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Black holes in asymptotically Lifshitz spacetime

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Abstract: A model of 3+1 dimensional gravity with negative cosmological constant coupled to abelian gauge fields has been proposed as a gravity dual for Lifshitz like critical phenomena in 2+1 dimensions. The finite temperature behavior is described by black holes that are asymptotic to the Lifshitz fixed point geometry. There is a one-parameter family of charged black holes, where the magnitude of the charge is uniquely determined by the black hole area. These black holes are thermodynamically stable and become extremal in the limit of vanishing size. The theory also has a discrete spectrum of localized objects described by non-singular spacetime geometries. The finite temperature behavior of Wilson loops is reminiscent of strongly coupled gauge theories in 3+1 dimensions, including screening at large distances.

Keywords: Field Theories in Lower Dimensions, Gauge-gravity correspondence, Black Holes

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Contents

1 Introduction

AdS/CFT duality [1–3] provides unique access to the physics of supersymmetric gauge theories at strong coupling. The original conjecture, and various refinements of it, have been tested to the extent that its validity is generally accepted in the theoretical high-energy physics community. By now AdS/CFT duality has been generalized in many directions and the notion of a gauge theory/gravity correspondence has become a standard item in the toolbox of high-energy theory. Recently, considerable effort has been put into extending AdS/CFT beyond high-energy physics by constructing gravity models that are conjectured to be dual to various condensed matter systems [4]–[15]. In this case there is no supersymmetry, or truncation of higher-dimensional supergravity, to justify the duality but this is not necessarily a problem. The gravity dual provides a phenomenological description of whatever strongly coupled physics that is being modeled, and, as such, it can be useful even if the connection to the underlying dynamics cannot be spelled out in detail. Ultimately, the usefulness of a dual description of a real physical system is to be judged by its success in explaining, or better yet, predicting experimental results.

In the present paper we develop further a recently proposed gravitational dual description [16] of a class of critical phenomena exhibiting unconventional scaling of the form

$$t \to \lambda^z t, \qquad \mathbf{x} \to \lambda \mathbf{x}, \tag{1.1}$$

with $z \neq 1$. The so called Lifshitz theory,

$$L = \int d^2x dt \left((\partial_t \phi)^2 - K \left(\nabla^2 \phi\right)^2 \right). \tag{1.2}$$

provides a simple example of a 2+1 dimensional field theory which is invariant under precisely this kind of scaling (with $z = 2$). This, and other related models, have been used to model quantum critical behavior in strongly correlated electron systems [17]–[20].

In [16] it was conjectured that strongly coupled systems with Lifshitz scaling can be modeled by a gravity theory with a spacetime metric of the following form

$$ds^2 = L^2 \left(-r^{2z} dt^2 + r^2 d^2\mathbf{x} + \frac{dr^2}{r^2} \right). \tag{1.3}$$

This metric is invariant under the transformation

$$t \to \lambda^z t, \quad r \to \frac{r}{\lambda}, \quad \mathbf{x} \to \lambda \mathbf{x}\,. \tag{1.4}$$

The coordinates (t, r, x^1, x^2) are dimensionless and the only characteristic length scale of the geometry is L. It was shown in [16] that an action coupling four-dimensional gravity, with a negative cosmological constant, to a simple complement of abelian gauge fields,

$$S = \int d^4x \sqrt{-g}\,(R - 2\Lambda) - \frac{1}{2} \int \left(*F_{(2)} \wedge F_{(2)} + *H_{(3)} \wedge H_{(3)} \right) - c \int B_{(2)} \wedge F_{(2)}, \tag{1.5}$$

can support such a metric with $z > 1$. Here $F_{(2)} = dA_{(1)}$ and $H_{(3)} = dB_{(2)}$ are a two-form and a three-form field strength respectively, and the length scale L is related to the cosmological constant $\Lambda = -5/L^2$.

Here we focus on the $z = 2$ case but we expect that most of our results can be generalized to other z values. In section 2, we extend the analysis of [16] to global coordinates, and demonstrate the existence of a discrete set of solutions that we call Lifshitz stars. We also introduce finite temperature by having a black hole at the center of an asymptotically Lifshitz space time. Such a black hole carries an electric charge that couples to the two-form gauge field strength. Interestingly, the magnitude of the charge is uniquely fixed by the black hole size and these black holes become extremal in the small size limit. In section 3 we study the thermodynamic properties of black holes in asymptotically Lifshitz spacetime, and conclude that, contrary to black holes in asymptotically AdS spacetime, they are thermodynamically stable for all black hole sizes. In section 4 we speculate on the nature of the boundary theory, and perform a calculation analogous to the evaluation of a Wilson loop in AdS/CFT.

A number of recent papers have analyzed black hole geometries in gravity duals of non-relativistic quantum systems [21]–[25] but those gravitational models are different from the one we consider here, leading to a different spectrum of black holes and different thermodynamic properties.

2 Asymptotically Lifshitz spacetime

In this section we look for global metrics that solve the equations of motion of (1.5) and approach the Lifshitz geometry (1.3) in an asymptotic limit. We find two types of spherically symmetric, static solutions. One is a black hole with a non-degenerate event horizon and the other is a smooth geometry that describes a non-singular, spherically symmetric concentration of the gauge fields.

The equations of motion of (1.5) are easily obtained. The gauge fields satisfy a pair of coupled equations,

$$d * F_{(2)} = -c H_{(3)}, \tag{2.1}$$

$$d * H_{(3)} = -c F_{(2)}, \tag{2.2}$$

and the Einstein equations are

$$G_{\mu\nu} - \frac{5}{L^2} g_{\mu\nu} = \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}^{\ \lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right) + \frac{1}{4}\left(H_{\mu\lambda\sigma}H_{\nu}^{\ \lambda\sigma} - \frac{1}{6}g_{\mu\nu}H_{\lambda\sigma\rho}H^{\ \lambda\sigma\rho}\right). \quad (2.3)$$

The Lifshitz metric (1.3) is a solution of the equations of motion if the topological coupling between the gauge fields is tuned to be

$$c = \frac{\sqrt{2z}}{L}. \quad (2.4)$$

We will assume this value for c in what follows.

Our ansatz for the metric and the two- and three-form fluxes generalizes the one employed in [16] to global coordinates,

$$ds^2 = L^2\left(-r^4 f(r)^2 dt^2 + r^2 d^2\Omega + \frac{g(r)^2}{r^2}dr^2\right), \quad (2.5)$$

$$F_{(2)} = \frac{2}{L}h(r)\,\theta_r \wedge \theta_t, \quad (2.6)$$

$$H_{(3)} = \frac{2}{L}j(r)\,\theta_r \wedge \theta_\theta \wedge \theta_\phi, \quad (2.7)$$

where

$$\theta_r = L\frac{g(r)}{r}dr, \quad (2.8)$$

$$\theta_t = Lr^2 f(r)\,dt, \quad (2.9)$$

$$\theta_\theta = Lrd\theta, \quad (2.10)$$

$$\theta_\phi = Lr\sin\theta d\phi. \quad (2.11)$$

The two-form field strength $F_{(2)}$, given by $h(r)$, is an electric field directed radially outwards from the origin at $r = 0$. The topological coupling between $F_{(2)}$ and $H_{(3)}$ in the action implies that the three-form flux is electrically charged and acts as a source of the electric field. The above ansatz for $H_{(3)}$ thus corresponds to a charged fluid whose density is governed by $j(r)$.

The Einstein equations and the field equations for the gauge fields reduce to a system of non-linear first order differential equations,

$$rf' = -\frac{5f}{2} + \frac{fg^2}{2}\left(5 + \frac{1}{r^2} + j^2 - h^2\right), \quad (2.12)$$

$$rg' = \frac{3g}{2} - \frac{g^3}{2}\left(5 + \frac{1}{r^2} - j^2 - h^2\right), \quad (2.13)$$

$$rj' = 2gh + \frac{j}{2} - \frac{jg^2}{2}\left(5 + \frac{1}{r^2} + j^2 - h^2\right), \quad (2.14)$$

$$rh' = 2gj - 2h. \quad (2.15)$$

In the $r \gg 1$ limit, we can replace the sphere metric, $d^2\Omega$, in (2.5) by a flat metric, or, equivalently, neglect the $1/r^2$ terms in equations (2.12) to (2.14). In this case we recover the Lifshitz solution, $f(r) = g(r) = h(r) = j(r) = 1$, as constructed by [16]. It is also easy to see that the AdS-Schwarzschild geometry,

$$g(r) = \frac{r}{\sqrt{\frac{5}{3}r^2 + 1 - \frac{\mu}{r}}}, \qquad f(r) = \frac{\sqrt{\frac{5}{3}r^2 + 1 - \frac{\mu}{r}}}{r^2}, \tag{2.16}$$

is a solution of the equations with $h(r) = j(r) = 0$ but we have not found explicit analytic solutions with non-trivial gauge fields. It is, however, straightforward to integrate the system of equations numerically and one can learn a lot about the behavior of solutions by a combination of numerics and asymptotic analysis.

It is convenient to first consider equations (2.13)–(2.15), which only involve the three functions $g(r)$, $h(r)$, and $j(r)$ and then, given a solution to this system, solve equation (2.12) for the remaining function $f(r)$. We are primarily interested in geometries that are asymptotically Lifshitz in the sense that $f, g, h, j \to 1$ as $r \to \infty$. In order to study the asymptotic behavior at large r we linearize the system (2.13)–(2.15) around the fixed point,

$$r\frac{d}{dr}\begin{bmatrix}\delta g\\ \delta h\\ \delta j\end{bmatrix} = \begin{pmatrix}-3 & 1 & 1\\ 2 & -2 & 2\\ -3 & 3 & -3\end{pmatrix}\begin{bmatrix}\delta g\\ \delta h\\ \delta j\end{bmatrix} - \frac{1}{2r^2}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}, \tag{2.17}$$

where $g = 1 + \delta g$, *etc.* By standard manipulations, the matrix that appears in the linear system can be brought into Jordan form,

$$\begin{pmatrix}-4 & 1 & 0\\ 0 & -4 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{2.18}$$

from which we read off that at the linear level there are two decaying modes, that behave as $1/r^4$ and $\log(r)/r^4$, and a zero mode that does not depend on r. In addition to the eigenmodes, the general solution of the linear system includes a universal $1/r^2$ mode that comes from the inhomogeneous term on the right hand side of (2.17). It is straightforward to find the general solution of the linear system and confirm that it has all these features, but, since we do not need the details of the full solution for what follows, we do not write it down here.

At large values of r the non-zero modes, including the universal $1/r^2$ mode, have decayed away leaving only the zero mode behind. The solution then has the form

$$g \approx 1 + \gamma, \qquad h \approx 1 + 2\gamma, \qquad j \approx 1 + \gamma, \tag{2.19}$$

with $\gamma \ll 1$, but this is not the whole story. When non-linear corrections are included, the zero mode is lifted and becomes either a marginally growing or marginally decaying mode.

To study the evolution of the zero mode at large r, we insert (2.19) into the original non-linear equations of motion (2.13)–(2.15), except that, since we are assuming that the non-zero modes have already decayed, we drop the $1/r^2$ terms in the equations. Working to leading non-vanishing order in γ then gives

$$rg'(r) \approx 7\gamma^2, \qquad rh'(r) \approx 2\gamma^2, \qquad rj'(r) \approx \gamma^2. \tag{2.20}$$

The leading order terms on the right hand side all come with a positive sign. This immediately implies that a zero mode of positive amplitude $\gamma > 0$ is a growing mode and a solution of the equations of motion where such a mode is present at large r cannot be asymptotic to the Lifshitz fixed point.

A zero mode of negative amplitude will, on the other hand, slowly decay. The decay can be described analytically in a simple fashion. Writing $\gamma(r) = -1/\log r$ it is straightforward to check that

$$\begin{aligned} g(r) &\approx 1 + \gamma(r) + \gamma^2(r) + \cdots , \\ h(r) &\approx 1 + 2\gamma(r) - \gamma^2(r) + \cdots , \\ j(r) &\approx 1 + \gamma(r) - 2\gamma^2(r) + \cdots , \end{aligned} \tag{2.21}$$

solves the non-linear equations of motion up to terms that are small compared to $1/(\log r)^2$. The solution thus appears to be asymptotic to the Lifshitz fixed point, at least as far as the functions g, h, and j go, but the approach to the fixed point is extremely slow. In order to decide whether the geometry is truly asymptotic to the Lifshitz geometry we have to consider the evolution of the remaining function $f(r)$, which enters into the g_{tt} component of the metric (2.5). Inserting a zero mode (2.19) with $\gamma(r) = -1/\log r$ into the $f(r)$ equation of motion (2.12) gives

$$r f'(r) = -\frac{4 f(r)}{\log r} + \cdots . \tag{2.22}$$

This integrates to

$$f(r) = \frac{\text{const}}{(\log r)^4}, \tag{2.23}$$

which goes to zero in the $r \to \infty$ limit. A solution with a marginally decaying zero mode is therefore not asympotic to the Lifshitz fixed point after all.

The upshot of all this is that the only solutions of our system that are asymptotic to the Lifshitz fixed point are those for which the amplitude of the zero mode of the linearized system happens to vanish in the asymptotic region. This requires fine-tuning of initial values when solving the non-linear equations of motion, which in turn reduces the number of free parameters in the solutions that are of most interest to us.

2.1 Black holes

We now describe results obtained by integrating the equations of motion numerically. We find a one-parameter family of black hole solutions with a non-degenerate horizon which are asymptotic to the Lifshitz geometry (1.3). The characteristic parameter of the black hole can be either taken as r_0, the value of the area coordinate r at the horizon, or $h_0 \equiv h(r_0)$, the value of the radial electric field at the horizon. At first sight, one would expect these two parameters to be independent as they are for an ordinary Reissner-Nordström black hole but, as discussed above, only a restricted set of geometries approaches the Lifshitz fixed point at $r \to \infty$. The restriction on the parameters can be understood in terms of the interaction between the charged fluid, represented by the three-form field strength, and the

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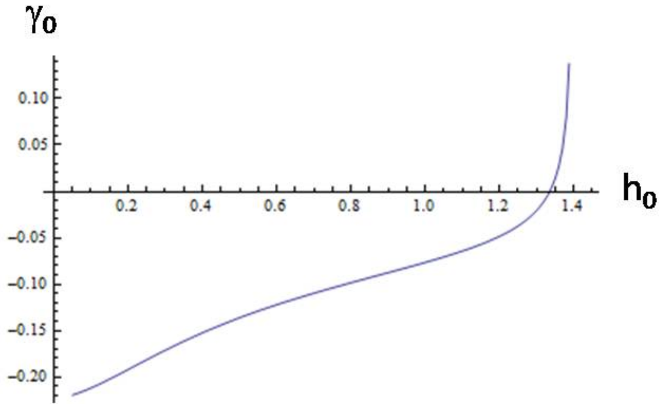

Figure 1. Large r behavior of numerical black hole solutions. The figure plots the amplitude of the zero mode γ_0 at $r = 10^6$ as a function of $h(r_0)$, the electric field at the horizon, for a black hole with $r_0 = 10$. The curve has a single zero at which the zero mode amplitude vanishes, and this uniquely determines a value of $h(r_0)$ for which the black hole is asymptotic to the Lifshitz fixed point geometry.

black hole. If a neutral black hole is placed within a charged fluid, then some of the fluid flows into the black hole and makes it charged. A static geometry describes an equilibrium configuration of the charged fluid outside a charged black hole, where the electric repulsion from the charge of the black hole precisely balances the gravitational pull.

Let us assume that there is a non-degenerate horizon at $r = r_0$. The g_{tt} component of the metric should then have a simple zero and the g_{rr} component a simple pole at the horzion. If we further assume that the electric field $h(r)$ has a finite value at the horizon, we find that the charged fluid density must go to zero at the horizon. This is in line with the equilibrium argument in the previous paragraph.

With these assumptions, we can develop a near-horizon expansion of the various fields,

$$\begin{aligned} f(r) &= \sqrt{r-r_0}\left(f_0 + f_1(r-r_0) + f_2(r-r_0)^2 + \cdots\right), \\ g(r) &= \frac{1}{\sqrt{r-r_0}}\left(g_0 + g_1(r-r_0) + g_2(r-r_0)^2 + \cdots\right), \\ j(r) &= \sqrt{r-r_0}\left(j_0 + j_1(r-r_0) + j_2(r-r_0)^2 + \cdots\right), \\ h(r) &= h_0 + h_1(r-r_0) + h_2(r-r_0)^2 + \cdots . \end{aligned} \tag{2.24}$$

Inserting this into the equations of motion and working order by order in $r - r_0$ one obtains

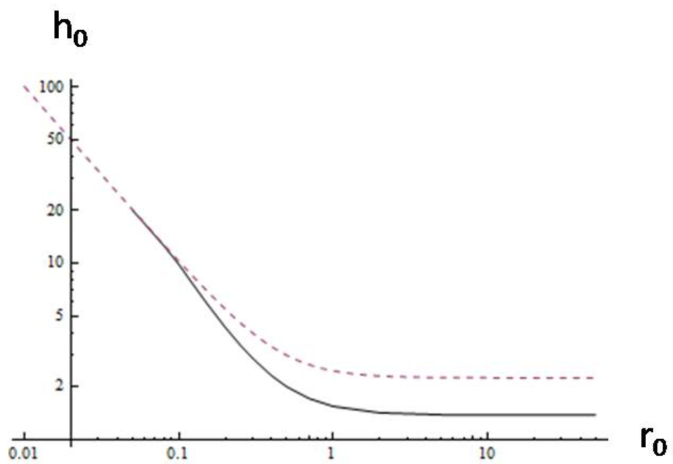

Figure 2. The charge of an asymptotically Lifshitz black hole is uniquely determined by the horizon area. The figure plots $h(r_0)$, the electric field strength at the horizon, as a function of r_0, the area coordinate at the horizon. The dashed curve is the upper bound on h_0 in equation (2.29). The bound is saturated in the small black hole limit.

relations between the various constant coefficients,

$$g_0 = \frac{r_0^{3/2}}{\sqrt{(5-h_0^2)r_0^2+1}}, \tag{2.25}$$

$$j_0 = \frac{2h_0\sqrt{r_0}}{\sqrt{(5-h_0^2)r_0^2+1}}, \tag{2.26}$$

$$h_1 = \frac{2h_0 r_0((h_0^2-3)r_0^2-1)}{(5-h_0^2)r_0^2+1}, \tag{2.27}$$

$$f_1 = \frac{f_0((6h_0^4-52h_0^2+100)r_0^4+(45-11h_0^2)r_0^2+5)}{2r_0((5-h_0^2)r_0^2+1)^2}, \tag{2.28}$$

$$\vdots$$

We notice that for a given black hole size, r_0, there is an upper bound on the electric field strength at the horizon,

$$|h_0| < \sqrt{5+\frac{1}{r_0^2}}, \tag{2.29}$$

beyond which g_0 and j_0 would be complex valued.

We now use the expansion to generate initial values for the numerical integration of the equations of motion (2.12)–(2.15), starting close to the horizon and integrating outwards in r. We have a two-parameter family of initial data, using r_0 and h_0 as the independent variables in the expansion. At first sight, f_0 also appears to be an independent free parameter but this is not really the case. Equation (2.12) is linear in f so the overall normalization of $f(r)$ is not determined. As discussed above, the solutions that are of most interest to us are those where f goes to a constant asymptotically, $f(r) \to f_\infty$ as $r \to \infty$. Such a solution

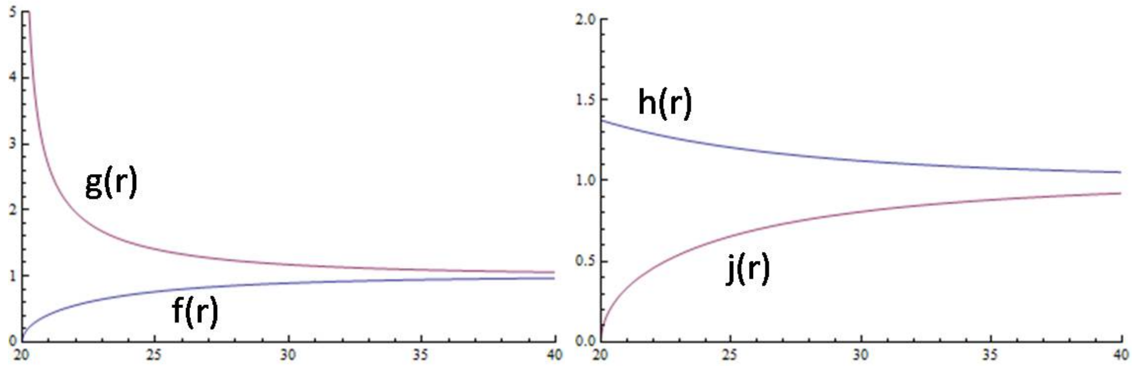

Figure 3. A large black hole with $r_0 = 20$. The figure on the left shows the metric functions $f(r)$ and $g(r)$, while the figure on the right shows the electric field $f(r)$ and the charge density $j(r)$.

can always be normalized to $f \to 1$ by dividing through by f_∞. We can therefore set $f_0 = 1$ in our numerical runs and take care of the normalization of f at the end of the day.

The next step is to look systematically for solutions that describe black holes in an asymptotically Lifshitz background. A convenient way to conduct the search is to fix r_0 and then prepare a sequence of initial data for different values of h_0. For each set of initial values, the equations of motion are integrated from the near-horizon region out to sufficiently large r so that the non-zero eigenmodes have decayed away. One then looks for a zero mode of the form (2.19) and notes how the amplitude γ_0 varies as a function of h_0, for a given value of r_0. The result of this procedure is shown in figure 1. There is a unique value of h_0, for which the amplitude of the zero mode vanishes, and this corresponds to the charge of an asymptotically Lifshitz black hole with horizon at $r = r_0$.

This can all be repeated for different sized black holes and figure 2 shows the critical value of h_0 as a function of r_0. This figure nicely summarizes the numerical results of this subsection. Starting from a two-parameter family of initial data, we have found a one-parameter family of black holes geometries that have the correct asymptotic behavior to be sitting in a Lifshitz background. The upper bound (2.29) on h_0 is given by the dashed curve in the figure. The bound is saturated, i.e. the black hole is becoming extremal, in the small black hole limit.

The actual metric and gauge field configurations for two such black holes, a large one and a small one, are shown in figures 3 and 4 respectively.

2.2 Lifshitz stars

In this subsection we turn our attention to a different kind of localized object in an asymptotically Lifshitz background, one described by a smooth geometry with no horizon or curvature singularity anywhere in spacetime. It has a spherically symmetric concentration of the charged fluid with $j(0)$ taking a finite value. Since the fluid density is finite at the origin the electric field strength must vanish there, which translates into $h(0) = 0$. If we also impose that the g_{tt} and g_{rr} components of the metric be finite at $r = 0$ we arrive at

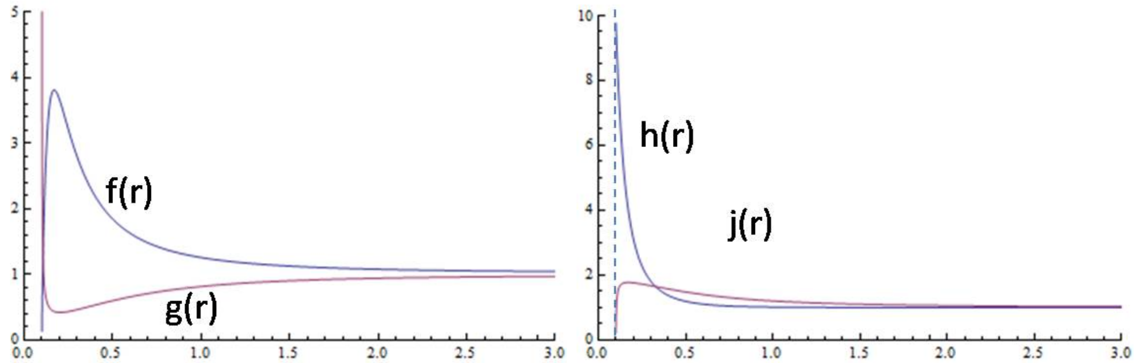

Figure 4. A small black hole with $r_0 = 0.1$. The figure on the left shows the metric functions $f(r)$ and $g(r)$, while the figure on the right shows the electric field $f(r)$ and the charge density $j(r)$.

the following expansion at small r,

$$j(r) = j_0 \left(1 - \frac{1}{6}(1 + 2j_0^2)r^2 + \frac{1}{360}(103 + 152j_0^2 + 36j_0^4)r^4 + \cdots\right), \tag{2.30}$$

$$h(r) = j_0 r \left(\frac{2}{3} - \frac{1}{15}(6 + j_0^2)r^2 + \frac{1}{1260}(528 + 96j_0^2 + 7j_0^4)r^4 + \cdots\right), \tag{2.31}$$

$$g(r) = r \left(1 + \frac{1}{6}(-5 + j_0^2)r^2 + \frac{1}{360}(375 - 146j_0^2 - 9j_0^4)r^4 + \cdots\right), \tag{2.32}$$

$$f(r) = \frac{1}{r^2} \left(1 + \frac{1}{6}(5 + 2j_0^2)r^2 + \frac{1}{360}(-125 + 16j_0^2 + 4j_0^4)r^4 + \cdots\right). \tag{2.33}$$

This can be used to generate a one-parameter family of initial value data, corresponding to different charge densities j_0 at the origin. We start the numerical integration at small r and integrate outwards to a large value of r, where the non-zero eigenmodes have decayed away. Figure 5 displays the amplitude of the remaining zero mode as a function of j_0. It reveals a discrete set of 'magic' values of j_0 for which the zero-mode vanishes and the geometry is asymptotic to the Lifshitz fixed point geometry.

We refer to the configurations with magic j_0 values as Lifshitz stars. They occur at $j_0 = 3.59,\ 21.8,\ 1.34 \times 10^2,\ 8.18 \times 10^2,\ 5.05 \times 10^3,\ 2.99 \times 10^4, \ldots$. These magic values can be seen in figure 5 and the shape of the curve in the figure suggests that the sequence continues. We eventually run out of numerical precision when we go to higher j_0 values.

The metric and gauge fields of the Lifshitz star with the lowest magic value of j_0 are shown in figure 6.

3 Black hole thermodynamics

In the original AdS/CFT correspondence, finite temperature is studied by replacing the AdS_5 part of the ten-dimensional spacetime by a five-dimensional AdS-Schwarzschild black hole [26] and we expect black holes to play a similar role here.

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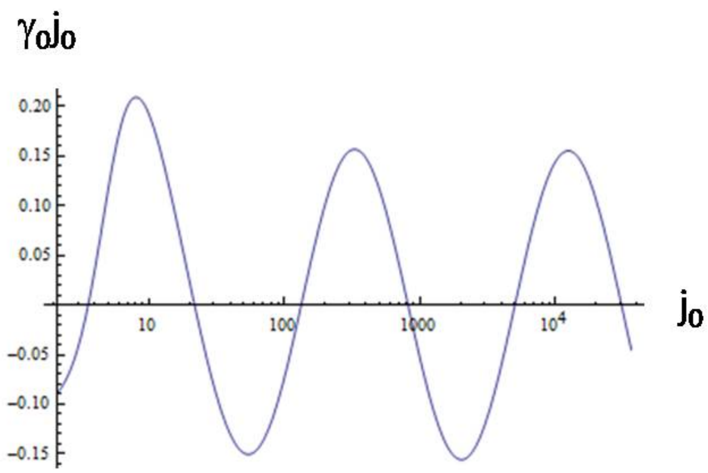

Figure 5. Large r behavior of the non-singular solutions discussed in section 2.2. The figure indicates the amplitude of the zero mode γ_0 at $r = 10^4$ as a function of j_0, the charge density at the origin. The zeroes correspond to Lifshitz stars which are asymptotic to the Lifshitz fixed point geometry. We plot the product $j_0\gamma_0$ rather than γ_0 itself. This does not affect the location of the zeroes but makes the plot more readable.

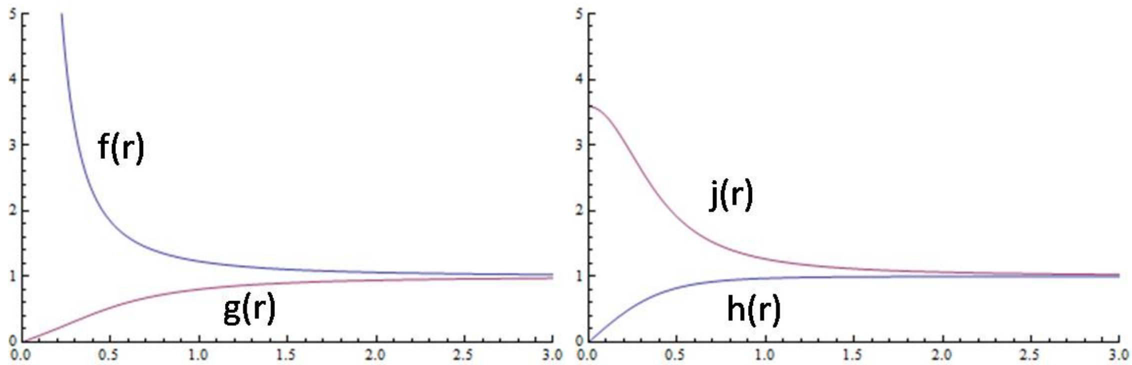

Figure 6. The Lifshitz star with the smallest allowed value of $j(0)$, the charge density at the origin. The figure on the left shows the metric functions $f(r)$ and $g(r)$, while the figure on the right shows the electric field $f(r)$ and the charge density $j(r)$.

The Hawking temperature can be obtained by going to a Euclidean metric and requiring regularity at the horizon,

$$T_H = \frac{1}{4\pi}\frac{f_0}{g_0}r_0^3. \tag{3.1}$$

where we have used the near-horizon expansion (2.24) for the metric. The coefficients g_0 and f_0 are easily determined from our numerical solutions for the metric and figure 7 shows the Hawking temperature as a function of black hole size.

We can immediately see important differences between the thermodynamic behavior of asymptotically Lifshitz black holes and that of asymptotically AdS black holes. Let us first

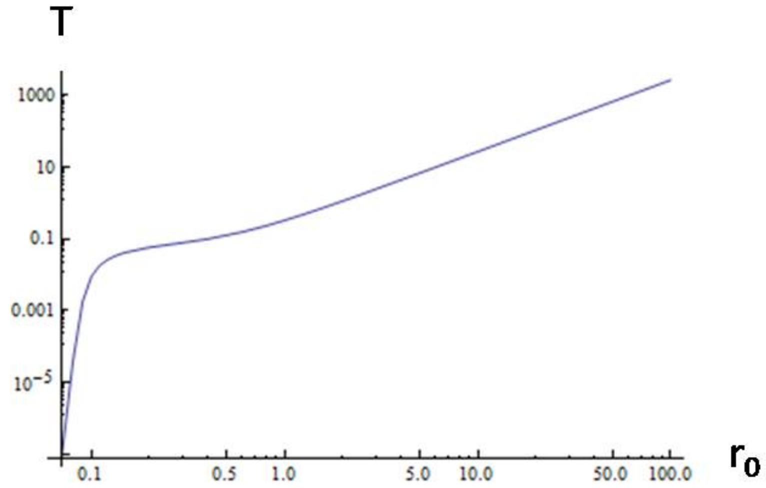

Figure 7. Numerical results for Hawking temperature as a function of black hole size.

consider black holes that are small compared to the length scale L, set by the cosmological constant. Small AdS black holes behave much like ordinary Schwarzschild black holes in asymptotically flat spacetime. In particular, their Hawking temperature increases as they get smaller leading to a thermodynamic instability. In the asymptotically Lifshitz case, on the other hand, the temperature is a monotonic function of black hole size and is rapidly falling at the smallest black hole sizes that our numerical calculations can handle. This supports our earlier claim that these black holes become extremal in the limit of vanishing black hole size. It also means that there is no analog of the Hawking-Page transition in the Lifshitz system.

Black holes that are large compared to L satisfy simple scaling relations. The form of the near-horizon expansion (2.24) suggests that $g_0 \sim r_0^{1/2}$ and $f_0 \sim r_0^{-1/2}$ for black holes with $r_0 \gg 1$, which in turn gives

$$S \propto T_H, \tag{3.2}$$

where $S \equiv \pi r_0^2$ is the Bekenstein-Hawking entropy (the corresponding behavior for large 3+1 dimensional AdS black holes is $S \propto T_H^2$). This is confirmed by our numerical results. We find that $g_0 \approx 0.57\, r_0^{1/2}$ and $f_0 \approx 1.96\, r_0^{-1/2}$, giving

$$S \approx 11.4\, T_H \tag{3.3}$$

for large asymptotically Lifshitz black holes. Figure 8 plots the black hole entropy as a function of temperature over the range of black hole sizes for which we have obtained numerical solutions.

4 Wilson loops

The bulk theory we are studying is conjectured to be dual to a boundary theory at the Lifshitz point. Following [20] we observe that the term $K\left(\nabla^2\phi\right)^2$ in the Lifshitz action (1.2)

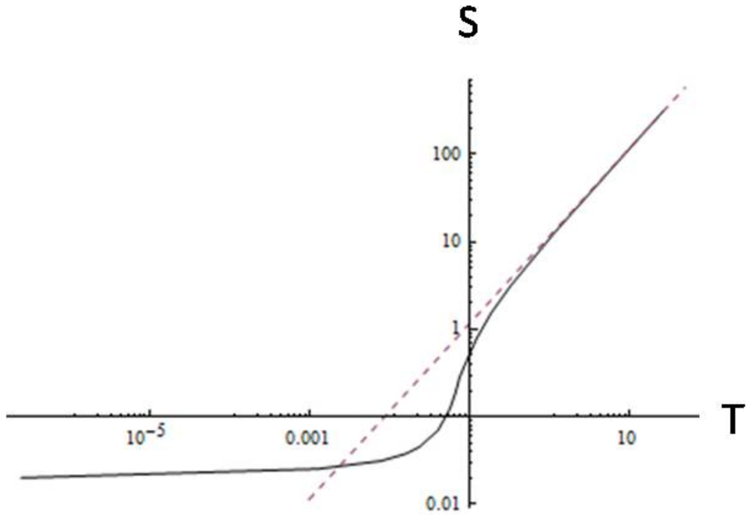

Figure 8. Black hole entropy as a function of Hawking temperature.

can be written $K\left|\bar{\nabla}\times\bar{E}\right|^2$, where $E_i=\varepsilon_{ij}\partial_j\phi$ automatically solves Gauss' law. Note also that the field ϕ is dimensionless under Lifshitz scaling (1.1), K is a dimensionless coupling constant, and an additional term in the action of the form $\left|\bar{E}\right|^2$ would be accompanied by a coupling with dimensions of mass squared. We conclude that the boundary theory can be viewed as a gauge theory in $2+1$ dimensions with a dimensionless coupling constant and an unusual action. The dimensionless coupling suggests that the theory perhaps has some nice features in common with conventional gauge theory in $3+1$ dimensions.

With the dual gauge theory in mind, we introduce Wilson loops on the gravity side. The Wilson loops contain information about the force acting between 'quarks', i.e. particles that are charged under the gauge fields. The recipe given in [27, 28] involves hanging a string from the boundary, with the end points of the string representing the quarks.

The action of the string for a rectangular Wilson loop, with initial and final Euclidean time separated by Δ, is given by

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{\det g_{MN}\partial_\alpha X^M\partial_\beta X^N} \\ &= \frac{1}{2\pi\alpha'}\int dxdt\sqrt{g_{tt}\left(g_{xx}+g_{rr}\left(\frac{dr}{dx}\right)^2\right)} \\ &= \frac{L^2\Delta}{2\pi\alpha'}\int dx\sqrt{f^2g^2r^{2z-2}\left(\frac{dr}{dx}\right)^2+f^2r^{2z+2}}, \end{aligned} \tag{4.1}$$

where we have put $\sigma=x$ and $\tau=t$ in static gauge. Extremizing the action leads to

$$\frac{f^2r^{2z+2}}{\sqrt{f^2g^2r^{2z-2}\left(\frac{dr}{dx}\right)^2+f^2r^{2z+2}}}=f_{\min}r_{\min}^{z+1}, \tag{4.2}$$

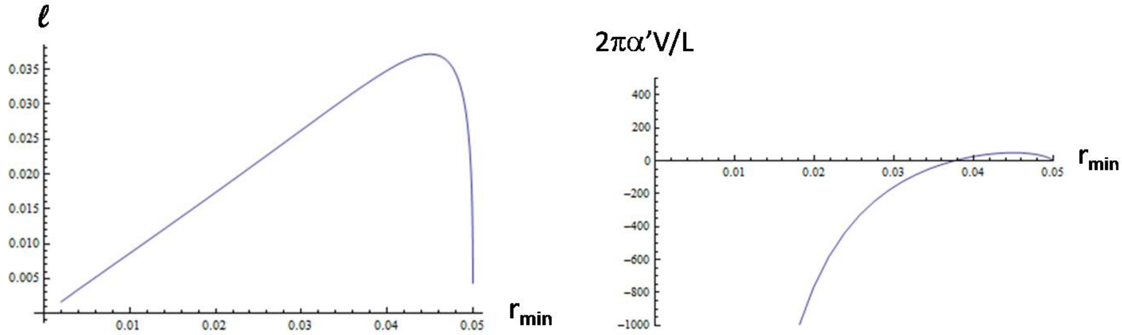

Figure 9. Screening behavior at finite temperature. The figure on the left shows the boundary distance between the endpoints of a hanging string as a function of $r_{\rm min}$, while the figure on the right plots the potential energy as a function of $r_{\rm min}$.

where $r_{\rm min}$ is the r coordinate of the midpoint of the hanging string and $f_{\rm min} \equiv f(r_{\rm min})$. The boundary distance between the end points of the string is then given by

$$\ell = 2\int_{r_{\rm min}}^{\infty} \frac{dr}{r^2} \frac{g}{\sqrt{\left(\frac{r}{r_{\rm min}}\right)^{2z+2}\left(\frac{f}{f_{\rm min}}\right)^2 - 1}}. \tag{4.3}$$

The energy of the string configuration is

$$V = \frac{L}{2\pi\alpha'}\left[2\int_{r_{\rm min}}^{\infty} dr \frac{r^{z-1} f g}{\sqrt{1-\left(\frac{r_{\rm min}}{r}\right)^{2z+2}\left(\frac{f_{\rm min}}{f}\right)^2}} - 2\int_{r_0}^{\infty} dr\, r^{z-1} f g\right], \tag{4.4}$$

where we have regularized the expression by subtracting the contribution of two straight strings going from the boundary down to the horizon of the black hole at $r = r_0$.

At vanishing temperature there is no black hole and the above expressions simplify considerably. In this case $f, g \to 1$ and $r_0 \to 0$, and the boundary distance between the endpoints of the strings is given by

$$\ell = 2\int_{r_{\rm min}}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{\left(\frac{r}{r_{\rm min}}\right)^{2z+2} - 1}} = \frac{2}{r_{\rm min}}\int_1^{\infty} \frac{dy}{y^2}\frac{1}{\sqrt{y^{2z+2}-1}} = \frac{2\sqrt{\pi}}{r_{\rm min}}\frac{\Gamma\left(\frac{2+z}{2+2z}\right)}{\Gamma\left(\frac{1}{2+2z}\right)}, \tag{4.5}$$

while the potential energy reduces to

$$\begin{aligned} V &= \frac{L r_{\rm min}^z}{\pi\alpha'}\left[\int_1^{\infty} dy\, y^{z-1}\left(\frac{1}{\sqrt{1-y^{-2z-2}}} - 1\right) - \frac{1}{z}\right] \\ &= -\frac{L^2}{\alpha'}\frac{1}{L}\left(\frac{2\sqrt{\pi}}{\ell}\right)^z \frac{1}{\sqrt{\pi} z}\left(\frac{\Gamma\left(\frac{2+z}{2+2z}\right)}{\Gamma\left(\frac{1}{2+2z}\right)}\right)^{z+1}. \end{aligned} \tag{4.6}$$

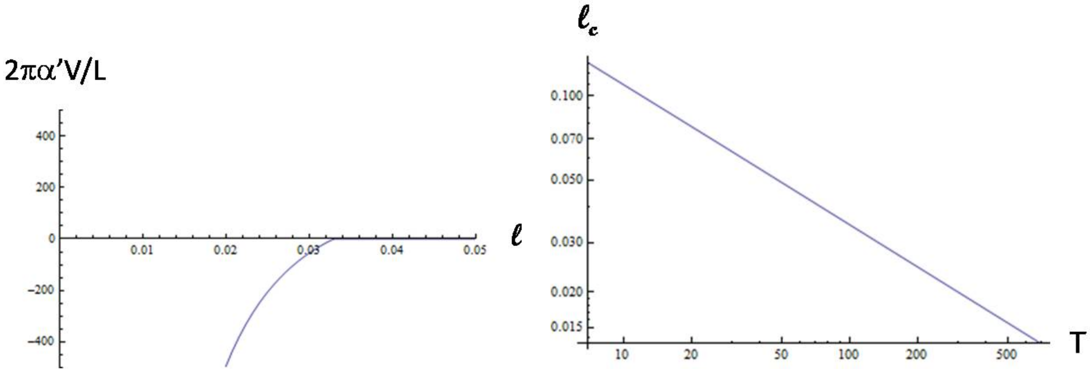

Figure 10. The figure on the left shows the potential energy as a function of the boundary distance between quarks represented by the endpoints of the hanging string. The figure on the right plots the critical distance, ℓ_c, where screening sets in, *vs.* temperature. The log-log graph shows a clear $T^{-1/2}$ dependence.

Here $\frac{L^2}{\alpha'}$ is a dimensionless coupling constant that we assume to be large. The energy V has dimensions of inverse length, hence the factor of $1/L$, and the dependence on ℓ can be traced to the unconventional scaling properties of the Lifshitz system. At $z = 1$ we recover the results of [27, 28]. At $z = 2$ it is tempting to relate the dimensionless coupling to the Lifshitz coupling K. One has to keep in mind, though, that our calculation can at best be expected to make sense at strong coupling whereas K is defined in a free theory.

At non-zero temperature we have to resort to a numerical evaluation of the integrals, using our numerically evalutated metric as input. Figure 9 shows the boundary distance between the string endpoints as a function of $r_{\min}$ and the potential energy of the hanging string, also as function of $r_{\min}$, for $z = 2$. The black hole is taken to be large so that $\ell \ll r_0$. The figure shows qualitatively the same behavior as was found for $3+1$ dimensional gauge theory in [29, 30]. At small separation the inter-quark potential is similar to the zero temperature potential but when the separation between the quarks becomes sufficiently large the gauge interaction is screened and the potential energy vanishes. The crossover occurs where the potential energy V becomes positive in figure 9, signaling that at large separation the configuration minimizing the energy is simply two straight strings stretching from the boundary down to the horizon. Since there is no interaction between such strings the potential simply vanishes in this case. Figure 10 plots the resulting potential as a function of the boundary distance between the string endpoints, and also shows how the crossover distance, where screening sets in, depends on temperature. The Lifshitz scaling is apparent in the $\ell_c \propto T^{-1/2}$ falloff as opposed to the T^{-1} behavior seen in 3+1 dimensional gauge theory.

We do not know whether the finite temperature behavior exhibited by the Wilson loops is in some way related to the ultra-locality discussed in [16, 31]. We hope to return to this issue in future work.

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5 Conclusions

In this paper we have explored further the recently proposed gravity dual description of Lifshitz type fixed points. We have mainly focused on the gravity side of the duality, finding non-trivial spacetime geometries that are asymptotic to the Lifshitz fixed point geometry, including black holes that provide a dual description of a Lifshitz system at finite temperature.

It is by no means obvious how to incorporate the bulk metrics that we have found into a solution of ten dimensional string theory. It is nevertheless tempting to proceed under the assumption that such a construction can be found, and that a duality analogous to AdS/CFT actually exists. Alternatively, we can view the gravity dual as a purely phenomenological description of the 2+1 dimensional physics. Either way, one is motivated to study the gravitational theory in more detail.

Our numerical black hole solutions in principle contain all the information that is needed to calculate finite temperature correlation functions in the dual system but this requires rather delicate numerical analysis which we leave for future work. It also requires a better understanding of holographic renormalization for non-Lorentz invariant field theories [32].

It would also be interesting to make contact with recent work [33] on non-relativistic, non-abelian gauge theories which exhibit $z = 2$ quantum critical behavior. A gravity dual of a large-N limit of such a theory would be on firmer theoretical ground than the models studied here.

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