Chiral Equations in Gravitational Theories

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Chiral equations appear often in theories of gravitation. One reason is that the field equations of an n-dimensional Riemannian Ricci flat manifold possessing $n-2$ Killing vectors reduce to the chiral equations plus a first order linear differential equation. More explicitly, let $\mathcal{M}$ be an n-dimensional Rimannian manifold with $n-2$ Killing vectors. We can then write the metric on $\mathcal{M}$ as

$$ds^2 = f(d\rho^2 + d\zeta^2) + g_{ij}dx^i dx^j$$

$$i,, j = 3, \cdots, n$$

The field equations $R_{AB} = 0, \quad A, B = 1, \cdots, n$ reduce to [1]

$$a) \qquad (\alpha g_{,z}g^{-1})_{,\bar{z}} + (\alpha g_{,\bar{z}}g^{-1})_{,z} = 0$$

$$b) \qquad (lnf)_{,z} = \frac{(ln\alpha)_{,zz}}{(ln\alpha)_{,z}} + \frac{1}{4(ln\alpha)_{,z}} tr(g_{,z}g^{-1})^2 \qquad (1)$$

where the $(n-2) \times (n-2)$ matrix g is defined in terms of the metric components $(g)_{ij} = g_{ij}$ and $z = \rho + i\zeta$ and $\bar{z} = \rho - i\zeta$ are null coordinates. The determinant of g is given by $det\ g = -\alpha^2$. If we take the trace of equation (1.a) this implies that

$$\alpha_{,z\bar{z}} = 0.$$

The general solution if this equation is $\alpha = Z_1(z) + Z_2(\bar{z})$. The choice $\alpha = \rho$ defines the canonical Weyl coordinates. Equation (1.a) is invariant under the transformation $g \rightarrow \alpha^p g$ for any real number p. If $\mathcal{M}$ is a manifold without torsion, then g is symmetric.

In any case, the set of all matrices g build a group. Because of the invariance $g \rightarrow \alpha^p g$ we can write $det\ g = \pm 1$. Without loss of generality we can suppose $g \epsilon SL(n-2, \mathcal{R})$, if $\mathcal{M}$ is a real manifold. On the other hand, the chiral equations imply the integrability of lnf in equations (1.b). We say that lnf is a superpotential defined by the chiral equation. Then the fields of $\mathcal{M}$ are the chiral fields and the intregration of their superpotential.

One example of this construction are the axisymmetric stationary vacuum Einstein's equations (ASVE) which reduce to the system (1) for the matrix $g\epsilon SL(2,\mathcal{R})$. In four dimensions there is another equivalent formalism for obtaining exact solutions by means of the Ernst potential $\mathcal{E}$. It is well known that the ASVE are equivalent to the Ernst equations [2]

$$f \triangle \mathcal{E} = (\nabla \mathcal{E})^2, \quad f = R_e \mathcal{E}$$

It is sorprising that the Ernst equations can be cast into a chiral form for a 2×2-matrix $g\epsilon SU(1,1)$, where g can be parametrized in terms of the Ernst potential as [3]

$$g = \frac{1}{R_e\mathcal{E}} \begin{pmatrix} \mathcal{E}\bar{\mathcal{E}} & -Im\mathcal{E} \\ -Im\mathcal{E} & 1 \end{pmatrix}$$

However, the fact that $SU(1,1)$ is isomorphic to $SL(2,\mathcal{R})$ tells us that it is also possible to write down the Ernst equation in a $SL(2,\mathcal{R})$ representation, i.e. the Ernst equation are equivalent to the chiral equation for the 2×2 matrix $g\epsilon SL(2,\mathcal{R})$ given by [2]

$$g = \frac{1}{f} \begin{pmatrix} f^2+\epsilon^2 & -\epsilon \\ -\epsilon & 1 \end{pmatrix} \tag{2}$$

where $\mathcal{E} = f + i\epsilon$, so we have the ASVE written in two different formalisms as chiral equations for the group $SL(2,\mathcal{R})$.

The extension of the Ernst equation to the Einstein-Maxwell theory gives rise also to another example of chiral equatons in general relativity. In the axisymmetric stationary case, the Einstein-Maxwell field equations in the potential formalism are given by [2]

$$\begin{aligned} f \triangle \mathcal{E} &= (\nabla\mathcal{E} + 2\bar{\Phi} \nabla \Phi) \nabla \mathcal{E} \\ f \triangle \Phi &= (\nabla\Phi + 2\bar{\Phi} \nabla \Phi) \nabla \Phi \\ f &= \mathcal{R}_e\mathcal{E} + \bar{\Phi}\Phi \end{aligned} \tag{3}$$

(3) can be also cast into chiral form for the group $g\epsilon SU(2,1)$. A sutible parametrization of g in terms of the Ernst and the electromagnetic potentials is [3]

$$g = -\frac{1}{\mathcal{E}+\bar{\mathcal{E}}+2\bar{\Phi}\Phi} \begin{pmatrix} 1+\mathcal{E}\bar{\mathcal{E}}+2\Phi\bar{\Phi} & 1-\mathcal{E}\bar{\mathcal{E}}+\mathcal{E}-\bar{\mathcal{E}} & 2i\Phi(1-\bar{\mathcal{E}}) \\ -1+\mathcal{E}\bar{\mathcal{E}}+\mathcal{E}-\bar{\mathcal{E}} & -1-\mathcal{E}\bar{\mathcal{E}}+2\Phi\bar{\Phi} & -2i\Phi(1+\bar{\mathcal{E}}) \\ 2i\bar{\Phi}(1-\mathcal{E}) & 2i\bar{\Phi}(1+\mathcal{E}) & \mathcal{E}+\bar{\mathcal{E}}-2\Phi\bar{\Phi} \end{pmatrix}$$

Higher dimensional potential formalisms are also known. The five-dimensional Jordan's extention of the Kaluza-Klein theory admits also a potential formalism. In the stationary case the five-dimensional space possesses two Killing vectors X and Y. The first is due to the action of the grup $U(1)$ on the manifold. The second is associated with stationarity. In terms of these Killing-vectors we can define the five potentials [4]

$$I^2 = X^\mu X_\mu, \qquad f = -IY^\mu Y_\mu + I^{-1}(X^\mu Y_\mu)^2, \quad \psi = I^{-2} X_\mu Y^\mu$$

$$X_{,\mu} = \epsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta X^{\gamma;\delta} \qquad \epsilon_{,\mu} = \epsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta Y^{\gamma;\delta}$$

If we write (1) in terms of these five potentials, equation (1.a) reduce to five independent second order non-linear differential equations for the five potential $\psi^A = (f, \epsilon, \psi, \chi, I)$ [4]. These five differential equations can be cast into chiral form with the matrix [5].

$$g = -\frac{2}{fI} \begin{pmatrix} f^2 + \epsilon^2 - fI^3\psi^2 & -\epsilon & -\frac{1}{2\sqrt{2}}(\epsilon\chi + fI^3\psi) \\ -\epsilon & 1 & \frac{1}{2\sqrt{2}}\chi \\ -\frac{1}{2\sqrt{2}}(\epsilon\chi + fI^3\psi) & \frac{1}{2\sqrt{2}}\chi & \frac{1}{8}(\chi^2 - I^3 f) \end{pmatrix} \tag{4}$$

Matrix (4) belongs to the group $SL(3, \mathcal{R})$. Then the situation in four-dimensional gravity repeats in five dimensions, i.e. the Ricci flat field equations can be cast into chiral form in spacetime and in the potential space formalisms for the same group. Note that if we make $\psi = \chi = 0$, $I = 1$ in (4) this matrix reduces just to matrix (2) which is in agreement with the fact that Kaluza-Klein theory reduces to the Einstein theory in vacuum for vanishing electromagnetic and scalar fields. In general it is possible to write potentials in d-dimensional gravity if the d-dimensional space posseses $n = d - 3$ Killing vectors. Let g_{ab} be the components of a d-dimensional space and $\xi_a = \xi_a^c \frac{\partial}{\partial x^c}$ the n-Killing vectors, one can define the $n \times n$ projections matrix

$$\lambda_{ik} = \xi_i^a \xi_k^b g_{ab}$$

and for the Ricci flat case, the curl-free twists

$$w_{ia} = \epsilon_{ab_1 \cdots b_n de}\, \xi_1^{b_1} \cdots \xi_n^{b_n} \nabla^d \xi_i^e$$

In terms of the matrix λ and the column vector $\mathbf{w} = (w_i)$ the field equations (1.a) can be cast into chiral form for the matrix [6]

$$g = \frac{1}{\tau} \begin{pmatrix} 1 & -\mathbf{w}^T \\ -\mathbf{w} & \lambda\tau + \mathbf{w}\mathbf{w}^T \end{pmatrix} \tag{5}$$

where $\tau = det\lambda$ This matrix is symmetric and belongs to the group $SL(d-2, \mathcal{R})$. It is easy to check that matrix (5) is matrix (4) and matrix (2) for $d = 5$ and $d = 4$ respectively after an apropriate rotation. So we have the general situation that the

Ricci field equations in a d-dimensional Riemannian space can be cast into a chiral form in the spacetime formalism (equation (1)) and in the potential space formalism (with matrix (5)) for the same group $SL(d-2,\mathcal{R})$.

Another interesting example are the field equations derived from the action

$$S = \int d^4x\sqrt{-g}\,[-R + e^{2\alpha\phi}\,F^2 \; + \; (\nabla\phi)^2] \qquad (6)$$

This action reduces to the low energy string action for $\alpha^2 = 1$ [7], to the Kaluza-Klein four dimensional action for $\alpha^2 = 3$, and to the Einstein-Maxwell action for $\alpha = 0$. The field equations of action (6) can be also cast into a chiral form for a matrix $g\epsilon SL(3,\mathcal{R}) \times \mathcal{R}$ for $\alpha^2 > 3$, $g\epsilon SL(3,\mathcal{R})$ for $\alpha^2 = 3$ and $g\epsilon SL(3,\mathcal{R}) \times U(1)$ for $\alpha^2 < 3$. To end this part we want to mention that Chern Simons equations reduce to a chiral form as well (see Ref. [8]).

Now we outline briefly a method for solving the chiral equations for any n-dimensional Lie group G. Let $g\epsilon G$ a matrix which depends on z and $\bar{z}$, i.e. $g = g(z,\bar{z})$ and fulfills the chiral equations (1.a). Let V_p be a p-dimensional Riemannian space with $p \leq n$. We will suppose that the manifold p is well-known. Let $\{\lambda^i\}_{i=1,\cdots,p}$ be a set of Harmonic maps on V_p,, i.e.

$$(\rho\lambda^i_{,\bar{z}})_{,\bar{z}} \; + \; (\rho\lambda^i_{,\bar{z}})_{,z} \; + \; 2\rho\Gamma^i_{jk}\;\lambda^j_{,z}\lambda^k_{,\bar{z}} = 0$$

We make use of the following ansatz [3]. Suppose that g can be parametrized by λ^i, *i.e.* $g = g(\lambda^i) = g(\lambda^i(z,\bar{z}))$. Then the chiral equations transform into a Killing equation on V_p for the elements of the matrix $A_i = (\partial_{\lambda^i}g)g^{-1}$. The matrices A_i belongs to the corresponding Lie algebra $\mathcal{G}$ of G because they are the Maurer-Cartan forms of g on G. So we can write A_i in terms of a basis $\{\sigma_j\}$ of the Lie algebra of $\mathcal{G}$ and a basis $\{\xi_i\}$ of the Killing vector space of V_p, i.e. $A_i = \xi^j_i\sigma_j$. We know V_p so, we know its Killing vectors. We known G so we know $\mathcal{G}$ and therefore a basis for it. Then we can obtain A_i from the linear combination of σ_i and ξ_j. We map $A_i\epsilon\mathcal{G}$ into the group G by exponentiation or by direct integration of the relation $A_i = (\partial_{\lambda^i}g)g^{-1}$. (For details see Ref. [9]).

This method has been used for obtaining exact solutions of the chiral equations for the group $SL(N,\mathcal{R})$. Explicit results for $SL(2,\mathcal{R})$ and $SL(3,\mathcal{R})$ are given in Ref. [10], and for $SL(4,\mathcal{R})$ in Ref. [11] . The application of the method for $SU(1,1)$ and $SU(2,1)$ is found in Ref. [3].

The results of the $SL(2,\mathcal{R})$ chiral fields can be used for the vacuum Einstein equations when spacetime possesses two Killing vectors. The method separates naturally the exact solutions in classes [12]. There exist three solutions of the chiral equations for a V_1 space and one for a V_2. In the spacetime the V_1 classes are the Weyl's class (with the Von Stockum class as limit), the Lewis and the degenerated classes respectivelly. In the potential space only one class has *det* $g = +1$, and corresponds to the Papapetrou class. The V_2 class in spacetime is, as far as we know, not studied and in the potential space corresponds to the Tomimatsu-Sato class.

The results of the $SL(3,\mathcal{R})$ chiral fields has been used for obtaining exact solutions of the five-dimensional Einstein equations when the spacetime possessess three Killing vectors, one for the $U(1)$ symmetry, one for stationarity and one for axisymmetry [13]. In this case we obtain six different classes for a V_1 space and two for a V_2 space. These results have been applied to the potential space and to the spacetime formalism [14].

To conclude, I want to say that the presence of Dr. Plebański in our Department of Physics has been very stimulating to carry out our work.

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