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Fragmentation function in non-equilibrium QCD using closed-time path integral formalism

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Abstract In this paper we implement the Schwinger–Keldysh closed-time path integral formalism in non-equilibrium QCD in accordance to the definition of the Collins–Soper fragmentation function. We consider a high-p_T parton in QCD medium at initial time τ_0 with an arbitrary non-equilibrium (non-isotropic) distribution function $f(\vec{p})$ fragmenting to a hadron. We formulate the parton-to-hadron fragmentation function in non-equilibrium QCD in the light-cone quantization formalism. It may be possible to include final-state interactions with the medium via a modification of the Wilson lines in this definition of the non-equilibrium fragmentation function. This may be relevant to the study of hadron production from a quark–gluon plasma at RHIC and LHC.

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1 Introduction

RHIC and LHC heavy-ion colliders are the best facilities to study a quark–gluon plasma in the laboratory. Since two nuclei travel almost at the speed of light, the QCD matter formed at RHIC and LHC may be in non-equilibrium. In order to make a meaningful comparison of the theory with the experimental data on hadron production, it may be necessary to study non-equilibrium–non-perturbative QCD at RHIC and LHC. This, however, is a difficult problem.

Non-equilibrium quantum field theory can be studied by using the Schwinger–Keldysh closed-time path (CTP) formalism [1–3]. However, implementing CTP in non-equilibrium at RHIC and LHC is a very difficult problem, especially due to the presence of gluons in non-equilibrium and hadronization etc. Recently, the one-loop resumed gluon propagator in non-equilibrium in covariant gauge has been derived in [4, 5].

High-p_T hadron production at high energy e^+e^-, ep and pp colliders is studied by using the Collins–Soper fragmentation function [6–10]. For a high-p_T parton fragmenting to a hadron, Collins and Soper derived an expression for the fragmentation function based on field theory and the factorization properties in QCD at high energy [11–18]. This fragmentation function is universal in the sense that, once its value is determined from one experiment it explains the data at other experiments.

The derivation of the parton-to-hadron fragmentation function in QCD medium based on a first principle calculation has not been done so far. This may be relevant at RHIC and LHC heavy-ion colliders to the study of hadron production from a quark–gluon plasma. A further complication arises because the partons at RHIC and LHC may be in non-equilibrium.

In this paper we note that one can implement the closed-time path integral formalism in non-equilibrium QCD to the definition of the Collins–Soper fragmentation function. We consider a high-p_T parton in medium at initial time τ_0 with an arbitrary non-equilibrium (non-isotropic) distribution function $f(\vec{p})$ fragmenting to a hadron. We formulate the parton-to-hadron fragmentation function in non-equilibrium QCD in the light-cone quantization formalism. The special case

$$f(\vec{p}) = \frac{1}{\mathrm{e}^{\frac{p_0}{T}} \pm 1}$$

corresponds to finite-temperature QCD in equilibrium. This fragmentation function may be relevant to the study of hadron production from a quark–gluon plasma at RHIC and LHC.

We find the following definition of the parton-to-hadron fragmentation function in non-equilibrium QCD by using the closed-time path integral formalism. For a quark (q) with arbitrary non-equilibrium distribution function $f_q(\vec{k})$ at initial time, the quark-to-hadron fragmentation function

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is given by

$$
\begin{aligned}
D_{\mathrm{H/q}}(z,P_{\mathrm{T}}) = {} & \frac{1}{2z[1+f_q(\vec{k})]} \\
& \times \int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{ik^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\
& \times \frac{1}{2}\,\mathrm{tr}_{\mathrm{Dirac}}\,\frac{1}{3}\,\mathrm{tr}_{\mathrm{color}}\big[\gamma^+ \langle \mathrm{in}|\psi\big(x^-,x_{\mathrm{T}}\big) \\
& \times \Phi\big[x^-,x_{\mathrm{T}}\big] a_{\mathrm{H}}^{\dagger}\big(P^+,0_{\mathrm{T}}\big) a_{\mathrm{H}}\big(P^+,0_{\mathrm{T}}\big) \\
& \times \Phi[0]\bar{\psi}(0)|\mathrm{in}\rangle\big],
\end{aligned} \tag{1.1}
$$

where $z\ (=\frac{P^+}{k^+})$ is the longitudinal momentum fraction of the hadron with respect to the parton and P_{T} is the transverse momentum of the hadron. $|\mathrm{in}\rangle$ is the initial state of the non-equilibrium QCD medium in the Schwinger–Keldysh in–in closed-time path formalism.

For a gluon (g) with arbitrary non-equilibrium distribution function $f_{\mathrm{g}}(\vec{k})$ at initial time, the gluon-to-hadron fragmentation function is given by

$$
\begin{aligned}
D_{\mathrm{H/g}}(z,P_{\mathrm{T}}) = {} & -\frac{1}{2zk^+[1+f_{\mathrm{g}}(\vec{k})]} \\
& \times \int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{ik^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\
& \times \frac{1}{8}\sum_{a=1}^{8}\big[\langle \mathrm{in}|F_a^{+\mu}\big(x^-,x_{\mathrm{T}}\big)\Phi\big[x^-,x_{\mathrm{T}}\big] \\
& \times a_{\mathrm{H}}^{\dagger}\big(P^+,0_{\mathrm{T}}\big) a_{\mathrm{H}}\big(P^+,0_{\mathrm{T}}\big)\Phi[0]F^{+}_{\mu a}(0)|\mathrm{in}\rangle\big].
\end{aligned} \tag{1.2}
$$

The path-ordered exponential

$$
\Phi\big[x^\mu\big]_{ab} = \mathcal{P}\exp\bigg[ig\int_0^\infty \mathrm{d}\lambda\, n\cdot A^c\big(x^\mu + n^\mu\lambda\big)T^c_{ab}\bigg] \tag{1.3}
$$

is the Wilson line [19]. It may be possible to include final-state interactions with the medium via modification of the Wilson lines in this definition of the non-equilibrium fragmentation function, similar to the p_{T} distribution of the parton distribution function studied in [20].

Equations (1.1) and (1.2) can be compared with the following definition of the Collins–Soper fragmentation function [6–8]:

$$
\begin{aligned}
D_{\mathrm{H/q}}(z,P_{\mathrm{T}}) = {} & \frac{1}{2z}\int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{ik^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\
& \times \frac{1}{2}\,\mathrm{tr}_{\mathrm{Dirac}}\,\frac{1}{3}\,\mathrm{tr}_{\mathrm{color}}\big[\gamma^+ \langle 0|\psi\big(x^-,x_{\mathrm{T}}\big) \\
& \times \Phi\big[x^-,x_{\mathrm{T}}\big] a_{\mathrm{H}}^{\dagger}\big(P^+,0_{\mathrm{T}}\big) \\
& \times a_{\mathrm{H}}\big(P^+,0_{\mathrm{T}}\big)\Phi[0]\bar{\psi}(0)|0\rangle\big]
\end{aligned} \tag{1.4}
$$

and

$$
\begin{aligned}
D_{\mathrm{H/g}}(z,P_{\mathrm{T}}) = {} & -\frac{1}{2zk^+}\int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{ik^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\
& \times \frac{1}{8}\sum_{a=1}^{8}\big[\langle 0|F_a^{+\mu}\big(x^-,x_{\mathrm{T}}\big) \\
& \times \Phi\big[x^-,x_{\mathrm{T}}\big] a_{\mathrm{H}}^{\dagger}\big(P^+,0_{\mathrm{T}}\big) a_{\mathrm{H}}\big(P^+,0_{\mathrm{T}}\big) \\
& \times \Phi[0]F^{+}_{\mu a}(0)|0\rangle\big].
\end{aligned} \tag{1.5}
$$

We shall present derivations of (1.1) and (1.2) in this paper.

The paper is organized as follows. In Sect. 2 we briefly review the definition of the Collins–Soper fragmentation function in vacuum, which is widely used at *pp*, *ep* and e^+e^- colliders. In Sect. 3 we give a brief description of the Schwinger–Keldysh closed-time path integral formalism in non-equilibrium QCD. We implement the closed-time path integral formalism in non-equilibrium QCD in accordance to the Collins–Soper fragmentation function in Sect. 4. Section 5 contains our conclusions.

2 Collins–Soper fragmentation function in vacuum

In this section we shall briefly review the Collins–Soper fragmentation function in vacuum, which is widely used at *pp*, *ep* and e^+e^- colliders. Consider a scalar gluon, for example, with four momentum k^μ in vacuum fragmenting to a hadron with four momentum P^μ. For application to collider experiments it is convenient to use the light-cone quantization formalism. The scalar gluon field $\phi(x)$ can be written as

$$
\begin{aligned}
\phi\big(x^-,x_{\mathrm{T}}\big) = {} & \frac{1}{(2\pi)^{d-1}}\int \frac{\mathrm{d}k^+}{\sqrt{2k^+}}\,\mathrm{d}^{d-2}k_{\mathrm{T}} \\
& \times \big[\mathrm{e}^{-\mathrm{i}k\cdot x}a(k) + \mathrm{e}^{\mathrm{i}k\cdot x}a^{\dagger}(k)\big]_{x^+=0},
\end{aligned} \tag{2.1}
$$

where $a^\dagger$ and a are the creation and annihilation operators, respectively. We have $d=4-2\epsilon$, where $3-2\epsilon$ is the space dimension. The single-particle parton state is given by

$$
\big|k^+,k_{\mathrm{T}}\big\rangle = a^{\dagger}\big(k^+,k_{\mathrm{T}}\big)|0\rangle, \quad \text{with } a\big(k^+,k_{\mathrm{T}}\big)|0\rangle = 0, \tag{2.2}
$$

with the normalization

$$
\begin{aligned}
& \big\langle k^+,k_{\mathrm{T}}\big|k'^+,k'_{\mathrm{T}}\big\rangle \\
& \quad = (2\pi)^{d-1}\delta\big(k^+-k'^+\big)\delta^{d-2}\big(k_{\mathrm{T}}-k'_{\mathrm{T}}\big).
\end{aligned} \tag{2.3}
$$

Note that the correct interpretation of the state $|k\rangle$ is created by an appropriate Fourier transform of the corresponding field operator and should not be associated with the on-shell condition $k^2 = 0$ of the massless quark or gluon.[1]

Consider inclusive production of a hadron, H, created in the out-state $|H, X\rangle$ from the parton a in the in-state $|k\rangle$ with the probability amplitude

$$\langle H, X|k\rangle. \tag{2.4}$$

The probability distribution $h_k(P)$ of the hadron with momentum P from the parton of momentum k can be found from the above amplitude. Explicitly, we have

$$\begin{aligned} h_k(P)\langle k|k'\rangle &= \sum_X \langle k|H, X\rangle\langle H, X|k'\rangle \\ &= \sum_X \langle k|a^\dagger_{\mathrm{H}}(P)|X\rangle\langle X|a_{\mathrm{H}}(P)|k'\rangle \\ &= \langle k|a^\dagger_{\mathrm{H}}(P)a_{\mathrm{H}}(P)|k'\rangle, \end{aligned} \tag{2.5}$$

where $a^\dagger_{\mathrm{H}}$ is the creation operator of hadron H. In the light-cone quantization formalism we find (by using (2.2) and (2.3))

$$\begin{aligned} &h(z, P_{\mathrm{T}})\langle k^+, k_{\mathrm{T}}|k'^+, k'_{\mathrm{T}}\rangle \\ &\quad= 2z(2\pi)^{d-1} D_{\mathrm{H}/a}(z, P_{\mathrm{T}})(2\pi)^{d-1} \\ &\qquad\times \delta(k^+ - k'^+)\delta^{d-2}(k_{\mathrm{T}} - k'_{\mathrm{T}}) \\ &\quad= \langle 0|a(k^+, k_{\mathrm{T}})a^\dagger_{\mathrm{H}}(P^+, P_{\mathrm{T}})a_{\mathrm{H}}(P^+, P_{\mathrm{T}})a^\dagger(k'^+, k'_{\mathrm{T}})|0\rangle, \end{aligned} \tag{2.6}$$

where $D_{\mathrm{H}/a}(z, P_{\mathrm{T}})$ is the fragmentation function and $z = \frac{P^+}{k^+}$ is the longitudinal momentum fraction of hadron H with respect to parton a. Using

$$\begin{aligned} &(2\pi)^{d-1}\langle 0|\phi(x^-, x_{\mathrm{T}})a^\dagger_{\mathrm{H}}(P^+, P_{\mathrm{T}})a_{\mathrm{H}}(P^+, P_{\mathrm{T}})\phi(0)|0\rangle \\ &\quad= \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}k^+}{\sqrt{2k^+}}\,\mathrm{d}^{d-2}k_{\mathrm{T}} \int \frac{\mathrm{d}k'^+}{\sqrt{2k'^+}}\,\mathrm{d}^{d-2}k'_{\mathrm{T}} \\ &\qquad\times \big[\langle 0|\mathrm{e}^{-\mathrm{i}k\cdot x} a(k^+, k_{\mathrm{T}})a^\dagger_{\mathrm{H}}(P^+, P_{\mathrm{T}})a_{\mathrm{H}}(P^+, P_{\mathrm{T}}) \\ &\qquad\times a^\dagger(k'^+, k'_{\mathrm{T}})|0\rangle\big]_{x^+=0}, \end{aligned} \tag{2.7}$$

we find from (2.6)

$$\begin{aligned} D_{\mathrm{H}/a}(z, P_{\mathrm{T}}) = \frac{k^+}{z} \int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{\mathrm{i}k^+x^- - \mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}} \\ \times \langle 0|\phi(x^-, x_{\mathrm{T}})a^\dagger_{\mathrm{H}}(P^+, P_{\mathrm{T}})a_{\mathrm{H}}(P^+, P_{\mathrm{T}}) \\ \times \phi(0)|0\rangle. \end{aligned} \tag{2.8}$$

It is convenient to rewrite the definition in a form analogous to the definition of the distribution of partons in a hadron. The transverse momentum is of the parton relative to the hadron rather than vice versa. For this purpose we make a Lorentz transformation to a frame where the hadron's transverse momentum is zero:

$$\begin{aligned} &(P^+, P_{\mathrm{T}}) \to (P^+, 0), \\ &(k^+, 0) \to (k^+, -P_{\mathrm{T}}/z). \end{aligned} \tag{2.9}$$

Hence we find from (2.8)

$$\begin{aligned} D_{\mathrm{H}/a}(z, P_{\mathrm{T}}) = \frac{k^+}{z} \int \mathrm{d}x^- \frac{\mathrm{d}^{d-2}x_{\mathrm{T}}}{(2\pi)^{d-1}} \mathrm{e}^{\mathrm{i}k^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\ \times \langle 0|\phi(x^-, x_{\mathrm{T}})a^\dagger_{\mathrm{H}}(P^+, 0_{\mathrm{T}})a_{\mathrm{H}}(P^+, 0_{\mathrm{T}}) \\ \times \phi(0)|0\rangle. \end{aligned} \tag{2.10}$$

The p_{T} integrated fragmentation function is given by

$$\begin{aligned} d_{\mathrm{H}/a}(z) &= \int \mathrm{d}^{d-2}P_{\mathrm{T}} D_{\mathrm{H}/g}(z, P_{\mathrm{T}}) \\ &= \frac{k^+ z^{d-3}}{2\pi} \int \mathrm{d}x^-\, \mathrm{e}^{\mathrm{i}P^+x^-/z} \\ &\quad\times \big[\langle 0|\phi(x^-)a^\dagger_{\mathrm{H}}(P^+, 0_{\mathrm{T}})a_{\mathrm{H}}(P^+, 0_{\mathrm{T}})\phi(0)|0\rangle\big]. \end{aligned} \tag{2.11}$$

2.1 Quark fragmentation function

Following similar steps as above, but performing the calculation for the quark we find the fragmentation function of the quark:

$$\begin{aligned} D_{\mathrm{H}/\mathrm{q}}(z, P_{\mathrm{T}}) = \frac{1}{2z(2\pi)^{d-1}} \int \mathrm{d}x^-\, \mathrm{d}^{d-2}x_{\mathrm{T}}\, \mathrm{e}^{\mathrm{i}k^+x^- + \mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\ \times \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}} \frac{1}{3}\mathrm{tr}_{\mathrm{color}}\big[\gamma^+ \langle 0|\psi(x^-, x_{\mathrm{T}}) \\ \times a^\dagger_{\mathrm{H}}(P^+, 0_{\mathrm{T}})a_{\mathrm{H}}(P^+, 0_{\mathrm{T}})\bar{\psi}(0)|0\rangle\big], \end{aligned} \tag{2.12}$$

where $\psi(x)$ is the quark field. The P_{T} integrated quark fragmentation function becomes

$$\begin{aligned} d_{\mathrm{H}/\mathrm{q}}(z) = \frac{z^{d-3}}{4\pi} \int \mathrm{d}x^-\, \mathrm{e}^{\mathrm{i}P^+x^-/z} \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}} \\ \times \frac{1}{3}\mathrm{tr}_{\mathrm{color}}\big[\gamma^+ \langle 0|\psi(x^-)a^\dagger_{\mathrm{H}}(P^+, 0_{\mathrm{T}}) \\ \times a_{\mathrm{H}}(P^+, 0_{\mathrm{T}})\bar{\psi}(0)|0\rangle\big]. \end{aligned} \tag{2.13}$$

2.2 Gluon fragmentation function

Following the above steps but now for gluons, we find the gluon fragmentation function:

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[1] I thank George Sterman for pointing this.

$$\begin{aligned} D_{\mathrm{H/g}}(z,P_{\mathrm{T}}) &= -\frac{1}{2zk^{+}(2\pi)^{d-1}} \\ &\quad\times\int \mathrm{d}x^{-}\,\mathrm{d}^{d-2}x_{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}k^{+}x^{-}+\mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\ &\quad\times\frac{1}{8}\sum_{a=1}^{8}\big[\langle 0|F_{a}^{+\mu}(x^{-},x_{\mathrm{T}})a_{\mathrm{H}}^{\dagger}(P^{+},0_{\mathrm{T}}) \\ &\quad\times a_{\mathrm{H}}(P^{+},0_{\mathrm{T}})F_{\mu a}^{+}(0)|0\rangle\big], \end{aligned} \tag{2.14}$$

where

$$F_{a}^{\mu\nu}=\partial^{\mu}A_{a}^{\nu}-\partial^{\nu}A_{a}^{\mu}. \tag{2.15}$$

The P_{T} integrated gluon fragmentation function becomes

$$\begin{aligned} d_{\mathrm{H/g}}(z) &= -\frac{z^{d-3}}{4\pi k^{+}}\int \mathrm{d}x^{-}\,\mathrm{e}^{\mathrm{i}P^{+}x^{-}/z} \\ &\quad\times\sum_{a=1}^{8}\big[\langle 0|F_{a}^{+\mu}(x^{-})a_{\mathrm{H}}^{\dagger}(P^{+},0_{\mathrm{T}}) \\ &\quad\times a_{\mathrm{H}}(P^{+},0_{\mathrm{T}})F_{\mu a}^{+}(0)|0\rangle\big]. \end{aligned} \tag{2.16}$$

2.3 Wilson lines and fragmentation functions

The quark and gluon fragmentation functions as defined above are not gauge invariant. Gauge-invariant parton fragmentation functions are obtained by incorporating Wilson lines $\Phi[x]_{ab}$ into the definition of the quark and gluon fragmentation functions [6–8, 11–19, 21–27]. We find

$$\begin{aligned} D_{\mathrm{H/q}}(z,P_{\mathrm{T}}) &= \frac{1}{2z(2\pi)^{d-1}}\int \mathrm{d}x^{-}\,\mathrm{d}^{d-2}x_{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}k^{+}x^{-}+\mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\ &\quad\times\frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}}\,\frac{1}{3}\mathrm{tr}_{\mathrm{color}}\big[\gamma^{+}\langle 0|\psi(x^{-},x_{\mathrm{T}}) \\ &\quad\times\Phi[x^{-},x_{\mathrm{T}}]a_{\mathrm{H}}^{\dagger}(P^{+},0_{\mathrm{T}})a_{\mathrm{H}}(P^{+},0_{\mathrm{T}}) \\ &\quad\times\Phi[0]\,\bar{\psi}(0)|0\rangle\big], \end{aligned} \tag{2.17}$$

where $\Phi[x^{-},x_{\mathrm{T}}]$ is given by (1.3) with T^{ab} in the fundamental representation of SU(3).

Similarly, incorporating Wilson lines, we find the gauge-invariant gluon fragmentation function:

$$\begin{aligned} D_{\mathrm{H/g}}(z,P_{\mathrm{T}}) &= -\frac{1}{2zk^{+}(2\pi)^{d-1}} \\ &\quad\times\int \mathrm{d}x^{-}\,\mathrm{d}^{d-2}x_{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}k^{+}x^{-}+\mathrm{i}P_{\mathrm{T}}\cdot x_{\mathrm{T}}/z} \\ &\quad\times\frac{1}{8}\sum_{a=1}^{8}\big[\langle 0|F_{a}^{+\mu}(x^{-},x_{\mathrm{T}}) \\ &\quad\times\Phi[x^{-},x_{\mathrm{T}}]a_{\mathrm{H}}^{\dagger}(P^{+},0_{\mathrm{T}})a_{\mathrm{H}}(P^{+},0_{\mathrm{T}}) \\ &\quad\times\Phi[0]\,F_{\mu a}^{+}(0)|0\rangle\big], \end{aligned} \tag{2.18}$$

where $\Phi[x^{-},x_{\mathrm{T}}]$ is given by (1.3) with $T_{c}^{ab}=f^{abc}$ in the adjoint representation of SU(3).

3 Non-equilibrium QCD using closed-time path formalism

Unlike pp collisions, the ground state at RHIC and LHC heavy-ion collisions (due to the presence of a QCD medium at the initial time, $t=t_{\mathrm{in}}$, (say $t_{\mathrm{in}}=0$) is not a vacuum state $|0\rangle$ any more. We denote by $|\mathrm{in}\rangle$ the initial state of the non-equilibrium QCD medium at t_{in}. The non-equilibrium distribution function $f(\vec{k})$ of a parton (quark or gluon), corresponding to such an initial state, is given by

$$\begin{aligned} \langle a^{\dagger}(\vec{k})a(\vec{k}')\rangle &= \langle \mathrm{in}|a^{\dagger}(\vec{k})a(\vec{k}')|\mathrm{in}\rangle \\ &= f(\vec{k})(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}-\vec{k}'), \end{aligned} \tag{3.1}$$

where we have assumed space translational invariance at initial time.

The finite-temperature field theory formulation is a special case of this when

$$f(\vec{k})=\frac{1}{\mathrm{e}^{\frac{k_0}{T}}\pm 1}.$$

Consider scalar gluons first. In the CTP formalism in non-equilibrium there are four Green's functions:

$$\begin{aligned} G{+}{+}(x,x') &= \langle \mathrm{in}|T\phi(x)\phi(x')|\mathrm{in}\rangle = \langle T\phi(x)\phi(x')\rangle, \\ G{-}{-}(x,x') &= \langle \mathrm{in}|\bar{T}\phi(x)\phi(x')|\mathrm{in}\rangle = \langle \bar{T}\phi(x)\phi(x')\rangle, \\ G{+}{-}(x,x') &= \langle \mathrm{in}|\phi(x')\phi(x)|\mathrm{in}\rangle = \langle \phi(x')\phi(x)\rangle, \\ G{-}{+}(x,x') &= \langle \mathrm{in}|\phi(x)\phi(x')|\mathrm{in}\rangle = \langle \phi(x)\phi(x')\rangle, \end{aligned} \tag{3.2}$$

where the $+$ $(-)$ sign corresponds to the upper (lower) time branch of the Schwinger–Keldysh closed-time path [1, 2]. T is the time-ordered product and $\bar{T}$ is the anti-time-ordered product. The field $\phi(x)$ is in the Heisenberg representation. Explicitly, we have

$$\begin{aligned} T\phi(x)\phi(x') &= \theta(t-t')\phi(x)\phi(x') \\ &\quad+\theta(t'-t)\phi(x')\phi(x), \\ \bar{T}\phi(x)\phi(x') &= \theta(t'-t)\phi(x)\phi(x') \\ &\quad+\theta(t-t')\phi(x')\phi(x). \end{aligned} \tag{3.3}$$

At initial time, $t=t_{\mathrm{in}}=0$, the Heisenberg picture coincides with the Schroedinger and interaction pictures. We write

$$\phi(x)=\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}\sqrt{2k^{0}}}\big[a(\vec{k})\mathrm{e}^{-\mathrm{i}k\cdot x}+a^{\dagger}(\vec{k})\mathrm{e}^{\mathrm{i}k\cdot x}\big], \tag{3.4}$$

where $a^\dagger(\vec{k})$ and $a(\vec{k})$ are creation and annihilation operators, respectively. The commutation relations are given by

$$\begin{aligned}&\big[a(\vec{k}), a^\dagger(\vec{k}')\big]_{t=0} = (2\pi)^{d-1}\delta^{(d-1)}(\vec{k}-\vec{k}'),\\ &\big[a(\vec{k}), a(\vec{k}')\big]_{t=0} = \big[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\big]_{t=0} = 0.\end{aligned} \tag{3.5}$$

We assume space translational invariance at the initial time and find

$$\big[G_{ij}(x,x')\big]_{t=0} = \bigg[\int \mathrm{d}^d k\, G_{ij}(k)\mathrm{e}^{-\mathrm{i}k\cdot(x-x')}\bigg]_{t=0}, \tag{3.6}$$

where i, j are $+, -$. Explicitly,

$$\begin{aligned}G_{-+}(x,x') &= \langle\phi(x)\phi(x')\rangle\\ &= \int \frac{\mathrm{d}^{d-1}k}{\sqrt{2k^0}(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}k'}{\sqrt{2k'^0}(2\pi)^{d-1}}\\ &\quad \times \langle \mathrm{in}|\big[a(\vec{k})\mathrm{e}^{-\mathrm{i}k\cdot x} + a^\dagger(\vec{k})\mathrm{e}^{\mathrm{i}k\cdot x}\big]\\ &\quad \times \big[a(\vec{k}')\mathrm{e}^{-\mathrm{i}k'\cdot x'} + a^\dagger(\vec{k}')\mathrm{e}^{\mathrm{i}k'\cdot x'}\big]|\mathrm{in}\rangle.\end{aligned} \tag{3.7}$$

Using a Bogolyubov transformation we can set

$$\langle \mathrm{in}|a(\vec{k})a(\vec{k}')|\mathrm{in}\rangle = \langle \mathrm{in}|a^\dagger(\vec{k})a^\dagger(\vec{k}')|\mathrm{in}\rangle = 0. \tag{3.8}$$

By using (3.1), (3.5) and (3.8) in (3.7) we find

$$\begin{aligned}\big[G_{-+}(x,x')\big]_{t=0} &= \bigg[\int \frac{\mathrm{d}^{d-1}k}{2k^0(2\pi)^{d-1}}\big[\big[1+f(\vec{k})\big]\\ &\quad \times \mathrm{e}^{-\mathrm{i}k\cdot(x-x')} + f(\vec{k})\mathrm{e}^{\mathrm{i}k\cdot(x-x')}\bigg]_{t=0}\\ &= \bigg[\int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}}\delta\big(k^2\big)\mathrm{e}^{-ik\cdot(x-x')}\\ &\quad \times \big[\theta(k_0)\big[1+f(\vec{k})\big]\\ &\quad + \theta(-k_0)f(-\vec{k})\big]\bigg]_{t=0}\\ &= \bigg[\int \mathrm{d}^d k\, G_{-+}(k)\mathrm{e}^{-\mathrm{i}k\cdot(x-x')}\bigg]_{t=0}.\end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned}\big[G_{+-}(k)\big]_{t=0} &= \delta\big(k^2\big)\big[\theta(-k_0) + \theta(k_0)f(\vec{k})\\ &\quad + \theta(-k_0)f(-\vec{k})\big],\\ \big[G_{++}(k)\big]_{t=0} &= \frac{1}{k^2+\mathrm{i}\epsilon} + \delta\big(k^2\big)\big[\theta(k_0)f(\vec{k})\\ &\quad + \theta(-k_0)f(-\vec{k})\big],\\ \big[G_{--}(k)\big]_{t=0} &= \frac{-1}{k^2-\mathrm{i}\epsilon} + \delta\big(k^2\big)\big[\theta(k_0)f(\vec{k})\\ &\quad + \theta(-k_0)f(-\vec{k})\big].\end{aligned} \tag{3.10}$$

3.1 Quarks in non-equilibrium

The non-equilibrium (massless) quark propagator at initial time $t = t_{\mathrm{in}}$ is given by (suppression of color indices is understood)

$$G(k)_{ij} = \not{k}\begin{pmatrix} \frac{1}{k^2+\mathrm{i}\epsilon} + 2\pi\delta(k^2)f_{\mathrm{q}}(\vec{k}) & -2\pi\delta(k^2)\theta(-k_0) + 2\pi\delta(k^2)f_{\mathrm{q}}(\vec{k}) \\ -2\pi\delta(k^2)\theta(k_0) + 2\pi\delta(k^2)f_{\mathrm{q}}(\vec{k}) & -\frac{1}{k^2-\mathrm{i}\epsilon} + 2\pi\delta(k^2)f_{\mathrm{q}}(\vec{k}) \end{pmatrix}, \tag{3.11}$$

where $i, j = +, -$ and $f_{\mathrm{q}}(\vec{k})$ is the arbitrary non-equilibrium distribution function of the quark.

3.2 Gluons in non-equilibrium

We work in the frozen-ghost formalism [4, 5], where the non-equilibrium gluon propagator at initial time, $t = t_{\mathrm{in}}$, is given by (the suppression of color indices is understood)

$$\begin{aligned}G^{\mu\nu}(k)_{ij} &= -\mathrm{i}\bigg[g^{\mu\nu} + (\alpha-1)\frac{k^\mu k^\nu}{k^2}\bigg]G^{\mathrm{vac}}_{ij}(k)\\ &\quad - \mathrm{i}T^{\mu\nu}G^{\mathrm{med}}_{ij}(k),\end{aligned} \tag{3.12}$$

where $i, j = +, -$. The transverse tensor is given by

$$T^{\mu\nu}(k) = g^{\mu\nu} - \frac{(k\cdot u)(u^\mu k^\nu + u^\nu k^\mu) - k^\mu k^\nu - k^2 u^\mu u^\nu}{(k\cdot u)^2 - k^2}, \tag{3.13}$$

with the flow velocity of the medium u^μ. The $G^{\mu\nu}_{ij}(k)$ are the usual vacuum propagators of the gluon:

$$G^{\mathrm{vac}}_{ij}(k) = \begin{pmatrix} \frac{1}{k^2+\mathrm{i}\epsilon} & -2\pi\delta(k^2)\theta(-k_0) \\ -2\pi\delta(k^2)\theta(k_0) & -\frac{1}{k^2-\mathrm{i}\epsilon} \end{pmatrix}, \tag{3.14}$$

and the medium part of the propagators is given by

$$G^{\mathrm{med}}_{ij}(k) = 2\pi\delta\big(k^2\big)f_g\big(\vec{k}\big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.15}$$

4 Fragmentation function in non-equilibrium QCD using closed-time path formalism

For simplicity, let us consider the scalar gluon first in the light-cone quantization formalism. Generalizing the vacuum analysis in [6–8] we define the state $|k^+, k_T\rangle$ of the fragmenting gluon in non-equilibrium QCD medium at the initial time $x^+ = x^+_{\rm in}$ (say at $x^+_{\rm in} = 0$):

$$\big|k^+, k_T\big\rangle = a^\dagger\big(k^+, k_T\big)|{\rm in}\rangle. \tag{4.1}$$

The non-equilibrium distribution function $f(k^+, k_T)$ of the fragmenting gluon is given by

$$\begin{aligned}&\langle{\rm in}|a^\dagger\big(k^+, k_T\big)a\big(k'^+, k'_T\big)|{\rm in}\rangle\\&\quad= f\big(k^+, k_T\big)(2\pi)^{d-1}\delta\big(k^+ - k'^+\big)\delta^{(d-2)}(k_T - k'_T),\end{aligned} \tag{4.2}$$

where we have assumed space (x^- and x_T) translational invariance at initial time, $x^+ = x^+_{\rm in}$. The commutation relations are given by

$$\begin{aligned}&\big[a\big(k^+, k_T\big), a^\dagger\big(k'^+, k'_T\big)\big]_{x^+=0}\\&\quad= (2\pi)^{d-1}\delta\big(k^+ - k'^+\big)\delta^{(d-2)}(k_T - k'_T),\\&\big[a\big(k^+, k_T\big), a\big(k'^+, k'_T\big)\big]_{x^+=0}\\&\quad= \big[a^\dagger\big(k^+, k_T\big), a^\dagger\big(k'^+, k'_T\big)\big]_{x^+=0} = 0.\end{aligned} \tag{4.3}$$

By using (3.1) and (3.5), we find

$$\begin{aligned}\big\langle k^+, k_T|k'^+, k'_T\big\rangle &= \langle{\rm in}|a\big(k^+, k_T\big)a^\dagger\big(k'^+, k'_T\big)|{\rm in}\rangle\\&= (2\pi)^{d-1}\delta\big(k^+ - k'^+\big)\delta^{d-2}(k_T - k'_T)\\&\quad\times\big[1 + f\big(k^+, k_T\big)\big].\end{aligned} \tag{4.4}$$

Consider the inclusive production of hadron H, created in the out-state $|H, X\rangle$ from a scalar gluon in non-equilibrium in the initial state $|k\rangle$ with the probability amplitude

$$\langle H, X|k\rangle. \tag{4.5}$$

X is for the other outgoing final-state particles. Similar to the vacuum case of the Collins–Soper fragmentation function, the correct interpretation of the above state, $|k\rangle$, is created by an appropriate Fourier transform of the corresponding field operator and should not be associated with the on-shell condition $k^2 = 0$ of the massless quark or gluon (see footnote 1). The distribution $h_k(P)$ of hadron H with momentum P from the parton of momentum k can be found from the above amplitude. We find

$$\begin{aligned}&\sum_X\langle k, k_T|H, X\rangle\big\langle H, X\big|k^{+\prime}, k'_T\big\rangle\\&\quad= h_k(P)\big\langle k^+, k_T\big|k^{+\prime}, k'_T\big\rangle.\end{aligned} \tag{4.6}$$

For the left-hand side we write

$$\begin{aligned}&\sum_X\big\langle k^+, k_T\big|H, X\big\rangle\big\langle H, X\big|k^{+\prime}, k'_T\big\rangle\\&\quad= \sum_X\big\langle k^+, k_T\big|a_H^\dagger(P)|X\rangle\langle X|a_H(P)\big|k^{+\prime}, k'_T\big\rangle\\&\quad= \big\langle k^+, k_T\big|a_H^\dagger(P)a_H(P)\big|k^{+\prime}, k'_T\big\rangle.\end{aligned} \tag{4.7}$$

Equating (4.6) and (4.7) and using (4.1), we find

$$\begin{aligned}&\langle{\rm in}|a\big(k^+, k_T\big)a_H^\dagger\big(P^+, P_T\big)a_H\big(P^+, P_T\big)a^\dagger\big(k'^+, k'_T\big)|{\rm in}\rangle\\&\quad= 2z(2\pi)^{d-1}D_{H/a}(z, P_T)\langle{\rm in}|a\big(k^+, k_T\big)a^\dagger\big(k'^+, k'_T\big)|{\rm in}\rangle.\end{aligned} \tag{4.8}$$

This expression is exactly similar to that of the jet fragmentation function in vacuum as given in (2.6), except that the vacuum expectation is replaced by a medium average at the initial time $x^+ = x^+_{\rm in}$. Using (4.1) and (4.2), we find

$$\begin{aligned}&\langle{\rm in}|a\big(k^+, k_T\big)a_H^\dagger\big(P^+, P_T\big)a_H\big(P^+, P_T\big)a^\dagger\big(k'^+, k'_T\big)|{\rm in}\rangle\\&\quad= 2z(2\pi)^{d-1}D_{H/a}(z, P_T)\big[1 + f\big(k^+, k_T\big)\big].\end{aligned} \tag{4.9}$$

From (3.4), we obtain

$$\begin{aligned}&(2\pi)^{d-1}\langle{\rm in}|\phi\big(x^-, x_T\big)a_H^\dagger\big(P^+, P_T\big)a_H\big(P^+, P_T\big)\phi(0)|{\rm in}\rangle\\&\quad= \frac{1}{(2\pi)^{d-1}}\int\frac{{\rm d}k^+}{\sqrt{2k^+}}\,{\rm d}^{d-2}k_T\int\frac{{\rm d}k'^+}{\sqrt{2k'^+}}\,{\rm d}^{d-2}k'_T\\&\qquad\times\big[\langle{\rm in}|{\rm e}^{-{\rm i}k\cdot x}a\big(k^+, k_T\big)a_H^\dagger\big(P^+, P_T\big)\\&\qquad\times a_H\big(P^+, P_T\big)a^\dagger\big(k'^+, k'_T\big)|{\rm in}\rangle\big]_{x^+=0}.\end{aligned} \tag{4.10}$$

Using this in (4.9), we find the expression of the fragmentation function in non-equilibrium QCD:

$$\begin{aligned}D_{H/a}(z, P_T) &= \frac{k^+}{z\,[1 + f(k^+, k_T)]}\\&\quad\times\int{\rm d}x^-\,\frac{{\rm d}^{d-2}x_T}{(2\pi)^{d-1}}\,{\rm e}^{{\rm i}k^+x^- - {\rm i}k_T\cdot x_T}\\&\quad\times\langle{\rm in}|\phi_a\big(x^-, x_T\big)a_H^\dagger\big(P^+, P_T\big)a_H\big(P^+, P_T\big)\\&\quad\times\phi_a(0)|{\rm in}\rangle.\end{aligned} \tag{4.11}$$

From (2.10) we find

$$\begin{aligned}D_{H/a}(z, P_T) &= \frac{k^+}{z[1 + f(k^+, k_T)]}\\&\quad\times\int{\rm d}x^-\,\frac{{\rm d}^{d-2}x_T}{(2\pi)^{d-1}}\,{\rm e}^{{\rm i}k^+x^- + {\rm i}P_T\cdot x_T/z}\\&\quad\times\langle{\rm in}|\phi_a\big(x^-, x_T\big)a_H^\dagger\big(P^+, 0_T\big)a_H\big(P^+, 0_T\big)\\&\quad\times\phi_a(0)|{\rm in}\rangle.\end{aligned} \tag{4.12}$$

In the above expression, $f(k^+, k_T)$ is the non-equilibrium distribution function of the fragmenting gluon at initial time, $x^+ = x^+_{\text{in}}$, and $|\text{in}\rangle$ is the initial state of the non-equilibrium QCD medium in the Schwinger–Keldysh in–in closed-time path formalism.

4.1 Quark fragmentation function in non-equilibrium

Following the above steps, but for quarks, we find the quark fragmentation function in non-equilibrium QCD:

$$\begin{aligned} D_{\text{H/q}}(z, P_T) = {} & \frac{1}{2z[1 + f_{\text{q}}(k^+, k_T)]} \\ & \times \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}} e^{ik^+x^- + iP_T\cdot x_T/z} \\ & \times \frac{1}{2}\text{tr}_{\text{Dirac}} \frac{1}{3}\text{tr}_{\text{color}}\big[\gamma^+\langle \text{in}|\psi(x^-, x_T) \\ & \times a^\dagger_{\text{H}}(P^+, 0_T)a_{\text{H}}(P^+, 0_T)\bar{\psi}(0)|\text{in}\rangle\big], \end{aligned} \tag{4.13}$$

where $f_{\text{q}}(k^+, k_T)$ is the non-equilibrium distribution function of the fragmenting quark at initial time.

4.2 Gluon fragmentation function in non-equilibrium

For the gluons, we consider the frozen-ghost formalism described above; see (3.12). Hence the whole analysis above can be applied. Carrying out similar algebraic manipulations as above we find the gluon fragmentation function

$$\begin{aligned} D_{\text{H/g}}(z, P_T) = {} & -\frac{1}{2zk^+[1 + f_{\text{g}}(k^+, k_T)]} \\ & \times \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}} e^{ik^+x^- + iP_T\cdot x_T/z} \\ & \times \frac{1}{8}\sum_{a=1}^{8}\big[\langle \text{in}|F_a^{+\mu}(x^-, x_T) \\ & \times a^\dagger_{\text{H}}(P^+, 0_T)a_{\text{H}}(P^+, 0_T)F^+_{\mu a}(0)|\text{in}\rangle\big], \end{aligned} \tag{4.14}$$

where $f_{\text{g}}(k^+, k_T)$ is the non-equilibrium distribution function of the fragmenting gluon at initial time.

4.3 Wilson lines

The above definition of the parton-to-hadron fragmentation function in non-equilibrium QCD is not gauge invariant. To make it gauge invariant we need to incorporate Wilson lines. Incorporating the Wilson lines (1.3) into the definition of the fragmentation function, (4.13), we find the gauge-invariant quark fragmentation function in non-equilibrium QCD:

$$\begin{aligned} D_{\text{H/q}}(z, P_T) = {} & \frac{1}{2z[1 + f_{\text{q}}(k^+, k_T)]} \\ & \times \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}} e^{ik^+x^- + iP_T\cdot x_T/z} \\ & \times \frac{1}{2}\text{tr}_{\text{Dirac}} \frac{1}{3}\text{tr}_{\text{color}}\big[\gamma^+\langle \text{in}|\psi(x^-, x_T) \\ & \times \Phi[x^-, x_T]a^\dagger_{\text{H}}(P^+, 0_T)a_{\text{H}}(P^+, 0_T) \\ & \times \Phi[0]\bar{\psi}(0)|\text{in}\rangle\big], \end{aligned} \tag{4.15}$$

where $f_{\text{q}}(k^+, k_T)$ is the non-equilibrium distribution function of the fragmenting quark at initial time, $x^+ = x^+_{\text{in}}$. $|\text{in}\rangle$ is the initial state of the non-equilibrium QCD medium in the Schwinger–Keldysh in–in closed-time path formalism. This reproduces (1.1).

Incorporating the Wilson lines (1.3) into the definition of the fragmentation function, (4.14), we find the gauge-invariant gluon fragmentation function in non-equilibrium QCD:

$$\begin{aligned} D_{\text{H/g}}(z, P_T) = {} & -\frac{1}{2zk^+[1 + f_{\text{g}}(k^+, k_T)]} \\ & \times \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}} e^{ik^+x^- + iP_T\cdot x_T/z} \\ & \times \frac{1}{8}\sum_{a=1}^{8}\big[\langle \text{in}|F_a^{+\mu}(x^-, x_T) \\ & \times \Phi[x^-, x_T]a^\dagger_{\text{H}}(P^+, 0_T)a_{\text{H}}(P^+, 0_T) \\ & \times \Phi[0]F^+_{\mu a}(0)|\text{in}\rangle\big], \end{aligned} \tag{4.16}$$

where $f_{\text{g}}(k^+, k_T)$ is the non-equilibrium distribution function of the fragmenting gluon at the initial time, $x^+ = x^+_{\text{in}}$. $|\text{in}\rangle$ is the initial state of the non-equilibrium QCD medium in the Schwinger–Keldysh in–in closed-time path formalism. This reproduces (1.2).

It may be possible to include final-state interactions with the medium via modification of the Wilson lines in this definition of the non-equilibrium fragmentation function, similar to the p_T distribution of the parton distribution function studied in [20].

5 Conclusions

In this paper we have implemented the closed-time path integral formalism in non-equilibrium QCD to the definition of the Collins–Soper fragmentation function. We have considered a high-p_T parton in QCD medium at initial time τ_0

with an arbitrary non-equilibrium (non-isotropic) distribution function $f(\vec{p})$ fragmenting to a hadron. We have formulated the parton-to-hadron fragmentation function in non-equilibrium QCD in the light-cone quantization formalism. This may be relevant to the study of hadron production from a quark–gluon plasma at RHIC and LHC. It may be possible to include final-state interactions with the medium via modification of the Wilson lines in this definition of the non-equilibrium fragmentation function, similar to the p_T distribution of the parton distribution function studied in [20]. This will be the subject of a future analysis.

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