Journal of Physics: Conference Series

PAPER • OPEN ACCESS

The theory of pseudo-differential operators on the noncommutative n-torus

To cite this article: J Tao 2018 *J. Phys.: Conf. Ser.* **965** 012042

The theory of pseudo-differential operators on the noncommutative n-torus

J Tao

Department of Mathematics, Caltech MC 253-37, Pasadena, CA, 91125, USA

E-mail: jtao@caltech.edu

Abstract. The methods of spectral geometry are useful for investigating the metric aspects of noncommutative geometry and in these contexts require extensive use of pseudo-differential operators. In a foundational paper, Connes showed that, by direct analogy with the theory of pseudo-differential operators on finite-dimensional real vector spaces, one may derive a similar pseudo-differential calculus on noncommutative n-tori, and with the development of this calculus came many results concerning the local differential geometry of noncommutative tori for n=2,4, as shown in the groundbreaking paper in which the Gauss–Bonnet theorem on the noncommutative two-torus is proved and later papers. Certain details of the proofs in the original derivation of the calculus were omitted, such as the evaluation of oscillatory integrals, so we make it the objective of this paper to fill in all the details. After reproving in more detail the formula for the symbol of the adjoint of a pseudo-differential operator and the formula for the symbol of a product of two pseudo-differential operators, we extend these results to finitely generated projective right modules over the noncommutative n-torus. Then we define the corresponding analog of Sobolev spaces and prove equivalents of the Sobolev and Rellich lemmas.

1. Introduction

The methods of spectral geometry are useful for investigating the metric aspects of noncommutative geometry [1–4] and in these contexts require extensive use of pseudo-differential operators. In the foundational paper [5], Connes showed that, by direct analogy with the theory [6–8] of pseudo-differential operators on $\mathbb{R}^n$, one may derive a similar pseudo-differential calculus on noncommutative n tori $\mathbb{T}^n_\theta$. With the development of this calculus came many results concerning the local differential geometry of noncommutative tori for $n = 2, 4$, as shown in the groundbreaking paper [9] in which the Gauss–Bonnet theorem on $\mathbb{T}^2_\theta$ is proved and later papers [10–14]. In these papers, the flat geometry of $\mathbb{T}^n_\theta$ which was studied in [5] is conformally perturbed using a Weyl factor given by a positive invertible smooth element in $C^\infty(\mathbb{T}^n_\theta)$. Connes' pseudo-differential calculus is critically used to apply heat kernel techniques to geometric operators on $\mathbb{T}^n_\theta$ to derive small time heat kernel expansions that encode local geometric information such as scalar curvature. As discovered in [10, 12, 13], a purely noncommutative feature that appears in the computations and in the final formula for the curvature is the modular automorphism of the state implementing the conformal perturbation of the metric.

Certain details of the proofs in the derivation of the calculus in [5] were omitted, such as the evaluation of oscillatory integrals, so we make it the objective of this paper to fill in all the details.

After reproving in more detail the formula for the symbol of the adjoint of a pseudo-differential operator and the formula for the symbol of a product of two pseudo-differential operators, we extend these results to finitely generated projective right modules over the noncommutative n torus. Then we define the corresponding analog of Sobolev spaces for which we prove the Sobolev and Rellich lemmas.

We list these results below.

Theorem 1.1. *Suppose P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M. Then the symbol of the adjoint P^* is of order M and satisfies $\sigma(P^*) \sim \sum_{\ell\in\mathbb{Z}^n_{\geq 0}} \frac{\partial^\ell\delta^\ell[(\rho(\xi))^*]}{\ell_1!\cdots\ell_n!}$.*

Theorem 1.2. *Suppose that P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M_1, and Q is a pseudo-differential operator with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order M_2. Then the symbol of the product QP is of order $M_1 + M_2$ and satisfies $\sigma(QP) \sim \sum_{\ell\in\mathbb{Z}^n_{\geq 0}} \frac{1}{\ell_1!\cdots\ell_n!}\partial^\ell\phi(\xi)\delta^\ell\rho(\xi)$, where $\partial^\ell := \prod_j \partial_j^{\ell_j}$ and $\delta^\ell := \prod_j \delta_j^{\ell_j}$.*

Theorem 1.3. *(a) For a pseudo-differential operator P with $r \times r$ matrix valued symbol $\sigma(P) = \rho = \rho(\xi)$, the symbol of the adjoint P^* satisfies*

$$\sigma(P^*) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho(\xi))^*}{\ell_1!\cdots\ell_n!}.$$

(b) If Q is a pseudo-differential operator with $r \times r$ matrix valued symbol $\sigma(Q) = \rho' = \rho'(\xi)$, then the product PQ is also a pseudo-differential operator and has symbol

$$\sigma(PQ) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}(\rho(\xi))\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho'(\xi))}{\ell_1!\cdots\ell_n!}.$$

Theorem 1.4. *For $s > k + 1$, $H^s \subseteq (A_\theta^k)^r e$.*

Theorem 1.5. *Let $\{\vec{a}_N\} \in (A_\theta^\infty)^r e$ be a sequence. Suppose that there is a constant C so that $||\vec{a}_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{\vec{a}_{N_j}\}$ that converges in H^t.*

2. Preliminaries

Fix some skew symmetric $n \times n$ matrix θ with upper triangular entries in $\mathbb{R}\backslash\mathbb{Q}$ that are linearly independent over $\mathbb{Q}$. Consider the irrational rotation C^*-algebra A_θ with n unitary generators $U_1, \ldots, U_n$ which satisfy $U_kU_j = e^{2\pi i\theta_{j,k}}U_jU_k$ and $U_j^* = U_j^{-1}$. Let $\{\alpha_s\}_{s\in\mathbb{R}^n}$ be a n-parameter group of automorphisms given by $\prod_j U_j^{m_j} \mapsto e^{is\cdot m}\prod_j U_j^{m_j}$. We define the subset A_θ^k of C^k elements of A_θ to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is C^k, and we define the subalgebra A_θ^∞ of smooth elements of A_θ to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is smooth. An alternative definition of the subalgebra A_θ^∞ of smooth elements is the elements in A_θ that can be expressed by an expansion of the form $\sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$, where the sequence $\{a_m\}_{m\in\mathbb{Z}^n}$ is in the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ in the sense that, for all $\alpha \in \mathbb{Z}^n$, $\sup_{m\in\mathbb{Z}^n}(\prod_j |m_j|^{\alpha_j}|a_m|) < \infty$. Define the trace $\tau : A_\theta \to \mathbb{C}$ by $\tau(\prod_j U_j^{m_j}) = 0$ for m_j not all zero and $\tau(1) = 1$ and define an inner product $\langle\cdot,\cdot\rangle : A_\theta \times A_\theta \to \mathbb{C}$ by $\langle a, b\rangle = \tau(b^*a)$ with induced norm $||\cdot|| : A_\theta \to \mathbb{R}_{\geq 0}$. Let $D_j = -i\partial_j$ and define derivations δ_j by the relations $\delta_j(U_j) = U_j$ and $\delta_j(U_k) = 0$ for $j \neq k$. For convenience, denote $\partial^\ell := \prod_j \partial_j^{\ell_j}$, $\delta^\ell := \prod_j \delta_j^{\ell_j}$, $D^\ell := \prod_j D_j^{\ell_j}$, and $\ell! = \prod_j \ell_j!$ for multi-indices $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$. We define a map $\psi : \rho \mapsto P_\rho$ assigning a pseudo-differential operator on A_θ^∞ to a symbol $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$.

Definition 2.1. For $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$, let P_ρ be the pseudo-differential operator sending arbitrary $a \in A_\theta^\infty$ to $P_\rho(a) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)\alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi$.

The integral above does not converge absolutely; it is an oscillatory integral. We define oscillatory integrals below as in [7].

Definition 2.2. Let q be a nondegenerate real quadratic form on $\mathbb{R}^n$, a be a C^∞ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1+|x|^2)^{-m/2}\partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$, and φ be a Schwartz function, i.e. the functions $x^\alpha \partial^\beta \varphi(x)$ are bounded on $\mathbb{R}^n$ for all pairs $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. Suppose further that $\varphi(0) = 1$. Then the limit $\lim_{\epsilon\to 0} \int e^{iq(x)} a(x)\varphi(\epsilon x)\,\mathrm{d}x$ exists, is independent of φ (as long as $\varphi(0) = 1$), and is equal to $\int e^{iq(x)} a(x)\,\mathrm{d}x$ when $a \in L^1$. When $a \notin L^1$, we continue to denote this limit by $\int e^{iq(x)} a(x)\,\mathrm{d}x$, and have an estimate $\left|\int e^{iq(x)} a(x)\,\mathrm{d}x\right| \leq C_{q,m} \max_{|\alpha|\leq m+n+1} \inf\{U \in \mathbb{R} : |(1+|x|^2)^{-m/2}\partial^\alpha a| \leq U \text{ almost everywhere}\}$ where $C_{q,m}$ depends only on the quadratic form q and the order m.

As shown in [7], oscillatory integrals behave essentially like absolutely convergent integrals in that one can still make changes of variables, integrate by parts, differentiate under the integral sign, and interverse integral signs. Given certain conditions on ρ, $P_\rho(a)$ satisfies the conditions of the oscillatory integral, and we can evaluate $P_\rho(a)$.

Definition 2.3. An element $\rho = \rho(\xi) = \rho(\xi_1, \ldots, \xi_n)$ of $C^\infty(\mathbb{R}^n, A_\theta^\infty)$ is a symbol of order m if and only if for all non-negative integers $i_1, \ldots, i_n, j_1, \ldots, j_n$, $||\delta_1^{i_1} \cdots \delta_n^{i_n}(\partial_1^{j_1} \cdots \partial_n^{j_n}\rho)(\xi)|| \leq C_\rho(1+|\xi|)^{m-|j|}$.

Example 2.6(i) of [7] gives a convenient formula for evaluating the oscillatory integrals that appear in our calculation of $P_\rho(a)$.

Proposition 2.4. *Suppose that, for some m, a is a C^∞ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1+|x|^2)^{-m/2}\partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. Then* $\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta} a(y)\,\mathrm{d}y\,\mathrm{d}\eta = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta} a(\eta)\,\mathrm{d}y\,\mathrm{d}\eta = a(0)$.

We apply Proposition 2.4 to get a basic result.

Lemma 2.5. *Let $a = \sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$ be an arbitrary element of A_θ^∞ and let $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$ be a symbol of order M. Then $P_\rho(a) = \sum_{m\in\mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j}$.*

Proof. First consider the case $a = \prod_j U_j^{m_j}$. We get

$$\begin{aligned}
P_\rho\left(\prod_{j=1}^n U_j^{m_j}\right) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)\alpha_s\left(\prod_{j=1}^n U_j^{m_j}\right)\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)e^{is\cdot m}\prod_{j=1}^n U_j^{m_j}\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot(\xi-m)}\rho(\xi)\,\mathrm{d}s\,\mathrm{d}\xi \prod_{j=1}^n U_j^{m_j} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\eta}\rho(\eta+m)\,\mathrm{d}s\,\mathrm{d}\eta \prod_{j=1}^n U_j^{m_j} = \rho(m)\prod_{j=1}^n U_j^{m_j},
\end{aligned}$$

as desired, having substituted $\eta = \xi - m$ and applied the result of Proposition 2.4. Now consider the general case $a = \sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$. Since α_s is an automorphism on A_θ, we get $P_\rho(a) = \sum_{m\in\mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j}$, and we are done. □

3. Asymptotic formula for the symbol of the adjoint of a pseudo-differential operator

Here we prove the formula for the symbol of the adjoint for the noncommutative n torus, adapting the proof of Lemma 1.2.3 of [6] to the noncommutative n torus.

Theorem 3.1. *Suppose P is a pseudo-differential operator with symbol $\sigma(P)=\rho=\rho(\xi)$ of order M. Then the symbol of the adjoint P^* is of order M and satisfies $\sigma(P^*)\sim\sum_{\ell\in\mathbb{Z}^n_{\geq 0}}\frac{\partial^\ell\delta^\ell[(\rho(\xi))^*]}{\ell_1!\cdots\ell_n!}$.*

Proof. Let $a,b\in A_\theta^\infty$. We have

$$\begin{aligned}\langle P_\rho(a),b\rangle &= \tau\left(b^*\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\xi}\rho(\xi)\alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi\right)\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\xi}\tau(b^*\rho(\xi)\alpha_s(a))\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\xi}\tau(\alpha_{-s}(\rho(\xi)^*b)^*a)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \tau\left(\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{+is\cdot\xi}\alpha_{-s}(\rho(\xi)^*b)\,\mathrm{d}s\,\mathrm{d}\xi\right)^*a\right)=\langle a,P_\rho^*(b)\rangle\end{aligned}$$

where

$$\begin{aligned}P_\rho^*(b) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{+is\cdot\xi}\alpha_{-s}(\rho(\xi)^*b)\,\mathrm{d}s\,\mathrm{d}\xi=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{+is\cdot\xi}\alpha_{-s}(\rho(\xi)^*)\alpha_{-s}(b)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \sum_{m,k}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{+is\cdot\xi}e^{+is\cdot m}\rho_m(\xi)^*e^{-is\cdot k}b_k\,\mathrm{d}s\,\mathrm{d}\xi\,U_n^{-m_n}\cdots U_1^{-m_1}U_1^{k_1}\cdots U_n^{k_n}\\ &= \sum_{m,k}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\eta}\rho_m((k-m)-\eta)^*b_k\,\mathrm{d}s\,\mathrm{d}\eta\,U_n^{-m_n}\cdots U_1^{-m_1}U_1^{k_1}\cdots U_n^{k_n}\\ &= \sum_{m,k}\rho_m(k-m)^*b_kU_n^{-m_n}\cdots U_1^{-m_1}U_1^{k_1}\cdots U_n^{k_n}\\ &= \sum_{m,k}(\rho_m(k-m)U_1^{m_1}\cdots U_n^{m_n})^*(b_kU_1^{k_1}\cdots U_n^{k_n}),\end{aligned}$$

so

$$\begin{aligned}\sigma(P_\rho^*)(\xi) &= \left[\sum_m\rho_m(\xi-m)\prod_{j=1}^n U_j^{m_j}\right]^*\\ &= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-ix\cdot y}\sum_m\rho_m(\xi-y)\alpha_x\left(\prod_{j=1}^n U_j^{m_j}\right)\mathrm{d}x\,\mathrm{d}y\right]^*\\ &= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-ix\cdot y}\alpha_x(\rho(\xi-y))\,\mathrm{d}x\,\mathrm{d}y\right]^*.\end{aligned}$$

We have $\rho(\xi-y)=\sum_{|\ell|<N_1}\frac{(-y)^\ell}{\ell!}(\partial^\ell\rho)(\xi)+R_{N_1}(\xi,y)$ and

$$\alpha_x(\rho(\xi-y))=\sum_{|\ell|<N_1}\sum_m\frac{(-y)^\ell}{\ell!}(\partial^\ell\rho_m)(\xi)e^{ix\cdot m}\left(\prod_{j=1}^n U_j^{m_j}\right)+\alpha_x(R_{N_1}(\xi,y))$$

where $R_{N_1}(\xi, y) = N_1 \sum_{|\ell|=N_1} \frac{(-y)^\ell}{\ell!} \int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell \rho)(\xi - y\gamma)\,\mathrm{d}\gamma$ with corresponding symbol

$$\sigma(P_\rho^*)(\xi) = \left[\sum_{|\ell|<N_1} \sum_m \frac{(-m)^\ell}{\ell!}(\partial^\ell \rho_m)(\xi) \prod_{j=1}^m U_j^{m_j} + T_{N_1}(\xi, y)\right]^* = \sum_{|\ell|<N_1} \frac{\partial^\ell \delta^\ell[(\rho(\xi))^*]}{\ell!} + T_{N_1}(\xi, y)^*$$

where $T_{N_1}(\xi, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \alpha_x(R_{N_1}(\xi, y))\,\mathrm{d}x\,\mathrm{d}y$. It remains to show that this symbol is of order M. Obviously, $\sum_{N\le \ell<N_1} \frac{\partial^\ell \delta^\ell[(\rho(\xi))^*]}{\ell!} \in S^{M-N}$, so we need to show that the remainder is of order $M - N_1$. Note that $\sigma(P_\rho^*)(\xi) - \sum_{|\ell|<N_1} \frac{\partial^\ell \delta^\ell[(\rho(\xi))^*]}{\ell!} = T_{N_1}(\xi, y)^*$. Integrating by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \frac{(-y)^\ell}{\ell!} \alpha_x \left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell \rho)(\xi - y\gamma)\,\mathrm{d}\gamma\right) \mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} (-D_x)^\ell \alpha_x \left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell \rho)(\xi - y\gamma)\,\mathrm{d}\gamma\right) \mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} (-\delta)^\ell \alpha_x \left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell \rho)(\xi - y\gamma)\,\mathrm{d}\gamma\right) \mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \int_0^1 (1-\gamma)^{N_1-1}(-\delta)^\ell \alpha_x((\partial^\ell \rho)(\xi - y\gamma))\,\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} (-1)^\ell \int_0^1 (1-\gamma)^{N_1-1} \alpha_x((\delta^\ell \partial^\ell \rho)(\xi - y\gamma))\,\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

where, for arbitrary $a = \sum_m a_m \prod_j U_j^{m_j}$,

$$\begin{aligned}
(-D_x)^\ell \alpha_x(a) = (-D_x)^\ell \alpha_x \left(\sum_m a_m \prod_{j=1}^n U_j^{m_j}\right) &= (-D_x)^\ell \sum_m a_m e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j} \\
&= \sum_m a_m (-m)^\ell e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j} = (-\delta)^\ell \alpha_x(a).
\end{aligned}$$

Since $\rho \in S^M$ and $|\ell| = N_1$ we have $\delta^\ell \partial^\ell \rho \in S^{M-N_1}$. We get the boundedness of $\ell^{N_1-M}(\xi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \alpha_x(R_{N_1}(\xi, y))\,\mathrm{d}x\,\mathrm{d}y$ because $\partial^\ell R_{N_1}(y, \xi)$ is the rest of index N_1 in the Taylor expansion of $\partial^\ell(\xi - y\gamma)$ for which one has $\partial^\ell \rho \in S^{M-N_1}$. $\square$

4. Asymptotic formula for the symbol of a product of two pseudo-differential operators

Next we prove the formula for the product or composition of symbols for the noncommutative n torus, adapting the proof of Theorem 7.1 of [8].

Theorem 4.1. *Suppose that P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M_1, and Q is a pseudo-differential operator with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order M_2. Then the symbol of the product QP is of order $M_1 + M_2$ and satisfies $\sigma(QP) \sim \sum_{\ell \in \mathbb{Z}^n_{\ge 0}} \frac{1}{\ell_1! \cdots \ell_n!} \partial^\ell \phi(\xi) \delta^\ell \rho(\xi)$, where $\partial^\ell := \prod_j \partial_j^{\ell_j}$ and $\delta^\ell := \prod_j \delta_j^{\ell_j}$.*

Proof. We want to show that if $\rho : \mathbb{R}^n \to A_\theta^\infty$ is of order M_1 and $\phi : \mathbb{R}^n \to A_\theta^\infty$ is of order M_2, $P_\phi \circ P_\rho = P_\mu$ where μ is of order $M_1 + M_2$ and has asymptotic expansion $\mu \sim \sum_\ell \frac{1}{\ell!} \partial^\ell \phi(\xi) \delta^\ell \rho(\xi)$. Let $\{\varphi_k\}$ be the partition of unity constructed in Theorem 6.1 of [8] and define $\phi_k(\xi) := \phi(\xi)\varphi_k(\xi)$. We have $P_{\phi_k}(a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{is\cdot\xi} \phi_k(\xi) \alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi$. Summing over k from zero to infinity and applying Fubini's Theorem, we get $\sum_{k=0}^\infty P_{\phi_k}(a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \sum_{k=0}^\infty \phi_k(\xi) \alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi =$

$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi(\xi)\alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi$, so $P_\phi(a) = \sum_{k=0}^\infty P_{\phi_k}(a)$ and the convergence of the series is absolute and uniform for all $a \in A_\theta^\infty$. We want to compute the symbol of $P_\phi \circ P_\rho$, but issues with convergence of integrals make it so we need to compute the symbol of $P_{\phi_k} \circ P_\rho$. Let $a \in A_\theta^\infty$ be arbitrary. Applying $P_{\phi_k} \circ P_\rho$ we get

$$\begin{aligned}
P_{\phi_k}(P_\rho(a)) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s(P_\rho(a))\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s\Big(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-it\cdot\eta}\rho(\eta)\alpha_t(a)\,\mathrm{d}t\,\mathrm{d}\eta\Big)\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi-it\cdot\eta}\phi_k(\xi)\alpha_s(\rho(\eta))\alpha_{s+t}(a)\,\mathrm{d}t\,\mathrm{d}\eta\Big\}\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi-i(y-x)\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\eta\Big\}\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-\eta)-iy\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\eta\Big\}\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma-iy\cdot\tau}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau\Big\}\,\mathrm{d}x\,\mathrm{d}\sigma \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\Big\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma\Big\}\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\mu_k(\tau)\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau
\end{aligned}$$

where $\mu_k(\tau) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau)))\,\mathrm{d}x\,\mathrm{d}\sigma$, having done the changes of variables $(x,y) = (s, s+t)$ and $(\sigma,\tau) = (\xi-\eta,\eta)$ and applied Proposition 2.4. This suggests that $P_\phi(P_\rho(a)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\mu(\tau)\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau$ where $\mu(\tau) = \sum_{k=0}^\infty \mu_k(\tau)$. We need to show that μ is a symbol in $S^{M_1+M_2}$ and has our desired asymptotic expression. Define μ_k by $\mu_k(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi_k(\xi+y)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y$ for all $\xi \in \mathbb{R}^n$. By Taylor's formula with integral remainder given in Theorem 6.3 of [8], we get $\phi_k(\xi+y) = \sum_{|\ell|<N_1}\frac{y^\ell}{\ell!}(\partial^\ell\phi_k)(\xi) + R_{N_1}(y,\xi)$ where $R_{N_1}(y,\xi) = N_1\sum_{|\ell|=N_1}\frac{y^\ell}{\ell!}\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma$ for all $y,\xi \in \mathbb{R}^2$. Substituting back into our expression for $\mu_k(\xi)$ we get $\mu_k(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\sum_{|\ell|<N_1}\frac{y^\ell}{\ell!}(\partial^\ell\phi_k)(\xi)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y + T_{N_1}^{(k)}(\xi)$ where $T_{N_1}^{(k)}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}R_{N_1}(y,\xi)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y$. Expressing $\rho(\xi)$ as $\rho(\xi) = \sum_m \rho_m(\xi)\prod_{j=1}^n U_j^{m_j}$, we see that $\alpha_x(\rho(\xi)) = \sum_m \rho_m(\xi)e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}$ so

$$\begin{aligned}
\mu_k(\xi) - T_{N_1}^{(k)}(\xi) &= \sum_{|\ell|<N_1}\sum_m \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\frac{y^\ell}{\ell!}(\partial^\ell\phi_k)(\xi)\rho_m(\xi)e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{|\ell|<N_1}\frac{1}{\ell!}(\partial^\ell\phi_k)(\xi)\sum_m \rho_m(\xi)\prod_{j=1}^n U_j^{m_j}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot(y-m)}y^\ell\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{|\ell|<N_1}\frac{1}{\ell!}(\partial^\ell\phi_k)(\xi)\sum_m \rho_m(\xi)\prod_{j=1}^n U_j^{m_j}m^\ell = \sum_{|\ell|<N_1}\frac{1}{\ell!}(\partial^\ell\phi_k)(\xi)(\delta^\ell\rho)(\xi).
\end{aligned}$$

Let $\mu(\xi) = \sum_{k=0}^\infty \mu_k(\xi)$. It remains to show that μ is of order M_1+M_2. Obviously, $\sum_{N\le|\ell|<N_1}\frac{y^\ell}{\ell!}(\partial^\ell\phi)(\xi)(\delta^\ell\rho)(\xi) \in S^{M_1+M_2-N}$, so we just need to show that the remainder is of order $M_1+M_2-N_1$. Note that $\mu(\xi) - \sum_{|\ell|<N_1}\frac{y^\ell}{\ell!}(\partial^\ell\phi)(\xi)(\delta^\ell\rho)(\xi) = \sum_{k=0}^\infty T_{N_1}^{(k)}(\xi)$ where $T_{N_1}^{(k)}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}R_{N_1}(y,\xi)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y$ and $R_{N_1}(y,\xi) = N_1\sum_{|\ell|=N_1}\frac{y^\ell}{\ell!}\int_0^1(1-$

$\gamma)^{N_1-1}(\partial^\ell\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma$. Integrating by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\frac{y^\ell}{\ell!}\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}y^\ell\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell(\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell(\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma D_x^\ell\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell(\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma\delta^\ell\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\int_0^1(1-\gamma)^{N_1-1}(\partial^\ell(\phi_k)(\xi+\gamma y)\,\mathrm{d}\gamma\alpha_x((\delta^\ell\rho)(\xi))\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

where for arbitrary $a=\sum_m a_m\prod_j U_j^{m_j}$ we have

$$\begin{aligned}
(D_x)^\ell\alpha_x(a)&=(D_x)^\ell\alpha_x\left(\sum_m a_m\prod_{j=1}^n U_j^{m_j}\right)=(D_x)^\ell\sum_m a_m e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}\\
&=\sum_m a_m m^\ell e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}=\delta^\ell\alpha_x(a).
\end{aligned}$$

Since $|\ell|=N_1$, we have $\partial^\ell(\phi_k)\in S^{M_2-N_1}$ and $\delta^\ell\rho\in S^{M_1}$. We get the boundedness of $\mu^{N_1-M_1-M_2}(\xi)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-ix\cdot y}R_{N_1}(y,\xi)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y$ since $\delta^\ell\rho\in S^{M_1}$ and $\partial^\ell R_{N_1}(y,\xi)$ is the rest of index N_1 in Taylor's expansion of $\partial^\ell\phi_k(\xi+\gamma y)$ for which on has $\partial^\ell\phi_k\in S^{M_2-N_1}$. □

5. The pseudo-differential calculus on finitely generated projective modules over the noncommutative n torus

We can generalize these results to arbitrary finitely generated projective right modules over the noncommutative n torus following p. 553 of [2], which considers finitely generated projective modules over an arbitrary unital $*$-algebra. Let E be a finitely generated projective right A_θ^∞-module. Since E is a finitely generated projective right A_θ^∞-module, we can write E as a direct summand $E=(A_\theta^\infty)^r e$ of a free module $(A_\theta^\infty)^r$ with direct complement $F=(A_\theta^\infty)^r(\mathrm{id}-e)$, where the idempotent $e\in M_r(A_\theta^\infty)$ is self-adjoint. Consider an $r\times r$ matrix valued symbol $\rho=(\rho_{j,k})$ where $\rho_{j,k}:\mathbb{R}^n\to A_\theta^\infty$ are scalar symbols and $\rho_{j,k}\in S^d$. Define the operator $P_\rho:E\to E$ as follows: $P_\rho(\vec{a}):=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\xi}\rho(\xi)\alpha_s(\vec{a})\,\mathrm{d}s\,\mathrm{d}\xi$. Define the inner product $\langle\vec{a},\vec{b}\rangle:E\times E\to\mathbb{C}$ sending $(\vec{a},\vec{b})\mapsto\tau(\vec{b}^*\vec{a})$. Since $P_\rho(\vec{a})_j=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-is\cdot\xi}\sum_{k=1}^r\rho_{j,k}(\xi)\alpha_s(a_k)\,\mathrm{d}s\,\mathrm{d}\xi$, Lemma 2.5 generalizes to E as follows after applying it to each component: $P_\rho(\vec{a})=\sum_m\rho(m)\vec{a}_m\prod_{j=1}^n U_j^{m_j}$. Theorems 3.1 and 4.1 generalize as follows.

Theorem 5.1. *(a) For a pseudo-differential operator P with $r\times r$ matrix valued symbol $\sigma(P)=\rho=\rho(\xi)$, the symbol of the adjoint P^* satisfies*

$$\sigma(P^*)\sim\sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq0})^n}\frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho(\xi))^*}{\ell_1!\cdots\ell_n!}.$$

(b) If Q is a pseudo-differential operator with $r\times r$ matrix valued symbol $\sigma(Q)=\rho'=\rho'(\xi)$, then the product PQ is also a pseudo-differential operator and has symbol

$$\sigma(PQ)\sim\sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq0})^n}\frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}(\rho(\xi))\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho'(\xi))}{\ell_1!\cdots\ell_n!}.$$

IOP Conf. Series: Journal of Physics: Conf. Series **965** (2018) 012042 doi:10.1088/1742-6596/965/1/012042

Proof. First let's prove part (a). Let ρ be an $r\times r$ matrix valued symbol of order M and $\vec{a},\vec{b}\in E$. We have

$$\begin{aligned}\langle P_\rho(\vec{a}),\vec{b}\rangle &= \tau\left(\vec{b}^*\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)\alpha_s(\vec{a})\,\mathrm{d}s\,\mathrm{d}\xi\right)\\ &= \tau\left(\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\alpha_{-s}(\rho(\xi)^*\vec{b})\,\mathrm{d}s\,\mathrm{d}\xi\right)^*\vec{a}\right) = \langle\vec{a},P^*_\rho(\vec{b})\rangle\end{aligned}$$

where

$$\begin{aligned}P^*_\rho(\vec{b})_j &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\sum_{k=1}^{r}\alpha_{-s}(\rho(\xi)^*_{j,k}b_k)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\sum_{k=1}^{r}\alpha_{-s}(\rho(\xi)^*_{j,k})\alpha_{-s}(b_k)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \sum_{m,p}\left[\rho_m(p-m)\prod_{h=1}^{n}U_h^{m_h}\right]^*\left(b_p\prod_{h=1}^{n}U_h^{p_h}\right)\end{aligned}$$

so

$$\begin{aligned}\sigma(P^*_\rho)(\xi) &= \left[\sum_m\rho_m(\xi-m)\prod_{h=1}^{n}U_h^{m_h}\right]^*\\ &= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\sum_m\rho_m(\xi-y)\alpha_x\left(\prod_{h=1}^{n}U_h^{m_h}\right)\,\mathrm{d}x\,\mathrm{d}y\right]^*\\ &= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\alpha_x(\rho(\xi-y))\,\mathrm{d}x\,\mathrm{d}y\right]^*.\end{aligned}$$

The rest of the proof reduces to the $r=1$ case, applying it to each entry in $\rho=(\rho_{j,k})$.

We proceed to part (b). Let ρ be an $r\times r$ matrix valued symbol of order m_1 and ϕ be an $r\times r$ matrix valued symbol of order m_2. Let $\{\varphi_k\}$ be a partition of unity and define $\phi_k(\xi):=\phi(\xi)\varphi_k(\xi)$. Let $\vec{a}\in E$. We have

$$\begin{aligned}P_{\phi_k}(P_\rho(\vec{a})) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s(P_\rho(\vec{a}))\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-it\cdot\eta}\rho(\eta)\alpha_t(\vec{a})\,\mathrm{d}t\,\mathrm{d}\eta\right)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi-it\cdot\eta}\phi_k(\xi)\alpha_s(\rho(\eta))\alpha_{s+t}(\vec{a})\,\mathrm{d}t\,\mathrm{d}\eta\right\}\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi-i(y-x)\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\eta\right\}\,\mathrm{d}x\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-\eta)-iy\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\eta\right\}\,\mathrm{d}x\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma-iy\cdot\eta}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau\right\}\,\mathrm{d}x\,\mathrm{d}\sigma\\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma\right\}\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\lambda_k(\tau)\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau\end{aligned}$$

where $\lambda_k(\tau) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma$ so

$$P_\phi(P_\rho(\vec{a})) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\lambda(\tau)\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau$$

where $\lambda(\tau) = \sum_{k=0}^{\infty}\lambda_k(\tau)$. Let $\lambda_k(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi_k(\xi+y)\alpha_x(\rho(\xi))\,\mathrm{d}x\,\mathrm{d}y$. Since

$$\begin{aligned}\lambda_k(\xi)_{\alpha,\gamma} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\sum_{\beta=1}^{r}\phi_k(\xi+y)_{\alpha,\beta}\alpha_x(\rho(\xi)_{\beta,\gamma})\,\mathrm{d}x\,\mathrm{d}y\\ &= \sum_{\beta=1}^{r}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi_k(\xi+y)_{\alpha,\beta}\alpha_x(\rho(\xi)_{\beta,\gamma})\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

the rest of the proof reduces to the $r = 1$ case, applying it to each summand in the above sum. □

6. Sobolev spaces on finitely generated projective modules over the noncommutative n torus

Let $\lambda(\xi) = (1+\xi_1^2+\cdots+\xi_n^2)^{1/2}\mathrm{id}_E$. Consider the following inner product on E.

Definition 6.1. Define the Sobolev inner product $\langle\cdot,\cdot\rangle_s : E\times E \to \mathbb{C}$ by $\langle\vec{a},\vec{b}\rangle_s := \langle P_{\lambda^s}(\vec{a}), P_{\lambda^s}(\vec{b})\rangle = \sum_{j,m}(1+|m_1|^2+\cdots+|m_n|^2)^s\overline{b_{j,m}}a_{j,m}$.

Note that for $s = 0$ this agrees with $\langle\cdot,\cdot\rangle$. This inner product induces the following norm.

Definition 6.2. Define the Sobolev norm $||\cdot||_s : E \to \mathbb{R}_{\geq 0}$ by $||\vec{a}||_s^2 := \langle P_{\lambda^s}(\vec{a}), P_{\lambda^s}(\vec{a})\rangle = \sum_{j,m}(1+|m_1|^2+\cdots+|m_n|^2)^s|a_{j,m}|^2$.

Using this norm, we can define the analog of Sobolev spaces on E.

Definition 6.3. Define the Sobolev space H^s to be the completion of E with respect to $||\cdot||_s$.

We can prove that a pseudo-differential operator of order $d\in\mathbb{R}$ continuously maps H^s into H^{s-d}. However we must first prove the case where $s = d$.

Theorem 6.4. *Suppose ρ is a matrix valued symbol of order d. Then, for any $\vec{a}\in E$, $||P_\rho(\vec{a})||_0 \leq C||\vec{a}||_d$ for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^d \to H^0$.*

Proof. Let F be an orthogonal eigenbasis of e normalized with respect to $\langle\cdot,\cdot\rangle$, and let $F_1 := \{f_h : 1\leq h\leq r_1\}$ be the subset of eigenvectors with eigenvalue 1. Note that $\{\prod_g U_g^{m_g}f_j : m\in\mathbb{Z}^n, 1\leq j\leq r_1\}$ is an orthogonal basis of E considered as a $\mathbb{C}$-vector space, with respect to $\langle\cdot,\cdot\rangle_s$. We have $||\prod_g U_g^{m_g}f_j||_0^2 = 1$, $||\rho_m(\xi)_{h,j}\prod_g U_g^{m_g}f_j||_0^2 = |\rho_m(\xi)_{h,j}|^2$, and $||\rho(\xi)_{h,j}f_j||_0^2 = \sum_m|\rho_m(\xi)_{h,j}|^2$. Since $\rho_{h,j}$ is of order d, we have $||\rho(\xi)_{h,j}||_0 \leq C_\rho(1+|\xi|)^d$, and since $(1-|\xi|)^2\geq 0$ gives us $(1+|\xi|)^2 \leq 2(1+|\xi|^2)$, we have $||\rho(\xi)_{h,j}||_0^2 \leq C_\rho^2(1+|\xi|)^{2d} \leq C_\rho^2 2^d(1+|\xi|^2)^d$. Let $k_\rho := C_\rho^2 2^d$. Then we have $\sum_m|\rho_m(\xi)_{h,j}|^2 \leq k_\rho(1+|\xi|^2)^d$. Let $e_{s,m} := (1+|m_1|^2+\cdots+|m_n|^2)^{-s/2}\prod_g U_g^{m_g}$ and $E_s := \{e_{s,m} \mid m\in\mathbb{Z}\}$. By definition we have E_sF_1 orthonormal with respect to $\langle\cdot,\cdot\rangle_s$. It suffices to prove this theorem for the case $\vec{a} = e_{d,m}f_j$ by the orthonormality of E_dF_1 since $||P_\rho(\vec{a})||_0^2 = \sum_{j,m}|a_{j,m}|^2||P_\rho(e_{d,m}f_j)||_0^2$ and $||\vec{a}||_d^2 = \sum_{j,m}|a_{j,m}|^2||e_{d,m}f_j||_d^2$. Since $||e_{d,m}f_j||_d^2 = 1$, it suffices to show that $||P_\rho(e_{d,m}f_j)||_0^2 \leq K$ for some constant $K > 0$. We have

$$\begin{aligned}||P_\rho(e_{d,m}f_j)||_0^2 = ||\rho(m)e_{d,m}f_j||_0^2 &= \left|\left|\rho(m)(1+|m_1|^2+\cdots+|m_n|^2)^{-d/2}\prod_g U_g^{m_g}f_j\right|\right|_0^2\\ &= \left|\left|\sum_k\rho_k(m)\prod_g U_g^{k_g}(1+|m_1|^2+\cdots+|m_n|^2)^{-d/2}\prod_g U_g^{m_g}f_j\right|\right|_0^2\end{aligned}$$

$$
\begin{aligned}
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \left\|\sum_k \rho_k(m)\prod_g U_g^{k_g}\prod_g U_g^{m_g} f_j\right\|_0^2 \\
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \left\|\sum_k \rho_k(m)w(m,k)\prod_g U_g^{k_g+m_g} f_j\right\|_0^2 \\
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \left\|\sum_k \rho_{k-m}(m)w(m,k-m)\prod_g U_g^{k_g} f_j\right\|_0^2 \\
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \left\|\sum_{h,k} \rho_{k-m}(m)_{h,j}w(m,k-m)\prod_g U_g^{k_g} f_j\right\|_0^2 \\
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \sum_{h,k} |\rho_{k-m}(m)_{h,j}|^2 \\
&= (1+|m_1|^2+\cdots+|m_n|^2)^{-d} \sum_{h,k} |\rho_{k}(m)_{h,j}|^2 \\
&\leq (1+|m_1|^2+\cdots+|m_n|^2)^{-d} r_1 k_\rho (1+|m_1|^2+\cdots+|m_n|^2)^{d} = r_1 k_\rho
\end{aligned}
$$

where $w(k,m) := \prod_{j=1}^n U_j^{m_j}\prod_{j=1}^n U_j^{k_j}\left(\prod_{j=1}^n U_j^{m_j+k_j}\right)^{-1} \in S^1 \subset \mathbb{C}$ so our desired constant is $K = r_1 k_\rho = r_1 C_\rho^2 2^d$ and we are done. □

For the general case $s \neq d$ we need to prove a lemma saying that $||\cdot||_s = ||\cdot||_{s-t} \circ P_{\lambda^t}$.

Lemma 6.5. *For any $\vec{a} \in E$ and $s, t \in \mathbb{R}$, $\vec{a} \in H^s$ if and only if $P_{\lambda^t}(\vec{a}) \in H^{s-t}$ with $||\vec{a}||_s = ||P_{\lambda^t}(\vec{a})||_{s-t}$.*

Proof. Suppose that $\vec{a} \in H^s$ or $P_{\lambda^t}(\vec{a}) \in H^{s-t}$. Then

$$
\begin{aligned}
||P_{\lambda^t}(\vec{a})||_{s-t}^2 &= \sum_{j,m}(1+|m_1|^2+\cdots+|m_n|^2)^{s-t}\lambda^{2t}(m)|a_{j,m}|^2 \\
&= \sum_{j,m}(1+|m_1|^2+\cdots+|m_n|^2)^{s-t}(1+|m_1|^2+\cdots+|m_n|^2)^{t}|a_{j,m}|^2 \\
&= \sum_{j,m}(1+|m_1|^2+\cdots+|m_n|^2)^{s}|a_{j,m}|^2 = ||\vec{a}||_s
\end{aligned}
$$

so we know that $\vec{a} \in H^s$ and $P_{\lambda^t}(\vec{a}) \in H^{s-t}$. □

Then the general case follows quite easily.

Corollary 6.6. *Suppose ρ is a matrix valued symbol of order d. Then $||P_\rho(\vec{a})||_{s-d} \leq C||\vec{a}||_s$ for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^s \to H^{s-d}$.*

Proof. By Lemma 6.5, we have $||P_\rho(\vec{a})||_{s-d} = ||P_{\lambda^{s-d}}(P_\rho(\vec{a}))||_0$. By Proposition 5.1(b), the matrix valued symbol $\sigma(P_{\lambda^{s-d}} \circ P_\rho)$ is of order $d + (s-d) = s$, so Theorem 6.4 gives us $||P_{\lambda^{s-d}}(P_\rho(\vec{a}))||_0 \leq C||\vec{a}||_s$ for some constant $C > 0$. □

We can also define an analog of the C^k norm on E.

Definition 6.7. Define the C^k norm $||\cdot||_{\infty,k} : E \to \mathbb{R}_{\geq 0}$ as follows: $||\vec{a}||_{\infty,k} := \sum_{|\ell|\leq k} ||\delta^\ell(\vec{a})||_{C^*}$ where the C^* norm $||\cdot||_{C^*}$ is given by $||\vec{a}||_{C^*}^2 := \sup\{|\lambda| : \vec{a}^*\vec{a} - \lambda\cdot 1 \text{ not invertible}\}$.

Since, for arbitrary $\vec{a} = \sum_{j,m} a_{j,m} \prod_g U_g^{m_g} f_j$,

$$D_s^\ell \alpha_s(\vec{a}) = (-i\partial_s)^\ell \sum_{j,m} e^{is\cdot m} a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j = \sum_{j,m} m^\ell e^{is\cdot m} a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j$$
$$= \sum_{j,m} \delta^\ell e^{is\cdot m} a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j = \delta^\ell \alpha_s(\vec{a})$$

we have $(A_\theta^k)^r e = C^k$. We can easily prove an analog of the Sobolev lemma on E as follows.

Theorem 6.8. *For $s > k+1$, $H^s \subseteq (A_\theta^k)^r e$.*

Proof. First consider the case $k = 0$. Note that $||\cdot||_{\infty,0} = ||\cdot||_{C^*}$ so for arbitrary $a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j$ we have $||a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j||_{\infty,0}^2 = \sup\{|\lambda| : |a_{j,m}|^2 - \lambda \cdot 1 \text{ not invertible}\} = |a_{j,m}|^2$ and for arbitrary $\vec{a} = \sum_{j,m} a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j$ we have $||\vec{a}||_{\infty,0}^2 \leq \sum_{j,m} ||a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j||_{\infty,0}^2 = \sum_{j,m} |a_{j,m}|^2 = ||\vec{a}||_0^2$ by the triangle inequality. We have $\vec{a} = \sum_{j,m} a_{j,m} \lambda^s(m)\lambda^{-s}(m) \prod_{g=1}^n U_g^{m_g} f_j$ so by the Cauchy-Schwarz inequality we get $||\vec{a}||_0^2 \leq ||\vec{a}||_s^2 \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}$. Since $2s > 2$, $(1+|m_1|^2+\cdots+|m_n|^2)^{-s}$ is summable over $m \in \mathbb{Z}^n$ and $j \in \{1, \ldots, r\}$ so $||\vec{a}||_0^2 \leq C||\vec{a}||_s$. Thus we get $||\vec{a}||_{\infty,0} \leq ||\vec{a}||_0 \leq C||\vec{a}||_s$ and $H^s \subseteq (A_\theta^0)^r e$.

Now suppose $k > 0$. Using what we've proven for the previous case, we have

$$||\delta^\ell(\vec{a})||_{\infty,0} \leq C||\delta^\ell(\vec{a})||_{s-|\ell|} = C \left|\left| \sum_{j,m} m^\ell a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j \right|\right|_{s-|\ell|}$$
$$< C \left|\left| \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^{|\ell|} a_{j,m} \prod_{g=1}^n U_g^{m_g} f_j \right|\right|_{s-|\ell|}$$
$$= C||P_{\lambda^{|\ell|}}(\vec{a})||_{s-|\ell|} = C||\vec{a}||_s$$

for $|\ell| \leq k$ since $s - |\ell| \geq s - k > 1$. Therefore, $||\vec{a}||_{\infty,k} = \sum_{|\ell|\leq k} ||\delta^\ell(\vec{a})||_{\infty,0} \leq \sum_{|\ell|\leq k} C||\vec{a}||_s \leq C||\vec{a}||_s(k+1)(k+2)/2$ and we get $H^s \subseteq (A_\theta^k)^r e$. □

We get the following corollary.

Corollary 6.9. *$\bigcap_{s\in\mathbb{R}} H^s = (A_\theta^\infty)^r e$.*

Proof. Suppose $a \in \bigcap_{s\in\mathbb{R}} H^s$. Then for any $k \in \mathbb{Z}_{\geq 0}$, $\vec{a} \in H^{k+2}$, so by the theorem we just proved, $\vec{a} \in (A_\theta^k)^r e$. Consequently $\vec{a} \in (A_\theta^\infty)^r e$, so $\bigcap_{s\in\mathbb{R}} H^s \subseteq (A_\theta^\infty)^r e$.

Suppose $a \in (A_\theta^\infty)^r e$. Then since H^s is the completion of $(A_\theta^\infty)^r e$ with respect to $||\cdot||_s$, $(A_\theta^\infty)^r e \subseteq H^s$ for all $s \in \mathbb{R}$, and $(A_\theta^\infty)^r e \subseteq \bigcap_{s\in\mathbb{R}} H^s$. □

We can also prove an analog of the Rellich lemma on E.

Theorem 6.10. *Let $\{\vec{a}_N\} \in (A_\theta^\infty)^r e$ be a sequence. Suppose that there is a constant C so that $||\vec{a}_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{\vec{a}_{N_j}\}$ that converges in H^t.*

Proof. Let F be an orthogonal eigenbasis of e normalized with respect to $\langle\cdot,\cdot\rangle$, and let $F_1 := \{f_h : 1 \leq h \leq r_1\}$ be the subset of eigenvectors with eigenvalue 1. Let $e_{s,m} := (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s/2} \prod_g U_g^{m_g}$ and $E_s := \{e_{s,m} \mid m \in \mathbb{Z}\}$. $E_s F_1$ is an orthogonal basis of E considered as a $\mathbb{C}$-vector space normalized with respect to $\langle\cdot,\cdot\rangle_s$, so we can write $\vec{a}_N := \sum_{h,k} a_{N,h,k} e_{s,k} f_h$. Then $|a_{N,h,k}|^2 \leq \sum_{h,k} |a_{N,h,k}|^2 \leq C^2$ and $|a_{N,h,k}| \leq C$. Applying the Arzela-Ascoli theorem to $\{a_{N,h,k}\}$ for some fixed (h,k), we can get a subsequence $\{a_{N_j,h,k}\}$ of

$\{a_{N,h,k}\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that $|a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Do this for all $1 \leq h \leq r_1$ and $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$, replacing $\{a_N\}$ with $\{a_{N_j}\}$ each time. Then we get a subsequence $\{a_{N_j}\}$ of $\{a_N\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that, for all $1 \leq h \leq r$ and $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$, $|a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Now consider the sum $||a_{N_i} - a_{N_j}||_t^2 = \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s}$. Decompose it into two parts: one where $|k_1|^2 + \cdots + |k_n|^2 > R^2$ and one where $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$. On $|k_1|^2 + \cdots + |k_n|^2 > R^2$ we estimate $(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s}$ so that $\sum_h \sum_{|k_1|^2 + \cdots + |k_n|^2 \geq R^2} |a_{N_i,h,k} - a_{N_j,h,k}|^2 (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s} \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 \leq 2r_1 C^2 (1 + R^2)^{t-s}$. If $\epsilon > 0$ is given, we choose R so that $2r_1 C^2 (1 + R^2)^{t-s} < \epsilon$. The remaining part of the sum is over $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$ and can be bounded above by $\epsilon' := \epsilon - 2r_1 C^2 (1 + R^2)^{t-s}$ if $i, j \geq M(\sqrt{\epsilon' / [r_1(2R+1)^n]})$ because an n-ball of radius R centered at the origin is contained in an n-cube of side length $2R$ that has $(2R+1)^n$ lattice points. Then the total sum is bounded above by ϵ, and we are done. □

Acknowledgement
I would like to thank Farzad Fathizadeh for suggesting the problem of filling in the details in the pseudo-differential calculus on the noncommutative n-torus. I would also like thank Matilde Marcolli for helping me make plans to take my candidacy exam. I would like to thank Vlad Markovic and Eric Rains for agreeing to be on my candidacy exam committee.

References

[1] Connes A 1985 *Inst. Hautes Études Sci. Publ. Math.* **62** 257–360
[2] Connes A 1994 *Noncommutative Geometry* (San Diego, CA: Academic Press)
[3] Connes A and Moscovici H 1995 *Geom. Funct. Anal.* **5** 174–243
[4] Connes A and Marcolli M 2008 *Noncommutative Geometry, Quantum Fields and Motives* (American Mathematical Society Colloquium Publications vol 55) (Providence, RI: American Mathematical Society)
[5] Connes A 1980 *C. R. Acad. Sci. Paris Sér. A-B* **290** A599–A604
[6] Gilkey P B 1974 *The Index Theorem and the Heat Equation* (Mathematics Lecture Series vol 4) (Boston, MA: Publish or Perish)
[7] Raymond X S 1991 *Elementary Introduction to the Theory of Pseudodifferential Operators* (Boston: CRC Press)
[8] Wong M W 2014 *An Introduction to Pseudo-differential Operators* 3rd ed (Series on Analysis, Applications and Computation vol 6) (Hackensack, NJ: World Scientific Publishing)
[9] Connes A and Tretkoff P 2011 *Noncommutative Geometry, Arithmetic, and Related Topics* (Baltimore, MD: Johns Hopkins Univ. Press) pp 141–58
[10] Connes A and Moscovici H 2014 *J. Amer. Math. Soc.* **27** 639–84
[11] Fathizadeh F and Khalkhali M 2012 *J. Noncommut. Geom.* **6** 457–80
[12] Fathizadeh F and Khalkhali M 2013 *J. Noncommut. Geom.* **7** 1145–83
[13] Fathizadeh F and Khalkhali M 2015 *J. Noncommut. Geom.* **9** 473–503
[14] Fathizadeh F and Khalkhali M 2013 *Lett. Math. Phys.* **103** 1–18