

Yukawa couplings in 6D gauge–Higgs unification on T^2/Z_N with magnetic fluxes

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 We discuss the Yukawa couplings in 6D gauge–Higgs unification models on T^2/Z_N in the presence of magnetic fluxes. We provide general formulae for them, and numerically evaluate their magnitude in a specific model on T^2/Z_3 . Thanks to the nontrivial profiles of the zero-mode wave functions, the top quark Yukawa coupling can be reproduced without introducing a large representation of the gauge group for matter fields. However, it is difficult to realize small Yukawa couplings only by the magnetic fluxes and the Wilson-line phases because of the complicated structure of the mode functions on T^2/Z_N ($N = 3, 4, 6$).

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1. Introduction

The gauge–Higgs unification (GHU) [1–5] is an interesting candidate for the new physics beyond the standard model. The Higgs fields are identified with extra components of higher-dimensional gauge fields, and we do not need to introduce elementary scalar fields. The higher-dimensional gauge symmetry governs the Higgs and the Yukawa sectors. Namely, the gauge invariance prohibits the Higgs masses at tree-level,¹ and the Yukawa couplings originate from the (higher-dimensional) gauge couplings. In particular, five-dimensional (5D) models have been extensively investigated [7–18] because they have the simplest extra-dimensional structure and the 5D gauge invariance protects the Higgs mass against large quantum corrections.

Six-dimensional GHU models are also phenomenologically attractive because the existence of Higgs quartic couplings at tree level makes a realization of the observed Higgs mass easier [6]. In our previous work [19], we investigated 6D GHU models on T^2/Z_N orbifolds, and searched for possible gauge groups, orbifolds, and representations of the matter fermions by requiring the theory to have the custodial symmetry and realize the top quark mass. By employing the group theoretical analysis, we found that the minimal candidate is an U(4) gauge theory on T^2/Z_3 and the third-generation quarks are embedded into $\mathbf{20}'$ of SU(4).

Six-dimensional models have another important feature. We can introduce magnetic fluxes that penetrate the compact space as a background. Such a background is phenomenologically interesting

¹ Six-dimensional (6D) models generically allow tadpole terms proportional to the field strength F_{45} at the orbifold fixed points. Such terms induce tree-level Higgs masses unless they are cancelled [6].

because it induces gauge symmetry breaking, chiral fermions in four-dimensional (4D) effective theories, and multiple zero-modes from a single bulk field [20–24]. Besides, since the magnetic flux deforms the flat profile of zero-mode wave functions in the extra dimensions, it can control 4D effective Yukawa couplings [25].

In this paper, we discuss the Yukawa couplings in 6D GHU models on T^2/Z_N in the presence of background magnetic fluxes. As mentioned above, the Yukawa couplings originate from higher-dimensional gauge couplings. Hence, they become flavor-universal in a simple setup. In 5D models on S^1/Z_2 , we can vary them by means of the bulk fermion masses that have kink profiles. Unfortunately, they cannot be extended to 6D models because we only have codimension 2 singularities on two-dimensional orbifolds. Instead, we can control them by the magnetic fluxes and the Wilson-line phases. Furthermore, the magnetic fluxes, which are quantized, can realize the generational structure of quarks and leptons. In Refs. [26–29], possibilities of reproducing the realistic Yukawa structure by magnetic fluxes are investigated in the context of ten-dimensional super Yang–Mills or superstring theories, and it is shown to be reproduced in some cases. Their success of the realization of the Yukawa hierarchy is supported by the following two points. One is that the gauge groups they considered are large and contain a lot of $U(1)$ subgroups that have the magnetic fluxes, which means that there exist a sufficient number of independent magnetic fluxes to control the Yukawa couplings. The other is that their models are compactified on T^2 or T^2/Z_2 .² Hence, the mode functions have simpler structures than those on T^2/Z_N ($N = 3, 4, 6$), and are easier to control. However, these properties are not necessary conditions for the GHU models. In this paper, we discuss realization of the Yukawa hierarchy in smaller gauge groups, and especially focus on a $U(3)$ model on T^2/Z_3 as a specific example.

The paper is organized as follows. In the next section, we explain our setup and introduce the magnetic fluxes. In Sect. 3, we show explicit forms of the mode functions on T^2 and T^2/Z_N . In Sect. 4, we provide a formula for the Yukawa coupling constants, and evaluate their numerical values in a specific model. Section 5 is devoted to the summary.

2. Setup

We consider a 6D gauge theory compactified on an orbifold T^2/Z_N ($N = 2, 3, 4, 6$). The gauge group is $G \times U(1)_X$, where G is a simple group that includes $SU(2)_L \times U(1)_Z$.³ The field content consists of the G gauge field A_M , the $U(1)_X$ gauge field B_M , where $M = 0, 1, \dots, 5$ is the 6D Lorentz index, and 6D Weyl fermions $\Psi_{\chi_6}^f$ ($f = 1, 2, \dots$), where $\chi_6 = \pm$ denotes the 6D chirality. The 6D Lagrangian is

$$\mathcal{L} = -\frac{1}{4g_A^2} \text{Tr} \left(F^{MN} F_{MN} \right) - \frac{1}{4g_B^2} B^{MN} B_{MN} + \mathcal{L}_{\text{gf}} + \sum_f i \bar{\Psi}_{\chi_6}^f \Gamma^M \mathcal{D}_M \Psi_{\chi_6}^f, \quad (2.1)$$

where \mathcal{L}_{gf} denotes the gauge-fixing terms, Γ^M are 6D gamma matrices, and g_A and g_B are the 6D gauge coupling constants for G and $U(1)_X$, respectively. The field strengths and the covariant

² In Ref. [28], the cases in which three generations are realized are discussed on T^2/Z_N ($N = 2, 3, 4, 6$). However, the numerical evaluations of the Yukawa couplings are performed only on T^2/Z_2 .

³ We do not consider the color group $SU(3)_C$ since it is irrelevant to the discussion, and $U(1)_X$ is introduced in order to adjust the Weinberg angle to the realistic value.

derivatives are defined as

$$\begin{aligned} F_{MN} &\equiv \partial_M A_N - \partial_N A_M - i[A_M, A_N], \\ B_{MN} &\equiv \partial_M B_N - \partial_N B_M, \\ \mathcal{D}_M \Psi_{\chi_6}^f &\equiv (\partial_M - iA_M - iq_f B_M) \Psi_{\chi_6}^f, \end{aligned} \quad (2.2)$$

where q_f is the $U(1)_X$ charge of $\Psi_{\chi_6}^f$.

2.1. Orbifold and boundary conditions

For the coordinates of the extra dimensions, it is convenient to use a complex (dimensionless) coordinate $z \equiv \frac{1}{2\pi R_1}(x^4 + ix^5)$, where $R_1 > 0$ is one of the radii of T^2 . Correspondingly, the extra-dimensional components of the gauge fields are written as

$$A_z = \pi R_1 (A_4 - iA_5), \quad B_z = \pi R_1 (B_4 - iB_5). \quad (2.3)$$

The orbifold T^2/Z_N is defined by identifying points in the extra space as

$$z \sim \omega z + n_1 + n_2 \tau, \quad (n_1, n_2 \in \mathbb{Z}), \quad (2.4)$$

where $\omega \equiv e^{2\pi i/N}$ and τ is a complex constant that satisfies $\text{Im } \tau > 0$. An arbitrary value of τ is allowed when $N = 2$, while it must be equal to ω when $N = 3, 4, 6$. The orbifold T^2/Z_N has the following fixed points in the fundamental domain [30]:

$$z = z_f \equiv \begin{cases} 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} & (\text{on } T^2/Z_2), \\ 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} & (\text{on } T^2/Z_3), \\ 0, \frac{1+\tau}{2} & (\text{on } T^2/Z_4), \\ 0. & (\text{on } T^2/Z_6). \end{cases} \quad (2.5)$$

We can introduce 4D fields or interactions at these fixed points. Fields at equivalent points on T^2/Z_N do not have to be equal as long as the Lagrangian is single-valued. The torus boundary conditions are expressed as

$$\begin{aligned} A_M(x, z+s) &= U_s(z) A_M(x, z) U_s^{-1}(z) + i(U_s \partial_M U_s^{-1})(z), \\ B_M(x, z+s) &= B_M(x, z) + \partial_M \Lambda_s(z), \\ \Psi_{\chi_6}^f(x, z+s) &= e^{iq_f \Lambda_s(z)} U_s(z) \Psi_{\chi_6}^f(x, z), \end{aligned} \quad (2.6)$$

where $s = 1, \tau$. Matrices $U_s(z) \in G$ and real functions $\Lambda_s(z)$ may depend on z . The orbifold boundary conditions are

$$\begin{aligned} A_\mu(x, \omega z) &= P A_\mu(x, z) P^{-1}, \quad A_z(x, \omega z) = \omega^{-1} P A_z(x, z) P^{-1}, \\ B_\mu(x, \omega z) &= B_\mu(x, z), \quad B_z(x, \omega z) = \omega^{-1} B_z(x, z), \\ \Psi_{\chi_6, \chi_4}^f(x, \omega z) &= \omega^{-\frac{\chi_4 \chi_6}{2}} e^{i\varphi_f} P \Psi_{\chi_6, \chi_4}^f(x, z), \end{aligned} \quad (2.7)$$

where χ_4 denotes the 4D chirality, and φ_f and $P \in G$ are a real constant and a constant matrix, respectively.

The G gauge field is decomposed as

$$A_M = \sum_i C_M^i H_i + \sum_{\alpha} W_M^{\alpha} E_{\alpha}, \quad (2.8)$$

where $\{H_i, E_{\alpha}\}$ are the generators of G in the Cartan–Weyl basis, i.e., H_i ($i = 1, 2, \dots, \text{Rank } G$) are the Cartan generators and α runs over all the roots of G . The generators are normalized as $\text{Tr}(H_i H_j) = \delta_{ij}$ and $\text{Tr}(E_{\alpha} E_{\beta}) = \delta_{\alpha, -\beta}$. We can always choose the generators in such a way that P in (2.7) is expressed as

$$P = \exp(ip \cdot H), \quad (2.9)$$

where $p \cdot H \equiv \sum_i p^i H_i$ (p^i : real constants). Since (2.7) is a Z_N -transformation, the following relations must hold:

$$\begin{aligned} e^{ip \cdot \alpha} &= \exp\left(\frac{2n_{\alpha} \pi i}{N}\right), \\ \omega^{-\frac{\chi_4 \chi_6}{2}} e^{i\varphi_f} e^{ip \cdot \mu} &= \exp\left(\frac{2n_{\mu f}^{\chi_4 \chi_6} \pi i}{N}\right), \end{aligned} \quad (2.10)$$

where $n_{\alpha}, n_{\mu f}^{\chi_4 \chi_6} \in \mathbb{Z}$.

2.2. Magnetic fluxes

We introduce the magnetic fluxes that penetrate T^2/Z_N as a background. For simplicity, we assume that W_M^{α} do not have nonvanishing background and the background values of the field strengths are constants. Then nonvanishing constant fluxes are

$$\begin{aligned} C^i &\equiv \int_{T^2/Z_N} dx^4 dx^5 \langle C_{45}^i \rangle = \mathcal{A} \langle C_{45}^i \rangle = -\frac{2i \text{Im } \tau}{N} \langle C_{z\bar{z}}^i \rangle, \\ B &\equiv \int_{T^2/Z_N} dx^4 dx^5 \langle B_{45} \rangle = \mathcal{A} \langle B_{45} \rangle = -\frac{2i \text{Im } \tau}{N} \langle B_{z\bar{z}} \rangle, \end{aligned} \quad (2.11)$$

where $C_{z\bar{z}}^i = \partial_z C_{\bar{z}}^i - \partial_{\bar{z}} C_z^i$, $B_{z\bar{z}} = \partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z$, and $\mathcal{A} \equiv (2\pi R_1)^2 \text{Im } \tau / N$ is the area of the fundamental domain of T^2/Z_N . This indicates that the vector potentials C_z^i and B_z have the following background values:

$$\langle C_z^i \rangle = -\frac{iN(c^i \bar{z} + \bar{c}^i)}{4 \text{Im } \tau}, \quad \langle B_z \rangle = -\frac{iN(\beta \bar{z} + \bar{b})}{4 \text{Im } \tau}, \quad (2.12)$$

where c^i and b are complex constants, which correspond to the Wilson-line phases [25,32]. From (2.12), we identify $U_s(z)$ and $\Lambda_s(z)$ ($s = 1, \tau$) in (2.6) as

$$\begin{aligned} U_s(z) &= \exp \left\{ i \sum_i \left(\frac{N c^i \text{Im}(\bar{s}z)}{2 \text{Im } \tau} + 2\pi \alpha_s^i \right) H_i \right\}, \\ \Lambda_s(z) &= \frac{N B \text{Im}(\bar{s}z)}{2 \text{Im } \tau} + 2\pi \beta_s, \end{aligned} \quad (2.13)$$

where α_s^i and β_s are real constants, which correspond to the Scherk–Schwarz (SS) phases [25,32]. The magnetic fluxes C^i and B are quantized as

$$\begin{aligned} N C \cdot \alpha &= 2k_{\alpha} \pi, \\ N (C \cdot \mu + q_f B) &= 2k_{\mu f} \pi, \end{aligned} \quad (2.14)$$

where α and μ are a root and a weight of G , and $k_{\alpha}, k_{\mu f} \in \mathbb{Z}$. The first and the second conditions originate from the requirement for the single-valuedness of W_z^{α} and Ψ^f on T^2/Z_N , respectively.

Using (2.14), the background gauge fields are expressed as

$$\begin{aligned}\langle C_z \cdot \alpha \rangle &= -\frac{k_\alpha \pi i (\bar{z} + \bar{\zeta}_\alpha)}{2\text{Im } \tau}, \\ \langle C_z \cdot \mu + q_f B_z \rangle &= -\frac{k_{\mu f} \pi i (\bar{z} + \bar{\zeta}_{\mu f})}{2\text{Im } \tau},\end{aligned}\quad (2.15)$$

where

$$\zeta_\alpha \equiv \frac{c \cdot \alpha}{c \cdot \alpha}, \quad \zeta_{\mu f} \equiv \frac{c \cdot \mu + q_f b}{c \cdot \mu + q_f b}.\quad (2.16)$$

We assume that the magnetic fluxes break G to $SU(2)_L \times U(1)_X \times U(1)^{r-2}$ (r : rank of G), and that $U(1)_Z \times U(1)_X$ is broken down to the hypercharge group $U(1)_Y$ at one of the orbifold fixed points by some dynamics. The generators of the unbroken $SU(2)_L$ and $U(1)_Z$ are expressed as

$$\left(T_L^\pm, T_L^3 \right) = \left(\frac{E_{\pm\alpha_L}}{|\alpha_L|}, \frac{\alpha_L \cdot H}{|\alpha_L|^2} \right), \quad Q_Z = \eta \cdot H,\quad (2.17)$$

where α_L is a root of $SU(2)_L \subset G$, and a constant real vector η satisfies $\eta \cdot \alpha_L = 0$. Then the hypercharge Y is expressed in terms of Q_Z and the $U(1)_X$ generator Q_X as

$$Y = Q_Z + Q_X.\quad (2.18)$$

3. Mode functions

In this section, we provide a brief review of the results in Refs. [25,28,31–33] in our notations, and show explicit forms of the mode functions on T^2 and T^2/Z_N .

3.1. Kaluza–Klein mode expansion

The 6D fields are expanded into the Kaluza–Klein (KK) modes as

$$\begin{aligned}C_\mu^i(x, z) &= \frac{g_A}{\sqrt{2\pi R_1}} \sum_n f_n^i(z) C_\mu^{i(n)}(x), \quad W_\mu^\alpha(x, z) = \frac{g_A}{\sqrt{2\pi R_1}} \sum_n f_n^\alpha(z) W_\mu^{\alpha(n)}(x), \\ B_\mu(x, z) &= \frac{g_B}{\sqrt{2\pi R_1}} \sum_n f_n^B(z) B_\mu^{(n)}(x), \\ C_z^i(x, z) &= \left\langle C_z^i \right\rangle(z) + g_A \sum_n g_n^i(z) \varphi_n^i(x), \quad W_z^\alpha(x, z) = g_A \sum_n g_n^\alpha(z) \varphi_n^\alpha(x), \\ B_z(x, z) &= \langle B_z \rangle(z) + g_B \sum_n g_n^B(z) \varphi_n^B(x), \\ \psi_\pm^f(x, z) &= \frac{1}{\sqrt{2\pi R_1}} \sum_n \sum_\mu h_{Rn}^{(\pm)\mu f}(z) |\mu\rangle \psi_{\pm n}^{\mu f}(x), \\ \bar{\lambda}_\pm^f(x, z) &= \frac{1}{\sqrt{2\pi R_1}} \sum_n \sum_\mu h_{Ln}^{(\pm)\mu f}(z) |\mu\rangle \bar{\lambda}_{\pm n}^{\mu f}(x),\end{aligned}\quad (3.1)$$

where $|\mu\rangle$ is a vector in the G representation space that corresponds to the weight μ . The fermion fields ψ_{\pm}^f and $\bar{\lambda}_{\pm}^f$ are the right- and the left-handed two-component spinors defined as

$$\begin{aligned}\Psi_+^f &= \begin{pmatrix} \hat{\Psi}_+^f \\ \mathbf{0}_4 \end{pmatrix}, & \hat{\Psi}_+^f &= \begin{pmatrix} \psi_{+\alpha}^f \\ \bar{\lambda}_+^{f\dot{\alpha}} \end{pmatrix}, \\ \Psi_-^f &= \begin{pmatrix} \mathbf{0}_4 \\ \hat{\Psi}_-^f \end{pmatrix}, & \hat{\Psi}_-^f &= \begin{pmatrix} \psi_{-\alpha}^f \\ \bar{\lambda}_-^{f\dot{\alpha}} \end{pmatrix}.\end{aligned}\quad (3.2)$$

All the mode functions are defined to be dimensionless, and normalized as

$$\int_{T^2/Z_N} dz d\bar{z} F_n^*(z) F_m(z) = \delta_{nm}, \quad (3.3)$$

where $F_n(z)$ denotes the mode functions. The coefficients in the KK expansion are determined so that the 4D KK modes have canonically normalized kinetic terms.⁴

From (2.6) and (2.13), the mode functions should satisfy

$$\begin{aligned}f_n^i(z+s) &= f_n^i(z), & f_n^B(z+s) &= f_n^B(z), \\ f_n^\alpha(z+s) &= \exp\left\{\frac{k_\alpha \pi i}{\text{Im } \tau} \text{Im}(\bar{s}z) + 2\pi i \phi_s^\alpha\right\} f_n^\alpha(z), \\ g_n^i(z+s) &= g_n^i(z), & g_n^B(z+s) &= g_n^B(z), \\ g_n^\alpha(z+s) &= \exp\left\{\frac{k_\alpha \pi i}{\text{Im } \tau} \text{Im}(\bar{s}z) + 2\pi i \phi_s^\alpha\right\} g_n^\alpha(z), \\ h_{\text{Rn}}^{(\pm)\mu f}(z+s) &= \exp\left\{\frac{k_{\mu f} \pi i}{\text{Im } \tau} \text{Im}(\bar{s}z) + 2\pi i \phi_s^{\mu f}\right\} h_{\text{Rn}}^{(\pm)\mu f}(z), \\ h_{\text{Ln}}^{(\pm)\mu f}(z+s) &= \exp\left\{\frac{k_{\mu f} \pi i}{\text{Im } \tau} \text{Im}(\bar{s}z) + 2\pi i \phi_s^{\mu f}\right\} h_{\text{Ln}}^{(\pm)\mu f}(z),\end{aligned}\quad (3.4)$$

where

$$\phi_s^\alpha \equiv \alpha_s \cdot \alpha, \quad \phi_s^{\mu f} \equiv \alpha_s \cdot \mu + q_f \beta_s, \quad (3.5)$$

and from (2.7), they also satisfy

$$\begin{aligned}f_n^i(\omega z) &= f_n^i(z), & f_n^\alpha(\omega z) &= e^{ip \cdot \alpha} f_n^\alpha(z), & f_n^B(\omega z) &= f_n^B(z), \\ g_n^i(\omega z) &= \omega^{-1} g_n^i(z), & g_n^\alpha(\omega z) &= \omega^{-1} e^{ip \cdot \alpha} g_n^\alpha(z), & g_n^B(\omega z) &= \omega^{-1} g_n^B(z), \\ h_{\text{Rn}}^{(\pm)\mu f}(\omega z) &= \omega^{\mp \frac{1}{2}} e^{i\varphi_f} e^{ip \cdot \mu} h_{\text{Rn}}^{(\pm)\mu f}(z), & h_{\text{Ln}}^{(\pm)\mu f}(\omega z) &= \omega^{\pm \frac{1}{2}} e^{i\varphi_f} e^{ip \cdot \mu} h_{\text{Ln}}^{(\pm)\mu f}(z).\end{aligned}\quad (3.6)$$

The SS phases $\phi_s = \phi_s^\alpha, \phi_s^{\mu f}$ in (3.4) are defined modulo 1. This means that a set of solutions to the mode equation is invariant under $\phi_s \rightarrow \phi_s + 1$. When $|K| > 1$ ($K = k_\alpha, k_{\mu f}$), however, each mode function is not invariant under such shifts. In fact, the shift $\phi_1 \rightarrow \phi_1 + 1$ changes an eigenstate to another degenerate eigenstate, and the shift $\phi_\tau \rightarrow \phi_\tau + 1$ rotates the phase of the mode function (see Sect. 2.2 of Ref. [32]). If we focus on a specific eigenstate among the degenerate mass eigenstates, the period of ϕ_s is $|K|$, rather than 1.

⁴ Note that $\int dx^4 dx^5 = 2(\pi R_1)^2 \int dz d\bar{z}$.

We should also note that the SS phases can be converted into the Wilson-line phases by a large gauge transformation, and vice versa [32]. The correspondence is

$$\begin{aligned}\phi_s^\alpha = 0, \quad \zeta_\alpha \longleftrightarrow \phi_s^\alpha &= \frac{k_\alpha}{2\text{Im } \tau} \text{Im}(\bar{s}\zeta_\alpha), \quad \zeta_\alpha = 0, \\ \phi_s^{\mu f} = 0, \quad \zeta_{\mu f} \longleftrightarrow \phi_s^{\mu f} &= \frac{k_{\mu f}}{2\text{Im } \tau} \text{Im}(\bar{s}\zeta_{\mu f}), \quad \zeta_{\mu f} = 0,\end{aligned}\quad (3.7)$$

or equivalently,

$$\begin{aligned}\phi_s^\alpha, \quad \zeta_\alpha = 0 \longleftrightarrow \phi_s^\alpha = 0, \quad \zeta_\alpha &= \frac{2}{k_\alpha} (\tau\phi_1^\alpha - \phi_\tau^\alpha), \\ \phi_s^{\mu f}, \quad \zeta_{\mu f} = 0 \longleftrightarrow \phi_s^{\mu f} = 0, \quad \zeta_{\mu f} &= \frac{2}{k_{\mu f}} (\tau\phi_1^{\mu f} - \phi_\tau^{\mu f}).\end{aligned}\quad (3.8)$$

In the following, we choose a gauge where all the SS phases are zero. As mentioned in Refs. [32,34–36], the Wilson-line phases can only take finite numbers (which are equal to the numbers of the orbifold fixed points) of values when the theory is compactified on T^2/Z_N (see Appendix A).

3.2. Mode equations

We choose the following gauge-fixing terms:

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2g_A^2} \text{Tr} \left\{ \left(D^M \tilde{A}_M \right)^2 \right\} - \frac{1}{2g_B^2} \left(\partial^M \tilde{B}_M \right)^2, \quad (3.9)$$

where $\tilde{A}_M \equiv A_M - \langle A_M \rangle$, $\tilde{B} \equiv B_M - \langle B_M \rangle$, and

$$D_M \tilde{A}_N \equiv \partial_M \tilde{A}_N - i \left[\langle A_M \rangle, \tilde{A}_N \right]. \quad (3.10)$$

Then, the mode equations are read off as

$$\begin{aligned}\partial_z \partial_{\bar{z}} f_n^i &= -\tilde{m}_n^2 f_n^i, \quad \mathcal{O}_\alpha f_n^\alpha = -\tilde{m}_n^2 f_n^\alpha, \quad \partial_z \partial_{\bar{z}} f_n^B = -\tilde{m}_n^2 f_n^B, \\ \partial_z \partial_{\bar{z}} g_n^i &= -\tilde{m}_n^2 g_n^i, \quad \left(\mathcal{O}_\alpha + \frac{k_\alpha \pi}{2\text{Im } \tau} \right) g_n^\alpha = -\tilde{m}_n^2 g_n^\alpha, \quad \partial_z \partial_{\bar{z}} g_n^B = -\tilde{m}_n^2 g_n^B, \\ D_{\bar{z}}^{(\mu f)} h_{\text{Rn}}^{(+)\mu f} &= -\tilde{m}_n h_{\text{Ln}}^{(+)\mu f}, \quad D_z^{(\mu f)} h_{\text{Ln}}^{(+)\mu f} = \tilde{m}_n^* h_{\text{Rn}}^{(+)\mu f}, \\ D_z^{(\mu f)} h_{\text{Rn}}^{(-)\mu f} &= -\tilde{m}_n h_{\text{Ln}}^{(-)\mu f}, \quad D_{\bar{z}}^{(\mu f)} h_{\text{Ln}}^{(-)\mu f} = \tilde{m}_n^* h_{\text{Rn}}^{(-)\mu f},\end{aligned}\quad (3.11)$$

where $\tilde{m}_n \equiv \pi R_1 m_n$ (m_n are the KK mass eigenvalues),⁵ and

$$\begin{aligned}\mathcal{O}_\alpha &\equiv \left(\partial_{\bar{z}} + \frac{k_\alpha \pi (z + \zeta_\alpha)}{2\text{Im } \tau} \right) \left(\partial_z - \frac{k_\alpha \pi (\bar{z} + \bar{\zeta}_\alpha)}{2\text{Im } \tau} \right) + \frac{k_\alpha \pi}{2\text{Im } \tau} \\ &= \left(\partial_z - \frac{k_\alpha \pi (\bar{z} + \bar{\zeta}_\alpha)}{2\text{Im } \tau} \right) \left(\partial_{\bar{z}} + \frac{k_\alpha \pi (z + \zeta_\alpha)}{2\text{Im } \tau} \right) - \frac{k_\alpha \pi}{2\text{Im } \tau}, \\ D_z^{(\mu f)} &\equiv \partial_z - \frac{k_{\mu f} \pi (\bar{z} + \bar{\zeta}_{\mu f})}{2\text{Im } \tau}, \quad D_{\bar{z}}^{(\mu f)} \equiv \partial_{\bar{z}} + \frac{k_{\mu f} \pi (z + \zeta_{\mu f})}{2\text{Im } \tau}.\end{aligned}\quad (3.12)$$

⁵ The eigenvalues m_n are in general complex for the fermionic fields, while they are real for the bosonic fields because of the Hermiticity of the corresponding differential operators.

3.3. Mode functions on T^2

Let us first find the mode functions defined on T^2 , which are denoted by letters with a tilde. They are obtained by solving (3.11) with (3.4) in the manner of Refs. [25,32].

3.3.1. Gauge fields

Since C_M^i and B_M do not feel the background gauge fields, their mode equations are easily solved, and the solutions are⁶

$$\begin{aligned} \tilde{f}_{n,l}^i(z), \tilde{g}_{n,l}^i(z) &= \mathcal{N}_{n,l}^{ci} \cos \frac{2\pi \operatorname{Im} \{(n+l\bar{\tau})z\}}{\operatorname{Im} \tau} + \mathcal{N}_{n,l}^{si} \sin \frac{2\pi \operatorname{Im} \{(n+l\bar{\tau})z\}}{\operatorname{Im} \tau}, \\ \tilde{f}_{n,l}^B(z), \tilde{g}_{n,l}^B(z) &= \mathcal{N}_{n,l}^{cB} \cos \frac{2\pi \operatorname{Im} \{(n+l\bar{\tau})z\}}{\operatorname{Im} \tau} + \mathcal{N}_{n,l}^{sB} \sin \frac{2\pi \operatorname{Im} \{(n+l\bar{\tau})z\}}{\operatorname{Im} \tau}, \end{aligned} \quad (3.13)$$

where $\mathcal{N}_{n,l}^{ci}$, $\mathcal{N}_{n,l}^{si}$, $\mathcal{N}_{n,l}^{cB}$, and $\mathcal{N}_{n,l}^{sB}$ are real constants, and the corresponding mass eigenvalues are

$$\tilde{m}_n = \frac{\pi |n+l\tau|}{\operatorname{Im} \tau}. \quad (3.14)$$

Note that the zero-mode functions are constant.

For W_M^α with $k_\alpha = 0$, the mode functions are affected only by the Wilson-line phases:⁷

$$\tilde{f}_{n,l}^\alpha(z), \tilde{g}_{n,l}^\alpha(z) = \mathcal{N}_{n,l}^\alpha \exp \left\{ \frac{2\pi i}{\operatorname{Im} \tau} \operatorname{Im} \left\{ \left(n+l\bar{\tau} - \frac{k_\alpha \bar{\zeta}_\alpha}{2} \right) z \right\} \right\}, \quad (3.15)$$

where $\mathcal{N}_{n,l}^\alpha$ are normalization constants, and the mass eigenvalues are

$$\tilde{m}_{n,l} = \frac{\pi}{\operatorname{Im} \tau} \left| n+l\tau - \frac{k_\alpha \zeta_\alpha}{2} \right|. \quad (3.16)$$

The other fields feel the magnetic fluxes,⁸ and there are degenerate mass eigenstates at each KK level. For W_μ^α with $k_\alpha \neq 0$, there are no zero-modes, i.e.,

$$\tilde{m}_n^2 = \left(n + \frac{1}{2} \right) \frac{|k_\alpha| \pi}{\operatorname{Im} \tau} \geq \frac{|k_\alpha| \pi}{2 \operatorname{Im} \tau} > 0. \quad (3.17)$$

As for W_z^α , only components with $k_\alpha > 0$ have zero-modes. The corresponding mode functions are

$$\tilde{g}_0^{\alpha(j)}(z) = \mathcal{F}^{(j)}(z; k_\alpha, \zeta_\alpha), \quad (3.18)$$

where $j = 1, 2, \dots, k_\alpha$, and

$$\mathcal{F}^{(j)}(z; K, \zeta) \equiv \begin{cases} (2K \operatorname{Im} \tau)^{\frac{1}{4}} e^{K\pi i(z+\zeta) \frac{\operatorname{Im}(z+\zeta)}{\operatorname{Im} \tau}} \vartheta \begin{bmatrix} \frac{j}{K} \\ 0 \end{bmatrix} (K(z+\zeta), K\tau) & (K > 0), \\ (2|K| \operatorname{Im} \tau)^{\frac{1}{4}} e^{K\pi i(\bar{z}+\bar{\zeta}) \frac{\operatorname{Im}(\bar{z}+\bar{\zeta})}{\operatorname{Im} \tau}} \vartheta \begin{bmatrix} \frac{j}{K} \\ 0 \end{bmatrix} (K(\bar{z}+\bar{\zeta}), K\bar{\tau}) & (K < 0). \end{cases} \quad (3.19)$$

⁶ For these modes, we label the KK level by a pair of integers.

⁷ Note that $k_\alpha \zeta_\alpha = Nc \cdot \alpha / 2\pi$ is independent of the flux c^i . It can take nonvanishing values even in the case of $k_\alpha = 0$.

⁸ For simplicity, we do not consider the case of $k_{\mu f} = 0$.

Here, $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ is the Jacobi theta function defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (Kz, K\tau) \equiv \sum_{l=-\infty}^{\infty} e^{\pi i(l+a)^2 K\tau} e^{2\pi i(l+a)(Kz+b)}. \quad (3.20)$$

The function $\mathcal{F}^{(j)}$ satisfies the relation

$$\left\{ \mathcal{F}^{(j)}(z; K, \zeta) \right\}^* = \mathcal{F}^{(-j)}(z; -K, \zeta), \quad (3.21)$$

and is normalized as

$$\int_{T^2} d^2z \left\{ \mathcal{F}^{(j)}(z; K, \zeta) \right\}^* \mathcal{F}^{(k)}(z; K, \zeta) = \delta_{jk}. \quad (3.22)$$

The mode functions for the KK excitation modes are

$$\tilde{g}_n^{\alpha(j)}(z) \propto \left(D_z^{(\alpha)} \right)^n \tilde{g}_0^{\alpha(j)}(z), \quad (3.23)$$

where

$$D_z^{(\alpha)} \equiv \partial_z - \frac{k_\alpha \pi (\bar{z} + \bar{\zeta}_\alpha)}{2\text{Im } \tau}, \quad (3.24)$$

and the mass eigenvalues are

$$\tilde{m}_0^2 = \frac{nk_\alpha \pi}{\text{Im } \tau}. \quad (3.25)$$

The components of W_z^α with $k_\alpha < 0$ do not have zero-modes, and

$$\tilde{m}_n^2 = \frac{(n+1)|k_\alpha| \pi}{\text{Im } \tau} \geq \frac{|k_\alpha| \pi}{\text{Im } \tau} > 0. \quad (3.26)$$

3.3.2. Fermions

For components of $\Psi_{\chi_6}^f$ with $k_{\mu f} > 0$, only ψ_+^f and $\bar{\lambda}_-^f$ have zero-modes whose mode functions are given by

$$\tilde{h}_{\text{R0}}^{(+)\mu f(j)}(z), \tilde{h}_{\text{L0}}^{(-)\mu f(j)}(z) = \mathcal{F}^{(j)}(z; k_{\mu f}, \zeta_{\mu f}), \quad (3.27)$$

where $j = 1, 2, \dots, k_{\mu f}$. For components of $\Psi_{\chi_6}^f$ with $k_{\mu f} < 0$, only ψ_-^f and $\bar{\lambda}_+^f$ have zero-modes whose mode functions are

$$\tilde{h}_{\text{R0}}^{(-)\mu f(j)}(z), \tilde{h}_{\text{L0}}^{(+)\mu f(j)}(z) = \mathcal{F}^{(j)}(z; k_{\mu f}, \zeta_{\mu f}), \quad (3.28)$$

where $j = 1, 2, \dots, |k_{\mu f}|$.

The mode functions for the KK excitation modes are obtained by operating $D_z^{(\mu f)}$ (for $k_{\mu f} > 0$) or $D_{\bar{z}}^{(\mu f)}$ (for $k_{\mu f} < 0$) on the above functions, and their mass eigenvalues are

$$\tilde{m}_n^2 = \frac{n|k_{\mu f}| \pi}{\text{Im } \tau}. \quad (3.29)$$

3.4. Mode functions on T^2/Z_N

As we have seen in the previous subsection, $\left\{ \tilde{f}_0^i(z), \tilde{f}_0^B(z) \right\}$ and $\left\{ \tilde{g}_0^i(z), \tilde{g}_0^B(z) \right\}$ are constants. The former satisfies the orbifold boundary conditions in (3.6), but the latter does not. Thus, C_μ^i and B_μ have zero-modes on T^2/Z_N while C_z^i and B_z do not.

As for W_M^α with $k_\alpha = 0$, zero-modes exist on T^2 only when $\zeta_\alpha = 0$; see (3.16).⁹ Since the corresponding mode functions are constants, they satisfy (3.6) only when $p \cdot \alpha = 0$ for $f_0^\alpha(z)$, and $p \cdot \alpha = 2\pi/N$ for $g_0^\alpha(z)$. These are the conditions for W_μ^α and W_z^α to have zero-modes on T^2/Z_N .

The other modes feel the magnetic fluxes. Thus, they have degenerate modes at each KK level. The orbifold boundary conditions in (3.6) have the form

$$F_n^{(j)}(\omega z) = \eta F_n^{(j)}(z), \tag{3.30}$$

where η is an N th root of unity, and $j = 1, 2, \dots, |K|$ discriminates the degenerate modes. Note that

$$\hat{F}_0^{(j)}(z) \equiv \frac{1}{N} \sum_{l=0}^{N-1} \eta^{-l} \tilde{F}_0^{(j)}(\omega^l z), \tag{3.31}$$

where $\tilde{F}_0^{(j)}(z)$ is a zero-mode function on T^2 , satisfies (3.30). Since $\tilde{F}_0^{(j)}(\omega^l z)$ is a solution of (3.11) that satisfies (3.4), it can be expressed as a linear combination of $\tilde{F}_0^{(k)}(z)$, i.e.,

$$\tilde{F}_0^{(j)}(\omega^l z) = \sum_{k=1}^{|K|} D_{jk}^{(\omega^l)} \tilde{F}_0^{(k)}(z), \tag{3.32}$$

where $D_{jk}^{(\omega^l)}$ are constants. Thus, $\hat{F}_0^{(j)}(z)$ in (3.31) is expressed as

$$\hat{F}_0^{(j)}(z) = \sum_{k=1}^{|K|} \mathcal{M}_{jk}^{(\eta)} \tilde{F}_0^{(k)}(z), \tag{3.33}$$

where

$$\mathcal{M}_{jk}^{(\eta)} \equiv \frac{1}{N} \sum_{l=0}^{N-1} \eta^{-l} D_{jk}^{(\omega^l)}. \tag{3.34}$$

Although j runs from 1 to $|K|$, not all of $\hat{F}_0^{(j)}(z)$ are independent mode functions [32]. In fact, the matrix $\mathcal{M}^{(\eta)}$ generically has zero eigenvalues. The number of zero-modes is equal to the rank of $\mathcal{M}^{(\eta)}$. Here, note that the matrix $\mathcal{M}^{(\eta)}$ is Hermitian because

$$\mathcal{M}^{(\eta)\dagger} = \frac{1}{N} \sum_{l=0}^{N-1} \eta^l D^{(\bar{\omega}^l)\dagger} = \frac{1}{N} \sum_{l'=0}^{N-1} \eta^{-l'} D^{(\omega^{l'})} = \mathcal{M}^{(\eta)}, \tag{3.35}$$

where $l' \equiv -l$ (see Appendix A). Thus, $\mathcal{M}^{(\eta)}$ can be diagonalized by a unitary matrix $V^{(\eta)}$:

$$V^{(\eta)} \mathcal{M}^{(\eta)} V^{(\eta)\dagger} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0), \tag{3.36}$$

where λ_j ($j = 1, 2, \dots, r$) are the non-zero (real) eigenvalues, and $r \equiv \text{Rank } \mathcal{M}^{(\eta)}$. Then we find that

$$\sum_{k=1}^{|K|} V_{jk}^{(\eta)} \hat{F}_0^{(k)}(z) = \begin{cases} \lambda_j \sum_k V_{jk}^{(\eta)} \tilde{F}_0^{(k)}(z) & (1 \leq j \leq r), \\ 0 & (r+1 \leq j \leq |K|). \end{cases} \tag{3.37}$$

Therefore, it is convenient to choose independent mode functions on T^2/Z_N as

$$F_0^{(j)}(z) \equiv \sqrt{N} \sum_{k=1}^{|K|} V_{jk}^{(\eta)} \tilde{F}_0^{(k)}(z), \tag{3.38}$$

⁹ Note that $k_\alpha \zeta_\alpha / 2$ is defined modulo 1 and τ as can be seen from (3.8).

where $j = 1, 2, \dots, r$. We can easily show that these satisfy the orthonormal condition

$$\int_{T^2/Z_N} d^2z \left\{ F_0^{(j)}(z) \right\}^* F_0^{(k)}(z) = \delta_{jk}, \quad (3.39)$$

which follows from the orthonormal condition of $\tilde{F}_n^{(j)}(z)$. The matrix $\mathcal{M}^{(\eta)}$ is expressed as

$$\mathcal{M}_{jk}^{(\eta)} = \int_{T^2} d^2z \left\{ \tilde{F}_0^{(j)}(z) \right\}^* \hat{F}_0^{(k)}(z). \quad (3.40)$$

In Ref. [33], analytic forms of the matrix $\mathcal{M}^{(\eta)}$ are derived implying the operator formalism. It is obtained from (3.34) with analytic forms of $D_{jk}^{(\omega^l)}$, which are collected in Appendix A.

The mode functions for the KK modes are obtained by operating $D_z = D_z^{(\alpha)}, D_z^{(\mu,f)}$ or $D_{\bar{z}} = D_{\bar{z}}^{(\alpha)}, D_{\bar{z}}^{(\mu,f)}$ on $F_0^{(j)}(z)$, just like those on T^2 . However, since

$$\begin{aligned} D_z \left(\tilde{F}_0^{(j)}(\omega^l z) \right) &= \omega^l \left(D_z \tilde{F}_0^{(j)} \right) (\omega^l z) \propto \omega^l \tilde{F}_1^{(j)}(\omega^l z) \quad (\text{when } K > 0), \\ D_{\bar{z}} \left(\tilde{F}_0^{(j)}(\omega^l z) \right) &= \bar{\omega}^l \left(D_{\bar{z}} \tilde{F}_0^{(j)} \right) (\omega^l z) \propto \bar{\omega}^l \tilde{F}_1^{(j)}(\omega^l z) \quad (\text{when } K < 0), \end{aligned} \quad (3.41)$$

the phase factor η in $\mathcal{M}_{jk}^{(\eta)}$ becomes $\eta\omega^{-1}$ (for $K > 0$) or $\eta\omega$ (for $K < 0$). Therefore, the expression corresponding to (3.38) for the KK modes is

$$F_n^{(j)}(z) = \begin{cases} \sqrt{N} \sum_{k=1}^K V_{jk}^{(\eta\omega^{-n})} \tilde{F}_n^{(k)}(z) & (\text{for } K > 0), \\ \sqrt{N} \sum_{k=1}^{|K|} V_{jk}^{(\eta\omega^n)} \tilde{F}_n^{(k)}(z) & (\text{for } K < 0). \end{cases} \quad (3.42)$$

The number of mass eigenstates at each KK level is given by the rank of $\mathcal{M}^{(\eta\omega^{-n})}$ (for $K > 0$) or that of $\mathcal{M}^{(\eta\omega^n)}$ (for $K < 0$).

Note that the constants $D_{jk}^{(\omega^l)}$ in Appendix A, which are functions of K and ζ , satisfy

$$D_{jk}^{(\omega^l)}[-K, \zeta] = D_{kj}^{(\bar{\omega}^l)}[K, \zeta], \quad (3.43)$$

where $\zeta = \frac{2}{K}(\tau\phi_1 - \phi_\tau)$. Thus, we find that

$$\begin{aligned} \mathcal{M}_{jk}^{(\eta)}[-K, \zeta] &= \frac{1}{N} \sum_{l=0}^{N-1} \bar{\eta}^l D_{jk}^{(\omega^l)}[-K, \zeta] = \frac{1}{N} \sum_{l=0}^{N-1} \bar{\eta}^l D_{kj}^{(\bar{\omega}^l)}[K, \zeta] \\ &= \frac{1}{N} \sum_{l'=0}^{N-1} \bar{\eta}^{-l'} D_{kj}^{(\omega^{l'})}[K, \zeta] = \mathcal{M}_{kj}^{(\bar{\eta})}[K, \zeta], \end{aligned} \quad (3.44)$$

where $l' \equiv -l$. This indicates that the number of zero-modes for a field that feels a magnetic flux $K < 0$ and an orbifold twist phase η is equal to that for a field with $|K|$ and $\bar{\eta}$.

4. Yukawa coupling constants

4.1. General expression

In the gauge–Higgs unification, the Yukawa couplings originate from the 6D gauge interactions:

$$\begin{aligned} S &= \int d^6x \left(\sum_{f_+} i \bar{\Psi}_+^{f_+} \gamma^M \mathcal{D}_M \Psi_+^{f_+} + \sum_{f_-} i \bar{\Psi}_-^{f_-} \gamma^M \mathcal{D}_M \Psi_-^{f_-} \right) + \dots \\ &= \int d^4x \int d^2z \, 2\pi R_1 \left(- \sum_{f_+} i \bar{\psi}_+^{f_+} A_z \bar{\lambda}_+^{f_+} + \sum_{f_-} i \lambda_-^{f_-} A_z \psi_-^{f_-} \right) + \text{h.c.} + \dots, \end{aligned} \quad (4.1)$$

where $d^2z \equiv dz d\bar{z}$. In the 4D effective theory, we have the following Yukawa couplings:

$$\begin{aligned} \mathcal{L}_{\text{yukawa}}^{(4D)} &= \sum_{\mu} \sum_{f_+} \sum_{i,j,k} y_{ijk}^{(+)\mu f_+} \bar{\psi}_{+0}^{(\mu+\alpha) f_+(i)} \varphi_0^{\alpha(k)} \bar{\lambda}_{+0}^{\mu f_+(j)} + \text{h.c.} \\ &\quad + \sum_{\mu} \sum_{f_-} \sum_{i,j,k} y_{ijk}^{(-)\mu f_-} \lambda_{-0}^{\mu f_-(j)} \varphi_0^{\alpha(k)} \psi_{-0}^{(\mu+\alpha) f_-(i)} + \text{h.c.}, \end{aligned} \quad (4.2)$$

where the indices i, j, k run over the degenerate zero-modes, and

$$\begin{aligned} y_{ijk}^{(+)\mu f_+} &\equiv - \frac{i g_A \langle \mu + \alpha | E_\alpha | \mu \rangle}{\pi R_1} \int_{T^2/Z_N} d^2z \left\{ h_{R0}^{(+)(\mu+\alpha) f_+(i)}(z) \right\}^* g_0^{\alpha(k)}(z) h_{L0}^{(+)\mu f_+(j)}(z) \\ &= - \frac{2i \bar{g}_A \sqrt{\text{Im} \tau}}{N \sqrt{N}} \langle \mu + \alpha | E_\alpha | \mu \rangle \int_{T^2} d^2z \left\{ h_{R0}^{(+)(\mu+\alpha) f_+(i)}(z) \right\}^* g_0^{\alpha(k)}(z) h_{L0}^{(+)\mu f_+(j)}(z) \\ &= -2i \bar{g}_A \sqrt{\text{Im} \tau} \langle \mu + \alpha | E_\alpha | \mu \rangle \sum_{i'=1}^{|K_1|} \sum_{j'=1}^{|K_2|} \sum_{k'=1}^{|K_3|} V_{ii'}^{(\eta_1)*} V_{jj'}^{(\eta_2)} V_{kk'}^{(\eta_3)} \\ &\quad \times \int_{T^2} d^2z \mathcal{F}^{(i')} (z; K_1, \zeta_1) \mathcal{F}^{(j')} (z; K_2, \zeta_2) \mathcal{F}^{(k')} (z; K_3, \zeta_3), \\ y_{ijk}^{(-)\mu f_-} &\equiv 2i \bar{g}_A \sqrt{\text{Im} \tau} \langle \mu + \alpha | E_\alpha | \mu \rangle \sum_{i'=1}^{|K_1|} \sum_{j'=1}^{|K_2|} \sum_{k'=1}^{|K_3|} V_{ii'}^{(\eta_1)} V_{jj'}^{(\eta_2)*} V_{kk'}^{(\eta_3)} \\ &\quad \times \int_{T^2} d^2z \mathcal{F}^{(i')} (z; K_1, \zeta_1) \mathcal{F}^{(j')} (z; K_2, \zeta_2) \mathcal{F}^{(k')} (z; K_3, \zeta_3), \end{aligned} \quad (4.3)$$

where $\bar{g}_A \equiv \frac{g_A}{\sqrt{A}} = \frac{\sqrt{N} g_A}{2\pi R_1 \sqrt{\text{Im} \tau}}$ is the 4D gauge coupling constant, $K_1 \equiv k_{(\mu+\alpha) f_\pm}$, $\zeta_1 \equiv \zeta_{(\mu+\alpha) f_\pm}$, $K_2 \equiv k_{\mu f_\pm}$, $\zeta_2 \equiv \zeta_{\mu f_\pm}$, $K_3 \equiv k_\alpha$, $\zeta_3 \equiv \zeta_\alpha$, and $\{\eta_1, \eta_2, \eta_3\}$ are the phase factors in the orbifold boundary conditions.¹⁰

4.1.1. Couplings to fermions with $\chi_6 = +$

From the gauge invariance of the Lagrangian, the following conditions hold:

$$K_1 = K_2 + K_3, \quad K_1 \zeta_1 = K_2 \zeta_2 + K_3 \zeta_3, \quad (4.4)$$

and from the condition that the zero-modes exist, it follows that

$$K_1 > 0, \quad K_2 < 0, \quad K_3 > 0. \quad (4.5)$$

¹⁰ The phase factors η_1 and η_2 depend on the flavor index f_+ or f_- .

Then we find that

$$\begin{aligned} & \left\{ \mathcal{F}^{(i')} (z; K_1, \zeta_1) \right\}^* \mathcal{F}^{(j')} (z; K_2, \zeta_2) \\ &= \frac{1}{\sqrt{K_3}} \sum_{m=1}^{K_3} \left\{ \mathcal{F}^{(i'-j'+K_1m)} (z; K_3, \zeta_3) \mathcal{F}^{(|K_2|i'+K_1j'+K_1|K_2|m)} \left(0; |K_1K_2K_3|, \frac{\zeta_1 - \zeta_2}{K_3} \right) \right\}^*, \end{aligned} \quad (4.6)$$

which follows from the formula [(5.8) in Ref. [25]]

$$\begin{aligned} & \vartheta \begin{bmatrix} \frac{i'}{K_1} \\ 0 \end{bmatrix} (K_1 (z + \zeta_1), K_1 \tau) \cdot \vartheta \begin{bmatrix} -\frac{j'}{|K_2|} \\ 0 \end{bmatrix} (|K_2| (z + \zeta_2), |K_2| \tau) \\ &= \sum_{l=1}^{K_1+|K_2|} \vartheta \begin{bmatrix} \frac{i'-j'+K_1l}{K_1+|K_2|} \\ 0 \end{bmatrix} \left((K_1 + |K_2|) \left(z + \frac{K_1\zeta_1 + |K_2|\zeta_2}{K_1 + |K_2|} \right), (K_1 + |K_2|) \tau \right) \\ & \quad \times \vartheta \begin{bmatrix} \frac{|K_2|i'+K_1j'+K_1|K_2|l}{K_1|K_2|(K_1+|K_2|)} \\ 0 \end{bmatrix} (K_1 |K_2| (\zeta_1 - \zeta_2), K_1 |K_2| (K_1 + |K_2|) \tau), \end{aligned} \quad (4.7)$$

with (3.21) and (4.4). Therefore, using the orthonormal condition (3.22), we obtain

$$\begin{aligned} & \int_{T^2} d^2z \left\{ \mathcal{F}^{(i')} (z; K_1, \zeta_1) \right\}^* \mathcal{F}^{(j')} (z; K_2, \zeta_2) \mathcal{F}^{(k')} (z; K_3, \zeta_3) \\ &= \frac{1}{\sqrt{K_3}} \sum_{m=1}^{K_3} \mathcal{F}^{(K_2i'-K_1j'+K_1K_2m)} \left(0, K_1K_2K_3, \frac{\zeta_1 - \zeta_2}{K_3} \right) \delta_{i'-j'+K_1m, k'}. \end{aligned} \quad (4.8)$$

Notice that $\delta_{i'-j'+K_1m, k'}$ is defined on Z_{K_3} , i.e.,

$$\delta_{i'-j'+K_1m, k'} \equiv \begin{cases} 1 & (i' - j' + K_1m = k' \bmod K_3), \\ 0 & (\text{other cases}). \end{cases} \quad (4.9)$$

As a result, we obtain the following expression for the Yukawa coupling constant:

$$\begin{aligned} y_{ijk}^{(+)} &= -\frac{2i\bar{g}_A\sqrt{\text{Im}\tau}}{\sqrt{K_3}} \langle \boldsymbol{\mu} + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu} \rangle \sum_{i'=1}^{K_1} \sum_{j'=1}^{|K_2|} \sum_{k'=1}^{K_3} V_{ii'}^{(\eta_1)*} [K_1, \zeta_1] V_{jj'}^{(\eta_2)} [K_2, \zeta_2] V_{kk'}^{(\eta_3)} [K_3, \zeta_3] \\ & \quad \times \sum_{m=1}^{K_3} \mathcal{F}^{(K_2i'-K_1j'+K_1K_2m)} \left(0, K_1K_2K_3, \frac{\zeta_1 - \zeta_2}{K_3} \right) \delta_{i'-j'+K_1m, k'}. \end{aligned} \quad (4.10)$$

Note that the matrix $V^{(\eta)}$ depends on the flux and the Wilson-line phase. The indices i , j , and k run from 1 to $\text{Rank } \mathcal{M}^{(\eta_1)}$, $\text{Rank } \mathcal{M}^{(\eta_2)}$, and $\text{Rank } \mathcal{M}^{(\eta_3)}$, respectively.

4.1.2. Couplings to fermions with $\chi_6 = -$

From the gauge invariance, K_a and ζ_a ($a = 1, 2, 3$) satisfy

$$K_2 = K_1 + K_3, \quad K_2\zeta_2 = K_1\zeta_1 + K_3\zeta_3, \quad (4.11)$$

and the zero-mode conditions are

$$K_1 < 0, \quad K_2 > 0, \quad K_3 > 0. \quad (4.12)$$

Following the same procedure as in the previous case, we obtain

$$y_{ijk}^{(-)} = \frac{2i\bar{g}_A\sqrt{\text{Im}\tau}}{\sqrt{K_3}} \langle \boldsymbol{\mu} + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu} \rangle \sum_{i'=1}^{K_1} \sum_{j'=1}^{K_2} \sum_{k'=1}^{K_3} V_{ii'}^{(\eta_1)} [K_1, \zeta_1] V_{jj'}^{(\eta_2)*} [K_2, \zeta_2] V_{kk'}^{(\eta_3)} [K_3, \zeta_3] \\ \times \sum_{m=1}^{K_3} \mathcal{F}^{(K_1 j' - K_2 i' + K_1 K_2 m)} \left(0, K_1 K_2 K_3, \frac{\zeta_2 - \zeta_1}{K_3} \right) \delta_{j'-i'+K_2 m, k'}. \quad (4.13)$$

4.2. Specific model

In this subsection, we evaluate the Yukawa coupling constants in a specific model. We consider the case that $G = \text{SU}(3)$,¹¹ $N = 3$, and the matter fermions consist of two $\chi_6 = -$ spinors (Ψ_-^1, Ψ_-^3) that belong to $\mathbf{3}$ of $\text{SU}(3)$ and two $\chi_6 = +$ spinors (Ψ_+^2, Ψ_+^4) that belong to $\bar{\mathbf{3}}$. The $U(1)_X$ charges are assigned as $(q_1, q_2, q_3, q_4) = (0, 1/3, -2/3, -1/3)$.

4.2.1. Symmetry breaking and irreducible decomposition

The roots of $\text{SU}(3)$ are

$$\boldsymbol{\alpha}_1 \equiv \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad \boldsymbol{\alpha}_2 \equiv \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad \boldsymbol{\alpha}_3 \equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1, 0), \\ -\boldsymbol{\alpha}_1, \quad -\boldsymbol{\alpha}_2, \quad -\boldsymbol{\alpha}_3; \quad (4.14)$$

the weights of $\mathbf{3}$ are

$$\boldsymbol{\mu}_1 \equiv \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \boldsymbol{\mu}_2 \equiv \boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1 = \left(0, -\frac{1}{\sqrt{3}} \right), \\ \boldsymbol{\mu}_3 \equiv \boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}} \right); \quad (4.15)$$

and the weights of $\bar{\mathbf{3}}$ are $\{-\boldsymbol{\mu}_1, -\boldsymbol{\mu}_2, -\boldsymbol{\mu}_3\}$.

We choose the direction of the G flux in (2.12) as

$$(c^1, c^2) = c^1 \left(1, -\frac{1}{\sqrt{3}} \right), \quad (4.16)$$

so that G is broken to $\text{SU}(2)_L \times U(1)_Z$. Then, $\boldsymbol{\alpha}_L$ and $\boldsymbol{\eta}$ in (2.17) are identified as

$$\boldsymbol{\alpha}_L = \boldsymbol{\alpha}_1, \quad \boldsymbol{\eta} = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}} \right). \quad (4.17)$$

The normalization of $\boldsymbol{\eta}$ is chosen in such a manner that the hypercharge of the Higgs doublet becomes $\pm 1/2$ [see (4.20)]. The fluxes c^1 and B are determined so that the quantization condition (2.14) is

¹¹ We do not consider the custodial symmetry, for simplicity.

satisfied for all the roots and the weights. In this model, (2.14) becomes

$$\begin{aligned}
0 &= 2k_{\pm\alpha_1}\pi, \quad \pm NC^1 = 2k_{\pm\alpha_2}\pi = 2k_{\pm\alpha_3}\pi, \\
\frac{NC^1}{3} &= 2k_{\mu_1 1}\pi = 2k_{\mu_2 1}\pi, \quad -\frac{2NC^1}{3} = 2k_{\mu_3 1}\pi, \\
N\left(-\frac{C^1}{3} + \frac{B}{3}\right) &= 2k_{-\mu_1 2}\pi = 2k_{-\mu_2 2}\pi, \quad N\left(\frac{2C^1}{3} + \frac{B}{3}\right) = 2k_{-\mu_3 2}\pi, \\
N\left(\frac{C^1}{3} - \frac{2B}{3}\right) &= 2k_{\mu_1 3}\pi = 2k_{\mu_2 3}\pi, \quad N\left(-\frac{2C^1}{3} - \frac{2B}{3}\right) = 2k_{\mu_3 3}\pi, \\
N\left(-\frac{C^1}{3} - \frac{B}{3}\right) &= 2k_{-\mu_1 4}\pi = 2k_{-\mu_2 4}\pi, \quad N\left(\frac{2C^1}{3} - \frac{B}{3}\right) = 2k_{-\mu_3 4}\pi.
\end{aligned} \tag{4.18}$$

These can be solved as

$$\begin{aligned}
NC^1 &= 6\kappa\pi, \quad NB = 6\kappa'\pi, \\
k_{\pm\alpha_1} &= 0, \quad k_{\pm\alpha_2} = k_{\pm\alpha_3} = \pm 3\kappa, \\
k_{\mu_1 1} &= k_{\mu_2 1} = \kappa, \quad k_{\mu_3 1} = -2\kappa, \\
k_{-\mu_1 2} &= k_{-\mu_2 2} = -\kappa + \kappa', \quad k_{-\mu_3 2} = 2\kappa + \kappa', \\
k_{\mu_1 3} &= k_{\mu_2 3} = \kappa - 2\kappa', \quad k_{\mu_3 3} = -2\kappa - 2\kappa', \\
k_{-\mu_1 4} &= k_{-\mu_2 4} = -\kappa - \kappa', \quad k_{-\mu_3 4} = 2\kappa - \kappa',
\end{aligned} \tag{4.19}$$

where κ and κ' are integers.

Under the unbroken $SU(2)_L$, the $SU(3)$ adjoint representation is decomposed as

$$\begin{aligned}
\{ |-\alpha_1\rangle, |\mathbf{0}\rangle_T, |\alpha_1\rangle \} &: \text{triplet } (Y = 0) \\
\{ |\alpha_2\rangle, |\alpha_3\rangle \} &: \text{doublet } (Y = 1/2) \\
\{ |-\alpha_3\rangle, |-\alpha_2\rangle \} &: \text{doublet } (Y = -1/2) \\
|\mathbf{0}\rangle_S &: \text{singlet } (Y = 0),
\end{aligned} \tag{4.20}$$

where $|\mathbf{0}\rangle_T$ and $|\mathbf{0}\rangle_S$ are the states that correspond to the Cartan generators, and Y is the hypercharge. Since the above states do not have the $U(1)_X$ charges, Y in (4.20) is equal to the $U(1)_Z$ charge. Thus, the Higgs doublets are identified as $(\varphi_0^{\alpha_2(k)}, \varphi_0^{\alpha_3(k)})$ or $(\varphi_0^{-\alpha_3(k)}, \varphi_0^{-\alpha_2(k)})$.

As for the matter sector, $\{|\mu_2\rangle, |\mu_1\rangle\}$ and $\{|\mu_3\rangle, |-\mu_3\rangle\}$ ($|\mu_3\rangle$ and $|-\mu_3\rangle$) are doublets (singlets) of $SU(2)_L$. From (2.18), the hypercharges of the components of $\Psi_-^{1,3}$ are

$$\begin{aligned}
(Y(\mu_1), Y(\mu_2), Y(\mu_3)) &= (\eta \cdot \mu_1, \eta \cdot \mu_2, \eta \cdot \mu_3) + (q_f, q_f, q_f) \\
&= \begin{cases} (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}) & (\text{for } \Psi_-^1), \\ (-\frac{1}{2}, -\frac{1}{2}, -1) & (\text{for } \Psi_-^3), \end{cases}
\end{aligned} \tag{4.21}$$

and those of $\Psi_+^{2,4}$ are

$$(Y(-\mu_1), Y(-\mu_2), Y(-\mu_3)) = \begin{cases} (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}) & (\text{for } \Psi_+^2), \\ (-\frac{1}{2}, -\frac{1}{2}, 0) & (\text{for } \Psi_+^4). \end{cases} \tag{4.22}$$

Thus $(\bar{\lambda}_{-0}^{\mu_2 f(j)}, \bar{\lambda}_{-0}^{\mu_1 f(j)})$ and $(\bar{\lambda}_{+0}^{-\mu_1 f(j)}, \bar{\lambda}_{+0}^{-\mu_2 f(j)})$ ($\psi_{-0}^{\mu_3 f(i)}$ and $\psi_{+0}^{-\mu_3 f(i)}$) are identified as the left-handed doublets (the right-handed singlets) in the standard model. They are denoted by

$$\begin{aligned} Q_L^j (\mathbf{2}_{1/6}), \quad d_R^i (\mathbf{1}_{-1/3}) & \quad (\text{from } \Psi_-^1); \\ Q_L^{\prime j} (\mathbf{2}_{1/6}), \quad u_R^i (\mathbf{1}_{2/3}) & \quad (\text{from } \Psi_+^2); \\ L_L^j (\mathbf{2}_{-1/2}), \quad e_R^i (\mathbf{1}_{-1}) & \quad (\text{from } \Psi_-^3); \\ L_L^{\prime j} (\mathbf{2}_{-1/2}), \quad \nu_R^i (\mathbf{1}_0) & \quad (\text{from } \Psi_+^4); \end{aligned} \tag{4.23}$$

where L and R denote the 4D chiralities.

4.2.2. Model parameters

We choose the matrix P in (2.7) in such a way that it does not affect the symmetry breaking caused by the magnetic fluxes. Then the possible choices are

$$p = \frac{2\pi n_p}{N} \left(1, -\frac{1}{\sqrt{3}} \right), \tag{4.24}$$

where $n_p = 0, 1, 2$.

In order for the components in (4.23) to have zero-modes, the integers κ and κ' in (4.19) should satisfy

$$\begin{aligned} \kappa, 2\kappa + \kappa', \quad \kappa - 2\kappa', \quad 2\kappa - \kappa' & \geq 1, \\ -2\kappa, -\kappa + \kappa', \quad -2\kappa - 2\kappa', \quad -\kappa - \kappa' & \leq -1, \end{aligned} \tag{4.25}$$

which are summarized as

$$\kappa \geq 1, \quad -\kappa + 1 \leq \kappa' \leq \frac{\kappa - 1}{2}. \tag{4.26}$$

Hence, the $(\varphi_0^{\alpha_2(k)}, \varphi_0^{\alpha_3(k)})$ are identified as the Higgs doublets H_k because $k_{\alpha_2} = k_{\alpha_3} = 3\kappa > 0$.

The values of the orbifold twist phase η in (3.30) for the relevant components are expressed as

$$\eta = \begin{cases} \omega^{-1} e^{ip \cdot \alpha_2} = \omega^{n_p - 1} & (\text{for } H_k) \\ \omega^{-\frac{1}{2}} e^{i\varphi_1} e^{ip \cdot \mu_1} = \omega^{n_1 + n_p} & (\text{for } Q_L^j) \\ \omega^{\frac{1}{2}} e^{i\varphi_1} e^{ip \cdot \mu_3} = \omega^{n_1 + 1} & (\text{for } d_R^i) \\ \omega^{\frac{1}{2}} e^{i\varphi_2} e^{-ip \cdot \mu_1} = \omega^{n_2 + 1} & (\text{for } Q_L^{\prime j}) \\ \omega^{-\frac{1}{2}} e^{i\varphi_2} e^{-ip \cdot \mu_3} = \omega^{n_2 + n_p} & (\text{for } u_R^i) \\ \omega^{-\frac{1}{2}} e^{i\varphi_3} e^{ip \cdot \mu_1} = \omega^{n_3 + n_p} & (\text{for } L_L^j) \\ \omega^{\frac{1}{2}} e^{i\varphi_3} e^{ip \cdot \mu_3} = \omega^{n_3 + 1} & (\text{for } e_R^i) \\ \omega^{\frac{1}{2}} e^{i\varphi_4} e^{-ip \cdot \mu_1} = \omega^{n_4 + 1} & (\text{for } L_L^{\prime j}) \\ \omega^{-\frac{1}{2}} e^{i\varphi_4} e^{-ip \cdot \mu_3} = \omega^{n_4 + n_p} & (\text{for } \nu_R^i), \end{cases} \tag{4.27}$$

where n_f ($f = 1, 2, 3, 4$) are integers [see (2.10)].

Table 1. The magnetic flux K and the orbifold twist phase η felt by each field, and $\phi \equiv K\zeta/2(\tau - 1)$, where ζ is the Wilson-line phase. The constant $2l$ (l') is even for even κ (κ'), and odd for odd κ (κ').

	H	Q_L	d_R	Q'_L	u_R	L_L	e_R	L'_L	ν_R
K	3κ	κ	-2κ	$-\kappa + \kappa'$	$2\kappa + \kappa'$	$\kappa - 2\kappa'$	$-2\kappa - 2\kappa'$	$-\kappa - \kappa'$	$2\kappa - \kappa'$
η	ω^{n_p-1}	$\omega^{n_1+n_p}$	ω^{n_1+1}	ω^{n_2+1}	$\omega^{n_2+n_p}$	$\omega^{n_3+n_p}$	ω^{n_3+1}	ω^{n_4+1}	$\omega^{n_4+n_p}$
ϕ	l	$\frac{l}{3}$	$-\frac{2l}{3}$	$-\frac{2l-l'}{6}$	$\frac{4l+l'}{6}$	$\frac{l-l'}{3}$	$-\frac{2l+l'}{3}$	$-\frac{2l+l'}{6}$	$\frac{4l-l'}{6}$

From (2.14), (2.16), (3.8), and (A3), the allowed values of the Wilson-line phases are expressed as

$$\begin{aligned}\zeta_\alpha &= \frac{Nc \cdot \alpha}{2k_\alpha \pi} = \frac{2}{k_\alpha} \phi^\alpha (\tau - 1), \\ \zeta_{\mu_f} &= \frac{N(c \cdot \mu + q_f b)}{2k_{\mu_f} \pi} = \frac{2}{k_{\mu_f}} \phi^{\mu_f} (\tau - 1),\end{aligned}\quad (4.28)$$

where

$$\phi^\alpha = \frac{l_\alpha}{3} + \frac{1}{4} \left\{ 1 - (-1)^{k_\alpha} \right\}, \quad \phi^{\mu_f} = \frac{l_{\mu_f}}{3} + \frac{1}{4} \left\{ 1 - (-1)^{k_{\mu_f}} \right\}, \quad (4.29)$$

with $l_\alpha, l_{\mu_f} = 0, 1, 2$. Noting that $\phi^{\alpha_1} = 0$ from the condition that $SU(2)_L$ is unbroken, the Wilson-line phase vectors in (2.12) should be

$$\vec{c} = \frac{4\pi l (\tau - 1)}{3} \begin{pmatrix} 1 \\ -1/\sqrt{3} \end{pmatrix}, \quad \vec{b} \equiv \frac{2\pi l' (\tau - 1)}{3}, \quad (4.30)$$

where l and l' are real constants. Then, ϕ^α and ϕ^{μ_f} are parametrized as

$$\begin{aligned}\phi^{\alpha_1} &= 0, & \phi^{\alpha_2} &= \phi^{\alpha_3} = l, \\ \phi^{\mu_1^1} &= \phi^{\mu_2^1} = \frac{l}{3}, & \phi^{\mu_3^1} &= -\frac{2l}{3}, \\ \phi^{-\mu_1^2} &= \phi^{-\mu_2^2} = -\frac{l}{3} + \frac{l'}{6}, & \phi^{-\mu_3^2} &= \frac{2l}{3} + \frac{l'}{6}, \\ \phi^{\mu_1^3} &= \phi^{\mu_2^3} = \frac{l}{3} - \frac{l'}{3}, & \phi^{\mu_3^3} &= -\frac{2l}{3} - \frac{l'}{3}, \\ \phi^{-\mu_1^4} &= \phi^{-\mu_2^4} = -\frac{l}{3} - \frac{l'}{6}, & \phi^{-\mu_3^4} &= \frac{2l}{3} - \frac{l'}{6}.\end{aligned}\quad (4.31)$$

These phases $\phi = \phi^\alpha, \phi^{\mu_f}$ are defined modulo $|K|$ ($K = k_\alpha, k_{\mu_f}$; see the comment below (3.6)). Comparing (4.31) with (4.29), we find that $2l$ (l') is even for even κ (κ'), and odd for odd κ (κ').

In summary, the Yukawa sector of this model is specified by nine integers: $\kappa, \kappa', l, l', n_p$, and n_f ($f = 1, 2, 3, 4$). The numbers of zero-modes and mode functions are determined by the magnetic flux the field feels K , the orbifold twist phase η , and the Wilson-line phase $\zeta = \frac{2}{K} \phi (\tau - 1)$, which are summarized in Table 1.

4.2.3. Realization of three generations

Here we consider the possibility that the three generations of quarks and leptons are realized by the magnetic fluxes. This occurs when $\kappa = 6, \kappa' = 0, n_p = 0, n_{1,3} = 0, n_{2,4} = 2$, and $l = l' = 0$.¹² In

¹² If we allow extra zero-modes in addition to (4.23), other parameter choices are also possible.

this case, we obtain the following terms in the 4D effective Lagrangian from the bulk:

$$\begin{aligned} \mathcal{L}^{(4D)} = & - \sum_{k=1}^5 \sum_{i,j=1}^3 \left(y_{ij}^{(k)D} \bar{Q}_L^j H_k d_R^i + y_{ij}^{(k)U} \bar{u}_R^i \epsilon H_k Q_L^j \right. \\ & \left. + y_{ij}^{(k)E} \bar{L}_L^j H_k e_R^i + y_{ij}^{(k)N} \bar{\nu}_R^i \epsilon H_k L_L^j + \text{h.c.} \right) + \dots, \end{aligned} \quad (4.32)$$

where $\epsilon H_k Q_L^{ij} \equiv \epsilon_{ab} H_k^a Q_L^{ijb}$ and $\epsilon H_k L_L^{ij} \equiv \epsilon_{ab} H_k^a L_L^{ijb}$ (a, b : $SU(2)_L$ -doublet indices), and

$$\begin{aligned} y_{ij}^{(k)D} = y_{ij}^{(k)E} &= \frac{ig}{\sqrt{2} \cdot 3^{\frac{3}{4}}} \sum_{i'=1}^{12} \sum_{j'=1}^6 \sum_{k'=1}^{18} V_{ii'}^{(\omega)} [-12, 0] V_{jj'}^{(1)*} [6, 0] V_{kk'}^{(\omega^2)} [18, 0] \\ &\times \sum_{m=1}^{18} \mathcal{F}^{(-12j' - 6i' - 72m)} (0, -1296, 0) \delta_{j' - i' + 6m, k'}, \\ y_{ij}^{(k)U} = y_{ij}^{(k)N} &= \frac{ig}{\sqrt{2} \cdot 3^{\frac{3}{4}}} \sum_{i'=1}^{12} \sum_{j'=1}^6 \sum_{k'=1}^{18} V_{ii'}^{(\omega^2)*} [12, 0] V_{jj'}^{(1)} [-6, 0] V_{kk'}^{(\omega^2)} [18, 0] \\ &\times \sum_{m=1}^{18} \mathcal{F}^{(-6i' - 12j' - 72m)} (0, -1296, 0) \delta_{i' - j' + 12m, k'}, \end{aligned} \quad (4.33)$$

where $g = \bar{g}_A \simeq 0.652$ is the 4D $SU(2)_L$ gauge coupling, and we have used that¹³

$$\begin{aligned} \langle -\mu_3 | E_{\alpha_2} | -\mu_2 \rangle &= \langle -\mu_3 | E_{\alpha_3} | -\mu_1 \rangle = -\frac{1}{\sqrt{2}}, \\ \langle \mu_2 | E_{\alpha_2} | \mu_3 \rangle &= \langle \mu_1 | E_{\alpha_3} | \mu_3 \rangle = \frac{1}{\sqrt{2}}. \end{aligned} \quad (4.34)$$

Extra $SU(2)_L$ -doublets in (4.32) can be made heavy by introducing the following brane-localized terms:

$$\begin{aligned} \mathcal{L}_{\text{brane}} = & \sum_{i=1}^3 \left[\bar{Q}_R^i(x) \left\{ c_Q^i Q_L(x, z) + c_Q^i Q_L'(x, z) \right\} \right. \\ & \left. + \bar{L}_R^i(x) \left\{ c_L^i L_L(x, z) + c_L^i L_L'(x, z) \right\} + \text{h.c.} \right] \delta^{(2)}(z), \end{aligned} \quad (4.35)$$

where \bar{Q}_R^i and \bar{L}_R^i are brane-localized 4D fields; and $Q_L, Q_L', L_L,$ and L_L' are $SU(2)_L$ -doublet components of $\Psi_-^1, \Psi_+^2, \Psi_-^3,$ and $\Psi_+^4,$ respectively. The parameters $c_Q^i, c_Q^i, c_L^i,$ and c_L^i are dimensionless constants. Focusing on the zero-modes, (4.35) is rewritten as

$$\begin{aligned} \mathcal{L}_{\text{brane}} = & \sum_{i,j=1}^3 \left[\bar{Q}_R^i(x) \left\{ m_{Q0}^{ij} Q_L^j(x) + m_{Q0}^{ij} Q_L^j(x) \right\} \right. \\ & \left. + \bar{L}_R^i(x) \left\{ m_{L0}^{ij} L_L^j(x) + m_{L0}^{ij} L_L^j(x) \right\} + \text{h.c.} + \dots \right] \delta^{(2)}(z), \end{aligned} \quad (4.36)$$

¹³ We can always redefine the phases of the fields so that the matrix elements in (4.34) are real.

where the ellipsis denotes terms involving non-zero KK modes, and

$$\begin{aligned} m_{Q0}^{ij} &\equiv \frac{c_Q^i h_{L0}^{(-)\mu_1 1(j)}(0)}{\sqrt{2\pi} R_1}, & m_{Q0}^{ij} &\equiv \frac{c_Q^i h_{L0}^{(+)-\mu_1 2(j)}(0)}{\sqrt{2\pi} R_1}, \\ m_{L0}^{ij} &\equiv \frac{c_L^i h_{L0}^{(-)\mu_1 3(j)}(0)}{\sqrt{2\pi} R_1}, & m_{L0}^{ij} &\equiv \frac{c_L^i h_{L0}^{(+)-\mu_1 4(j)}(0)}{\sqrt{2\pi} R_1} \end{aligned} \quad (4.37)$$

are effective mass parameters. If these mass parameters are large enough, only the following linear combinations remain in the 4D effective theory:¹⁴

$$q_L^i \equiv V_Q^{i+3,j} Q_L^j + V_Q^{i+3,j+3} Q_L^{j'}, \quad l_L^i \equiv V_L^{i+3,j} L_L^j + V_L^{i+3,j+3} L_L^{j'}, \quad (4.38)$$

where $i = 1, 2, 3$, and V_Q and V_L are 6×6 unitary matrices that satisfy

$$\begin{aligned} U_Q(m_{Q0}, m'_{Q0}) V_Q^{-1} &= \begin{pmatrix} \lambda_Q^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_Q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_Q^3 & 0 & 0 & 0 \end{pmatrix}, \\ U_L(m_{L0}, m'_{L0}) V_L^{-1} &= \begin{pmatrix} \lambda_L^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_L^3 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (4.39)$$

with 3×3 unitary matrices U_Q and U_L . After the extra modes are decoupled, we obtain

$$\begin{aligned} \mathcal{L}^{(4D)} &= - \sum_{k=1}^5 \sum_{i,j=1}^3 \left(\tilde{y}_{ij}^{(k)D} \bar{q}_L^j H_k d_R^i + \tilde{y}_{ij}^{(k)U} \bar{u}_R^i H_k q_L^j \right. \\ &\quad \left. + \tilde{y}_{ij}^{(k)E} \bar{l}_L^j H_k e_R^i + \tilde{y}_{ij}^{(k)N} \bar{\nu}_R^j H_k l_L^i + \text{h.c.} \right) + \dots, \end{aligned} \quad (4.40)$$

where

$$\begin{aligned} \tilde{y}_{ij}^{(k)D} &\equiv y_{ij'}^{(k)D} \left(V_Q^{-1} \right)^{j',j+3}, & \tilde{y}_{ij}^{(k)U} &\equiv y_{ij'}^{(k)U} \left(V_Q^{-1} \right)^{j'+3,j+3}, \\ \tilde{y}_{ij}^{(k)E} &\equiv y_{ij'}^{(k)E} \left(V_L^{-1} \right)^{j',j+3}, & \tilde{y}_{ij}^{(k)N} &\equiv y_{ij'}^{(k)N} \left(V_L^{-1} \right)^{j'+3,j+3}. \end{aligned} \quad (4.41)$$

In order to avoid large flavor-changing processes, we assume that only one Higgs doublet H_{k_0} acquires a nonvanishing vacuum expectation value (VEV). Then, the fermion masses are obtained as eigenvalues of the mass matrices given by

$$M_{ij}^D = \tilde{y}_{ij}^{(k_0)D} v, \quad M_{ij}^U = \tilde{y}_{ij}^{(k_0)U} v, \quad M_{ij}^E = \tilde{y}_{ij}^{(k_0)E} v, \quad M_{ij}^N = \tilde{y}_{ij}^{(k_0)N} v, \quad (4.42)$$

where $v \equiv \langle H_{k_0} \rangle$. We can control the mass spectrum by tuning the parameters c_Q^i , c_Q^i , c_L^i , and c_L^i through the unitary matrices V_Q and V_L . For example, if we choose those parameters in a manner such that $V_Q \simeq \mathbf{1}_6$, we can realize the hierarchy $m_t \gg m_b$. In such a case, the eigenvalues of the Yukawa matrix $\tilde{y}_{ij}^{(k_0)U}$ are approximately given by those of $y_{ij}^{(k_0)U}$, whose absolute value $|\lambda_i^{(k_0)U}|$ ($i = 1, 2, 3$) are shown in Appendix C1. From (C1), we find that the top quark Yukawa coupling, which is close

¹⁴ Here we neglect the mixing effect with the KK modes, which is expected to be small. In order to take it into account, we need to solve the modified mode equations that include contributions from (4.35).

to one, can be obtained when $k_0 = 2, 5$. However, large hierarchies among the Yukawa couplings cannot be realized.

Besides the Yukawa hierarchy, the existence of the five Higgs doublets may be problematic because it seems difficult to hide so many extra Higgs bosons from the collider experiment. Therefore, in the next subsection we focus on the case that only one Higgs doublet appears.

4.2.4. One-Higgs-doublet case

Here we evaluate the magnitude of the Yukawa coupling constants in the case where only one Higgs doublet appears. This occurs when $(\kappa, n_p) = (1, 2), (2, 0)$. As an example, we focus on the case $(\kappa, n_p) = (2, 0)$. The Yukawa couplings are more restricted in the other case. From (4.26), possible values of κ' are -1 and 0 . In these cases, each component of (4.23) has at most one zero-mode. Hence we will omit the ‘‘flavor indices’’ i and j in the following. The Yukawa coupling constants are expressed as follows:

(i) $\kappa' = 0$ case:

$$\begin{aligned} y^D &= Y^{(-)}\left(n_1, -\frac{2l}{3}, \frac{l}{3}\right), & y^U &= Y^{(+)}\left(n_2, \frac{4l+l'}{6}, -\frac{2l-l'}{6}\right), \\ y^E &= Y^{(-)}\left(n_3, -\frac{2l+l'}{3}, \frac{l-l'}{3}\right), & y^N &= Y^{(+)}\left(n_4, \frac{4l-l'}{6}, -\frac{2l+l'}{6}\right), \end{aligned} \quad (4.43)$$

where l is an integer, l' is an even number, and

$$\begin{aligned} Y^{(+)}(n, \phi_1, \phi_2) &\equiv \frac{ig}{\sqrt{2} \cdot 3^{\frac{1}{4}}} \sum_{i'=1}^4 \sum_{j'=1}^2 \sum_{k'=1}^6 V_{1i'}^{(\omega^n)*} [4, \phi_1] V_{1j'}^{(\omega^{n+1})} [-2, \phi_2] V_{1k'}^{(\omega^{-1})} [6, \phi_1 - \phi_2] \\ &\quad \times \sum_{m=1}^6 \mathcal{F}^{(-2i'-4j'-8m)} \left(0, -48, \frac{(\phi_1 + 2\phi_2)(\tau - 1)}{12}\right) \delta_{i'-j'+4m, k'}, \\ Y^{(-)}(n, \phi_1, \phi_2) &\equiv \frac{ig}{\sqrt{2} \cdot 3^{\frac{1}{4}}} \sum_{i'=1}^4 \sum_{j'=1}^2 \sum_{k'=1}^6 V_{1i'}^{(\omega^{n+1})} [-4, \phi_1] V_{1j'}^{(\omega^n)*} [2, \phi_2] V_{1k'}^{(\omega^{-1})} [6, \phi_2 - \phi_1] \\ &\quad \times \sum_{m=1}^6 \mathcal{F}^{(-2i'-4j'-8m)} \left(0, -48, \frac{(\phi_1 + 2\phi_2)(\tau - 1)}{12}\right) \delta_{j'-i'+2m, k'}, \end{aligned} \quad (4.44)$$

where ϕ_a ($a = 1, 2$) are defined by $\zeta_a = \frac{2\phi_a}{K_a}(\tau - 1)$, and here we choose them as the second argument of $V_{ij}^{(n)}$ instead of ζ_a . The possible values of n, ϕ_1 , and ϕ_2 in (4.44) are

$$\begin{aligned} n &= 0, 1, 2 \pmod{3}, \\ \phi_1 &= \phi_2 - \text{floor}(\phi_2) + u \pmod{4}, \\ \phi_2 &= 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3} \pmod{2}, \end{aligned} \quad (4.45)$$

where $u = 0, 1, 2, 3$. Numerical values of $|Y^{(\pm)}|$ are listed in Table C1 of Appendix C. From the table, we can see that possible values of the Yukawa coupling constants are

$$\left|y^{D,U,E,N}\right| = 0.191, 0.270, 0.369, 0.522, 0.573, 0.811. \quad (4.46)$$

(ii) $\kappa' = -1$ case:

$$\begin{aligned}
y^D &= Y^{(-)} \left(n_1, -\frac{2l}{3}, \frac{l}{3} \right), \\
y^U &= \frac{ig}{\sqrt{2} \cdot 3^{\frac{1}{4}}} \sum_{i'=1}^3 \sum_{j'=1}^3 \sum_{k'=1}^6 V_{1i'}^{(\omega^{n_2})*} \left[3, \frac{4l+l'}{6} \right] V_{1j'}^{(\omega^{n_2+1})} \left[-3, -\frac{2l-l'}{6} \right] V_{1k'}^{(\omega^{-1})} [6, l] \\
&\quad \times \sum_{m=1}^6 \mathcal{F}^{(-3i'-3j'-9m)} \left(0, -54, -\frac{2(l+l')(\tau-1)}{54} \right) \delta_{i'-j'+3m, k'}, \\
y^E &= Y^{(+)} \left(n_3, \frac{l-l'}{3}, -\frac{2l+l'}{3} \right), \\
y^N &= \frac{ig}{\sqrt{2} \cdot 3^{\frac{1}{4}}} \sum_{i'=1}^5 \sum_{k'=1}^6 V_{1i'}^{(\omega^{n_4})*} \left[5, \frac{4l-l'}{6} \right] V_{11}^{(\omega^{n_4+1})} \left[-1, -\frac{2l+l'}{6} \right] V_{1k'}^{(\omega^{-1})} [6, l] \\
&\quad \times \sum_{m=1}^6 \mathcal{F}^{(-i'-5-5m)} \left(0, -30, -\frac{(l-l')(\tau-1)}{45} \right) \delta_{i'-1+5m, k'}. \tag{4.47}
\end{aligned}$$

Numerical values of these are summarized in Tables C2 and C3 of Appendix C. From the tables, we can see that the Yukawa coupling constants take the following values:

$$\begin{aligned}
|y^{D,E}| &= 0.191, 0.270, 0.369, 0.522, 0.573, 0.811; \\
|y^U| &= 0.365, 0.430, 0.461, 0.667, 0.798; \\
|y^N| &= 0.101, 0.176, 0.188, 0.288, 0.533, 0.541, 0.559, 0.924. \tag{4.48}
\end{aligned}$$

In each case of Sect. 4.2.3 and Sect. 4.2.4, the eigenvalues of the Yukawa matrices are within the region $[0.1, 1]$, and we cannot realize small Yukawa couplings only by means of the magnetic fluxes and the Wilson-line phases. We need an additional mechanism to obtain them. This is mainly due to the matrices $V_{ij}^{(\eta)}$ in (4.10) and (4.13). In order to see this, let us define the quantity

$$\tilde{y}_{ij}^{(k)}(\kappa) \equiv \frac{ig}{3^{\frac{1}{4}}\sqrt{\kappa}} \sum_{m=1}^{3\kappa} \mathcal{F}^{(-\kappa i - 2\kappa j - 2\kappa^2 m)} \left(0, -6\kappa^3, 0 \right) \delta_{i-j+2\kappa m, k}, \tag{4.49}$$

which is obtained from (4.10) in the case of our model by taking $\kappa' = l' = 0$ and replacing the $V_{ij}^{(\eta)}$ matrices with δ_{ij} . The indices i and j are assumed to run from 1 to κ . Then, we can see that the eigenvalues of (4.49), $\tilde{\lambda}_i^{(k)}(\kappa)$ ($i = 1, \dots, \kappa$), can take small values. For example, $|\lambda_i^{(k)}(2)|$ take values in the range $[6.04 \times 10^{-4}, 0.843]$, and $|\lambda_i^{(k)}(4)|$ are in $[2.05 \times 10^{-7}, 1.12]$.

We should also note that the top quark Yukawa coupling, which is close to 1, can be reproduced in our model, which only has the small representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. This is in contrast to a model without the magnetic fluxes. In the absence of the magnetic fluxes, the zero-mode wave functions are constants unless the brane-localized terms exist. In such a case, the Yukawa couplings are equal to $1/\sqrt{2}$. Thus, we need an enhancement factor, which is roughly $\sqrt{2}$, in order to obtain the top quark mass. This can be accomplished by embedding the quark fields in a larger representation of $SU(3)$. In the presence of the magnetic fluxes, on the other hand, such an enhancement factor is obtained as an overlap integral of the mode functions that have nontrivial profiles.

5. Summary

We have studied the Yukawa couplings in 6D gauge–Higgs unification models compactified on an orbifold T^2/Z_N in the presence of background magnetic fluxes. The effects of the magnetic fluxes are multiplication of zero-modes for each 6D field and deformation of the constant mode functions for the zero-modes. The former opens up the interesting possibility that the generational structure of quarks and leptons is realized, and the latter is essential to controlling the magnitude of the Yukawa coupling constants.

We considered a $G \times U(1)_X$ gauge theory, where G is a simple group, and introduced the magnetic fluxes for $U(1)_X$ and the Cartan part of G . The number of zero-modes are determined by the orbifold boundary conditions, and the fluxes and the Wilson-line phases that the 6D field actually feels. It should be emphasized that all these quantities are quantized. Thus, the Yukawa sector is controlled by a finite number of integers. As a specific model, we consider an $SU(3) \times U(1)_X$ gauge theory on T^2/Z_3 with four 6D Weyl fermions belonging to $\mathbf{3}$ or $\bar{\mathbf{3}}$. We evaluated the Yukawa coupling constants in cases where three generations are realized, and where only one Higgs doublet appears in the 4D effective theory. The Yukawa sector of our model is specified by nine integers. Due to this property and the symmetric structure of the Yukawa coupling formula, the coupling constants can take only limited numbers of values. They are all within the region $[0.1, 1]$. This stems from the fact that the mode functions on T^2/Z_3 are given by mixtures of those on T^2 . This mixing effect makes the profiles of the mode functions complicated. Thus it is difficult to realize the observed large hierarchy among the fermion masses only by means of the magnetic fluxes and the Wilson-line phases. We need an additional mechanism to obtain it. The situation is similar in models on T^2/Z_4 or T^2/Z_6 . In the case of T^2/Z_2 , the mixing matrices $V_{ij}^{(\eta)}$ in (4.10) and (4.13) become diagonal, and thus small Yukawa couplings can easily be obtained [28]. We should also note that there is an advantageous feature of a model with magnetic fluxes. We can realize the top quark Yukawa coupling without introducing a large representation of G , thanks to the nontrivial profiles of the zero-mode wave functions.

In this work, we neglected the mixing with the KK modes induced by the brane-localized terms and the Higgs VEVs. Such effects are important in evaluating the deviation of each coupling constant from the standard model value. They can be taken into account by solving the mode equations in the presence of the brane-localized terms and the W_z^α background. This will be discussed in a subsequent paper.

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Appendix A. Analytic forms of $D_{jk}^{(\omega')}$ in (3.32)

Here we collect the analytic forms of $D_{jk}^{(\omega')}$ in (3.32) obtained in Ref. [33]. In the following formulae, we choose a gauge in which the Wilson-line phases are zero. The correspondence to the

Wilson-line phases in the text can be read off from (3.7) or (3.8). Here, K , ϕ_1 , and ϕ_τ collectively denote $\{k_\alpha, k_{\mu f}\}$, $\{\phi_1^\alpha, \phi_1^{\mu f}\}$, and $\{\phi_\tau^\alpha, \phi_\tau^{\mu f}\}$, respectively. The SS phases can only take discrete values on T^2/Z_N from the consistency conditions [32]. This is equivalent to only discrete values of the Wilson-line phases being allowed [34–36].

Note that $D_{jk}^{(1)} = \delta_{jk}$ by definition. The other coefficients $D_{jk}^{(\omega^l)}$ ($l \neq 0$) are shown in the following.

T^2/Z_2 :

The allowed values of the SS phases are

$$(\phi_1, \phi_\tau) = (0, 0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right). \quad (\text{A1})$$

The explicit form of $D_{jk}^{(-1)}$ is

$$\begin{aligned} D_{jk}^{(-1)} &= \exp\left\{-\frac{4\pi i}{K}\phi_\tau(\phi_1 + j)\right\} \delta_{-2\phi_1 - j, k} \\ &= \exp\left\{\frac{4\pi i}{K}\phi_\tau(\phi_1 + k)\right\} \delta_{-2\phi_1 - k, j} = \left\{D^{(-1)\dagger}\right\}_{jk}. \end{aligned} \quad (\text{A2})$$

T^2/Z_3 :

The allowed values of the SS phases are

$$\phi \equiv \phi_1 = \phi_\tau = \begin{cases} 0, \frac{1}{3}, \frac{2}{3} & (K : \text{even}), \\ \frac{1}{6}, \frac{1}{2}, \frac{5}{6} & (K : \text{odd}). \end{cases} \quad (\text{A3})$$

The explicit forms of $D_{jk}^{(\omega^l)}$ are

$$\begin{aligned} D_{jk}^{(\omega)} &= \frac{e^{-\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp\left\{\frac{\pi i}{K}(3\phi^2 + k(k + 6\phi) + 2jk)\right\}, \\ D_{jk}^{(\omega^2)} &= \frac{e^{\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp\left\{-\frac{\pi i}{K}(3\phi^2 + j(j + 6\phi) + 2jk)\right\} = \left\{D^{(\omega)\dagger}\right\}_{jk}. \end{aligned} \quad (\text{A4})$$

T^2/Z_4 :

The allowed values of the SS phases are

$$\phi \equiv \phi_1 = \phi_\tau = 0, \frac{1}{2}. \quad (\text{A5})$$

The explicit forms of $D_{jk}^{(\omega^l)}$ are

$$\begin{aligned} D_{jk}^{(\omega)} &= \frac{1}{\sqrt{|K|}} \exp\left\{\frac{2\pi i}{K}(\phi^2 + 2\phi k + jk)\right\}, \\ D_{jk}^{(\omega^2)} &= \exp\left\{-\frac{4\pi i}{K}\phi(\phi + j)\right\} \delta_{-2\phi - j, k} = \left\{D^{(\omega^2)\dagger}\right\}_{jk}, \\ D_{jk}^{(\omega^3)} &= \frac{1}{\sqrt{|K|}} \exp\left\{-\frac{2\pi i}{K}(\phi^2 + \phi j + jk)\right\} = \left\{D^{(\omega)\dagger}\right\}_{jk}. \end{aligned} \quad (\text{A6})$$

T^2/Z_6 :

The allowed values of the SS phases are

$$\phi \equiv \phi_1 = \phi_\tau = \begin{cases} 0 & (K : \text{even}), \\ \frac{1}{2} & (K : \text{odd}). \end{cases} \quad (\text{A7})$$

The explicit forms of $D_{jk}^{(\omega^l)}$ are

$$\begin{aligned} D_{jk}^{(\omega)} &= \frac{e^{\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp \left\{ \frac{\pi i}{K} (\phi^2 - k(k - 2\phi) + 2jk) \right\}, \\ D_{jk}^{(\omega^2)} &= \frac{e^{-\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp \left\{ \frac{\pi i}{K} (3\phi^2 + j(j + 2\phi) + 2k(j + 2\phi)) \right\}, \\ D_{jk}^{(\omega^3)} &= \exp \left\{ -\frac{4\pi i}{K} \phi(\phi + j) \right\} \delta_{-2\phi-j,k} = \left\{ D^{(\omega^3)\dagger} \right\}_{jk}, \\ D_{jk}^{(\omega^4)} &= \frac{e^{\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp \left\{ -\frac{\pi i}{K} (3\phi^2 + k(k + 2\phi) + 2j(k + 2\phi)) \right\} = \left\{ D^{(\omega^2)\dagger} \right\}_{jk}, \\ D_{jk}^{(\omega^5)} &= \frac{e^{-\text{sgn}(K)\frac{\pi i}{12}}}{\sqrt{|K|}} \exp \left\{ -\frac{\pi i}{K} (\phi^2 - j(j - 2\phi) + 2jk) \right\} = \left\{ D^{(\omega)\dagger} \right\}_{jk}. \end{aligned} \quad (\text{A8})$$

The sign function $\text{sgn}(K)$ comes from the formula

$$\sum_{s=0}^{|K|-1} \exp \left\{ \frac{\pi i}{K} (s + \beta)^2 \right\} = \sqrt{|K|} e^{\text{sgn}(K)\frac{\pi i}{4}}, \quad (\text{A9})$$

where β is an integer (half-integer) when K is even (odd).

Appendix B. Normalizations of KK modes

In this appendix, we identify the coefficients in (3.1). Here we focus on those for W_μ^α and W_z^α . The other normalization factors are obtained similarly. The 6D Lagrangian (2.1) includes the following terms:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4g_A^2} \text{Tr} \left\{ F^{\mu\nu} F_{\mu\nu} + \frac{2}{(\pi R_1)^2} F^{\mu\bar{z}} F_{\mu z} \right\} + \dots \\ &= -\frac{1}{4g_A^2} \sum_\alpha \left\{ (W^{\alpha\mu\nu})^* W_{\mu\nu}^\alpha + \frac{2}{(\pi R_1)^2} (W^{\alpha\mu z})^* W_{\mu z}^\alpha \right\} + \dots, \end{aligned} \quad (\text{B1})$$

where

$$\begin{aligned} W_{\mu M}^\alpha &\equiv \partial_\mu W_M^\alpha - \partial_M W_\mu^\alpha - i \left\{ \sum_i \alpha^i (C_\mu^i W_M^\alpha - W_\mu^\alpha C_M^i) + \sum_\beta N_{\beta,\alpha-\beta} W_\mu^\beta W_M^{\alpha-\beta} \right\}, \\ N_{\beta,\gamma} &\equiv \langle \beta + \gamma | E_\beta | \gamma \rangle. \end{aligned} \quad (\text{B2})$$

The KK expansion is expressed as

$$\begin{aligned} W_\mu^\alpha(x, z) &= \mathcal{N}_W \sum_n f_n^\alpha(z) W_\mu^{\alpha(n)}(x), \\ W_z^\alpha(x, z) &= \mathcal{N}_\varphi \sum_n g_n^\alpha(z) \varphi_n^\alpha(x), \end{aligned} \quad (\text{B3})$$

where \mathcal{N}_W and \mathcal{N}_φ are positive constants, and the mode functions are normalized as

$$\int_{T^2/Z_N} d^2z \{f_n^\alpha(z)\}^* f_m^\alpha(z) = \int_{T^2/Z_N} d^2z \{g_n^\alpha(z)\}^* g_m^\alpha(z) = \delta_{nm}. \quad (\text{B4})$$

Substituting (B3) into (B1), we obtain the 4D effective Lagrangian:

$$\begin{aligned} \mathcal{L}^{(4D)} &= \int_{T^2/Z_N} dx^4 dx^5 \mathcal{L} = 2(\pi R_1)^2 \int_{T^2/Z_N} d^2z \mathcal{L} \\ &= -\frac{2(\pi R_1)^2}{4g_A^2} \int_{T^2/Z_N} \left\{ 2|W_{\mu\nu}^{\alpha_1}|^2 + \frac{2}{(\pi R_1)^2} \left(|\partial_\mu W_z^{\alpha_2} - iN_{-\alpha_1, \alpha_3} W_\mu^{-\alpha_1} W_z^{\alpha_3} + \dots|^2 \right. \right. \\ &\quad \left. \left. + |\partial_\mu W_z^{\alpha_3} - iN_{\alpha_1, \alpha_2} W_\mu^{\alpha_1} W_z^{\alpha_2} + \dots|^2 \right) \right\} + \dots \\ &= -\frac{\mathcal{N}_W^2 (\pi R_1)^2}{g_A^2} \left| \partial_\mu W_\nu^{\alpha_1(0)} - \partial_\nu W_\mu^{\alpha_1(0)} \right|^2 - \frac{\mathcal{N}_\varphi^2}{g_A^2} \left(|\mathcal{D}_\mu \varphi_0^{\alpha_2}|^2 + |\mathcal{D}_\mu \varphi_0^{\alpha_3}|^2 \right) + \dots, \quad (\text{B5}) \end{aligned}$$

where α_1 and $\{\alpha_2, \alpha_3\}$ are the roots such that $W_\mu^{\pm\alpha_1}$ and $W_z^{\alpha_{2,3}}$ have zero-modes that are identified with the W boson and the Higgs doublet fields respectively, and

$$\begin{aligned} \mathcal{D}_\mu \varphi_0^{\alpha_2} &\equiv \partial_\mu \varphi_0^{\alpha_2} - iN_{-\alpha_1, \alpha_3} \mathcal{N}_W f_0^{-\alpha_1}(z) W_\mu^{-\alpha_1(0)} \varphi_0^{\alpha_3}, \\ \mathcal{D}_\mu \varphi_0^{\alpha_3} &\equiv \partial_\mu \varphi_0^{\alpha_3} - iN_{\alpha_1, \alpha_2} \mathcal{N}_W f_0^{\alpha_1}(z) W_\mu^{\alpha_1(0)} \varphi_0^{\alpha_2}. \quad (\text{B6}) \end{aligned}$$

We have used that

$$f_0^{\pm\alpha_1}(z) = \sqrt{\frac{N}{2\text{Im}\tau}}. \quad (\text{B7})$$

Comparing (B5) with the standard model,

$$\begin{aligned} \mathcal{L}_{\text{SM}} &= -\frac{1}{2} \text{Tr} \left\{ \left(\sum_a F_{\mu\nu}^a \frac{\sigma^a}{2} \right)^2 \right\} - \left| \left(\partial_\mu - ig A_\mu^a \frac{\sigma^a}{2} \right) \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \right|^2 + \dots \\ &= -\frac{1}{4} \left(|F_{\mu\nu}^1|^2 + |F_{\mu\nu}^2|^2 + \dots \right) - \left| \partial_\mu \varphi^0 - ig \left(A_\mu^1 + iA_\mu^2 \right) \varphi^+ \right|^2 \\ &\quad - \left| \partial_\mu \varphi^+ - ig \left(A_\mu^1 - iA_\mu^2 \right) \varphi^0 \right|^2 + \dots \\ &= -\frac{1}{2} \left| \partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \right|^2 - \left| \partial_\mu \varphi^0 - \frac{ig}{\sqrt{2}} W_\mu^- \varphi^+ \right|^2 - \left| \partial_\mu \varphi^+ - \frac{ig}{\sqrt{2}} W_\mu^+ \varphi^0 \right|^2 + \dots, \quad (\text{B8}) \end{aligned}$$

where A_μ^a ($a = 1, 2, 3$) are the $\text{SU}(2)_L$ gauge fields, $F_{\mu\nu}^a$ are their field strengths, and $W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$, the constants \mathcal{N}_W and \mathcal{N}_φ should be chosen as

$$\frac{\mathcal{N}_W^2 (\pi R_1)^2}{g_A^2} = \frac{1}{2}, \quad \frac{\mathcal{N}_\varphi^2}{g_A^2} = 1, \quad (\text{B9})$$

and the 4D gauge coupling constant \bar{g}_A is identified from (B6) as

$$\frac{\bar{g}_A}{\sqrt{2}} = N_{-\alpha_1, \alpha_3} \mathcal{N}_W \sqrt{\frac{N}{2\text{Im}\tau}} = N_{\alpha_1, \alpha_2} \mathcal{N}_W \sqrt{\frac{N}{2\text{Im}\tau}}. \quad (\text{B10})$$

Solving these, we obtain

$$\mathcal{N}_W = \frac{g_A}{\sqrt{2}\pi R_1}, \quad \mathcal{N}_\varphi = g_A, \quad \bar{g}_A = \frac{g_A}{\sqrt{A}}. \quad (\text{B11})$$

We have used that

$$N_{-\alpha_1, \alpha_3} = N_{\alpha_1, \alpha_2} = \frac{1}{\sqrt{2}} \quad (\text{B12})$$

after appropriate phase redefinitions of the fields.

Appendix C. Magnitude of Yukawa coupling constants

C.1. Three-generation case

Here we collect numerical values of the Yukawa coupling constants in (4.33). The eigenvalues of the matrices $y^{(k)F}$ ($F = D, U, E, N$) are denoted by $\lambda^{(k)F}$. Their absolute values are calculated as

$$\begin{aligned} |\lambda^{(1)D}| &= |\lambda^{(1)E}| = (0.845, 0.274, 0.057), \\ |\lambda^{(2)D}| &= |\lambda^{(2)E}| = (0.921, 0.350, 0.321), \\ |\lambda^{(3)D}| &= |\lambda^{(3)E}| = (0.821, 0.517, 0.358), \\ |\lambda^{(4)D}| &= |\lambda^{(4)E}| = (0.644, 0.524, 0.208), \\ |\lambda^{(5)D}| &= |\lambda^{(5)E}| = (0.799, 0.259, 0.155), \\ |\lambda^{(1)U}| &= |\lambda^{(1)N}| = (0.731, 0.279, 0.0644), \\ |\lambda^{(2)U}| &= |\lambda^{(2)N}| = (0.921, 0.350, 0.321), \\ |\lambda^{(3)U}| &= |\lambda^{(3)N}| = (0.665, 0.579, 0.394), \\ |\lambda^{(4)U}| &= |\lambda^{(4)N}| = (0.769, 0.415, 0.220), \\ |\lambda^{(5)U}| &= |\lambda^{(5)N}| = (0.945, 0.315, 0.108). \end{aligned} \quad (\text{C1})$$

C.2. One-Higgs-doublet case

Here we collect numerical values of the Yukawa coupling constants in Sect. 4.2.4.

C.2.1. $\kappa' = 0$ case

The possible values of n , ϕ_1 , and ϕ_2 in (4.44) are

$$\begin{aligned} n &= 0, 1, 2 \pmod{3}, \\ \phi_1 &= \phi_2 - \text{floor}(\phi_2) + u \pmod{4}, \\ \phi_2 &= 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3} \pmod{2}, \end{aligned} \quad (\text{C2})$$

where $u = 0, 1, 2, 3$. For these values, only one generation is realized for each component. The absolute values of $Y^{(+)}(n, \phi_1, \phi_2)$ are listed in Table C1. Those of $Y^{(-)}$ are related to $|Y^{(+)}|$ through

$$\left| Y^{(-)}(n, \phi_1, \phi_2) \right| = \left| Y^{(+)}(-n + 2, \phi_1, \phi_2) \right|. \quad (\text{C3})$$

C.2.2. $\kappa' = -1$ case

The absolute values of $y^{D,E}$ can be read off from Table C1. Those of $y^U(n_2, l, l')$ ($l = 0, 1$) are shown in Table C2. When $n_2 = 0$, Q'_L does not have a zero-mode. The coupling constants for the

Table C1. The absolute values of $Y^{(+)}(n, \phi_1, \phi_2)$. The “—” denote cases in which the left- or the right-handed components do not have zero-modes.

(n, u)	ϕ_2					
	0	1/3	2/3	1	4/3	5/3
(0,0)	0.573	—	—	0.191	—	—
(0,1)	0.369	—	—	0.369	—	—
(0,2)	0.191	—	—	0.573	—	—
(0,3)	0.369	—	—	0.369	—	—
(1,0)	—	0.522	0.270	—	0.522	0.811
(1,1)	—	0.270	0.522	—	0.811	0.522
(1,2)	—	0.522	0.811	—	0.522	0.270
(1,3)	—	0.811	0.522	—	0.270	0.522
(2,0)	0.811	0.369	0.191	0.270	0.369	0.573
(2,1)	0.522	0.191	0.369	0.522	0.573	0.369
(2,2)	0.270	0.369	0.573	0.811	0.369	0.191
(2,3)	0.522	0.573	0.369	0.522	0.191	0.369

Table C2. The absolute values of $y^U(n_2, l, l')$. The “—” denote cases in which the left- or the right-handed components do not have zero-modes.

(n_2, l)	l'							
	1	3	5	7	9	11	13	15
(1,0)	—	0.365	—	—	0.667	—	—	0.461
(1,1)	—	—	0.430	—	—	0.798	—	—
(2,0)	0.667	—	0.461	0.365	—	0.667	0.365	—
(2,1)	0.798	0.430	—	0.430	0.430	—	0.430	0.430
(n_2, l)	l'							
	17	19	21	23	25	27	29	
(1,0)	—	—	0.667	—	—	0.365	—	
(1,1)	0.430	—	—	0.430	—	—	0.430	
(2,0)	0.365	0.667	—	0.365	0.461	—	0.667	
(2,1)	—	0.430	0.798	—	0.430	0.430	—	

other values of l are related to those in Table C2 by

$$\begin{aligned}
 |y^U(n_2, 2u, l')| &= |y^U(n_2, 0, l' + 8u)|, \\
 |y^U(n_2, 2u + 1, l')| &= |y^U(n_2, 1, l' + 8u)|,
 \end{aligned}
 \tag{C4}$$

where $u = 0, 1, 2, \dots, 14$.¹⁵

¹⁵ Note that l and l' are defined modulo 30.

Table C3. The absolute values of $y^N(n_4, l, l')$. The “—” denote cases in which the left- or the right-handed components do not have zero-modes.

(n_4, l)	l'								
	1	3	5	7	9	11	13	15	17
(0,0)	0.559	—	0.101	0.176	—	0.176	0.101	—	0.559
(0,1)	—	0.541	0.188	—	0.288	0.288	—	0.188	0.541
(1,0)	0.559	0.533	0.101	0.176	0.533	0.176	0.101	0.533	0.559
(1,1)	0.924	0.541	0.188	0.924	0.288	0.288	0.924	0.188	0.541
(2,0)	0.559	—	0.101	0.176	—	0.176	0.101	—	0.559
(2,1)	—	0.541	0.188	—	0.288	0.288	—	0.188	0.541

The absolute values of $y^N(n_4, l, l')$ ($l = 0, 1$) are shown in Table C3. The coupling constant for the other values of l are related to those in Table C3 by

$$\begin{aligned} |y^N(n_4, 2u, l')| &= |y^N(n_4, 0, l' - 2u)|, \\ |y^N(n_4, 2u + 1, l')| &= |y^N(n_4, 1, l' - 2u)|, \end{aligned} \quad (\text{C5})$$

where $u = 0, 1, 2, \dots, 8$.¹⁶

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