Session IV

THEORETICAL PHYSICS
Chairman: M. L. Goldberger

SCHWINGER: Structure of Green's functions

I shall make some general comments on a comparatively simple approach to the problem of finding the structure of Green's functions for the quantized field.

The Green's functions are a mathematical technique for describing all the properties of the quantized field system, the energy values of systems, the scattering properties and so on. In particular, therefore, the problem of finding the general structure of Green's functions also includes the question of finding the general structure of scattering amplitudes and a special application of such an approach is that particular question of structure which is discussed under the heading of dispersion relations. This approach to the problem of deriving or understanding dispersion relations is used with the full understanding that it stays within the framework of present field theory. To that extent it is to be somewhat distinguished from the approach making use of the idea of causality. But I think it has now begun to be realized that, by the time it is through, the derivation of dispersion relations from the point of view of causality is, in a sense, fully equivalent to conventional field theory. So to this extent we are not really doing anything very different.

The first question we want to discuss is what are the Green's functions, how are they to be constructed in a formal way, and what can one say in a general manner about their structure. We want to use only those properties which, although they may depend upon dynamics in a specific way, are obtainable at the state where one is not yet actually making specific calculations about the numerical quantities that enter into the structure of the formalism. To do this, let me remind you that what one means by a Green's function is a function of n space-time points which is defined to be a vacuum expectation value of the time ordered product of field operators:

$$G(x_1, \cdots x_n) = \pm i \langle (\chi(x_1) \cdots \chi(x_n))_+ \rangle \quad . \qquad (1)$$

$\chi(x)$ is a general field; it may be a Bose field or it may be a Fermi field. Correspondingly there are various kinds of Green's functions. The subscript "+" means positive time ordering in the sense that the operators are to be written out in a sequence of increasing times read from right to left. In addition, the Green's functions, as conventionally defined, contain various plus or minus signs in order to compensate the anti-commutative nature of the Fermi fields, and there are various factors of "i" in order to simplify the structure of the differential equations, and so on.

From the specific set of field equations, the Green's functions will obey an infinite set of interlocking differential equations. The principal point I want to direct attention to, however, is the boundary conditions that accompany those differential equations, which have to do with the fact that these are vacuum expectation values of operators written in a certain sequence of time ordering. To see that boundary condition, we may consider what happens, for example, if the time coordinates are in a definite sequence. Suppose x^0 is the time coordinate which is greater than all of the other time coordinates. Corresponding to the definition in terms of positive time ordering, that means that the Green's function will contain the field at x^0 entirely to the left of all the other fields, which will then in turn occupy a sequence depending upon their particular time ordering. Let me remind you that this is an expectation value for the vacuum states of the fully interacting system of fields.

Now the space-time variation of the field χ is, of course, symbolically expressed by a unitary operator in terms of the energy momentum operator that would produce the field at the point x from the field at some conventional point, say the origin:

$$\chi(x) = e^{-iPx}\, \chi(0)\, e^{iPx} .$$

Since this acts on a vacuum state, for which the energy and momentum values have conventionally the value zero, the exponential on the left simply disappears, and the dependence upon the space-time point referring to the latest of all times is described by the eigenvalues of the exponential operator on the right. This means, of course, that what we will have here will be a spectrum of eigenvalues, and the Green's function will be represented by a sum or integral of Fourier terms, in which the various frequencies which occur are the energy-momentum eigenvalues of those states that are produced by the action of this particular field component acting upon the vacuum state. This will be a certain selection of the possible states of the whole system. It is, in fact, the point of the description in terms of the

Green's functions, or propagation functions, that it represents a certain selection out of the totality of states, of those that can be conveniently studied as being produced by the action of one or more field operators acting in a certain sequence. The point, of course, is that all that appears are the energy values of the system that are produced in this way.

Since the vacuum energy is the lowest of all physically acceptable energies, the spectrum of values that you have here will be equal to, possibly, but in general greater than the vacuum energy. That is, there will be a spectrum of positive frequencies only. Therefore, the Green's function of any number of points is characterized by the fact that its dependence upon that time coordinate which is greater than all the rest of them, is such that only positive frequencies are contained in it. That, of course, is true, no matter which particular field component we are dealing with, although, depending upon the nature of the field you will have various choices of the energy-momentum spectrum, characteristic of whatever particular field we're talking about.

In the same way, if we consider the space-time point x to be not the latest of all the points, but the earliest, then the field operator χ (x) will appear not entirely on the left but rather entirely on the right. The space-time variation is represented as before, but the right-hand exponential now acts on the vacuum state and has the value one. In the representation of the space-time variation we will again have a sequence of eigenvalues, harmonic terms with various frequencies characteristic of the energy states that can be produced in this way, and corresponding to the sign change in the exponential, the frequencies will be entirely negative.

We have, therefore, the boundary condition that the Green's function, in its dependence upon the latest of all times, contains only positive frequencies, and in its dependence upon the earliest of all times, contains only negative frequencies. In effect we have a description in terms of waves which can be considered as moving in the space-time region in such a way that if we have a number of such points in space-time, the waves are moving always out of the region in question. When we are on the boundary of the region in the sense of considering the time coordinate that is later than all the others, the frequencies are positive and the waves move out; if it is the earliest of all times, the frequencies are negative, and the waves move again out. In short, we are dealing with a generalization of the Green's function originally introduced by Feynman which corresponds precisely to the boundary condition of outgoing

waves. The waves are, in a time sense, running out of the region in question.

This is the basic boundary condition, then, that characterizes the general Green's functions, and the boundary condition that I want to exploit as a general physical requirement will have to be taken into account in examining the structure of the Green's functions. Incidentally, of course, while I considered this as referring to a single field component, the same thing would hold equally well if we had a group of points held together. In other words, suppose we consider two space-time points x_1 and x_2, and these together are later than any other of the field components. If we then considered a common translation of these two fields, rigidly maintaining a fixed space-time interval between them, then the same argument would apply to the translation of both together. As long as these two points are later than any of the others, you would have only positive frequencies. Those frequencies, of course, would refer to yet another subgroup of the energy mass spectrum, namely, those states that can be produced by the action of two fields. In a certain sense, the positive frequencies you get by having the single field act, pick out one spectrum of single particle states, if you like, another spectrum of two particle states and so on. Again, the indication of how the Green's functions selectively pick out states from the totality of states, can be described in terms of energy values or equally well in terms of scattering.

Now the problem is, for the purpose of selecting solutions of the differential equations that govern the Green's functions, to replace the boundary condition of outgoing waves by a regularity requirement; in other words, to look for the solutions of the differential equations that are regular in a certain complex variable domain, as being equivalent to this choice of boundary conditions. This is all familiar, and the result will perhaps be clear if I say that the selection of outgoing wave boundary conditions is equivalent to the requirement that the Green's function, defined as a function of the space-time coordinates, should remain a regular function when you make the time coordinate complex in a specific way, and that you never find an exponential that becomes unlimitedly large. This regularity requirement is equivalent to a selection of positive or negative frequencies in the two situations. The regularity requirement is that when I take all time coordinates and multiply them by a complex number,

$$x^0 \longrightarrow x^0(1 - i\epsilon), \quad \epsilon > 0, \tag{2}$$

in which ϵ is an arbitrary fundamentally positive quantity, which can be considered to be infinitesimal for our present purposes, the Green's function, as a function of the time coordinate, remains regular. That is fully equivalent to the particular choice of boundary conditions of outgoing waves. I think that you will see that, if I consider just a simple example of two points:

$$x_1,\ x_2;\ x_1^{\,0} > x_2^{\,0};\ e^{-iP^0(x_1^{\,0} - x_2^{\,0})} \longrightarrow e^{-iP^0(x_1^{\,0} - x_2^{\,0})}(1 - i\epsilon)$$

You recognize that this substitution, which multiplies equally well the time interval by $1 - i\epsilon$, forces me, if I am to deal with a quantity that remains bounded no matter how great this positive time difference is, to choose these numbers to be positive only, so that the real part is negative. In short, with this sequence of time differences, the substitution above forces me to pick positive frequencies. On the other hand, it is clear that if the time interval were negative, then I should have to take negative frequencies. So the distinction between positive frequencies and negative frequencies, in accordance with the sign of the time difference, is equally well expressed by the requirement of regularity of these Green's functions under the substitution $x^0 \longrightarrow x^0(1 - i\epsilon)$.

In effect, our problem now is to examine the most general structure of Green's functions depending upon n coordinates which remain finite under this extension of the time coordinate into the complex plane. This appears now simply as an equivalent and simpler way of presenting the fundamental boundary condition that accompanies the Green's function definition. To see how this goes, let's consider the simple case again of 2 points. Just to take the simplest example, suppose we thought of the Green's function of the spin zero field. The complications produced by spin are not essential. They simply mean that any Green's function describing a Dirac field, for example, would be produced from a kind of equivalent spin zero Green's function by multiplication by an invariant produced out of Dirac gammas and differentiations. The basic structure is spin zero. Any finite spin is generated from that in a perfectly straightforward way.

From its definition, the Green's function is supposed to be invariant under translation, invariant under arbitrary Lorentz rotations. It is a function defined without restriction on the points x and x'. The only invariant function that you can produce is constructed from the fundamental invariant, the space-time distance between the points. In other words, it must be a function of

$(x - x')^2$ (where I define this to mean the spacial distance minus the time distance).

$$G(x, x') = G((x - x')^2), \tag{3}$$

Now what we have to do, of course, is to pick that particular function of the space-time interval which remains regular under the transformation $x^0 \longrightarrow x^0 (1 - i\epsilon)$, $x^{0\prime} \longrightarrow x^{0\prime} (1 - i\epsilon)$. Under this extension into the complex plane the square of the time interval becomes

$$(x - x')^2 = (\underset{\sim}{X} - \underset{\sim}{X}')^2 - (x^0 - x^{0\prime})^2 \longrightarrow (x - x')^2 + i\epsilon . \tag{4}$$

(Of course, a positive numerical factor on ϵ doesn't matter, just the sign, or in which half of the complex plane you demand regularity). So the statement is that G is to be a function of the invariant distance which remains regular when the argument is extended into the upper half plane. That's the boundary condition that accompanies the physical choice of outgoing waves.

We now say that any such function can be written, at least in a formal way, as

$$G(x, x') = \int_0^\infty d\lambda \, e^{i\lambda (x-x')^2} f(\lambda) . \tag{5}$$

The statement that this function is regular in the upper half plane, where $(x - x')^2$ acquires a positive imaginary part, is equivalent to the familiar statement that the parameter λ ranges over positive values only. So the integral runs from zero to infinity. In coordinate space this represents the determination of the structure of the Green's function going with the boundary condition of outgoing waves.

It is more convenient to make use of this in the momentum representation:

$$G(x, x') = \int \frac{d_4 p}{(2\pi)^4} e^{i p (x-x')} \int_0^\infty d s \, e^{-i s p^2} g(s) , \tag{6}$$

where $s = 1/4 \lambda$ and $g(s) = f(1/4 s)$.
The regularity requirement in the upper half plane for the square of the coordinate difference becomes transformed into a regularity requirement in the lower half plane for the square of the invariant momentum. This is a straightforward mathematical transformation.

No physics has been added beyond the boundary condition that accompanies the structure of the Green's function. Then we can, if we wish, write g(s) as a formal Fourier integral,

$$g(s) = \int d\varkappa^2 \, e^{-is\varkappa^2} g(\varkappa) . \tag{7}$$

We call the variable of integration $\varkappa^2$ in anticipation of the physical identification. For the moment, $\varkappa^2$ ranges from minus to plus infinity. It is intended just as a formal integration.

Having written it in this way, I can now carry out the s integration to get the final and familiar form

$$G(p) = \int (d(x - x')) G(x - x') \, e^{-ip(x - x')} = -i \int d\varkappa^2 \frac{g(\varkappa)}{p^2 + \varkappa^2 - i\epsilon} . \tag{8}$$

At this stage, of course, we are on familiar ground. We know that, on performing the momentum integrations, you would get back the various frequencies with the appropriate sense of outgoing waves that represent the proper choice of boundary conditions. The number $\varkappa$ then become interpreted as the spectrum of mass values that describe invariantly the energy momentum relations. In other words, in order that a particle with a given momentum have a real energy, the numbers $\varkappa^2$ have to be positive. In short, we must not only satisfy the boundary conditions with outgoing waves, but the outgoing waves have to carry real frequencies, not complex frequencies. That means, therefore, that $\varkappa^2$ is identified mathematically as a measure of the invariant energy-momentum that a particular solution carries and therefore becomes represented as a positive number.

The result, then, is, so to speak, the general functional form that one obtains. One can, incidentally, add the statement: the amplitudes $g(\varkappa)$ are entirely real. In a certain standard way of representing the field equations, nothing but hermitian operators and real numbers appear. The only way in which complex numbers enter is in the deliberate choice of boundary conditions, which the $i\epsilon$ is designed to represent; i.e., a selection of waves traveling in one sense rather than in the other. In a domain, for example, outside the light cone, where no real propagation can occur, everything is real and the amplitudes $g(\varkappa)$ are real numbers. The $i\epsilon$ represents completely the complex nature of the Green's function.

The real problem begins when we now wish to go on to a

Green's function that depends upon several coordinates. All the major problems, of course, occur as soon as we think of the Green's function, let's say, still for the spin zero field, from which everything else can be constructed.

We are now looking for a Green's function depending upon three space-time coordinates. It is to be invariant under translations and invariant under Lorentz transformations without restriction upon the location of the point. Again the most general invariant is a function of the three invariant space-time intervals that one can construct from these coordinates:

$$G\left(x_1, x_2, x_3\right) = G\left[(x_1 - x_2)^2, (x_1 - x_3)^2, (x_2 - x_3)^2\right]. \quad (9)$$

Until we come to a number of points equal to five, the invariant space-time intervals between the points represent a full counting of the independent invariants. When the number of points goes beyond five, however, there are then relations among the space-time differences. The number of invariants essentially grows linearly with the number of points, the number of intervals grows quadratically. You eventually reach a point where there are relations among them. But if we stick to the simpler cases of three and four, which are the most interesting physically, we need not worry about that.

We are now to find that function of these invariants which remains regular when all of them are simultaneously extended into the complex plane in the sense of adding $+ i \epsilon$. Epsilons, of course, can vary, but the point is that they are all positive. In a purely formal, mathematical way we can write the dependence upon each one of these variables in a sequence of Fourier transformations. This would be designed to recast this in terms of the mass spectrum. If such a transformation were permissible, then

$$G(x_1, x_2, x_3) = \int d\lambda_1 d\lambda_2 d\lambda_3 \times \exp\left\{-i\left[\lambda_1 (x_2 - x_3)^2 + \lambda_2 (x_3 - x_1)^2 + \lambda_3 (x_1 - x_2)^2\right]\right\} \rho(\lambda_1, \lambda_2, \lambda_3). \quad (10)$$

Strictly speaking, of course, each λ runs from minus infinity to plus infinity. We could imagine breaking each λ -integration up into three parts. A part which is discontinuously associated with $\lambda = 0$ would be necessary to represent the possibility that the Green's function does not depend upon this particular variable at all. If you want to write it as a Fourier integral, of course, you

have to move aside that exceptional possibility. So there would be a Delta function for $\lambda = 0$ if you like, in the amplitude. The range of integration over positive λ exhibits that part of the function that is regular in the upper half plane, and the range over negative values of λ represents that part of the function which is regular in the lower half plane. This is a decomposition of the general function into three parts. Treating the simple possibility of no dependence on one coordinate difference at all separately, you can imagine each λ -integral as a sum of two, one referring to $\lambda > 0$ exactly analogous to the case we were considering, and one referring to $\lambda < 0$. For each of these you can go over to the equivalent Fourier integral representation:

$$G(p_{23}, p_{31}, p_{12}) = \int ds_1 ds_2 ds_3 \, e^{-i(s_1 p_{23}^2 + s_2 p_{31}^2 + s_3 p_{12}^2)} g(s_1, s_2, s_3) . \tag{11}$$

When $\lambda > 0$, s runs over positive values, and when $\lambda < 0$, s runs over negative values. In other words, depending upon these two possibilities, you have a function of p^2 that is regular in the lower half plane or regular in the upper half plane.

The weight factor $g(s_1, s_2, s_3)$ in turn is represented in a Fourier integral form, which is just a conventional way of writing it:

$$g(s_1, s_2, s_3) = \int d\alpha_{12}^2 \, d\alpha_{23}^2 \, d\alpha_{31}^2 \, e^{-i(s_1 \alpha_{23}^2 + s_2 \alpha_{31}^2 + s_3 \alpha_{12}^2)} g(\alpha_{23}, \alpha_{31}, \alpha_{12}) . \tag{12}$$

α^2 is merely a mathematical parameter that at this stage runs from minus to plus infinity. The performance of the integration with respect to s brings you to the stage of writing it

$$G(p_{23}, p_{31}, p_{12}) = i \int d\alpha_{12}^2 \, d\alpha_{23}^2 \, d\alpha_{31}^2 \, \frac{g(\alpha_{23}, \alpha_{31}, \alpha_{12})}{(p_{23}^2 + \alpha_{23}^2 \pm i\epsilon)(p_{31}^2 + \alpha_{31}^2 \pm i\epsilon)(p_{12}^2 + \alpha_{12}^2 \pm i\epsilon)} \tag{13}$$

the plus or minus $i\epsilon$, or regularity domain, depending upon which part of the function of x^2 you began with. That is only to say that I can isolate the dependence upon each variable and write it in this way

by a sequence of Fourier transformations, replacing it by an amplitude that depends upon a mathematical parameter $\varkappa^2$. Since I can do this for all three variables, the whole thing is a composite Fourier integral, and, depending upon which part of the spacial dependence I began with, the regularity domain is in the upper or lower half plane. I leave the question open, because I want to impose the boundary condition finally upon the whole Green's function.

I know nothing, at this stage, about the analyticity of the function and its dependence upon each one of the space-time coordinates. There are three relative momenta here of which one is redundant in the sense that the absolute momentum that is associated with the space-time point x_1 is in fact given by $p_1 = p_{12} + p_{13}$, and similarly, $p_2 = -p_{12} + p_{23}$, $p_3 = -p_{13} - p_{23}$. The sum of these three momenta is zero. That just expresses the translational symmetry that we built into the theory, and it indicates, for example, that if you specify p_1 and p_2, then there is one relative momentum that is left free which we have to integrate over in order to get the final structure of the actual Green's function in its dependence upon the physical momentum. That, of course, is a straightforward integration which can be performed by the usual parametrization. Let me write the results in the following way:

$$G(x_1,x_2,x_3) = \text{const.} \times \int dp_1\, dp_2\, dp_3 \int d\varkappa_{23}^2\, d\varkappa_{31}^2\, d\varkappa_{12}^2 \qquad (14)$$

$$\times \int_0^\infty dz_1 \int_0^\infty dz_2 \int_0^\infty dz_3 \; \frac{e^{i(p_1x_1+p_2x_2+p_3x_3)}\, g(\varkappa_{23},\varkappa_{31},\varkappa_{12})\, \delta(p_1+p_2+p_3)\, \delta(z_1+z_2+z_3-1)}{p_1^2 z_2 z_3 + p_2^2 z_1 z_3 + p_3^2 z_1 z_2 + \varkappa_{23}^2 z_1 + \varkappa_{31}^2 z_2 + \varkappa_{12}^2 z_3 - i\epsilon}$$

That's the general structure you get.

At this point one imposes the physical requirement of regularity, which is that if you extend the time coordinates into the complex plane in the indicated way, you must have a regular function. In terms of the momentum transform, that appears in the form of the requirement that if the invariant squares of the momenta are moved into the lower half plane, the function must be regular. That therefore means, in a sense, that we must have a regularity domain that remains consistent, which is not crossed, when every p^2 is replaced by $p^2 - i\epsilon$, which is just the Fourier transformation of replacing every x^2 by $x^2 + i\epsilon$. All the ϵ being positive, this means that the whole domain of regularity is expressed by adding $-i\epsilon$ to the whole denominator.

We have, then, the general form of the Green's function meeting all of the requirements, and in which again the complex nature of the Green's function is entirely expressed in the usual formal way by the addition of the $-i\epsilon$. The amplitudes in terms of the parameters $\varkappa$ are real numbers again. You recognize, of course, that the three invariant squares of the momenta appear combined together in a single denominator. The result shows, so to speak, the general content of the information fed in. Translation and Lorentz invariance, the reality properties and the boundary conditions imply that the Green's function is simply transformed into a function of certain invariant mass parameters. The physics is all concealed in $g(\varkappa_{23}, \varkappa_{31}, \varkappa_{12})$.

So we have the general structure for the case of three points. You recognize it as following a general pattern that could be applied, with more and more algebraic complications of course, to larger numbers. Particularly, the whole thing goes through in exactly the same way for four points, which is what you need in order to have a physical application, say, to scattering; meson nucleon scattering, for example. We are dealing with very complicated structures, and when we finally apply this to a simple problem like the derivation of the dispersion relations for forward scattering, the apparatus is much too ponderous for that particular application. But that is only because we are describing the full content of the field theory, and inevitably we deal with many more complications because much more is in principle contained in this formulation.

The physical problem then becomes transferred to the question of finding the amplitudes g and their dependence upon the parameters $\varkappa$, and, in particular, in finding for any particular physical application what the range of values of the quantities $\varkappa$ is. That comes down to the question of what you are going to apply this structure formula to, that is, what particular Green's function will one use it for. In any actual application one would apply it not to the Green's function itself, but to quantities derived from it which characterize, not the full propagation characteristics of the particles in interaction together with their subsequent free propagation without interaction, but rather the really fundamental aspects of the interaction of several particles. That means replacing the Green's function, if you like, by another one, depending upon the same number of space-time points but limited to the space-time behavior in the region where the particles are all fully interacting with each other. It means to isolate the relatively trivial aspect that refers to the self-interaction properties. So in actual application, this structure would be applied to Green's

functions derived from the ones written down, referring to all points associated with fields, or the corresponding particles, in full interaction, spacially and temporally limited to the space-time points where they approach each other. Let me try to illustrate this in the simple case again of two points, where all the ideas are present.

Suppose again we had a Green's function of two points referring to a spin zero field. Let's make it very explicitly now a Bose field:

$$G(x,x') = i\langle (\varphi(x)\,\varphi(x'))_+ \rangle .$$

We have to make use very explicitly of the field equations that this field obeys. These will be equations, say,

$$(k^2 + \mu_0^2)\,\varphi = j . \tag{15}$$

One begins with the field equations in the absence of interaction, and j will be the coupling. If you think of φ as the meson field, this will be the coupling with the nucleon system. j will be some effective current, an operator referring to the second field that φ is coupled with. Now we will want to make use of the field equation in the following way.

Suppose we write this so as to bring in the propagation equation that the meson field obeys in its lowest mass state. That is, among the effects of the interaction is that of replacing the mass parameter μ_0 to begin with by the mass parameter μ that refers to the actual propagation of the particle in full interaction. The point of this formalism is to refer the interaction properties of the particles to the propagation characteristics as they would actually be observed in terms of fully interacting particles, not the mathematical idealization of the adiabatic decoupling which is so often employed. We want to represent the scattering interactions and the scattering characteristics of particles, identifying them by their spatial separation rather than by their time separation. So we deal in this formalism always with fully interacting particles and attempt to separate the fully interacting free propagation characteristics from the scattering interactions, for example.

In this case we are not talking about scattering, but these same

ideas apply. You would therefore write

$$(k^2+\mu^2)G = 1 + \langle(((\mu^2-\mu_0^2)\varphi + j)\varphi)_+\rangle . \tag{16}$$

What we have applied is the operator referring to the actual observed mass μ, not the original mass μ_0. The difference will then appear on the right. This is the first stage in the process I referred to; to replace the Green's function referring to the actual field by a Green's function referring to those combinations that describe the interaction properties of the field, the change in mass and the effects of the coupling with the second Fermi-Dirac field, or whatever it may be. In other words, we replace the original fields by a new set of fields, all referring, of course, to a single space-time point:

$$\Phi = (\mu^2-\mu_0^2)\varphi + j . \tag{17}$$

You would then repeat this derivation; you would apply the differential operator again in order to replace the original field by a modified field that really describes the local interaction properties rather than the free propagation characteristics. So you would apply the differential operator $k^2 + \mu^2$ on the right, and there would appear finally this structure of the Green's function (skipping the intermediate steps):

$$G = \frac{1}{k^2+\mu^2} + \frac{1}{k^2+\mu^2}\left[\langle(\Phi\Phi)_+ + (\mu^2-\mu_0^2)\rangle\right]\frac{1}{k^2+\mu^2} . \tag{18}$$

In short, the original Green's function has been replaced by something having the same general structure, but with an altered meaning of the field. The point is that from this Green's function all reference to the lowest mass state has been removed. We have separated that part of the information in the Green's function for two points that simply says that in there is contained the propagation of a particle of mass μ. From that we have separated the part referring to all of the local disturbances that are occasioned by the creation of a particle, the disturbances that do not propagate to a very great distance, the disturbances that are not particularly physically meaningful as far as

the single particle Green's function is concerned. We have their analogue in the local disturbance that produces the physical scattering.

You recognize that if the vacuum expectation value in G remains constant when k^2 is replaced by $-\mu^2$ you would have not only a single pole, which is the mark of a state of energy-momentum connected by the mass μ, but you would have a double pole. Such double poles, of course, cannot exist. In fact, if the Green's function $\langle (\underline{\Phi}\,\underline{\Phi})_+ \rangle$ had a singularity at that point, it would correspond to a triple pole. All reference to the lowest mass state, however, has been removed from this Green's function. It therefore describes only the more complicated local excitations. The free propagation of the particle is isolated in the term $(k^2+\mu^2)^{-1}$. For this new Green's function you would use a general representation of the kind we discussed:

$$\langle (\underline{\Phi}\,\underline{\Phi})_+ \rangle = \int^{\infty} \frac{d\varkappa\, B(\varkappa)}{k^2+\varkappa^2} \quad . \tag{19}$$

The mass spectrum would not begin with mass μ, but would begin higher, with all of the additional states that could be created accompanying the operation of the field operator once. If you were talking about the pseudo-scalar field, for example, this mass spectrum would begin at 3μ, which would be the smallest mass that would be coupled with the mass μ. This means that you have a physical condition derived from the meaning of the Green's function as a compact expression of all the energy-momentum states. We say, for example, that $\langle (\underline{\Phi}\,\underline{\Phi})_+ \rangle$ does not have a pole at μ. That is expressed by the range of the mass spectrum. More than that; the double pole does not exist, which is to say that when you evaluate $\langle (\underline{\Phi}\,\underline{\Phi})_+ \rangle$, $k^2 = -\mu^2$, the coefficient of the double pole in G has to vanish. That gives you a formula for the mass μ_0 in the absence of interaction in terms of the actual mass μ and all the local interaction properties.

From the result of all of this, one gets the actual Green's function. The facts that there is a renormalization, and that the single mass is present only a fraction of the time, then come out in the re-expression of the entire Green's function.

Of course, we would want to go through the same kind of rearrangement in the Green's functions referring to several points, four

points in particular, to describe scattering. If you are interested in meson-nucleon scattering, for example, you are concerned with the Green's function referring to two Boson points and two Fermion points:

$$G_{11} = \pm i \langle (\varphi\varphi\psi\psi)_+ \rangle .$$

This Green's function, referring basically to one nucleon and one meson, would then be separated into the product of the Green's functions referring to one nucleon, one meson non-interacting but containing all physical properties, and the rest describing the additional scattering owing to the interaction that is produced only when these particles physically overlap. If they do not, you have the free propagation of each without interaction with each other, but each interacting locally, giving them their actual physical properties. Then, following a calculation, you will have

$$G_{11} = G_1 G_1 + \frac{1}{\gamma p + m} \frac{1}{k^2+\mu^2} [\cdots] \frac{1}{\gamma p + m} \frac{1}{k^2+\mu^2} .$$

The whole structure of the theory is now transferred to what appears in the bracket. And what appears there contains a number of different parts, of which the most complicated one is again a four point Green's function referring to the fields as they are modified by the effective interaction:

$$\langle (\bar{\Phi}\bar{\Phi}\bar{\Psi}\bar{\Psi})_+ \rangle .$$

This term exists only by virtue of the fact that the physical meson and nucleon are overlapping in space and time and are physically interacting. It is to this quantity that the general structure of the Green's function referring to four space-time points is to be applied. It is this quantity also that gives you the nature of the scattering amplitude and its complex structure, from which in particular the dispersion relations will follow. Of course, in principle, it contains very much more because you now have an expression for the scattering amplitude in terms of field operators from which equations of motion can be derived, and so on. It may enable you, in principle, to obtain not only the most general consequences, such as dispersion relations, but the most specific connections between the physical properties that are involved in scattering and the physical properties involved in other phenomena in which the same kind of field operator combination might occur.

DISCUSSION

VAN KAMPEN: The irregularity condition on functions of the square of the time differences does not seem to me to be equivalent to the original boundary condition.

SCHWINGER: Well, I simply said that if the Green's functions are functions only of the invariant space-time distances, then the regularity requirement on the time intervals becomes transformed into a regularity requirement on the squares of the time differences in the lower half-plane. The latter follows directly from the former, since only these invariant combinations occur because of Lorentz invariance.

VAN KAMPEN: The function you write down is regular as a function of $(x - x')^2$, but not necessarily as a function of $(x - x')$.

SCHWINGER: The implication is that it will be. I satisfied all of my requirements. I may then, if I wish, go back and recognize explicitly that when I construct the function it is true. We can certainly do this explicitly in the simplest cases, and it is true. For three and four points it is, of course, algebraically complicated. Let me say, incidentally, that I have actually carried this work only to the point of recognizing the general forms. I have never gone through the labor of reducing the algebraic expressions down to the point where they are usefully applicable to scattering.

LEHMANN: Are these forms you wrote down for three or four points conjectured, or have they been derived by actual proofs?

SCHWINGER: I have told you the basis of the proof.

LEHMANN: Well, yes, but....

SCHWINGER: I contend that the structures can be nothing more than what can be obtained by direct application of the regularity requirements which, I say, are equivalent to the boundary conditions. They are correct insofar as the formal manipulations with Fourier integrals are justified. You can certainly point out generally mathematical functions that will not meet these requirements; that I cannot possibly prevent. But the statement would be that a physical theory ought to fall into this general framework.

LEHMANN: Unless you know what the ranges of your parameters are, these formulas are not very useful. It is very essential to know these.

SCHWINGER: I meant to say at the end that the range of the $\varkappa$'s depends upon which particular function you apply this structure formula to. The range of the parameters would always be such as to begin at the first excited groups of particles, whose energy spectra could be added to the lowest physical state; that is, the lowest physical states are always removed.

FEYNMAN: If you have three points, isn't there some kind of triangle inequality? That is, if among three points two are separated by one space, and two by another, then the third pair cannot be separated by more space than the sum of the other two. There is, therefore, a limitation on the range of the three coordinate differences. Does that have any effect on your results?

SCHWINGER: I don't think so. This is a physical limitation on the range in which you can freely assign numbers, but it does not change the counting of independent variables. Certainly every invariant is reducible to the three invariant coordinate differences squared. There is no algebraic restriction on these three. That is particularly true, of course, if you make the extension into the complex plane, where the reality restrictions no longer hold. That doesn't effect anything locally, which is all we really make use of. In a local region, these invariants are freely assignable.

KÄLLÉN: Structure of the vacuum expectation value of three field operators

The previous speaker has most eloquently told us why it is so interesting to look at the vacuum expectation values of products of field operators. I should like to discuss the same problem from a slightly different point of view. Thereby, I should like to concentrate on the analytic properties of the functions involved. In particular, I should like to emphasize that we really have to deal with two analytic functions, one in x-space and one in p-space. Let me now get down to details.

We have two scalar field operators (I agree with the previous speaker that spin, etc., are entirely irrelevant) and take the expectation value for the physical vacuum:

$$\langle 0|A(x)B(x')|0\rangle = \int d_p\, e^{ip(x-x')}\theta(p)\,G(p^2) = F(x-x'),$$

where

$$\theta(p) = \begin{cases} 1 \; ; \; p_0 > 0 \\ 0 \; ; \; p_0 < 0 \end{cases} .$$

This is an ordinary product, not time ordered. That's why you have the θ (p) in the Fourier transform. The function F in x space is the boundary value of a certain analytic function F(z), where $z = -x^2$. (I use the minus sign here for convenience). We all know, as was so beautifully explained before, that this function is regular in the plane cut along the real positive axis, and therefore we can write it in terms of these dispersion-like relations:

$$F(z) = \int_0^\infty \frac{da\, f(a)}{a - z} .$$

This is one function we have to do with, and for convenience I will in the future refer to it as the analytic function in x-space. The mathematical reason for its analytic properties is, of course, the appearance of only positive time-like vectors in the Fourier transform.

There is another analytic function which one gets by taking

$$\langle 0 | [A(x), B(x')] | 0 \rangle \theta(x - x') = \int dp\, e^{ip(x-x')} H(p) .$$

Here, H(z) is again the boundary value of a certain analytic function This z, of course, is $-p^2$. This analytic function in p-space, as I shall refer to it in the future, is also analytical in the cut plane. So much for the two-vacuum expectation values and the appearance of the two different analytic functions, one in x-space and the other one in p-space.

Now let us take a three-fold vacuum expectation value. This we write in a similar way:

$$\langle 0 | A(x) B(x') C(x'') | 0 \rangle = \int dp\, dp'\, e^{ip(x-x')} e^{ip'(x'-x'')} \times \theta(p)\, \theta(p')\, G(p^2, p'^2, p \cdot p')$$

$$= F^{ABC}(z_1, z_2, z_3) ,$$

$$z_1 = -(x - x')^2$$

$$z_2 = -(x' - x'')^2$$

$$z_3 = -(x - x'')^2$$

F^{ABC} is again the boundary value of a certain analytic function of z_1, z_2, z_3. It is the generalization of F(z), and I will refer to it as the analytic function in x-space.

In quite a similar way, you can generalize the commutator times the θ- function, which, for all practical purposes, is the same as the time ordered product. Just to bring out the mathematics, I will show you what it is:

$$\theta(x-x')\,\theta(x'-x'')\langle 0|[C,[B,A]]|0\rangle + \theta(x-x'')\,\theta(x''-x')\langle 0|[B,[C,A]]|0\rangle$$

$$= \int dp\, dp'\, H^A(p,p')\, e^{ip(x-x')}\, e^{ip'(x-x'')}\,.$$

H is now the analytic function in p-space. If you take the time ordered product, you have exactly the same analytic function in p-space, only different branches of it, (I mean, on different sides of the cut).

The point I would like to emphasize is that, for the three-fold vacuum expectation value that we have here, also, the domains of regularity of these two analytic functions, the analytic function in x-space and the analytic function in p-space, must be identically the same. I should also like to emphasize that the domain of analyticity for these new functions, F and H, in the three-fold vacuum expectation values, is not just the product of the 3 cut planes. So I completely agree with the discussion remarks made by Feynman, that the restrictions on the vectors are extremely important, and that they mean very severe restrictions on the domain of analyticity of these functions.

The proof that those two functions, F and H, must have exactly the same domain of analyticity is not quite trivial. I will skip it here, but I will be glad to give all of the details afterwards to anyone interested. It involves some calculations. I will now try to discuss the domain of analyticity that we have for both these functions.

The property of these functions to be analytic, is the direct consequence of the structure of the Fourier transform, namely, that the function G is different from zero only for positive time-like vectors. I should like to emphasize that a very, very important property, for these domains of analyticity, is the commutation property of the scalar field. The domains of analyticity for these

functions are quite different for a theory where the field operators commute for space-like distances and for a theory where they do not. To see this, we remark that if we take a theory where the operators commute and take instead of the original vacuum expectation value, this one:

$$\langle 0 | BAC | 0 \rangle = F^{BAC},$$

we are led to another function, which in this notation will be F^{BAC}. From the property that A and B commute for space-like distances we can conclude that F^{ABC} and F^{BAC} are really the same function in a certain restricted domain of the variables z_1, z_2, and z_3. But then there is the well-known property of analytic functions that if they are equal on a rather small domain, say, along a finite piece of a line, however small that piece is, they are really the same function. It is one of the most powerful properties of analytic functions in general. This allows us to say that everywhere where F^{ABC} is analytic, there the same is true for F^{BAC}. If you sit down and investigate the domains of analyticity that really follow from the vector properties, you find that the domain for the function F^{ABC} and the domain for the function F^{BAC} are not exactly the same. In a very rough general picture, things look like this:

There is a certain piece where the three domains overlap and that contains the space-like separations. Therefore we know that the functions are equal there, and hence we can continue the function F^{ABC} from the domain where it is, so to speak, trivially defined to the larger domain which is the union of the original domain and the two permuted domains. The local commutativity is therefore a very, very powerful method for defining the domain of analyticity of these functions.

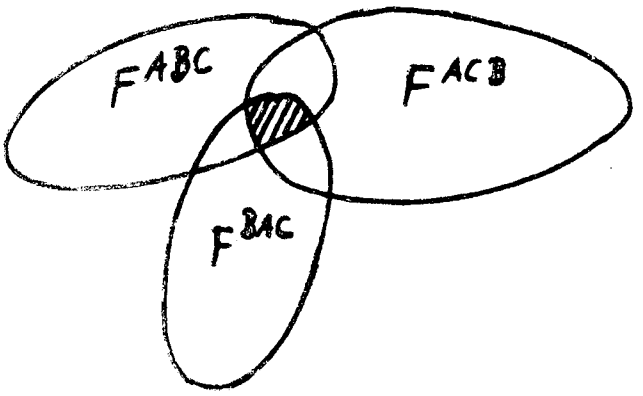


Fig. 1

As far as these domains are concerned, I now have to do the discussion in two steps. I have to discuss the original domain

that is sketched on page 20 for F^{ABC}, then I have to commute z_1 and z_2 to get the permuted domain, and then finally I have to take the union of the three domains and I can claim that the function must be analytic in the whole of these domains. Although this discussion requires quite a lot of formal manipulation, the algebra is perfectly straightforward. I will skip all the practical details here. I shall mention only that all the calculations have been done in close collaboration with A. Wightman, now at Copenhagen.

In giving a detailed picture of these domains, I will meet with some practical difficulties. We have three complex variables, which means a six dimensional real space. So I have to make drawings in that six dimensional space, and everybody who ever tried to draw a three dimensional space, and everybody who ever tried to draw a three dimensional picture on a two dimensional blackboard will perhaps appreciate the difficulty. One way I can do it is by fixing four of the degrees of freedom once and for all, giving the corresponding coordinate numbers, and draw a curve for the remaining two variables. For each pair of points z_1 and z_2, I have to draw a new curve. I have to apologize for the quality of the following sketches but I hope that they will be good enough to see something at least. These pictures are to be understood in the following way: you give two complex points, z_1 and z_2, and then you show the complex plane for the third point, where you have a shaded domain where the function might have singularities and an unshaded, white domain where the function must be regular.

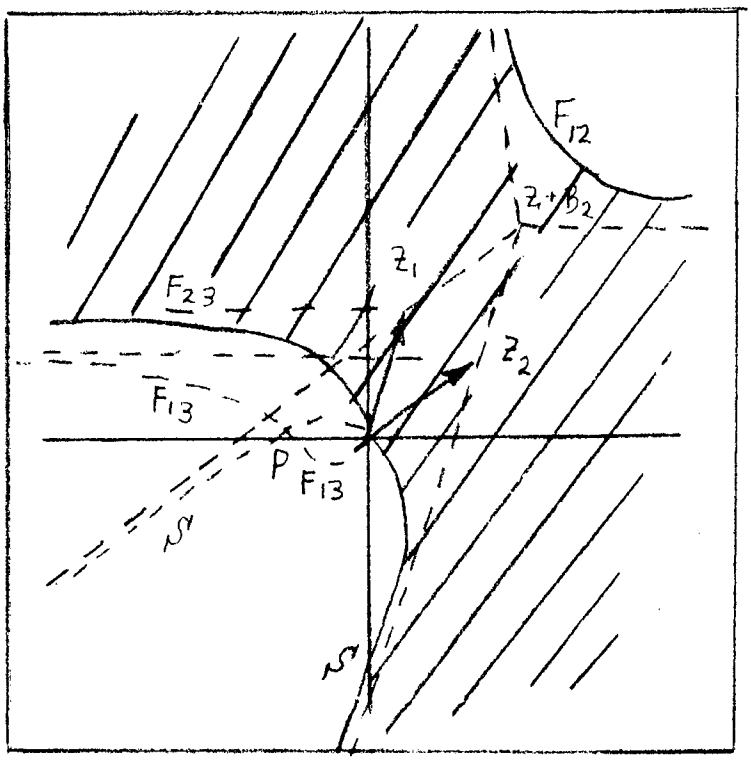


Fig. 2
$\arg z_1 + \arg z_2 < \pi$

Figure 2 shows the situation when the two points z_1 and z_2 are both in the first quadrant. Actually, the important point is that they are both in the same half plane, e.g. the upper, and that the sum of the two arguments is smaller than π . Then the white domain, as you see, consists of two large disconnected pieces. The boundary of the domain is made up of several pieces of different curves. F_{12}, for instance, the curve in the upper right-hand corner, is a hyperbola centered at the point $z_1 + z_2$, with one asymptote horizontal, and with the other asymptote in the direction defined by the sum of the arguments of z_1 and z_2. The equation for that curve is:

$$z_3 = z_1 + z_2 + K + \frac{z_1 z_2}{K}, \quad 0 < K < \infty .$$

For each real value K you get a certain complex number z_3, and when K is varied, z_3 describes the hyperbola F_{12}.

I should like to mention that if you do not have local commutativity and therefore the permutation symmetry, then the domain of analyticity for z_3 is just the one white region in the upper right-hand corner. However, by virtue of local commutativity the domain is very much extended and you get also the piece in the lower left, which is bounded by two curves, F_{23} and S. F_{23}, the lower boundary in the upper half plane, might look like a hyperbola, but actually it is not. It is obtained from F_{12} by an interchange of z_1 and z_3. In the lower half plane, the boundary is an entirely different curve, S, which again happens to be a hyperbola with its center in the same point $z_1 + z_2$, but with different asymptotes. Its equation is

$$z_3 = z_1\left(1 - K\right) + z_2\left(1 - \frac{1}{K}\right), \quad -\infty < K < \infty .$$

Fig. 3 shows a similar picture The curve F_{13} starts at the origin, goes around in this absolutely crazy way and then off to the left. The intersection of the S curve with F_{13} is no longer the origin, but the point P: to the right of it. The region of singularities is now the shaded domain, with the further addition of a prong on the real axis from the intersection point P to the origin. So the allowed domain is now the white domain, with the exception of that part with the real axis that goes out like a prong from the shaded domain to the origin.

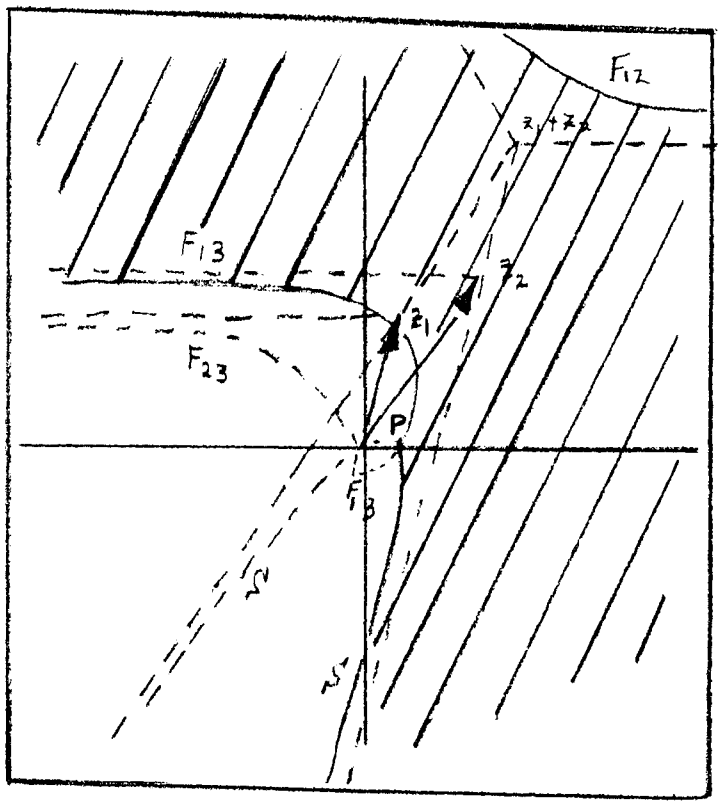


Fig. 3
$\arg z_1 + \arg z_2 < \pi$

As I said before, the direction of the second asymptote of F_{12} is given by the argument of the product of the two complex numbers z_1 and z_2. As long as the sum of the two arguments is less than π, this asymptote goes up, and we have two dis-connected domains. However, if we vary, say, z_2 it will sooner or later happen that the sum of the two arguments becomes larger than π. Then this asymptote points downward. That situation is illustrated in Fig. 4.

You see that the white domain has become connected, because the F_{23} curve and the F_{12} curve intersect. This is the case when the sum of the two arguments is larger than π. You see this is a rather complicated domain, and there is nothing at all as simple as just the product of the three cut planes. There is also a prong of the real axis from P to the origin that has to be added to the shaded area.

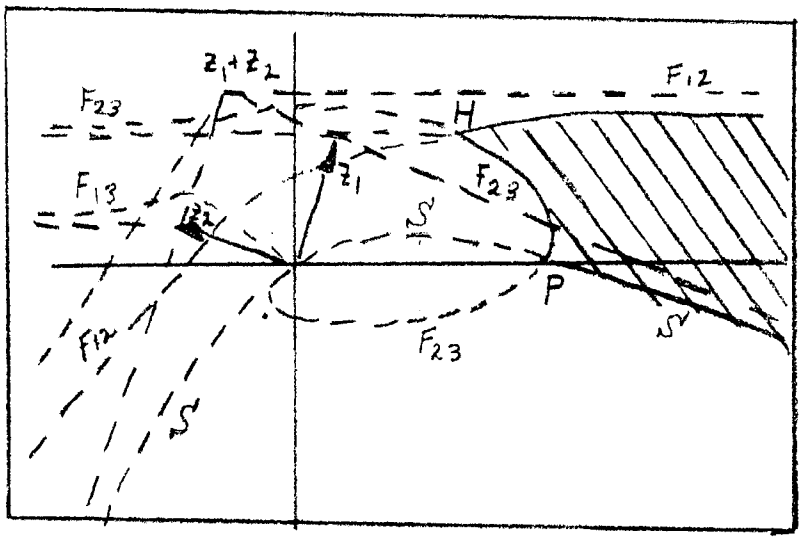


Fig. 4
$\arg z_1 + \arg z_2 > \pi$

As the point z_2 approaches the negative axis, the point P and the horn H both run to plus infinity. When z_2 emerges in the lower half plane, we have the situation illustrated in Fig. 5. You see that the whole plane is

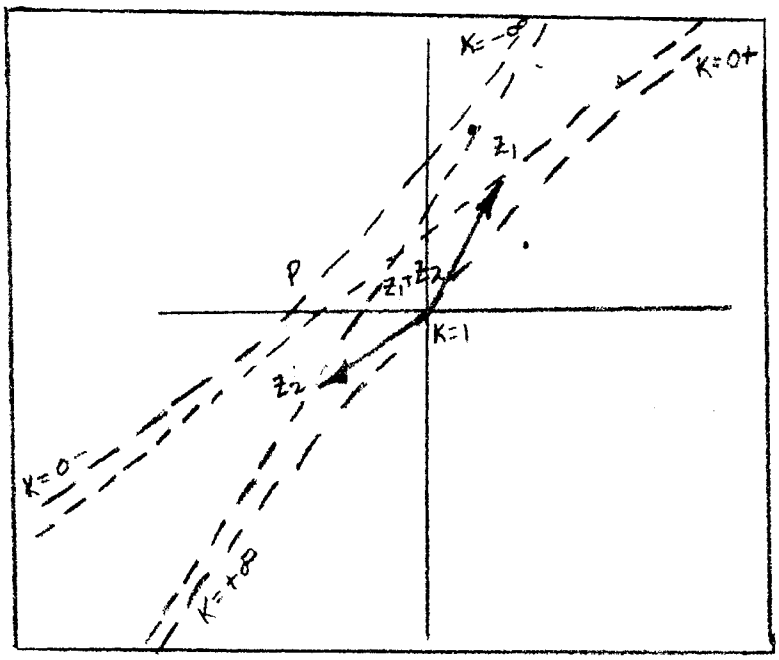


Fig. 5
z_1 and z_2 in opposite half planes. $\arg z_2 - \arg z_1 < \pi$

allowed, except the cut. Since it now never happens that z_1, z_2 and z_3 are all in the same half plane, it is always the S curve that is relevant. In Fig. 5 the S curve is the upper hyperbola, where K runs from minus infinity to zero, for some reason which I don't care to explain. Actually what happens is that without using the local commutativity everything on the left side of the S curve is allowed. If you do the permutation, it so happens that the S curve goes over into itself, essentially, plus pieces of the other branch. That is why I call it the symmetric curve or the S curve. After the permutation, then, one sees that everything to the right of S is allowed, and the only exception is the cut.

This happy situation prevails until the two branches of the hyperbola cross, and that happens when the two points z_1 and z_2 are on the same line through the origin. Fig. 6 shows what happens after that. The area between the two branches of S is now the excluded domain, and in addition there is a cut from P to plus infinity.

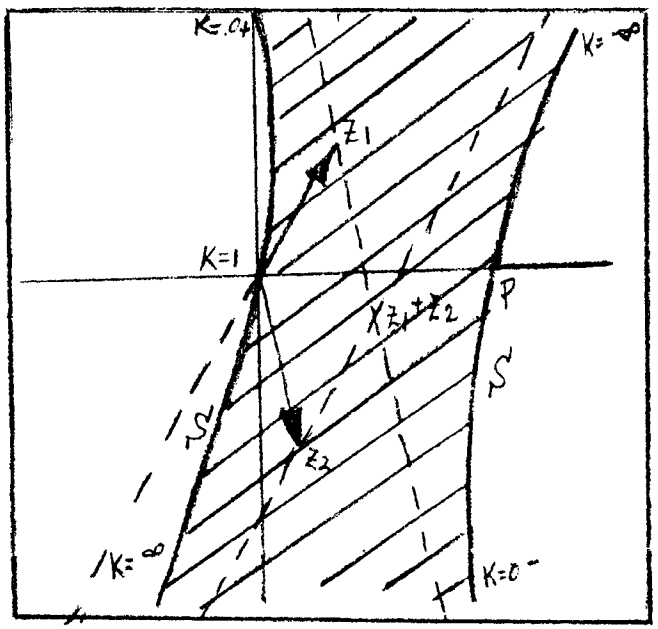


Fig. 6
z_1 and z_2 in opposite half planes.

$$\arg z_2 - \arg z_1 > \pi$$

These pictures together, I hope, give you some idea, perhaps not exactly of how the domain of regularity looks, but about how complicated it is. The next mathematical problem is to find the most general function which is regular in the white domain, but which has a singularity anywhere in the shaded domain.

Actually, this problem has a very simple mathematical answer. There is no such function whatsoever. In other words, every function that is regular in the white domain can be continued a bit into the shaded domain. The mathematical reason for this is

a well-known (to mathematicians) theorem. I didn't know it until about a year ago when I first started investigating these things. Roughly speaking, the situation is that an arbitrary region in a 2 n-dimensional space like this can in general not be the domain of regularity for an analytic function.

If we take a very simple example, Fig. 7 shows one z-plane, say, and the slanting line might represent all the other dimensions. Then you want to find a function that is singular only in a domain A around the origin but regular everywhere else. You can put a path C around the vertical axis and parallel to the z_1 plane and look at the integral

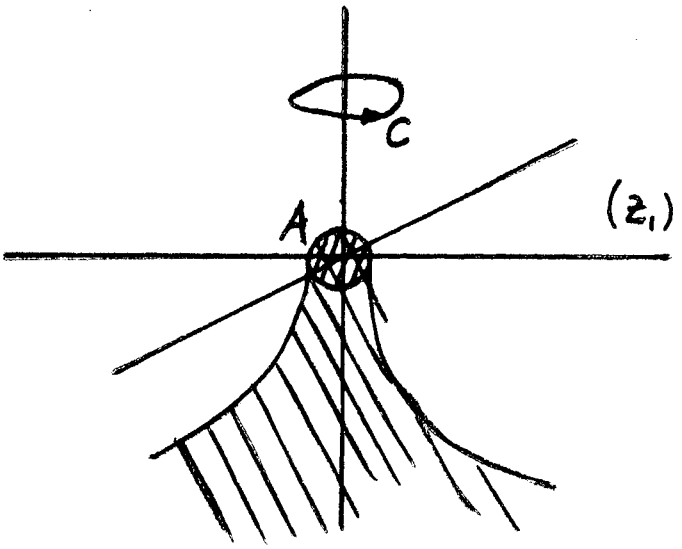


Fig. 7

$$\frac{1}{2\pi i}\int_C d\zeta\ \frac{F(\zeta, z_2, \cdots)}{\zeta - z_1}\ .$$

C being entirely inside the domain of regularity of the analytic function, the ordinary Cauchy formula tells us that this integral is equal to the analytic function F. Then you can start moving the path of integration downward, and if you are moving it inside the domain of analyticity, it still represents the same function. You can start pulling it over the region A, and with this technique you can continue the function also into a small isolated domain like A. So, roughly speaking, one point in this 2n-dimensional space can never be a manifold of singularity for the analytic function.

It is clear, I think, that if you have something like a horn sticking out, you can use the same technique, and you can chip the horn off the domain of singularity. Now, in our pictures we saw lots of horns of the domain of singularity. Therefore you

know that you can continue the function analytically over those horns.

Actually I do not know how far you can do this continuation. There is quite a lot of algebra involved in getting the final answer. We do not have it yet and I apologize for speaking of this uncompleted work. It seems to be complete enough, however, to tell about a few things that we must not do. One conclusion that can be drawn from it is the statement that our domain of analyticity is not just the product of the three cut planes. Historically, this was first shown by a counterexample by Jost and Lehmann, but we can immediately see it here from the representation (14) of the previous talk. If you take this representation and compute the analytic function F in x space you find that it is simply a product of three $\Delta^{(+)}$ functions. (The sign "+" here means positive frequencies and not time ordering!) Therefore, the function F is analytic in the product of the three cut planes and the representation is certainly a possible one. However, the corresponding function H can also be computed and is rather directly given by the kernel of the representation (14). It turns out that the domain of analyticity for this function is the product of the cut planes except for the situation illustrated in Fig. 6. In that case, the H function has singularities in a domain around the positive real axis between the origin and the point P. The fact that it has singularities anywhere except on the cuts makes the statement that it is the most general representation of the three fold vacuum expectation value into a self contradiction. Nevertheless it is certainly a possible form for this quantity but does not follow from the very general arguments used here today.

DISCUSSION

(Editors' note: Part of this discussion was lost)

GOLDBERGER: You said that the naive formula which Schwinger wrote down was, in fact, a possible representation of the three-fold vacuum expectation value. To the extent that it represents, for example, what comes out of lowest order perturbation theory, it is indeed a possible formula. Nonetheless, you made a number of cryptic remarks that the formula was in itself self-contradictory. How can it be both possible and self-contradictory?

KÄLLÉN: What is self-contradictory is the statement that the formula is the most general representation with the required

regularity. The particular formula you are referring to is supposed to represent the analytic function in x-space, regular in the cut plane. The corresponding function in p-space has a more complicated domain of regularity. I think I said it twice, maybe three times, even if I didn't prove it, that it follows from very general arguments that the domain of analyticity for the function in x-space and the function in p-space must be exactly the same. Therefore, as the domains here are different, it simply can't be the most general function. Is that clear?

GOLDBERGER: Yes. Thank you.

(Editors' note: The following supplementary remarks were not presented at the conference but were submitted by Schwinger after the conference.)

SCHWINGER: What I wanted to emphasize was the importance of the outgoing wave boundary condition characteristic of the Green's function, and its (at least partial) restatement by a regularity requirement. The latter, however, was used rather imperfectly in my talk, and the following is an improvement.

For a Green's function of three points, invariant under the inhomogeneous Lorentz group, i.e., Eq. (9), the outgoing wave boundary condition is to be imposed in the form of the requirement of regularity under the complex extension Eq. (2). We then write it in the form Eq. (10) and observe that the complex substitution Eq. (4) produces the additional factor

$$\exp\left\{-\epsilon\left[\lambda_1(x_2^0-x_3^0)^2+\lambda_2(x_3^0-x_1^0)^2+\lambda_3(x_1^0-x_2^0)^2\right]\right\}$$

in the integrand. We must therefore require that the bracket above be positive for all x^0, which implies

$$\lambda_1+\lambda_2>0,\quad \lambda_2+\lambda_3>0,\quad \lambda_3+\lambda_1>0,$$
$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1>0. \tag{20}$$

Alternatively, we may go into momentum space and write

$$G(x_1,x_2,x_3)=\int d p_1 d p_2 d p_3\, e^{i(p_1x_1+p_2x_3+p_3x_3)} \times \delta(p_1+p_2+p_3)\,G(p_1,p_2,p_3)\,, \tag{21}$$

$$G(p_1,p_2,p_3)=\int \frac{ds_1 ds_2 ds_3}{(s_1+s_2+s_3)^2}\, g(s_1,s_2,s_3)\exp\left\{-i\left[\frac{p_1^2 s_2 s_3+p_2^2 s_3 s_1+p_3^2 s_1 s_2}{s_1+s_2+s_3}\right]\right\},$$

where $s_i = 1/4\,\lambda i$. The equivalent of the extension Eq. (2) in coordinate space is

$$p^0 \longrightarrow p^0\,(1+i\epsilon)$$

in momentum space. This implies the requirement

$$\frac{p_1^{0\,2} s_2 s_3 + p_2^{0\,2} s_3 s_1 + p_3^{0\,2} s_1 s_2}{s_1+s_2+s_3} > 0 \tag{22}$$

in addition to

$$p_1^0 + p_2^0 + p_3^0 > 0.$$

We then regain the conditions Eq. (20) on the λ's.

We now include a factor

$$\int ds\, \delta(s-s_1-s_2-s_3)$$

in the integrand of Eq. (21), and introduce the new variables

$$z_1 = s_1/s\,, \quad z_2 = s_2/s\,, \quad z_3 = s_3/s\,.$$

Then

$$G(p_1,p_2,p_3)=\int ds\, dz_1 dz_2 dz_3\, \delta(z_1+z_2+z_3-1) \times g(sz_1,sz_2,sz_3)\exp\left\{-is\left[p_1^2 z_2 z_3 + p_2^2 z_3 z_1 + p_3^2 z_1 z_2\right]\right\}, \tag{23}$$

where s and the z_i are to be integrated over freely, subject only to the regularity requirement Eq. (22), which now reads

$$s z_1(1-z_1)>0,\ s z_2(1-z_2)>0,\ s z_3(1-z_3)>0,$$
$$z_1 z_2 z_3 > 0. \tag{24}$$

In this form we recognize that there are the fundamentally different domains:

a) $s > 0$; here $z_i(1-z_i)>0$, and hence
$0 < z_i < 1$;
This is the normal domain.

b) $s < 0$; here every z_i must be negative or greater than unity. Not all the z_i can be greater than unity, since their sum must be unity. Furthermore, the number of negative z_i's must be even, by Eq. (24). Hence two of the z_i's are negative without restriction on their magnitude, and the third is then automatically positive and greater than one. This defines the abnormal or Källén domain.

For the normal domain we write Eq. (12), and if we return to coordinate space, Eq. (14), for the normal contribution to G. If we use

$$\delta(p_1+p_2+p_3)(s_1+s_2+s_3)^{-1}\exp\left\{-i\left[\frac{p_1^2 s_2 s_3 + p_2^2 s_3 s_1 + p_3^2 s_1 s_2}{s_1+s_2+s_3}\right]\right\}$$
$$\propto \int d\ell_1 d\ell_2 d\ell_3\, e^{-i(s_1\ell_1^2+s_2\ell_2^2+s_3\ell_3^2)}\delta(\ell_2-\ell_3+p_1)\delta(\ell_3-\ell_1+p_2)\delta(\ell_1-\ell_2+p_3)$$

then we can write the normal contribution

$$G_{norm.} = \int dx_1^2 dx_2^2 dx_3^2\, g(x)\int d\ell_1 \frac{e^{i\ell_1(x_2-x_3)}}{\ell_1^2+x_1^2-i\epsilon}\int d\ell_2 \frac{e^{i\ell_2(x_3-x_1)}}{\ell_2^2+x_2^2-i\epsilon}$$
$$\times\int d\ell_3 \frac{e^{i\ell_3(x_1-x_2)}}{\ell_3^2+x_3^2-i\epsilon} \tag{25}$$
$$= \int dx_1^2 dx_2^2 dx_3^2\, g(x_1,x_2,x_3)\Delta_+(x_2-x_3,x_1^2)\Delta_+(x_3-x_1,x_2^2)\Delta_+(x_1-x_2,x_3^2).$$

We recognize that this satisfies the outgoing wave boundary condition, provided that each $\varkappa_i^2 > 0$ to insure oscillatory rather than exponentially damped time behavior. The energy-momentum spectra associated with the various coordinates are characterized by masses m_i which for given $\varkappa_i$'s, obey $m_1 \geq \varkappa_2 + \varkappa_3$ etc. Hence, if m_i denotes the smallest mass in the $\varkappa_i$ spectrum, then the $\varkappa_i$ range upward from the minimum values

$$\varkappa_1 = \tfrac{1}{2}(m_2 + m_3 - m_1), \quad \varkappa_2 = \tfrac{1}{2}(m_3 + m_1 - m_2), \quad \varkappa_3 = \tfrac{1}{2}(m_1 + m_2 - m_3). \tag{26}$$

This discussion assumed that $G(x_1, x_2, x_3)$ depends explicitly on all three of the invariant coordinate differences. Possible terms that are functions of x_1, x_2, x_3 but of only two of the distances, e.g., $G[(x_2-x_3)^2, (x_3-x_1)^2]$, are best dealt with separately. The same procedure as before leads to

$$G(p_1, p_2) = \int ds\, dz_1\, dz_2\, \delta(z_1 + z_2 - 1)\, g(sz_1, sz_2)\, e^{-is(p_1^2 z_1 + p_2^2 z_2)}$$

and the regularity requirement

$$s(p_1^{0^2} z_1 + p_2^{0^2} z_2) > 0$$

or

$$s z_1 > 0, \quad s z_2 > 0.$$

For $s > 0$ we have again the normal domain, $0 < z_i < 1$. The abnormal domain, however, is now excluded, since $z_1 < 0$ and $z_2 < 0$ contradicts $z_1 + z_2 = 1$. Consequently,

$$G(p_1, p_2) = \int_0^\infty dz_1 \int_0^\infty dz_2\, \delta(z_1 + z_2 - 1) \int d\varkappa_1^2\, d\varkappa_2^2\, \frac{g(\varkappa_1, \varkappa_2)}{p_1^2 z_1 + p_2^2 z_2 + \varkappa_1^2 z_1 + \varkappa_2^2 z_2 - i\epsilon}$$

or, in coordinate space, (with an altered meaning for g)

$$G(x_1, x_2, x_3) = \int d\varkappa_1^2\, d\varkappa_2^2\, g(\varkappa_1, \varkappa_2)\, \Delta_+(x_2 - x_3, \varkappa_1^2)\, \Delta_+(x_3 - x_1, \varkappa_2^2).$$

Such a structure certainly satisfies the outgoing wave boundary condition if $\alpha_i^2 > 0$. The minimal values are now

$$\underline{\alpha}_1 = \underline{m}_1 \ , \ \underline{\alpha}_2 = \underline{m}_2 \ , \ \underline{m}_3 = \underline{m}_1 + \underline{m}_2 \ .$$

In the abnormal region we write

$$g(s_1, s_2, s_3) = \int d\alpha_1^2 d\alpha_2^2 d\alpha_3^2 \, e^{i(s_1\alpha_1^2 + s_2\alpha_2^2 + s_3\alpha_3^2)} \, g(\alpha_1, \alpha_2, \alpha_3)$$

and obtain

$$G(p_1, p_2, p_3)_{abn.} = \int \frac{dz_1 dz_2 dz_3 \, \delta(z_1+z_2+z_3-1) \, d\alpha_1^2 d\alpha_2^2 d\alpha_3^2 \, g(\alpha_1, \alpha_2, \alpha_3)}{-p_1^2 z_2 z_3 - p_2^2 z_3 z_1 - p_3^2 z_1 z_2 + \alpha_1^2 z_1 + \alpha_2^2 z_2 + \alpha_3^2 z_3 - i\epsilon} \ ,$$

which is the sum of contributions from the three regions in which two of the z's are negative. On fixing two of the coordinates, which leaves us with a function of a single variable, we find that the boundary condition is explicitly satisfied, provided that

$$\alpha_1^2 z_1 + \alpha_2^2 z_2 + \alpha_3^2 z_3 > 0 .$$

Unlike the normal case, we cannot conclude here, that each α_i^2 is positive. If $z_1 < 0$, $z_2 < 0$, then we learn that $\alpha_3^2 > 0$, $\alpha_3^2 - \alpha_1^2 > 0$, $\alpha_3^2 - \alpha_2^2 > 0$. The general form of the abnormal terms in coordinate space is sufficiently intertwined, e.g. for $z_3 > 0$,

$$\int d\alpha_1^2 d\alpha_2^2 d\alpha_3^2 \, g(\alpha_1, \alpha_2, \alpha_3) \int d k_1 d k_2 d k_3$$

$$\times \frac{e^{i k_1 (x_2 - x_3)} \, e^{i k_2 (x_3 - x_1)} \, e^{i k_3 (x_1 - x_2)}}{\left[k_1^2 - k_3^2 + \alpha_3^2 - \alpha_1^2 - i\epsilon\right]\left[k_2^2 - k_3^2 + \alpha_3^2 - \alpha_2^2 - i\epsilon\right]\left[-k_3^2 + \alpha_3^2 - i\epsilon\right]} \ ,$$

that it is not clear at the moment whether or not the outgoing wave boundary condition is satisfied in detail. Hence it is temporarily undecided if abnormal terms can appear in the Green's function.

It is worth mentioning that multiplicative composition of normal forms cannot produce an abnormal one. That is to say, abnormal terms do not occur in a perturbation expansion.

(Editors' note: the following remark was added by Källén in response to Schwinger's addendum.)

KÄLLÉN: One idea that naturally suggest itself is to try to generalize Eq. (14) by extending the domains of the mass parameters and/or the "Feynman variables". It is, however, clear that any extension involving negative values of the mass squares is not allowed. This is intuitively clear from the following argument. The function H in p-space from Eq. (14) has singularities at the points corresponding to the "thresholds" e.g. $-p_I^2 = (\varkappa_2 + \varkappa_3)^2$. For positive mass squares these singularities are on the cuts. For negative mass squares these thresholds are complex. However, it follows from e.g. Fig. 5 that if our function has a singularity, the position of which depends only on one coordinate, that singularity must be on the cut. Therefore, negative mass squares are not allowed.

LEHMANN: Dispersion relations, general remarks

I wish to make some general remarks on dispersion relations and their connection with microscopic causality. Let us consider the forward amplitude for two particle scattering. So far this is the only quantity for which a dispersion relation may be compared with experiment. As is known, such a dispersion relation follows from a causality condition for the field operators describing the interacting particles. In a model of self-interacting scalar particles described by a field A(x) this condition is

$$[A(x), A(y)] = 0 \qquad \text{if } (x-y) \text{ spacelike.}$$

What I want to point out is that relativistic field theories exist (with finite S-matrix elements) which satisfy the two-particle dispersion relations, even though they do not satisfy the above causality requirement.

To illustrate this, consider a scalar field

$$A(x) = A_{in}(x) - g\int d^4y\, \Delta_R(x-y)\int_0^1 d\lambda\, e^{-i\lambda\eta} : A_{in}^3(\eta) : e^{i\lambda\eta}$$

where $A_{in}(x)$ is a free field,

$$\eta = g \cdot \int d^4z : A_{in}^4(z) : \quad ,$$

and g a coupling constant. This defines a field which is invariant and finite but certainly does not satisfy the causality condition. It yields the unitary scattering matrix

$$S = e^{i\eta}$$

with the above η . (Such a matrix has been studied before; I believe e.g. by Moshinski). The exact forward scattering amplitude is (apart from a factor)

$$T(\omega) = \frac{1}{x}\left(e^{igx} - 1\right) ; \quad x = \sqrt{\frac{\omega - m}{\omega + m}} \quad .$$

It is analytic in the ω -plane with a cut along the real axis, except from $-m < \omega < m$. There are no other singularities and therefore everything is satisfied.

It is possible to construct non-local field theories which satisfy those dispersion relations which can be checked experimentally. I do not know how one could detect the acausalities contained in such a theory.

DISCUSSION

SYMANZIK: Your S matrix does not satisfy crossing symmetry.

LEHMANN: One can also give more complicated models which have this symmetry.

KÄLLÉN: Out of pure ignorance, I have a question here which has to do with the acausal properties of the theory. One way the acausality could possibly show up is in the vacuum fluctuations of the field. Physically you expect that the vacuum expectation value of the square of the field, averaged over a space-time volume, should be positive. Is that the case in this model?

LEHMANN: I don't know, but I don't know how the vacuum fluctuations can be measured.

KÄLLÉN: I was thinking, of course, of electrodynamics where the field is observable. That is the great swindle of field theory, that everything is treated as though the field were really observable.

LEHMANN: I just don't know, but perhaps I wasn't cautious enough. I don't say that I can prove that you cannot detect the acausality of the field.

VALLARTA: I just want to say that Moshinski's paper was published in Revista Mexicana de Fisica about two years ago.

NAMBU: The size of the region in which the field components fail to commute is of the order of magnitude of 1/m, it seems to me.

LEHMANN: I don't really know. In order to find out, you would have to calculate the more complicated matrix elements. However, since the mass m is the only constant with a dimension in this theory, I would guess that it is of that order.

OEHME: Dispersion relations in N-N scattering

I would like to report on some work on dispersion relations for nucleon-nucleon scattering. This work has been done by Goldberger, Nambu and myself. It is still in progress and so I will restrict my talk to a few remarks.

It is well-known that the scattering amplitudes for neutron-proton and proton-proton scattering in the center of mass system can be characterized by five scalar functions of energy and angle, or of energy and momentum transfer. We have obtained the relativistic dispersion relations for these functions in terms of the energy, for fixed restricted values of the momentum transfer.

I would like to talk at first about the relatively simple case of forward scattering. In this case the scattering amplitudes in the center of mass system can be written in terms of three parameters:

$$T = \frac{m}{E}\left\{\alpha + \beta\,\underset{\sim}{\sigma_1}\cdot\underset{\sim}{\sigma_2} + (\epsilon-\beta)\,\underset{\sim}{\sigma_1}\cdot\underset{\sim}{k}\;\underset{\sim}{\sigma_2}\cdot\underset{\sim}{k}\right\},$$

where $\underline{k}$ is a unit vector in the direction of motion of the projectile. I would like to consider α, not as a function of energy in the center of mass sytem, but rather as a function of the variable λ, where

$$\lambda = (2E^2 - m^2)/m \quad .$$

This is the energy in the laboratory system. The only reason is that the dispersion relation is somewhat simpler in this variable.

Now let me write down the dispersion relation for α. I will not write down those for β and ϵ; they are all of the same structure. The dispersion relation for α is the most important one, because α is the complete trace of the scattering amplitude, so that the imaginary part of α is related to the total cross section by the optical theorem. I introduce the notation α_p for the trace of the neutron-proton scattering amplitude, and $\alpha_{\bar{p}}$ for the trace of the neutron-anti-proton amplitude. I'll talk later about the proton-proton amplitude. For $\alpha_{p,\bar{p}}$ we get the following dispersion relation for the forward scattering, assuming for convenience and because of the size of the blackboard, that the total cross section goes to zero as the energy goes to infinity. Then we get

$$\alpha_{\substack{p\\ \bar{p}}} = \frac{1}{2}f^2\frac{1}{\lambda_\mu \pm \lambda} + \Gamma_D\frac{1}{\lambda_D \mp \lambda} + \frac{1}{8\pi^2}\int_m^\infty d\lambda'\sqrt{\lambda'^2-m^2}\left(\frac{\sigma_p}{\lambda' \mp \lambda} + \frac{\sigma_{\bar{p}}}{\lambda' \pm \lambda}\right) + \frac{1}{\pi}\int_{\lambda_{2\mu}}^m d\lambda'\,\frac{\mathrm{Im}\,\alpha_{\bar{p}}(\lambda')}{\lambda' \pm \lambda}$$

where

$$\lambda_\mu = -m + \frac{\mu^2}{2m}\,,\quad \lambda_{2\mu} = -m + \frac{4\mu^2}{2m}\,,\quad \lambda_D = -m + \frac{(2m-B)^2}{2m}\,,$$

and B is the deuteron binding energy.

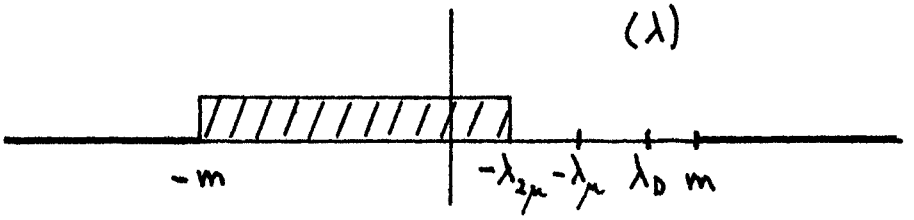
Γ_D can be calculated approximately by nonrelativistic methods which should be sufficiently accurate:

$$\Gamma_D \simeq 3\sqrt{\frac{B}{m}}\;\frac{1 - B/2m}{1 - \sqrt{mB}\,r_{eff}} \quad .$$

The constant f^2 is a well-known constant, named in this conference the Yukawa constant. It is exactly the same constant which appears in the pion-nucleon dispersion relation. Unfortunately, there is also

the last term, which is a continuous contribution from the unphysical region. In the integrand of this term I wrote, to the horror of the experts, $\mathrm{Im}\ \alpha_{\bar{p}}$. I am aware that this function has to be defined in a completely indirect way, and this has been done. But it does have certain physical properties. What one actually has to do in order to apply these dispersion relations is to introduce more powers of λ in the denominators, for instance, by making two subtractions.

Let me now draw the spectrum (or the singularities) of $\alpha_p(\lambda)$ in the complex λ-plane.



There are cuts from m to infinity and from - m to minus infinity. Due to the last integral, there is a cut in the unphysical region, from -m to $-\lambda_{2\mu}$. ($\lambda_{2\mu}$ is a negative quantity.) Furthermore, There are poles at $-\lambda_\mu$ and at λ_D . You get the physical amplitude by taking an improper limit from above to the real axis for $\lambda > m$. If you go to the real axis from above for $\lambda < -m$, you get $\alpha_p(-\lambda^*)$. By use of charge conjugation invariance you can show that this must be equal to $\alpha_{\bar{p}}^*(\lambda)$, which is commonly referred to as the crossing theorem.

One thing one can very easily see from these relations by comparing α_p and $\alpha_{\bar{p}}$. The contribution from the one meson intermediate state to the neutron-proton scattering amplitude is very much enhanced compared to that in the neutron anti-proton scattering amplitude. In the first case, the denominator of the first terms is $\lambda_\mu + \lambda$, and if λ is of the order of m at relatively low energy, then this is of the order of $\mu^2/2m$. In the case of neutron anti-proton scattering, we have $\lambda_\mu - \lambda$ and you get a denominator of the order of 2m. So there is a ratio of about 100. Furthermore, if you look at the integral in the unphysical region you see that in the neutron-proton case it is weighted toward the lower end because of the sum in the denominator, whereas in the neutron anti-proton case it is weighted toward the upper end. This means, in a sense, that in the anti-particle scattering the intermediate states with more mesons are more important compared to states with fewer mesons, whereas it is opposite in the case of particle scattering. However, we know, of course, nothing definite about $\mathrm{Im}\ \alpha_{\bar{p}}(\lambda')$ for $|\lambda'| < m$,

and so, a priori, we cannot say very much.

When you come to the corresponding equation for non-forward scattering, take the nonrelativistic limit, and talk in the language of potentials, you find that the first term corresponds to space exchange potentials of range $1/\mu$, whereas the last term describes the forces of shorter range, shorter than $1/2\mu$. In the case of neutron anti-proton scattering the concept of a potential is much more involved, and I do not want to go into that. From the present dispersion relation you do not get any information about the non-exchange forces, but we know that we can also write down other dispersion relations from which you can get some information about those. This work is still in progress and I cannot report on it.

One would like to apply the forward dispersion relations to the experimental data. This would, of course, have to be done for the subtracted equations, where you have a sufficient number of denominators for convergence. The integral which contains the total cross section of the anti-particle scattering is not so well-known, but it contains the denominator $\lambda' + \lambda$. This is no principle value integral, and it is conceivable that it is relatively small. Notice that it is positive definite.

Let me briefly talk about the proton-proton case. All you have to change is to put the deuteron contribution Γ_D to zero, replace the 1/2 in the first term by 1/4, and put into the first integral the corresponding proton-proton, proton-anti-proton cross sections, and you get the proton-proton dispersion relations. These will probably be most effectively applied to experiments. In the subtracted equations the last integral has three powers of λ' in the denominator, and the only thing I can say is that it may perhaps be worthwhile to put some numbers in and see what its order of magnitude is. Then you may get a feeling whether one could make sense by approximating it in one way or another, or even by neglecting it. I would also like to say that if you take a specific model of a field theory, you may be able to get some information about the imaginary part of α in the unphysical region, and you may get some kind of a theory of potentials which is perhaps better than the Tamm-Dancoff approximation.

Finally, I would like to make a remark about non-forward dispersion relations. As I said, we have dispersion relations also for the other coefficients which appear only in the non-forward case.

One quite interesting example is the case of a term of the form

$$\gamma(\underset{\sim}{\sigma}_1 + \underset{\sim}{\sigma}_2)\cdot\underset{\sim}{n} ,$$

where $\underset{\sim}{n}$ is the unit vector normal to the plane of scattering. For this γ we get, in the non-relativistic limit, a fairly simple dispersion relation, in which the one meson term is very small, down by the factor of $(1/m^2)$ compared to the corresponding term in other dispersion relations. Now γ is very important for the polarization of an unpolarized beam scattered by an unpolarized target. This polarization is something like

$$P \propto Re\left[(\alpha+\beta)\gamma^*\right]$$

and the smallness of the one-pion contribution may have some implications concerning the short range ($\lesssim 1/2\mu$) of forces responsible for polarization, but we have no details on that.

DISCUSSION

GOLDBERGER: I would like to take advantage of my position as chairman and also as an interested co-worker in this, to make one or two remarks myself.

From the standpoint of interpreting this in terms of potential, you can be bold or not bold, as you like. If you take a bold approach, you might be tempted to say something like this. The dispersion relation contains the amplitude for neutron-proton scattering. Let's put the one for anti-protons to sleep. There is the total cross section for neutron-proton scattering, then the extraneous thing - the cross section for neutron - anti-proton scattering, and this contribution from a rather horrible unphysical region. There is a strong analogy and a very strong temptation from the general structure of this equation and the corresponding relations which can be deduced in non-relativistic quantum mechanics, to lump the integrals together, and if you would push me hard I would even take out the inelastic processes and call everything except the first two terms the Fourier component of the potential corresponding to forward scattering. That is, it is the matrix element $\langle \underset{\sim}{k} | V | \underset{\sim}{k} \rangle$.

Now this can't be quite complete at this stage, because of a rather artifical way in which we have written this formula. The first term, for example, is the one meson charge exchange contribution. There is certainly a one meson non-charge exchange contribution,

which we can exhibit explicitly but have not done as yet. I think it is quite feasible to make a fairly accurate estimate of the integral, as Dr. Oehme mentioned, by committing yourself to what theory you're talking about. We will talk about conventional pseudoscalar theory, wherein the mesons which are scattered and responsible for the forces are pi-mesons. Using suggestions made by Dr. Chew, it seems quite possible to estimate the two-meson intermediate state contribution to the integral, and I think this should provide a numerically very sensible calculation of what you might term the two-nucleon potential.

I am fully aware of the fact that there is a great deal of arbitrariness in what you decide to call a potential. Also, needless to say, one has to discuss the full non-forward scattering dispersion relations in order to characterize the problem completely. These are things that are in progress, and certainly by next year we will know the answer.

RUDERMAN: Do you have a dispersion relation for scattering by a non-relativistic potential?

GOLDBERGER: Yes. This problem has recently been discusses in great detail in a very careful and rigorous fashion by Mr. Khuri at Princeton University. Using the Fredholm technique of Jost and Pais (Phys. Rev. 82, 840, 1951), he has been able to show that if the potential decreases as rapidly as an exponential, then you may deduce dispersion relations for the scattering amplitude regarded as a function of the wave number for a fixed value of the momentum transfer, which value of the momentum transfer must not exceed the exponential fall-off associated with this potential. If the potential falls off faster than any exponential, say, like a Gaussian, then you get a dispersion relation for arbitrary fixed momentum transfer.

WENTZEL: In the proton-proton case, what role does the Pauli principle play?

GOLDBERGER: An algebraic role. This is a problem which disturbed us for a long time. It was a gigantic red herring; it is simply trivial, a technical matter of substituting the correct thing into the dispersion relation. There is no fundamental difficulty in the identical particle problem.

KLEIN: Dispersion relation for fixed source theories and theory of the nucleon-nucleon interaction

I would like to take ten minutes to discuss two different contributions. The first piece of work that I will describe was done in collaboration with Mr. Richard E. Norton.

At the last Rochester Conference a great measure of optimism was expressed concerning the future role of dispersion relations in field theory. By this I mean their role as a basis for complete calculations, not their role in experimental physics. The latter part of their role has been well established. In particular it was suggested that they might serve as a complete scheme for calculation. In order to test this idea on a tractable yet not completely trivial model, we have studied the dispersion theoretic formulation of a fixed source theory with the Hamiltonian:

$$H = H_0 + H_1 \quad ,$$

where H_0 describes the free meson Hamiltonian for neutral scalar mesons, and the interaction H_1 has the following structure:

$$H_1 = \sum_{\mu=1}^{N} \lambda_\mu \left\{ \int d\underset{\sim}{r}\, \rho(|\underset{\sim}{r}|)\, \varphi(\underset{\sim}{r}) \right\}^\mu .$$

In each one of these terms the meson field operator $\varphi(\underset{\sim}{r})$ is averaged over the spherically symmetrical source distribution which represents the fixed source nucleon, and therefore this is a theory which contains only S-waves. This is a great simplification. However, it describes vertices in which the fundamental interactions are n-fold in nature, and we can have the simultaneous emission or absorption of up to N mesons at a time. In particular, for $N > 2$, this theory contains multiple meson production to all orders.

First of all, for a general fixed source Hamiltonian, not restricted to the particular model shown above, we have extended the Low-Wick formulation of scattering for fixed source theories and obtained a formula for S matrix elements for the most general process in any such theories, that is, any number of mesons incident and any number of mesons emerging from the source. In order to have something to which the dispersion theoretic formulation can be applied, one has to subtract from the general S matrix element all contributions which describe the product of two or more independent processes, that is, contributions in which one can have two or more

separately energy conserving processes taking place. One then has a suitable object to study:

$$\left(\omega(p_1)\cdots\omega(p_n)\,|\,G\,|\,\omega(k_1)\cdots\omega(k_m)\right)$$

$$= \frac{2^{m+n}\,\omega(k_1)\cdots\omega(p_n)}{\rho(k_1)\cdots\rho(p_n)}\left(n\,|\,T\,|\,m\right).$$

The resulting quantity, from which all products of independent processes have been removed, is called $(n|T|m)$ for m incident and n emerging mesons. After removal of a convenient factor which contains the product of the energies of each of the mesons and the product of the Fourier transforms of the source functions, one ends up with the quantity which we call G, which is a function of all the $m+n$ energies.

We study G as a function of the total energy E of the system, which is the sum of all the energies of the incident or emerging mesons and for fixed values of the ratios

$$\mu_i = \frac{\omega(p_i)}{E}, \quad \nu_i = \frac{\omega(k_i)}{E},$$

a distribution of variables which was suggested by some work of Polkinghorne. We then find that as a function of E for fixed μ_i and ν_i, the matrix G can be continued suitably into the complex plane and has suitable symmetry properties there, so that one can ultimately write down an infinite number of dispersion relations, one for every such matrix element. The form of the dispersion relations is

$$\mathrm{Re}\,G_{nm}(E) = \delta(\lambda_{nm},0)\,\mathrm{Re}\,G_{nm}(E_0)$$

$$+\frac{(E-E_0)^{1-2\lambda_{nm}}}{\pi}P\int_{\mu}^{\infty}dE'\left[\frac{(E'+E)(E'-E_0)^{2\lambda_{nm}-1}}{E'^2-E^2}-\frac{(E'-E)(E'+E_0)^{2\lambda_{nm}-1}}{E'^2-E^2}\right]\mathrm{Im}\,G_{nm}(E'),$$

where λ_{nm} is the non-negative integer determined by

$$(2\lambda_{nm}-1)N \leq n+m \leq (2\lambda_{nm}+1)N.$$

One sees that this involves an integral from the meson rest energy, which is the beginning of the continuum in this theory, to infinity. That means that for processes other than elastic scattering one has a contribution from a non-physical region, that is, from μc^2 up to the threshold for that particular process. Now this is, of course, a difficulty if one wants to compare such things with experiments, but it is no objection from our present point of view, which is to consider these relations as integral equations of the field theory when amplified by a suitable unitarity condition. Such a condition can be derived directly from the same general formulation which gives rise to these relations. Then we can consider the set of coupled integral equations defined by these relations plus the unitarity condition. (They are much too complicated to exhibit.)

The conclusions are the following: first of all, one may ask how many of those arbitrary constants $Gnm(E_o)$ are there? It turns out that there are just N - 1, which is one fewer than the number of terms in the Hamiltonian. In general, one might have expected N. The special reason for N - 1 in this case may be that the linear term in the Hamiltonian is the trivial neutral scalar term, which has no dynamical consequences.

Next, one wants to examine the existence and the nature of a solution to these equations. Now the first point to be made in this connection is the following: if there exists a power series solution which converges for some value of the coupling constant and converges uniformly with respect to the energy (the uniformity of convergence being required by the fact that we have to carry out an integration), then that solution is determined uniquely by this set of equations. However, when one asks whether these relations permit a unique solution aside from the power series solution, then one can easily show that this is not the case. There are solutions which are not uniquely determined by the N - 1 arbitrary parameters. This can be proved directly using the mathematics of Castillejo, Dalitz and Dyson (Phys. Rev. 101, 453, 1956). However, one can see it more physically by writing down explicitly an infinite number of Hamiltonians all of which give rise to the same dispersion relations. Rather than writing down any of these Hamiltonians, I will describe the characteristics of the prototype of such a Hamiltonian.

In the theory we have discussed, we have assumed that the source has only one state, that the expression consists of the ground state of the source plus the continuum of μc^2 and upward. One writes down a Hamiltonian in which the undisturbed part, before one has turned on the coupling to the meson field, permits the source to have one or more excited states which lie above the continuum, and such that when one turns on the coupling, these are not depressed below the continuum and give rise to actual discreet states, that is, such that these states then become, in the common terminology, decaying states and in the mathematical sense do not form a part of the actual spectrum of the full system. This is an example, and there are an infinite number of them, which gives rise to the same dispersion relations, but where the solution as a function of energy is not uniformly convergent. The Hamiltonian then contains more information than the dispersion relations, in that there are additional parameters associated with these extra energy levels which make no appearance whatsoever in the dispersion relations.

The following work was done in collaboration with Bruce McCormick and independently by Miyazawa.

The basic idea is the following: we stick to the idea of trying to construct a static potential of a finite range of validity from the meson theory, and we wish to obtain a formulation which represents, in a mathematical form, the maximum information from the observed pion-nucleon interactions. Now it turns out that one can obtain an infinite expansion of a potential, $\sum_{N=1}^{\infty} V_{2N}$, in which the N^{th} order term describes an exchange of N mesons. However, in this definition we insist that the given order potential, say the fourth order potential, is defined only by the exchange of mesons and that all virtual interactions of each of the mesons be included, in principle, exactly. Formally, V_{2N} can be exhibited in principle exactly, within the approimations made. For example, the second order potential is given to a very good approximation in the adiabatic limit, by the usual second order potential, the only replacement being that one has put in the renormalized coupling constant, the f^2 of which we heard so much yesterday. One then turns to the fourth order potential. It turns out that this can be exhibited completely in terms of f^2 plus certain integrals over the pion-nucleon cross section in given channels. Now I should also add that when one finally tries to do something with this formalism, one limits oneself to S and P-waves. The sixth order potential could be written in terms of f^2 and the pion-nucleon cross sections, if one knew the cross section for single meson production, and so on.

What we've done up to date is as follows. It can be shown that the S-wave contributions, which have been evaluated fully by Miyasawa and partially by us some time ago, are quite small. Let us limit ourselves to P-wave interactions, basically the cut-off formalism of Chew and Low. We evaluate the second and fourth order potentials arising from P-wave interactions. One other small point, if one makes an estimate of the radius of convergence of the series for V (basically the same estimate that Chew made yesterday except for a factor of 2) then one finds:

$$\left| \frac{V_{2N}(r)}{V_{2(N-1)}(r)} \right| \sim \frac{f^2}{4\pi} \frac{2}{x^2} e^{-x} < 1.$$

Hence $x > 1/3$ (or 0.5×10^{-13} cm),

where $x = \mu r$ and $f^2/4\pi = 0.080$.

Now the preliminary results are shown in Figures 7, 8, and 9. What I plotted is the potential in units of the meson rest energy as a function of $X = \mu r$ over the limited range $0.5 \times 10^{-13}\,\text{cm} < x < 1.5 \times 10^{-13}\,\text{cm}$ on a semi-logarithmic scale. The emphasis here is on the nature of the corrections which we have found compared to previous calculations. Similar results apparently have been found in Japan under Miyazawa. The negative of the

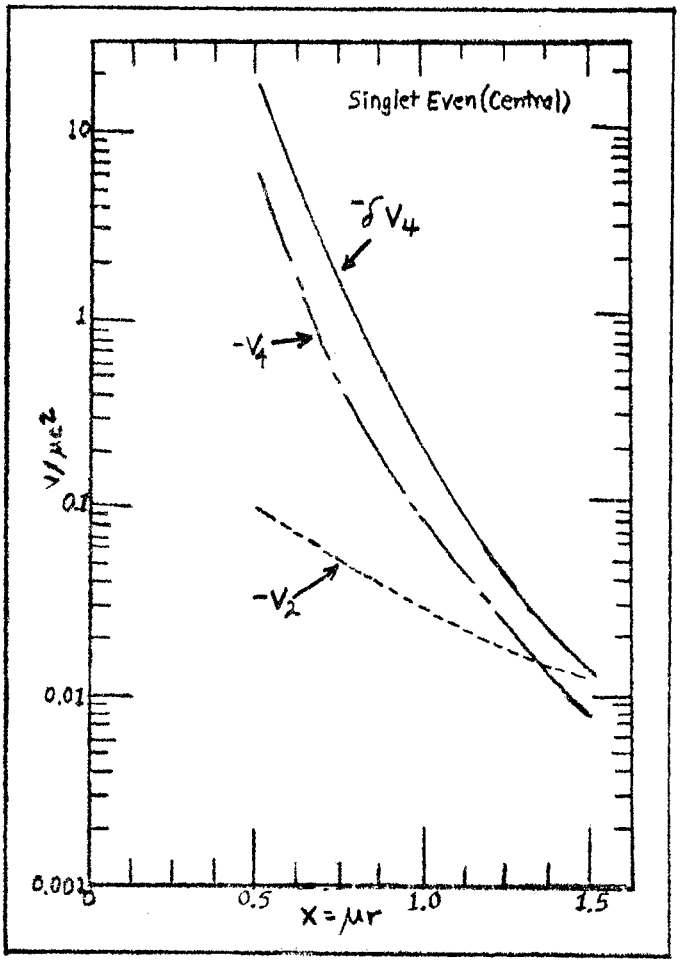


Fig. 7

conventional second order potential is plotted. The Brueckner-Watson contribution to the fourth order potential is shown as V_4, and then what I call δV_4 is the correction we found in each case, arising from the observed pion-nucleon cross section compared to Born's approximation. That is, V_4 is basically the potential which one would obtain if one said that pion-nucleon scattering was well described by the Born approximation. We know that this is not the case. We know that the actual scattering, say, in the three-three state, is greater than the Born approximation by a factor of two or more.

The correction as computed so far may be somewhat overestimated, but not by very much. They are quite substantial. One has qualitatively the same situation in all the even states.

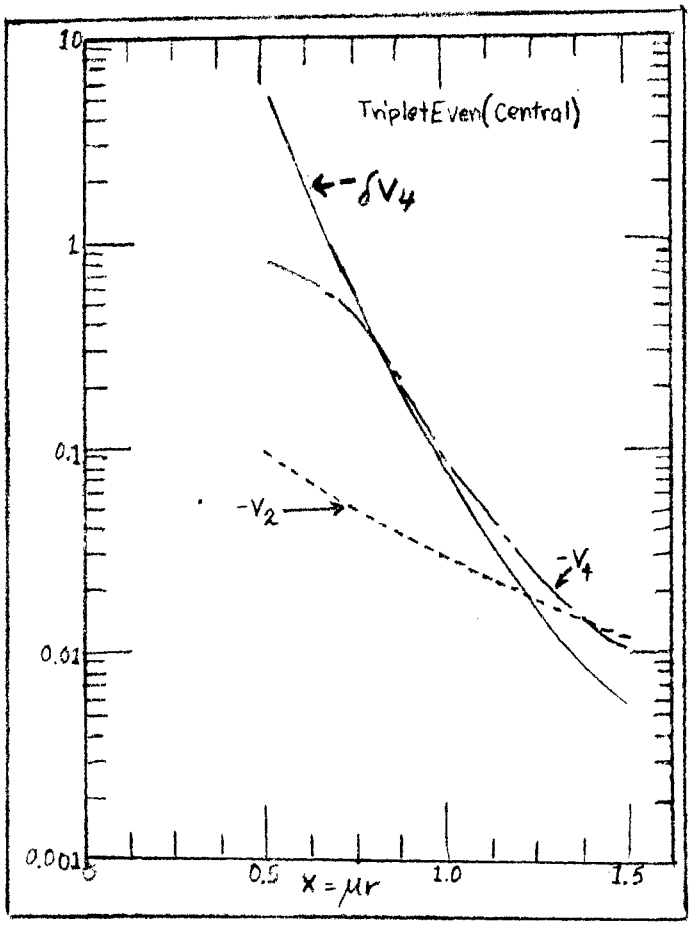


Fig. 8

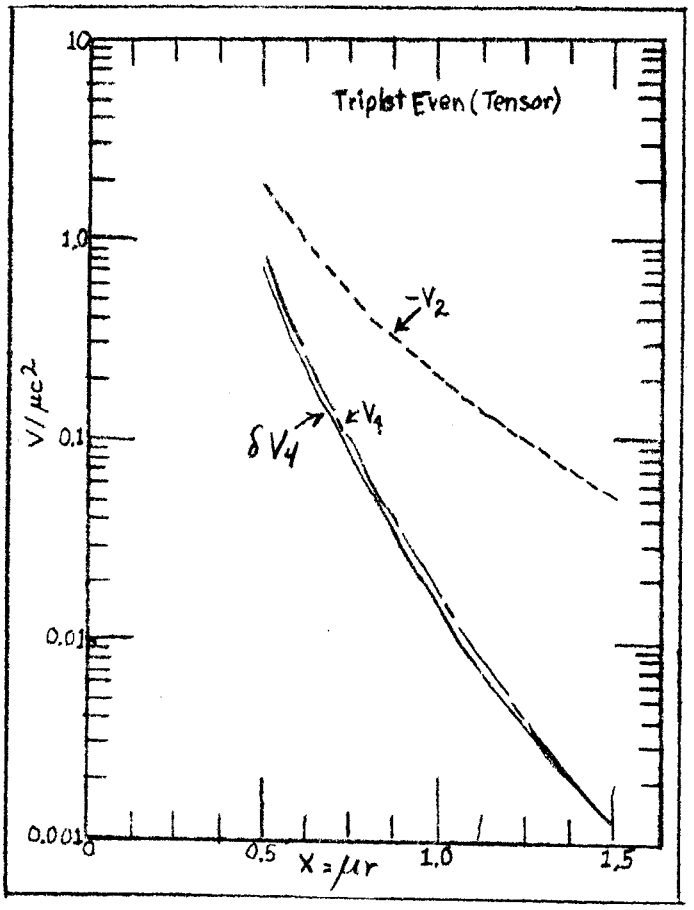


Fig. 9

As remarked on several occasions, even with the corrections, the second order tensor force is still the predominant tensor force over most of the range of interest. Nevertheless, the corrections are substantial compared to the fourth order of the perturbations results.

We still have, I think, considerable work to do to make sure that the evaluation of the corrections is good. I think that we have a good analytical method, but the question which remains, in integrating over the pion-nucleon cross sections, is: how far shall we go? I went all the way up past the second resonance. Probably one should stop after the first resonance, and we're going to do this.

Finally, I have a number. Mr. Younger is examining the comparison with experiments at low energy. He fitted the singlet even central force to the scattering length and found an effective range in the singlet state which is 2.77×10^{-13}cm., which, if it is correct, I think is a really promising start.

DISCUSSION

RUDERMAN: Gasiorowicz and I, in collaboration, found that the dispersion relations for the scattering of a particle by a square well potential also do not possess unique solutions.

VAN KAMPEN: The Hamiltonian is clearly not uniquely defined by the dispersion relations. It is not even defined by the S-matrix itself.

RUDERMAN: But if you restrict yourself to square well potentials, or potentials of a finite range, then the S-matrix does define the Hamiltonian uniquely. The dispersion relations, however, do not even have unique solutions if you restrict yourself to square well potentials.

GOLDBERGER: I think it is probably correct to say that the method for calculating potentials which Klein has just described is rather closely related to the method which I suggested in connection with the relativistic dispersion relations, except for the fact that I believe, in the relativistic dispersion relations, one has a slight advantage. One always deals with quantities on the mass shell in contrast to the sort of computations which are involved in Klein's approach. The actual evaluation, though, may require a very similar function.

BREIT: In connection with the nucleon-nucleon problem, it seems to me that the power of the dispersion relation approach is limited on account of the computations pointed out by the chairman. For that reason, it seems to me that we ought to consider a slightly different viewpoint, namely that of classifying states of the same total angular momentum and parity, and isolating them from the rest of the system. For these states one can then consider the nuclear reaction matrix. There is always the possibility that at very high energies there are other particles created which, in some way, participate in the formal aspects of the dispersion theory speculations. Perhaps one might make progress with the nuclear reaction matrix approach. Then one deals with rather simple things such as, let us say, the logarithmic derivative of a certain function as a function of energy. (That is not a quite rigorous thing, but for potentials it is rigorous). In this method one would, therefore, bring in the production of particles in the scattering matrix. Such a viewpoint can be regarded as almost as general as the dispersion relations.

GOLDBERGER: I think that the difficulty that we face in your suggestion is simply one of technical ignorance. The discussion of the dispersion approach in terms of the reaction matrix as opposed to the S-matrix is something which does not seem to be convenient in quantum field theory, as it is in nuclear reaction theory. If you have billions of channels effectively, which is the case here, then the problem of going from the reaction matrix to the S-matrix involves a matrix inversion problem of awesome proportions. I, at least, don't know how to do it.

BREIT: Suppose you are interested in nucleon-nucleon scattering in the energy range up to 400 Mev. If you take, say, angular momentum 2, then, in that range, pions have only a limited importance. The higher energy scattering, as in nuclear reaction theories, could be represented by a background term.

GOLDBERGER: I think it is simply a contrast in disciplines. I don't know that approach, and consequently I think along these lines. There is no reason to believe, however, that the two are necessarily exclusive.

POLKINGHORNE: Dispersion relations for many meson processes (Some of this material appears in Nuovo Cimento 4, 216)

The causality condition is mainly applied to scattering where you have a single particle and scatterer in the initial state and a single particle and scatterer in the final state. I want to discuss a formalism which extends this to an arbitrary situation where, say, in the case of pions and nucleons, you have M pions in the initial state and N pions in the final state. As far as the experiment goes, the case $M = 1$ is the only one that is really of interest with present techniques. But if you want to use things like the unitarity condition, you have to have matrix elements of this more general type appearing. So I want to construct a formalism which will produce these dispersion relations, exploiting the causality condition.

The work on dispersion relations really stems from a rather remarkable observation first made by Gell-Mann, Goldberger, and Thirring. You have the scattering amplitudes expressed essentially as matrix elements of the time ordered product of two current operators. This is a rather complicated thing to think about. But it turns out that for positive energy the Fourier transform of this is the same as the matrix element of a much simpler thing, the retarded commutator. Now going from one to the other:

$$\langle | T(j(x)j(0)) | \rangle \rightarrow \langle | \theta(x)[j(x), j(0)] | \rangle$$

is called the comparison theorem. It's useful because the retarded commutator is a very simple thing to think about, essentially a classical object, and it satisfies a causality condition in a very simple classical way. It vanishes unless x lies in the forward light cone of the origin. So you can get a dispersion relation in a simple way.

There are, then, two steps: the comparison theorem and the causality condition. Now I want to construct something which looks like a classical causality condition ("the bell not ringing until the button is pressed" sort of thing, generalized to more complicated cases).

Consider first of all the case $M = 1$ and N having general values, or more particularly the case where $N = 2$. This was actually written down by Källén earlier. The object you have is the time ordered product

$$T\left(j(x_1)\, j(x_2)\, j(0)\right).$$

Now you can prove that this

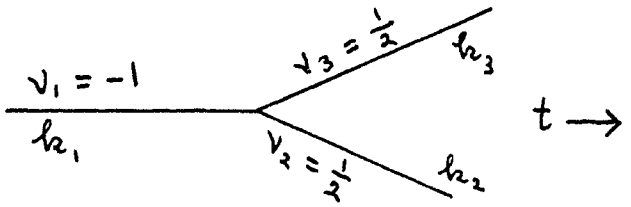
$$\theta(x_2-x_1)\,\theta(x_1)\left[j(x_1),\left[j(x_2),\, j(0)\right]\right]$$

plus a term with x_1 and x_2 interchanged, has the same Fourier transform as the time ordered product. It's quite simple; you prove it in the usual way by bringing in sums over intermediate states. It also is very simple to see that it satisfies a simple causality condition since it vanishes unless x_1 and x_2 are both in the forward light cone. In this way you get a dispersion relation.

Another thing I think one learns from the single meson case is that one has to be a little careful about one's description of the kinematics. For example, you have to fix the momentum transfer and not the scattering angle, although it would be much more convenient to fix the scattering angle. I just want to say something about how one fixes the kinematics for this general case. It is a rather curious description, but it seems pretty much forced on one. First of all one has to decide how to describe the nucleons and, of course, one chooses the frame of reference in which both the incoming and the outgoing nucleons have equal energy. Now let us consider the number of mesons and their energy momentum vectors k_i. Of course, they satisfy $k_i^2 = -\mu^2$. Let us write the fourth components of these vectors $k_i^4 = \omega \nu_i$. This is what Klein used in the particular case he solved. The ν_i are going to be fixed; they fix the ratios of the energies. ω , of course, is the variable which one is going to integrate over in the dispersion relation. Now $\sum \nu_i = 0$, since the energy transfer from nucleons to mesons is zero. The ν 's are still arbitrary to the extent of an arbitrary factor. We'll fix the sign of the factor by requiring that the outgoing mesons should have positive ν 's, the incoming mesons, negative ν 's. And we'll fix, if you like, the scale factor which remains by requiring $\sum_+ \nu_i = 1$. That fixes the energies. The reason for using these variables is that it will produce exponents which have a pretty straightforward dependence on ω . Another important property is that when you change the sign of ω you get a crossing theorem more readily because everything changes sign.

The thing that remains for us to do in fixing the kinematics is to fix the orientation of the spatial part. (The magnitude is determined since we are on the energy shell.) The way one does this is

rather curious. For given fixed ν 's you can construct certain vectors, δ^α , which have the property that as the ν 's are fixed and ω varies, they always have zero fourth component. A simple example of this, of course, is the momentum transfer of the nucleon to the meson system. Another example would be:



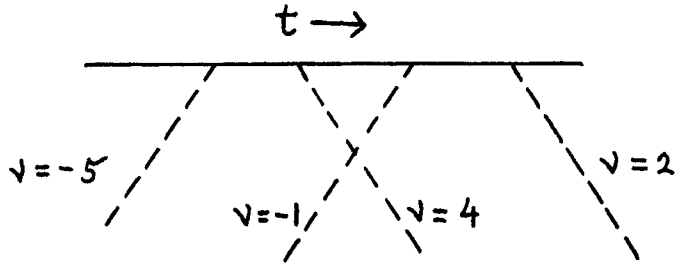
You have one incident meson with $\nu=-1$ and two outgoing with $\nu = 1/2$. The vector $(1/2)\, k_1 + k_2$ always has zero fourth component for all values of ω . Now you construct a set of the δ^α in spin and space, and you fix the magnitudes of as many of them as are necessary to complete the specifications of the kinematics. There aren't as many as I said in my paper. I forgot about the dimensionality of space. There are 2N - 4 that you have to fix. It then turns out in a very straightforward way that those remaining vectors, δ^α , which are not fixed in magnitude, are bounded in magnitude as ω varies. This is very important when you are trying to write dispersion relations because I shall do it simply by changing orders of integration. For a particular experiment, one has an expression with some direct dependence on ω . For example, one has

$$\exp\left\{ i \left[(\cdots)\,\omega + \text{Junk}(\delta) \right] \right\} .$$

The coefficient of ω turns out to be positive; that's what we want. Then there is a lot of junk left over which depends upon the deltas only. If these are bounded, then the behavior of the expression on the sem-circle at infinity in the upper half plane is just what we want. That's really the reason for using the deltas to complete the specifications of the kinematics.

The final thing I would like to say is a little bit about the causality condition in the case where $M \neq 1$. Suppose you have two incident particles and two outgoing particles. Quite clearly, a possible condition would be that no particles should be emitted until both particles have been absorbed. That would be a really rather surprisingly strong condition and an unreasonable condition, because the nucleon could absorb one meson and if it had sufficient energy it could then produce one of the outgoing mesons before the second one was absorbed. So if we're going to be just simple, naive and classical, which seems to be the right thing to do, we would expect the

causality condition to depend on the energies involved. That means, of course, that it is going to depend upon the ν 's. That's another reason for having them in the first place. So let us just consider a case of two incident mesons and two outgoing mesons:



If - 5 is absorbed first, then it's clearly possible for, say, 4 to be emitted. But having emitted 4 there isn't enough energy to emit 2. So 2 can't be emitted until -1 has been absorbed. So you see in this sort of diagram one forms correlations between the outgoing particles and the incoming particles. Here one correlates 4 with -5; everyone would say that the meson 4 is created entirely by the meson -5. Meson 2 is correlated both with -5 and with -1, because both of these, in this particular ordering, have to play a part in creating 2. For given values of the ν 's you have a great variety of possible correlations that can be formed in this way, and to each correlation there corresponds a natural ordering and so a natural causality condition. The causality condition for the more general case will now become simply this: given the ν 's, you look at the expression and you write down all the correlations energetically possible. Then, if for none of these correlations the space-time points involved have the correct ordering, that is, lie in the correct light cone, then the thing will have to vanish. This is the causality condition, and it gives a dispersion relation.

I want to make two observations in closing. I said that these dispersion relations were proved to the extent of exchanging integrations. Mr. Kibble at Edinburgh has been investigating the case of M = 1 and N = 2 by the same sort of technique that Bogoliubov used in his proof for the single meson case. It gets very involved, as you might imagine. The sort of thing that turns up looks rather like what Källén showed. One has to use similar techniques of analytic continuation, and functions of many complex variables. Kibble made quite remarkable progress. And though it isn't completed, I think there's quite a good chance he will succeed in constructing a proof, or at least in exhibiting what theorems mathematicians should prove in order to make these dispersion relations true.

The second point I would like to make is: what use are these? Well, that is always an embarrassing question with a complicated formalism. So far the only use I know of was made by Corinaldesi at Glasgow, who has used them to analyze the τ decay. We have two parameters in the τ decay, one of which looks like an energy, and you can write down a dispersion relation. Unfortunately the experimental data don't seem to be sufficient to draw a very definite conclusion, but there doesn't seem to be any inconsistency.

EKSTEIN: Representationless formalism in field theory

I would like to mention that this work was done in cooperation with Swihart and Tanaka. It is an attempt to do field theory by choosing as objects of the theory matrix elements of operators between eigenstates of the Hamiltonian. The idea, of course, is to avoid the traditional approach, where one tries to expand some physical state in eigenfunctions of the unperturbed Hamiltonian. It has become clear in recent years that this is not really a meaningful expansion. For instance, in the simplest case of a static point nucleon, it can be shown that all the physical states are orthogonal to the unperturbed states. So this doesn't work except if one introduces an infinite wave function normalization constant and this is an unnecessary complication. So, by considering matrix elements between physical states of some operators, one can hope to avoid at least some of the problems.

Contrary to the covariant formulations about which Schwinger and Källén have talked, we have considered Schrödinger operators. Also, as far as the physical states are concerned, we have not considered vacuum states as is now mostly done in the covariant calculations, but rather eigenstates of the Hamiltonian which represent single particles, for instance of momenta p and p'. One loses, of course, covariance in this manner, but one gains in tractability of the equations because the fourth variable is definitely useless in these calculations. One conveniently chooses two physical states which have different momenta but equal energies. This has the advantage that the self-energy of the state does not occur in the equation and one does not have to renormalize, to subtract an infinite self-energy. By a Lorentz transformation one can then obtain matrix elements between two general states, p and p'.

The cases which so far have been considered by this method are the following: a static nucleon without cut-off interacting with scalar charge-symmetric mesons; then a moving nonrelativistic nucleon with the same interaction; then the static case where the inter-

action is pseudo-vector with pseudo-scalar mesons (here, of course, we need a cut-off); and finally, this is now in progress, the moving but nonrelativistic case with pseudo-vector coupling and without cut-off.

One starts with the Heisenberg equation of motion and one takes matrix elements between eigenstates of the Hamiltonian. Since we have chosen eigenstates with the same energy, the time derivative term vanishes and we have the expectation value of the commutator between two states of the same energy:

$$\left(\Psi, [A, H]\,\Psi'\right) = 0,$$

where A is a product of Schrödinger creation and destruction operators. By putting in the Hamiltonian explicitly, one obtains a linear equation which then has to be renormalized. That is, one has to take out unobserved quantities. In particular, the Hamiltonian contains the unrenormalized coupling constant f_0 which one wants to replace explicitly by something observable. This can be done arbitrarily by conveniently defining what one means by the observable coupling constant in terms of physical matrix elements. So one obtains an explicit expression for f_0 in terms of the renormalized coupling constant and physical matrix elements.

In the case of the moving nonrelativistic nucleon there is another unobservable quantity which must be eliminated: the mass renormalization, Δm, for which one must also give an explicit expression in terms of observable finite quantities. Then all the equations are written in terms of only observable quantities and can presumably be treated by any number of approximation methods. They can, for instance, be iterated and one can obtain an expansion in terms of the renormalized coupling constant. But one is now constrained to do only perturbation theory.

Of course, the information containing such expectation values does not give the complete answer. This is the price that one has to pay for not using Heisenberg operators but Schrödinger operators. The information is sufficient for such quantities as, for instance, a magnetic moment, but it is not sufficient for scattering. In order to obtain information for scattering one needs another equation in which these quantities enter. This equation is of the following type: if the S-matrix is written in the usual manner,

$$S = 1 - 2\pi i \, \delta(\ldots) R$$

then the equation for the R-matrix is:

$$R = (\text{inhomog.}) + \int \frac{RR}{(\text{resonance den.})} \, d\mathcal{E}' .$$

This equation looks somewhat similar to the Chew-Low equations, but it is not the same because the inhomogeneity is not just a simple observable number or something like that, but it is a rather complicated quantity which must be obtained by solving the equations for these elements. It is a two-stage process which is very reminiscent of the manner in which one approaches, for instance, nonrelativistic scattering problems of composite particles. Proceeding in this manner, one obtains, at least so far, finite quantities. We have, in particular, considered the perturbation expansion in terms of f which seem to be, so far, in good agreement with anything that one might expect.

DISCUSSION

Källén: Could the speaker say in a few words what he feels is the advantage of the Schrödinger representation compared to the covariant Heisenberg representation?

EKSTEIN: I believe that the four variables are a real headache in solving equations. It is certainly easier to work with three variables.

LUEDERS: Invariance under TCP

I want to talk about some considerations by Zumino and myself on invariance under the product TCP, where T stands for time reversal, C for charge conjugation, and P not for Pauli, but for parity or reflections in space. We put

$$\textcircled{H} = \text{TCP}$$

We assume the theorem on invariance under $\textcircled{H}$ to be valid and study which relations between properties of particles and anti-particles follow from it.

Application to decay properties are non-trivial in some sense, since Θ reverses the time order of a process and so transforms a decay process for particles into a build-up process for the corresponding anti-particles. We split the total Hamiltonian, H, in the form

$$H = H_S + H_W \quad ,$$

where H_S contains both strong (in the usual sense) and electromagnetic interactions, and where H_W represents the weak interactions.

First we assume that H_S has been taken into account correctly and treat the decay to first order in H_W. As initial state we have an eigenstate $|0\rangle$ of H_S; as final states we have to use particular scattering eigenstates $\langle -a|$ of H_S; it is important to pick the correct "-" states. Transition probabilities are given essentially in terms of the matrix element $\langle -a | H_W | 0 \rangle$. To be more specific, let us assume that the scattering states have been normalized with respect to the continuous and discreet variables "a" so that

$$\langle -a | b - \rangle = \delta(a-b) .$$

As new variables we then introduce the total energy E_a, total momentum $\underset{\sim}{P}_a$, and further variables $\tilde{a}$. We define a function g(a) by

$$da = g(a)\, dE_a\, d\underset{\sim}{P}_a\, d\tilde{a}$$

and a reduced interaction matrix element by

$$\langle -a | H_W | 0 \rangle = [g(a)]^{-\frac{1}{2}} \delta(\underset{\sim}{P}_a) \langle -\tilde{a} | H_W | \tilde{0} \rangle .$$

The differential transition probability per unit time is then given by the Golden Rule

$$dw = \frac{2\pi}{\hbar} \left| \langle -\tilde{a} | H_W | \tilde{0} \rangle \right|^2 d\tilde{a} .$$

Now we assume that Θ and H_W commute. But when applying Θ to $|a-\rangle$ we obtain a "+" scattering state for the anti-particles which cannot be used directly to calculate transition probabilities. Since, however, both types of scattering eigenstates are related to each other

through the S-matrix of the strong interaction, one has

$$\langle -\Theta\tilde{a}\,|\,H_w\,|\,\Theta\tilde{O}\rangle^* = \int d\tilde{b}\,\langle\tilde{a}\,|\,S\,|\,\tilde{b}\rangle\langle -\tilde{b}\,|\,H_w\,|\,\tilde{O}\rangle,$$

Here the reduced S-matrix on the energy momentum shell is defined by

$$\langle +a\,|\,b-\rangle = \left[q(a)\,q(b)\right]^{-\frac{1}{2}}\delta(E_a - E_b)\,\delta(\underset{\sim}{P}_a - \underset{\sim}{P}_b)\,\langle\tilde{a}\,|\,S\,|\,\tilde{b}\rangle$$

We can draw two conclusions: 1) To obtain the reciprocal lifetime we have to integrate in the Golden Rule over all variables $\tilde{a}$. From the unitarity of $\langle\tilde{a}\,|\,S\,|\,\tilde{b}\rangle$ it then follows that lifetimes of particles and anti-particles are equal, as already shown by Lee, Oehme, and Yang. The same is true for the lowest order correction to the mass of the decaying particle. 2) Branching ratios into corresponding channels are the same if the strong interaction does not lead to a scattering between the various decay channels. This is especially true if there is no strong interaction.

The equality of masses and lifetimes of particles and anti-particles can be shown in a very simple way without any explicit use of perturbation expansions. Mass and lifetimes can (for simple exponential decay) be read from the analytic behavior of the resolvent, i.e. the following matrix element

$$\langle 0\,|\,(\lambda - H)^{-1}\,|\,0\rangle\ .$$

Here $|0\rangle$ is the eigenstate of H_s giving the decaying particle at rest; H is the total Hamiltonian and λ a complex parameter. For $H_w = 0$ the above matrix element shows a pole at the rest energy of the stable particle; for $H_w \neq 0$ the pole moves into a region which can be reached by analytic continuation. The position of the pole gives the rest mass and reciprocal lifetime of the decaying particle. From the commutativity of H and Θ, the hermiticity of H, and the anti-linearity of Θ, it follows that

$$\langle 0\,|\,(\lambda - H)^{-1}\,|\,0\rangle = \langle\Theta\,0\,|\,(\lambda - H)^{-1}\,|\,\Theta\,0\rangle.$$

Consequently, both matrix elements have the same analytic properties, and mass and lifetime are the same for particle and antiparticle. It is not claimed that the decaying particle is correctly given by the eigenstate $|0\rangle$ of H_S; this particular state is rather used to identify the relevant singularity of the operator $(\lambda - H)^{-1}$. Both in first order and in the general treatment it has not been assumed that H_S is invariant with respect to T, P, and C separately.

Then there are a few applications:

1. π-μ-decay. From the experimentally equal lifetimes of π^+ and π^- it cannot be concluded that H_S is invariant under C.

2. μ-e-decay. If electromagnetic interactions are neglected, the decay spectra of μ^+ and μ^- are equal, as already pointed out by Lee and Yang. Equality of the radiative corrections could not be concluded if only TCP would hold. Incidentally, the S-matrix of the strong interaction in this case is that for Compton scattering.

3. β-decay of aligned nuclei. The Coulomb term in the asymmetry changes sign if one goes over from nuclei to anti-nuclei, as can easily be checked from the explicit formula. This term is only present if T is violated but TCP holds.

4. K^+ and K^-, π modes. a) Assume K has spin O and H_S is invariant under P. Then the 2π and 3π states have opposite parity and the strong interaction does not mix them. The branching ratios into both channels are the same for K^+ and K^-. b) This is approximately also true for spin greater than zero if H_S is also invariant under C and under rotations in isobaric spin space. Then a generalized Furry theorem forbids transitions between states with two and three π mesons; this selection rule is violated by the electromagnetic interactions. To check more than only TCP in the case of charged K-particles one has to look into the τ-τ' branching ratio or into the spectrum of the τ decay for K^+ and K^-.

DISCUSSION

SYMANZIK: If charge conjugation is not conserved even in the strong interactions, I don't believe that it is a well-defined quantity.

LUEDERS: If the strong interaction does not lead to decay of the particle, the operator Θ can be applied to a particle state and then gives the antiparticle.

LEE: The examination of possible consequences of non-invariance under T, C, and P separately of the strong interactions, I think, is somewhat academic. There is good experimental evidence that the strong interactions are, in fact, invariant under T, C, and P separately.

I would also like to ask, how do you define the lifetime?

ZUMINO: In order to define the lifetime, the thing that counts is that the imaginary part of the pole is small, not that the coupling constant is small. Then the imaginary part is the lifetime.

LEE: Unfortunately, a lifetime implies a physical measurement, and not just a mathematical definition.

ZUMINO: Yes, of course; it is a question of how the state will develop in time. It is possible to see that the development will not be rigorously exponential, but if the pole is very close to the real axis, it will be exponential to a very good approximation for a large region of time. This is true even if the interaction is large, provided that the pole is not far from the real axis.

DYSON: Is there any reason to believe that there really is a pole? I am of the contrary opinion.

ZUMINO: Well, I don't want to give a very sophisticated answer to this. This is what generally happens if you have a bound state that becomes embedded in the continuum. If you switch on a perturbation, then a previous bound state will no longer be a bound state. The pole, which corresponds to the energy level, shifts outside the real physical line. I'm talking about a perturbation and not about perturbation theory, that is, not about a perturbation expansion.