NISHIJIMA : I do not know. The correspondence between this solution and the ordinary perturbation theory solution is that of a non-local interaction in perturbation theory.

SYMANZIK : The break in that correspondence is possible, because otherwise one would really lose many Lagrangian schemes. One may break that simple connection, for instance, in those examples discussed this morning after the first talk.

NISHIJIMA : Yes, for the moment I cannot say any more than this.

OPPENHEIMER : As I understand it, this is primarily a scheme, that is, a method of constructing the τ function. It is not meant to be looked at abstractly but as a way of going to higher and higher τ functions systematically. Therefore, you will always have to start with the simplest τ functions, and you will always encounter the fact that either τ remains zero or it is an ambiguous object. And that ambiguity will always have the character of involving ratios of the p's or not involving them, so the whole content of causality is in that first step and, perhaps, some later steps where also you have constants.

NISHIJIMA : The requirement of causality is implicitly involved in this definition. First of all, after writing down this set of equations and the dispersion relations, we can forget about definitions of the τ functions. If we remember the definition of a τ function, the fourier transform of the ρ is a function of scalar products alone only when microscopic causality is satisfied.

A NEW REDUCTION TECHNIQUE IN QUANTUM FIELD THEORY (*)

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A new technique is presented for reduction of the scattering matrix or other physical quantities in quantum field theory. These quantities are expressed as integrals involving vacuum expectation values of products of field variables. The procedure is similar to that of Lehmann, Symanzik and Zimmermann but differs from LSZ by the explicit introduction of functions which vanish outside the future light cone. The new technique is more cumbersome than the LSZ method and inferior to it for many applications but has the advantage of yielding a larger *primitive* domain of analyticity and of *not requiring local commutativity* nor any other form of causality assumption beyond those requirements that are implicit in the modified asymptotic condition. The method is illustrated for the case of two particle scattering and for the vertex function. It has been checked in simple cases of examples in perturbation theory in lowest order. It is shown that, without additional assumptions, the reduction technique has only trivial consequences, for it is proved that any matrix element can be chosen as an *arbitrary* invariant function of the energies and momentum transfers involved and that it can still be extended off the mass shell to satisfy the analyticity and mass spectrum conditons. Thus, useful restrictions involving only quantities at points of direct physical meaning have not yet been obtained; to gain such restrictions, the unitarity condition or some explicit form of causality assumption would be required.

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1. INTRODUCTION

In this paper, I will present a new reduction technique that has been developed in collaboration with S. Aks and K. Chadan at the University of Maryland. The check of this technique for two particle scattering in perturbation theory and the proof that an arbitrary function on the mass shell can be extended into a function with the required analyticity and mass support conditions, has been carried out in conjunction with Professor Gunnar Källén of the University of Lund, whose stimulating comments on other phases of this work are gratefully acknowledged. We have also benefitted from comments of members of the University of Colorado Summer 1960 Seminars, especially A. Bohr, N. Burgoyne, R. Jost, K. Symanzik, and A. Wightman, on possible tests and applications of this reduction proocedure.

Reduction techniques which express the physical quantities of quantum field theory in terms of integrals involving vaccum expectation values of products of the field operators were first introduced by Umezawa and Kamefuchi [1] and by Källén [2]. This work was extended by many investigators and a general reduction procedure, depending only on basic axioms of quantum field theory, including especially the asymptotic condition, was then formulated by Lehmann, Symanzik and Zimmermann [3]. This work has been the starting point for most of the proofs of dispersion relations [4] in quantum theory, for the proofs of analyticity of the scattering amplitude as a function of momentum transfer [5], for the Mandelstam conjecture on the representation of the scattering amplitude [6], etc.

As a result of the LSZ technique, one can express any element of the scattering matrix between states of ingoing and outgoing particles with well-defined momenta $\{k_j\}$ as multiple fourier transforms of configuration space expressions involving vacuum expectation values of products of field operators. (The number of field operators in any product and the number of configuration space 4-vectors x_j in this fourier integral is just equal to the total number of ingoing plus outgoing particles for this scattering matrix element.)

LSZ showed that the scattering matrix could be expressed as a fourier transform involving either the time-ordered product or the retarded product; the latter is superior for many applications, for the retarded function $r(x)$ vanishes unless the preferred co-ordinate (let us call it x_1) is in the future of all the other integration vectors $x_2 \ldots x_n$. In contrast, the time-ordered product in general has support dense in configuration space. This fact that the retarded product vanishes unless all the $x_1 - x_j$ $(j > 1)$ are in the future light cone leads immediately to the result that the S-matrix is the boundary value of an analytic function in the momentum variables k_j in a certain primitive domain, and this is the first property needed for proofs of dispersion relations, etc.

It might at first seem curious that a given function, the S-matrix, can be expressed as the fourier transform of either of two quite different expressions. The answer is that the S-matrix has physical meaning only *on the mass shell* where the real vectors k_j satisfy $k_j^2 = k_{jo}^2 - \mathbf{k}_j^2 = m_j^2$. For almost all other real values of the k_i, the two fourier transforms will differ.

This well-known fact suggests the question: What other convenient fourier transforms might be found for the scattering matrix in addition to the two types of LSZ? In particular, one can ask whether fourier transforms can be found which have even smaller supports in configuration space than the retarded functions and which therefore lead to extensions of the scattering matrix off the mass shell which have even larger primitive domains of analyticity in momentum space. We will find in the investigation below that it is possible to reduce the support in configuration space to a much smaller domain than that of the retarded function. The retarded function is non-vanishing when x_1 is in the future of each of the other x_j, but we find transforms that vanish unless the x_j are in a particular ordered time sequence; e.g., the transform can be made to vanish unless *each* $x_j - x_{j+1}$ is in the future light cone.

Furthermore, this reduction will be accomplished without assuming that the field operators commute for space-like separations. (This assumption is essential in the LSZ reductions in terms of the retarded function). To many the axiom of local commutativity may appear to be the best way to introduce causality. However, it seems to us desirable to investigate whether a more intuitive (and possibly weaker) form of the causality assumption can be introduced (in place of local commutativity) which involves only conditions on the asymptotic signals in the distant past and future of the scattering event. The reduction technique developed in this report is thus conceived as appropriate for investigating such alternative formulations of strict causality [7].

However we have not been able to complete the introduction of such a "strict asymptotic causality" assumption into this work. Without any explicit causality assumption or use of the unitarity condition it is natural to expect that only trivial consequences will result. Such is indeed the case, for it is proved that any S-matrix element can be chosen as an arbitrary invariant function of the physical variables (e.g., momentum transfers and energies) on the mass shell and can still be extended to satisfy both the required analyticity and appropriate mass spectrum conditions. Thus this work has as yet led to no useful restrictions on quantities of direct physical interest. It should be stressed that this reduction technique is much more cumbersome than that of LSZ and clearly inferior to LSZ for most applications. The main interest of the present work is only to show that the S-matrix can be reduced without use of local commutativity and to show the triviality of analyticity in a tube domain in momentum space.

It is easy to understand why there is such great freedom possible in the choice of the reduction of any matrix element between ingoing and outgoing states. The outgoing (or ingoing) wave packets of particles are represented as integrals over the field operators in the distant future (or past). The difference between the infinite future and past is then represented as the integral over time of the derivative of any function which goes to these past and future limits; the way in which one interpolates between these asymptotic values is arbitrary. Thus no direct physical meaning in terms of localized processes should be attached to the integrands resulting from the reduction procedure.

2. THE MODIFIED ASYMPTOTIC CONDITION

An essential axiom in the work of LSZ is the asymptotic condition, namely for a scalar field $A(X)$:

$$\langle \Phi | A^f_{\substack{\text{out}\\ \text{(in)}}} | \Psi \rangle = i \lim_{x_0 \to \pm\infty} \int d^3x f^*(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} \langle \Phi | A(x) | \Psi \rangle \tag{2.1}$$

Here $|\phi\rangle$ and $|\psi\rangle$ are any two normalizeable states of the system and A^f_{out} is the annihilation operator for an outgoing particle corresponding to the field A with mass m in a state determined by $f(x)$, a normalizeable positive energy solution of the Klein-Gordon equation:

$$f(x) = \int d^4q \delta(q^2 - m^2) \theta(q) e^{-iqx} \phi(q) \tag{2.2}$$

Here

$$\frac{\overleftrightarrow{\partial}}{\partial x_0} = \frac{\overrightarrow{\partial}}{\partial x_0} - \frac{\overleftarrow{\partial}}{\partial x_0} \tag{2.3}$$

In our work we wish to be able to modify this asymptotic condition, when appropriate, to the form:

$$\langle \Phi | A^f_{\substack{\text{out}\\ \text{(in)}}} | \Psi \rangle = i \lim_{x_0 \to \pm\infty} \int d^3x f^*(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} \times \times \{ s(\pm x - y) \langle \Phi | A(x) | \Psi \rangle \} \tag{2.4}$$

where y is any fixed space-time point during the integration and the limit in x and $s(x)$ is any function that we wish to select which vanishes outside the future light cone and which approaches 1 as $x_0 \to +\infty$ for fixed $\mathbf{x}$. It is normally convenient to choose $s(x)$ to be invariant under orthochronous homogeneous Lorentz transformations. A possible simple choice is the characteristic function of the future light cone V_+, namely $s(x) = \theta(x_0)\theta(x^2)$

$$\text{or:} \quad s_0(x) = \begin{cases} 1 & \text{if } x \text{ is in } V_+ \\ 0 & \text{if } x \text{ is not in } V_+ \end{cases} \tag{2.5}$$

However, in some applications one may want to guarantee that $s(x)$ and certain of its derivatives are all bounded, in which case one can choose within the future light cone any function of x^2 which rises sufficiently smoothly from zero on the light cone to 1 for large x^2:

$$s(x) = \frac{1}{2\pi^3} \int d^4q e^{-i(q+i\eta)x} \sigma((q+i\eta)^2) \tag{2.6}$$

Here, η is any real vector within the future light cone (it drops out after the integration) and $\sigma(z)$ is an analytic function of z in the z-plane cut along the positive z axis; thus the integrand is an analytic function of q when Im q is in V_+ which guarantees that $s(x) = 0$ when x is not in V_+. For the characteristic function $s_0(x)$ of Eq. (2-5), $\sigma(z) = z^{-2}$. The behavior of $s(x)$ for large x_0 is determined by the $\sigma(z)$ for small z and to insure that $s(x) \to 1$ as $x_0 \to \infty$ we must set $\sigma(z)z^2 \to 1$ for small z. Otherwise, we need only require that $\sigma(z)z^2$ be bounded for $z = (q + i\eta)^2$, with q varying over all real vectors; and we can choose any analytic function in the cut plane consistent with these requirements. For example, one can use

$$\sigma_n(z) = z^{-2} \Lambda^{2n} (\Lambda^2 - z^2)^{-n} \tag{2.7}$$

where n is any non-negative integer chosen so that the resulting $s_n(x)$, which goes as $(x^2)^n$ for small x^2, (x in V_+) is as smooth as desired in the particular problem. (We find $n = 0$ is sufficient in all calculations that we have done so far provided one combines terms in appropriate manner, but that $n \geq 2$ may be required if terms are separated.)

The asymptotic condition (2·4) is an intuitively reasonable modification of the LSZ condition (2·1). Furthermore, we have checked that it does hold to lowest order in a perturbation theory example. In a general theory these asymptotic conditions are just assumed to be part of the basic axioms, and we will examine their consequences [8].

3. REDUCTION OF TWO PARTICLE SCATTERING TERM

The reduction procedures can be applied to arbitrary S-matrix elements. We will illustrate our technique by applying it to the simple case of two incoming and two outgoing particles. To avoid unnecessary complications, we will assume all the particles are uncharged and scalar. Then the scattering matrix S is defined by:

$$\langle p'k' \text{ out} | pk \text{ in} \rangle - \langle p'k' \text{ in} | pk \text{ in} \rangle = \langle p'k' | S - 1 | pk \rangle \tag{3.1}$$

(Here the ingoing and outgoing particles are thought of as given by wave packets which are nearly plane waves except for a damping modulation at very large distances; hence the wave packets are assumed to have appreciable contributions only for momenta about certain central values of p', k', p, and k, respectively. The scattering process is assumed to depend continuously on the momenta (within momentum conservation) and, although the resonances in the scattering may be very sharp compared with their central values, the spread in any momentum component in each wave packet is assumed to be far smaller than the width of any resonance. For brevity we indicate the states by only the central momentum of each packet).

The first step in the reduction procedure gives only the momentum conservation and is most conveniently done following LSZ exactly:

$$\langle p'k' | S - 1 | pk \rangle = \langle p' | A^{k'}_{\text{out}} - A^{k'}_{\text{in}} | pk \text{ in} \rangle$$

$$= \int_{-\infty}^{+\infty} dx_0 \frac{\partial}{\partial x_0} i \int d^3x f_{k'}{}^*(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} \langle p' | A(x) | pk \text{ in} \rangle \tag{3.2}$$

$$= i \int d^4x f_{k'}{}^*(x) \overleftrightarrow{\frac{\partial^2}{\partial x_0^2}} \langle p' | A(x) | pk \text{ in} \rangle \tag{3.3}$$

Here we have used that for single particle states:

$$|p' \text{in}\rangle = |p' \text{out}\rangle = |p'\rangle \tag{3.4}$$

The wave packet $f_{p'}(x)$, is a solution of the Klein-Gordon equation; hence:

$$\frac{\partial^2}{\partial x_0^2} f_{k'}{}^*(x) = (\nabla^2 - m^2) f_{k'}{}^*(x) \tag{3.5}$$

The wave packet $f_{k'}(x)$, although it is approximately proportional to $\exp(-ik'x)$ for moderate values of x, is damped by its modulation so that $f_{k'}(x) \to 0$ as $|\mathbf{x}|$ becomes very large for any fixed x_0. Hence Greens' theorem can be applied in the integral to transfer $\overleftarrow{\nabla}^2$ into $\overrightarrow{\nabla}^2$, with the boundary terms all zero. Thus we obtain:

$$\langle p'k' | S - 1 | pk \rangle = i \int d^4x f_{k'}{}^*(x) | \overrightarrow{K}_x \langle p' | A(x) | pk \text{ in} \rangle$$

$$= i \int d^4x f_{k'}{}^*(x) \langle p' | j(x) | pk \text{ in} \rangle \tag{3.6}$$

where

$$K_x A(x) = j(x) = \left(\frac{\partial^2}{\partial x_0^2} - \nabla^2 + m^2 \right) A(x) \tag{3.7}$$

If we now let each packet approach a plane wave $f_{k'} \to (2\pi)^{-3/2} \exp(-ik'x)$.) and use that, by translation invariance of the theory

$$j(x) = e^{+iPx} j(0) e^{-iPx} \tag{3.8}$$

where P is the four-momentum operator, we obtain:

$$\langle p'k'|S-1|pk\rangle \to i(2\pi)^{5/2}\delta^{(4)}(p'+k'-p-k)\langle p'|j(0)|pk\rangle \quad (3.9)$$

Thus, apart from the delta function of momentum conservation, our S-matrix gives $i\langle p'|j(0)|pk\text{ in}\rangle$ which we now reduce further; here we use the same technique again but diverge from LSZ for the first time by using the asymptotic condition (2.4) instead of (2.1):

$$i\langle p'|j(0)|pk\text{ in}\rangle = i\langle p'|j(0)B^{k+}_{\text{in}}|p\rangle$$

$$= \lim_{x_0\to-\infty}\int d^3x f_k(x)\frac{\overleftrightarrow{\partial}}{\partial x_0}\{s(-x)\langle p'|j(0)B(x)|p\rangle\} \quad (3.10)$$

$$= T = -\int d^4x f_k(x)K_x s(-x)\langle p'|j(0)B(x)|p\rangle \quad (3.11)$$

We can similarly reduce the remaining particles in terms of their corresponding field operators C and D to obtain

$$T = \int d^4x d^4x' d^4x'' f_{p'}{}^*(x')f_k(x)f_p(x'')K_xK_{x'}K_{x''}s(x')\times$$
$$\times s(-x)s(x-x'')\langle 0|C(x')j(0)B(x)D(x'')|0\rangle \quad (3.12)$$

where each Klein-Gordon operator K includes the mass of the corresponding particle. In expressions (3.11) and (3.12), we can now let the wave packets approach plane waves (with $f_k(x)\to(2\pi)^{-3/2}\exp(-ikx)$), so that (3.11) becomes:

$$T = -(2\pi)^{-3/2}\int d^4x e^{-ikx}K_x s(-x)\langle p'|j(0)B(x)|p\rangle \quad (3,13)$$

and (3.12) becomes:

$$T = \int\frac{d^4x d^4x' d^4x''}{(2\pi)^{9/2}}e^{ip'x'-ikx-ipx''}K_xK_{x'}K_{x''}s(x')\times$$
$$\times s(-x)s(x-x'')\langle 0|C(x')j(0)B(x)D(x'')|0\rangle \quad (3.14)$$

We will find it convenient to alter the notation by introducing:

$$\xi_1 = x-x'' \qquad k_1 = p$$
$$\xi_2 = -x \qquad k_2 = k$$
$$\xi_3 = x' \qquad k_3 = -k' = p'-p-k$$
$$k_4 = -p' = -(k_1+k_2+k_3) \quad (3.15)$$

Then (3.14) becomes:

$$T = \int\frac{d^4\xi_1 d^4\xi_2 d^4\xi_3}{(2\pi)^{9/2}}e^{i(\xi_1k_1+\xi_2(k_2+k_1)+\xi_3[k_3+k_2+k_1])}\times$$
$$\times K_{\xi_3}K_{\xi_2}K_{\xi_1}s(\xi_3)s(\xi_2)s(\xi_1),$$
$$\langle 0|C(\xi_3)j(0)B(-\xi_2)D(-\xi_2-\xi_1)|0\rangle \quad (3.16)$$

4. PRIMITIVE ANALYTICITY DOMAIN OF THE TWO-PARTICLE SCATTERING MATRIX

In the expression 3.16, the integrand vanishes unless the integration variables ξ_1, ξ_2, and ξ_3 are all within the future light cone. Thus it follows that the integral will be an analytic function when each of the vectors k_1, k_2+k_1, and $k_3+k_2+k_1$ all have imaginary parts within the future light cone. This domain in the three complex vectors is normally called the "tube" in these three vectors and we will use the following notation to designate this domain:

$$T(v_1,v_2,v_3) = T(k_1,k_2+k_1,k_3+k_2+k_1)$$
$$= \{k|\,\text{Im}\,k_1, \text{Im}\,(k_2+k_1)_1\,\text{Im}\,(k_3+k_2+k_1)\text{ are in }V_+\} \quad (4.1)$$

By a slight alteration of the reduction technique one can get an even larger domain of analyticity than the tube mentioned above. For example, when the function $s(y-x)$ is introduced in the integration under the variable x to give an incoming field, the point y can depend in any way we choose on previous integration variables. Using arbitrary linear combinations of previous integration variables, it is possible to show directly by the above technique that the scattering matrix can be extended into a function analytic in the tube in the vectors k_1, k_2+ak_1, and $k_3+bk_2+ck_1$, where a, b and c are any three positive numbers. It should be stressed that the same function is not analytic in the tubes obtained for all values of these three constants. One must first choose the constants a, b and c and then one can find an appropriate analytic function which is analytic in this tube domain (and of course all such domains for smaller values of a, b and c) but this function will not normally be analytic in the larger tubes obtained as a, b and c increase.

There is still considerable freedom in the function that one obtains by the reduction procedure. For example, the form of the $s(x)$ is still to be chosen as one wishes and the three functions s in equation (3.16) can all be different if one desires.

The order in which the various particles are reduced is obviously arbitrary, and thus one is able to obtain tube domains in which the roles of the different momenta are permuted. Hence one can extend the scattering matrix to be an analytic function in each of several different tube domains. This situation is reminiscent of that for the vacuum expectation values of products of field operators ("Wightman functions") as discussed by Källén and Wightman [9]. However, there is one very important difference. The extension of the S-matrix into each of these tube domains is obtained by a *different* analytic function; these functions do not agree on a dense set but only agree on the mass shell which is not of sufficiently high density to permit identification of the two analytic functions. Thus we are *not* able to extend our tube domain into a permuted tube domain but instead are limited to the primitive domain of analyticity obtained directly from any one of the tubes.

In some respects this analyticity domain in momentum space is similar to that obtained from the retarded function of the LSZ reduction technique. The retarded function gives a fourier transform $\tilde{r}$ which, with appropriate labeling of the momentum, is analytic in the tube in the vectors k_1, k_2, and k_3. Obviously any function which is analytic in this tube domain of the $\tilde{r}$ function is analytic in the larger tube in k_1, k_2+ak_1, and $k_3+bk_2+ck_1$ where a, b and c are positive. However, the contrary is not true. Thus, as a result of our smaller support in configuration space, we obtained a larger tube of analyticity in momentum space.

It should be noted that the physical points of the mass shell do not lie within this tube. It is easily shown that the point of the mass shell

$$k_0^2 = m_j^2 + i0 \to m_j^2 > 0 \tag{4.2}$$

cannot be obtained by vectors k_j with imaginary parts within the future light cone. However, each of these points on the mass shell is a boundary point of the analyticity domain.

5. EXAMPLE OF THE VERTEX FUNCTION

Now we will discuss an application of this reduction technique for the simple case of a vertex function. For example, let us consider the following matrix element of a current operator between two one-particle states:

$$\begin{aligned} V &= \langle p'|j(0)|p\rangle \\ &= -\int d^4x d^4x' e^{ip'x'-ipx} K_x K_{x'} s(x') s(-x) \times \\ &\quad \times \langle 0|A(x')j(0)B(x)|0\rangle \end{aligned} \tag{5.1}$$

where the reduction has been carried out explicitly by our technique and it is readily seen that this vertex function is the boundary value of the analytic function in the tube of the two momentum vectors p and p'. By Lorentz invariance this analytic function V can be expressed in terms of the three inner products: $V = V(z_1, z_2, z_3)$, where

$$z_1 = p^2, \quad z_2 = p'^2, \quad z_3 = (p+p')^2 \tag{5.2}$$

On the mass shell,

$$z_1 = m_1^2, \quad z_2 = m_2^2, \quad \text{and } z_3 = x_3 \geq (m_1+m_2)^2,$$

where the function V takes on physical meaning. Both z_1 and z_2 approach definite points on the positive real axis from above and the physically meaningful vertex function is a function of only the one real variable x_3. (V actually becomes double-valued on this boundary, depending on the way in which the boundary point is approached.)

The analyticity domain in the three variables z_1, z_2, and z_3 has been thoroughly studied by Källén and Wightman [9] and is sketched in Fig. 1.

We see that, as z_1 and z_2 approach the mass shell, the domain of analyticity in z_3 shrinks into the region bounded by a small sling above the real axis and disappears in the limit. Thus the function $V(m_1^2, m_2^2, z_3)$ is not itself analytic in z_3 in *any* domain.

The situation discussed above for the vertex function extends immediately to the general case. For example, any point in the tube $T(k_1, k_2, \ldots k_N)$ must be such that k_j and k_n lie in the tube $T(k_j, k_n)$ and thus the region of analyticity in the variable $(k_j+k_n)^2$ necessarily shrinks to zero as k_j^2 and k_n^2 pass onto the mass shell.

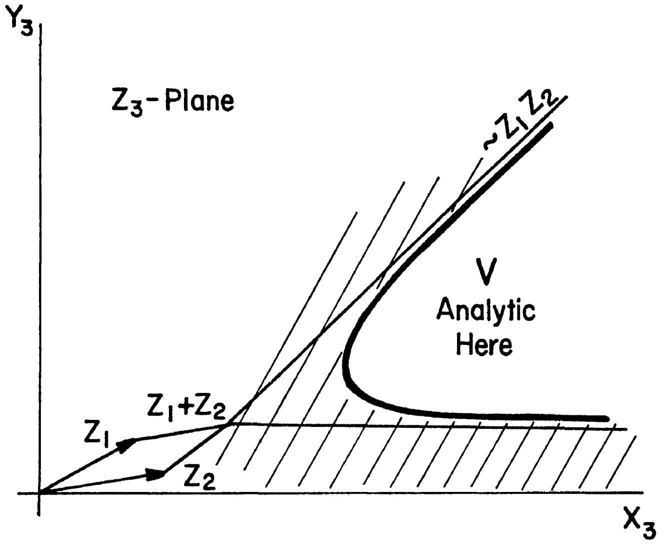

Fig. 1 Analyticity domain of the vertex function in z_1,z_2, and z_3 .

6. PROOF THAT ARBITRARY FUNCTIONS ON THE MASS SHELL EXTEND INTO ANALYTIC FUNCTIONS IN THE TUBE AND SATISFY MASS SPECTRUM CONDITIONS

In this section we will show that the analyticity in tube domains that we have derived is, by itself, no physical restriction. Let us consider a general matrix element of any operator $j(0)$ between a state of M outgoing particles and a state of N ingoing particles :

$$F=\langle -k_{-M},\dots,-k_{-1}\,,\mathrm{out}\,|j(0)|k_N\dots k_1\,,\mathrm{in}\rangle \qquad (6.1)$$

This matrix element is defined on the mass shell, where

$$\left.k_j{}^2=m_j{}^2\right\}\text{and}\begin{cases}k_j \text{ is in } V_+ \text{ for } 1\leq j\leq N\\ -k_j \text{ is in } V_+ \text{ for } -M\leq j\leq -1\end{cases} \qquad (6.2)$$

(If F came from an S-matrix element; then $k_0=-\sum_{\substack{j=-M\\ j(\neq 0)}}^{N}$ is also on a mass shell, but we consider here even the more general case of varying k_0^2.)

F is a function of the invariant inner products

$$b_{jn}=k_jk_n-k_nk_j\,,\; j<n \qquad (6.3)$$

By a technique similar to that leading to equation (3.16), we can express F as :

$$F=\left[\prod_{\substack{j=-M\\(j\neq 0)}}^{N}\int\frac{id^4x_j}{(2\pi)^{3/2}}e^{-ik_jx_j}K_{x_j}s\left(\varepsilon(j)\left(\sum_{e=-M}^{j-1}\alpha_{je}x_e-x_j\right)\right)\right]$$
$$\cdot\langle 0|A_{-M}(x_{-M})\dots A_{-1}(x_{-1})j(0)A_1(x_1)\dots A_N(x_N)|0\rangle \qquad (6.4)$$

(Here the symbol $[\Pi_jE_j]$ designates the formal product of the expressions E_j.) We have reduced the outgoing particles first and then the ingoing particles and we have set y_j in $s(y_j-x_j)$ to be an arbitrary real linear combination of the previous integration variables. Thus the matrix α is an arbitrary real triangular matrix satisfying the conditions :

$$\begin{Bmatrix}\alpha_{jn}=0 \text{ for } j\leq n\\ \alpha_{0j}=\alpha_{j0}=\delta_{j0}\end{Bmatrix} \qquad (6.5)$$

and defined for $-M\leq j,\,n\leq N$

The integrand obviously vanishes unless each x_j are in V_+ where

$$\Psi_j=\varepsilon(j)\left(\sum_n\alpha_{jn}x_n-x_j\right) \qquad (6.6)$$

Because of the simple properties of triangular matrices, we can easily solve the system (6.6) to obtain

$$x_j=\sum_e B_{je}\Psi_e \qquad (6.7)$$

where the matrix B is

$$B=\sum_{P=O}^{M+N}(-\alpha^P\varepsilon),\;\varepsilon_{ij}=\delta_{ij}\varepsilon(j),\;\varepsilon(j)=\begin{cases}1 \text{ for } j\geq 0\\ -1 \text{ for } j<0\end{cases}$$

Thus the exponentials in (6.5) involve

$$\sum_{j=-M}^{N}k_jx_j=\sum_{n=-M}^{N}v_n\Psi_n$$

where

$$v_n=\sum_j k_jB_{jn} \qquad (6.8)$$

Here B is also triangular :

$$B_{jn}=0 \quad \text{for} \quad j<n$$
$$B_{jj}=\varepsilon(j);B_{n0}=B_{0n}=0 \quad \text{for} \quad n\neq 0 \qquad (6.9)$$

and the other B_{jn}'s can be made *any* set of real numbers by changing the α_{jk}'s.

Thus equation (6.4) obviously defines an extension of F off the mass shell which is analytic in the $(N+M)$ momentum vectors in the tube

$$T(v_{-M}, \dots, v_{-1}, v_1, \dots v_N) .$$

The vacuum expectation value in (6.4) must satisfy certain mass spectrum conditions [9] in its fourier transform which are conveniently summarized [10] by the representation in terms of Δ^+_{N+M+1}:

$$\langle 0|A_{-M}(x_{-M}) \dots A_{-1}(x_{-1})j(0)A_1(x_1) \dots A_N(x_N)|0\rangle =$$

$$\int [\prod_{i \leq j} (da_{ij})] \Delta^+_{N+M+1}(\{x_n - x_{n+1}\}; \{a_{ij}\}) G(\{a_{ij}\})$$

(where $x_0 = 0$). (6.10)

We will now show that an arbitrary function $F(\{k_i k_j\})$ of the inner products over their range of variation on the mass shell can be represented in the form (6.4) and (6.10), where the $G(\{a_{ij}\})$ weight in the $[(N+M)(N+M+1)/2]$ masses a_{ij} satisfies the support conditions appropriate to the mass spectra for the fields $A_n(x_n)$. This can be done by choosing the particular weight function :

$$G(\{a_{ij}\}) = \left[\prod_{\substack{j=-M \\ (j \neq 0)}}^{N} \delta(b_{jj} - m_j^2)(2\pi)^{3/2}/i \right] F(\{b_{jk}, j < k\}) \tag{6.11}$$

where the a_{ij} are chosen as functions of the b_{jk} by the equations :

$$a_{ij} = \sum_{m=i}^{N} \sum_{n=j}^{N} b_{mn}, \qquad -M \leqslant i, j \leqslant N \tag{6.12}$$

Here the b_{jk} are to represent the inner products of the vectors k_j, k_k and hence are assumed to satisfy the conditions necessary in order that the b_{jk} can be obtained by real vectors on the mass shell. Thus :

$$b_{ij} = b_{ji}, \; b_{jj} = m_j^2, \; b_{0j} = 0 \tag{6.13}$$

$$m_j m_k \leq \varepsilon(jk) b_{jk} < \infty \tag{6.14}$$

(Since only four vectors can be linearly independent, for $N+M>4$ obvious linear relations among the b_{jk}'s result, which are assumed to be satisfied, along with the positivity of the necessary determinants that guarantee real vectors can give the inner product values. The weight G can be set equal to zero for other values.)

From (6.12), we find :

$$b_{ij} = a_{i,j} + a_{i+1,j+1} - a_{i+1,j} - a_{j+1,i} \tag{6.14}$$

The weight function $G(\{a_{ij}\})$ defined by (6.11) can be seen to have the support that would result if the $A_j(x_j)$ were free fields creating successively free particles of mass m_j, for $j>0$, and then the fields $A_j(x_j)$, for $j<0$, destroyed successively quanta of mass m_j; the free field expressions are changed, however, by making each non-zero contribution arbitrary rather than giving it the particularly simple value that results when free fields are used. Thus this weight function automatically satisfies the mass support conditions appropriate for the succession of fields.

The equation (6.4), into which (6.10) and (6.11) are now inserted, will be shown to be an identity, so that an arbitrary function $F(b_{jn}, j<n)$ on the mass shell is explicitly extended into an analytic function in the tube domain. To prove that the integral in (6.12) does actually reduce to $F(\{b_{jn}\})$ on the mass shell, it is convenient to give each of the momentum vectors k_j an appropriate small imaginary part so as to damp the integral in the time variables; we also note that each plane wave really represents a wave packet that decreases in amplitude for fixed time as the spatial distance becomes very large. Hence we can apply Green's theorem so that the Klein-Gordon operators all act to the left instead of to the right. We can now choose for $s(x)$ the simplest expression $s_0(x)$ of equation (2.5). (The result would be the same for other choices). Then we let the imaginary part of each k_j approach zero so that $k_j^2 \to m_j^2$, beginning with $j = N$, and use repeatedly the identity:

$$\lim \frac{[(k+i\eta)^2 - m^2]\delta(Q^2 - m^2)\theta(Q)}{i\pi^2[(k+i\eta-Q)^2]^2} = \delta^{(4)}(k-Q) \tag{6.15}$$

η in $V_+ \to 0$

$(k^2 = m^2)$

$k_0 > 0$

This yields the general identity :

$$\lim_{\substack{\mathrm{Im}\, k_j \to 0 \\ k_j^2 \to m_j^2}} \left[\prod_{\substack{j=-M \\ (j \neq 0)}}^{N} \int i d^4 x_j e^{-ik_j x_j} K_{x_j} s\big(\varepsilon(j)(\sum_n \alpha_{jn} x_n - x_j)\big) \right] .$$

$$\Delta^+(\{x_n - x_{n+1}\}, \{a_{ij}\}) = \left[\prod_{\substack{j=-M \\ (j \neq 0)}}^{N} \prod_{\substack{n=j+1 \\ (n \neq 0)}}^{N} \delta(k_j k_n - b_{jn}) \right] \tag{6.16}$$

Using this result it is easily verified that the integral in (6.4) does indeed reduce to $F(\{k_j k_n, j<n\})$ on the mass shell.

7. DIFFICULTIES OF THIS REDUCTION PROCEDURE

It is interesting to compare this reduction procedure with that of LSZ, and to see why the LSZ technique is superior. In our procedure we obtain a larger primitive domain of analyticity in momentum space. However, a direct copy of the further techniques used in the usual retarded function scheme is not possible in our case; in the usual approach the imaginary part of the scattering amplitude is directly a fourier transform of the expectation value of a commutator, and the Jost-Lehmann-Dyson representation [11] gives a summary of the mutual support in both x and p-space of the commutator and leads to a direct proof of the extension of analyticity needed to prove dispersion relations [5]. However, our multiplication by the factor $s(x)$ leads to a convolution with its transform which is analytic in p-space, hence extends all over the p-space and thus destroys any momentum support properties. The introduction of the mass spectrum properties in our case led to no enlargement of the domain of analyticity, yet in the case of the usual retarded function one does get an enlarged domain in this way, *which includes physical points* and therefore leads to proofs of dispersion relations, etc. This illustrates the great power of the strict micro-causality assumption (local commutativity). Nevertheless, the situation is not entirely hopeless in our approach; it may be possible to derive some real restrictions on physical quantities when further conditions on the weight $G(\{a_{ij}\})$ are imposed by the unitarity condition; however, the difficulties of completing such a program are too well known to discuss it further here. The application of the strict asymptotic causality requirement, along with the mass spectrum conditions, is the problem that we are concentrating on at the present [12].

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7. This whole investigation was in fact motivated by the aim of extending to quantum field theory earlier investigations on wave packet formulations of strict causality in simpler relativistic theories. See Annals of Physics **1**, p. 91 (1957) and **3**, p. 49 (1958). A similar aim is shown in the studies of Bohr, A. and Mottleson, B., reported at the 1958 CERN Conference on High Energy Physics. Asymptotic states in quantum field theory which are the analog of classical signals with a sharp front have been studied by James Knight (Ph.D. dissertation, University of Maryland, 1960, to be published).
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12. We are indebted to Professor Wightman for an account of similar work; Hardy has noted independently that reduction procedures can be altered to include the function $\theta(x^2)$ as a factor.

DISCUSSION

LEHMANN : I entirely agree with your last statement. The question really is one of having a convenient technique to further analyze and put in the mass spectrum. The only tool which we have been able to design for that is a function which vanishes both in some part of x space and in some part of p space and this is the commutator. Now you must find some substitute for such a procedure. You need not work with a function having such properties but so far there is simply no way of keeping track of the smearing out with the s function. Isn't this a " very unsolved " problem in your whole approach?

OPPENHEIMER : If you were unscrupulous about the s function and did not take the smearing into account, what would you get then?

TOLL : Even the s function which is of the simplest form, namely the characteristic function of the light cone, still smears the support over all of momentum space.

LEHMANN : Another point that you stressed is that the primitive domain that you get here is larger than the one for the retarded function. That is, of course, true, but on the mass shell you certainly get no more properties than by using the retarded function because on the mass shell the functions are identical. What is not known is how much stronger your asymptotic condition really is in comparison with the usual one. Does it incorporate part of the locality or does it not?

TOLL : I agree completely with Lehmann's first point that indeed the problem is to incorporate the mass restrictions and a causality assumption simultaneously, that this is solved in the usual case by the Jost-Lehmann-Dyson representation and that it is very difficult to find any equally good approach for our case where local commutativity is not assumed. However, there are still some representations that do not utilize local commutativity. Källén and I have studied representations in terms of the Bergmann-Weil technique, as well as the Δ_+ function technique, and we are trying to use these now. These representations conveniently summarize the information that is known about the mass spectrum and we hope to utilize these to find out just what the restrictions on the mass shell boundary values are. We do assume less than the usual procedure and we will get out less. In fact, the motivation in this whole research was to see if we could not relax slightly the causality asumption. It started out of an attempt to use a wave packet formulation in quantum field theory for the causality assumption and therefore it is of interest to see if we can get anything at all. We do not claim that we will get as much as the local commutativity assumption might imply.

The second point is that the asymptotic condition really in effect does include some elements of asymptotic causality. To speak very non-rigorously, why is the asymptotic condition supposed to be valid? It is because in the distant past or future the wave packets, which describe particles in a particular state, diffuse away from each other and the interaction is over large space-like distances : hence one hopes to approach a free particle picture (which includes the self interactions). But if the interaction did not decrease for space-like distances, then obviously one would not expect to have an asymptotic condition; so implicit in the asymptotic condition is a small amount of causality already. We do not think that there is much more in our asymptotic condition than in the usual one, but there may be a little more. There may be a theory in which the one asymptotic condition is satisfied and not the other.

Actually Knight at the University of Maryland has worked out what are the equivalent of wave packets in quantum fields theory—what we call strictly localized states. That is, outside of a region of localization, all measurements for a system in such a state give the same result as for the vacuum. The aim is then to use these states as incoming signal states, detect them with corresponding outgoing states, and say that no influence can get beyond its light cone. This is the " asymptotic but strict causality condition." I cannot report on this today because these localized states are terribly complicated. They involve an indefinite number of particles. For any number N, there is always a non-zero probability that there will be more than N particles present. To go over to the regular S matrix one must take a limit of weak signals. We are in the midst of this and have no result yet. In other words, one can introduce the causality assumption separately from local commutativity to see whether it is much weaker or whether it does imply most of the analytic properties that one desires.

LEHMANN : I am not sure I like these localized states very much because you yourself stressed in the beginning that you do not attach any physical meaning to the interpolation procedure. I would prefer just to take a non-local field equation which contains a form factor, do the lowest order perturbation theory on that and analyze it with respect to both types of asymptotic condition and see whether both of them hold or whether they hold only for some special form factors or things like that. I mean a relativistic non-local form factor theory such as the theories which were proposed many years ago. In lowest order perturbation theory one can certainly check that.

TOLL: We have not done this. For example, we have not tested the two asymptotic conditions in a theory of the Møller-Christensen type. We have only tested it in standard perturbation theories at present. My guess is that the theories which Haag calls "almost-local-theories" will probably satisfy both asymptotic conditions.

OPPENHEIMER: There may be also some question in multi-channel problems about the identity of the analytic functions for which the S matrices are different boundary values; for instance the question of crossing symmetry.

TOLL: Of course we always get the S matrix on the mass shell. So we satisfy all symmetry properties that are satisfied there. It is only the extension that is different. Anything that is built into the S matrix is still there.

OPPENHEIMER: But the connection between the S matrices for two different processes as different boundary values of the same analytic function does not seem to me to be implied. And this is a very deep thing somehow and a very useful one.

TOLL: That is right. There is no way of proving such properties out of this procedure.

OPPENHEIMER: And that does not seem to me to follow from the mass spectrum. That seems to be really a special consequence of causality.

TOLL: I stress that this is a clumsy procedure; it has many defects compared to the usual one.

OPPENHEIMER: It is terribly interesting how much it gives, but then it is also important to notice what it does not give.

TOLL: Yes, and it does not give many properties. The only point was to see what it could give as an alternative to the usual causality assumption. We still do not know whether it is trivial. Our expectation is that this result may not be entirely trivial.

WIGHTMAN: I would just like to make one remark about what Hardy did that was different. Hardy expressed the S matrix elements in terms of a function similar to Toll's but symmetrized in all the operators. This is the antithesis of Toll's procedure because you get no support properties at all. However, you get a neat reduction formula for all S matrix elements. Now, if you look in the old LSZ paper you find that the matrix element for two particles going into n is very neatly given in terms of one of the retarded functions, but at the time when Hardy began work nobody knew how to write in a similar way the matrix element for three particles going into n for $n \geqslant 3$. Since that time this problem has been solved by Steinman and Ruelle, so there is no more need for this symmetrized thing to give a neat reduction formula for the general S matrix element. The solution of Steinman and Ruelle is that you just must add to the list of retarded functions given by LSZ the ones given by Steinman and Ruelle. *All* of these are different boundary values of the *same* analytic function.