SLAC-PUB-2765
June 1981
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FRACTIONAL QUANTUM NUMBERS ON SOLITONS*

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ABSTRACT

A method is proposed to calculate quantum numbers on solitons in quantum field theory. The method is checked on previously known examples and, in a special model, by other methods. We find, for example, that the fermion number on kinks in one dimension or on magnetic monopoles in three dimensions is, in general, a transcendental function of the coupling constant of the theories.

Submitted to Physical Review Letters

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* Work supported in part by the Department of Energy under contract DE-AC03-SF00515 and by the NSF Grant PHY 77-27084.

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Peculiar quantum numbers have been found to be associated with solitons in several contexts:

(i) The soliton provides, of course, a different background than the usual vacuum around which to quantize other fields. The difference between these "vacuum polarizations" may induce unusual quantum numbers localized on the soliton.[1-3]

(ii) Solitons may require unusual boundary conditions on the fields interacting with them, in particular leading to conversion of internal quantum numbers into rotational quantum numbers.[4-6]

(iii) In the case of dyons, there is classically a family of solitons with arbitrary electric charge. The determination of which of these are in the physical spectrum requires quantum-mechanical considerations and brings in the θ-parameter of non-Abelian gauge theories.[7,8]

At present all these phenomena seem distinct although there are suggestive relationships. In this note, we shall concentrate on (i), proposing a general method of analysis and working out a few examples.

Polymer Chains: An intuitively appealing, and perhaps physically realizable, example of the phenomena we are addressing are the fractionally charged solitons on polyacetylene.[2,3,9] A caricature model of a polyacetylene molecule is shown in Fig. 1(a) - in the ground state we have alternating single and double bonds, which may be arranged in two inequivalent but degenerate forms A and B. If there is an imperfection, as shown in Fig. 1(b), we go from A on the left to B on the right. This configuration cannot be brought to either pure A or pure B by any finite rearrangement of electrons, so it will relax to a stable configuration -

a soliton. If we put two imperfections together, as in Fig. 1(c), we find a configuration which begins and ends as A. Compared to the corresponding segment of pure A, it is missing one bond. If we add an electron to the two-imperfection strand, we can deform this configuration by a finite rearrangement into a pure A strand. (We are pretending, for simplicity, that each bond represents a single electron instead of a pair.) Interpreting this, we see a two-soliton state is equivalent to the ground state if we add an electron. Thus, by symmetry, each separated soliton must carry electron number -1/2 (and electric charge +1/2 e).

We can relate these stick-figure pictures of polyacetylene to field theory as follows: Let $d_1 > d_2$ be the internuclear distances characterizing single, respectively double, bonds. Define a scalar field which is a function of the link i by $\phi_i = (-1)^i(d - \frac{1}{2}d_1 - \frac{1}{2}d_2)$, where d is the internuclear distance for link i. Thus in the A configuration $\phi_i = \frac{1}{2}(d_1 - d_2)$ (independent of i), in the B configuration $\phi_i = -\frac{1}{2}(d_1 - d_2)$, and in the soliton configuration ϕ_i interpolates between these values. Now we can show that it makes sense to approximate ϕ_i by a continuum field and the interactions of the electrons with ϕ (a charge-density wave) by $\mathscr{L}_I = g\bar{\psi}\gamma^5\phi\psi$, furthermore the electrons can be treated for present purposes (near the Fermi energy) as relativistic particles.

In this formulation, we make contact with the work of Jackiw and Rebbi.[1] They found that the spectrum of the Dirac equation in the presence of a soliton contains a zero-energy solution. By symmetry, this solution is composed of (projects onto) half a positive-energy and half a negative-energy solution with respect to the normal ground state. Thus if we fill the zero-energy level, we have a soliton state with electron number +1/2, if we leave it empty, the electron number is -1/2.

Su and Schrieffer have described a generalization,[10] which occurs in a chain with a repeating unit of single-single-double bonds, as in Fig. 2. A slight modification of the discussion of Fig. 1 shows that we now have solitons which can be added in triples to give the normal ground state, deficient by one electron. We expect the electron number of a single soliton to be -1/3.

A field theoretic model must now have essentially new features. Jackiw and Rebbi emphasized that in their model the Dirac equation in the presence of a soliton has a charge conjugation symmetry, and then their interpretation of the zero modes cannot account for any charges other than half-integral. Thus we will consider models where the background destroys all symmetries which interchange positive and negative energy solutions of the Dirac equation.

Adiabatic Charge Flow: Our method of calculating the soliton quantum number will be to imagine building up the soliton by slow changes in fields, starting from the ground state. In order to reach the solitons by slow changes, we may have to enlarge the field space during intermediate stages, as we shall see. In any case, for slow variations of fields in space and time, we can readily compute the flow of the appropriate charge in the no-particle state. We then simply integrate to find the accumulated charge on the soliton.

Let us illustrate these remarks on a concrete example. We consider, in 1+1 dimensions, massless fermions interacting with two scalar fields ϕ_1 and ϕ_2 as follows:

$$\mathscr{L}_I = g\bar{\psi}\left(\phi_1 + i\gamma_5\phi_2\right)\psi \quad . \tag{1}$$

Now if ϕ_1 and ϕ_2 are slowly varying in space and time, i.e., their gradients are $\ll g\left(\phi_1^2+\phi_2^2\right)^{\frac{1}{2}}$, we may conveniently calculate the change in the expected value of $j^\mu = \bar{\psi}\gamma^\mu\psi$ in the no-particle state by considering the Feynman graph of Fig. 3. Since the interaction (1) is chirally invariant, we may first suppose that only $\phi_1 \neq 0$ at a given point, and then express the result in a chirally symmetric form. We then need only do a very simple calculation for an effectively massive fermion to find

$$\langle j^\mu \rangle = \frac{1}{2\pi}\,\varepsilon^{\mu\nu}\,\varepsilon_{ab}\,\frac{\phi_a\,\partial_\nu\,\phi_b}{|\phi|^2} = \frac{1}{2\pi}\,\varepsilon^{\mu\nu}\,\partial_\nu \tan^{-1}\frac{\phi_2}{\phi_1} \quad . \tag{2}$$

If the scalar fields do not propagate (they represent very massive particles) more complicated graphs need not be considered.

If in the end we reach the soliton state by slow changes, we need only to evaluate (2) to find the fermion number charge on the soliton. It is important to remark that the resulting state will be a true eigenstate of the charge, not a superposition of states of different charge (even though we only derived an expectation value). For this it is only necessary to note that there are no degenerate states of different charge. In this the localized charge on a soliton differs from, for instance, the "localized charges" of 1/2 on the top and bottom of an ammonia ion.

Two general features of the result deserve comment. First, the divergence $\partial_\mu j^\mu$ vanishes identically, reflecting the conservation of fermion number. Second, the charge $Q = \int j^o\, dx^1 = \frac{1}{2\pi}\,\Delta\left(\tan^{-1}\phi_2/\phi_1\right)$ is independent of the coupling constant g and depends only on the values of ϕ_1 and ϕ_2 at spatial infinity.

We can represent a massive fermion by fixing $\phi_1 = m/g$. If the theory supports a soliton for which $\phi_2(x) \to \pm v$ as $x \to \pm\infty$, we find

$$Q = \frac{1}{\pi} \tan^{-1} \frac{gv}{m} \quad . \tag{3}$$

Notice this is a transcendental function of the couplings! As $m \to 0$, we find $Q \to 1/2$; this is the Jackiw-Rebbi case of a single (linear) scalar coupling. The limit $m \to 0$ is delicate just because there are two degenerate states of charge $\pm 1/2$ in the limit. If we take $m = 0$ from the beginning adiabatic changes will fill these equally on an average. The current would vanish. A slight perturbation lifts the degeneracy. Of course the charge $-1/2$ state is reached by letting $m \to 0$ through negative values.

A field theory version of the chains of Figs. 1 and 2 is the interaction

$$\mathscr{L}_I = g\bar{\psi} e^{i\theta\gamma_5} \psi \tag{4}$$

for which we find

$$\langle j^\mu \rangle = \frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \theta \quad , \qquad Q = \frac{1}{2\pi} \Delta\theta \quad . \tag{5}$$

The solitons with θ varying from 0 to π (so two together give $0 \to 2\pi \sim 0$, equivalent to vacuum) have charge 1/2, with θ varying from 0 to $2\pi/3$ charge 1/3, etc.

<u>Bosonization</u>: Some 1+1 dimensional models become especially transparent if the method of bosonization is employed. In 1+1 dimensions, one can rewrite fermion fields as nonlocal expressions in boson fields.[11] Some bilinears transform in a simple local way, however:

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi \to \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$\bar{\psi}\gamma^\mu\psi \to \varepsilon^{\mu\nu}\partial_\nu\phi/\sqrt{\pi}$$

$$\bar{\psi}\psi \to \mu\cos 2\sqrt{\pi}\;\phi$$

$$i\bar{\psi}\gamma_5\psi \to \mu\sin 2\sqrt{\pi}\;\phi$$

(μ = arbitrary scale parameter). Thus the interaction (4) becomes in this representation $\mathscr{L}_I = g\mu\cos(2\sqrt{\pi}\,\phi - \theta)$. Now if θ in a soliton varies by $\Delta\theta$ from $-\infty$ to $+\infty$, the potential $-\mathscr{L}$ is minimized when $\phi = \theta/2\sqrt{\pi}$; in particular, $\Delta\phi = \Delta\theta/2\sqrt{\pi}$. Integrating $\bar{\psi}\gamma^0\psi = \partial_1\phi/\sqrt{\pi}$, we find the charge $\Delta\theta/2\pi$, as from our earlier derivation.

<u>3 + 1 Dimensional σ-Model</u>: Although the σ-model proper does not support finite-energy solitons, we can consider a fermion interacting with external fields of this type. This proves useful as a warm-up for the gauge theory monopoles to be discussed shortly.

The interaction Lagrangian is of standard form:

$$\mathscr{L}_I = g\bar{\psi}\left(\phi_0 + i\vec{\phi}\cdot\vec{\tau}\gamma_5\right)\psi$$

with ψ an isodoublet fermion field.

We compute the induced current as in the 1+1 dimensional examples, from graphs as in Fig. 3. A straightforward calculation leads to

$$\langle j^\mu\rangle = \frac{1}{12\pi^2|\phi|^4}\,\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{dabc}\,\phi_d\partial_\alpha\phi_a\partial_\beta\phi_b\partial_\gamma\phi_c \quad . \tag{6}$$

With this form, $\partial_\mu\langle j^\mu\rangle \equiv 0$. This, of course, indicates that only the behavior at spatial infinity determines the charge, since changes in the fields in a finite volume lead only to current flows in a finite volume and therefore do not change the total charge. In fact, if we take $\phi_0 = m/g$; $\phi_a = \hat{\phi}_a(\vec{x})f(t)$, $a = 1,2,3$, where $\hat{\phi}_a(\vec{x}) \to vx_a/|x|$ as $|x| \to \infty$, and evaluate the current flow at infinity, we find a fermion number

$$\frac{1}{\pi}(\theta - \sin\theta\cos\theta) \quad , \qquad \tan\theta = \frac{gv}{m} \tag{7}$$

which $\to 1/2$ as $m \to 0$.

Gauge Model, Magnetic Monopoles: We may extend this analysis in a simple way to the monopole solutions of non-Abelian gauge theories by simply gauging the SU(2) × SU(2) chiral symmetry of our σ-model. In the end, we can specialize by setting the axial gauge fields to zero, and fixing a fermion mass (ϕ_0 = constant).

The expression (6) for the current is changed in the first instance by the conversion of ordinary to covariant derivatives, $\partial \to \nabla \equiv \partial + eA$. This is not sufficient, however, since this minimally modified current is not conserved. The current

$$\langle j^\mu \rangle = \frac{1}{12\pi^2}\,\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{dabc}\left[\frac{\phi_d}{|\phi|^4}(\nabla_\alpha\phi)_a(\nabla_\beta\phi)_b(\nabla_\gamma\phi)_c + \frac{3}{4}\,eF_{\alpha\beta,ab}\,\frac{\phi_d}{|\phi|^2}(\nabla_\gamma\phi)_c\right] \tag{8}$$

obeys $\partial_\mu\langle j^\mu\rangle \equiv (-e^2/128\pi^2)\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcd}F_{\alpha\beta,ab}F_{\gamma\delta,cd}$. This is the expected anomaly and vanishes when we have only vector gauge fields as in the monopole. The coefficient of the second term in (8) can be checked by evaluation of the diagram in Fig. 3 with one gauge field vertex inserted.

We now take ϕ as before and $A_{ab} = \hat{A}_{ab}(\vec{x})$; $A_{ao} = 0$, $a,b = 1,2,3$, where $\hat{\phi}$ and $\hat{A}$ are the monopole fields, and find the current flow at infinity. Since $(\nabla_i\phi)_a = 0$ at infinity, the only contribution comes from taking

$\gamma = d = 0$ in the second term of (8) and gives for the fermion number

$$\frac{e\Phi}{4\pi^2} \tan^{-1} \frac{gv}{m} \tag{9}$$

where Φ is the magnetic flux out of the sphere at infinity. Since $e\Phi = 4\pi$, this gives fermion number 1/2 when $m \to 0$!

Remarks: The direct utility of our results for particle physics is highly problematical. Even if magnetic monopoles were found, their fermion number is not a reasonable quantity in standard theories. (In principle, we could imagine coupling a U(1) gauge field to the fermion number, so the calculation is not entirely content-free!) We do think the results are an interesting curiosity in quantum field theory and as such may eventually be useful. It is likely that kindred, but experimentally accessible, effects do arise in condensed matter systems.

ACKNOWLEDGEMENTS

We are especially grateful to J. R. Schrieffer for interesting us in this problem and to L. Susskind for reminding us of the bosonization method. This work was started while the first author was at the Santa Barbara Institute.

REFERENCES

1. R. Jackiw and C. Rebbi, Phys. Rev. D13, 3398 (1976).
2. W. P. Su, J. R. Schrieffer and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979), and Phys. Rev. B22, 2099 (1980).
3. R. Jackiw and J. R. Schrieffer, Santa Barbara preprint NSF-ITP-81-01.
4. P. Hasenfratz and G. 't Hooft, Phys. Rev. Lett. 36, 1119 (1976).
5. R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1116 (1976).
6. F. Wilczek, in preparation (discussion of fractional angular momentum around strings).
7. E. Witten, Phys. Lett. 86B, 283 (1979).
8. F. Wilczek, in preparation (dyons in the presence of fermionic matter).
9. For a discussion of both theoretical and experimental aspects of polyacetylene, see A. Heeger, Comments on Solid State Physics X, 53 (1981).
10. W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. 46, 738 (1981).
11. See, e.g., S. Mandelstam, Phys. Rev. D11, 3026 (1975) and S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. 93, 267 (1975).

FIGURE CAPTIONS

Fig. 1. (a) The two degenerate ground states for electronic structure of polyacetylene.

(b) An imperfection interpolating between the two ground states.

(c) A chain with two imperfections.

Fig. 2. A form of polymer with single-single-double bond pattern in the ground state.

Fig. 3. Vacuum polarization graphs for evaluation of induced currents.

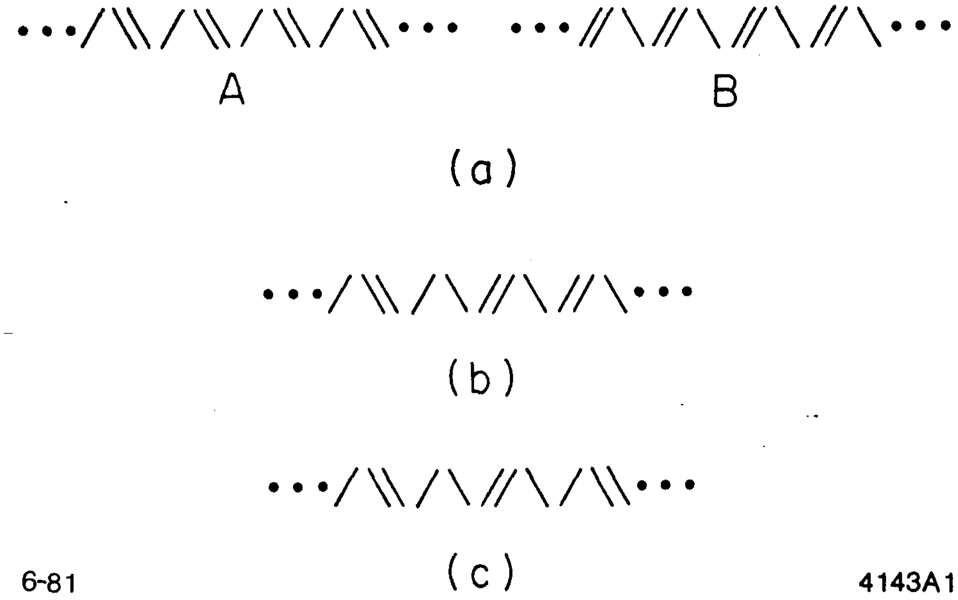

Fig. 1

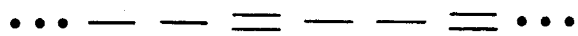

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Fig. 2

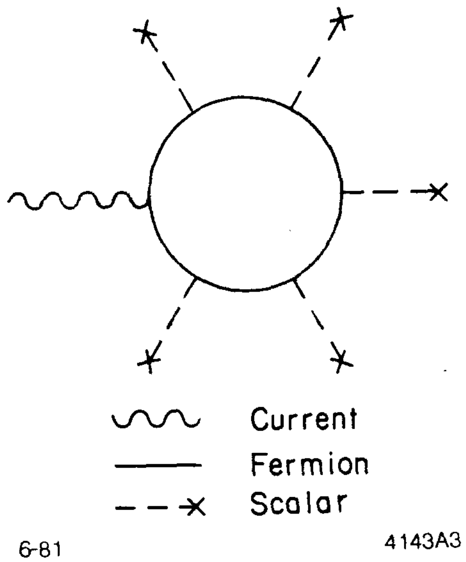

Fig. 3