

Topological ordered spaces as a foundation for a quantum spacetime theory

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Abstract. I outline some recent progress on the theory of topological preordered spaces, and comment on the role of these spaces for an approach to a quantum theory of spacetime.

1. Ideas for a quantum spacetime theory

Physical theories rely in a hierarchy of specialized mathematical structures. Our current model for spacetime describes it as a 4-dimensional time-oriented Lorentzian manifold, usually denoted (M, g) . Here we can recognize the following progressively less restrictive structures: (a) that of a Lorentzian manifold with its induced causal order determined by the distribution of light cones; (b) the deeper differentiable structure, typical of manifolds, which serves to give a meaning to differentiability; and the even deeper (c) topological structure, which gives a meaning to the notions of closeness and continuity.

The history of Science suggests that some of the deepest and more general structures can actually turn out to be dynamical. For example, the old rigid Euclidean space has been replaced by a dynamical Lorentzian manifold, and many other old rigid concepts such as distant simultaneity have lost their absolute status. We expect this process to continue and hence we are led to ask which concepts are more likely to survive when we remove mathematical specialization to our present model of spacetime.

Trying to answer this question leads us to consider quantum physics, and with it the idea that spacetime may lose its smooth character, which would be, therefore, just an apparent approximate feature for low energy processes. If we accept that smoothness should be lost at a quantum regime then we must also be ready to accept that the concept of a Lorentzian metric should not make sense in that limit. We would like, however, to retain the causal relation as this seems to be a fundamental concept without which the mathematics would become too poor to lead to any interesting physical consequence, as even the interpretation of the theory would become too obscure. We are therefore led to the idea that the following ingredients should be present in a fundamental spacetime theory

- Topology: gives a notion of continuity to work with.
- Order: to be interpreted in a causal fashion; essential for the interpretation of the theory.
- Measure: most of this element is already contained in the topology which determines the σ -algebra.



These ingredients are indeed those selected in other approaches to quantum gravity such as *Causal Set Theory* [1, 2]. However, in our approach we are not going to infer topology from order, as in that theory. Rather, we are going to impose compatibility conditions between topology and order. The approach by Causal Set Theory is, therefore, more restrictive as the known topologies that one can construct from the order are quite few in number, e.g. Alexandrov, Scott, etc.. We wish to show that our approach is indeed more natural since topology and order might be considered as two faces of the same entity: the *quasi-uniformity*. The theory that studies this connection is that of *topological preordered spaces*, which is still unknown to physicists and in fact also to most mathematicians. In this contribution I will give an introduction to this theory. First, however, let me mention how I came to study this theory and its relevance for relativity.

2. Some motivation from mathematical relativity

In mathematical relativity two events $p, q \in M$ are said to be causally related, in formula $p \leq q$, if there is a future directed causal curve from p to q or $p = q$. The causal relation is a subset of $M \times M$ made by all the causally related pairs and is denoted J^+ . It turns out that J^+ is not necessarily closed (e.g. consider Minkowski spacetime minus a point), and that closing it to get $\overline{J^+}$ might spoil transitivity. Early relativists looked for a better causal relation and Seifert [3] came up with the idea of considering $J_S^+ := \bigcap_{g' > g} J_{g'}^+$, where $g' > g$ if the timelike cones of g' contain the lightlike cones of g . The Seifert relation is indeed closed and transitive and it is a partial order if and only if the spacetime is stably causal, namely if there is $g' > g$ such that (M, g') is causal, i.e. opening the light cones does not spoil causality. Under stable causality J_S^+ is also the smallest closed and transitive relation which contains J^+ , see [4].

Hawking proved that every stably causal spacetime admits a time function, namely a continuous function which increases over every causal curve [5]. His proof is technically quite complicated and uses in an essential way the Lorentzianity of the spacetime manifold. I realized that the existence of a time function had instead more to do with the existence of the closed partial order J_S^+ . After some research I found [6] the existence of the following result in the theory of topological preordered spaces (I removed the Hausdorff condition [7])

Theorem 2.1 (Levin [8, 9]). *Every second countable locally compact space endowed with a closed order (e.g. a stably causal spacetime endowed with the Seifert relation) admits a strictly increasing continuous function.*

We are therefore able to infer the existence of time functions without using the Lorentzianity of the manifold or even the smoothness of the manifold. This is one of those results which would make perfect sense in a quantum theory of spacetime.

3. The mathematical theory

A topological preordered space is a triple (E, \mathcal{T}, \leq) where (E, \mathcal{T}) is a topological space and \leq is a reflexive and transitive relation over E (preorder), namely its graph $G(\leq) := \{(x, y) : x \leq y\}$ satisfies $G \circ G \subset G$, and $\Delta \subset G$. Here \circ is the composition of relations, and $\Delta := \{(x, y) : y = x\}$ is the diagonal of $E \times E$, which plays the role of the identity for the composition. If $G \cap G^{-1} = \Delta$ we say that \leq is an order (or partial order).

Given $x \in E$ the set $i(x) := \{y : x \leq y\}$ is the increasing hull of x and $d(x) := \{y : y \leq x\}$ is the decreasing hull. A set $S \subset E$ for which $i(S) = S$ is said *increasing*, and dually, a set for which $d(S) = S$ is said *decreasing*.

Given a topological preordered space one has the upper topology \mathcal{U} , whose open sets are the open increasing sets, and the lower topology \mathcal{L} which is defined dually. Sometimes from the triple $(E, \mathcal{U}, \mathcal{L})$ one can recover (E, \mathcal{T}, \leq) , thus there is a connection between the theory of *bitopological spaces* and that of *topological preordered spaces*. A topological preordered space is said to be *convex* if $\mathcal{T} = \sup(\mathcal{U}, \mathcal{L})$, see [10].

Apart from convexity, one can place several inequivalent conditions between topology and order. We speak of *closed preordered space* if G is closed as a subset of $E \times E$ with the product topology. Let us observe that the usual topology theory can be understood as equivalent to this theory where the order is the discrete one, namely $G = \Delta$. In topology it is known that the Hausdorff condition is equivalent to the closure of Δ , thus any Hausdorff topological space is a *closed preordered space* for which the relation is the trivial one. It is also useful to observe that if \leq is a closed order then as $G \cap G^{-1} = \Delta$ is closed, that is, \mathcal{T} is Hausdorff.

We speak of *normally preordered space* if G is closed and if any disjoint pair made of a closed increasing set B and a closed decreasing set A , can be separated by a disjoint pair made of an open increasing set $V \supset B$ and an open decreasing set $U \supset A$. In this case, for A and B as above, there is always a non-decreasing (isotone) continuous function $f : E \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. This is Nachbin's generalization of Urysohn's lemma [11, 12, 13].

Let \mathcal{F} be the family of isotone continuous functions in $[0, 1]$. Another compatibility condition is that defining a *Tychonoff (or completely regularly) preordered space*. This is a closed preordered space for which (a) \mathcal{F} determines the topology (the coarsest topology which makes all the functions in \mathcal{F} continuous is \mathcal{T}), and (b) \mathcal{F} determines the order, that is, $x \leq y$ iff for every $f \in \mathcal{F}$, $f(x) \leq f(y)$. These spaces are important because they can be compactified in a precise way [14, 15].

One of the reasons for the success of topology lies in the possibility of improving the separability properties of the space using countability or compactness conditions. Consider for instance the two results

Theorem 3.1. *Every locally compact Hausdorff space is completely regular (Tychonoff).*

and

Theorem 3.2 (Urysohn). *Every second countable regular space is metrizable.*

Under local compactness and second countability one can therefore prove that the Hausdorff condition is improved to metrizability which implies normality.

In the preordered case these theorems do not hold and in any case, rather puzzlingly, a completely regularly preordered space need not be regularly preordered [16, Example 1]. One could prove that compact preordered spaces are normally preordered [11, 17] but this result is rather limited. For these difficulties the theory could not be applied to practical problems, as the manifolds which are usually encountered in applications are quite general, and certainly not compact. Therefore, I was led to follow a different path proving [17]

Theorem 3.3. *Every locally compact σ -compact space equipped with a closed preorder is normally preordered.*

Since manifolds satisfy these assumptions, this theorem proved that topological preordered spaces appearing in applications share good separability properties. These applications are in fact not restricted to relativity, as Thermodynamics, Microeconomics and Computer Science present situations in which topology and order are the basic elements of the theory.

Convex normally preordered spaces are completely regularly preordered spaces, thus one would like to prove convexity. I proved that a locally compact σ -compact closed ordered space for which the order is compactly generated is indeed convex and hence completely regularly preordered [18]. Here an order is compactly generated if it can be recovered from its specification over compact sets, that is if, so to speak, the information required to determine the order is local in nature. Cones structures over topological manifolds determine orders which are of this type.

In order to understand the relation between topology and preorder let us mention the notion of *uniformity* as it was introduced by Weyl. A uniformity is a kind of topology for which two far away open sets can be compared. Metric spaces induce a uniformity.

More precisely, a uniformity \mathcal{U} is a filter of neighborhoods V of the diagonal $\Delta \subset E \times E$, such that (a) if $V \in \mathcal{U}$ then $V^{-1} := \{(x, y) : (y, x) \in V\} \in \mathcal{U}$; and (b) if $V \in \mathcal{U}$ then there is $U \in \mathcal{U}$ such that $U \circ U \subset V$. Every uniformity determines a topology over E , whose neighborhoods of x admits the base $U(x) := \{y : (x, y) \in U\}$ with $U \in \mathcal{U}$. The topology is Hausdorff if $\bigcap \mathcal{U} = \Delta$.

Nachbin [11] realized that one can actually drop axiom (a) getting a very interesting concept: the *quasi-uniformity*. Quasi-uniformities allow us to naturally define not only a topology, as the topology \mathcal{T} of the symmetric uniformity $\sup(\mathcal{U}, \mathcal{U}^{-1})$, but also a preorder through $G = \bigcap \mathcal{U}$. It is easy to verify that this is indeed a preorder, which is an order if and only if the topology is Hausdorff. Thus topology and preorder are two aspects of the same entity, the quasi-uniformity. Also it turns out that the quasi-uniformizable topological preordered spaces are the completely regularly preordered spaces introduced above [11].

One can also introduce the concept of quasi-pseudo-metric $p(x, y)$ which satisfies: (i) $p(x, x) = 0$ and (ii) $p(x, z) \leq p(x, y) + p(y, z)$. The question is whether a topological preordered space (E, \mathcal{T}, \leq) admits a quasi-pseudo-metric, in the sense that the topology \mathcal{T} comes from the symmetric pseudo-metric $d(x, y) = p(x, y) + p(y, x)$, and the preorder satisfies $x \leq y$ if and only if $p(x, y) = 0$. I proved that the answer is affirmative if (E, \mathcal{T}, \leq) is a quasi-uniformizable space with a second countable topology [19].

4. Conclusion

I have given a short account of the theory of topological preordered spaces. A study of this theory reveals that topology and order are two aspects of the same mathematical object and, therefore, should be studied jointly. This result is particularly interesting because these are exactly the mathematical ingredients which are believed to make sense in a quantum spacetime theory, namely at a limit in which the differentiability of the manifold is lost but the concepts of causal order and continuity still make sense.

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