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RESEARCH ARTICLE

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Approximate Noether symmetries of the geodesic equations for the charged-Kerr spacetime and rescaling of energy

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Abstract Using approximate symmetry methods for differential equations we have investigated the exact and approximate symmetries of a Lagrangian for the geodesic equations in the Kerr spacetime. Taking Minkowski spacetime as the exact case, it is shown that the symmetry algebra of the Lagrangian is 17 dimensional. This algebra is related to the 15 dimensional Lie algebra of conformal isometries of Minkowski spacetime. First introducing spin angular momentum per unit mass as a small parameter we consider first-order approximate symmetries of the Kerr metric as a first perturbation of the Schwarzschild metric. We then consider the second-order approximate symmetries of the Kerr metric as a second perturbation of the Minkowski metric. The approximate symmetries are recovered for these spacetimes and there are no non- trivial approximate symmetries. A rescaling of the arc length parameter for consistency of the trivial second-order approximate symmetries of the geodesic equations indicates that the energy in the charged-Kerr metric has to be rescaled and the rescaling factor is r-dependent. This re-scaling factor is compared with that for the Reissner–Nordström metric.

Keywords Kerr, Charged-Kerr spacetimes, Perturbed Lagrangian, First- and second-order approximate symmetries, Energy

1 Introduction

In general a spacetime may not be stationary (and especially may not be static) and hence local (global) energy conservation may be lost. Due to this fact there is

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a long standing problem of the definition of energy (or mass) in general relativity [1]. If the spacetime is static there is a timelike *isometry* or *Killing Vector* (KV). Further energy conservation in a spacetime is guaranteed in the frame using a timelike KV to define the time direction. However, in the absence of a timelike KV the energy of a test particle is not defined and hence the energy in the gravitational field is not well defined (of course, one could use the quasilocal energy defined for a Lagrangian for a *field theory* using an ADM foliation see [2; 3]).

If there does not exist a timelike KV, energy is not conserved. Since gravitational wave spacetimes are non-static vacuum solutions of the Einstein Field Equations (EFEs), for which a timelike KV does not exist, the problem of defining the energy content of gravitational waves is particularly severe. Different people have tried different *approximate symmetry* approaches [4; 5; 6] to define the energy content of gravitational waves but there is no clear solution to the problem. We use approximate symmetry methods for differential equations (DEs) [7] with the hope of finding approximate timelike KVs to look at the solution of the problem. It is obvious that we need to learn how to physically interpret the results that will emerge from the approximate symmetry calculations. For this purpose first the approximate symmetries of the Schwarzschild metric were investigated [8]; next we studied the Reissner–Nordström (RN) metric [9]; and here we consider the Kerr metric. We compare our results for the energy with those of Komar [10] and discuss the difference. In a subsequent paper we plan to investigate the approximate symmetries of time-varying spacetimes and hence try to identify what this approach would give as the energy content of gravitational waves.

The 10 generators of the Poincarè isometry algebra $so(1,3)\oplus_s \mathbb{R}^4$, (where $\oplus_s$ denotes semi-direct sum) for the Minkowski spacetime (which is maximally symmetric) [11; 12] gives conservation laws for energy, linear momentum and spin angular momentum. Going from Minkowski to non flat spacetimes like Schwarzschild, RN and Kerr spacetimes some of the conservation laws are lost because of the gravitational field. Using Lie symmetry methods [13], first-order approximate symmetries of the system of the geodesic equations for the Schwarzschild metric were discussed in [8] and second-order approximate symmetries of the system of the geodesic equations for the RN metric were given in [9]. For the first-order and also for the second-order approximate symmetries, the lost conservation laws of spin angular momentum and linear momentum are recovered as trivial approximate conservation laws. In the case of second-order approximate symmetries of the RN spacetime one finds that it is necessary to rescale the energy of test particles.

In this paper we start by using symmetries of the *Lagrangian*, rather than those of the geodesic equations. In particular we explore first and second-order approximate symmetries of a Lagrangian of the Kerr spacetime. First, we consider the Kerr metric as a first perturbation of the Schwarzschild metric with spin as a small parameter, ε. The isometry algebra for the Schwarzschild spacetime [11] is $so(3)\oplus\mathbb{R}$ while the symmetry algebra for the Lagrangian is $so(3)\oplus\mathbb{R}\oplus d_1$ (where d_1 is the Lie algebra generated by $\partial/\partial s$). Retaining terms of first order in ε and neglecting its higher powers we show that there is no "non-trivial" (in the technical sense explained in the next section) first-order approximate symmetry for the Lagrangian of this perturbed Schwarzschild metric. We only recover the two symmetry generators of angular momentum as "trivial" first-order approximate symmetry generators which were lost in going from Schwarzschild to the Kerr spacetime. We then consider the Kerr metric as a second perturbation of the Minkowski metric. Taking Minkowski spacetime as an exact case we obtain a seventeen dimensional Lie algebra, which contains the ten dimensional isometry algebra (Poincaré algebra). The significance of the remaining seven symmetry generators will be discussed in Sect. 3. Regarding mass as a small parameter, ε, for the approximate Schwarzschild metric as a first perturbation of the Minkowski spacetime, we recover all the lost symmetries as "trivial" first-order approximate symmetries. The isometry algebra of the unperturbed Kerr spacetime is two dimensional [11] and the symmetry algebra of the Lagrangian for this spacetime is three dimensional, i.e. the two KVs $\partial/\partial t$, $\partial/\partial\phi$ and the translation in the geodetic parameter $\partial/\partial s$. Now introducing the spin as a small parameter, ε and retaining terms of order ε^2 in the approximate Kerr spacetime as second perturbation of the Minkowski spacetime we recover all the lost symmetries of the Lagrangian as "trivial" second-order approximate symmetries.

A problem arises in the search for a scaling factor for the energy of test particles in the Kerr metric. Whereas, in the RN-case the energy rescaling was by $(1-Q^2/2Gm^2)$, there is a simple multiplicative factor for the Kerr metric. In the absence of the constant (unity in this case), it is not clear what significance to attach to the rescaling. So as to relate that factor to the factor arising in the RN-case, we investigate second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime. For this purpose we take mass, charge and angular momentum per unit mass as small parameters, of order ε, and only retain the second power, neglecting its higher powers. More specifically, in the set of determining equations for second-order approximate geodesic equations, the coefficient of $\partial/\partial s$ (in the point transformation generator given in Sect. 2) collects a rescaling factor (given in Sect. 4). Since s is the proper time and energy conservation is related to time translation, the energy of a test particle in the charged-Kerr spacetime rescales. This scaling factor consists of two terms, one due to charge and the other due to the spin of the gravitating source. We then compare this scaling factor with that of the RN spacetime. We also give a comparison of the scaling factor obtained here with the already existing expressions in the literature [14; 15; 16; 17] for the mass (energy) of the charged-Kerr spacetime.

The plan of the paper is as follows. In the next section we briefly review the definitions of symmetries and approximate symmetries of a Lagrangian. In Sect. 3, approximate symmetries of the Lagrangian for the Kerr spacetime are considered. In Sect. 4 we briefly discuss second-order approximate symmetries of the geodesic equations for the charged-Kerr metric. Finally a summary and discussion are given in Sect. 5. In Sect. 5 the comparison of the scaling factors is also given.

2 Symmetries and approximate symmetries of a Lagrangian

The significance of variational symmetries is clear from the celebrated Noether's theorem [18]. According to this theorem there is a procedure which relates the constants of the motion of a given Lagrangian system to its symmetry transformations [7; 19]. Symmetry generators of a Lagrangian of a manifold form a Lie algebra [20]. Geometrically, KVs characterize the isometries of a manifold [21].

In general a manifold does not possess any exact symmetry but may do so approximately. It is worth exploring the approximate symmetries of a manifold, which form an approximate Lie algebra [22]. Lie symmetries (and approximate Lie symmetries) of the system of the geodesic equations for a spacetime yield conserved quantities but there are also non-Noether symmetries that are not related to conservation laws and therefore are of no interest for our purpose. To calculate symmetries of a system of geodesic equation is tedious, as it involves the second prolongation of the symmetry generator. On the other hand the symmetries of a Lagrangian directly give us the conserved quantities in which we are interested and here only the first prolongation of the symmetry generator is required. Methods for obtaining exact symmetries and first-order approximate symmetries of a Lagrangian are available in the literature [7; 20; 23; 24]. In this paper we extend the procedure of calculating the approximate symmetries of a Lagrangian to the second order.

Noether symmetries, or symmetries of a Lagrangian, are defined as follows. Consider a vector field defined on a real parameter fibre bundle over the manifold [7]

$$\mathbf{X} = \xi(s,x^{\mu})\frac{\partial}{\partial s} + \eta^{\nu}(s,x^{\mu})\frac{\partial}{\partial x^{\nu}}, \tag{1}$$

where $\mu, \nu = 0,1,2,3$. The first prolongation of the above vector field defined on the real parameter fibre bundle over the tangent bundle to the manifold, is

$$\mathbf{X}^{[1]} = \mathbf{X} + (\eta^{\nu}_{,s} + \eta^{\nu}_{,\mu}\dot{x}^{\mu} - \xi_{,s}\dot{x}^{\nu} - \xi_{,\mu}\dot{x}^{\mu}\dot{x}^{\nu})\frac{\partial}{\partial \dot{x}^{\nu}}. \tag{2}$$

Generally one takes first-order Lagrangians as the corresponding Euler–Lagrange equations are second-order ordinary differential equations (ODEs). In particular, we take $L(s,x^{\mu},\dot{x}^{\mu})$, where "$\cdot$" denotes differentiation with respect to the arc length parameter s, which yields a set of second-ODEs

$$\ddot{x}^{\mu} = g(s,x^{\mu},\dot{x}^{\mu}). \tag{3}$$

Then $\mathbf{X}$ is a Noether point symmetry of this Lagrangian if there exists a gauge function, $A(s,x^{\mu})$, such that

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_sA, \tag{4}$$

where

$$D_s = \frac{\partial}{\partial s} + \dot{x}^{\mu}\frac{\partial}{\partial x^{\mu}}, \tag{5}$$

which is defined on the real parameter fibre bundle over the tangent bundle to the manifold. For more general considerations and a discussion of generalized symmetries see [7; 25]. The significance of Noether symmetries is clear from the following theorem [18].

Theorem 1 *If* $\mathbf{X}$ *is a Noether point symmetry corresponding to a Lagrangian* $L(s,x^\mu,\dot{x}^\mu)$ *of* (3)*, then*

$$I = \xi L + (\eta^\mu - \dot{x}^\mu \xi)\frac{\partial L}{\partial \dot{x}^\mu} - A, \tag{6}$$

is a first integral of (3) *associated with* $\mathbf{X}$*. For the proof of this theorem see for example* [26].

For a second-order (in ε) perturbed system of ODEs

$$\mathbf{E} = \mathbf{E}_0 + \varepsilon \mathbf{E}_1 + \varepsilon^2 \mathbf{E}_2 = O(\varepsilon^3), \tag{7}$$

second-order approximate symmetries of the first-order Lagrangian

$$L(s,x^\mu,\dot{x}^\mu,\varepsilon) = L_0(s,x^\mu,\dot{x}^\mu) + \varepsilon L_1(s,x^\mu,\dot{x}^\mu) + \varepsilon^2 L_2(s,x^\mu,\dot{x}^\mu) + O(\varepsilon^3), \tag{8}$$

are defined as follows. The functional $\int_V L ds$ is invariant under the one-parameter group of transformations with approximate Lie symmetry generator

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + O(\varepsilon^3), \tag{9}$$

up to gauge

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2, \tag{10}$$

where

$$\mathbf{X}_j = \xi_j \frac{\partial}{\partial s} + \eta_j^\mu \frac{\partial}{\partial x^\mu} \quad (j = 0,1,2), \tag{11}$$

$$\mathbf{X}_0^{[1]} L_0 + (D_s \xi_0) L_0 = D_s A_0, \tag{12}$$

$$\mathbf{X}_1^{[1]} L_0 + \mathbf{X}_0^{[1]} L_1 + (D_s \xi_1) L_0 + (D_s \xi_0) L_1 = D_s A_1 \tag{13}$$

and

$$\mathbf{X}_2^{[1]} L_0 + \mathbf{X}_1^{[1]} L_1 + \mathbf{X}_0^{[1]} L_2 + (D_s \xi_2) L_0 + (D_s \xi_1) L_1 + (D_s \xi_0) L_2 = D_s A_2. \tag{14}$$

For the first-order perturbed case (13) corresponding to a single equation, see for example [24].

Here $\mathbf{X}_0$ is the exact symmetry generator, $\mathbf{X}_1$ is the first-order approximate part, $\mathbf{X}_2$ is the second-order approximate part of the approximate symmetry generator, L_0 is the exact Lagrangian corresponding to the exact equations $\mathbf{E}_0 = 0$, and $L_0 + \varepsilon L_1$ the first-order approximate Lagrangian corresponding to the first-order perturbed equations $\mathbf{E}_0 + \varepsilon \mathbf{E}_1 = 0$. The perturbed equations (13) and (14) always have the approximate symmetry generators $\varepsilon \mathbf{X}_0$ which are known as “trivial” approximate symmetries and $\mathbf{X}$ given by (9) with $\mathbf{X}_0 \neq 0$ is called a “nontrivial” approximate symmetry.

3 Symmetries and approximate symmetries of a Lagrangian for the Kerr spacetime

The Kerr spacetime is an axially symmetric, stationary solution of the Einstein vacuum field equations. The line element for this spacetime in Boyer–Lindqust coordinates is given by [1]

$$ds^2 = \left(1 - \frac{2Gmr}{\rho^2 c^2}\right) c^2 dt^2 - \left(\frac{\rho^2}{\Delta}\right) dr^2 - \rho^2 d\theta^2 - \Lambda \frac{\sin^2\theta}{\rho^2} d\phi^2 + \left(\frac{2Gmra\sin^2\theta}{\rho^2 c^2}\right) dt d\phi, \tag{15}$$

where

$$\rho^2 = r^2 + \frac{a^2}{c^2}\cos^2\theta, \quad \Lambda = \left(r^2 + \frac{a^2}{c^2}\right)^2 - \frac{a^2}{c^2}\Delta\sin^2\theta, \quad \Delta = r^2 + \frac{a^2}{c^2} - \frac{2Gmr}{c^2},$$

with m the mass and a the angular momentum per unit mass of the gravitating source. This metric reduces to the Schwarzschild metric when $a = 0$. This spacetime has two KVs which give the energy and azimuthal angular momentum conservation laws. Besides, there is a non-trivial Killing tensor for this spacetime [27] which yields the square of the total angular momentum [28].

We consider the Lagrangian for minimizing the arc-length (written from the square of the arc length for convenience) which yields the *geodesic equations* as the Euler–Lagrange equations,

$$L[x^\mu, \dot{x}^\mu] = g_{\mu\nu}(x^\sigma) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}. \tag{16}$$

For the metric given by (15) it becomes

$$L = \left(1 - \frac{2Gmr}{\rho^2 c^2}\right) c^2 \dot{t}^2 - \frac{\rho^2}{\Delta}\dot{r}^2 - \rho^2\dot{\theta}^2 - \Lambda\frac{\sin^2\theta}{\rho^2}\dot{\phi}^2 + \frac{2Gmra\sin^2\theta}{\rho^2 c^2}\dot{t}\dot{\phi}. \tag{17}$$

Using (17) in (4) we obtain the 19 determining (partial differential) equations for 6 unknown functions ξ, η_μ and A, where each of these is a function of 5 variables, i.e. s, t, r, θ and ϕ. Solving these equations we get the isometries for the Kerr metric, the geodesic parameter translation and the gauge function, i.e.

$$\mathbf{Y}_0 = \frac{\partial}{\partial t}, \quad \mathbf{Y}_3 = \frac{\partial}{\partial \phi}, \quad \mathbf{W}_0 = \frac{\partial}{\partial s} \quad \text{and} \quad A = c \quad (\text{constant}). \tag{18}$$

Thus, here we see that the isometries form a sub-algebra of the symmetries of the Lagrangian. Use of (18) in (6) will provide the first integrals of the geodesic equations for the Kerr metric.

For the approximate symmetries of a Lagrangian for the geodesic equations in the Kerr spacetime we first consider the Kerr metric as a first perturbation of the

Schwarzschild metric by introducing the spin angular momentum per unit mass a/c^2 as a small parameter ε. This first-order perturbed Lagrangian is given by

$$L = \left(1 - \frac{2Gm}{rc^2}\right) c^2 \dot{t}^2 - \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \varepsilon \frac{2Gm}{r} \sin\theta \dot{t}\dot{\phi} + O(\varepsilon^2). \tag{19}$$

For $\varepsilon = 0$ we recover the Lagrangian of the unperturbed Schwarzschild metric. The symmetry algebra of the Lagrangian is 5 dimensional, given by $so(3) \oplus \mathbb{R} \oplus d_1$, and it properly contains the isometry algebra. The gauge function A is just a constant. From this information and (6) one can obtain the first integrals of the geodesic equations for the Schwarzschild metric. Using the 5 exact symmetry generators in (13) we get the set of determining equations whose solution gives us no non-trivial symmetry but only exact symmetries are recovered as trivial first-order approximate symmetries. Here we have recovered the conservation laws of angular momentum as trivial first-order approximate conservation laws which were lost in going from the Schwarzschild to the Kerr spacetime.

Next we take the Kerr spacetime as a second perturbation of the Minkowski spacetime. For this purpose we set

$$m = \varepsilon\mu, \quad a = \varepsilon\alpha, \tag{20}$$

where $\mu = c^2/2G$ and $\alpha = c\sqrt{k_1}$. For the Kerr black hole (see, e.g. [29]) we have $0 < k_1 \leq 1/4$. Here the second-order perturbed Lagrangian is given by

$$L = \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 - \frac{1}{r}\varepsilon(\dot{t}^2 + \dot{r}^2) - \varepsilon^2 \left[\frac{1}{r^2}\left(1 - \frac{k_1^2}{4}\sin^2\theta\right)\dot{r}^2 + k_1^2\cos^2\theta\dot{\theta}^2 + k_1^2\sin^2\theta\dot{\phi}^2 - \frac{\sqrt{k_1}}{r}\sin^2\theta\dot{t}\dot{\phi}\right] + O(\varepsilon^3). \tag{21}$$

For the exact case, $\varepsilon = 0$, i.e. no mass or angular momentum per unit mass, the Lagrangian (21) reduces to that of the Minkowski spacetime. It has a 17 dimensional Lie algebra spanned by the symmetry generators: 10 $\mathbf{Y}_i$'s, which are generators of the Poincaré algebra $so(1,3)\oplus_s\mathbb{R}^4$,

$$\mathbf{Y}_0=\frac{\partial}{\partial t},\quad \mathbf{Y}_1=\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi},\tag{22}$$

$$\mathbf{Y}_2=\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi},\quad \mathbf{Y}_3=\frac{\partial}{\partial\phi},\tag{23}$$

$$\mathbf{Y}_4=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi},\tag{24}$$

$$\mathbf{Y}_5=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi},\tag{25}$$

$$\mathbf{Y}_6=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},\tag{26}$$

$$\mathbf{Y}_7=\frac{r\sin\theta\cos\phi}{c}\frac{\partial}{\partial t}+ct\left(\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi}\right),\tag{27}$$

$$\mathbf{Y}_8=\frac{r\sin\theta\sin\phi}{c}\frac{\partial}{\partial t}+ct\left(\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi}\right),\tag{28}$$

$$\mathbf{Y}_9=\frac{r\cos\theta}{c}\frac{\partial}{\partial t}+ct\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right),\tag{29}$$

and 7 other generators, whose significance is discussed below

$$\mathbf{W}_0=\frac{\partial}{\partial s},\quad \mathbf{W}_1=s\frac{\partial}{\partial s}+\frac{1}{2}\left(t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\right),\tag{30}$$

$$\mathbf{Z}_0=s\mathbf{Y}_0,\quad \mathbf{Z}_1=s\mathbf{Y}_4,\quad \mathbf{Z}_2=s\mathbf{Y}_5,\quad \mathbf{Z}_3=s\mathbf{Y}_6,\tag{31}$$

$$\mathbf{Z}_4=\frac{1}{2}s\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\right).\tag{32}$$

As before, the generator $\mathbf{W}_0$ gives translation in s and always exists for a Lagrangian of the type (16) [30], $\mathbf{W}_1=[\mathbf{W}_0,\mathbf{Z}_4]$ which is a scaling symmetry in s,t,r that can be used to get rid of the s dependence in the generators given by (31) and (32). This is reasonable as symmetries of a Lagrangian always form a sub-algebra of the symmetries of the Euler–Lagrange (geodesic) equations [31] and the algebra of the Euler–Lagrange equations for Minkowski spacetime is $sl(6,\mathbb{R})$ which is 35 dimensional [32]. As mentioned above, using $\mathbf{W}_1$, we can write $s=t^2$ or $s=r^2$ and

$$\mathbf{Z}_4=\frac{r^2}{4}\left[\frac{1}{t}(r^2+2t^2)\frac{\partial}{\partial t}+3r\frac{\partial}{\partial r}\right].\tag{33}$$

Now, every flat spacetime is conformally flat, i.e. for which all components of the Weyl tensor are zero [21]. The Lie algebra of the Conformal Killing Vectors (CKVs) for a conformally flat spacetime is 15 dimensional [33]. Therefore for the Minkowski spacetime we already know that there are 15 CKVs. The 5 symmetry generators, i.e. $\mathbf{Z}_i$ for $i=0,\ldots,4$ given by (31) and (32), are proper CKVs with conformal factor $\psi=(c_0t^2+c_1)/2$. Thus we see that not only the KVs but also the CKVs form a sub-algebra of the symmetries of the Lagrangian for the Minkowski spacetime. The extra 2 generators, $\mathbf{W}_0, \mathbf{W}_1$, essentially provide the translation and appropriate scaling in the geodetic parameter.

The gauge function is

$$A = \frac{1}{2}c_0(t^2 - r^2) + 2tc_3 + c_4 - 2r(c_{14}\sin\theta\cos\phi + c_{15}\sin\theta\sin\phi + c_{16}\sin\theta), \quad (34)$$

where $c_0, \ldots, c_{16}$ are the arbitrary constants of integration associated with the symmetry generators.

Retaining terms of first-order in ε and neglecting $O(\varepsilon^2)$, the Lagrangian (21) becomes a first-order perturbed Lagrangian for the Schwarzschild metric considered as a first perturbation of the Minkowski metric. Using (13) and the exact symmetry generators given by (22)–(32) we get a new set of determining equations. In these equations only 12 of the 17 exact symmetry generators appear. These 12 generators of the exact symmetry have to be eliminated for consistency of these determining equations, making them homogeneous. The resulting system is the same as for the Minkowski spacetime, yielding 17 first-order approximate symmetry generators given by (22)–(32). Thus for the Schwarzschild metric as a first-order approximate case, we recover all the lost conservation laws as trivial first-order approximate conservation laws. Beside energy and angular momentum which always remain conserved for the Schwarzschild metric (for both the exact and perturbed cases) we see approximate conservation of linear momentum and spin angular momentum. This was also observed for the first-order approximate symmetries of the geodesic equations for the Schwarzschild metric [8].

Going from Minkowski to the Kerr spacetime we are left with only two KVs which give conservation of energy and azimuthal angular momentum. For the Lagrangian of the geodesic equations for exact (unperturbed) Kerr metric there are only three symmetry generators given by (18). We see that there is no non-trivial approximate symmetry in the first-order approximation. To check whether we can see non-trivial approximate symmetries from the definition of second-order approximate symmetries of a Lagrangian, which will hopefully give us a non-trivial conservation law, we take the Kerr metric as a second-order perturbation of the Minkowski spacetime. In the second approximation, that is when we retain terms quadratic in ε, we have the Lagrangian given by (21). From (14) we have a new system of 19 determining equations. In these equations now 14 of the 17 exact (also first-order approximate) symmetry generators appear. The two symmetry generators that arise here, which did not occur in the set of determining equations for first-order approximation, are $\mathbf{Y}_1$ and $\mathbf{Y}_2$ given in (22) and (23). At first sight it seems that these two new symmetry generators may yield some non-trivial second-order approximate symmetries. But for the consistency of the determining equations all the 14 constants have to be eliminated and the system again becomes homogeneous. The resulting system is once more the same as for the Minkowski spacetime, yielding 17 second-order approximate symmetry generators given by (22)–(32). Thus there is no non-trivial second-order approximate symmetry generator. In the second-order approximation we recover all the lost conservation laws as trivial second-order approximate conservation laws for the Kerr spacetime. Hence we have recovered the Lorentz covariance using approximate symmetries of the Lagrangian.

4 Second-order approximate symmetries of the geodesic equations for the charged-Kerr metric and rescaling of energy of a test particle

We have studied approximate symmetries of a Lagrangian for the Kerr spacetime in which we recovered trivial first-order and second-order approximate conservation laws. The rescaling of energy of test particles was seen from the approximate symmetries of the geodesic equations [9]. Therefore we now consider approximate symmetries of the geodesic equations. In [8] the definition of first-order approximate symmetries of DEs was used to calculate first-order approximate symmetries of the geodesic equations for the Schwarzschild metric. There, in the perturbed equations (given by 40 here), instead of the perturbed system (given as subscript in 40) the exact system of geodesic equations was used and no energy rescaling was forthcoming. The interesting result of energy rescaling of test particles for RN spacetime [9] was seen by application of the definition of the second-order approximate symmetries of DEs, wherein the perturbed system of geodesic equations was used. It was further remarked that it should be checked if the result of rescaling also holds for the Kerr metric. Here we investigate this question.

In the RN metric the charge appears as a second-order perturbation of the Minkowski metric [9]. The quadratic term in charge appears in the scaling factor. Hence, here we investigate the charged-Kerr metric to keep the charge up to the same second-order and relate the scaling factor for this metric with that of the RN spacetime.

In the charged-Kerr metric we have

$$\begin{aligned} g_{00} &= 1-\frac{G(2c^2mr-Q^2)}{\rho^2c^4}, \quad g_{03}=\frac{a}{\rho^2c^2}G\left(2mr-\frac{Q^2}{c^2}\right)\sin^2\theta, \\ \Delta &= \frac{a^2}{c^2}+r^2-\frac{G}{c^2}\left(2mr-\frac{Q^2}{c^2}\right). \end{aligned} \tag{35}$$

Setting $Q=\varepsilon\chi$, where $\chi=c^2\sqrt{k/G}$ and ε is defined by (20), we have the second-order approximate geodesic equations for the Kerr metric

$$\ddot{t}+\varepsilon\frac{1}{r^2}\dot{t}\dot{r}+\varepsilon^2\left[\frac{1}{r^3}(1-2k)\dot{t}\dot{r}-\frac{2\sqrt{k_1}}{r^2}\sin^2\theta\dot{r}\dot{\phi}\right]+O(\varepsilon)^3=0, \tag{36}$$

$$\begin{aligned} &\ddot{r}-r(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)+\varepsilon\left[\frac{1}{2r^2}(\dot{t}^2-\dot{r}^2)+(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\right] \\ &\quad-\varepsilon^2\left[\frac{1}{2r^3}(1+2k)\dot{t}^2+\frac{\sqrt{k_1}}{r^2}\sin^2\theta\dot{t}\dot{\phi}-\frac{1}{r^3}(2(k_1\sin\theta+k)-1)\dot{r}^2\right. \\ &\quad\left.+\frac{k_1}{r^2}\sin^2\theta\dot{r}\dot{\theta}+\frac{1}{r}(k_1\sin^2\theta+k)(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\right]+O(\varepsilon)^3=0, \end{aligned} \tag{37}$$

$$\begin{aligned} &\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\cos\theta\dot{\phi}^2+\varepsilon^2\left[\frac{\sqrt{k_1}}{r^3}\sin 2\theta\dot{t}\dot{\phi}-\frac{k_1}{2r^4}\sin 2\theta\dot{r}^2-\frac{2k_1}{r^3}\cos^2\theta\dot{r}\dot{\theta}\right. \\ &\quad\left.-\frac{k_1}{2r^2}\sin 2\theta(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\right]+O(\varepsilon)^3=0, \end{aligned} \tag{38}$$

$$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}+\varepsilon^2\left[\frac{\sqrt{k_1}}{r^4}\dot{t}\dot{r}+\frac{2\sqrt{k_1}}{r^3}\cot\theta\dot{t}\dot{\theta}-\frac{2}{r^3}k_1\dot{r}\dot{\phi}\right]+O(\varepsilon)^3=0. \tag{39}$$

If $\varepsilon = 0$, these equations reduce to those of the Minkowski metric. When we retain terms only up to order ε and neglect higher orders, they reduce to the first-order approximate geodesic equation of the Schwarzschild metric. If we put $k_1 = 0$, they further reduce to those of the RN-metric. We now apply the definition of second-order approximate symmetries of a system of ODEs,

$$(\mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2)\,(\mathbf{E}_0 + \varepsilon \mathbf{E}_1 + \varepsilon^2 \mathbf{E}_2)\big|_{\mathbf{E}_0+\varepsilon\mathbf{E}_1+\varepsilon^2\mathbf{E}_2=O(\varepsilon^3)} = O(\varepsilon^3), \quad (40)$$

(see [9] and references therein) to (36)–(39), where $\mathbf{X}_0$ is the exact symmetry generator, $\mathbf{X}_1$, $\mathbf{X}_2$ are the first-order and second-order approximate parts of the approximate symmetry generator respectively, $\mathbf{E}_0$ is the exact part, $\mathbf{E}_1$ is the first-order perturbed part and $\mathbf{E}_2$ is the second order perturbed part of the system of ODEs respectively. The exact symmetry algebra includes the generators of the dilation algebra, $\partial/\partial s$, $s\partial/\partial s$ corresponding to

$$\xi(s) = c_0 s + c_1. \quad (41)$$

In the determining equations for the first-order approximate symmetries [8] the terms involving $\xi_s = c_0$ cancel out. Taking the RN metric as a second perturbation of the Minkowski metric [9], it was seen that the terms involving ξ_s do not automatically disappear but collect a scaling factor of $(1 - Q^2/2Gm^2)$ in order to cancel out. In the case of the charged-Kerr spacetime, as a second perturbation of the Minkowski spacetime, the terms involving ξ_s in the set of determining equations also do not disappear automatically but collect a scaling factor

$$(1/r^3)(1-2k)\dot{t} - (2/r^2)(\sqrt{k_1}\sin^2\theta)\dot{\phi}, \quad (42)$$

so as to cancel out, where $k = Q^2/4Gm^2$ and $k_1 = a^2c^2/4G^2m^2$. From (1) one can see that ξ is the coefficient of $\partial/\partial s$ in the point transformations. This scaling factor involves the derivatives of the coordinates t and ϕ, which can be replaced by the first integrals of the geodesic equations and involve constants that are the mass and the spin. As such, we put them in as m and a. Thus we get (taking $G = 1$, $c = 1$)

$$M_{c-K} = m - \frac{Q^2}{2m} + \frac{ma}{2r}. \quad (43)$$

For $a = 0$, (43) reduces to m-times of the expression for the RN spacetime [9].

Komar, using his definition of approximate symmetry [4; 5], wrote down an integral for the mass in a spacetime [10]

$$M = \frac{1}{8\pi}\int_{s^2} *d\tilde{\xi}, \tag{44}$$

where $\tilde{\xi}$ is the time-like Killing 1-form for the exact symmetry, $*d\tilde{\xi}$ the dual of the 2-form $d\tilde{\xi}$ and s^2 is the 2-surface [14; 15; 34]. Using the Komar integral (44) Cohen and de Felice considered ξ as the stationary Killing 1-form over a charged-Kerr background metric [14]. They obtained a formula for the effective mass (and hence energy) for the charged-Kerr spacetime

$$M_{c-K} = m - \frac{Q^2}{r} - \frac{Q^2(r^2+a^2)}{ar^2}\tan^{-1}\left(\frac{a}{r}\right). \tag{45}$$

In the above expression (45) a does not appear explicitly and only appears in a product with Q. When $Q \longrightarrow 0$ in the above expression (45) the effects of rotation also disappear. This does not seem reasonable. In the limit of $a \longrightarrow 0$ expression (45) reduces to that of the RN spacetime given in [35; 36].

Chellathurai and Dadhich modified the Komar integral and obtained an expression for the effective mass of the charged-Kerr black hole [15]

$$M_{c-K} = m - \frac{Q^2}{r} - \frac{(12m^2+Q^2)a^2}{3r^3} + \frac{14ma^2Q^2}{3r^4} + \cdots. \tag{46}$$

This expression (46) reduces to that of the RN spacetime in the limit $a \longrightarrow 0$ and in the limit $Q \longrightarrow 0$ reduces to that for the Kerr spacetime [37]. However, it is not clear that this modification satisfactorily adjusts for the approximate symmetry of Komar.

Qadir and Quamar [16; 17] obtained an expression for the ψN-potential of the charged-Kerr spacetime,

$$\varphi = -\frac{mr - Q^2/2}{(r^2 + a^2\cos^2\theta)}. \tag{47}$$

In the limit $a \longrightarrow 0$ (47) reduces to that for the RN spacetime [38; 39]. This yields the approximate modification of the mass to be

$$M_{c-K} = m - \frac{Q^2}{2r} - \frac{ma^2\cos^2\theta}{r^2} + \frac{a^2Q^2\cos^2\theta}{2r^3} + \cdots. \tag{48}$$

The significance and comparison of our expression with (45), (46) and (48) will be discussed further in the next section.

5 Summary and discussion

In this paper we have discussed exact and approximate symmetries of a Lagrangian for the geodesic equations in the Kerr spacetime. Minkowski spacetime is maximally symmetric having 10 KVs. Going from Minkowski to the Kerr spacetime we are left only with two KVs which correspond to energy and azimuthal angular momentum conservation. The unperturbed Lagrangian for the geodesic equations in the Kerr spacetime has an additional symmetry $\partial/\partial s$ and the unperturbed Lagrangian for the Schwarzschild metric has a 5 dimensional algebra which contains the four KVs of this metric and $\partial/\partial s$. Taking the Kerr spacetime as a first perturbation of the Schwarzschild metric with spin as a small parameter we recovered the conservation laws as trivial first-order approximate conservation laws which were lost in going from the Schwarzschild spacetime to the Kerr spacetime.

Retaining terms of $O(\varepsilon^2)$ in the Kerr spacetime we have a second-order perturbed Lagrangian given by (21). This Lagrangian reduces to that of Minkowski spacetime if $\varepsilon = 0$ and if we retain terms of first-order in ε and neglecting $O(\varepsilon^2)$, we get a Lagrangian for the perturbed Schwarzschild metric which is a first perturbation of the Minkowski metric. For the exact case (Minkowski spacetime) symmetries of the Lagrangian form a 17 dimensional Lie algebra, which also holds in Cartesian coordinates and thus there is no coordinate dependence. [It may be mentioned here that the symmetries of the Minkowski metric Lagrangian were first discussed in [20], where the metric taken was $ds^2 = \cosh(x/a)dt^2 - dx^2 - dy^2 - dz^2$, which is not Minkowski, as it has $R^0_{101} \neq 0$. The calculation was left incomplete, giving an impression that the algebra is infinite dimensional, and it was shown that the isometry algebra is a sub-algebra of the symmetries of the Lagrangian. We pointed these errors out to the authors. This problem was revisited in [40] with the correct metric, but the symmetry algebra of the Lagrangian was given as 12 dimensional and the gauge function as zero, which was again erroneous.]

For the first-order approximate case (perturbed Schwarzschild) there is no non-trivial first-order approximate symmetry of the Lagrangian. However all the exact 17 symmetry generators are recovered as first-order approximate symmetry generators. In the second-order approximate case, i.e. when we retain terms quadratic in ε, which is the second perturbation of the Minkowski metric, we again have no non-trivial second-order approximate symmetry of the Lagrangian and only 17 symmetry generators of the exact case are recovered as second-order approximate symmetry generators. Thus we see that in going from Minkowski to Schwarzschild and Kerr metrics the conservation laws which were lost are now recovered as approximate conservation laws. It was shown [30] that a Lagrangian possesses at least one additional symmetry generator, $\partial/\partial s$, apart from the isometry algebra. This is verified for the Schwarzschild and Kerr spacetimes. As in the case of the Minkowski metric the CKVs form a sub-algebra of the symmetries of the Lagrangian which include $\partial/\partial s$. We conjecture that *the CKVs form a sub-algebra of the symmetries of the Lagrangian that minimize the arc length, for any spacetime*.

For both the Schwarzschild and Kerr spacetimes the unperturbed Lagrangian has only the one additional symmetry $\partial/\partial s$. For both the metrics the gauge function A is a constant. It remains an open question, whether this is true in general for all 4 dimensional curved spacetimes. In Minkowski spacetime there are 7 additional symmetries and the gauge function A is a function of 4 variables t, r, θ and ϕ given by (34). In these additional 7 symmetry generators of the Minkowski metric Lagrangian, which are also recovered as first-order and second-order approximate symmetries generators for the Schwarzschild and Kerr metrics respectively, $\mathbf{W}_0$ is the translation in the geodetic parameter s and $\mathbf{W}_1$ is used to replace s by t^2 in

$\mathbf{Z}_i$, $(i = 0, \ldots, 4)$ to obtain the CKVs. In the exact (unperturbed) case, the symmetries of a Lagrangian form a sub-algebra of symmetries of the Euler–Lagrange equations [31]. Here we conjecture that *approximate symmetries of a perturbed Lagrangian also form a sub-algebra of the approximate symmetries of the perturbed Euler–Lagrange equations*.

We also looked at the second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime to find a rescaling factor. Since the rescaling comes in the derivative relative to proper time, it was argued [9] that it gives a rescaling of the energy in this spacetime. In the RN spacetime [9], the rescaling was independent of r while for the charged-Kerr metric the rescaling factor given by (43) consists of two parts - one is due to charge and the other is due to spin of the gravitating source which depends on r. The charge comes in *quadratically* compared to unity in one term. The spin comes in *linearly*. It does not come with a constant term to compare. However, taken as a whole, we see that the spin has an effectively *lower order* effect.

In all three expressions (45), (46) and (48), the charge and spin appear at the same order (quadratically). The last one comes with a θ-dependent part, which arises from the θ-dependence of the "force" experienced by a body in the Fermi-Walker frame [41]. This θ-dependence does not seem reasonable for the defining the energy in the Kerr spacetime. As mentioned earlier, (45) seems unreasonable as the rotational effect depends on the presence of a charge! In (43) in the absence of charge, the effect is to *enhance* the mass. This seems reasonable as the frame-dragging effect also appears to lead to an enhanced mass - "friction" of the rotating mass with the background spacetime, as it were. Recall that one can extract rotational energy from a rotating black hole and hence the rotation should *add* into the mass. As would be expected, this effect decreases with r. The other three expressions give a *reduction* of the rotating mass. Also notice that (43) gives a change in the mass due to charge that is position independent. That this should be so is not so clear to us. However, nor is it clear to us that it *should* be position dependent. The force experience by a particle in the field of a charged gravitational source would be position dependent, but this does not say that the mass should be modified by a position dependent expression. It might be that in (43) the modification is due to the electromagnetic self-energy to the gravitational self-energy. As such, we conclude that the other three expressions have definite drawbacks to be considered reliable and that (43) seems to be free of those problems.

It would be of interest to analyse the Kerr-AdS and other solutions using approximate Noether symmetries. One could use [42; 43] and those cited therein for the purpose. In particular, there is no good definition of energy for spacetimes containing gravitational waves, because of the lack of a timelike KV. There is a proposal for a definition using superpotentials [44; 45], whose relationship to the definition using approximate symmetries would be worth exploring. It is of interest to apply this method of approximate symmetries of a Lagrangian to gravitational waves in the hope of finding an approximate timelike KV which will give energy conservation up to a certain approximation. This matter will be discussed in detail elsewhere. A preliminary discussion is given in [9].

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