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Noncommutative CP^N and CH^N and their physics

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Abstract. We study noncommutative deformation of manifolds by constructing star products. We start from a noncommutative $\mathbb{R}^d$ and discuss more genaral noncommutative manifolds. In general, star products can not be described in concrete expressions without some exceptions. In this article we introduce new examples of noncommutative manifolds with explicit star products. Karabegov's deformation quantization of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ with separation of variables gives explicit calulable star products represented by gamma functions. Using the results of star products between inhomogeneous coordinates, we find creation and anihilation operators and obtain the Fock representation of the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$.

1. Introduction

Phyisical theories in noncommutative spaces are realized in several situations. For example, in string theories, low energy effective theories on D-branes in a constant background NS-NS B fields are given as noncommutative gauge theories. In the theories, B field plays a role of an inverse of a noncommutative parameter matrix that represents noncommutativity of space coordinates. Another example in string theories is a series of matrix models which are constructed so that all dynamical variables are given by matricies. Spacetime coordinates themselves are also described as matrices, and then the spacetime becomes a noncommutative space.

By the way, how can we obtain the noncommutative space? Let us consider a simple example, the noncommutative $\mathbb{R}^d$ using the Moyal product. The Moyal product in $\mathbb{R}^d$ is defined as

$$(f * g)(x) = e^{\frac{i}{2}\theta^{ij}\partial_i^x\partial_j^y} f(x)g(y)\Big|_{y=x}.$$

Here θ^{ij} is an element of a skew symmetric constant matrix θ and is called a noncommutative parameter. The right hand side is denoted by $f(x)e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j}g(x)$, for simplicity. Note that the commutation relation of the coordinates are given by

$$[x^i,\ x^j]_* = x^i * x^j - x^j * x^i = i\theta^{ij}.$$

In the usual commutative case, the ordinary commutative product is employed to define a ring of functions. If we chose the Moyal product to define a ring of functions, then the considering space become noncommutative because all products between arbitrary functions are replaced by the Moyal product. This is known as noncommutative $\mathbb{R}^d$.

In the following, we move to more general spaces. To make a notion of noncommutative deformation of space be mathematically rigor, we introduce a definition of deformation quantization.

Definition 1. Deformation Quantization (strong sense)
Deformation Quantization is defined as follows. $\mathcal{F}$ is defined as a set of formal power series of $\hbar$:

$$\mathcal{F} := \Big\{ f \;\Big|\; f = \sum_k f_k \hbar^k \Big\}.$$

A star product of functions f and g is defined as

$$f * g = \sum C_k(f,g)\hbar^k$$

such that the product satisfies the following conditions.

(i) $*$ is associative product.
(ii) C_k is a bidifferential operator.
(iii) C_0 and C_1 are defined as

$$C_0(f,g) = fg, \qquad C_1(f,g) = \frac{i}{2}\{f,g\},$$

where $\{f,g\}$ is the Poisson bracket.
(iv) $f * 1 = 1 * f$.

Deformation quantization of manifolds are given by this star product in a form of a formal power series in a deformation parameter $\hbar$. The power series is obtained as solutions of an infinite system of differential equations in general. The existence of the solution is proved for a wide class of manifolds, however explicit expressions of deformation quantizations are constructed for few kinds of manifolds. For example, Euclidean spaces are deformed by using the Moyal product as we saw above, and on manifolds with spherically symmetric metrics explicit star products are given as concrete calcurable forms in the context of the Fedosov's deformation quantization [1]. The Fedosov's Deformation Quantization is constructed as the Moyal product on the Weyl algebra bundle in essentials.

As mentioned above, there are few noncommutative manifolds that have explicit expression of star products, for example 2 dimensional manifolds with circular symmetric metrics. Following three examples are given by the methods of Fedosov's Deformation Quantization.

(1). $\mathbb{R}^2$

$$f(r,\varphi) * g(r,\varphi) = f(\sqrt{r^2 + 2y^1 r}, \varphi + \frac{y^2}{r}) \exp\Big(-\frac{i\hbar}{2}\overleftarrow{\frac{\partial}{\partial y^i}}\omega^{ij}\overrightarrow{\frac{\partial}{\partial y^j}}\Big)$$
$$g(\sqrt{r^2 + 2y^1 r}, \varphi + \frac{y^2}{r})\Big|_{y=0},$$

where (r,φ) is a polar coordinate and $\omega^{12} = -\omega^{21} = 1, \omega^{11} = \omega^{22} = 0$.
(2). $\mathbb{C}P^1$

$$f(r,\varphi) * g(r,\varphi) = f(\sqrt{\frac{2y^1 r(1+r^2) + r^2}{-2y^1 r(1+r^2) + 1}}, \varphi + \frac{y^2}{r})$$
$$\exp\Big(-\frac{i\hbar}{2}\overleftarrow{\frac{\partial}{\partial y^i}}\omega^{ij}\overrightarrow{\frac{\partial}{\partial y^j}}\Big) g(\sqrt{\frac{2y^1 r(1+r^2) + r^2}{-2y^1 r(1+r^2) + 1}}, \varphi + \frac{y^2}{r})\Big|_{y=0}.$$

(3). $\mathbb{C}H^1$

$$
\begin{aligned}
&f(r,\varphi)*g(r,\varphi)\\
=&f\Big(\sqrt{\frac{2y^1r(1-r^2)+r^2}{2y^1r(1-r^2)+1}},\varphi+\frac{y^2}{r}\Big)\exp\Big(-\frac{i\hbar}{2}\overleftarrow{\frac{\partial}{\partial y^i}}\omega^{ij}\overrightarrow{\frac{\partial}{\partial y^j}}\Big)\\
&g\Big(\sqrt{\frac{2y^1r(1-r^2)+r^2}{2y^1r(1-r^2)+1}},\varphi+\frac{y^2}{r}\Big)\Big|_{y=0}.
\end{aligned}
$$

The purpose of this article is to introduce another way to have explicit expressions of star products, in particular for Kähler manifolds, and to construct star products for $\mathbb{C}P^N$ and $\mathbb{C}H^N$ by using special functions.

2. Deformation quantization of Kähler manifolds

In this article, noncommutative space is constructed by using deformation quantization. In the following, we employ slight differenct deformation quantization from the one in the previous section. The deformation quantization is defined as follows.

Definition 2. Let $\mathcal{F}$ be a set of formal power series of $\hbar$:

$$\mathcal{F} := \Big\{ f \ \Big|\ f = \sum_k f_k \hbar^k \Big\}.$$

A star product of $f, g \in \mathcal{F}$ is defined as

$$f * g = \sum C_k(f,g)\hbar^k,$$

such that the product satisfies the following conditions.

(i) $*$ is associative product.
(ii) C_k is a bidifferential operator.
(iii) C_0 and C_1 is defined as

$$C_0(f,g) = fg, \qquad C_1(f,g) - C_1(g,f) = i\{f,g\}.$$

where $\{f,g\}$ is the Poisson bracket.

(iv) $f * 1 = 1 * f$.

The difference from the definition of the deformation quantization in the previous section is in the condition for C_1.

There is a star product called star product with separation of variables, which is defined on Kähler manifolds. A Kähler manifold have a Kähler potential Φ and a Kähler 2-form ω

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \qquad g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^j}.$$

$*$ is called a star product with separation of variables when

$$a * f = af$$

for holomorphic function a and

$$f * b = fb$$

for anti-holomorphic function b. Karabegov showed that for arbitrary ω, there exists a star product with separation variables $*$ [2] (see also [3, 4]). In this method of deformation qunatization, a star product is constructed as a formal power series of differential operators. Let L_f be a differential operator corresponding to a left $*$ multiplication by f:

$$L_f g := f * g.$$

Then L_f has the following form:

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n,$$

, where formal power series of differential operators

$$A_n = a_{n,\alpha}(f) \prod_i \left(D^{\bar{i}}\right)^{\alpha_i}, \qquad (D^{\bar{i}} = g^{\bar{i}j}\partial_j).$$

It is required that L_f satisfies

$$\begin{aligned} L_f 1 &= f * 1 = f, \\ L_f (L_g h) &= f * (g * h) = (f * g) * h = L_{L_f g} h. \end{aligned}$$

L_f which has the properties described above is determined by the following condition.

$$[L_f, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0,$$

and $A_0 = f$. This condition is equivalent to the recursion relations

$$[A_n,\ \partial_{\bar{i}}\Phi] = [\partial_{\bar{i}},\ A_{n-1}].$$

If one obtains the operator $L_{\bar{z}^i}$ $(L_{\bar{z}^l} f = \bar{z}^l * f)$, L_f is given by

$$L_f = \sum_\alpha \frac{1}{\alpha} \left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f \left(L_{\bar{z}} - \bar{z}\right)^\alpha.$$

Here, α is a multi-index, $\alpha = (\alpha_1, \cdots, \alpha_m)$. It is not easy to derive explicit expressions of star products in all order of $\hbar$ by solving the recursion relation. In the next section, this recursion relation is solved by using gamma functions for the case of $\mathbb{C}P^N$.

3. Noncommutative deformation of $\mathbb{C}P^N$

From this section to the end of this article, we studied noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$. The detail derivations of the following results are found in [5].

Let z^i $(i = 1, 2, \cdots, N)$ be inhomogeneous coordinates of $\mathbb{C}P^N$. Then, the Kähler potential of $\mathbb{C}P^N$ is given by

$$\Phi = \ln\left(1 + |z|^2\right), \qquad (|z|^2 = \sum_i z^i \bar{z}^i).$$

The complex metric $(g_{i\bar{j}})$ is derived from the Kähler potential as

$$ds^2 = 2g_{i\bar{j}} dz^i d\bar{z}^j,$$
$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1+|z|^2)\delta_{ij} - z^j \bar{z}^i}{(1+|z|^2)^2}$$

Its inverse metric $(g^{\bar{i}j})$ is

$$g^{\bar{i}j} = (1+|z|^2)\left(\delta_{ij} + z^j \bar{z}^i\right).$$

The following relations simplify our calculations of L_f in the case of $\mathbb{C}P^N$,

$$\partial_{\bar{i}_1}\partial_{\bar{i}_2}\cdots\partial_{\bar{i}_n}\Phi = (-1)^{n-1}(n-1)!\ \partial_{\bar{i}_1}\Phi\partial_{\bar{i}_2}\Phi\cdots\partial_{\bar{i}_n}\Phi,$$
$$\text{Riemann tensor:}\ \ R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}}.$$

Let us construct $L_{\bar{z}^l}$ $(L_{\bar{z}^l} f = \bar{z}^l * f)$. Recall that $L_{\bar{z}^l}$ is described as

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n A_n,$$

where A_n $(n \geq 2)$ is a formal series of $D^{\bar{k}}$. Then we would like to have the concrete form of A_n. We assume that A_n has the following form,

$$A_n = \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}} D^{\bar{l}},$$

where the coefficients $a_m^{(n)}$ do not depend on z^i and $\bar{z}^i$. From $[L_{\bar{z}^l}, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0$, A_n are recursively determined by

$$[A_n, \partial_{\bar{i}}\Phi] = [\partial_{\bar{i}}, A_{n-1}], \qquad (n \geq 2)$$

where $A_1 = D^{\bar{l}}$. After some calculations, we found the following recursion relation

$$a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1)a_m^{(n-1)}.$$

and $a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1$. To solve these equations, we introduce a generating function

$$\alpha_m(t) \equiv \sum_{n=m}^{\infty} t^n a_m^{(n)}, \qquad (m \geq 2).$$

From the recursion relation, $\alpha_m(t)$ is determined as

$$\alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1-t},$$
$$\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1-m+\frac{1}{t})}{\Gamma(1+\frac{1}{t})}, \qquad (m \geq 2).$$

The coefficient $a_m^{(n)}$ is related to the Stirling number of the second kind $S(n,k)$,

$$a_m^{(n)} = S(n-1, m-1).$$

Summarizing the above calculations, $L_{\bar{z}^l}$ becomes

$$\begin{aligned} L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\ &= \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\hbar) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}. \end{aligned}$$

Using these results, star products among z^i and $\bar{z}^i$ are obtained as

$$z^i * z^j = z^i z^j, \qquad z^i * \bar{z}^j = z^i \bar{z}^j, \qquad \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \tag{1}$$

$$\begin{aligned} \bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1+|z|^2) {}_2F_1 \left(1, 1; 1 - 1/\hbar; -|z|^2\right) \\ &\quad + \frac{\hbar}{1-\hbar} \bar{z}^i z^j (1+|z|^2) {}_2F_1 \left(1, 2; 2 - 1/\hbar; -|z|^2\right). \end{aligned} \tag{2}$$

Furthermore, we can derive an explicite formula for L_f for an arbitrary function f

$$L_f = \sum_{n=0}^{\infty} \frac{\alpha_n(\hbar)}{n!} g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) D^{\bar{k}_1} \cdots D^{\bar{k}_n}.$$

This satisfies $[L_f, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0$. This star product on $\mathbb{C}P^N$ is characterized by a function of $\hbar$, $\alpha_n(\hbar)$.

4. Fock representation of $\mathbb{C}P^N$

We have considered noncommutative $\mathbb{C}P^N$ in the formalism of deformation quantization until the previous section. In the context of deformation quantization, all objects in noncommutative $\mathbb{C}P^N$ are described as formal power series. In physical theories, phyisical quantities schould be expressed as convergent power seiries. Since much of finiteness or convergency in deformation quantization are not known, we change our strategy here. To introduce a calcurable framework, we consider a Fock representaion of $\mathbb{C}P^N$ in this section.

$\{z^i, \partial_j\Phi \mid i,j = 1,2,\cdots,N\}$ and $\{\bar{z}^i, \partial_{\bar{j}}\Phi \mid i,j = 1,2,\cdots,N\}$ constitute $2N$ sets of the creation-annihilation operators under the star product,

$$\begin{aligned} &\left[\partial_i\Phi,\ z^j\right]_* = \hbar\delta_{ij}, \qquad \left[z^i,\ z^j\right]_* = 0, \qquad \left[\partial_i\Phi,\ \partial_j\Phi\right]_* = 0, \\ &\left[\bar{z}^i,\ \partial_{\bar{j}}\Phi\right]_* = \hbar\delta_{ij}, \qquad \left[\bar{z}^i,\ \bar{z}^j\right]_* = 0, \qquad \left[\partial_{\bar{i}}\Phi,\ \partial_{\bar{j}}\Phi\right]_* = 0. \end{aligned}$$

So we can ascribe $\partial_i\Phi$, $\bar{z}^j$ as annihilation operators, and z^i, $\partial_{\bar{j}}\Phi$ as creation operators. Then, $e^{-\Phi/\hbar} = (1+|z|^2)^{-1/\hbar}$ plays the role of the vacuume projection :

$$\begin{aligned} &\partial_i\Phi * e^{-\Phi/\hbar} = \bar{z}^j * e^{-\Phi/\hbar} = 0, \quad e^{-\Phi/\hbar} * z^i = e^{-\Phi/\hbar} * \partial_{\bar{j}}\Phi = 0, \\ &e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar}. \end{aligned}$$

Let us introduce a class of functions which is constructed by acting the creation-operators on the vacuume projection:

$$\begin{aligned} M_{i_1\cdots i_m; j_1\cdots j_n} &:= c_{mn} z^{i_1} * \cdots * z^{i_m} * e^{-\Phi/\hbar} * \bar{z}^{j_1} * \cdots * \bar{z}^{j_n} \\ &= c_{mn} z^{i_1} \cdots z^{i_m} \bar{z}^{j_1} \cdots \bar{z}^{j_n} e^{-\Phi/\hbar}, \end{aligned}$$

where we choose $c_{mn} = 1/\sqrt{m!n!\alpha_m(\hbar)\alpha_n(\hbar)}$. These functions form a closed algebra:

$$M_{i_1\cdots i_m;j_1\cdots j_n} * M_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr}\delta^{k_1\cdots k_n}_{j_1\cdots j_n} M_{i_1\cdots i_m;l_1\cdots l_s},$$

where $\delta^{k_1\cdots k_n}_{j_1\cdots j_n}$ is defined as

$$\delta^{k_1\cdots k_n}_{j_1\cdots j_n} = \frac{1}{n!}\left[\delta^{k_1}_{j_1}\cdots\delta^{k_n}_{j_n} + \text{permutations of } (j_1,\cdots,j_n)\right].$$

We list the results of the operation of creation and anihilation operators to $M_{i_1\cdots i_m;j_1\cdots j_n}$ here.

$$\begin{aligned}
z^k * M_{i_1\cdots i_m;j_1\cdots j_n} &= \sqrt{\frac{m+1}{-m+1/\hbar}} M_{ki_1\cdots i_m;j_1\cdots j_n},\\
\partial_{\bar{k}}\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} &= \hbar\sqrt{(m+1)(-m+1/\hbar)} M_{ki_1\cdots i_m;j_1\cdots j_n},\\
\partial_k\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} &= \hbar\sqrt{\frac{-m+1+1/\hbar}{m}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n},\\
\bar{z}^k * M_{i_1\cdots i_m;j_1\cdots j_n} &= \frac{1}{\sqrt{m(-m+1+1/\hbar)}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n},\\
M_{i_1\cdots i_m;j_1\cdots j_n} * z^k &= \frac{1}{\sqrt{n(-n+1+1/\hbar)}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n},\\
M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_{\bar{k}}\Phi &= \hbar\sqrt{\frac{-n+1+1/\hbar}{n}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n}\\
M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_k\Phi &= \hbar\sqrt{(n+1)(-n+1/\hbar)} M_{i_1\cdots i_m;j_1\cdots j_n k},\\
M_{i_1\cdots i_m;j_1\cdots j_n} * \bar{z}^k &= \sqrt{\frac{n+1}{-n+1/\hbar}} M_{i_1\cdots i_m;j_1\cdots j_n k}.
\end{aligned}$$

Using this, we can construct field theories on noncommutative $\mathbb{C}P^N$ by replacing all field elements of usual commutative field theories in $\mathbb{C}P^N$ by matrices valued on these $M_{i_1\cdots i_m;j_1\cdots j_n}$. For example, we can define scalar field theories in this noncommutative $\mathbb{C}P^N$, and it is possible to construct scalar soliton solutions like GMS soliton in this formulation.

5. Noncommutative deformation of $\mathbb{C}H^N$

In this section, we consider the noncommutative defromation of $\mathbb{C}H^N$ by using Karabegov's deformation quantization with separation of varaibles. Any explicit expression of noncommutative $\mathbb{C}H^N$ ($N \geq 2$) is not given until now.

The Kähler potential of $\mathbb{C}H^N$ is given by

$$\Phi = -\ln\left(1 - |z|^2\right).$$

The metric $g_{i\bar{j}}$ and the inverse metric $g^{\bar{i}j}$ are defined by

$$\begin{aligned}
g_{i\bar{j}} &= \partial_i\partial_{\bar{j}}\Phi = \frac{(1-|z|^2)\delta_{ij} + \bar{z}^i z^j}{(1-|z|^2)^2},\\
g^{\bar{i}j} &= (1-|z|^2)\left(\delta_{ij} - \bar{z}^i z^j\right).
\end{aligned}$$

These are similar to the ones of $\mathbb{C}P^N$. The way to construct the star product is also as same as the one in section 3, so we skip the detail of the derivation. As similar to (1)-(2), star products between inhomogeneous coordinates are given as

$$\begin{aligned}
z^i * z^j =& z^i z^j, \\
z^i * \bar{z}^j =& z^i \bar{z}^j, \\
\bar{z}^i * \bar{z}^j =& \bar{z}^i \bar{z}^j, \\
\bar{z}^i * z^j =& \bar{z}^i z^j + \hbar\delta_{ij}(1-|z|^2)_2F_1\left(1,1;1+1/\hbar;|z|^2\right) \\
&- \frac{\hbar}{1+\hbar}\bar{z}^i z^j(1-|z|^2)_2F_1\left(1,2;2+1/\hbar;|z|^2\right).
\end{aligned}$$

The explicit representation of the star product with separation of variables on $\mathbb{C}H^N$ is given by

$$L_{\bar{z}^l} = \bar{z}^l + \sum_{m=1}^{\infty}(-1)^{m-1}\beta_m(\hbar)\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}},$$

with

$$\beta_n(t) = (-1)^n\alpha_n(-t) = \frac{\Gamma(1/t)}{\Gamma(n+1/t)}.$$

A Fock representation of $\mathbb{C}H^N$ is also given, similary. As in the case of $\mathbb{C}P^N$, sets $\{z^i, \partial_j\Phi\}$ and $\{\bar{z}^i, \partial_{\bar{j}}\Phi\}$ satisfy the commutation relations for the creation-annihilation operators. Also $e^{-\Phi/\hbar}$ is the vacuum projection operator,

$$\partial_i\Phi * e^{-\Phi/\hbar} = 0, \tag{3}$$
$$\bar{z}^i * e^{-\Phi/\hbar} = 0, \tag{4}$$
$$e^{-\Phi/\hbar} * \partial_{\bar{i}}\Phi = 0, \tag{5}$$
$$e^{-\Phi/\hbar} * z^i = 0, \tag{6}$$

and

$$e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar}. \tag{7}$$

As in the case of $\mathbb{C}P^N$, we consider a class of functions

$$N_{i_1\cdots i_m;j_1\cdots j_n} = \frac{z^{i_1}\ldots z^{i_m}\bar{z}^{j_1}\ldots\bar{z}^{j_n}}{\sqrt{m!n!\beta_m(\hbar)\beta_n(\hbar)}}e^{-\Phi/\hbar}$$

$N_{i_1\cdots i_m;j_1\cdots j_n}$ is totally symmetric under permutations of i's and j's, respectively. Then we can show that these functions form a closed algebra

$$N_{i_1\cdots i_m;j_1\cdots j_n} * N_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr}\delta^{k_1\cdots k_n}_{j_1\cdots j_n}N_{i_1\cdots i_m;l_1\cdots l_s}. \tag{8}$$

Moreover, the star products between $N_{i_1\cdots i_m;j_1\cdots j_n}$ and one of $z^k, \partial_k\Phi, \bar{z}^k$ and $\partial_{\bar{k}}\Phi$ are calculated as follows,

$$z^k * N_{i_1\cdots i_m;j_1\cdots j_n} = \sqrt{\frac{m+1}{m+1/\hbar}} N_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{9}$$

$$\partial_k\Phi * N_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{\frac{m-1+1/\hbar}{m}} \sum_{l=1}^{m} \delta_{ki_l} N_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{10}$$

$$\bar{z}^k * N_{i_1\cdots i_m;j_1\cdots j_n} = \frac{1}{\sqrt{m(m-1+1/\hbar)}} \sum_{l=1}^{m} \delta_{ki_l} N_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{11}$$

$$\partial_{\bar{k}}\Phi * N_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{(m+1)(m+1/\hbar)} N_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{12}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * z^k = \frac{1}{\sqrt{n(n-1+1/\hbar)}} \sum_{l=1}^{n} \delta_{kj_l} N_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n}, \tag{13}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \partial_k\Phi = \hbar\sqrt{(n+1)(n+1/\hbar)} N_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{14}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \bar{z}^k = \sqrt{\frac{n+1}{n+1/\hbar}} N_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{15}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \partial_{\bar{k}}\Phi = \hbar\sqrt{\frac{n-1+1/\hbar}{n}} \sum_{l=1}^{n} \delta_{kj_l} N_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n}. \tag{16}$$

6. Summary

In this article, we studied noncommutative deformation of several manifolds by using deformation quantization. We started from a noncommutative $\mathbb{R}^d$ by using the Moyal product and we moved into more genaral manifolds deformed with star products. For most manifolds, star products can not be described in explicit expression, but they are determined recursively. As exceptions, $\mathbb{R}^2$, $\mathbb{C}P^1$ and $\mathbb{C}H^1$ cases were reviewd, and their star products are given by the method of Fedosov's deformation quantization. In this article we introduced new examples of noncommutative manifolds with explicit star products, that is Karabegov's deformation quantization of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ with separation of variables. Explicit calulable star products were given by using gamma functions. Using the results of star products between the inhomogeneous coordinates, we found the creation and anihilation operators and obtained the Fock representations of the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$. Noncommutative $\mathbb{C}P^N$ is also studied in [6, 7, 8, 9, 10], and noncommutative $\mathbb{C}H^1$ is constructed in [11]. This is the first time to construct an explicit noncommutative $\mathbb{C}H^N$ for $N \geq 2$.

How can we construct physical theories on the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$? There are a lot of problems about finiteness, convergency and so on, in deformation quantization. Fock representations discussed in this article provide one of good ways to make physical field theories on the noncommutative manifolds. Indeed, few soliton solutions are given on such noncommutative manifolds. It is important to construct frameworks of physics in the noncommutative manifolds defined by deformation quantization.

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