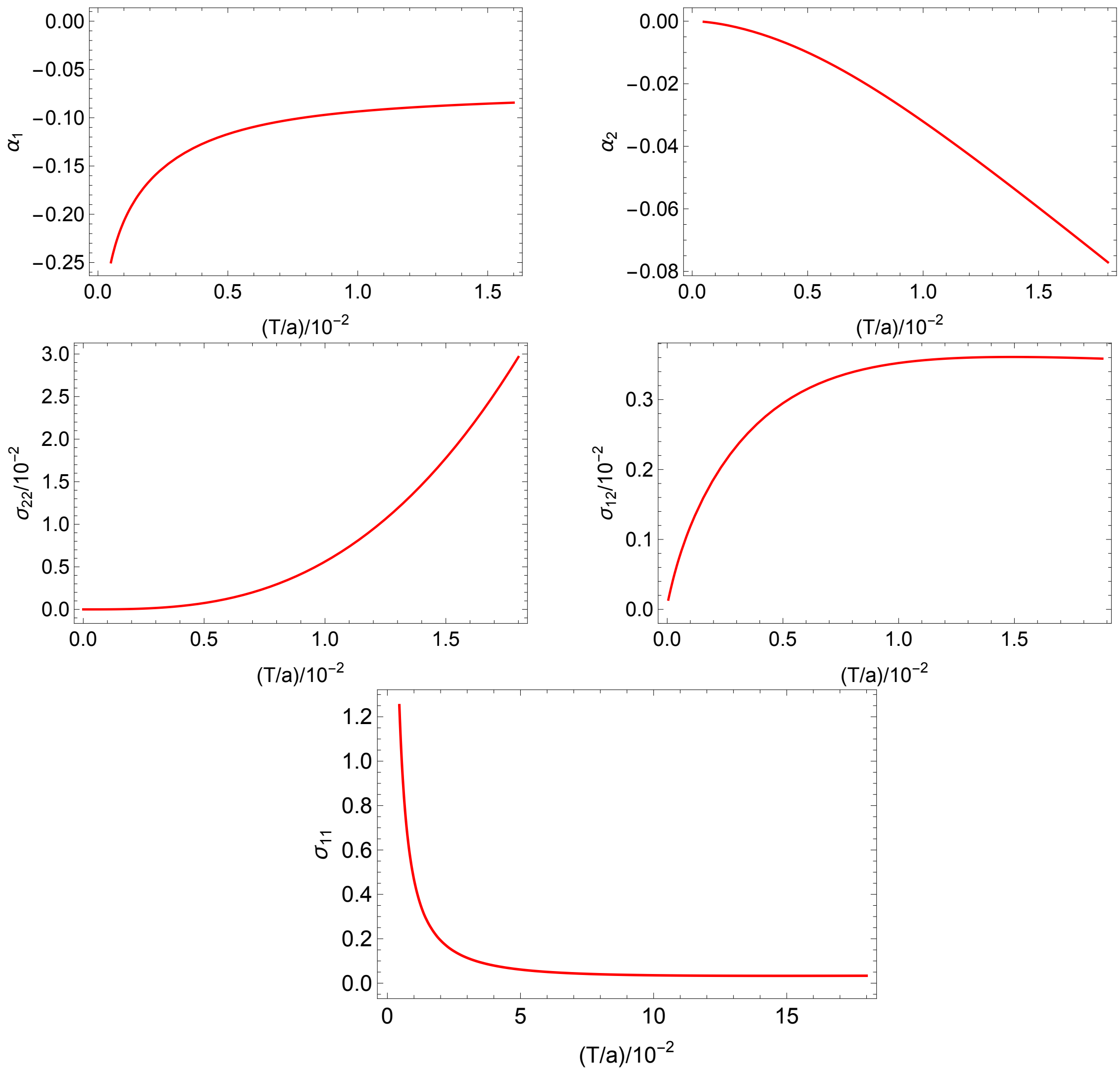


Figure 15: Plot of the DC thermoelectric conductivity for the low temperature charged phase, with $\mu/a = 0.02$. We see that both σ_{11} and α_1 diverge as $T \to 0$, as the coupling between the $U(1)$ gauge field, A_1, and gravity becomes infinite in the bulk theory. Note that the conductivities here have been rescaled by a factor of $16\pi G_N$ for the ease of presentation.

various values of μ/a, and the low temperature scaling appears to give the same values. It is interesting to compare these results to the results from the neutral case [4], where it was shown that the neutral black holes have low temperature scaling of $\bar{\kappa} \sim T^{10/3}$.

We also find that the electrical conductivities σ_{11} and α_1 diverge as $T \to 0$, despite the fact that translational invariance has been explicitly broken in the z direction. To understand this, we note that as $T \to 0$, $\psi_{1h} \to \infty$. Since the coupling of A_1 in the action is $\exp 2\psi/\sqrt{6}$, as $T \to 0$ this coupling will diverge on the black hole horizon. Therefore, any perturbation of a electric field should lead to an infinite response from the system, and hence an infinite conductivity. Diverging conductivity as $T \to 0$ has been seen before in the case of AdS-RN black holes, but here it is the coupling, rather than the geometry which is driving the instability. A similar result to this was seen in [58].

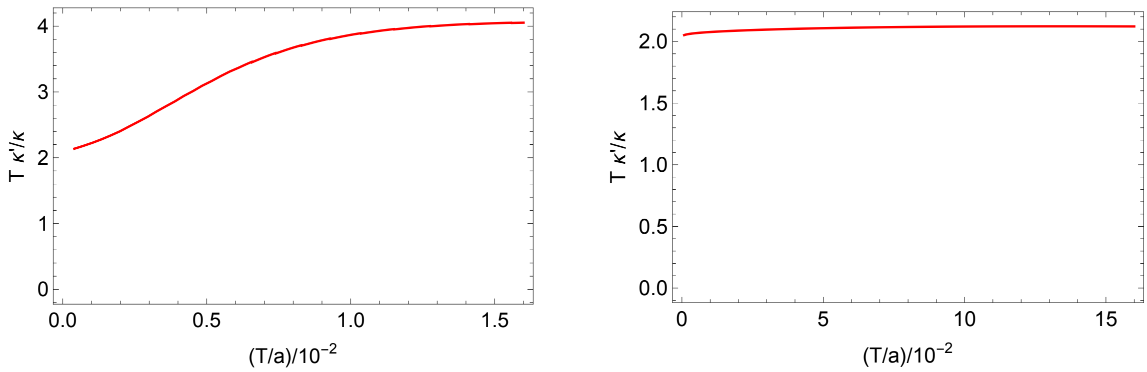


Figure 16: Plot showing the scaling of the DC thermal conductivity, $\bar{\kappa}$ against temperature, T/a, at $\mu/a = 0.02$ (left) and $\mu/a = 1$ (right). In both cases, as $T \to 0$, $\bar{\kappa} \sim T^c$, with $c \sim 2$.

4 Discussion

We have extended the analysis of the last chapter, and constructed a more complete class of black hole solutions that are consistent truncations of the Kaluza-Klein reduction of type IIB supergravity on S^5 [175], and dual to an anisotropically deformed $N = 4$ super Yang-Mills plasma. We have shown that the low temperature phase that we previously constructed is actually unstable below a critical temperature, corresponding to an unstable black hole . Whilst the phase transition we previous found is also present when a finite $U(1)$ chemical potential, μ, is switched on, at $\mu > \mu_c \sim 0.015$, this instability vanishes, and the plasma appears to be stable right down to zero temperature. This charged low temperature plasma has interesting properties, including a divergent electrical conductivity despite the explicit breaking of translational invariance, similar to examples considered in [58].

We first considered a consistent truncation that retained five, rather than one, of the 20 scalar fields that transform in the $\mathbf{20'}$ of $SO(6)$. At low temperatures, the presence of additional scalar fields in the truncation gives rise to further normalisable static modes in the black hole solution constructed in [4], and hence new branches of black hole solutions from a critical point T_{c2}. Rather than undergoing a phase transition to a thermodynamically preferred phase below T_{c2}, however, these new branches of black hole solution only exist at $T > T_{c2}$, and so is an example of retrograde condensation, which has been seen in other top-down holographic models [85, 86]. By analysing perturbations of this scalar mode, we can conclude that the solution is unstable below this critical temperature. It would be interesting to understand more generally when retrograde condensation occurs in a top-down setting.

Following this analysis, we turned on a finite chemical potential, μ, in the dual field theory, using a consistent truncation that contains two scalar fields and three $U(1)$ gauge fields. The phase transition first seen in [4] is also present in the case with a finite chemical potential, with the same order and critical exponents in both cases. In this case, the black hole in [63] undergoes a phase transition to a new charged branch of solution.

Interestingly, whilst the black hole in [63] reaches zero temperature at some finite radius - an extremal black hole - the new branch of solution is not extremal at $T = 0$. Whilst the phase transition from the black hole studied in [63] occurs at all values of the chemical potential, once $\mu > \mu_c \sim 0.015$, the lower temperature instability discussed above is no longer seen - instead, the solution appears to be stable right down to zero temperature solutions.

However, these two truncations are not the end of the story. As with any top-down model, one is always free to include more of the scalar and gauge fields to analyse the full theory. It would be particularly interesting to understand whether the supergravity solutions of [60,61,63] are truly unstable at low temperatures, or if the full matter content leads to a stable phase. It would also be interesting to understand why the presence of a chemical potential removes this instability. To answer both of these questions would require the full consistent truncation from [175], which preserves 20 scalar fields and 15 gauge fields. Whilst a technically challenging task, this would allow us to further understand the phase diagram of the anisotropically deformed $N = 4$ super Yang-Mills plasma. However, in order to fully understand the phase diagram, one would have to construct all the solutions of the dual supergravity theory, not just those with a consistent truncation, and so one would ultimately construct black hole solutions in the full $D = 10$ theory.

Furthermore, it can be shown that the $D = 5$ model preserving the metric, axion, dilaton as well as a single $U(1)$ gauge field can come from a consistent truncation on any five-dimensional Sasaki-Einstein (SE) manifold, not just the five-sphere [181]. Therefore, the CGS and Mateos-Trancanelli solutions describe the high temperature phase of a whole class of dual $\mathcal{N} = 1$ spatially anisotropic plasmas. For a consistent truncation on an arbitrary SE manifold, the unstable scalar modes described here may not exist, and so the plasma would have a different phase diagram. It would be interesting to understand precisely how the results here change for different SE manifolds - since a truncation on an arbitrary SE manifold will not have the same scalar field structure, one may have to construct the black holes directly in ten dimensions.

We studied the thermoelectric response of the new branch of black hole solutions at finite chemical potential. We numerically determined the thermal conductivity, $\bar{\kappa}$, and showed that the plasma is a thermal insulator. Furthermore, we determined that $\bar{\kappa}$ scales at low temperatures by T^c, with $c \sim 2$ in the case where $\mu/a = 1$. In the last chapter, were able to construct the IR behaviour of the zero temperature black hole solution, and found it obeyed a scaling relation with a metric that was both Liftshitz and hyperscaling violating. It would be an interesting avenue of further work to try to find a similar solution in this case. This could possibly shed light on the reason for why a chemical potential stabilises the low temperature theory.

At low temperatures, part of the DC thermoelectric conductivity matrix diverges, despite the explicit breaking of translational invariance in the model. This is because the gauge field coupling in the theory diverges as $T \to 0$, and so any perturbation of the electric field will give an infinite response. This highlights a further challenge in modeling physical systems in a top down setting. In order to introduce finite DC conductivity in a holographic setting, one has to break translational invariance, such as through a spatially dependent source term. However, many top down models contain non-linear couplings between gauge and scalar fields, and so these couplings may well diverge at certain temperatures. Therefore, even when there is a mechanism for momentum dissipation, top down models can still have infinite DC thermoelectric conductivities.

All of the previous works have focused on the case of static black hole solutions, which correspond to electrically charged field theories. However, it would be interesting to understand how this system behaves when there is also a magnetic field present. In this case, the black holes would no longer be static, but simply stationary, and the Chern-Simons term from (7.13) would no longer be zero. Whilst this would be technically challenging, a starting point could be to construct black hole solutions similar to [63] that contains magnetic fields.

Part IV

Conclusions

Chapter 8

Discussion and final thoughts

Understanding the behaviour of strongly coupled systems remains one of the outstanding challenges in modern theoretical physics, and developing new tools to solve these problems could help us to explain high temperature superconductivity, model quark gluon plasmas and perhaps even shed light on the earliest moments in our universe. In this thesis, we have used gauge/gravity duality to develop novel tools to attack these problems, and improved our understanding of holographic lattice models. In particular, we have focused on two themes that are important in real physical systems - phase transitions and linear response.

The central idea in the first part of this thesis was the DC linear response of a holographic material. Firstly, we considered a general class of electrically charged black holes in Einstein-Maxwell-scalar theory. By analysing the linearised perturbations about the background solutions, we demonstrated that the thermoelectric DC conductivity matrix could be determined by solving the linearised Navier-Stokes equation for a charged, incompressible fluid on the event horizon. By solving the Navier-Stokes equations, one can determine the relevant current and thermal fluxes as function of the applied source, which can then be related to the current fluxes of the dual field theory. One can then determine the appropriate two-point functions of the theory, and hence the DC conductivity matrix. Much like black hole thermodynamics, we see that information about the dissipation of the holographic system is actually encoded on the event horizon of the black hole.

Having developed a general formalism to determine the DC thermoelectric conductivity of holographic lattices, we showed that, quite incredibly, for a wide range of systems the Navier-Stokes equations can be actually be solved explicitly. Firstly, we considered Q-lattices [57], where the scalar fields are schematically of the form $\psi = k_j x^j$, for some constant k in the spatial x^i directions. It turns out that in this case the Navier-Stokes equations can be solved exactly, leading to exact results for these black holes. Similarly, one-dimensional lattices, where translational invariance is only broken in one spatial direction, have a DC thermoelectric conductivity that can be written as an integral of background quantities on the event horizon, and so are also completely solved, once the background black hole has been determined. Historically, both of these solutions were determined earlier, using a brute force method to calculate conserved currents [55, 66]. It is therefore pleasing to see that these results fit into a general framework for all holographic lattices.

We then considered a perturbative lattice, where the periodic spatial deformations are constructed as a perturbative expansion about a background AdS-RN black hole. Physically, this corresponds to weak momentum dissipation of some perfect metal. It turns out that the Navier-Stokes equations can be solved order by order in a perturbative expansion, and we explicitly constructed the leading order terms. In particular, by analysing the leading order behaviour of the thermoelectric conductivities, we were able to obtain a Wiedemann-Franz law for strongly coupled holographic lattices.

Having developed a formalism to determine thermoelectric conductivities, we extended these results to include scalar perturbations in a few special cases. We showed that for holographic Q-lattices, a more general DC response matrix can be obtained by considering perturbations of the scalar operators, in addition to electric field and stress energy, perturbations. In this case the DC response could be written exactly in terms of horizon data. We then showed that, for certain theories, namely with a marginal massless scalar operator that doesn't couple to the gauge field, linear in time perturbations of the scalar field lead to new terms in the Navier-Stokes equations.

The fact that black holes encode information about dissipation is not new. This has origins in the membrane paradigm [68], and, more recently, was seen in the hydrodynamic limit [67, 128], and there has been suggestion that these results are linked [182]. Whilst in many ways these earlier results are similar, there are subtle differences in exactly how the two sets of fluid equations are built up. A natural question, and important area of further work, is therefore to try and understand exactly how these results relate to each other. In some ways, however, the fact that these results are related to a membrane paradigm is of secondary importance. The key point here is not that a set of fluid equations *exist*, but that this closed set of equations comes from a *subset* of metric and gauge field perturbations, and that the solution to these equations allows one to calculate thermoelectric fluxes on the horizon, and hence the thermoelectric response of the dual field theory. Fluid flow on black hole horizons thus has a natural place in holography, and is intrinsically linked with linear response of the dual field theory.

We then considered the DC response in the hydrodynamic limit, and showed that, for the simpler case of Einstein gravity, the *local* currents, rather than current fluxes could be determined in terms of boundary data. In addition, in the hydrodynamic regime the entire stress energy tensor response could be determined. On comparing our results to the fluid/gravity approach, we found that the results were the same. In addition, we were able to see an analogous expansion for thermoelectric DC conductivities to the perturbative lattices considered above. Note, however, that these two regimes are fundamentally different. In particular, the perturbative lattice corresponds to weak momentum dissipation, whereby the deformation itself is small but at any wavelength, whilst the hydrodynamic regime corresponds to any momentum dissipation but at long wavelengths (high temperature).

In summary, we know that, in general, at any order we can get DC response of a holographic black hole via the Navier-Stokes equations on the horizon. If we take the hydrodynamic limit, we can use fluid/gravity to get *any* linearised response (i.e AC or DC), which we have shown, is consistent with the Navier-Stokes approach in the hydrodynamic limit. The final piece in the puzzle is therefore a generalised approach to AC conductivity. Whilst this can clearly be solved in terms of the full bulk geometry, it is a natural question to ask if we can determine AC conductivity in terms of horizon data in a small ω limit, and build up some expansion order by order. This would effectively be a derivative expansion in time, rather than the typical hydrodynamic derivative expansion.

Keeping with the theme of hydrodynamics, we then considered general conformal field theories on curved spacetimes that have been deformed by spatially dependent and periodic local temperature variations, at the level of first order hydrodynamics. We argued that, when we considered linear response, thermal transport is governed by forced Navier-Stokes equations in curved space. We were then able to demonstrate that, under certain conditions, these systems exhibit thermal backflow when driven by a DC source - the thermal currents can locally flow in the opposite direction to the applied source. Note that while backflow in hydrodynamics isn't something new, and can occur due to momentum dissipating processes at boundaries, the key point here is that the CFT has been deformed by local periodic stresses. As far as we can tell, this is a new result. Electric current backflow has been observed experimentally [140, 141], so the next natural question is to ask if can we find similar results with electrically charged systems, by solving the charged Navier-Stokes equations.

The second major topic we studied was phase transitions. Specifically, we considered black hole solutions that are dual to a N=4 super Yang-Mills plasma, with anisotropic spatial deformations in four dimensions that were previously constructed in [60, 61], as well as the solution with a finite chemical potential considered in [63]. By considering additional scalar modes that are preserved in the consistent truncation of type IIB supergravity Kaluza-Klein reduced on a five-sphere, we showed that these black hole solutions are unstable at low temperatures, leading to new branches of solution that are thermodynamically preferred. We carried on this process further, truncating out fewer scalar fields, and showed that, at low temperatures, this potentially leads to problems with the theory. This is an important lesson in constructing top-down holographic models. Whilst you may have constructed nice, well behaved solutions in your reduced spacetime, you always have to be careful that modes from your full theory won't lead to further states in the phase diagram at low temperatures.

In the phase transitions described above, the critical exponents are different to the standard quadratic Ginzburg-Landau theory. This can be explained by the cubic term that is in the Lagrangian. It would be interesting to see if this is observed here. Whilst some work has gone into understanding critical exponents of phase transitions [74], there are still some outstanding questions. In particular, we were unable to find an appropriate Ginzburg-Landau expansion to characterise our phase transition. The lack of a Ginzburg-Landau expansion would suggest that our system is not mean field. Shedding light on this could give insights into the general nature of holographic phase transitions. In particular, are holographic phase transition naturally mean field, or can one capture a wider range of phase transitions.

The solutions considered in [60, 61, 63] are actually part of a wider class of solutions, dubbed "τ-lattices" [65]. Many of the features from our earlier work that led to the instabilities are present in the other solutions that can appear in this class, so a natural piece of further work would be to ask if the instabilities described above are also seen here. In particular, in the work of [65], black holes were constructed that approached AdS in both the UV and IR, but had some intermediate scaling regime that approached a Liftshitz scaling. This Liftshitz scaling was the cause of the phase transitions that were observed in our results, so it would be interesting to see whether this scaling regime will lead to holographic instabilities.

In conclusion, in this thesis we have demonstrated several new results relating to the DC linear response of strongly coupled systems, by considering the holographic dual of the field theory. In particular, we showed that the DC thermoelectric conductivity of a black hole could be understood in terms of black hole horizon data. This has opened up new questions on the relationship between fluid dynamics and gravity. We then considered top-down holographic models, and showed that existing holographic models were unstable at low temperatures. We were able to construct new black hole solutions, corresponding to new phases of the strongly coupled theory. These results may go some way to helping us answer one of physics biggest questions - in general, how can we mathematically understand and model strongly coupled systems.

Part V

Appendices and Bibliography

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Appendix A

Chapter 2 appendix

A.1 Radial Hamiltonian formalism

In this section we rewrite the equations of motion corresponding to the Lagrangian density:

$$\mathcal{L} = \sqrt{-g}\left(R - V(\phi) - \frac{Z(\phi)}{4}F^2 - \frac{1}{2}(\partial\phi)^2\right), \tag{A.1}$$

using a Hamiltonian decomposition with respect to the radial variable. We will follow the notation of [183], *mutatis mutandi*, generalising to include the gauge-field and the scalar field. A useful reference is [184] and we note that closely related work independently appeared recently in [185].

We consider a foliation by slices of constant r. We introduce the normal vector n^μ, satisfying $n^\mu n_\mu = 1$. The D-dimensional metric $g_{\mu\nu}$ induces a $(D-1)$-dimensional Lorentzian metric on the slices of constant r via $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The lapse and shift vectors are given by $n_\mu = N(dr)_\mu$ and $N^\mu = h^\mu{}_\nu r^\mu = r^\mu - Nn^\mu$ where $r^\mu = (\partial_r)^\mu$. In a local coordinate system we can write

$$ds^2 = N^2 dr^2 + \gamma_{ab}(dx^a + N^a dr)(dx^b + N^b dr), \tag{A.2}$$

where the shift vector has components $N^\mu = (0, N^a)$ and $h_{\mu\nu}$ has components $h_{rr} = N^a N^b \gamma_{ab}$, $h_{ra} = \gamma_{ab}N^b$ and $h_{ab} = \gamma_{ab}$. Note also that $N_\mu = (N^b N^c \gamma_{bc}, \gamma_{ab}N^a)$.

We will decompose the gauge-field components via

$$b_\mu = h_\mu{}^\nu A_\nu, \qquad \Phi = -Nn^\mu A_\mu, \tag{A.3}$$

and hence $A_\mu = b_\mu - N^{-1}\,\Phi\, n_\mu$. In the local coordinates we have $b_r = N^a A_a$, $b_a = A_a$ and $\Phi = -A_r + N^a A_a$.

The radial Hamiltonian formulation can be obtained by first rewriting the Lagrangian density as follows

$$\mathcal{L} = N\sqrt{-h}\left({}^{(D-1)}R + K^2 - K_{\mu\nu}K^{\mu\nu} - V - \frac{1}{4}Zf_{\mu\nu}f_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - \frac{1}{2}ZX_\mu h^{\mu\nu}X_\nu \right.$$
$$\left. - \frac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu - \frac{1}{2}(n^\mu\partial_\mu\phi)(n^\nu\partial_\nu\phi)\right), \tag{A.4}$$

where we have neglected total divergences. Here $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} = h_\mu{}^\rho\nabla_\rho n_\nu$ is the extrinsic curvature, $K = g^{\mu\nu}K_{\mu\nu}$ and

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu\,, \\ X_\mu &= f_{\mu\nu}n^\nu + \frac{\Phi}{N}n^\nu\nabla_\nu n_\mu - D_\mu\left(\frac{\Phi}{N}\right)\,, \\ &= f_{\mu\nu}n^\nu - \frac{1}{N}D_\mu\Phi\,, \end{aligned} \tag{A.5}$$

where the second expression utilises the fact that $n^\mu\nabla_\mu n_\nu = -\frac{1}{N}D_\nu N$. This latter result follows from writing $n_\nu = N\nabla_\nu r$ and using $\nabla_{\mu\nu}r = \nabla_{\nu\mu}r$. We also recall that D_μ is the Levi-Civita connection associated with the metric h and, for example, $D_\mu\Phi = h_\mu{}^\nu\nabla_\nu\Phi$.

With $\dot{h}_{\mu\nu} = \mathcal{L}_r h_{\mu\nu}$, $\dot{b}_\mu = \mathcal{L}_r b_\mu$, $\dot{\phi} = \mathcal{L}_r\phi$ we can show that

$$\begin{aligned} \dot{h}_{\mu\nu} &= 2NK_{\mu\nu} + D_\mu N_\nu + D_\nu N_\mu\,, \\ \dot{b}_\mu &= r^\rho f_{\rho\mu} + \nabla_\mu(b_\rho N^\rho)\,, \\ &= Nn^\rho f_{\rho\mu} + N^\rho f_{\rho\mu} + \nabla_\mu(b_\rho N^\rho)\,, \\ \dot{\phi} &= Nn^\mu\partial_\mu\phi + N^\mu\partial_\mu\phi\,. \end{aligned} \tag{A.6}$$

The corresponding conjugate momenta are then given by

$$\begin{aligned} \pi^{\mu\nu} &= \frac{\delta\mathcal{L}}{\delta\dot{h}_{\mu\nu}} = -\sqrt{-h}\,\left(K^{\mu\nu} - K\,h^{\mu\nu}\right)\,, \\ \pi^\mu &= \frac{\delta\mathcal{L}}{\delta\dot{b}_\mu} = \sqrt{-h}h^{\mu\rho}ZX_\rho\,, \\ &\qquad = \sqrt{-h}ZF^{\mu\rho}n_\rho\,, \\ \pi_\phi &= \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = -\frac{\sqrt{-h}}{N}\left(\dot{\phi} - N^\nu\partial_\nu\phi\right)\,, \\ &\qquad = -\sqrt{-h}n^\nu\partial_\nu\phi\,, \end{aligned} \tag{A.7}$$

where the second expressions for π^μ and π_ϕ, which are not written in the canonical variables, are useful.

The Hamiltonian density, defined as $\mathcal{H} = \pi^{\mu\nu}\dot{h}_{\mu\nu} + \pi^\mu\dot{a}_\mu - \mathcal{L}$, can be written as a sum of constraints

$$\mathcal{H} = N\,H + N_\mu\,H^\mu + \Phi\,C\,, \tag{A.8}$$

with

$$\begin{aligned}
H =& -(-h)^{-1/2}\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-2}\,\pi^2\right) - \sqrt{-h}\,\left({}^{(D-1)}R - V\right)\\
&- \frac{1}{2}\,(-h)^{-1/2}\,Z^{-1}h_{\mu\nu}\,\pi^\mu\,\pi^\nu + \frac{1}{4}\,\sqrt{-h}\,Z f_{\mu\nu}\,f_{\rho\sigma}\,h^{\mu\rho}\,h^{\nu\sigma}\\
&- \frac{1}{2}(-h)^{-1/2}\pi_\phi^2 + \frac{1}{2}\sqrt{-h}h^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\,, \qquad (A.9)\\
H^\nu =& -2\sqrt{-h}\,D_\mu\left((-h)^{-1/2}\pi^{\mu\nu}\right) + h^{\nu\sigma}f_{\sigma\rho}\pi^\rho\\
&- h^{\nu\sigma}a_\sigma\sqrt{-h}D_\rho\left((-h)^{-1/2}\,\pi^\rho\right) + h^{\nu\sigma}\partial_\sigma\phi\pi_\phi\,, \qquad (A.10)\\
C =& \sqrt{-h}\,D_\mu\left((-h)^{-1/2}\,\pi^\mu\right)\,, \qquad (A.11)
\end{aligned}$$

where $\pi = \pi^\mu{}_\mu$ and we have ignored total divergences[1].

The equations of motion are given by

$$\begin{aligned}
\dot{h}_{\mu\nu} =& -2N(-h)^{-1/2}\left(\pi_{\mu\nu} - \frac{1}{d}\,\pi\,h_{\mu\nu}\right) + 2\,D_{(\mu}\,N_{\nu)}\,,\\
\dot{\pi}^{\mu\nu} =& -N\,\sqrt{-h}\,\left({}^{(d+1)}R^{\mu\nu} - \frac{1}{2}{}^{(d+1)}R\,h^{\mu\nu} + \frac{1}{2}V\,h^{\mu\nu}\right)\\
&- \frac{1}{2}N\,(-h)^{-1/2}\,h^{\mu\nu}\left(\pi_{\gamma\delta}\pi^{\gamma\delta} - \frac{1}{d}\,\pi^2\right) + 2N(-h)^{-1/2}\left(\pi^{\mu\gamma}\,\pi^\nu{}_\gamma - \frac{1}{d}\,\pi\,\pi^{\mu\nu}\right)\\
&+ \sqrt{-h}\,\left(D^\mu D^\nu N - h^{\mu\nu}\,D^\gamma D_\gamma N\right)\\
&+ \frac{1}{2}N(-h)^{-1/2}Z^{-1}\left(\pi^\mu\pi^\nu - \frac{1}{2}h^{\mu\nu}(h_{\rho\sigma}\pi^\rho\pi^\sigma)\right)\\
&+ \frac{1}{2}N\sqrt{-h}Z\left(h^{\mu\lambda}h^{\nu\gamma}h^{\rho\sigma}f_{\lambda\rho}f_{\gamma\sigma} - \frac{1}{4}h^{\mu\nu}(h^{\rho\sigma}h^{\gamma\delta}f_{\rho\gamma}f_{\sigma\delta})\right)\\
&- \frac{1}{4}N(-h)^{-1/2}\pi_\phi^2 h^{\mu\nu} + \frac{1}{2}N\sqrt{-h}\left(h^{\mu\rho}h^{\nu\sigma}\partial_\rho\phi\partial_\sigma\phi - \frac{1}{2}(h^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi)h^{\mu\nu}\right)\\
&+ \sqrt{-h}\,D_\gamma\left((-h)^{-1/2}N^\gamma\pi^{\mu\nu}\right) - 2\,\pi^{\gamma\,(\mu}\,D_\gamma N^{\,\nu)} \qquad (A.12)
\end{aligned}$$

as well as

$$\begin{aligned}
\dot{b}_\mu =& D_\mu(N^\nu b_\nu) - N\,(-h)^{-1/2}\,\pi_\mu + N^\nu\,f_{\nu\mu} - D_\mu\Phi\,,\\
\dot{\pi}^\mu =& \sqrt{-h}\,D_\sigma\left(N Z\,h^{\sigma\rho}h^{\mu\delta}f_{\rho\delta}\right) + 2\sqrt{-h}\,D_\sigma\left((-h)^{-1/2}\,N^{[\sigma}\,\pi^{\mu]}\right)\,, \qquad (A.13)
\end{aligned}$$

where we have dropped pieces proportional to the constraints in $\dot{\pi}^{\mu\nu}$ and $\dot{\pi}^\mu$. We also

[1] In passing that we note that in the local coordinates (A.2) we have $\Phi - N^\mu a_\mu = -A_r$ and one sees that using explicit coordinates one could uses A_r as a Lagrange multiplier instead of Φ.

have

$$\begin{aligned}\dot{\phi} =& -(-h)^{-1/2} N\pi_\phi + N^\nu \partial_\nu \phi\,,\\ \dot{\pi}_\phi =& \sqrt{-h} D_\mu(\sqrt{-h}\pi_\phi N^\mu) + \sqrt{-h} D_\mu(N h^{\mu\nu} D_\nu \phi)\\ &- N\left(\sqrt{-h}V' + \frac{1}{2}(-h)^{-1/2}\,Z^{-2} Z' h_{\mu\nu}\,\pi^\mu\,\pi^\nu + \frac{1}{4}\sqrt{-h}\,Z' f_{\mu\nu}\,f_{\rho\sigma}\,h^{\mu\rho}\,h^{\nu\sigma}\right). \end{aligned} \tag{A.14}$$

A.2 Generalised Stokes equations from the constraints

In this appendix we show how the Stokes equations (2.40)-(2.42) arise from the constraint equations (A.9)-(A.11) in a radial decomposition. More precisely we will examine the constraints for the perturbed metric, at linearised order, focussing on the leading terms of an expansion at the black hole horizon. In other words, we evaluate the constraints on a hypersurface of constant r and then take the limit $r \to 0$.

We begin by noting that for the perturbed metric the unit normal vector has components

$$\begin{aligned} n^i &= -U^{1/2} F^{-1/2}\, g_d^{ij}\, \delta g_{rj}, \qquad n^t = G^{-1}\,(FU)^{-1/2}\,\delta g_{tr},\\ n^r &= U^{1/2} F^{-1/2}\left(1 - \frac{U}{2F}\,\delta g_{rr}\right). \end{aligned} \tag{A.15}$$

Furthermore, the corresponding shift and lapse functions are given by

$$\begin{aligned} N^j &= g_d^{ij}\,\delta g_{ri}, \qquad N^t = -\frac{1}{GU}\,\delta g_{rt}\,,\\ N &= F^{1/2} U^{-1/2}\left(1 + \frac{1}{2}\frac{U}{F}\,\delta g_{rr}\right), \end{aligned} \tag{A.16}$$

The components of the extrinsic curvature take the form

$$\begin{aligned} K^{tt} =& \frac{1}{2}\,G^{-2} U^{-3/2} F^{-1/2}\left(-\partial_r\,(GU) + \frac{1}{2}\frac{U}{F}\,\partial_r\,(GU)\;\delta g_{rr}\right)\\ &+ \frac{1}{2}\,G^{-2} U^{-3/2} F^{-1/2}\left((GU)^2\,\partial_r\left(\frac{\delta g_{tt}}{(GU)^2}\right) + \partial_j (GU)\,N^j\right),\\ K^{ti} =& \frac{1}{2}\,U^{1/2} F^{-1/2}\left(-\partial_r\left(\frac{1}{GU}\,g_d^{ij}\;(\delta g_{tj} - tGU\zeta_j)\right) + g_d^{ij}\,\partial_j\left(\frac{1}{GU}\,\delta g_{rt}\right)\right),\\ K^{ij} =& -U^{1/2} F^{-1/2}\,\nabla^{(i} N^{j)} + \frac{1}{2}\,U^{1/2} F^{-1/2}\left(\frac{U}{2F}\partial_r g_d^{ij}\,\delta g_{rr} - \partial_r g_d^{ij} + g_d^{ik} g_d^{jl}\partial_r \delta g_{kl}\right)\\ &- U^{1/2} F^{-1/2}\,g_d^{l(i} g_d^{j)m} g_d^{kn} \delta g_{mn}\,\partial_r g_{dkl}\,. \end{aligned} \tag{A.17}$$

where here ∇ is the covariant derivative compatible with the d-dimensional metric g_{dij}.

Expanding the extrinsic curvature close to the horizon we find

$$
\begin{aligned}
K^{tt} &\to -\frac{1}{2}\frac{1}{(4\pi T)^{1/2}}\frac{1}{r^{3/2}}\frac{1}{G^{(0)3/2}}\left(1+\frac{\delta g^{(0)}_{tt}}{G^{(0)}}-\frac{1}{2}\frac{\delta g^{(0)}_{rr}}{G^{(0)}}+\frac{1}{4\pi T}v^i\,\partial_i \ln G^{(0)}\right)\,,\\
K^{ti} &\to -\frac{1}{2}\frac{1}{(4\pi T)^{1/2}}\frac{1}{r^{3/2}}\frac{1}{G^{(0)3/2}}\,v^i\,,\\
K^{ij} &\to \frac{1}{(4\pi T)^{1/2}}\frac{1}{r^{1/2}}\frac{1}{G^{(0)1/2}}\,\nabla^{(i}v^{j)}\,,\\
K &\to \frac{1}{2}\frac{(4\pi T)^{1/2}}{G^{(0)1/2}r^{1/2}}\left(1-\frac{1}{2}\frac{\delta g^{(0)}_{rr}}{G^{(0)}}+\frac{1}{4\pi T}\,\nabla_i v^i+\frac{1}{4\pi T}\,v^i\nabla_i \ln G^{(0)}\right)\,,
\end{aligned}
\tag{A.18}
$$

and we also note that

$$
K^t{}_i \to \frac{1}{2}\frac{(4\pi T)^{1/2}}{G^{(0)1/2}r^{1/2}}\,\zeta_i\, t\,. \tag{A.19}
$$

We now consider the following quantity which appears in the momentum constraint (A.10)

$$
\begin{aligned}
W_\nu =& D_\mu\left((-h)^{-1/2}\pi^\mu{}_\nu\right) = -D_\mu K^\mu{}_\nu + D_\nu K\\
=& -(-h)^{-1/2}\,\partial_\mu\left(\sqrt{-h}\,K^\mu{}_\nu\right)+\frac{1}{2}\,\partial_\nu h_{\kappa\lambda}K^{\kappa\lambda}+\partial_\nu K\,.
\end{aligned}
\tag{A.20}
$$

Expanding at the horizon we find the following individual components

$$
\begin{aligned}
W_t &\to -\frac{1}{2}\frac{(4\pi T)^{1/2}}{G^{(0)1/2}}\frac{1}{r^{1/2}}\,\nabla_i v^i\,,\\
W_i &\to \frac{1}{G^{(0)1/2}}\frac{1}{(4\pi T)^{1/2}}\frac{1}{r^{1/2}}\left(-\nabla^j\nabla_{(j}v_{i)}-2\pi T\,\zeta_j+\frac{1}{2}\,\nabla_i p\right)
\end{aligned}
\tag{A.21}
$$

where

$$
p = -2\pi T\frac{1}{G^{(0)}}\left(\delta g^{(0)}_{tt}+g^{(0)}_{rr}\right)-\delta g^{(0)}_{it}\,g^{ij}_{(0)}\nabla_j\,\ln G^{(0)}\,. \tag{A.22}
$$

Notice that after imposing the boundary condition constraints (2.14) this definition of p is identical to the definition of pressure given in (2.39). Another quantity that enters the constraints is the momentum of the scalar field. At leading order in r we have

$$
\pi_\phi \to -\sqrt{g_{(0)}}\,v^i\partial_i\phi\,. \tag{A.23}
$$

We now turn to the gauge field. From the second expression in (A.7) we have

$$
\pi^\mu = \sqrt{-h}\,Z\,F^{\mu\lambda}\,n_\lambda = \sqrt{-g}\,Z\,F^{\mu r} = 16\pi G_N J^\mu\,. \tag{A.24}
$$

After expanding near the horizon we also find

$$f_{t\mu}\pi^{\mu} = \frac{1}{16\pi G_N} f_{ti} J^i \to 0\,,$$
$$f_{i\mu}\pi^{\mu} = \frac{1}{16\pi G_N}\left(f_{it}J^t + f_{ij}J^j\right) \to \frac{1}{16\pi G_N}\left(\partial_i w + E_i\right) J^t = \frac{1}{16\pi G_N} g_{(0)}^{1/2}\, Z^{(0)}\, a_t^{(0)}\, \left(\partial_i w + E_i\right)\,. \tag{A.25}$$

We now now consider the constraint equations. Substituting (A.24) into the Gauss constraint (A.11), $\partial_\mu \pi^\mu = 0$, we obtain the current continuity equation $\partial_\mu J^\mu = 0$. When evaluated at the horizon this leads to $\partial_i J^i_{(0)} = 0$. We next consider the momentum constraints (A.10) with lowered indices. Using the above results we find that the t component gives $\nabla_i v^i = 0$ while the i component gives the Stokes equation (2.42).

Finally, examining the Hamiltonian constraint (A.9), we find that the leading order expansion at the horizon for the terms involving the linearised perturbation implies $\nabla_i v^i = 0$, and hence gives no further conditions. To see this we consider H in (A.9) as a sum of six terms and it is convenient to divide by $\sqrt{-h}$. Using (A.23) and (A.24) we immediately see that the third and fifth terms vanish at linearised order. It turns out that the leading order power of r that appears is r^{-1}. We can show that the sixth term and, with a bit more effort, the fourth terms are of order r^0. Next we consider the second term. The potential term is clearly of order r^0. After examining the leading terms in the Christoffel symbols we can also show that the Ricci scalar term is also of this order. Finally, we need to examine the first term. To do so it is convenient to note that using (A.18) we have

$$\begin{aligned}
(-h)^{-1/2}\pi^{tt} &\to -\frac{1}{2}\frac{1}{(4\pi T)^{3/2} G^{(0)3/2}}\frac{1}{r^{3/2}}\nabla_i v^i\,,\\
(-h)^{-1/2}\pi^{ij} &\to \frac{1}{2}\frac{1}{(4\pi T)^{1/2} G^{(0)1/2}}\frac{1}{r^{1/2}}\Big(-2\nabla^{(i}v^{j)} - 4\pi T \delta g^{(0)}_{kl}\, g^{ik}_{(0)} g^{jl}_{(0)}\\
&\qquad + 4\pi T g^{ij}_{(0)}\big(1 - \frac{1}{2G^{(0)}}\delta g^{(0)}_{rr} + \frac{1}{4\pi T}\nabla_i v^i + \frac{1}{4\pi T} v^i \nabla_i \ln G^{(0)}\big)\Big)\,,
\end{aligned} \tag{A.26}$$

Continuing to evaluate the first term we are eventually led to the result that the leading term in the Hamiltonian constraint can be written

$$(-h)^{-1/2} H \to \frac{1}{2rG^{(0)}}\nabla_i v^i. \tag{A.27}$$

A.3 Holographic currents

On-shell we have

$$\delta S = \int d^3x \sqrt{-h}\left[\tfrac{1}{2}\left(r^{-(D+1)} t^{\mu\nu}\right)\delta h_{\mu\nu} + \left(r^{1-D} j^\mu\right)\delta A_\mu\right]\,. \tag{A.28}$$

where $t^{\mu\nu}$ and j^μ are the radially independent, holographic stress tensor and current, respectively. After substituting the time dependent sources given in (F.11), (2.12) we find

$$\delta S = \int d^3x \bar{G}^{1/2}\sqrt{\bar{g}_d}\left[(\bar{G}t^{ti} - \mu j^i)(-t\zeta_i) + j^i(-tE_i)\right] . \tag{A.29}$$

We thus see that $-tE_i$ is a source for the operator density $\bar{G}^{1/2}\sqrt{\bar{g}_d}j^i$ and $-t\zeta_i$ is a source for the operator density $\bar{G}^{1/2}\sqrt{\bar{g}_d}(\bar{G}t^{ti} - \mu j^i)$.

It is possible to show that the expectation values of these holographic tensor densities are given by

$$\begin{aligned} \bar{G}^{1/2}\sqrt{\bar{g}_d}j^i &= J^i|_\infty , \\ \bar{G}^{1/2}\sqrt{\bar{g}_d}(\bar{G}t^{ti} - \mu j^i) &= Q^i|_\infty - t\bar{G}^{3/2}\sqrt{\bar{g}_d}t^{ij}\zeta_j . \end{aligned} \tag{A.30}$$

Thus $J^i|_\infty$ and $Q^i|_\infty$ are the time-independent parts of the expectation values of the vector and tensor densities. To establish the first equation in (A.30) is straightforward. The second is a little more involved. Firstly, from the expression for Q^i given in (4.12) and using (A.17) we can show that at linearised order we have

$$Q^i = \frac{1}{16\pi G_N}F(GU)^{3/2}\sqrt{g_d}\left(-2K^{ti} + 2K^{ij}\frac{1}{GU}\left(\delta g_{tj} - tGU\zeta_j\right)\right) - a_t J^i . \tag{A.31}$$

Recall that if we write

$$16\pi G_N \tilde{t}^{\mu\nu} = -2K^{\mu\nu} + Xh^{\mu\nu} + Y^{\mu\nu} , \tag{A.32}$$

where $X = 2K + f(\phi) + \ldots$ and Y corresponds to additional terms arising from the counterterms, then we obtain the stress tensor if we evaluate $\tilde{t}^{\mu\nu}$ at the AdS boundary. Observing that at linearised order we have $g_{tt}h^{ti} = h^{ij}\left(\delta g_{tj} - tGU\zeta_j\right)$ we can therefore write

$$Q^i = \frac{1}{16\pi G_N}(GU)^{3/2}\sqrt{g_d}\left((\tilde{t}^{ti} - Y^{ti}) - (\tilde{t}^{ij} - Y^{ij})\frac{1}{GU}\left(\delta g_{tj} - tGU\zeta_j\right)\right) - a_t J^i . \tag{A.33}$$

We next want to take a limit as $r \to \infty$. If the combination of Y^{ti} and Y^{ij} that appear make sub-leading contributions, then we have

$$Q^i|_\infty = \lim_{r\to\infty}\left[r^{D+1}\bar{G}^{3/2}\sqrt{g_d}\left(\tilde{t}^{ti} + \tilde{t}^{ij}t\zeta_j\right) - a_t J^i\right] \tag{A.34}$$

and we recover (A.30) after using that $r^{D+1}\tilde{t}^{\mu\nu} = t^{\mu\nu}$. We have explicitly checked for particular cases, eg $D = 4$ with $Y^{\mu\nu} \sim R^{(3)\mu\nu}$ that this does indeed occur. It would be interesting to find a universal argument that this is always true.

The time dependent piece on the right hand side of (A.30) is associated with the static susceptibility for the heat current two point function, as explained in appendix C of [66]. It can also be understood by noting that if we start with the background black hole geometries, with in particular $t^{ti} = j^i = 0$, then the time independent linear perturbation that is generated by the coordinate transformation $t \to t + \zeta_i x^i$, induces the transformation $t^{ti} \to -\zeta_i t^{ij}$. Promoting this perturbation to one that is linear in time leads to the time dependence as in (A.30).

A.4 Alternative derivation of the Stokes equations

We discussed in section 3.3 how the generalised Stokes system of equations given in (2.40)-(2.42) can also be obtained from the equations (2.18), (2.30) for J^i and Q^i. Indeed evaluating the first of the two equations in each of (2.18), (2.30) at the black hole horizon we immediately obtain $\partial_i J^i_{(0)} = \partial_i Q^i_{(0)} = 0$. These comprise two of the three Stokes equations, given in (2.40), (2.41). The third Stokes equation, given in (2.42), can be obtained from the second equation of (2.30).

To obtain it we consider the pieces of Q^i that are linear in r obtaining

$$\frac{\sqrt{g_{(0)}}}{16\pi G_N}\left[-G^{(0)} g^{ij}_{((0))}\partial_j\left(4\pi T\frac{\delta g^{(0)}_{rt}}{G^{(0)}}\right) - \delta g^{(0)}_{jt} M^{ij} - 4\pi T G^{(0)}\zeta_j\right] - a^{(0)}_t J^i_{(0)}\,, \tag{A.35}$$

where we have defined the matrix

$$M^{ij} = g^{ij}_{(0)}\left[4\pi T\left(\frac{3G^{(1)}}{2G^{(0)}} - \frac{F^{(1)}}{2G^{(0)}}\right) + 2U^{(1)}\right] + 4\pi T\, g_{(0)}^{-1/2}\left(\sqrt{g}g^{ij}\right)^{(1)}\,. \tag{A.36}$$

Notice that this matrix depends on next to leading order terms in the expansion at the black hole horizon. Equation (2.30) then implies that (A.35) should equal

$$-\frac{1}{16\pi G_N}\partial_j\left[G^{(0)2}\sqrt{g_{(0)}}g^{jk}_{(0)}g^{il}_{(0)}\left(\partial_k\left(\frac{\delta g^{(0)}_{lt}}{G^{(0)}}\right) - k\leftrightarrow l\right)\right]\,. \tag{A.37}$$

To obtain an equation at the black hole horizon, we need to be able to express the matrix M in terms of leading order horizon data. After a long calculation, which we outline below, using the equations of motion for the background black hole we can show the key result

$$\begin{aligned} M^{ij} =& -2G^{(0)}\,{}^{(d)}R^{ij} + 4\,G^{(0)1/2}\,\nabla^i\nabla^j G^{(0)1/2} - g^{ij}_{(0)}\Box G^{(0)} \\ &+ Z\,G^{(0)}a^{(0)2}_t\, g^{ij}_{(0)} + G^{(0)}g^{ik}_{(0)}g^{jl}_{(0)}\partial_k\phi^{(0)}\partial_l\phi^{(0)}\,, \end{aligned} \tag{A.38}$$

and this leads to the final Stokes equation in (2.42).

We use the radial Hamiltonian presentation of the equations of motion for the background black hole solutions (2.4). The unit normal vector is $n = U^{1/2}\,F^{-1/2}\,\partial_r$. The lapse function is given by $N = U^{-1/2}F^{1/2}$ and the shift vector vanishes, $N^\mu = 0$. The non-vanishing components of the extrinsic curvature are given by

$$
\begin{aligned}
K_{tt} &= -\frac{1}{2}\,U^{1/2}\,F^{-1/2}\,\partial_r\,(UG)\ ,\\
K_{ij} &= \frac{1}{2}\,U^{1/2}\,F^{-1/2}\,\partial_r g_{ij}\ ,
\end{aligned}
\tag{A.39}
$$

and hence the non-vanishing components of the conjugate momentum are given by

$$
\begin{aligned}
\pi^{tt} &= -\,(FG)^{-1/2}\;\partial_r\,g^{1/2}\,,\\
\pi^{ij} &= -\frac{1}{2}U\;(G\,g)^{1/2}\;F^{-1/2}\,g^{ik}\,g^{jl}\,\partial_r\,g_{kl} + U^{1/2}\,F^{-1/2}\,g^{ij}\,\partial_r\;(UGg)^{1/2}\ ,\\
\pi^{t} &= (FG)^{-1/2}\,g^{1/2}Z\,\partial_r a_t\,,\\
\pi_\phi &= -G^{1/2}F^{1/2}g^{1/2}\dot{\phi}\,.
\end{aligned}
\tag{A.40}
$$

We also have

$$
\pi = h_{\mu\nu}\pi^{\mu\nu} = d\,U^{1/2}\,F^{-1/2}\,\partial_r\,(U\,G\,g)^{1/2}\,. \tag{A.41}
$$

It is convenient to rewrite the equation of motion for $\pi^{\mu\nu}$ given in (A.12) in the form

$$
\begin{aligned}
&\dot{\pi}^{\mu\nu} - 2N(-h)^{-1/2}\left(\pi^{\mu\gamma}\,\pi^{\nu}{}_{\gamma} - \frac{1}{d}\,\pi\,\pi^{\mu\nu}\right) = \\
&-N\sqrt{-h}\,\left({}^{(d+1)}R^{\mu\nu} - {}^{(d+1)}R\,h^{\mu\nu} + V\,h^{\mu\nu}\right) + \sqrt{-h}\,\left(D^\mu D^\nu N - h^{\mu\nu}\,D^\gamma D_\gamma N\right)\\
&+\frac{1}{2}NZ\sqrt{-h}\left(h^{\mu\lambda}h^{\nu\gamma}h^{\rho\sigma}f_{\lambda\rho}f_{\gamma\sigma} - \frac{1}{2}h^{\mu\nu}(h^{\rho\sigma}h^{\gamma\delta}f_{\rho\gamma}f_{\sigma\delta})\right) + \frac{1}{2}N(-h)^{-1/2}Z^{-1}\pi^\mu\pi^\nu\\
&+\frac{1}{2}N\sqrt{-h}\,(h^{\mu\rho}h^{\nu\sigma}\partial_\rho\phi\partial_\sigma\phi - h^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\,h^{\mu\nu})\ ,
\end{aligned}
\tag{A.42}
$$

where we used the fact that $N^\mu = 0$ as well as the constraint $H = 0$ with H as in (A.9).

We now wish to plug in the background expansions (2.6) in the equations of motion (A.42). Taking the $r \to 0$ limit of the left hand side yields

$$
\begin{aligned}
&\dot{\pi}^{tt} - 2N(-h)^{-1/2}\left(\pi^{t\gamma}\,\pi^{t}{}_{\gamma} - \frac{1}{d}\,\pi\,\pi^{tt}\right) \to -\frac{1}{r}\,\frac{1}{G^{(0)}}\;(\sqrt{g})^{(1)}\ ,\\
&\dot{\pi}^{ij} - 2N(-h)^{-1/2}\left(\pi^{i\gamma}\,\pi^{j}{}_{\gamma} - \frac{1}{d}\,\pi\,\pi^{ij}\right) \to \frac{\sqrt{g_{(0)}}}{2}\,M^{ij} + 4\pi T\,g^{ij}_{(0)}\;(\sqrt{g})^{(1)}\ ,
\end{aligned}
\tag{A.43}
$$

with M^{ij} as defined in equation (A.36). For the right hand side of (A.42) we find the following leading order behaviour as $r \to 0$:

$$
\begin{aligned}
(tt) \to & -\frac{1}{r}\frac{1}{4\pi T}\, g_{(0)}^{1/2}\, \left({}^{(d+1)}R^{tt}_{(0)}\, G^{(0)}\, 4\pi T r + {}^{(d+1)}R_{(0)} - V_{(0)}\right) \\
& + \frac{1}{r}\frac{1}{4\pi T}\, g_{(0)}^{1/2}\, G^{(0)1/2}\, \left(-\frac{1}{2}\frac{1}{G^{(0)2}} g^{ij}_{(0)}\, \partial_i G^{(0)}\, \partial_j G^{(0)1/2} + \frac{1}{G^{(0)}}\, D^\gamma D_\gamma G^{(0)1/2}\right) \\
& + \frac{1}{r}\frac{1}{4\pi T}\frac{1}{2\, G^{(0)2}}\, g_{(0)}^{1/2}\, Z a_t^{(0)2} + \frac{1}{4\pi r T}\frac{1}{2} g_{(0)}^{1/2} g^{ij}_{(0)} \partial_i \phi^{(0)} \partial_j \phi^{(0)}\,,
\end{aligned}
\tag{A.44}
$$

$$
\begin{aligned}
(ij) \to & - G^{(0)} g_{(0)}^{1/2}\, \left({}^{d+1}R^{ij}_{(0)} - {}^{d+1}R_{(0)}\, g^{ij}_{(0)} + V_{(0)}\, g^{ij}_{(0)}\right) \\
& + \sqrt{G^{(0)}\, g^{(0)}}\, \left(D^i D^j\, G^{(0)1/2} - g^{ij}_{(0)}\, D_\gamma D^\gamma\, G^{(0)1/2}\right) \\
& + \frac{1}{2} G^{(0)} g_{(0)}^{1/2}\, \left(g^{ik}_{(0)} g^{jl}_{(0)} \partial_k \phi^{(0)} \partial_l \phi^{(0)} - g^{kl}_{(0)} \partial_k \phi^{(0)} \partial_l \phi^{(0)}\, g^{ij}_{(0)}\right)\,.
\end{aligned}
\tag{A.45}
$$

We now decompose the $d+1$ dimensional Ricci tensor scalar via:

$$
\begin{aligned}
{}^{(d+1)}R_{ij} &= {}^{(d)}R_{ij} - \frac{1}{2}\, \nabla_i \left(\frac{\nabla_j G}{G}\right) - \frac{1}{4}\, G^{-2}\, \nabla_i G\, \nabla_j G\,, \\
{}^{(d+1)}R_{tt} &= \left(\frac{1}{2}\, \nabla_i \left(\frac{\nabla^i G}{G}\right) + \frac{1}{4}\, G^{-2}\, \nabla_i G\, \nabla^i G\right)\, G U\,, \\
{}^{(d+1)}R &= {}^{(d)}R - \nabla_i \left(\frac{\nabla^i G}{G}\right) - \frac{1}{2}\, G^{-2}\, \nabla_i G\, \nabla^i G\,.
\end{aligned}
\tag{A.46}
$$

We also have

$$
D^\gamma D_\gamma G^{(0)1/2} = \frac{1}{2}\, G^{(0)-1}\, g^{ij}_{(0)}\, \partial_i G^{(0)}\, \partial_j G^{(0)1/2} + \nabla^i \nabla_i G^{(0)1/2}\,, \tag{A.47}
$$

where ∇ is the covariant derivative with respect to the d dimensional horizon metric. Putting these ingredients together we finally obtain the expression for M^{ij} given in (A.38).

Appendix B

Chapter 3 appendix

B.1 Scalar expectation value from the Ward identity

We will now show how the scalar VEV from the thermoelectric perturbations can be calculated directly from the Ward identities. Recall that the Ward identity is given by

$$\nabla_\mu \bar{T}^{\mu\nu} - \bar{J}^\mu F_\mu{}^\nu - \sum_i \langle \bar{\mathcal{O}}_{\chi_i} \rangle \nabla^\nu \psi_i = 0 \ , \tag{B.1}$$

where the i refer to the various scalar fields in the theory, and ψ_i is the source corresponding to that field. If we take the x_1 component of the identity, and note that the source of χ_1 is $k_1 x_1$ we find

$$\nabla_\mu \bar{T}^{\mu x_1} - \bar{J}^\mu F_\mu{}^{x_1} - \langle \bar{\mathcal{O}}_{\chi_1} \rangle g^{x_1 x_1} k_1 = 0 \ . \tag{B.2}$$

First consider the term containing the stress energy tensor. This can be written as

$$\partial_t \bar{T}^{t x_1} + \partial_{x_1} \bar{T}^{x_1 x_1} + \Gamma^t_{it} \bar{T}^{i x_1} + 2\Gamma^{x_1}_{i x_1} \bar{T}^{i x_1} + \Gamma^{x_1}_{it} \bar{T}^{it} \ , \tag{B.3}$$

as the term $\bar{T}^{x_1 x_2}$ is zero. Noting that $\bar{T}^{t x_1} = \bar{T}^{t x_1}_0 - \zeta t \bar{T}^{x_1 x_1}$, and that the only non zero Christoffel symbols at leading order is $\Gamma^{x_1}_{tt} = (e^{-2V1} \delta f_2 - \frac{1}{2} \delta h_{r x_1} U U')$, only the first and last terms will contribute.

Similarly, for the current piece of the Ward identity, we have

$$\bar{J}^\mu F_\mu{}^{x_1} = \bar{J}^t g^{x_1 x_1} F_{t x_1} + \bar{J}^{x_1} F_{x_1 t} g^{t x_1} \tag{B.4}$$

$$= U^{-1/2} a' \delta f_1 e^{-2V1} \tag{B.5}$$

to first order. Combining these terms and expanding asymptotically (demanding that $\delta h_{r x_1}$ vanishes) we see that

$$- \frac{Eq + \zeta (T^{x_1 x_1} + T^{tt} - \mu q)}{r^3} - \langle \bar{\mathcal{O}}_{\chi_1} \rangle k_1 = 0 \ . \tag{B.6}$$

If we now identify the VEV as the leading term in the expansion, we have $\langle \mathcal{O}_{\chi_1} \rangle = r^3 \langle \bar{\mathcal{O}}_{\chi_1} \rangle$, and hence we obtain

$$\langle O_{\chi_1} \rangle = -\frac{Eq + \zeta(T^{tt} + T^{x_1 x_1} - \mu q)}{k_1}. \tag{B.7}$$

The Smarr relation

$$T^{tt} + T^{x_1 x_1} - \mu q = sT \tag{B.8}$$

then gives equation (3.25).

Appendix C

Chapter 4 appendix

C.1 Sub-leading corrections of the linearised perturbation

We consider a metric of the form

$$ds^2 = -U\,G\,\left(dt + \delta\chi\right)^2 + \frac{F}{U}\,dr^2 + g_{ij}dx^i dx^j\,, \tag{C.1}$$

as in (4.2), but now with an additional linear perturbation $\delta\chi(r,x)$. We would like to understand the behaviour of $\delta\chi$ as a perturbative expansion about the high temperature background solution (4.27) as well as the sub-leading perturbative corrections that are polynomial in ϵ. Later, when we combine this with the DC thermal gradient source as in (4.28), we will then see an elegant interplay between solutions of the Stokes equations and regularity of the combined perturbation at the horizon.

Without loss of generality we work in a coordinate system where $\delta\chi_r = 0$ (this can be achieved via the coordinate transformation $t \to t + f(r,x)$) and take $\delta\chi = \delta\chi_i(r,x)dx^i$. The equation of motion of $\delta\chi$ is then given by

$$\partial_r(U^2 G^{3/2} F^{-1/2}\sqrt{g_d}\,g^{ij}\,\partial_r\delta\chi_j) + \partial_k[U\,G^{3/2}F^{1/2}\sqrt{g_d}\,g^{kl}g^{ij}\,(\partial_l\delta\chi_j - \partial_j\delta\chi_l)] = 0\,, \tag{C.2}$$

as well as

$$\partial_i(U^2\,G^{3/2}F^{-1/2}\sqrt{g_d}\,g^{ij}\,\partial_r\delta\chi_j) = 0\,, \tag{C.3}$$

where $\sqrt{g_d}$ is the volume element associated with g_{ij}. One solution of these equations is to take an arbitrary closed form given by $\delta\chi = \chi_i(x)dx^i$. This solution gives rise to a source in the dual field theory and is not what we are interested in here.

We now consider the background solution as a derivative expansion in the high-temperature limit:

$$\begin{aligned}
U &= r_H^2\,\rho^2\,u(\rho) = r_H^2\,\rho^2\,(1-\rho^{1-D})\,,\\
G &= \left(1+\epsilon^2\,G^{(2)}(\rho,x)+\cdots\right)\,,\\
F &= \left(1+\epsilon^2\,F^{(2)}(\rho,x)+\cdots\right)\,,\\
g_{ij} &= r_H^2\,\rho^2\,\left(h_{ij}(x)+\epsilon^2\,h^{(2)}_{ij}(\rho,x)+\cdots\right)\,,
\end{aligned} \tag{C.4}$$

where $\epsilon = k/T$, with k the largest wavenumber of the background, and $r = r_H\rho$. We note that, in general, there will also be corrections that are non-perturbative in ϵ which we will ignore. As before $h_{ij}(x)$ is the UV deformation of the metric and since we want this to be the only deformation of the dual field theory, we demand that $G^{(2)}$, $F^{(2)}$ and $h^{(2)}_{ij}$ all vanish as $\rho\to\infty$. Note that regularity at the horizon imposes $F^{(2)}|_{\rho=1} = G^{(2)}|_{\rho=1}$ and that the metric on the horizon is $r_H^2(h^{(0)}_{ij}+\epsilon^2\,h^{(2)}_{ij}|_{\rho=1}+\cdots)$.

We want to solve (C.2), (C.3) perturbatively in ϵ by postulating an expansion of the form

$$\delta\chi_i = \epsilon^\nu\,(\chi_i^{(0)}+\epsilon^2\,\chi_i^{(2)}+\cdots)\,, \tag{C.5}$$

with ν an exponent that we will eventually have to fix. It is clear that the homogeneous, linear equation (C.2) cannot fix this exponent. It must be fixed by an inhomogeneous constraint involving the boundary and or the horizon. As we will see it is fixed by ensuring the perturbation is regular at the horizon when it is combined with the perturbation associated with the DC thermal gradient source as in (4.28).

To carry out the expansion in ϵ we note that any spatial derivative ∂_i is of order ϵ. Now the second term in (C.2) comes with two spatial derivatives so it is order ϵ^2. Thus, at leading order in ϵ we have the simple ODE

$$\partial_\rho(\rho^D\,u^2\,\partial_\rho\chi_i^{(0)}) = 0\,, \tag{C.6}$$

and solutions are given by

$$\chi_i^{(0)}(\rho,x) = c_i^{(0)}(x) + \frac{\rho^{1-D}}{r_H^2\,u(\rho)}\,v_i^{(0)}(x)\,. \tag{C.7}$$

To ensure that there are no additional sources at infinity we set

$$c_i^{(0)} = 0\,. \tag{C.8}$$

From equation (C.3) we also have

$$\nabla_i v^i_{(0)} = 0\,, \tag{C.9}$$

where ∇ is the Levi-Civita connection for the spatial metric h_{ij} and indices have been raised with h^{ij}.

At second order in ϵ, equation (C.2) implies

$$r_H^{D-2}\sqrt{h}\,h^{ij}\partial_\rho(\rho^D u^2\,\partial_\rho\chi_j^{(2)}) + r_H^{D-2}\,\partial_\rho(\rho^D u^2\sqrt{h}\,N^{ij}\,\partial_\rho\chi_j^{(0)})\,, \\ +\epsilon^{-2}\,r_H^{D-4}\,\rho^{D-4}\,\partial_k(u\sqrt{h}\,h^{kl}\,h^{ij}\,(\partial_l\chi_j^{(0)} - \partial_j\chi_l^{(0)})) = 0\,, \tag{C.10}$$

where we defined the matrix $N^{ij}(\rho, x)$:

$$N^{ij} = h^{ij}\left[\frac{3}{2}\,G^{(2)} - \frac{1}{2}\,F^{(2)}\right] + \frac{1}{2}\,h^{ij}\,h^{kl}h_{(2)kl}(\rho, x^i) - h_{(2)}^{ij}\,, \tag{C.11}$$

and the indices on the last term have been raised with h^{ij}. After substituting the zeroth order solution (C.7), (C.8) we can rewrite (C.10) as

$$h^{ij}\partial_\rho(\rho^D u^2\,\partial_\rho\chi_j^{(2)}) - \frac{D-1}{r_H^2}\,\partial_\rho N^{ij}\,v_j^{(0)} + \frac{2\rho^{-3}}{r_H^2(\epsilon r_H)^2}\,\nabla_k\nabla^{[k}v_{(0)}^{i]} = 0\,. \tag{C.12}$$

The general solution to this equation is of the form

$$\chi_i^{(2)}(\rho, x) = c_i^{(2)}(x) + \frac{\rho^{1-D}}{r_H^2\,u(\rho)}\,v_i^{(2)}(x) + q_i^{(2)}(\rho, x)\,. \tag{C.13}$$

The first two terms are the solutions to the homogeneous equation (with $v_{(0)}^i = 0$) and the third term is a particular solution of the inhomogeneous equation. The solution that we are interested in, to ultimately ensure that we have non-singular behaviour near the horizon, will be such that $q_j^{(2)}$ has no $(\rho-1)^{-1}$ term close to the horizon, but instead a $\log(\rho-1)$ behaviour. The function $c_j^{(2)}$ is fixed so that we don't have a source at infinity at the given order in the ϵ expansion; as we will see this implies that $c_j^{(2)} = 0$. We will see that these requirements uniquely fix $q_j^{(2)}$. On the other hand, the new function of integration $v_i^{(2)}(x)$ will be fixed at the next order in the expansion, a point we will return to later.

To proceed, we demand that in an expansion close to horizon the leading term of $q_j^{(2)}$ is given by

$$q_j^{(2)}(\rho, x^i) = q_j(x^i)\,\log(\rho-1) + \cdots\,. \tag{C.14}$$

From equation (C.12) we deduce

$$(4\pi T)^2\,h^{ij}\,q_j = (D-1)\,v_j^{(0)}\,\partial_\rho N^{ij}|_{\rho=1} - 2(\epsilon\,r_H)^{-2}\,\nabla_k\nabla^{[k}v_{(0)}^{i]}\,. \tag{C.15}$$

Now a key point is that the leading term in the expansion of the matrix N^{ij} near the horizon is actually fixed by horizon data. This result can be extracted[1] from calculations presented in [1] and we have

$$\partial_\rho N^{ij}|_{\rho=1} = -\frac{2}{(D-1)}\,\frac{1}{(\epsilon\, r_H)^2}\, R^{ij}\,, \tag{C.16}$$

where R_{ij} is the Ricci tensor of the d-dimensional UV metric h_{ij}, and again the indices have been raised using h^{ij}. Using (C.9) we can then deduce

$$q_j = -\frac{2}{(4\pi T)^2}\,\frac{1}{(\epsilon\, r_H)^2}\,\nabla^i\nabla_{(i}\, v^{(0)}{}_{j)}\,. \tag{C.17}$$

Having established how the expansion of the perturbation $\delta\chi$ works, at this point we now recall that the full perturbation that we are interested in also has the time dependent piece given in (4.28). By switching to the ingoing coordinate, $v = r_H(t + \ln u/(4\pi T))$, we see that the $\log(\rho-1)$ factors will be eliminated in the full perturbation provided that we choose the exponent in (C.5) to be

$$\nu = -2\,, \tag{C.18}$$

and, in addition, demand that $v_i^{(0)}$ satisfies the linearised Navier-Stokes equation:

$$-\frac{2}{(\epsilon r_H)^2}\,\nabla^i\nabla_{(i} v_{j)}^{(0)} = 4\pi T\zeta_j - \partial_j p\,, \tag{C.19}$$

Now a consideration of (C.1), (C.4), (C.5) and comparing with (4.31), reveals that the combined perturbation given in (4.28)-(4.32) is regular at the horizon, to leading order, provided that we have

$$V_i(x) = \epsilon^{-2}\, v_i^{(0)}(x)\,, \tag{C.20}$$

with $v_i^{(0)}$ satisfying (C.19). Furthermore, since we know that, in general, the Stokes equations are satisfied at the horizon we learn that we must have

$$p = \epsilon^{-1} p^{(0)} + \dots\,. \tag{C.21}$$

and that $4\pi T\zeta$ is the same order as $p^{(0)}$ and $v_i^{(0)}$.

Having established the behaviour at the horizon, we now give the full integrated ex-

[1]To compare we have $r^{there} = r_H(\rho - 1)$. Then in the notation of eq. (2.5) of [1] we have, with objects on the left hand side in the notation of [1], $U^{(1)} = 4\pi T(4-D)/(2r_H)$, $G^{(0)} = F^{(0)} = 1 + \dots$, $G^{(1)} = (r_H)^{-1}\epsilon^2\partial_\rho G^{(2)}|_{\rho=1} + \dots$ and $F^{(1)} = (r_H)^{-1}\epsilon^2\partial_\rho F^{(2)}|_{\rho=1} + \dots$. We then use (D.2) and (D.4) of [1] to get (C.16).

pression for $q_j^{(2)}(\rho, x)$:

$$\begin{aligned} q_j^{(2)} =& \frac{D-1}{r_H^2}\, h_{ij}\, v_l^{(0)} \int_\infty^\rho \frac{N^{il}(\rho', x)}{u^2(\rho')}\, \rho'^{-D}\, d\rho' + \frac{\rho^{1-D}}{r_H^2 u(\rho)} h_{ij}\, v_l^{(0)}\, N^{il}|_{\rho=1} \\ &+ (\epsilon\, r_H)^{-2} \left[\left(\int_\infty^\rho \frac{\rho'^{-D-2}}{u^2(\rho')} d\rho' \right) + \frac{1}{(D-1)} \frac{\rho^{1-D}}{u(\rho)} \right] \nabla^k \nabla_{[k} v_{j]}^{(0)} \,. \end{aligned} \tag{C.22}$$

Note that the radial dependence in $q_j^{(2)}$ can be made explicit provided that we can find the back-reaction of the background at order ϵ^2 that is packaged in N^{ij}. Observe that the last term in the first line and the last line of (C.22) are simply solutions of the homogeneous equation (their radial dependence is the same with the second term of (C.13)) with the corresponding function of integration chosen such that we get the behaviour (C.14) without a potential $(\rho - 1)^{-1}$ term, as we mentioned earlier. Furthermore one can check that the leading behaviour close to the horizon of the expression above is indeed given by (C.14).

From this expression we can now extract the asymptotic behaviour at $\rho = \infty$. We first note that N^{ij} goes to zero close to the boundary since it is constructed from higher order corrections to the background and by assumption these do not change the conformal boundary metric. We find

$$\chi_j^{(2)} = \rho^{1-D}\, r_H^{-2}\, v_j^{(2)} + \rho^{1-D} r_H^{-2}\, h_{ij}\, v_l^{(0)}\, N^{il}|_{\rho=1} + \frac{\rho^{1-D}}{(D-1)}\, (\epsilon\, r_H)^{-2}\, \nabla^k \nabla_{[k} v_{j]}^{(0)} + \cdots \,. \tag{C.23}$$

where we have set $c_i^{(2)} = 0$, which we now see does indeed correspond to having vanishing source term for the perturbation.

At this point one might wonder how $v_j^{(2)}$ is fixed. The next order in the expansion will include a function $q_j^{(4)}(\rho, x)$. Once again, near the horizon the $(\rho - 1)$ behaviour will be eliminated leaving $\log(\rho - 1)$ behaviour. Regularity at the horizon will imply that this is in turn be fixed by the corrected Navier-Stokes equation at the horizon (note that the perturbation (4.28) is also corrected because g_{tt} will also receive corrections). While this procedure can be carried out in detail, for the main results we want to present here, we will not need to. An additional point is that that we also need to satisfy (C.3) at next order. This condition reads

$$\partial_i \left(\sqrt{h} h^{ij} \partial_\rho \chi_j^{(2)} + \sqrt{h} N^{ij} \partial_\rho \chi_j^{(0)} \right) = 0 \,. \tag{C.24}$$

After substituting the expressions in and using the fact that $\nabla_i \nabla_j \nabla^{[i} v_{(0)}^{j]} = 0$, we find that we must impose

$$\partial_i \left(\sqrt{h} h^{ij} v_j^{(2)} + \sqrt{h} N^{ij}|_{\rho=1} v_j^{(0)} \right) = 0 \,. \tag{C.25}$$

At this point we have established that this perturbation has an expansion which can be written

$$\begin{aligned} \delta g_{ti}(\rho, x) &= -UG\delta\chi_i \\ &= -\epsilon^{-2}\rho^{3-D} \left(v_i^{(0)}(x) + \epsilon^2 V_i^{(2)}(\rho, x) + \dots \right) , \end{aligned} \tag{C.26}$$

where

$$V_i^{(2)}(\rho, x) = G^{(2)} v_i^{(0)}(x) + v_i^{(2)}(x) + \frac{r_H^2 u}{\rho^{1-D}} q_i^{(2)}(\rho, x) \,, \tag{C.27}$$

with $q_i^{(2)}(\rho, x)$ given in (C.22) and an expression for $v_i^{(2)}(x)$ can be explicitly obtained by continuing to higher orders. We have also shown that as we approach the horizon we have

$$V_i^{(2)}(\rho, x) \to \frac{r_H^2}{(4\pi T)^2} u \log(\rho - 1) \left(-\frac{2}{(\epsilon r_H)^2} \nabla^i \nabla_{(i} v_{j)}^{(0)} \right) , \qquad \rho \to 1 \,. \tag{C.28}$$

It is also worth noting that $v_i(x)$ that appears in the general Stokes equations on the black hole horizon (see (4.16)) is given by $v_i(x) = -\delta g_{ti}|_{\rho=1}$.

An important objective is to obtain the local heat current density of the dual field theory Q^i_{QFT}. We conclude this appendix by showing how that at leading order in the expansion we have $Q^i_{QFT}(x) = Q^i_{BH}(x)$, as well as indicating the structure of the sub-leading corrections. We first return to (C.2) and observe that this equation can be rewritten as

$$\partial_\rho Q^i = \frac{1}{16\pi G_N} r_H \partial_k [U\, G^{3/2} F^{1/2} \sqrt{g_d}\, g^{kl} g^{ij} \left(\partial_l \delta\chi_j - \partial_j \delta\chi_l \right)] \,, \tag{C.29}$$

where $Q^i = -\frac{1}{16\pi G_N}(r_H)^{-1} U^2 G^{3/2} F^{-1/2} \sqrt{g_d}\, g^{ij}\, \partial_\rho \delta\chi_j$ is the bulk thermal current, defined in (3.20) of [1]. By integrating in the radial direction we deduce that

$$Q^i_{QFT} - Q^i_{BH} = \frac{1}{16\pi G_N} r_H \int_{\rho=1}^{\infty} d\rho\, \partial_k [U\, G^{3/2} F^{1/2} \sqrt{g_d}\, g^{kl} g^{ij} \left(\partial_l \delta\chi_j - \partial_j \delta\chi_l \right)] \,, \tag{C.30}$$

which for the current densities gives, up to second order in ϵ,

$$\begin{aligned} Q^i_{QFT} - Q^i_{BH} &= \frac{1}{16\pi G_N} r_H^{D-3}\,\epsilon^{-2} \int_{\rho=1}^{\infty} d\rho\, u\, \rho^{D-4}\, \partial_k(\sqrt{h}\, h^{kl} h^{ij}\, (\partial_l \chi_j^{(0)} - \partial_j \chi_l^{(0)}))\,, \\ &= \frac{1}{16\pi G_N} 2 r_H^{D-3}\,(\epsilon\, r_H)^{-2}\sqrt{h} \int_{\rho=1}^{\infty} d\rho\ \rho^{-3}\, \nabla_k \nabla^{[k} v^{i]}_{(0)}\,, \\ &= \frac{1}{16\pi G_N} r_H^{D-3}\,(\epsilon\, r_H)^{-2}\sqrt{h}\, \nabla_k \nabla^{[k} v^{i]}_{(0)}\,. \end{aligned} \tag{C.31}$$

Observe that because of the two spatial derivatives the term on the right hand side of (C.31) is of order ϵ^0. Thus, we conclude that at leading order the heat current at the horizon is the same as at that of the dual field theory and moreover we also have obtained the leading order correction in the ϵ expansion. We also note that the heat current at the horizon is given by

$$Q^i_{BH} = \frac{1}{16\pi G_N} r_H^{d-3} 4\pi T \sqrt{h} \left[\epsilon^{-2} h^{ij} v_j^{(0)} + h^{ij} v_j^{(2)} + N^{ij}|_{\rho=1} v_j^{(0)} \right]\,, \tag{C.32}$$

where the sub-leading corrections involve corrections to the background via $N^{ij}|_{\rho=1}$ (see (C.11) and (C.4)) as well as the sub-leading terms in the perturbation, $v_j^{(2)}$, which can be obtained by the method discussed above.

Finally, we note that since the right hand side of (C.31) is a total derivative, this result is clearly consistent with the universal result of [97] that the total heat current flux of the field theory is always the same as the total heat current flux on the boundary, $\bar{Q}^i_{BH} = \bar{Q}^i_{QFT}$.

Appendix D

Chapter 5 appendix

D.1 General quantum field theories

We now consider a general relativistic quantum field theory, relaxing the constraint of conformal invariance. The set-up is very similar to that in section 2 and we again use the material in [151]. We now just impose the Ward identity $D_\mu T^\mu{}_\nu = 0$. For the constitutive relation we write

$$T_{\mu\nu} = P g_{\mu\nu} + (\varepsilon + P)\, u_\mu u_\nu + \tau_{\mu\nu}\,, \tag{D.1}$$

where

$$\tau_{\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta_b(g_{\mu\nu} + u_\mu u_\nu) D_\rho u^\rho\,, \tag{D.2}$$

Here $\sigma_{\mu\nu}$ is the same as in (5.3) and ζ_b is the bulk viscosity and should not be confused with the external thermal source one-form $\zeta = \zeta_\mu dx^\mu = d\phi$. For CFTs we have $\zeta_b = 0$. We also have the local thermodynamic relation and first law, which take the form

$$\varepsilon + P = Ts\,, \qquad dP = s\, dT\,. \tag{D.3}$$

To simplify the presentation, we will again just consider static backgrounds with Killing vector ∂_t. As we will see, background metrics with ∂_t having non-constant norm, *i.e.* $g_{tt} \equiv -f^2$ non-constant, will play an interesting role. In considering the perturbation about the background we note that P, ε, S, η and ζ_b are all functions of the local temperature. They can depend on other dimensionful parameters, but these will all be held fixed in the perturbations we are interested in. Thus, we can write $\varepsilon_0 \equiv \varepsilon(T_0)$, $\delta\varepsilon \equiv (\partial_T \varepsilon)_0 \delta T$

etc. For the perturbed metric and fluid velocity we write

$$\begin{aligned} ds^2 &= -f^2(x)(1-2\phi)\,dt^2 + g_{ij}(x)dx^i dx^j\,, \\ u_t &= -f(x)(1-\phi), \qquad u_j = \delta u_j\,. \end{aligned} \tag{D.4}$$

where ϕ and δu_i are both functions of (t,x) as before. A calculation then gives the stress tensor

$$\begin{aligned} T_{tt} &= \varepsilon_0\, f^2\,(1-2\phi) + \delta\varepsilon\, f^2\,, \\ T_{ti} &= -f\,(\varepsilon_0 + P_0)\,\delta u_i\,, \\ T_{ij} &= (P_0 + \delta P)\,g_{ij} - 2\eta_0\, f^{-1}\,\nabla_{(i}\left(f\delta u_{j)}\right) + \left(\frac{2\eta_0}{(d-1)} - \zeta_{b0}\right) g_{ij}\, f^{-1}\,\nabla_k(f\,\delta u^k)\,. \end{aligned} \tag{D.5}$$

The heat current, defined in (4.18) is given by

$$Q^i = \sqrt{g} f^2(\varepsilon_0 + P_0)\delta u^i = \sqrt{g} f^2 T_0 s_0 \delta u^i\,. \tag{D.6}$$

We next note that in order to ensure that the Ward identity is satisfied for the unperturbed background we must have

$$f^{-1}\partial_i f(\varepsilon_0 + P_0) + \nabla_i P_0 = 0\,. \tag{D.7}$$

Using the equation of state and the first law in (D.3) for the background we can then integrate (D.7) to find

$$T_0 = \bar{T}_0 f^{-1}\,, \tag{D.8}$$

where $\bar{T}_0$ is a constant. In particular, we see that in general T_0 depends on the spatial coordinates.

Returning now to the perturbed stress tensor, for the time component of the Ward identity we obtain

$$f\partial_t\delta\varepsilon + \nabla_i(f^2(\varepsilon_0 + P_0)\delta u^i) = 0\,. \tag{D.9}$$

For the spatial component, and using (D.7), we find

$$\begin{aligned} &f^{-1}(\varepsilon_0 + P_0)\,\partial_t\delta u_i + f^{-1}\partial_i f(\delta\varepsilon + \delta P) - (\varepsilon_0 + P_0)\zeta_i + \partial_i\delta P \\ &\quad - 2f^{-1}\,\nabla^j\left(\eta_0\,\nabla_{(j}\left(f\delta u_{i)}\right)\right) + f^{-1}\,\nabla_i\left(\left(\frac{2\eta_0}{d-1} - \zeta_{b0}\right)\nabla_k(f\delta u^k)\right) = 0\,. \end{aligned} \tag{D.10}$$

Notice that the time component of the four-vector ζ_t again does not appear. The perturbations $\delta\varepsilon$ and δP can both be expressed in terms of δT since we are holding all other dimensionful parameters fixed. In fact, using (D.3) we have $\delta P = s_0 \delta T$ and $\delta\varepsilon = T_0 (\partial_T s)_0 \delta T$. Thus these equations should again be solved for δT and δu_i.

When $f = 1$, from (D.8) we have that T_0 is a constant. As a consequence $P_0, \varepsilon_0, s_0, \eta_0$ and ζ_{b0} are then also constants. In this case the Ward identities simplify to the following linearised Navier-Stokes equations

$$\begin{aligned} &T_0^{-1} \partial_t \delta T + c_s^2 \nabla_i \delta u^i = 0\,, \\ &T_0 s_0\, \partial_t \delta u_i + s_0 \partial_i \delta T - 2\eta_0\, \nabla^j \nabla_{(j} \delta u_{i)} + \left(\frac{2\eta_0}{d-1} - \zeta_{b0} \right) \nabla_i \nabla_k \delta u^k = T_0 s_0 \zeta_i\,. \end{aligned} \tag{D.11}$$

In the first equation we have introduced the speed of sound squared, $c_s^2 = (\partial_\epsilon P)_0 = s_0/(T_0(\partial_T s)_0)$. For a CFT we have $c_s^2 = 1/(d-1)$. Moreover, to study DC response we can set the time derivatives to zero and we obtain the Stokes equations for an incompressible fluid

$$\nabla_i \delta u^i = 0\,, \qquad \partial_i \delta T - 2\frac{\eta_0}{s_0}\, \nabla^j \nabla_{(j} \delta u_{i)} = T_0 \zeta_i\,. \tag{D.12}$$

D.2 Numerical integration

We want to solve the system of equations (5.14) for the variables v_i, p for a specified constant $\hat{\zeta}_i$ on a torus with unit periods and metric g_{ij}. In order to numerically solve this boundary value problem for a two dimensional horizon, we will discretise our domain on $N_x \times N_y$ points. Given the periodicity of the problem and the fact that we expect to find smooth solutions, we use Fourier pseudo-spectral methods to approximate the derivatives of our functions on our computational grid. The problem then reduces to a $(3\, N_x\, N_y) \times (3\, N_x\, N_y)$ inhomogeneous linear system which we can write in matrix form as

$$\mathbb{M} \cdot \mathbf{v} = \mathbf{s}\,. \tag{D.13}$$

The $3\, N_x\, N_y$ dimensional vector $\mathbf{v}$ is used to store the values of the functions p, v_x and v_y on the grid. In more detail

$$\mathbf{v}_i = \begin{cases} p_{\,i \bmod N_x, \left[\frac{i}{Nx}\right]}, & 1 \le i \le N_x N_y \\ (v_x)_{(i-N_x N_y) \bmod N_x, \left[\frac{i-N_x N_y}{Nx}\right]}, & N_x N_y < i \le 2N_x N_y \\ (v_y)_{(i-2N_x N_y) \bmod N_x, \left[\frac{i-2N_x N_y}{Nx}\right]}, & 2N_x N_y < i \le 3N_x N_y \end{cases} \tag{D.14}$$

where $\left[\frac{a}{b}\right]$ denotes the integer part of the division between a and b. The vector $\mathbf{s}$ is reserved for the inhomogeneous part of system (5.14) and it does depend on the direction of the temperature gradient. For example, when the temperature gradient is just along the x direction, and unit valued, we have

$$(\mathbf{s}^x)_i = \begin{cases} 1, & N_x N_y < i \leq 2N_x N_y \\ 0, & \text{otherwise} \end{cases} \tag{D.15}$$

It is easy to see that we only have to do a single inversion of the matrix $\mathbb{M}$ and the solution for sources in different directions can simply be found by a matrix multiplication of $\mathbb{M}^{-1}$ with the corresponding source vector $\mathbf{s}^x$ or $\mathbf{s}^y$.

We have implemented the method outlined above in `C++` taking advantage of the language's templates to write code which can be used with various data types. However, we found that `double` precision was enough to obtain accurate solutions for our purposes. We did find though that we had to use quite large resolutions of the order of $N_x = N_y \sim 181$. This need is becoming obvious from our plots since there is small scale features we have to resolve. One example is the sharp peaks in the plots of p. The linear solver we used was the version of `PARDISO` included with Intel's `MKL BLAS` suite. The specific solver can take advantage of `OpenMP` at several stages of the solution of the linear system which proved useful when we ran our code on multicore systems.

Appendix E

Chapter 6 appendix

E.1 Smarr relation

We explain how to obtain the Smarr relation (6.33) via a direct calculation of the on-shell action. The bulk Euclidean bulk action is given by

$$I_{bulk} = -\frac{1}{16\pi G_N}\Delta\tau \mathrm{vol}_3 \int_0^{u_h} \mathrm{d}u \mathcal{L}_{bulk}\,, \tag{E.1}$$

where the Lagrangian density integrand is given by

$$\mathcal{L}_{bulk} = \sqrt{-g}\left(R - 3X^{-2}(\partial X)^2 + 4(X^2 + 2X^{-1}) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right)\,. \tag{E.2}$$

We would like to rewrite this as a total derivative in u after using the equations of motion. To achieve this we found it helpful to use the fact that after contraction of equation (6.2), we can write the integrand of the action as

$$\mathcal{L}_{bulk} = -\frac{8\sqrt{\mathcal{B}}e^{-\frac{7\phi}{4}}}{3u^5}\left(X^2 + 2X^{-1}\right)\,. \tag{E.3}$$

After some work we find that after using the equations of motion (6.12)-(6.15) the integrand can be written as

$$\mathcal{L}_{bulk} = \left(2\frac{\sqrt{\mathcal{B}}\mathcal{F}e^{-5\phi/4}}{u^4} - \frac{\mathcal{F}e^{-5\phi/4}\mathcal{B}'}{u^3\sqrt{\mathcal{B}}} - \frac{\sqrt{\mathcal{B}}e^{-5\phi/4}\mathcal{F}'}{u^3} + \frac{1}{2}\frac{\sqrt{\mathcal{B}}\mathcal{F}e^{-5\phi/4}\phi'}{u^3}\right)'\,, \tag{E.4}$$

where the prime indicates differentiation with respect to the u coordinate. Using this expression we will get contributions to the on-shell action both from the horizon and the boundary. Using the near horizon and boundary expansions of the fields given in (7.18),(7.19), and combining with the boundary counter terms we deduce that the free energy density can be expressed as

$$w = E - sT \,, \tag{E.5}$$

as in (6.32).

On the other hand, using (6.12)-(6.15) we can also write the integrand in the form

$$\mathcal{L}_{bulk} = \left(2\frac{\sqrt{\mathcal{B}}\mathcal{F}e^{-5\phi/4}}{u^4} + \frac{\sqrt{\mathcal{B}}\mathcal{F}e^{-5\phi/4}\phi'}{2u^3} \right)' \,. \tag{E.6}$$

This only gives contributions from the boundary leading to

$$w = -T^{xx} \,. \tag{E.7}$$

Combining these gives these expressions gives the Smarr relation (6.33).

E.2 Critical exponents for a cubic free energy[1]

Suppose we have a Landau-Ginzburg free energy functional for a scalar order parameter, m, of the form

$$f = f_0 + \frac{am^2}{2} + \frac{bm^3}{3} \,, \tag{E.8}$$

with f_0 a constant, $a = t^n$, with $t = (T - T_c)/T_c$, and b is a temperature dependent constant which we take to be positive. We choose n so that $a < 0$ for $T < T_c$ and we will be especially interested in the case $n = 1$. For the moment let us ignore the global instability for $m < 0$ and focus on the extrema at $m = 0$ and $m = -a/b$ which exists when $a < 0$ i.e for $T < T_c$. For the latter minimum we have $m \propto t^n$ and hence we conclude that $\beta = n$. To obtain α we want to differentiate the minimum value of the free energy with respect to T. Below T_c we have $f = f_0 + a^3/6b^2$ and hence we deduce that $T\partial^2 f/\partial T^2 \propto t^{3n-2}$ and thus $\alpha = 2-3n$. Note that above T_c the free energy is constant and hence the specific heat vanishes. To determine δ we add $-mh$ to the free energy where h is a background source. We now have $\partial f/\partial m = am + bm^2 - h$ and at $T = T_c$, where $a = 0$, we deduce that the equilibrium configuration has $m \propto h^{1/2}$ and hence $\delta = 2$. Finally, we consider the susceptibility $\chi = \partial m/\partial h$. At equilibrium we have $am + bm^2 - h = 0$ and differentiating we deduce that $\chi = 1/(a + 2bm)$. For $T > T_c$ we have $m = 0$ and $\chi = t^{-n}$, while for $T < T_c$ we have $m = -a/b$ and hence $\chi = -t^{-n}$. We thus deduce that $\gamma = n$. When $n = 1$ the critical exponents are thus given by $(\alpha, \beta, \gamma, \delta) = (-1, 1, 1, 2)$, exactly as we saw in our holographic phase transition.

[1] We would like to thank Makoto Natsuume for helpful discussions on this section.

We now return to the issue of the global instability for $m < 0$. We first note that the instability would be eliminated if we were restricted to configurations with $m \geq 0$. Interestingly, the critical exponents that we have obtained were discussed in the context of a continuum generalisation of the Ashkin-Teller-Potts models associated with percolation problems, by imposing such a restriction [186]. Note that we have no restrictions on the sign of the expectation value $\langle \mathcal{O}_\psi \rangle$, so this perspective is not available for our holographic phase transition.

It is also worth pointing out that if we try to stabilise the free energy with higher powers of m, a quartic for example, then the model has a first order transition, again unlike what we see in our holographic transition. More explicitly we can add a term $cm^4/4$ to the free energy in (E.8) with $c > 0$. Now for high temperatures, $a > b^2/(4c)$, the free energy has a minimum at $m = 0$. For $2b^2/(9c) < a < b^2/(4c)$ there is an additional minimum at $m = m_1 \equiv -b/(2c) - [b^2 - 4ac]^{1/2}/(2c)$, which, has higher free energy than the minimum at $m = 0$. For $0 < a < 2b^2/(9c)$ the minimum at m_1 has lower free energy than the minimum at $m = 0$ and there is a first order transition at $a = 2b^2/(9c)$. For $a < 0$, $m = 0$ becomes a maximum of the free energy with a new minimum appearing at $m = m_2 \equiv -b/(2c) + [b^2 - 4ac]^{1/2}/(2c)$. This m_2 minimum is the one associated with the critical exponents for the cubic with $c = 0$ that we discussed above, but it is simple to see that the m_1 minimum is always preferred.

In summary, we see that while the cubic Landau-Ginzburg model for a single scalar order parameter in a certain sense gives rise to the critical exponents we see in our holographic phase transition it does not capture key features. Perhaps a model containing more fields might be more effective.

Appendix F

Chapter 7 appendix

F.1 Zero temperature charged black hole solutions

F.1.1 Extremal black hole

We will now discuss some low temperature features of the CGS solution from [63]. It is interesting to note that we did not observe the Hawking-Page transition that the authors claimed appeared in their work. There, it was argued that there are different black hole radii corresponding to the same temperature, and hence there is a minimum black hole temperature and the solution is unstable below this temperature. However, we have not found this. Rather, for a fixed chemical potential, one can cool the solution down to arbitrarily low temperatures, as shown in the left hand plot of figure 17. We note that this is a different to the results of [187], where a model containg a linear axion, a dilaton and U(1) gauge field was studied, and no extremal horizon was found. However, in that case the gauge field coupled to the dilaton, whereas here there is no coupling between the gauge field and dilaton.

Futhermore, the zero temperature limit of the CGS solution is at a finite black hole radius, and so the black hole is an extremal black hole. As shown in the right plot of figure 17, the leading order scaling is $s \sim T^0$, and hence the entropy of the black hole tends to a constant value in the zero temperature limit. In addition, the functions $\mathcal{F}, \mathcal{F}', \mathcal{B}, \mathcal{B}'$ all vanish on the horizon, indicating a second order pole in the metric on the horizon and an extermal black hole. Whilst this is worrying from a physical perspective, as discussed in the main text, the black hole is unstable and therefore this extremal black hole would never be realised in reality.

F.1.2 Weak anisotropic limit

We now wish to study this zero temperature extremal black hole in the limit where $\mu/a >> 1$, i.e the limit of weak anisotropy. At zero temperature and $a = 0$, our solution

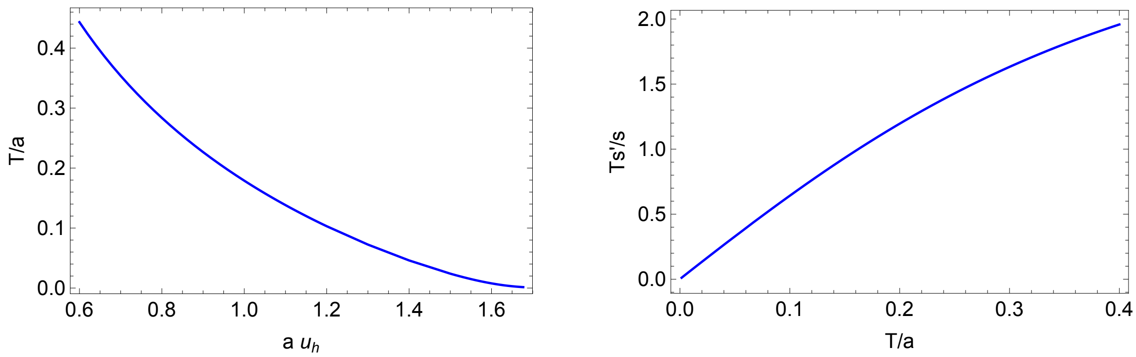

Figure 17: Plot showing charged CGS solution forming an extremal black hole, when $\mu/a = 1$. The right plot shows how T/a changes with $u_h a$. As we as we can tell, there is a one to one mapping, and hence there is no Hawking-Page transition. The left plot shows the temperature scaling of entropy. As $T/a \to 0$, the entropy tends to a constant value, and hence the black hole is extremal.

is just the extremal AdS-Reissner-Nordstrom black hole, and we therefore look for an expansion of the form

$$\begin{aligned}
\mathcal{F} &= \left(1 - \frac{(\mu u)^4}{12} + \frac{(\mu u)^6}{108}\right) + a^2 \mathcal{F}_2(u) + \mathcal{O}(a^4)\,, \\
\mathcal{B} &= 1 + a^2 \mathcal{B}_2(u) + \mathcal{O}(a^4)\,, \\
\phi &= a^2 \phi_2(u) + \mathcal{O}(a^4)\,, \\
b &= \mu\left(1 - \frac{(\mu u)^2}{6}\right) + a^2 b_2(u) + \mathcal{O}(a^4)\,,
\end{aligned} \tag{F.1}$$

where μ from (7.18) has been rescalled by $\mu \to \mu/\sqrt{3}$, as per the discussion in section (2.2). Note that only even powers of a are allowed, due to the symmetry $z \to -z$. In addition, the location of the extremal horizon will also have an expansion, given by

$$u_h = \frac{\sqrt{6}}{\mu} + a^2 u_h^{(2)} + \mathcal{O}(a^4)\,. \tag{F.2}$$

Now we substitute these expansions into the equations of motion, and solve order by order in a. We impose the boundary conditions that all the terms of $\mathcal{O}(a^2)$ and higher vanish at the UV boundary, which ensures the solution approaches AdS_5 in the UV. We also require b and $\mathcal{F}$ to vanish on the horizon at each order in a. We fix the $u_h^{(i)}$ by setting the temperature of the black hole to be zero at each order of a. After solving the

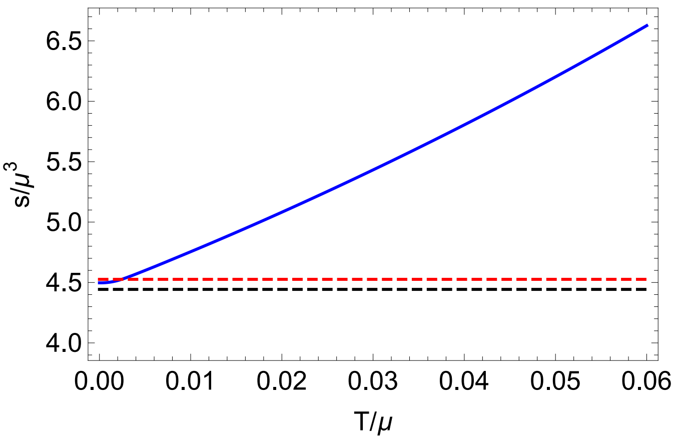

Figure 18: Plot showing charged entropy density versus temperature, when $\mu = \sqrt{3}$ and $a = 1/10$. The blue line is the entropy density, whilst the red dashed line is the analytical expression at zero temperature for s to $\mathcal{O}(a^2)$, and the black line is the entropy of the extremal AdS-Reissner-Nordstrom black hole. The result is in good agreement with the analytical expression.

equations of motion, we find analytical solutions at leading order

$$\begin{aligned}
\mathcal{B}_2(u) &= \frac{90\mu^2u^2 + (\mu^2u^2-6)(\mu^2u^2+3)\log(6-\mu^2u^2) + (3\mu^2u^2+18-\mu^4u^4)\log(2\mu^2u^2+6)}{12\mu^2(\mu^2u^2-6)(\mu^2u^2+3)}, \\
\mathcal{F}_2(u) &= \frac{u^6\mu^4\left(20+24\tanh^{-1}\left(\frac{1}{9}(3-2\mu^2u^2)\right)-2\left(5\log\left(\frac{5a^2}{9\mu^2}\right)+\log 2\right)\right)}{1296}+ \\
&+\frac{u^4\mu^2\left(60\log\left(\frac{5a^2}{\mu^2}\right)-126\tanh^{-1}\left(\frac{1}{9}(3-2\mu^2u^2)\right)-192+\log 8-120\log 3\right)}{1296} \\
&+\frac{\log\left(\frac{18+6\mu^2u^2}{18-3\mu^2u^2}\right)}{4\mu^2}+\frac{u^2}{3}, \\
b_2(u) &= \frac{5}{72}\mu u^2\left(\log\left(\frac{5a^2}{\mu^2}\right)+\log\left(\frac{\mu^2u^2+3}{54-9\mu^2u^2}\right)-2\right), \\
\phi_2(u) &= \frac{1}{2\mu^2}\log\left(\frac{6-\mu^2u^2}{6+2\mu^2u^2}\right), \qquad \text{(F.3)}
\end{aligned}$$

with $u_h^{(2)} = -5/(2\sqrt{6}\mu^3)$. Using these expressions, we can then get an expansion for the entropy density of the extremal black hole

$$16\pi G_N \frac{s}{\mu^3} = \sqrt{\frac{2}{3}}\frac{\pi}{3} + \frac{5\left(2-\log\left(\frac{5a^2}{18\mu^2}\right)\right)}{48\sqrt{6}}\left(\frac{a}{\mu}\right)^2 + \mathcal{O}\left(\frac{a}{\mu}\right)^4. \qquad \text{(F.4)}$$

This gives us the leading correction to the zero temperature entropy density, in the weak anisotropic limit. In figure 18, we plot entropy against temperature, and see that for $a/\mu << 1$, the entropy in the zero temperature limit is consistent with the analytic expansion. This is further evidence that the low temperature CGS solution is extremal and does not undergo a Hawking-Page type transition.

F.1.3 Fixed point solution

To understand zero temperature black holes, important physics can often come from scaling solutions. To understand the extremal CGS solution for arbitrary a/μ, we will therefore look for scaling solutions to the CGS equations of motion of the form

$$e^{\phi} = e^{\phi_0} u^{\phi_c}, \;\; \mathcal{F} = \mathcal{F}_0 u^{\mathcal{F}_c}, \;\; \mathcal{B} = \mathcal{B}_0 u^{\mathcal{B}_c}, \;\; b = b_0 u^{b_c} \,, \tag{F.5}$$

where b is the single gauge field that we have in the CGS solution, and recall that there are no additional scalar fields in the CGS solution.

Remarkably, we find that there is an exact solution to the equations of motion

$$e^{\phi} = \phi_0 (au)^{12/5}, \;\; \mathcal{F} = \frac{25}{192}\phi_0^3 (au)^{26/5}, \;\; \mathcal{B} = \frac{32}{75}\left(\frac{\mu}{a}\right)^2 \phi_0 (au)^6, \;\; b = \mu (au)^{16/5} \,, \tag{F.6}$$

where μ and ϕ_0 are constants. After writing $u = c\rho^{5/16}$ for some constant, c, this solution can be written as

$$\begin{aligned} ds^2 \sim & \rho^{\frac{-2(3-\theta)}{3}} \left(d\rho^2 - d\bar{t}^2 + \rho^{-2(z_1-1)}\left(d\bar{x}^2 + d\bar{y}^2\right) + \rho^{-2(z_2-1)} d\bar{z}^2\right) \,, \\ & e^{\phi} \sim \rho^{3/16}, \qquad a \sim \rho \,, \end{aligned} \tag{F.7}$$

with $\theta = 3$, $z_1 = 9/8$, $z_2 = 3/4$ and the bars indicate we have rescaled the coordinates. This is reminiscent of hyper-scaling solutions with hyper-scaling violation exponent θ, but here the spatial directions also scale with Liftshitz exponents z_1 and z_2. Under the scaling $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{\rho}) \to (\lambda\, \bar{t}, \lambda^{9/8}\bar{x}, \lambda^{9/8}\bar{y}, \lambda^{3/4}\bar{z}, \bar{\rho})$, we find that the metric transforms as $ds \to \lambda^{\theta/3} ds$.

Interestingly, the determinant of this metric does not depend on the radial coordinate. Therefore, if this solution can be generalised to a finite temperature black hole, then the area of this black hole would be the same at any radius and so the solution would have constant entropy. This might suggest that this scaling solution is related to the extremal black hole solution discussed above.

To see whether this solution can be heated to finite temperature, we consider static perturbations about the fixed point solution

$$\begin{aligned} e^{\phi} &= e^{\phi_0} u^{\phi_c}(1 + c_1 u^{\delta}), \qquad & \mathcal{F} &= \mathcal{F}_0 u^{\mathcal{F}_c}(1 + c_2 u^{\delta}), \\ \mathcal{B} &= \mathcal{B}_0 u^{\mathcal{B}_c}(1 + c_3 u^{\delta}), \qquad & b &= b_0 u^{b_c}(1 + c_4 u^{\delta}) \,. \end{aligned} \tag{F.8}$$

Substituting this ansatz into the equations of motion, and keeping terms linear in c_i, we find three solutions. Two of these are marginal modes with $\delta = 0$, and correpond to the scaling symmetries (7.17), whilst the third corresponds to the gauge transformation $b \to b + c$, for some constant c. This suggests that there are insufficient parameters to develop an IR expansion, and hence create a solution that interpolates between this scaling solution in the IR and AdS_5 in the UV.

Therefore, it appears that this scaling solution is not the fixed point solution in the extremal limit of the CGS black hole. It would be an interesting topic of futher work to understand the extremal CGS black hole for arbitrary charge, as it may shed light on the mechanism for the low temperature instability of the CGS black hole.

F.2 Thermoelectric DC conductivity with multiple gauge fields

We will now derive the DC thermoelectric conductivity as outlined in section 3.3. To start with, we will consider a more general theory, and derive its general conductivity matrix. Specifically, we generalise [1] to the case where there are multiple $U(1)$ gauge fields, as well as additional scalars. Using the same notation as [1, 97], we generalise the Lagrangian from [1] so that we consider a general Lagrangian

$$S = \frac{1}{16\pi G_N}\int d^D x\sqrt{-g}\left(R - V(\phi) - \sum_a \frac{Z_a(\phi)}{4}(F^a)^2 - \frac{1}{2}\mathcal{G}_{IJ}\partial\phi^I\partial\phi^J\right). \tag{F.9}$$

where a is a finite number of $U(1)$ gauge fields, and the only restriction on each of the Z_a is that $Z_a(0)$ is constant. We consider black hole solutions of the form

$$\begin{aligned} ds^2 &= -UG\,dt^2 + \frac{F}{U}\,dr^2 + ds^2(\Sigma_d)\,, \\ A^a &= a_t^a\,dt\,, \end{aligned} \tag{F.10}$$

where $ds^2(\Sigma_d) \equiv g_{ij}(r,x)dx^i dx^j$ is a metric on a $(d \equiv D-2)$-dimensional manifold, Σ_d, at fixed r. In addition, $U = U(r)$, while G, F, a_t and ϕ are all functions of (r, x^i).

The boundary conditions are chosen to ensure that the solution approaches AdS_D as $r \to \infty$. These are the same black hole solutions as in [1]. We now perturb this solution with a linear perturbation

$$\begin{aligned} \delta\left(ds^2\right) &= \delta g_{\mu\nu}dx^\mu dx^\nu - 2tM\zeta_i dt dx^i\,, \\ \delta A^a &= \delta a_\mu^a dx^\mu - tE_i^a dx^i + tN^a\zeta_i dx^i\,, \\ \delta\phi^I&\,. \end{aligned} \tag{F.11}$$

The calculation of the DC conductivity now proceeds in very similar way to [1]. Rather than repeat the entire calculation, we just quote the final important results, keeping the same notation as in the original paper. The Hamiltonian constraints evaluated on the black hole horizon lead to the Stokes equations, which are now

$$\nabla_i v^i = 0\,, \tag{F.12}$$

$$\nabla_i(Z_a^{(0)}\nabla^i w^a) + v^i\,\nabla_i\left(Z_a^{(0)} a^{a(0)}_{\ t}\right) = -\nabla_i(Z_a^{(0)} E^{ai})\,, \tag{F.13}$$

$$-2\,\nabla^i\nabla_{(i}v_{j)} - \sum_a \frac{Z_a^{(0)} a^{a(0)}_{\ t}}{G^{(0)}}\nabla_j w^a + \mathcal{G}_{IJ}(\phi^{(0)})\nabla_j\phi^{I(0)}\nabla_i\phi^{J(0)}v^i$$

$$+\nabla_j\, p = 4\pi T\,\zeta_j + \sum_a \frac{Z_a^{(0)} a^{a(0)}_{\ t}}{G^{(0)}} E^a_j\,, \tag{F.14}$$

where the a indicies are only summed over explicity, and

$$v_i \equiv -\delta g^{(0)}_{it}\,, \qquad w^a \equiv \delta a^{a(0)}_{\ t}\,, \qquad p \equiv -4\pi T\frac{\delta g^{(0)}_{rt}}{G^{(0)}} - \delta g^{(0)}_{it}\, g^{ij}_{(0)}\nabla_j\,\ln G^{(0)}\,, \tag{F.15}$$

where the (0) indicates the leading order term in the expansion of the field about the horizon, and $F^{(0)} = G^{(0)}$.

The heat current and electric currents are given by

$$16\pi G Q^i_{(0)} = 4\pi T\sqrt{g_{(0)}}v^j\,,$$
$$16\pi G J^{ai}_{\ (0)} = \sqrt{g_{(0)}}g^{ij}_{(0)}Z_a^{(0)}\left(\partial_j w^a + \frac{a^{a(0)}_{\ t}}{G^{(0)}}v_j + E^a_j\right)\,. \tag{F.16}$$

We now explicitly consider black hole solutions where there are scalars associated with a shift symmetry. In our explicit example that is the axion field. In general, these scalar fields take the form

$$\phi^{I_\alpha} = \mathcal{C}^{I_\alpha}{}_j\, x^j\,, \tag{F.17}$$

everywhere in bulk with $\mathcal{C}$ a constant n by d matrix. For simplicity in this general case, we assume that all spatial coordinates are involved and hence the DC conductivity in all spatial directions is finite. The metric, the gauge fields and the remaining scalar fields will depend on the radial direction but will be independent of the spatial coordinates x^i. The metric on the black hole horizon is flat and in addition, $Z^{(0)}$, $G^{(0)}$ and $a^{(0)}_t$ are all constant.

There is a solution to the fluid equations (F.12)-(F.14), with v^i, p and w all constant on the horizon. The fluid velocity is given by

$$v^i = 4\pi T\,\left(\mathcal{D}^{-1}\right)^{ij}\left(\zeta_j + \frac{1}{Ts}\sum_a \rho_a E^a_j\right)\,, \tag{F.18}$$

with constant E_i, ζ_i and we have defined the $d \times d$ matrix:

$$\mathcal{D}_{ij} = G_{I_{\alpha_1} I_{\alpha_2}} \mathcal{C}^{I_{\alpha_1}}{}_i \mathcal{C}^{I_{\alpha_2}}{}_j \,. \tag{F.19}$$

Furthermore, the averaged charge density, ρ, and the entropy density, s, are given by

$$16\pi G_N \rho_a = \sqrt{g_{(0)}} \frac{Z_a^{(0)} a^{a\,(0)}_{t}}{G^{(0)}}, \qquad 16\pi G s = \sqrt{g_{(0)}} \,. \tag{F.20}$$

The current densities J^i, Q^i are independent of the radius and are given by their horizon values:

$$\begin{aligned} J^{ai} =& \frac{s\, Z_a^{(0)}}{4\pi} \, g^{ij}_{(0)} \, E^a_j + \frac{4\pi\rho_a}{s} \sum_b \rho_b \left(\mathcal{D}^{-1}\right)^{ij} \, E^b_j + 4\pi T \rho_a \left(\mathcal{D}^{-1}\right)^{ij} \zeta_j \,, \\ Q^i =& 4\pi T s \left(\mathcal{D}^{-1}\right)^{ij} \left(\zeta_j + \sum_a \rho_a E^a_j\right) \,. \end{aligned} \tag{F.21}$$

The DC conductivities are thus given by

$$\begin{aligned} \sigma^{ij}_{ab} =& \frac{s\, Z_a^{(0)}}{4\pi} g^{ij}_{(0)} \delta_{ab} + \frac{4\pi \rho_a \rho_b}{s} \, \left(\mathcal{D}^{-1}\right)^{ij} \,, \\ \alpha^{ij}_a =& 16\pi G \bar{\alpha}^{ij}_a = 4\pi\rho_a \, \left(\mathcal{D}^{-1}\right)^{ij} \,, \\ \bar{\kappa}^{ij} =& 4\pi T s \, \left(\mathcal{D}^{-1}\right)^{ij} \,. \end{aligned} \tag{F.22}$$

We now return to the black hole solution in section (3.3), which preserves two gauge fields and one scalar field (in addition to the axion and the dilaton). In this case, the only the z direction will have a finite conductivity matrix. In the notation of [1], have $G = F = \frac{e^{-\phi/2}}{u^2}\sqrt{\mathcal{B}}$ and $U = \mathcal{F}\sqrt{\mathcal{B}}$. We also have $Z_1^{(0)} = Z_2^{(0)} = e^{2\psi_{1h}/\sqrt{6}}$ and $Z_3^{(0)} = e^{-4\psi_{1h}/\sqrt{6}}$, which leads to the charge densities

$$\rho_1 = \rho_2 = \frac{1}{16\pi G_N} \frac{a_{1h} e^{2\psi_{1h}/\sqrt{6}}}{u_h e^{3\phi_h/4}\sqrt{\mathcal{B}_h}}, \qquad \rho_3 = \frac{1}{16\pi G_N} \frac{a_{3h} e^{-4\psi_{1h}/\sqrt{6}}}{u_h e^{3\phi_h/4}\sqrt{\mathcal{B}_h}}, \tag{F.23}$$

while the entropy density is given as in (7.21). Finally, the matrix $\mathcal{D}$ is given by

$$\mathcal{D}_{33} = a^2 e^{2\phi} \tag{F.24}$$

and so the conductivity matrix can now be determined.