Physical Significance of Correlated and Squeezed States

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During last two decades the so called *squeezed* states were the subject of increasing flow of papers in various fields of quantum physics. It seems rather difficult to cite the first paper, since the history of these states goes back to Schrödinger [1], who first investigated Gaussian wave packets in quantum mechanics. Another important step was made by Glauber [2] who introduced the concept of *coherent* states. A lot of papers were devoted to various generalizations of Glauber's states. We mention only a few of them [3]–[8]. More complete lists of references can be found, e.g., in [9]–[12]. Different authors invented different names for new types of quantum states: *minimum uncertainty states*, *two-photon states*, etc. Now these states are usually called *squeezed states.* They are in fact nothing but Gaussian wave packets in corresponding representations, and different names describe different characteristic features of these states.

In ref. [13] the concept of *correlated states* was introduced. These states correspond to the minimal possible value of the lefthand side of the Robertson–Schrödinger uncertainty relation [14,15]

$$\sigma_{\mathrm{p}}\,\sigma_{\mathrm{q}}\,(\,1-\mathrm{r}^2\,) \geq \hbar^2/4 \ . \tag{1}$$

Here σ_{p} and σ_{q} are variances of the momentum and coordinate operators: $\sigma_{\mathrm{q}} = \langle \widehat{\mathrm{q}}^2 \rangle - \langle \widehat{\mathrm{q}} \rangle^2$, while r is the dimensionless *correlation coefficient*:

$$\mathrm{r} = \frac{\sigma_{\mathrm{pq}}}{(\,\sigma_{\mathrm{p}}\sigma_{\mathrm{q}}\,)^{1/2}}\,,\ \sigma_{\mathrm{pq}} = \frac{1}{2}\,\langle \widehat{\mathrm{q}}\widehat{\mathrm{p}} + \widehat{\mathrm{p}}\widehat{\mathrm{q}} \rangle - \langle \widehat{\mathrm{q}} \rangle \langle \widehat{\mathrm{p}} \rangle\,,\ |\mathrm{r}| < 1 \tag{2}$$

In the special case of $\mathrm{r} = 0$ (1) turns into the more familiar Heisenberg–Weyl uncertainty relation.

The explicit form of the correlated state wave function in the coordinate representation is as follows,

$$\Psi_\beta\,(\,x|\mathrm{r}\,,\,\eta\,) = \mathrm{N}_\beta\ \exp\left[-\frac{x^2}{4\eta^2}\left(1-\frac{i\mathrm{r}}{\sqrt{1-\mathrm{r}^2}}\right)+\frac{\beta x}{\eta}\right] \ . \tag{3}$$

Here N_β is the coordinate–independent part of the wave function (including the normalization factor), $\eta \equiv \sigma_{\mathrm{q}}$, β is a complex parameter. Function (3) satisfies the equation

$$\widehat{\mathrm{b}}\,\Psi_\beta = \beta\,\Psi_\beta \tag{4}$$

with

$$\widehat{\mathrm{b}} = \frac{1}{2\eta}\left(1 - \frac{\mathrm{ir}}{\sqrt{1-\mathrm{r}^2}}\right)\widehat{\mathrm{q}} + \mathrm{i}\eta\widehat{\mathrm{p}} \tag{5}$$

Beginning with eq. (3) we use dimensionless variables and assume $\hbar = 1$. If we define the *basic* annihilation operator as

$$\widehat{\mathrm{a}} = \frac{\widehat{\mathrm{q}} + \mathrm{i}\widehat{\mathrm{p}}}{\sqrt{2}} \tag{6}$$

then the relation between two systems of operators is as follows,

$$\widehat{\mathrm{b}} = u\widehat{\mathrm{a}} + v\widehat{\mathrm{a}}^+ \,,\ |u|^2 - |v|^2 = 1\,,\ [\,\widehat{\mathrm{a}}\,,\widehat{\mathrm{a}}^+\,] = [\,\widehat{\mathrm{b}}\,,\widehat{\mathrm{b}}^+\,] = 1\,, \tag{7}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}\,\eta}\left(1 - \frac{\mathrm{ir}}{\sqrt{1-\mathrm{r}^2}}\right) \pm \frac{\eta}{\sqrt{2}} \tag{8}$$

Just relations like (5) or (7) were the bases for introducing the new types of coherent states (*new*, *generalized*, *minimum uncertainty*, *two-photon*, *squeezed*, etc.) in refs. [4]–[12]. Therefore the mathematical grounds of all investigations in this direction are the same — linear canonical transformations and their properties.

Here we would like to elucidate the physical significance of correlated states and to give in this connection a brief review of our recent papers. (The mathematical interrelations between different generalizations of Glauber's coherent states were given in ref. [16].)

The main field of applications of both usual and generalized coherent states is quantum optics. But in quantum optics experiments people measure usualy not variances of quadrature components σ_{q} and σ_{p} but distribution functions of quanta. If quantum number eigenstates (Fock's states) are defined in terms of the *basic* operators (6):

$$\widehat{\mathrm{a}}^+\,\widehat{\mathrm{a}}\,|\,\mathrm{n}\,\rangle = \mathrm{n}\,|\,\mathrm{n}\,\rangle\,, \tag{9}$$

then the distribution function of quanta in a state Ψ is nothing but $\mathrm{W_n} = |\mathrm{c_n}|^2$, where $\mathrm{c_n}\,,\ \mathrm{n} = 0\,,1\,,2\,,\ldots$ are coefficients of the expansion

$$\Psi = \sum \mathrm{c_n}\,|\,\mathrm{n}\,\rangle \tag{10}$$

In the case of the Glauber state Ψ_α satisfying the equation

$$\widehat{\mathrm{a}}\,\Psi_\alpha = \alpha\,\Psi_\alpha \tag{11}$$

we have the simple Poisson distribution function

$$\mathrm{W_n} = \frac{|\alpha|^{2\mathrm{n}}}{\mathrm{n}!}\,\mathrm{e}^{-|\alpha|^2} \tag{12}$$

This distribution possesses the single peak at the value $\mathrm{n_{max}} \approx \langle \mathrm{n} \rangle = |\alpha|^2$. Quite different situation takes place for correlated or squeezed states satisfying eqs. (4), (5), (7), (8). In this case [17]–[19]

$$\mathrm{W_n} = \frac{1}{|u|\,\mathrm{n}!}\,|\frac{v}{2u}|^{\mathrm{n}}\,|\mathrm{H_n}\left(\frac{\beta}{\sqrt{2uv}}\right)|^2\,\exp\left[-|\beta|^2 + \mathrm{Re}\left(\frac{\beta^2\,v^*}{u}\right)\right]\,, \tag{13}$$

$H_n(x)$ being the Hermite polynomial. In spite of a simple analytic expression distribution (13) as a function of the integral variable n shows rather irregular behaviour. For certain combinations of parameters u, v, β (or η, r, β) its graph has many peaks separated by intervals in the n–axis with almost or even exactly zero values of the probabilities. Moreover, the situations are possible when the probability to detect the number of quanta which is equal to their average value exactly equals zero. Besides, the distribution function usually highly oscillates with the period $\Delta n = 2$, and both situations when the probability is large for even values of n and for odd values of n are possible. For the details see refs. [17]–[19]. Such a complicated structure of the distribution function can be used for detecting the system in the correlated or squeezed states.

One of the possible ways to create the correlated or squeezed state is to use some parametric process. Indeed, if we consider the Schrödinger equation with a time–dependent Hamiltonian like

$$\widehat{H}(t) = \frac{1}{2}\widehat{p}^2 + \frac{1}{2}\Omega^2(t)\widehat{q}^2 , \tag{14}$$

then its general solution (first obtained by Husimi [20]) is as follows,

$$\Psi(x, t) \sim \exp\left(\frac{i\dot{\epsilon}}{2\epsilon}x^2 + \cdots\right) , \tag{15}$$

where $\epsilon(t)$ is a solution of the classical equation of motion

$$\ddot{\epsilon}(t) + \Omega^2(t)\epsilon(t) = 0 \tag{16}$$

satisfying the subsidiary condition

$$\dot{\epsilon}\epsilon^* - \dot{\epsilon}^*\epsilon \equiv 2i \tag{17}$$

Comparing this expression with eq. (3) we see that solution (15) is just the correlated state with the correlation coefficient

$$r^2 = 1 - |\epsilon\dot{\epsilon}|^{-2} \tag{18}$$

and the squeezing coefficient

$$k = (\sigma_q / \sigma_p)^{1/2} = |\epsilon / \dot{\epsilon}| \tag{19}$$

Using eqs. (16), (18) we can find such a dependence $\Omega(t)$ for which, e.g., the correlation coefficient does not depend on time [21]:

$$\Omega(t) = (2rt + \mathrm{const})^{-1} \tag{20}$$

The corresponding quantum state is in fact unsqueezed, because

$$k_\Omega = \Omega(t)(\sigma_q / \sigma_p)^{1/2} \equiv 1 , \ \sigma_q = (2\Omega)^{-1}(1 - r^2)^{-1/2} . \tag{21}$$

Note that just the modified squeezing coefficient k_Ω describes the squeezing properly, since the ground–state variances of the oscillator with the unit mass and frequency Ω are as follows: $\sigma_q = \hbar / 2\Omega$, $\sigma_p = \Omega\hbar / 2$. This example shows distinctly the difference between the correlated and squeezed states: the state may be correlated but unsqueezed. If we shall look for the cases when $k = \text{const}$, then two dependences $\Omega(t)$ are possible. The first one is $\Omega = k^{-1}$: then we have unsqueezed ($k_\Omega = 1$) and noncorrelated ($r = 0$) states, i.e., usual coherent states. The second case corresponds to the unstable system with $\Omega^2 = -k^2$, when

$$\sigma_q = \frac{1}{2} k \cosh(2t/k), \; \sigma_p = (2k)^{-1} \cosh(2t/k), \; r = \tanh(2t/k) \qquad (22)$$

Eq. (16) itself can be considered as the one–dimensional Helmholtz equation describing the propagation of some *wave* through the *potential barrier* $\Omega^2(x)$. If the corresponding energy reflection coefficient from this barrier R is known (it is supposed that $\Omega(x) = \text{const}$ when $x \to \pm\infty$), then relations (17)–(19) lead to the following limitations on the possible values of the correlation and squeezing coefficients [17,21]:

$$|r| \le \frac{2\sqrt{R}}{1+R}, \; \frac{1-\sqrt{R}}{1+\sqrt{R}} \le k \le \frac{1+\sqrt{R}}{1-\sqrt{R}}, \; k_{max} = \sqrt{\frac{1+r_{max}}{1-r_{max}}} \qquad (23)$$

Suppose that the final frequency $\Omega_f = \Omega(t \to +\infty) = 1$ (this means that we normalize all quantities by the value of Ω_f). Introducing the energy of fluctuations in the final states $E_f = \frac{1}{2}(\sigma_p + \sigma_q)$ and taking into account the relations [22]

$$\sigma_q = \frac{1}{2} |\epsilon(t)|^2, \; \sigma_p = \frac{1}{2} |\dot{\epsilon}(t)|^2 \qquad (24)$$

one can obtain from eq. (18) the following inequality:

$$r \le \left(1 - \frac{1}{4E_f^2}\right)^{1/2} \qquad (25)$$

Linear canonical transformation (7) generating correlated or squeezed states is determined by two complex parameters satisfying one subsidiary condition. Thus we have three real parameters. However, one parameter is trivial — it is a common phase of complex numbers u and v which has no physical significance. Hence only two parameters determine in fact transformation (7) and wave function (3). In the most general case of a time–dependent Hamiltonian like (14) these parameters are independent, as was demonstrated above. But in the special case of the oscillator with a time–independent frequency the additional symmetry connected with the arbitrariness of choosing the initial moment of time arises. Therefore in this case parameters k and r are not independent: they are periodic functions of time and can be expressed, e.g., through the maximum values of the squeezing or correlation coefficients k_{max} or r_{max} (these coefficients, in turn, are related by the last equality in (23)). So in the stationary case there is no essential difference between correlated and squeezed states: they periodically turn into each other.

A simple geometrical picture illustrates the situation. If we calculate the Wigner function corresponding to the wave function (3), it will be again the Gaussian exponential. Consequently the curve $W(q, p)) = \text{const}$ is an ellipse.

When the axes of this ellipse are parallel to the coordinate axes in the phase space, then the correlation coefficient equals zero, so that we have a squeezed noncorrelated state. The correlation and squeezing coefficients are connected with the angle φ between the main axis of the ellipse and the coordinate axis by means of the relation

$$\tan(2\varphi) = \frac{\mathrm{kr}}{\mathrm{k}^2 - 1} \tag{26}$$

But in the case of the constant oscillator frequency the ellipse of constant values of the quasiprobability rotates as the rigid body, i.e., without changing its shape, with the angular velocity Ω. Thus squeezed and correlated states periodically turns into each other.

Another situation holds in the case of a time-dependent frequency. Then the shape of the ellipse also changes in time, but its area remains constant due to the conservation of the so-called *universal quantum invariant* [23]

$$\mathrm{I} = \sigma_\mathrm{p}\,\sigma_\mathrm{q} - \sigma_\mathrm{pq}^2 = \mathrm{const} \tag{27}$$

(the quantum analogue of the classical Liouville theorem on the conservation of the phase volume of Hamiltonian systems).

There are many different methods of generating squeezed states of some distinguished mode of electromagnetic field inside a resonator [11]. Recently we have considered a new approach to this problem based on the parametric excitation due to the periodic motion of resonator's walls (with the twice frequency with respect to the mode eigenfrequency) [24]. The following expression for the variance of some quadrature components was found:

$$\sigma_\mathrm{p} = \frac{1}{2}\exp(\pm 2z),\; z = \frac{1}{4}\pi \mathrm{aN} \ll 1\ , \tag{28}$$

where a is the relative amplitude of wall's vibrations (it is assumed to be small enough), and N is the number of half-periods of vibrations. The correlations coefficient in this case is nonzero, but approximately constant in time: $|\mathrm{r}| \approx \mathrm{a}/4$.

In ref. [25] the power of the spontaneous electromagnetic radiation from an oscillator or system of oscillators moving along an arbitrary trajectory was calculated in the most general case of a quite arbitrary quantum state of the oscillators, and the following formula was obtained for an oscillator in the correlated squeezed state:

$$\mathrm{P} = \mathrm{P}_{\mathrm{classical}} + \frac{\mathrm{e}^2\omega^3\hbar}{6\mathrm{m}\mathrm{c}^3}\left[\xi + \frac{1}{\xi(1-\mathrm{r}^2)} - 2\right],$$
$$\xi = \frac{2\mathrm{m}\omega}{\hbar}\sigma_\mathrm{q} = \frac{\sigma_\mathrm{q}}{\sigma_\mathrm{q}\,(\mathrm{groundstate})}\ . \tag{29}$$

Here $\mathrm{P}_{\mathrm{classical}}$ is the radiation power of the classical oscillator whose coordinates coincides with the average value of the coordinate of the quantum oscillator. We see that the correlation coefficient affects essentially on the quantum part of the radiation power:

$$\Delta\mathrm{P}_{\mathrm{quant}} \geq \frac{\mathrm{e}^2\hbar\omega^3}{3\mathrm{m}\mathrm{c}^3}\left[\frac{1}{\sqrt{1-\mathrm{r}^2}} - 1\right] \tag{30}$$

Another group of physical phenomena in which correlation and squeezing may manifest themselves corresponds to the wave packets propagation and tunneling through potential barriers. If we consider the spreading of the wave packet (3) in the empty space (without external potentials), then for sufficiently large values of time its shape will be as follows,

$$|\Psi(x, t \to \infty)|^2 = \left[\frac{2\eta^2(1-r^2)}{\pi t^2 \hbar^2}\right]^{1/2} \exp\left[-\frac{2\eta^2(1-r^2)}{t^2\hbar^2}(x - vt)^2\right] \quad (31)$$

We see that correlated wave packets with $r = 0$ looks like uncorrelated one, provided Planck's constant $\hbar$ is replaced by *effective* Planck's constant $\hbar_* = \hbar / \sqrt{1-r^2}$ in full accordance with generalized uncertainty relation (1). Therefore one might suppose that quantum effects may manifest themselves in correlated states more distinctly than in uncorrelated ones. In particular one could imagine that tunneling through potential barriers for correlated states is more effective than for uncorrelated ones. The reality, however, is more complicated. For example, the transmission coefficient through a rectangular potential barrier does not depend on the correlation coefficient at all (at any rate in the leading terms of the WKB formulae). The exact solution can be obtained for the parabolic barrier $U(x) = -\frac{1}{2}\Omega^2 x^2$. However, in this case the transmission coefficient defined as

$$T = \int_0^\infty |\Psi(x, t)|^2 dx \quad (32)$$

tends to 1/2 for any initial Gaussian wave packet of the form

$$\Psi_{in} = N \exp\left[-\frac{(x - x_0)^2}{4\eta^2} + i p_0 x\right] \quad (33)$$

in the limit

$$t \to \infty, \; x_0 \to -\infty, \; p_0 \to \infty, \; -\frac{1}{2}\Omega^2 x_0^2 + \frac{1}{2}p_0^2 = E_{classical} = const \quad (34)$$

Such an independence of the transmission coefficient both on the classical part of packet's Energy $E_{classical}$ and the enrgy of fluctuations (determined by the parameter η) is explained by the extremely rapid diffusion of the packet in the potential discussed.

We have considered in detail in ref. [21] the tunneling through the potential $U(x) = \frac{1}{2}\omega^2(x^2 - \gamma x^3)$. In this case highly squeezed and correlated ($k \gg 1$, $r \to 1$) states have a greater tunneling probability than usual coherent states due to a greater energy of fluctuations. However, for moderate values of squeezing and correlation coefficients both increasing and decreasing of tunneling rates are possible, depending on the relations between phases of complex numbers u and v in transformation (7).

There exists also an interesting relation between correlated states and Berry's phase. For one-dimensional quadratic Hamiltonians Berry's phase is nonzero only in the case when

$$\widehat{H}(t) = \frac{1}{2}\left[\mu(t)\widehat{t}^2 + \vartheta(t)x^2 + \rho(t)(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})\right] \quad (35)$$

and $\rho(t) \not\equiv 0$. But the instantaneous eigenfunctions of Hamiltonian (35) possess the nonzero correlation coefficient $r = -\rho / \sqrt{\mu\vartheta}$ [26]. Thus Berry's phase can be nonzero only for correlated states (at least for quadratic Hamiltonians).

References

1. E. Schrödinger: Naturwissenschaften **14** 664 (1926)
2. R. J. Glauber: Phys. Rev. **131** 2766 (1963)
3. H. Takahasi: Advances in Communication Systems. Theory and Applications. Ed. A. V. Balakrishnan (Acad. Press, N. Y., 1965), vol. 1, p. 227
4. M. M. Miller, E. A. Mishkin: Phys. Rev. **152** 1110 (1966)
5. D. Stoler: Phys. Rev. **D1** 3217 (1970)
6. P. P. Bertrand, K. Moy, E. A. Mishkin: Phys. Rev. **D4** 1909 (1971)
7. E. Y. C. Lu: Lett. Nuovo Cim. **2** 1241 (1971)
8. H. P. Yuen: Phys. Rev. **A13** 2226 (1976)
9. D. F. Walls: Nature **306** 141 (1983)
10. B. L. Schumaker: Phys. Rept. **135** 317 (1986)
11. *Squeezed States of the Electromagnetic Field*, special issue of JOSA B4 N10 (1987)
12. M. C. Teich, B. E. A. Saleh: Quantum Optics **1** 153 (1989)
13. V. V. Dodonov, E. V. Kurmyshev, V. I. Man'ko: Phys. Lett. **79A** 150 (1980)
14. E. Schrödinger: Ber. Akad. Wiss. Berlin 296 (1930)
15. H. P. Robertson: Phys. Rev. **35** 667 (1930)
16. V. V. Dodonov, E. V. Kurmyshev, V. I. Man'ko: Lebedev Physics Institute Proceedings **176** 169 (1988) (English translation — Nova Science, Commack)
17. V. V. Dodonov, A. B. Klimov, V. I. Man'ko: Phys. Lett. **A134** 211 (1989)
18. W. Schleich, J. A. Wheeler: Nature **326** 574 (1987)
19. A. Vourdas, R. M. Weiner: Phys. Rev **A36** 5866 (1987)
20. K. Husimi: Progr. Theor. Phys. **9** 381 (1953)
21. V. V. Dodonov, A. B. Klimov, V. I. Man'ko: Lebedev Physics Institute Proceedings **200** (1990 — to be published)
22. V. V. Dodonov, V. I. Man'ko: Lebedev Physics Institute Proceedings **183** 103 (1989) (English translation — Nova Science, Commack)
23. V. V. Dodonov, V. I. Man'ko: Group Theoretical Methods in Physics, Proceedings of the Second International Seminar (Zvenigorod, 24–26 November 1982), Harwood Academic Publishers, Chur–London–Paris–New York, 1985, vol. I, p. 591
24. V. V. Dodonov, A. B. Klimov, V. I. Man'ko: Phys. Lett. **A149** 225 (1990)
25. V. V. Dodonov, O. V. Man'ko, V. I. Man'ko: Lebedev Physics Institute Proceedings **192** 204 (1988)
26. V. V. Dodonov, V. I. Man'ko: *Topological Phases in Quantum Theory*. Eds. B. Markovski and S. I. Vinitsky (World Scientific, Singapore, 1989), p. 74

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