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Symmetries and dynamics for non-AdS backgrounds in three-dimensional gravity

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Acknowledgments

Mes parents.

Alors que certains cherchent le Graal, la composition de la Pierre Philosophale, les pépites d'or en Eldorado,

D'autres l'âme sœur, la Vérité, leur identité, le Temps Perdu, à plaire,

D'autres encore midi à quatorze heures, une aiguille dans une botte de foin, la petite bête, des poux sur la tête de leur petite sœur,

Toi, tu cherches... mais au fond que cherches-tu? Et puis c'est quoi chercher?

Chercher, est-ce-que c'est creuser de plus en plus profond, dans des terriers de plus en plus étroits, jusqu'à les trouver tellement minces qu'il est impossible d'y pénétrer?

Chercher, est-ce se mettre en mouvement, se montrer dynamique, agressif, se déplacer de plus en plus vite, pour arriver avant les autres, et dévoiler la trouvaille en franchissant tel un vainqueur la ligne d'arrivée?

Est-ce se faire si léger et s'élever tellement haut que les mystères semblent de loin soudain compréhensibles?

Chercher c'est en tous les cas rassembler, regrouper, structurer, faire des liens, mettre en évidence.

C'est aussi oser la confrontation, l'idée qui fâche, l'hypothèse scandaleuse, qui peut-être fera de vous un paria, un reclus, un exilé, un hérétique.

Chercher c'est penser et s'amuser, sentir vibrer un cerveau humain construit sur le modèle de l'univers : il ne s'arrête jamais, n'est jamais fatigué, même quand il semble dormir, ondes lentes, ondes rapides, sommeil profond et paradoxal.

Et finalement chercher c'est surtout ne jamais avoir peur, peur de ce que l'on pourrait trouver, ou peur de ne rien trouver.

"Résoudre le conflit cognitif en contredisant les apparences trompeuses et maîtriser les probabilités, équivaut fantasmatiquement à retrouver le paradis perdu de l'évidence, nostalgie du temps mythique où les limites de la condition humaine étaient inconnues.

C'est le fantasme de la possibilité de reconstruire le monde d'avant l'écroulement des certitudes qui soutient l'activité de pensée de l'enfant, et celle du chercheur.[1]*"*

M.P. Vanesse

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[1]Lu in *Bulletin de psychologie,* Clinique du fonctionnement mental des enfants à haut potentiel.

Contents

CHAPTER 1

Introduction

The puzzle of black hole entropy

1.3 billion years ago, two black holes of about 30 times the mass of the Sun, after orbiting around each other, collided at nearly one-half the speed of light and merged into a single black hole. In a fraction of second, about three times the mass of the Sun was converted in gravitational waves, with a peak of power of about 50 times greater than the combined power of all light radiated by all the stars in the observable universe. This cataclysmic event was recorded by detectors of the LIGO and Virgo scientific collaboration in September 2015, leading to the first direct observation of gravitational waves [1]. This discovery also demonstrated the existence of binary stellar black hole systems, namely pairs of black holes formed by the gravitational collapse of massive stars. There exists strong evidence that much heavier, supermassive black holes can form as well and exist at the center of many galaxies. At the core of our own Milky Way lies a supermassive black hole of about 4.3 million solar masses [2]. It is fair to say that black holes are the most intriguing astrophysical objects in the universe and that they challenge our intuition as no other phenomenon in nature, compelling us to reconsider the fundamental laws of physics.

Black holes are known as solutions of Einstein's fields equations since the very advent of General Relativity, and their first observational evidence arrived in the sixties. It was later shown in the early seventies by Bardeen, Bekenstein, Carter and Hawking that black holes are thermal objects, and that the laws of black hole mechanics share more than a mere similarity with the laws of thermodynamics [3, 4, 5]: A black hole radiates at a small temperature[2]

$$T = \frac{\hbar \kappa}{2\pi},\tag{1.1}$$

proportional to the surface gravity κ of the horizon, and possesses a large entropy given by

$$S = \frac{A}{4\hbar G},\tag{1.2}$$

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[2]The speed of light c and the Boltzmann constant k_B are set to 1.

proportional to the area A of the horizon, G being the Newton constant. One expects that, as it is usual for a thermodynamical system, the entropy of a black hole counts microscopic degrees of freedom. However, despite all efforts that have been made in this direction in the last decades, the very nature of these degrees of freedom still remains a mystery and is one of the main challenges that a theory of quantum gravity should address.

Some progress has been made in supplying a microscopic derivation of (1.2) in Ref. [6] for certain black holes in string theory, the latter being at this day the most promising candidate for a theory of quantum gravity. However, the black holes studied by Strominger and Vafa are five-dimensional black holes that enjoy supersymmetry, and their methods strongly rely on this property, while one wants instead to describe more realistic, non supersymmetric black holes. Astrophysical black holes are generally rotating and have almost zero electromagnetic charge. Therefore, the ultimate goal would be a microscopic derivation of the thermodynamics of the four-dimensional Kerr or Schwarzschild black hole.

Holographic dualities

Recently, the holographic principle has provided us with a powerful tool to address questions of high energy theoretical physics that remained outside the scope of our rudimentary understanding before its formulation in the mid 1990s [7, 8], due to their non-perturbative and intrinsic gravitational nature. The most concrete realization of the holographic principle is the AdS/CFT correspondence [9, 10, 11], which establishes a connection between quantum gravity in D-dimensional asymptotically Anti-de Sitter spaces (AdS) and conformally invariant quantum field theories formulated at the $(D-1)$-dimensional boundary of AdS. This is a weak-strong duality, meaning that when the latter of these theories is weakly coupled, the former is in its strong coupling regime, and vice versa. The first example was originally derived in string theory, where $AdS_5 \times S^5$ spaces were shown to be dual to the maximally supersymmetric $SU(N)$ Yang-Mills theory in four dimensions. Since then, the idea has been generalized to many different set-ups and dimensions, and this gauge/gravity duality is believed to hold in a broad context. The advent of AdS/CFT has risen the hope to explicitly work out the details of non-perturbative effects in gravity, such as black hole thermodynamics, in terms of a dual conformal field theory (CFT) description in one dimension less. In fact, the power of holographic dualities goes beyond that context, since nowadays it is understood that holography has interesting applications in condensed matter, nuclear, or atomic physics as well.

As a matter of fact, in three dimensions, the relation AdS_3/CFT_2 was discovered a long time before the holographic correspondence was formulated: In the work [12], which, according to Witten [13], can be considered as the precursor of AdS/CFT correspondence, Brown and Henneaux showed that the symmetry algebra of asymptotically AdS_3 spaces is generated by two copies of Virasoro algebra with non-vanishing central charge, namely the algebra of local conformal transformations in two dimensions. The appearance of this remarkably rich, infinite-dimensional algebra living at the boundary of a theory presumed

sterile had a great impact on our understanding of black hole thermodynamics. In particular, Strominger realized that the value of the central charge was exactly accounting for the macroscopic entropy of the three-dimensional black hole by means of a computation in the CFT side [14] (see also [15, 16] for earlier works). This example clearly shows that string theory is not the only route to explore holographic dualities: the general study of asymptotic symmetries at the classical level has proved to be an excellent tool, and this thesis will be constantly aimed at showing the utility and power of this latter approach. In fact, what the asymptotic analysis of Brown and Henneaux showed is that, unlike what could have been thought previously, the specific details of string theory were not responsible for the matching of the results obtained in [6] with the Bekeinstein-Hawking formula (1.2). Rather, any consistent quantum theory of gravity containing black holes that have in their near-horizon limit an AdS$_3$ factor must reproduce the matching. In order words, the details of the ultraviolet completions of quantum gravity are not strictly necessary for this purpose.

One of the lessons we have learned in the last years is that the infinite-dimensional nature of asymptotic algebras is not only a curiosity of AdS$_3$ gravity models, but rather a recurrent aspect of holographic scenarios in diverse number of dimensions and in diverse spacetimes. A first example of this is given by the Kerr/CFT correspondence [17], i.e. the proposal to extend AdS/CFT correspondence to the near-horizon region of rapidly rotating (extremal) four-dimensional Kerr black holes, close models of observed astrophysical black holes. Indeed, it has been argued that the thermodynamics and other physical phenomena occurring close to such black holes is also governed by an infinite-dimensional algebra that includes a Virasoro algebra. These results strongly suggested that the near-horizon quantum states can be identified with those of a chiral half of a two-dimensional CFT. The second example, which will be closely related, as we will see, to this thesis, concerns the case of four-dimensional asymptotically flat spacetimes. As shown in the seminal work of Bondi, van der Burg, Metzner and Sachs in the early sixties, the algebra of asymptotically flat spacetimes at null infinity turns out to be extremely rich since it enhances the usual Poincaré algebra to an infinite-dimensional one, the so-called 𝔟𝔪𝔰 algebra [18, 19, 20]. The latter consists of a semi-direct sum between the Lorentz transformations and the infinite-dimensional group of supertranslations, and has recently attracted considerable attention from many new perspectives [21, 22, 23, 24, 25, 26, 27].

Non-AdS spaces and holography

These examples show that holographic tools seem to be available beyond the standard case of Anti-de Sitter spaces, and indeed, since the formulation of AdS/CFT, but especially in the last decade, there have been many attempts to extend the AdS/CFT holographic correspondence to more generic, non-AdS backgrounds. On top of the Kerr/CFT and the BMS/CFT correspondence already mentioned, the most prominent examples of this are the dS/CFT correspondence [28], the WAdS/CFT correspondence [29], and extensions of the correspondence to non-relativistic systems [30, 31, 32]. The dS/CFT correspondence proposes to extend the holographic duality to the case of a positive cosmological constant,

namely to de Sitter (dS) spacetimes. In this context, asymptotic symmetry techniques turn out to be very useful since all attempts to embed de Sitter space as a solution of string theory have failed so far, preventing therefore the hope to use string dualities in this context. The WAdS/CFT consists of another nice set-up to extend holographic tools to a deformation of AdS backgrounds. Warped AdS spaces are squashed or stretched deformations of AdS [33] and, among others, have the very interesting feature that they admit black holes [34], permitting to explore black hole thermodynamics from the holographic point of view in a setup that goes beyond the asymptotically AdS examples.

Applying asymptotic symmetry techniques to investigate extensions of AdS/CFT correspondence to gravitational scenarios that involve non-AdS spaces will be the main goal of this thesis. The backgrounds that we will consider will include the case of asymptotically flat spacetimes (in supergravity), de Sitter spaces (in Einstein gravity), and Warped spaces (in massive gravity). In order to achieve this goal, we will be focused on the study of *three-dimensional* spaces (apart from an incursion in four spacetime dimensions). It is indeed well-known that three-dimensional gravity is interesting in its own right as it provides us with an interesting toy model to investigate diverse aspects of gravity [35, 36, 37] which, otherwise, would lie beyond our current understanding. While its dynamics is substantially simpler than the one of its four-dimensional analog, three-dimensional Einstein gravity or its massive deformations still exhibits several phenomena that are present in higher dimensions and are still poorly understood, such as black hole thermodynamics: Remarkably, Einstein gravity in $2 + 1$ spacetime dimensions admits black hole solutions [38, 39] whose properties resemble very much those of the four-dimensional black holes, as for instance the fact of having an entropy obeying the Bekenstein-Hawking area law. Although a fully satisfactory quantum version of three-dimensional general relativity has not yet been accomplished [40], promising results have been obtained, specially in the context of black hole physics [14], as we have already pointed out. Recall that the relation $\text{AdS}_3/\text{CFT}_2$ was discovered a long time before the holographic correspondence was formulated [12]. A key feature of three-dimensional gravity, which may *a priori* make look the theory trivial, is the fact of being purely topological; it does not contain any local degrees of freedom. Instead, it should be rather seen as a strength, since it allows its reformulation as a Chern-Simons gauge theory [41, 42]. The latter simplifies substantially both the structure of the action and equations of motion. In fact, having at hand the Chern-Simons formulation, one can perform a so-called Hamiltonian reduction, which permits to go further than the asymptotic symmetry analysis by explicitly constructing the classical action of the two-dimensional dual CFT. In particular, this powerful approach was used in [43] to show that the asymptotic dynamics of Einstein gravity around AdS_3 space is governed by the Liouville action, a non-trivial two-dimensional conformal field theory whose central charge coincides with the one found in [12].

Outline of this thesis

This thesis is organized as follows: As an invitation, we recall in chapter 2 the main features of three-dimensional gravity in AdS spaces. After reviewing the action of gravity in $2 + 1$ spacetime dimensions and the absence of local degrees of freedom, we will

recall the famous features of the three-dimensional black hole solution. We will introduce the Chern-Simons formulation of gravity by means of the vielbein and spin connection formalism, and then present the Brown-Henneaux boundary conditions and compute the associated asymptotic symmetry algebra in the Chern-Simons formalism. We will then move to the description of the asymptotic dynamics for this case of a negative cosmological constant by reviewing in details the result of Coussaert-Henneaux-van Driel showing that the asymptotic dynamics is described at the classical level by a Liouville theory. We will see how boundary conditions implement the asymptotic reduction in two steps: the first set reducing the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons action to a non-chiral $SL(2,\mathbb{R})$ Wess-Zumino-Witten model (WZW), while the second set imposes constraints on the WZW currents that reduce further the action to Liouville theory. We conclude this chapter by discussing the issues of considering the latter as an effective description of the dual conformal field theory describing AdS$_3$ gravity beyond the semi-classical regime.

In chapter 3, we will extend the analysis of asymptotic dynamics in three-dimensional gravity and supergravity to the case of asymptotically flat spacetimes; namely, the geometries that asymptote to Minkowski space at null infinity. In this case, the asymptotic symmetries are governed by the so-called Bondi-Metzner-Sachs symmetries. We will present a supersymmetric generalization of the $\mathfrak{bms}_3$ algebra. In order to do so, a consistent set of asymptotic conditions for the simplest supergravity theory without cosmological constant in three dimensions will be proposed. The canonical generators associated to the asymptotic symmetries will be shown to span a supersymmetric extension of the $\mathfrak{bms}_3$ algebra with an appropriate central charge. The energy will be seen to be manifestly bounded from below, with the ground state given by the null orbifold or Minkowski spacetime for periodic, respectively antiperiodic boundary conditions on the gravitino. These results will be related to the corresponding ones in AdS$_3$ supergravity by a suitable flat limit. The analysis will then be generalized to the case of flat supergravity with additional parity odd terms for which the Poisson algebra of canonical generators form a representation of the same algebra but with an additional central charge. Finally, the two-dimensional super-$\mathfrak{bms}_3$ invariant theory describing the boundary dynamics of the three-dimensional asymptotically flat $\mathcal{N} = 1$ supergravity will be constructed. It will be shown to be described by a constrained or gauged chiral Wess-Zumino-Witten action based on the super-Poincaré algebra in the Hamiltonian, respectively the Lagrangian formulation, whose reduced phase space description corresponds to a supersymmetric extension of flat Liouville theory.

In chapter 4, we will extend the discussion of asymptotically flat boundary conditions to another region of interest: the near-horizon region of non-extremal black holes. We will show that the asymptotic symmetries in that case are generated by an extension of the algebra of supertranslations; more precisely, a semi-direct sum of Virasoro and Abelian currents. We will discuss the differences and relations between this algebra and the $\mathfrak{bms}_3$ algebra. As a proof that three-dimensional gravity can actually serve as a toy model to learn about higher-dimensional gravity, we will explain how the results can be extended to four dimensions, also yielding infinite-dimensional symmetries in the near-horizon region of Kerr black holes. Both in three and four dimensions, when considering the special case of a stationary black hole, the zero mode charges correspond to the angular momentum

and the entropy at the black hole horizon.

In chapter 5, we will discuss the case of a cosmological horizon: We will investigate the asymptotic symmetries and asymptotic dynamics of three-dimensional gravity in de Sitter space. The dS/CFT correspondence postulates the existence of a Euclidean CFT dual to a suitable gravity theory with Dirichlet boundary conditions asymptotic to de Sitter space-time. A semi-classical model of such a correspondence consists of Einstein gravity with positive cosmological constant and without matter which is dual to Euclidean Liouville theory defined at the future conformal boundary. We will show that Euclidean Liouville theory also describes the dual dynamics of Einstein gravity with Dirichlet boundary conditions on a fixed timelike slice in the static patch. As a prerequisite of this correspondence, we will show that the asymptotic symmetry algebra which consists of two copies of the Virasoro algebra extends everywhere into the bulk.

In chapter 6, we will explore other non-AdS setups in which infinite-dimensional symmetries appear and happen to provide a description of gravitational physics and in particular of black holes. We will study the case of Warped AdS spaces (WAdS), which are stretched and squashed deformations of AdS that appear in several setups such as string theory, supergravity and massive gravity in three dimensions. We will consider the latter as our working example. For a specific choice of asymptotic boundary conditions, we will show that the algebra of charges is infinite-dimensional and coincides with the semi-direct sum of Virasoro algebra with non-vanishing central charge and an affine $\hat{u}(1)_k$ Kac-Moody algebra. We will show that the asymptotically WAdS$_3$ black hole configurations organize in terms of two commuting Virasoro algebras. We will identify the Virasoro generators that expand the associated representations in the dual conformal field theory and, by applying a Cardy formula, we prove that the microscopic CFT computation exactly reproduces the entropy of black holes in WAdS space. The relation with the so-called Warped CFT will be also discussed. Finally, we will extend the computation to a different set of asymptotic boundary conditions that, while still gathering the WAdS$_3$ black holes, also allow for new solutions that are not locally equivalent to WAdS$_3$ space, and therefore are associated to the local degrees of freedom of the theory (bulk massive gravitons). After presenting explicit examples of such geometries, we will compute the asymptotic charge algebra and show that it is also generated by the semi-direct sum of Virasoro algebra and an affine Kac-Moody algebra. Once again, the value of the central charge turns out to be exactly the one that leads to reproduce the entropy of the WAdS$_3$ black holes, probing the WAdS$_3$/CFT$_2$ correspondence in presence of bulk gravitons.

Original contributions

The original contributions of this thesis are based on the following publications, listed by order of appearance in the main text:

1. G. Barnich, L. Donnay, J. Matulich and R. Troncoso, "Asymptotic symmetries and dynamics of three-dimensional flat supergravity", *JHEP* **08** (2014) 071, `arXiv:hep-th/1407.4275`.

2. G. Barnich, L. Donnay, J. Matulich and R. Troncoso, "Super-BMS$_3$ invariant boundary theory from three-dimensional flat supergravity", `arXiv:hep-th/1510.08824`.

3. L. Donnay, G. Giribet, H. A. González and M. Pino, "Supertranslations and superrotations at the horizon", *Phys. Rev. Lett.* **116** no.9 (2016) 091101, `arXiv:1511.08687[hep-th]`.

4. G. Compère, L. Donnay, P.-H. Lambert and W. Schulgin, "Liouville theory beyond the cosmological horizon", *JHEP* **03** (2015) 158, `arXiv:hep-th/1411.7873`.

5. L. Donnay, J.J. Fernandez-Melgarejo, G. Giribet, A. Goya and E. Lavia, "Conserved charges in timelike warped AdS$_3$ spaces", *Phys. Rev.* **D 91** (2015) 125006, `arXiv:hep-th/1504.05212`.

6. L. Donnay and G. Giribet, "Holographic entropy of Warped-AdS$_3$ black holes", *JHEP* **06** (2015) 099, `arXiv:hep-th/1504.05640`.

7. L. Donnay and G. Giribet, "WAdS$_3$/CFT$_2$ correspondence in the presence of bulk massive gravitons", published in the Proceedings of 14th Marcel Grossmann Meeting, Rome, July 2015, `arXiv:hep-th/1511.02144`.

The introductory chapter 2 is based on the lecture notes of a course given at the eleventh *Modave Summer School in Mathematical Physics*:

8. L. Donnay, "Asymptotic dynamics of three-dimensional gravity", published in the Proceedings of Science for the Eleventh Modave Summer School in Mathematical Physics, Modave, September 2015, `arXiv:1602.09021 [hep-th]`.

CHAPTER 2

Asymptotic symmetries and dynamics of three-dimensional gravity

In this chapter, we start by reviewing the Einstein-Hilbert action of gravity in $2+1$ spacetime dimensions and the absence of local degrees of freedom inherent to this simplified model. We then introduce in section 2.2 the black hole solution hosted in the case of negative cosmological constant. In section 2.3, we will show how the Einstein-Hilbert action in three dimensions can be written as a Chern-Simons action for the appropriate gauge group, using the vielbein and spin connection formalism. In section 2.4, we will present the Brown-Henneaux AdS$_3$ boundary conditions and compute the associated asymptotic symmetry algebra in the Chern-Simons formalism. Section 2.5 contains a brief introduction on Wess-Zumino-Witten (WZW) models. The two steps of the reduction of the Chern-Simons action to, first, a non-chiral WZW model, and then to a Liouville action, are detailed in sections 2.6 and 2.7. After a brief introduction on Liouville theory, in section 2.8, we discuss in section 2.9 the possibility of the latter to account for the microstates of the BTZ black hole.

2.1 Gravity in $2+1$ dimensions

Pure gravity in 2+1 spacetime dimensions is defined by the three-dimensional Einstein-Hilbert action (where we set $c \equiv 1$):

$$S_{\text{EH}}[g] \equiv \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g}\,(R - 2\Lambda) + B, \tag{2.1}$$

with G the three-dimensional Newton constant, $g \equiv \det g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2$), with a metric $g_{\mu\nu}$ of signature $(-, +, +)$, $R \equiv R_{\mu\nu} g^{\mu\nu}$ is the curvature scalar, and $R_{\mu\nu}$ the Ricci tensor. $\mathcal{M}$ is a three-dimensional manifold, and Λ is the cosmological constant, which can be positive, negative, or null, yielding respectively locally de Sitter (dS), Anti-de Sitter (AdS), or flat spacetimes. Here, we will be concerned with Anti-de Sitter spacetime, for which the cosmological constant is related to the AdS radius ℓ through $\Lambda = -1/\ell^2$. Action (2.1) is defined up to a boundary term B, which is there in order to ensure that the action has a well-defined action principle.

Extremizing the action with respect to the metric $g_{\mu\nu}$ yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2.2}$$

An important property of general relativity in $2+1$ dimensions is that any solution of the vacuum Einstein equations (2.2) with $\Lambda < 0$ is locally Anti-de Sitter (locally de Sitter if $\Lambda > 0$ and locally flat if $\Lambda = 0$). This can be verified by realizing that the full curvature tensor in three dimensions is totally determined by the Ricci tensor[1]

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{2.3}$$

As a consequence, any solution of Einstein equations (2.2) has constant curvature; namely

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{2.4}$$

Physically, this means that on three-dimensional Einstein spacetimes there are no local propagating degrees of freedom: there are no gravitational waves in this theory. Another way to see that gravity in $D = 3$ dimensions has no degrees of freedom (d.o.f.) is counting them explicitly: Out of the $D(D+1)/2$ components of a symmetric tensor $g_{\mu\nu}$ in D spacetime dimensions, one can always remove D of them using diffeomorphism invariance (one removes one d.o.f. per coordinate). Moreover, D components of the metric appear in the Lagrangian with no temporal derivative, they are therefore no true d.o.f. but Lagrange multipliers. This counting leads therefore to $\frac{D(D+1)}{2} - D - D = 0 \quad (D = 3)$.

A priori, this property may look very disappointing: How could this theory be a realistic model to study four-dimensional gravity if there is no graviton at all? However, despite the absence of local d.o.f., it turns out that three-dimensional gravity is in fact far from being sterile. This is because of two fundamental reasons: First, as we will see later, even though every spacetime is locally equivalent to a constant curvature spacetime, it may differ from the maximally symmetric solution by *global* properties, and this allows for interesting geometrical properties such as non-trivial causal structures. The second reason is that, unexpectedly, in the case $\Lambda < 0$, there exist black hole solutions.

2.2   The three-dimensional black hole

To everyone's surprise, Bañados, Teitelboim and Zanelli (BTZ) showed in 1992 that $2+1$-dimensional gravity admits a black hole solution [38] that shares many physical properties with the four-dimensional Kerr black hole. The BTZ black hole[2] of mass M and angular momentum J is described, in Schwarzschild type coordinates, by the metric

$$ds^2 = -(N(r))^2 \, dt^2 + (N(r))^{-2} \, dr^2 + r^2 \left(d\varphi + N^\varphi(r)dt \right)^2, \tag{2.5}$$

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[1]In three dimensions, the Weyl tensor vanishes identically.
[2]For a review, see [16].

where the lapse and shift functions are given by[1]

$$N(r) \equiv \sqrt{-8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}} \quad , \quad N^\varphi(r) \equiv -\frac{4GJ}{r^2}. \tag{2.6}$$

with $-\infty < t < +\infty$, $0 < r < +\infty$ and $0 \le \varphi \le 2\pi$. It solves the Einstein equation (2.2) with cosmological constant $\Lambda = -1/\ell^2$.

The BTZ metric (2.5) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_φ. This metric exhibits a (removable) singularity at the points $r = r_\pm$ where $N(r_\pm) = 0$; that is,

$$r_\pm = \ell \left[4GM \left(1 \pm \sqrt{1 - (J/M\ell)^2} \right) \right]^{1/2}. \tag{2.7}$$

When $|J| \le M\ell$, the BTZ possesses an event horizon at r_+ and an inner Cauchy horizon (when $J \ne 0$) at r_-. In terms of $r_\pm$, the mass and angular momentum read

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \qquad J = \frac{r_+ r_-}{4G\ell}. \tag{2.8}$$

In the case $|J| = M\ell$, both horizons coincide $r_+ = r_-$; this case corresponds to the so-called extremal BTZ. If $M < 0$ (or if $|J|$ becomes too large), the horizon at $r = r_+$ disappears, leading therefore to a naked singularity at $r = 0$. Relation $|J| \le M\ell$ plays therefore the role of a cosmic censorship condition. There is, however, a special case: when $M = -1/(8G)$ and $J = 0$, both the horizon and the singularity disappear! At this point, the metric exactly coincides with (the universal covering of) AdS$_3$ spacetime; namely

$$ds^2_{M=-\frac{1}{8G}, J=0} = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2.$$

Therefore, AdS spacetime is separated from the continuous spectrum of the BTZ black holes by a mass gap of $\Delta_0 = 1/(8G)$, see Figure 2.1. The solution with $-1/(8G) < M < 0$ corresponds to naked singularities, with conical singularity at the origin. These solutions exhibit an angular deficit around $r = 0$ and admit to be interpreted as particle-like objects [36]. Therefore, one cannot continuously deform a black hole state to the AdS$_3$ vacuum, since it would imply to go through the regions with naked singularities. The solutions with $M < -1/(8G)$ also correspond to naked singularities, in this case with angular excesses around $r = 0$.

Since, as we saw above, any solution of pure gravity in three dimensions is locally of constant curvature, the BTZ solution (2.5) is locally Anti-de Sitter: every point of the black hole has a neighborhood isometric to AdS$_3$ spacetime, and therefore the whole black hole can be expressed as a collection of patches of AdS assembled in the right way. The fact that BTZ differs from AdS only by global properties suggests that the black hole metric can be obtained by identifying points of AdS spacetime by a sub-group of its isometry group. That is actually what Henneaux, Bañados, Teitelboim and Zanelli showed in [39], where these identifications were given explicitly.

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[1]Notice that many authors set $8G \equiv 1$.
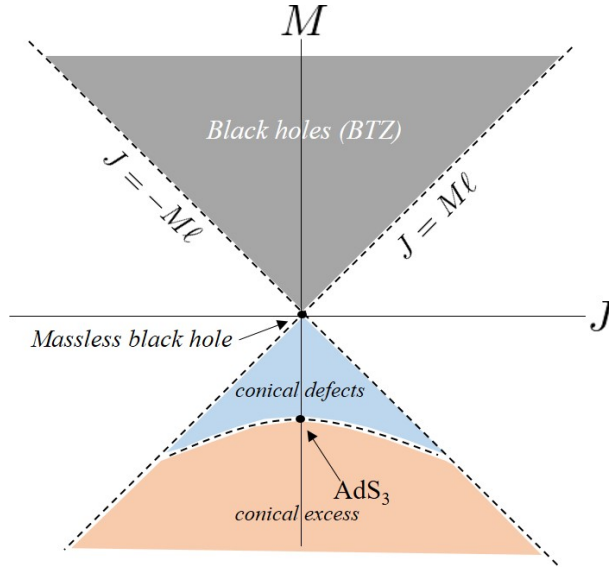
Figure 2.1: Spectrum of the BTZ black hole. Black holes exist for $M \geq 0$, $|J| \leq M\ell$. The vacuum state $M = -\frac{1}{8G}$, $J = 0$, separated by a gap from the continuous spectrum, corresponds to AdS$_3$.

Very far away from the black hole, namely when $r \gg r_+$, the metric reduces to

$$ds^2_{M=J=0} = -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\varphi^2. \tag{2.9}$$

The asymptotic behavior[1] of the BTZ black hole and AdS$_3$ is thus the same, this is why the BTZ is said to be asymptotically AdS. This is in contrast with the Schwarzschild and Kerr black holes which are asymptotically flat. In fact, there is no black hole asymptotically flat, nor asymptotically de Sitter in three-dimensions (for pure gravity) [44].

The g_{00} component of the BTZ is zero at $r = r_{\mathrm{erg}}$, with

$$r_{\mathrm{erg}} = \sqrt{r_+^2 + r_-^2} = \ell\sqrt{8GM}. \tag{2.10}$$

The $r < r_{\mathrm{erg}}$ region is called *ergosphere*, meaning that all observers in this region are unavoidably dragged along by the rotation of the black hole. The existence of an event horizon and of an ergosphere region make the BTZ extremely similar to the four-dimensional Kerr black hole. Another feature that BTZ shares with the Kerr solution is that, in both spaces, the surface $r = r_-$ is a Killing horizon. In fact, even though BTZ has no curvature singularity at the origin, it is quite similar to realistic (3+1)-dimensional black holes: it is the final state of gravitational collapse [45], and possesses similar properties also at quantum level. Indeed, remarkably, the BTZ exhibits non-trivial thermodynamical properties; it radiates at a Hawking temperature

$$T_{\mathrm{BH}} = \frac{\hbar(r_+^2 - r_-^2)}{2\pi\ell^2 r_+}, \tag{2.11}$$

---

[1]By asymptotic, here we simply mean far away. We will give a more precise definition in section 2.4.

and has a Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{A}{4\hbar G}, \tag{2.12}$$

with $A = 2\pi r_+$ the horizon size. That is, three-dimensional black holes also obey the *area* law, whose full microscopic understanding is one of the main questions in quantum gravity. Besides, these thermodynamical quantities satisfy the first law of black hole thermodynamics

$$dM = T_{\text{BH}} dS_{\text{BH}} + \Omega \, dJ, \tag{2.13}$$

where $\Omega = r_-/(r_+\ell)$ is the angular velocity at the horizon. Notice that the BTZ also exhibits a Hawking-Page transition at $r_+ \sim \ell$.

Finally, it is worth mentioning that, besides pure gravity, the BTZ solution appears in many other frameworks, such as supergravity [46], string theories [47], and higher-spins [48]. Moreover, the BTZ turned out to appear in the near-horizon limit of higher-dimensional solutions [49]; all of this showing the relevance of this black hole solution in more general set-ups.

2.3   3D **gravity as a gauge theory**

A crucial property of three-dimensional gravity action (2.1) is that it can be rewritten in terms of ordinary gauge fields, in such a way that both the structure of the action and equations of motion simplify substantially. This fact was discovered by Achúcarro and Townsend [41], and latter by Witten [42], and holds for any sign of the cosmological constant. The validity of this result can be extended to supergravity actions [41, 50], as well as higher-spins [51, 52].

2.3.1   Vielbein and spin connection formalism

This result is based on the *first-order*, or Palatini formulation of general relativity. This consists in the following: Instead of working as we usually do with the metric $g_{\mu\nu}$, we will use an auxiliary quantity e_μ^a (with a frame index $a = 0, 1, 2$), called *frame field*, or *vielbein*[1], which can be thought of as the square root of the metric[2]; namely

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^b(x), \tag{2.14}$$

where η_{ab} is the metric of flat $3D$ Minkowski spacetime.

Relation (2.14) can be simply seen as the transformation of a tensor under a change of coordinates described by the matrix e_μ^a. Since e_μ^a is a non-singular matrix, with $e \equiv \det e_\mu^a = \sqrt{-\det g} \neq 0$, there is an inverse frame field $e_a^\mu(x)$ such that $e_\mu^a e_b^\mu = \delta_b^a$ and $e_a^\mu e_\nu^a = \delta_\nu^\mu$. Notice that, for a given metric, the frame field is not unique; indeed, all frame

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[1] In three dimensions, it receives the name *dreibein*; *vierbein* or *tetrad* in four dimensions.

[2] Its existence comes from the fact that the metric tensor can be diagonalized by an orthogonal matrix O_μ^a with positive eingeinvalue λ^a, the vielbein is therefore defined as $e_\mu^a = \sqrt{\lambda^a}O_\mu^a$.

fields related by a local Lorentz transformation $e'^a_\mu = \Lambda^{-1a}_{\ \ b}(x)e^b_\nu(x)$ with $\Lambda \in SO(2,1)$ are equivalent (the transformation is local since it affects only the frame indices, while the spacetime indices do not see such transformation).

We can use the vielbein to define a basis in the space of differential forms. We define the one-form $e^a \equiv e^a_\mu dx^\mu$ and the Levi-Civita in frame components ϵ_{abc} in the following way:

$$\begin{aligned}
\epsilon_{\mu\nu\rho} &\equiv e^{-1}\epsilon_{abc}e^a_\mu e^b_\nu e^c_\rho, \\
\epsilon^{\mu\nu\rho} &\equiv e\,\epsilon^{abc}e^\mu_a e^\nu_b e^\rho_c.
\end{aligned}\tag{2.15}$$

In the tetrad formalism, the role of the connection is played by one-forms $\omega^{ab} = \omega^{ab}_\mu dx^\mu$, with $\omega^{ab} = -\omega^{ba}$. This quantity permits to construct a quantity that transforms as a local Lorentz vector. Indeed, unlike the 2-form de^a, the following quantity, called the *torsion* 2-form of the connection,

$$T^a \equiv de^a + \omega^a_{\ b} \wedge e^b \ ,\tag{2.16}$$

does transform as a vector under local Lorentz transformation, namely $T^a \to \Lambda^{-1a}_{\ \ b}T^b$, provided the quantity $\omega^a_{\ b}$, whose components ω^{ab}_μ are called *spin connections*, transforms as

$$\omega^a_{\ b} \to \Lambda^{-1a}_{\ \ c}d\Lambda^c_{\ b} + \Lambda^{-1a}_{\ \ c}\omega^c_{\ d}\Lambda^d_{\ b}.\tag{2.17}$$

Equation (2.16) is called the first Cartan structure equation. The second Cartan structure equation is given by

$$d\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb} = R^{ab},\tag{2.18}$$

with

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^{ac}_\mu\omega^b_{\nu c} - \omega^{ac}_\nu\omega^b_{\mu c}\tag{2.19}$$

the curvature tensor, and

$$\begin{aligned}
R^{ab} &= \frac{1}{2}R^{ab}_{\mu\nu}(x)dx^\mu \wedge dx^\nu, \\
R^{\lambda\sigma}_{\ \ \mu\nu} &= e^\lambda_a e^\sigma_b R^{ab}_{\ \ \mu\nu}.
\end{aligned}\tag{2.20}$$

Let us now go back to our three-dimensional Einstein-Hilbert action. In terms of the quantities we have defined above, (2.1) reads (we will take care of the boundary term later)

$$S_{\text{EH}}[e,\omega] = \frac{1}{16\pi G}\int_{\mathcal{M}}\epsilon_{abc}\left(e^a \wedge R^{bc}[\omega] - \frac{\Lambda}{3}e^a \wedge e^b \wedge e^c\right).\tag{2.21}$$

Indeed, using (2.15), we notice that

$$d^3x\sqrt{-g} = \frac{1}{3!}\epsilon_{abc}e^a \wedge e^b \wedge e^c;\tag{2.22}$$

and, using (2.20), we have

$$d^3x\sqrt{-g}R = \epsilon_{abc}\, e^a \wedge R^{bc}. \tag{2.23}$$

In what follows, we will adopt the so-called *dual* notation (valid only in three dimensions),

$$R_a \equiv \frac{1}{2}\epsilon_{abc}R^{bc} \leftrightarrow R^{ab} \equiv -\epsilon^{abc}R_c,$$

$$\omega_a \equiv \frac{1}{2}\epsilon_{abc}\omega^{bc} \leftrightarrow \omega^{ab} \equiv -\epsilon^{abc}\omega_c. \tag{2.24}$$

With this, we notice that the gravity action (2.21) can finally be rewritten as

$$S_{\text{EH}}[e,\omega] = \frac{1}{16\pi G}\int_{\mathcal{M}} \left(2\, e^a \wedge R_a[\omega] - \frac{\Lambda}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c \right). \tag{2.25}$$

2.3.2 The Chern-Simons action

Now that we have at hand the gravity action in terms of the vielbein and the spin connection, we are ready to prove, as announced above, that three-dimensional gravity is equivalent to a gauge theory with a specific kind of interaction, called Chern-Simons theory. Let us first introduce this very interesting model (for a more complete introduction to Chern-Simons in $3D$, see for instance [53]).

A *Chern-Simons action* for a compact gauge group G is given by

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \text{Tr}\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A \right], \tag{2.26}$$

where k is a constant called *level*, while the gauge field A represents a Lie algebra-valued one-form $A = A_\mu dx^\mu$, and Tr represents a non-degenerate[1] invariant bilinear form on the Lie algebra (of the gauge group G).

Integrating by parts, the variation of action (2.26) takes the form

$$\delta S_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \text{Tr}\left[2\delta A \wedge (dA + A \wedge A)\right] - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \text{Tr}\left[A \wedge \delta A\right]. \tag{2.27}$$

If δA is chosen such that its value on the boundary $\partial\mathcal{M}$ is such that the second term vanishes, we obtain

$$F \equiv dA + A \wedge A = 0, \tag{2.28}$$

where F is the usual field strength 2-form. These equations imply that, locally,

$$A = G^{-1}dG, \tag{2.29}$$

which means that A is a gauge transformation of the trivial field configuration; in other words, A is *pure gauge*. Therefore, a Chern-Simons theory has no true propagating degrees of freedom: it is purely topological. Indeed, all the physical content of the theory is contained in non-trivial topologies, which prevent relation (2.29) to hold everywhere on the manifold $\mathcal{M}$.

If we write $A = A^a T_a$, with T_a a basis[2] of the Lie algebra of the gauge group G, then

---

[1]This is asked in order that all gauge fields have a kinetic term in the action; this is always true for semisimple Lie algebras.

[2]This has nothing to do with the torsion T^a mentioned in the previous subsection.

one has, for the first term of (2.26),

$$\text{Tr}\left[A \wedge dA\right] = \text{Tr}(T_a T_b)\left[A^a \wedge dA^b\right]. \tag{2.30}$$

We then see that $d_{ab} \equiv \text{Tr}(T_a T_b)$ plays the role of a metric on the Lie algebra, and therefore should be non-degenerate. The existence of a Chern-Simons action, and the form it will take, relies on whether the gauge group one wants to consider admits such an invariant non-degenerate form. Notice that one can make use of this bilinear form to define an inner product $\langle \cdot, \cdot \rangle$.

2.3.3 $\Lambda < 0$ gravity as a Chern-Simons theory for $SO(2,2)$

What Achúcarro, Townsend and Witten discovered [41, 42] is that three-dimensional gravity action and equations of motions are equivalent to a Chern-Simons theory for an appropriate gauge group. More precisely, their result states that pure gravity (Einstein-Hilbert action) is equivalent to a three-dimensional Chern-Simons theory based on the gauge group $SO(2,2)$ for $\Lambda < 0$, $ISO(2,1)$ for $\Lambda = 0$, or $SO(3,1)$ for $\Lambda > 0$.

We will prove this result for the case $\Lambda < 0$, since we are interested in Anti-de Sitter spaces. In this case, the Lie algebra involved is $so(2,2)$, whose commutation relations are given by

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad [J_a, P_b] = \epsilon_{abc}P^c, \qquad [P_a, P_b] = \epsilon_{abc}J^c, \tag{2.31}$$

where the indices $a, b, c = 0, 1, 2$ are raised and lowered with the three-dimensional Minkowski metric η_{ab} and its inverse η^{ab}. In (2.31), we have used the three-dimensional rewriting

$$J_a \equiv \frac{1}{2}\epsilon_{abc}J^{bc} \leftrightarrow J^{ab} \equiv -\epsilon^{abc}J_c, \tag{2.32}$$

where the J_{ab} are the usual Lorentz generators, while the P_a are the generators of the translations. This Lie algebra admits the following non-degenerate invariant (symmetric and real) bilinear form[1]

$$\langle J_a, P_b \rangle = \eta_{ab}, \qquad \langle J_a, J_b \rangle = 0 = \langle P_a, P_b \rangle. \tag{2.33}$$

One then constructs the gauge field A living on this Lie algebra as

$$A_\mu \equiv \frac{1}{\ell}e_\mu^a P_a + \omega_\mu^a J_a. \tag{2.34}$$

Notice that the Lie algebra indices are identified with the frame indices of the vielbein and spin connection; this is crucial for the gravity $\leftrightarrow$ gauge theory relation that we are about to show. Equipped with the gauge field (3.10) and with the non-degenerate invariant form (2.33) one can write the Chern-Simons action (2.26) for the gauge group $G = SO(2,2)$. The first term is

$$\text{Tr}[A \wedge dA] = \frac{2}{\ell}e^a \wedge d\omega_a, \tag{2.35}$$

---

[1]This is not the only one, though; see the comments at the end of this section.

while the second term is found to be

$$\frac{2}{3}\text{Tr}[A \wedge A \wedge A] = \frac{1}{3}\text{Tr}[[A, A] \wedge A]$$
$$= \frac{1}{3\ell}\left(\frac{1}{\ell^2}e^a \wedge e^b \wedge e^c + 3\epsilon_{abc}e^a \wedge \omega^b \wedge \omega^c\right). \tag{2.36}$$

Therefore, we find that the Chern-Simons action for the group $SO(2,2)$ is equal to

$$S_{\text{CS}}[e, \omega] = \frac{k}{4\pi\ell}\int_{\mathcal{M}}\left(2e^a \wedge R_a[\omega] + \frac{1}{3\ell^2}\epsilon_{abc}\,e^a \wedge e^b \wedge e^c\right), \tag{2.37}$$

where we have remembered that $R_a = d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c$. We have thus shown that the Chern-Simons action for $SO(2,2)$ exactly matches the Einstein-Hilbert action (2.25) with $\Lambda = -1/\ell^2$, provided that the level acquires the value[1]

$$k = \frac{\ell}{4G}. \tag{2.38}$$

The fact that Einstein gravity is merely a Chern-Simons action reminds us that there is no propagating degree of freedom in the theory, and thus no graviton in three-dimensions, since we saw above that this gauge theory is purely topological. However, even though there are no local excitations, its dynamical content is far from being trivial due to the existence of boundary conditions[2]. We will see in section 2.4 that, under an appropriate choice of boundary conditions, there is in fact an *infinite number of degrees of freedom living on the boundary*. Boundary conditions are necessary in order to ensure that the action has a well-defined variational principle, but the choice of such conditions is not unique. In fact, the dynamical properties of the theory are extremely sensitive to the choice of boundary conditions. In this context, the residual gauge symmetry on the boundary is called global symmetry or *asymptotic symmetry*. The breakdown of gauge invariance at the boundary has the effect of generating this infinite amount of degrees of freedom.

Before concluding this section, let us mention a very useful fact, namely the isomorphism $so(2,2) \approx sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ (recall that $so(2,2)$ is semi-simple). Defining $J_a^{\pm} \equiv \frac{1}{2}(J_a \pm P_a)$, algebra (2.31) reads

$$[J_a^+, J_b^+] = \epsilon_{abc}J^{+c}, \qquad [J_a^-, J_b^-] = \epsilon_{abc}J^{-c}, \qquad [J_a^+, J_b^-] = 0. \tag{2.39}$$

Thanks to this splitting, one can rewrite the Chern-Simons action for the $so(2,2)$ connection[3] Γ as the sum of two Chern-Simons actions, each having their connections A, $\bar{A}$ in the first and second chiral copy of $sl(2,\mathbb{R})$ respectively:

$$A = (e^a/\ell + \omega^a)T_a, \qquad \bar{A} = (e^a/\ell - \omega^a)T_a, \tag{2.40}$$

---

[1] The sign of k depends on the identity $\sqrt{-g} = \pm e$, namely depends on the choice of relative orientation of the coordinate basis and the frame basis.

[2] In other words, the relevant d.o.f. are *global*. Introducing boundary conditions is not the only way to generate global d.o.f., one can also consider holonomies (we will not study them here). Notice however that holonomies generate only a finite number of degrees of freedom in the theory.

[3] which we previously called A

with T_a now being the generators of $sl(2, \mathbb{R})$. One can show that the decomposition of the action then reads

$$S_{\mathrm{CS}}[\Gamma] = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] \equiv S_{\mathrm{CS}}[A, \bar{A}], \tag{2.41}$$

that is, can be rewritten as the difference of a chiral and anti-chiral Chern-Simons action.

Finally, let us mention that Einstein's equations of motion are equivalent to the ones in the Chern-Simons formalism, namely $F^a = 0$, $\bar{F}^a = 0$. More precisely, varying the action with respect to e^a gives the constant curvature equation

$$F^a + \bar{F}^a = 0 \iff R^{ab} + \frac{1}{\ell^2} e^a \wedge e^b = 0, \tag{2.42}$$

while varying with respect to ω^a leads to the torsion free equation

$$F^a - \bar{F}^a = 0 \iff T^a = de^a + \omega^a{}_b \wedge e^b = 0. \tag{2.43}$$

We thus verify that solving the equations of motion in the Chern-Simons formalism is considerably simpler than solving Einstein's equations.

2.3.4   Some comments on Chern-Simons theories

Before concluding this section, let us make some remarks on Chern-Simons theories that can be relevant for their gravity application.

Let us begin by noticing that the Chern-Simons description of gravity is valid when the vielbein is invertible, which is true for classical solutions of gravity. However, from the gauge theory point of view, this is not entirely natural. This is relevant because, despite the identity between the actions of the two theories, it is not obvious that gravity and Chern-Simons are equivalent at quantum level where, besides the action, one has to provide a set of configurations over which to perform the functional sum. Perturbatively, close to classical saddle points, the relation between the gauge theory and three-dimensional gravity may remain valid; however, it is not clear that the relation still holds non-perturbatively [13]. Moreover, to claim that Chern-Simons and gravity theories are equivalent, we have to prove that the gauge transformations and the diffeomorphisms do match (up to a local Lorentz transformation). It is shown in [42] that this matching occurs only when the equations of motion are satisfied, namely on-shell.

A second comment regards the definition of the invariant bilinear form appearing in (2.26). In addition to (2.33), $so(2, 2)$ admits a second one, given by

$$\langle J_a, J_b \rangle = \eta_{ab} \, , \qquad \langle J_a, P_b \rangle = 0 \, , \qquad \langle P_a, P_b \rangle = \eta_{ab}, \tag{2.44}$$

consequence of the isomorphism $so(2, 2) \approx sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. One can use this new form to construct an alternative Chern-Simons action: the so-called *exotic* action [42], which corresponds to

$$S_{\mathrm{E}}[\Gamma] = S_{\mathrm{CS}}[A] + S_{\mathrm{CS}}[\bar{A}]. \tag{2.45}$$

This action is relevant in the construction of the so-called Topologically Massive Gravity [54].

Finally, let us mention an interesting feature of the Chern-Simons coupling constant, the level k. Generally, a Chern-Simons theory admits to be constructed by starting from a gauge invariant action S_P defined on a four-dimensional manifold $\mathcal{M}_4$ whose boundary is the three-dimensional space $\mathcal{M}$ where the Chern-Simons theory is defined. Here, we think of a gauge field in four dimensions that is an extension of the three-dimensional gauge field of Chern-Simons theory. The extension of the three-dimensional gauge field is, generically, non-unique, and this carries information about topology. In fact, the four-dimensional action S_P corresponds to a topological invariant; its Lagrangian density is a total derivative[1]. Topological invariant actions exist only in even dimension, and that is the reason why Chern-Simons actions exist only in odd dimensions. The four-dimensional action is given by

$$S_P = \frac{k}{4\pi} \int_{\mathcal{M}_4} \text{Tr}(F \wedge F) = \frac{k}{4\pi} \int_{\mathcal{M}_4} P, \tag{2.46}$$

with $P \equiv d_{ab} F^a \wedge F^b$, F being the curvature associated to the four-dimensional gauge field that extends the A appearing in (2.26). $P \equiv d_{ab} F^a \wedge F^b$ is called the Pontryagin form, and is a total derivative; more precisely $P = dL_{\text{CS}}$, where L_{CS} is the Chern-Simons Lagrangian of (2.26). Therefore, if $\partial \mathcal{M}_4 = \mathcal{M}$, one can, after using Stokes' theorem, rewrite S_P as an integral over $\mathcal{M}$, and one is left with the three-dimensional Chern-Simons action (2.26). Being a topological invariant, S_P takes discrete values. In fact, one can show that $\int \text{Tr}(F \wedge F) = 4\pi^2 n$, with $n \in \mathbb{Z}$. This, together with asking that the action is defined modulo 2π (so that e^{iS}, which appears in the path integral, is single-valued), leads to the conclusion that, for the theory to be well defined, the Chern-Simons level has to be quantized, namely $k \in \mathbb{Z}$ [13].

2.4 Asymptotically AdS$_3$ spacetimes

In this section, we will make more precise what we mean by *asymptotically Anti-de Sitter* spacetimes. We will consider a set of metrics which tend to the metric of AdS$_3$ in a specific way. Giving such information is actually equivalent to prescribing fall-off conditions on the metric components at large distances, the so-called *boundary conditions*. Before that, we need to specify what is the boundary of our spacetime.

Our three-dimensional manifold $\mathcal{M}$ is taken to have the topology of a cylinder $\mathbb{R} \times D_2$, where $\mathbb{R}$ is parametrized by the time coordinate $x^0 \equiv \tau \equiv t/l$ and D_2 is the two-dimensional spatial manifold parametrized by coordinates r, $x^1 \equiv \varphi$, with periodicity $\varphi \sim \varphi + 2\pi$. We introduce light-cone coordinates $x^\pm \equiv \tau \pm \varphi$ with $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\varphi)$. The boundary $\partial \mathcal{M}$ of the spacetime at spatial infinity ($r = \infty$) is thus a timelike cylinder of coordinates t, φ, see Fig. 2.2.

---

[1]A similar example is the Einstein-Hilbert action in two dimensions, which coincides with the Euler characteristic $\Xi = \frac{1}{2\pi} \int_{\mathcal{M}_2} \sqrt{-g} R d^2 x = 2 - 2g$, where g is the genus of the closed manifold $\mathcal{M}_2$.
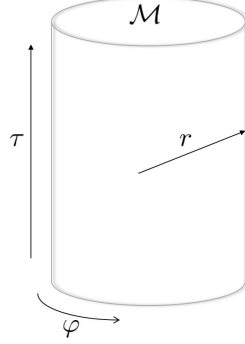
Figure 2.2: We consider the manifold $\mathcal{M}$ having the topology of a solid cylinder. Its boundary is taken to be the timelike cylinder at spatial infinity.

2.4.1   Boundary conditions and phase space

We adopt Fefferman-Graham coordinate system where the metric is given by ($i = 0, 1$)

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \gamma_{ij}(r, x^k)dx^i dx^j, \tag{2.47}$$

with the expansion, close to the the boundary $r \to \infty$, $\gamma_{ij} = r^2 g_{ij}^{(0)}(x^k) + \mathcal{O}(1)$. We call asymptotically AdS$_3$ spaces, in the sense of Brown-Henneaux [12], metrics of the form (2.47), where the boundary metric $g_{ij}^{(0)}$ is fixed as

$$g_{ij}^{(0)}dx^i dx^j = -dx^+ dx^-. \tag{2.48}$$

These *Brown-Henneaux boundary conditions* are Dirichlet boundary conditions with a flat boundary metric (2.48) on the cylinder located at spacial infinity.

It was shown in [55] that the most general solution (up to trivial diffeomorphisms) to Einstein's equations with $\Lambda = -1/\ell^2$ with boundary conditions (2.47), (2.48) is

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - \left(rdx^+ - \frac{\ell^2}{r}L(x^-)dx^-\right)\left(rdx^- - \frac{\ell^2}{r}\bar{L}(x^+)dx^+\right), \tag{2.49}$$

where $L(x^-)$ and $\bar{L}(x^+)$ are two single-valued arbitrary functions of x^- and x^+, respectively. In this gauge, one recovers well known geometries when these functions are constant; AdS$_3$ in global coordinates is recovered when $L = \bar{L} = -1/4$, $L = \bar{L} = 0$ corresponds to the massless BTZ, while generic positive values of $L, \bar{L}$ correspond to generic BTZ geometries of mass $M = (L + \bar{L})/(4G)$ and angular momentum $J = \ell(L - \bar{L})/(4G)$.

Let us now translate these boundary conditions in the Chern-Simons formalism. We choose a dreibein e^a which satisfies $ds^2 = \eta_{ab}e^a e^b$ with an off-diagonal metric η_{ab} ($a, b = 0, 1, 2$); see Appendix A for our conventions. One can check that

$$e^0 = -\frac{r}{\sqrt{2}}dx^- + \frac{\ell^2}{\sqrt{2}r}\bar{L}(x^+)dx^+, \; e^1 = \frac{r}{\sqrt{2}}dx^+ - \frac{\ell^2}{\sqrt{2}r}L(x^-)dx^-, \; e^2 = \frac{\ell}{r}dr, \tag{2.50}$$

reproduces $ds^2 = 2e^0 e^1 + (e^2)^2$, as desired. Then, the torsion free first Cartan structure equation (2.16) determines uniquely the associated spin connections

$$\omega^0 = \frac{r}{\sqrt{2}\ell} dx^- + \frac{\ell}{\sqrt{2}r} \bar{L}(x^+) dx^+, \ \omega^1 = \frac{r}{\sqrt{2}\ell} dx^+ + \frac{\ell}{\sqrt{2}r} L(x^-) dx^-, \ \omega^2 = 0. \quad (2.51)$$

The corresponding chiral Chern-Simons flat[1] connections are given by $A = (\omega^a + e^a/\ell) j_a$, $\bar{A} = (\omega^a - e^a/\ell) j_a$, where The gauge connections thus read

$$A = \begin{pmatrix} \dfrac{dr}{2r} & \dfrac{\ell}{r}\bar{L}(x^+)dx^+ \\ \dfrac{r}{\ell}dx^+ & -\dfrac{dr}{2r} \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} -\dfrac{dr}{2r} & \dfrac{r}{\ell}dx^- \\ \dfrac{\ell}{r}L(x^-)dx^- & \dfrac{dr}{2r} \end{pmatrix}. \quad (2.52)$$

A very useful trick is to notice that one can factorize out the r-dependance of the gauge fields by performing the following gauge transformation:

$$a = b^{-1} A\, b + b^{-1} db\,, \qquad \bar{a} = b\, \bar{A}\, b^{-1} + b\, db^{-1}, \quad (2.53)$$

with

$$b(r) = \begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix}. \quad (2.54)$$

Indeed, one can check that the reduced connections $a, \bar{a}$ are r-independent;

$$a = \begin{pmatrix} 0 & \ell\,\bar{L}(x^+)dx^+ \\ dx^+/\ell & 0 \end{pmatrix}, \qquad \bar{a} = \begin{pmatrix} 0 & dx^-/\ell \\ \ell\, L(x^-)dx^- & 0 \end{pmatrix}.$$

In analogy with the on-shell reduced connections (2.53), we define the off-shell reduced gauge connections $a = a^a_\mu j_a dx^\mu$ and $\bar{a} = \bar{a}^a_\mu j_a dx^\mu$ as

$$a = b^{-1} A\, b + b^{-1} db\,, \qquad \bar{a} = \bar{b}^{-1}\, \bar{A}\, \bar{b} + \bar{b}^{-1}\, d\bar{b}, \quad (2.55)$$

such that $a_r = 0 = \bar{a}_r$. We impose our boundary conditions in the following way; they come in two sets:

$(i) \quad a_- = 0 = \bar{a}_+,$

$(ii) \quad a_+ = \dfrac{\sqrt{2}}{\ell} j_1 + 0\, j_2 + \sqrt{2}\ell L(x^+) j_0, \quad \bar{a}_- = \sqrt{2}\ell \bar{L}(x^-) j_1 + 0\, j_2 + \dfrac{\sqrt{2}}{\ell} j_0.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.56)$

The phase space is clearly contained in these boundary conditions, with $\bar{b} = b^{-1}$. We will see that the first set of boundary conditions (i) will reduce the Chern-Simons action to a sum of chiral $SL(2, \mathbb{R})$ Wess-Zumino-Witten (WZW) actions. The remaining set (ii) will be used to further reduce the WZW model to Liouville theory.

---

[1]The fact that $A, \bar{A}$ are flat (i.e. $F = \bar{F} = 0$) is ensured by the fact that $L(x^-), \bar{L}(x^+)$ are chiral.

2.4.2   Asymptotic symmetry algebra

The asymptotic symmetries correspond to the set of gauge transformations[1]

$$\delta a = d\lambda + [a, \lambda] \, , \qquad \delta \bar{a} = d\bar{\lambda} + [\bar{a}, \bar{\lambda}] \tag{2.57}$$

that preserve the asymptotic behavior of the connections a, $\bar{a}$, namely equations (2.56). Writing the gauge parameters $\lambda = \lambda^a j_a$, $\bar{\lambda} = \bar{\lambda}^a j_a$ we find[2] that the latter have to be of the form

$$\begin{aligned}
\lambda &= \ell^2 \left(L\lambda^1 - \frac{1}{2}\partial_+^2 \lambda^1 \right) j_0 + \lambda^1 j_1 - \frac{\ell}{\sqrt{2}}\partial_+ \lambda^1 j_2, \\
\bar{\lambda} &= \bar{\lambda}^0 j_0 + \ell^2 \left(\bar{L}\bar{\lambda}^0 - \frac{1}{2}\partial_-^2 \bar{\lambda}^0 \right) j_1 + \frac{\ell}{\sqrt{2}}\partial_- \bar{\lambda}^0 j_2,
\end{aligned} \tag{2.58}$$

where the arbitrary functions λ^1, $\bar{\lambda}^0$ depend only on x^+, x^- respectively[3]. Writing $Y \equiv \ell\lambda^1/\sqrt{2}$, $\bar{Y} \equiv \ell\bar{\lambda}^0/\sqrt{2}$, we find

$$\begin{aligned}
\delta L &= Y\partial_+ L + 2L\,\partial_+ Y - \frac{1}{2}\partial_+^3 Y, \\
\delta \bar{L} &= \bar{Y}\partial_- \bar{L} + 2\bar{L}\,\partial_- \bar{Y} - \frac{1}{2}\partial_-^3 \bar{Y}.
\end{aligned} \tag{2.59}$$

At this stage, we can already notice that L and $\bar{L}$ transform in the same way as a two-dimensional CFT energy-momentum tensor does under generic infinitesimal conformal transformations, and one can already see that the last term, associated to the Schwarzian derivative, indicates the presence of a central extension.

The variation of the canonical generators associated to the asymptotic symmetries spanned by λ take a very simple form in the Chern-Simons formalism [56, 57, 58]; they are given by[4]

$$\delta\!\!\!/\, Q[\lambda] = -\frac{k}{2\pi} \int_0^{2\pi} \langle \lambda, \delta a_+ \rangle \, d\varphi \, , \qquad \delta\!\!\!/\, \bar{Q}[\bar{\lambda}] = -\frac{k}{2\pi} \int_0^{2\pi} \langle \bar{\lambda}, \delta \bar{a}_- \rangle \, d\varphi. \tag{2.60}$$

One then find that expressions in (2.60) become linear in the deviation of the fields, so that they can be directly integrated as

$$Q_Y = -\frac{k}{2\pi} \int_0^{2\pi} Y\, L \, d\varphi, \quad \bar{Q}_{\bar{Y}} = -\frac{k}{2\pi} \int_0^{2\pi} \bar{Y}\, \bar{L} \, d\varphi. \tag{2.61}$$

---

[1] The analysis of asymptotic boundary conditions can also be performed from the geometrical point of view, in terms of the metric (or the vielbein and the spin connection). In that case, instead of studying the gauge transformations one studies the asymptotic Killing vectors that preserve the form of the metric (2.48) at large r.

[2] We compute for instance $\delta a_+ = \partial_+\lambda + [a_+, \lambda]$ and identify the components of left and right hand sides along the generators j_a.

[3] This follows directly from $a_- = 0 = \bar{a}_+$.

[4] The charge may be non integrable, hence the $\delta\!\!\!/\,$ notation.

The Poisson brackets fulfill $\delta_{Y_1} Q_{Y_2} = \{Q_{Y_2}, Q_{Y_1}\}$; therefore, the algebra of the canonical generators can be directly computed from the transformation laws (2.59). Defining the modes ($m \in \mathbb{Z}$)

$$L_m \equiv \frac{k}{2\pi} \int_0^{2\pi} e^{im\varphi} L \, d\varphi \; ; \qquad \bar{L}_m \equiv \frac{k}{2\pi} \int_0^{2\pi} e^{im\varphi} \bar{L} \, d\varphi, \qquad (2.62)$$

one finds

$$
\begin{aligned}
i\{L_m, L_n\} &= (m-n)L_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}, \\
i\{L_m, \bar{L}_n\} &= 0, \\
i\{\bar{L}_m, \bar{L}_n\} &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12} m^3 \delta_{m+n,0},
\end{aligned}
\qquad (2.63)
$$

with central elements given by

$$c = \bar{c} = 6k = \frac{3\ell}{2G}. \qquad (2.64)$$

This shows that the charge algebra associated to the symmetry transformations that preserve the AdS₃ asymptotic form consists of a direct sum of two copies of the *Virasoro algebra*, with c being the *central charge*. The Virasoro algebra (2.63) is the algebra of local conformal transformations in two-dimensions; it is the central extension of the Witt algebra and is *infinite*-dimensional (recall that $m \in \mathbb{Z}$). Notice that the standard redefinitions $L_m \to L_m + \frac{c}{24}$, $\bar{L}_m \to \bar{L}_m + \frac{\bar{c}}{24}$ change the central terms in the algebra to $\frac{c}{12} m(m^2 - 1)\delta_{m+n,0}$ and $\frac{\bar{c}}{12} m(m^2 - 1)\delta_{m+n,0}$. From the CFT perspective, the existence of a non-trivial central extension is the result of a conformal (or Weyl) anomaly in the quantum theory. However, since this derivation was purely classical, finding a central extension at the classical level is quite remarkable. In the context of AdS/CFT, this is interpreted as a classical bulk (AdS₃) computation that describes a property of the effective action of the quantum boundary (CFT₂) theory. Algebra (2.63) was first shown in the seminal paper of Brown and Henneaux [12] in 1986, and this is the reason why this result is considered as the precursor of the AdS/CFT correspondence; notice that for this reason the central charge (2.64) is often called the Brown-Henneaux central charge.

It is worth mentioning that in the semi-classical limit $\ell \gg G$ (recall that the Planck length in three dimensions is $\ell_P \sim G$), the central charge (2.64) tends to infinity. Also, notice that the quantization of the Chern-Simons level k implies that the quantum theory seems to be well defined only for discrete values of the dimensionless ratio ℓ/G and, hence, discrete values of the central charge. This discretization of c is also understood from the dual CFT point of view, since the Zamolodchikov c-theorem prohibits the central charge to be continuous [59].

We will see in section 5.1 that the asymptotic symmetries of three-dimensional gravity with Brown-Henneaux boundary conditions can be defined everywhere into the bulk of spacetime, promoting in this sense the two copies of Virasoro algebra (2.63) to a new kind of symmetries, the so-called symplectic symmetries [60].

2.5   A brief introduction to Wess-Zumino-Witten models

In this section, we will present an important ingredient in our discussion, the Wess-Zumino-Witten model, which appears as an intermediate step in the connection between Chern-Simons and Liouville actions.

2.5.1   The nonlinear sigma model

In quantum field theory, a *nonlinear sigma model*[1] describes scalar fields ϕ^i ($i = 1, ..., n$) as maps from a flat spacetime to a target manifold. The latter is a n-dimensional Riemannian manifold $\mathcal{M}_n$ equipped with a metric $g_{ij}(\phi)$ which depends on the fields, this is why the model is intrinsically nonlinear. In other words, the coordinates on $\mathcal{M}_n$ are the scalar fields $\phi(x)$, in which the x^μ, $\mu = 1, ...D$ are the Cartesian coordinates of a flat spacetime. An action for this model is given by

$$S_\sigma[\phi] = \frac{1}{4a^2} \int d^D x \, g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i, \tag{2.65}$$

with $a^2 > 0$ a dimensionless coupling constant.

The so-called Wess-Zumino-Witten model[2] involves a particular two-dimensional σ-model in which the role of the target space is played by a semi-simple Lie group G and the fields are matrix fields living on G, noted $g(x)$. For a two-dimensional manifold Σ with coordinates $x^0 = \tau$, $x^1 = \varphi$ ($\mu, \nu = 0, 1$), the action of this nonlinear sigma model takes the form[3] [61]

$$S_\sigma[g] = \frac{1}{4a^2} \int_\Sigma d^2 x \, \mathrm{Tr}\left[\eta^{\mu\nu}\partial_\mu g \, \partial_\nu(g^{-1})\right]. \tag{2.66}$$

The group G has to be semi-simple to ensure the existence of the trace Tr, but can be either compact or non-compact.

This theory is conformally invariant only at the classical level. Indeed, under quantization, the coupling a acquires a scale dependance, leading therefore to a non-vanishing β-function (the quantum theory is in fact asymptotically free). Furthermore, even at the classical level, this theory is not totally satisfactory since it does not possess two conserved currents that factorize into a left (or holomorphic) and a right (antiholomorphic) part; this is the fundamental property of holomorphic factorization of a CFT. Indeed, the equations of motion are[4]

$$\partial^\nu(g^{-1}\partial_\nu g) = 0, \tag{2.67}$$

---

[1]The name comes from the fact that this model appeared for the first time in the description of a spinless meson called σ.

[2]It is important not to mistake the Wess-Zumino-Witten model with the Wess-Zumino model that describes four-dimensional supersymmetric interactions. The former is usually referred to as the Wess-Zumino-Novikov-Witten model.

[3]We take $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$

[4]Notice the useful relation $\delta(g^{-1}) = -g^{-1}\delta g \, g^{-1}$, which can be derived from $\delta(gg^{-1}) = 0$.

which read in light-cone coordinates $x^{\pm} \equiv \tau \pm \varphi^1$,

$$\partial_+ J_- + \partial_- J_+ = 0, \tag{2.68}$$

where we have defined the currents as $J_+ \equiv g^{-1}\partial_+ g$, $J_- \equiv g^{-1}\partial_- g$. We thus see that the equations of motion derived from (2.66) do not lead to the independent conservation of the left and right currents $J_{\pm}$. If one is conserved, (2.68) implies that the other current has to be conserved as well.

2.5.2 Adding the Wess-Zumino term

In order to have two independently conserved currents, it has been observed in [62, 63] that one has to consider, instead, the more involved action

$$S = S_\sigma[g] + k\Gamma[G], \tag{2.69}$$

with k an integer[2], and with the Wess-Zumino term $\Gamma[G]$ being

$$\begin{aligned}\Gamma[G] &= \frac{1}{3} \int_V d^3 x\, \epsilon^{\mu\nu\rho} \mathrm{Tr}\left[G^{-1}\partial_\mu G\, G^{-1}\partial_\nu G\, G^{-1}\partial_\rho G \right] \\ &\equiv \frac{1}{3} \int_V \mathrm{Tr}\left[(G^{-1}dG)^3 \right],\end{aligned} \tag{2.70}$$

where V is a three-dimensional manifold having Σ as a boundary, $\partial V = \Sigma$, and G is the extension of the element g on V. Notice that there are of course several choices for a V extending Σ, leading therefore to a potential ambiguity in the definition of Γ. In fact, for the quantum theory to be well-defined, and depending on the Lie group considered, this can imply a quantization condition for k. However, in the case of $SL(2, \mathbb{R})$ we are interested in, this issue does not appear.

Action (2.69) may look surprising since it mixes a nonlinear sigma model in two dimensions with the three-dimensional action (2.70). However, the Wess-Zumino term has the fundamental property that its variation under $g \to g + \delta g$ yields a two-dimensional functional. Actually, one can show that its variation is a total derivative, leading to the result (using Stokes' theorem)

$$\delta\Gamma[G] = \int_\Sigma d^2 x\, \mathrm{Tr}\left[\epsilon^{\mu\nu} \delta g g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g g^{-1} \right], \tag{2.71}$$

where we take as convention $\epsilon^{01} = 1$. Using this result, one can see that the equations of motion derived from (2.69) read

$$\frac{1}{2a^2}\eta^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) - k\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0. \tag{2.72}$$

---

[1]We have $\eta^{+-} = -2 = \eta^{-+}$, $\eta^{++} = 0 = \eta^{--}$.

[2]The name is of course not innocent, since we will see that it is indeed related to the Chern-Simons level k.

In light-cone coordinates $x^{\pm}$, with $\epsilon^{+-} = -2$, they become

$$(1 - 2a^2 k)\partial_+(g^{-1}\partial_- g) + (1 + 2a^2 k)\partial_-(g^{-1}\partial_+ g) = 0. \tag{2.73}$$

Therefore, for $a^2 = -1/(2k)$, which implies $k < 0$, one finds the conservation of the current $\partial_+ J_- = 0$, while for $a^2 = 1/(2k)$, one finds the conservation of the dual current $\partial_- J_+ = 0$. For the same conditions, one can show that the beta-function vanishes [62], representing a conformal invariant fixed point.

Taking $a^2 = -1/(2k)$, one obtains the *Wess-Zumino-Witten* (WZW), or *Wess-Zumino-Novikov-Witten* (WZNW) action; namely

$$S_{\text{WZW}}[g] = \frac{k}{2} \int d^2 x \, \text{Tr}\left[\eta^{\mu\nu} g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g\right] + k\Gamma[G]. \tag{2.74}$$

This action is sometimes called *non-chiral* WZW action, since it does not distinguish between left and right movers (it is symmetric under $x^+ \leftrightarrow x^-$), unlike the chiral action we will present later on. The solution of the equations of motion derived from (2.74), namely $\partial_+(g^{-1}\partial_- g) = 0$, is simply

$$g = \theta_+(x^+)\theta_-(x^-), \tag{2.75}$$

where $\theta_+(x^+)$ and $\theta_-(x^-)$ are arbitrary functions. Equation (2.75) means that left and right movers do not interfere between each others. One checks that the model described by (2.74) has the two conserved currents

$$J_- \equiv g^{-1}\partial_- g \,, \qquad \bar{J}_+ \equiv -\partial_+ g \, g^{-1}. \tag{2.76}$$

The independent conservation of the two currents implies that action (2.74) is invariant under $g \to \Theta_+(x^+)g\Theta_-^{-1}(x^-)$, with $\Theta_\pm$ two arbitrary matrices valued in G. Therefore, one sees that the global $G \times G$ invariance of the sigma model has been promoted to a local $G(x^+) \times G(x^-)$ invariance.

2.6   From Chern-Simons to Wess-Zumino-Witten

Chern-Simons theories have been shown to reduce to Wess-Zumino-Witten theories on the boundary [64, 65]. In particular, we will see explicitly in this section that the Chern-Simons theory

$$S_E[A, \bar{A}] = S_{CS}[A] - S_{CS}[\bar{A}], \tag{2.77}$$

with[1]

$$S_{CS}[A] = -\frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr}\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right], \tag{2.78}$$

---

[1]Notice that we have changed the overall sign for latter convenience.

describing $(2+1)$-dimensional gravity with $\Lambda = -1/\ell^2$ reduces under our boundary conditions to the $SL(2, \mathbb{R})$ WZW model on the cylinder at spatial infinity. From now on, we will use, instead of the level $k = \ell/(4G)$, the constant

$$\kappa \equiv \frac{k}{4\pi} = \frac{\ell}{16\pi G}. \tag{2.79}$$

Again, let us emphasize the advantage of the Chern-Simons formulation: Instead of working with a second order action in terms of the metric, we work with two flat gauge connections $A, \bar{A}$. In this section, we want to show explicitly how the first set of boundary conditions (2.56) implements the reduction of the Chern-Simons action (2.77) to a sum of two chiral Wess-Zumino-Witten actions.

2.6.1 Improved action principle

At this stage, we have a small issue to solve: our boundary conditions (2.56) do not lead to a well-defined action principle. To see that, let us first rewrite our Chern-Simons action (2.78) explicitly in coordinates r, τ, φ:

$$S_{CS}[A] = -\kappa \int_{\mathcal{M}} dx^\mu \wedge dx^\nu \wedge dx^\rho \operatorname{Tr} \left[A_\mu \partial_\nu A_\rho + \frac{1}{3} A_\mu [A_\nu, A_\rho] \right]$$

$$= -\kappa \int_{\mathcal{M}} dr d\tau d\varphi \operatorname{Tr} \left[A_r (\partial_\tau A_\varphi - \partial_\varphi A_\tau) + A_\tau (\partial_\varphi A_r - \partial_r A_\varphi) + A_\varphi (\partial_r A_\tau - \partial_\tau A_r) \right.$$

$$\left. + 2 A_\tau [A_\varphi, A_r] \right]. \tag{2.80}$$

Integrating by parts the second and fifth terms, and keeping only the radial boundary terms (the ones on φ vanish because of periodicity), one has, using Stokes' theorem,

$$S_{CS}[A] = -\kappa \int_{\mathcal{M}} dr d\tau d\varphi \operatorname{Tr} \left[A_r \dot{A}_\varphi - A_\varphi \dot{A}_r + 2 A_\tau (\partial_\varphi A_r - \partial_r A_\varphi + [A_\varphi, A_r]) \right]$$

$$+ \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \operatorname{Tr} \left[A_\tau A_\varphi \right], \tag{2.81}$$

where the dot stands for ∂_τ. The last boundary contribution being irrelevant, one can reabsorb it in the definition of the action. Therefore, the final form for the Chern-Simons action in terms of coordinates is given by

$$S_{CS}[A] = -\kappa \int_{\mathcal{M}} dr d\tau d\varphi \operatorname{Tr} \left[A_r \dot{A}_\varphi - A_\varphi \dot{A}_r + 2 A_\tau F_{\varphi r} \right], \tag{2.82}$$

where $F = dA + A \wedge A$ is the curvature two-form associated to the connection. However, this action does not have a well-defined action principle. Indeed, computing the variation of the total action (2.77), one finds

$$\delta S_E = (\text{EOM}) + 2\kappa \int_{\partial\mathcal{M}} d\tau d\varphi \operatorname{Tr} \left[A_\tau \delta A_\varphi - \bar{A}_\tau \delta \bar{A}_\varphi \right], \tag{2.83}$$

which is not zero on-shell when A_- and $\bar{A}_+$ are required to vanish on the boundary. Therefore, in order to have a well-defined variational principle, one must add the following surface term to the action

$$I = -\kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, \text{Tr} \left[A_\varphi^2 + \bar{A}_\varphi^2 \right] , \tag{2.84}$$

which is such that the improved action

$$S[A, \bar{A}] \equiv S_E + I = S_{CS}[A] - S_{CS}[\bar{A}] - \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, \text{Tr} \left[A_\varphi^2 + \bar{A}_\varphi^2 \right] \tag{2.85}$$

satisfies $\delta S[A, \bar{A}] = 0$ (recall that $A_- = 0 = \bar{A}_+$ on the boundary imply $A_\tau = A_\varphi$, $\bar{A}_\tau = \bar{A}_\varphi$).

2.6.2 Reduction of the action to a sum of two chiral WZW actions

Now that we have an action with a well-defined variational principle, we are ready to reduce the Chern-Simons to the WZW model. First, we will focus on the chiral sector, and then we will do a similar computation for the anti-chiral sector to finally compose the full WZW action.

The chiral part of the improved action (2.85) is given by

$$S[A] \equiv S_{CS}[A] - \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, \text{Tr} \left[A_\varphi^2 \right] , \tag{2.86}$$

with $S_{CS}[A]$ given by (2.82). Looking at the last term of (2.82), we realize that the component A_τ of the connection merely plays the role of a Lagrange multiplier, implementing the constraint $F_{r\varphi} = 0$. Therefore, assuming no holonomies (i.e. no holes in the spatial section), one can solve this constraint as

$$A_i = G^{-1}\partial_i G , \qquad (i = r, \varphi), \tag{2.87}$$

where G is an $SL(2, \mathbb{R})$ group element. Indeed, the solution to $dA + A \wedge A = 0$ is locally[1] given by $A = G^{-1}dG$. The condition $F_{r\varphi} = 0$, as a first class constraint, generates gauge transformations. One can partially fix the gauge by imposing[2] $\partial_\varphi A_r = 0$, which allows to factorize the general solution into

$$G(\tau, r, \varphi) = g(\tau, \varphi) \, h(r, \tau). \tag{2.88}$$

---

[1]Since we do not consider holonomies, we assume that this will also hold globally. In general, one should consider the more general case with holonomies, which appear as additional zero-modes that one should take into account if one wants to describe three-dimensional black holes. A treatment of holonomies was considered in [66, 67]. However, notice that holonomies do not affect neither the asymptotic symmetry nor the central charge.

[2]In fact, this consists of a gauge fixing condition only in an off-shell formulation, since this relation is obviously satisfied for the on-shell connection (2.52).

This implies $A_r = h^{-1}\partial_r h$ and $A_\varphi = h^{-1}g^{-1}g'h$, with the prime standing for the derivative with respect to φ. We will assume that, as it happens for the solutions of interest, $\dot{h}|_{\partial\mathcal{M}} = 0$. We are now ready to reduce the chiral action (2.86). Plugging (2.87) and (2.88), the boundary term simply reads

$$-\kappa \int_{\partial\mathcal{M}} d\tau d\varphi \mathrm{Tr}\left[(g^{-1}\partial_\varphi g)^2\right], \tag{2.89}$$

while the three-dimensional term gives explicitly (using the constraint $F_{r\varphi} = 0$)

$$S_{CS}[A] = \kappa \int_\mathcal{M} dr d\tau d\varphi \,\mathrm{Tr}[\partial_r hh^{-1}\dot{h}h^{-1}g^{-1}g' + \partial_r hh^{-1}g^{-1}\dot{g}g^{-1}g' \\ - \partial_r hh^{-1}g^{-1}\dot{g}' - h^{-1}\partial_r hh^{-1}g^{-1}g'\dot{h} - h^{-1}g^{-1}g'\dot{h}h^{-1}\partial_r h + h^{-1}g^{-1}g'\partial_r\dot{h}]. \tag{2.90}$$

On the other hand, one can see that, with the convention $\epsilon^{rt\varphi} \equiv 1$,

$$\frac{1}{3}\kappa \int_\mathcal{M} \mathrm{Tr}\left[(G^{-1}dG)^3\right] = \kappa \int_\mathcal{M} dr d\tau d\varphi \,\mathrm{Tr}[\partial_r hh^{-1}\dot{h}h^{-1}g^{-1}g' \\ + \partial_r hh^{-1}g^{-1}\dot{g}g^{-1}g' - \partial_r hh^{-1}g^{-1}g'g^{-1}\dot{g} - h^{-1}\partial_r hh^{-1}g^{-1}g'\dot{h}]. \tag{2.91}$$

Integrating by parts the third term in (2.90), one finds

$$S_{CS}[A] = \kappa\Gamma[G] + \kappa \int_\mathcal{M} \mathrm{Tr}[-h^{-1}g^{-1}g'\dot{h}h^{-1}\partial_r h + h^{-1}g^{-1}g'\partial_r\dot{h}]. \tag{2.92}$$

Finally, realizing that the last two terms are nothing but

$$\partial_r(G^{-1}\partial_\varphi GG^{-1}\partial_\tau G) = \partial_r(h^{-1}g^{-1}g'h(h^{-1}g^{-1}\dot{g}h + h^{-1}\dot{h})), \tag{2.93}$$

and recalling that $\dot{h} = 0$ on $\partial\mathcal{M}$, we have (using Stokes' theorem again)

$$S_{CS}[A] = \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \,\mathrm{Tr}\left[g^{-1}\partial_\varphi gg^{-1}\partial_t g\right] + \frac{\kappa}{3} \int_\mathcal{M} \mathrm{Tr}\left[(G^{-1}dG)^3\right]. \tag{2.94}$$

Therefore, we have shown that the Chern-Simons action for the chiral copy (2.86) reduces to

$$\begin{aligned} S[A] &= \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \,\mathrm{Tr}\left[g^{-1}\partial_\varphi g(g^{-1}\partial_t g - g^{-1}\partial_\varphi g)\right] + \kappa\Gamma[G]. \\ &\equiv S_{\mathrm{WZW}}^R[g], \end{aligned} \tag{2.95}$$

This is a *chiral* Wess-Zumino action action for the group element g. In light-cone coordinates $x^\pm = \tau \pm \varphi$, (2.95) reads

$$S[A] = 2\kappa \int_{\partial\mathcal{M}} d\tau d\varphi \,\mathrm{Tr}\left[g^{-1}\partial_\varphi g\, g^{-1}\partial_- g\right] + \kappa\Gamma[G]. \tag{2.96}$$

This first order action describes a *right-moving* group element g; this is the reason for the name "chiral", which means that the action distinguishes between left and right movers. Indeed, the equations of motion are $\partial_-(g^{-1}g') = 0$, whose solution is given by $g = f(\tau)k(x^+)$,

which is the equation for an element moving along the x^+ direction[1].

Similarly, the anti-chiral action

$$S[\bar{A}] \equiv S_{CS}[\bar{A}] + \kappa \int_{\partial \mathcal{M}} d\tau d\varphi \mathrm{Tr}\left[\bar{A}_\varphi^2\right], \tag{2.97}$$

after solving $\bar{F}_{r\phi} = 0$ by $\bar{A}_i = \bar{G}^{-1}\partial_i\bar{G}$, $\bar{G}(t, r, \varphi) = \bar{g}(t, \varphi)\,\bar{h}(r, t)$ (with $\dot{\bar{h}} = 0$ on $\partial\mathcal{M}$), can be written as

$$S[\bar{A}] = -\kappa \int_{\mathcal{M}} dr d\tau d\varphi \, \mathrm{Tr}\left[\bar{A}_r \dot{\bar{A}}_\varphi - \bar{A}_\varphi \dot{\bar{A}}_r\right] + \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \mathrm{Tr}\left[\bar{A}_\varphi^2\right]. \tag{2.98}$$

The only difference with the chiral action being the sign of the two-dimensional term, one finds easily that

$$\begin{aligned}S[\bar{A}] &= \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, \mathrm{Tr}\left[\bar{g}^{-1}\partial_\varphi \bar{g}(\bar{g}^{-1}\partial_t\bar{g} + \bar{g}^{-1}\partial_\varphi\bar{g})\right] + \kappa\Gamma[\bar{G}] \\ &\equiv S_{\mathrm{WZW}}^L[\bar{g}],\end{aligned} \tag{2.99}$$

where $S_{WZW}^L[\bar{g}]$ denotes a WZW action for a left-moving element $\bar{g}$. Indeed, the equations of motion $\partial_+(\bar{g}^{-1}\bar{g}') = 0$ imply $\bar{g} = \bar{f}(\tau)\bar{k}(x^-)$.

Therefore, combining left and right sectors, we have shown that the total Chern-Simons action is given by

$$S[A, \bar{A}] = S_{\mathrm{WZW}}^R[g] - S_{\mathrm{WZW}}^L[\bar{g}]. \tag{2.100}$$

2.6.3   Combining the sectors to a non-chiral WZW action

In order to recover the standard (non-chiral) WZW action (2.74), one can use the Hamiltonian form, since the chiral and anti-chiral actions are linear and of first order in time derivative. We combine left and right movers as $k \equiv g^{-1}\bar{g}$, $K = G^{-1}\bar{G}$; we define as well

$$\Pi \equiv -\bar{g}^{-1}\partial_\varphi g g^{-1}\bar{g} - \bar{g}^{-1}\partial_\varphi \bar{g}, \tag{2.101}$$

and observe that

$$\Gamma[K] = -\Gamma[G] + \Gamma[\bar{G}] - \int_{\partial\mathcal{M}} \mathrm{Tr}\left(d\bar{g}\bar{g}^{-1}dg g^{-1}\right). \tag{2.102}$$

We are allowed to change the variables from g and $\bar{g}$ to k and Π. In terms of the latter, the action (2.100) reads

$$S[k, \Pi] = \kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, \mathrm{Tr}\left[\Pi k^{-1}\dot{k} - \frac{1}{2}(\Pi^2 + (k^{-1}k')^2)\right] - \kappa\Gamma[K]. \tag{2.103}$$

---

[1] A right mover is often denoted $g(x^+)$; the notation meaning that g moves along the x^+ direction.

Eliminating the auxiliary variable Π by using its equation of motion, one finally gets

$$S[k] = \kappa \int_{\partial \mathcal{M}} d\tau d\varphi \, \text{Tr} \left[2k^{-1}\partial_+ k k^{-1}\partial_- k \right] - \frac{\kappa}{3} \int_{\mathcal{M}} \text{Tr} \left[(K^{-1}dK)^3 \right], \qquad (2.104)$$

which is the standard non-chiral $SL(2,\mathbb{R})$ WZW action for an element k.

Notice that the above change of variables above is not well-defined for the zero modes [68]. As a consequence, the equivalence of the sum of two chiral models with the non-chiral theory is not valid in that sector.

2.7   From the WZW model to Liouville theory

So far, we have shown that the asymptotic dynamics of three-dimensional gravity with $\Lambda < 0$ is described by the (non-chiral) WZW action for $SL(2,\mathbb{R})$. However, only the first set of boundary conditions (2.56) was used so far. In this section, we will see how the use of the second set further reduces the WZW model to eventually get Liouville field theory. Liouville theory is a two-dimensional conformal invariant field theory whose origin can be traced back to the work of Joseph Liouville in the 19th century. We will give a brief introduction to the classical and quantum Liouville theories in the last section.

2.7.1   The Gauss decomposition

In order to perform the reduction at the level of the action, it is useful to express the WZW action (5.56) in local form upon performing a Gauss decomposition of the form

$$\begin{aligned}
K &= e^{\sqrt{2}X j_0} e^{\phi j_2} e^{\sqrt{2}Y j_1} \\
&= \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\phi} & 0 \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix},
\end{aligned} \qquad (2.105)$$

where X, Y, ϕ are fields that depend on u, φ and r. We assume that the decomposition holds globally (for subtleties in the presence of global obstructions, see [69]). The Gauss decomposition allows to rewrite the three-dimensional integral in (5.56) as a two-dimensional integral using the relation

$$-\frac{1}{3}\text{Tr}(K^{-1}dK)^3 = dr d\tau d\varphi \, \epsilon^{\alpha\beta\gamma} \, \partial_\alpha \left(e^{-\phi}\partial_\beta X \, \partial_\gamma Y \right). \qquad (2.106)$$

Therefore, one finds (keeping only the radial boundary term)

$$-\frac{1}{3} \int_{\mathcal{M}} \text{Tr}(K^{-1}dK)^3 = \int_{\partial \mathcal{M}} d\tau \, d\varphi \, 2e^{-\phi}(\partial_- X \partial_+ Y - \partial_+ X \partial_- Y). \qquad (2.107)$$

The two-dimensional integral in (2.104) can be rewritten equivalently by replacing k by $K|_{\partial \mathcal{M}}$ since all factors of $h, \bar{h}$ exactly cancel in the trace. Therefore, one finds

$$\text{Tr} \left[2k^{-1}\partial_+ k k^{-1}\partial_- k \right] = \left(\partial_- \phi \partial_+ \phi + 2e^{-\phi}(\partial_+ X \partial_- Y + \partial_- X \partial_+ Y) \right). \qquad (2.108)$$

One can then combine all terms and find that (5.56) reduces to

$$S_{\text{red}} = 2\kappa \int_{\partial\mathcal{M}} d\tau \, d\varphi \left(\frac{1}{2}\partial_-\phi \, \partial_+\phi + 2e^{-\phi}\partial_-X \, \partial_+Y \right), \tag{2.109}$$

where all fields X, Y, ϕ have been pull-backed on $\partial\mathcal{M}$.

2.7.2   Hamiltonian reduction to the Liouville theory

The second set of boundary conditions (2.56) on the gauge fields set the currents of the WZW model to constants. This is the well-known Hamiltonian reduction of the WZW model to Liouville [70, 71, 72].

Let us begin by considering the left and right moving WZW currents. They are given by[1]

$$J_a = k^{-1}\partial_a k, \qquad \bar{J}_a = -\partial_a k k^{-1}. \tag{2.110}$$

Using the definition of k, we deduce (recall that $a = g^{-1}dg, \bar{a} = \bar{g}^{-1}d\bar{g}$):

$$J_- = -k^{-1}a_- k + \bar{a}_-, \qquad \bar{J}_+ = a_+ - k\bar{a}_+ k^{-1}. \tag{2.111}$$

Then, using the first set of boundary conditions (i), $a_- = \bar{a}_+ = 0$, we obtain a simple relation between k, the WZW currents, and the gauge fields: $J_- = k^{-1}\partial_- k = \bar{a}_-$, $\bar{J}_+ = -\partial_+ k k^{-1} = a_+$. Implementing the second set of boundary conditions (ii), $a_+ = (\sqrt{2}/\ell)j_1 + 0\, j_2 + \sqrt{2}\ell L(x^+)j_0$, $\bar{a}_- = \sqrt{2}\ell \bar{L}(x^-)j_1 + 0\, j_2 + (\sqrt{2}/\ell)j_0$, one finds

$$
\begin{aligned}
J_-^0 = [k^{-1}\partial_- k]^0 = \frac{\sqrt{2}}{\ell}, &\qquad J_-^2 = 0, \\
\bar{J}_+^1 = [-\partial_+ k k^{-1}]^1 = \frac{\sqrt{2}}{\ell}, &\qquad \bar{J}_+^2 = 0.
\end{aligned}
\tag{2.112}
$$

We thus see that the second set of boundary conditions (2.56) has for effect to set the WZW currents to constants. The set of constraints (2.112) is equivalent, in terms of the ϕ, X, Y fields introduced above, to the set

$$
\begin{aligned}
e^{-\phi}\partial_- X = \frac{1}{\ell}, &\qquad e^{-\phi}\partial_+ Y = -\frac{1}{\ell}, \\
X = 2\ell\partial_+\phi, &\qquad Y = -2\ell \, \partial_-\phi.
\end{aligned}
\tag{2.113}
$$

Before solving these constraints in the action, one has to be sure that the latter has a well-defined variational principle. To achieve so, one needs to add an improvement term to the action (2.109) as follows[2]

$$S_{\text{impr}} \equiv S_{\text{red}} - 2\kappa \int_0^{2\pi} d\varphi \left(e^{-\phi}\left(X\partial_+ Y + Y\partial_- X \right) \right)\Big|_{\tau_1}^{\tau_2}. \tag{2.114}$$

---

[1] In this section, all the k symbols appearing denote the group element $k = g^{-1}\bar{g}$, which has not to be confused with the level, which is a constant labeled by the same letter.

[2] This boundary term comes from the fact that the constraints restrict $\partial_+ Y$ and $\partial_- X$ rather than X and Y themselves.

After inserting the constraints, we are left with the Liouville action

$$S_{\text{Liouville}}[\phi] = 2\kappa \int_{\partial\mathcal{M}} d\tau \, d\varphi \, \left(\frac{1}{2}\partial_+\phi \, \partial_-\phi + \frac{2}{\ell^2}\exp(\phi) \right) \, . \tag{2.115}$$

Notice that the boundary term in (2.114) contributes as $2\kappa \int_{\partial\mathcal{M}} d\tau d\varphi \, (4/\ell^2)e^\phi$. Also, notice that by shifting ϕ by a constant, one can set $2/\ell^2$ to any (positive) value.

We have therefore shown that the boundary dynamics of AdS gravity in $D = 3$ dimensions is described by the two-dimensional Liouville action (we will briefly introduce Liouville actions in the next section). Liouville theory consist of a conformal field theory and, therefore, has associated two mutually commuting sets of Virasoro generators L_m and $\bar{L}_m$. Then, identifying these generators with the ones appearing in the asymptotic analysis carried out in section 2.4 is very tempting. Does it mean that Liouville theory *is* the dual conformal theory of three-dimensional gravity with $\Lambda < 0$? Actually not. The reduction we have presented is a classical computation, working at the level of the actions through the prescription of specific boundary conditions; to establish a correspondence between the two theories one would also need a full understanding at the quantum level. Nevertheless, the connection between Einstein theory on AdS$_3$ and Liouville theory on the boundary we have described, shows that the latter theory is, at least, an effective theory of the holographic dual CFT$_2$.

2.7.3   Gauged WZW point of view

Before concluding this section, let us mention an interesting observation made by Balog et al. in [73], where it was shown that Toda theories (and Liouville as a particular case) can be regarded as certain gauged Wess-Zumino-Witten models. We review the general procedure in the Lagrangian framework in Appendix B, and apply it here directly for the case of $G = SL(2,\mathbb{R})$.

Following this procedure, we consider the gauged WZW action

$$\begin{aligned} S[k, A_-, A_+] = S[k] + \kappa \int d^2x \text{Tr}[&g^{-1}\partial_- g A_+ - \partial_+ g g^{-1} A_- - A_- g A_+ g^{-1} \\ &+ A_-\mu_M - A_+\nu_M], \end{aligned} \tag{2.116}$$

where $\mu_M = \mu E_-$, $\nu_M = \nu E_+$ and $A_- = a_-(x^+, x^-)E_+$, $A_+ = a_+(x^+, x^-)E_-$ (with $a_\pm$ some functions and $E_\pm$ are the Chevalley-Serre generators, see Appendix (A)). Action (2.116) consists of the WZW action $S[k]$ given by (2.104) supplemented by a new piece involving two gauge fields A_- and A_+ in the adjoint representation of the subgroups E_+ and E_- respectively (therefore, they are nilpotent matrices). Action (2.116) is invariant under the gauge transformations

$$\begin{aligned} g \to \alpha g \beta^{-1} \, , \qquad \alpha = \alpha(x^+, x^-) \in E_+, \ \beta = \beta(x^+, x^-) \in E_-, \\ A_- \to \alpha A_- \alpha^{-1} + \alpha \partial_- \alpha^{-1} \, , \qquad A_+ \to -\beta A_+ \beta^{-1} - (\partial_+ \beta)\beta^{-1}. \end{aligned} \tag{2.117}$$

One can solve the equations of motion for the gauge fields (they are Lagrange multipliers), which gives:

$$
\begin{aligned}
a_+ &= \frac{\mu - \mathrm{Tr}[E_+(\partial_+ g)g^{-1}]}{\mathrm{Tr}[E_+ g E_- g^{-1}]}, \\
a_- &= \frac{-\nu + \mathrm{Tr}[E_- g^{-1}(\partial_- g)]}{\mathrm{Tr}[E_+ g E_- g^{-1}]}.
\end{aligned}
\tag{2.118}
$$

Using these relations in (2.116) amounts to implement the constraints inside the action; indeed, in terms of the fields of the Gauss decomposition, (2.116) becomes

$$
S[\phi] = \kappa \int d^2 x \, (\partial_+ \phi \partial_- \phi - \mu \nu \, e^\phi),
\tag{2.119}
$$

which is just the Liouville action.

2.8   Liouville field theory

2.8.1   At classical level

The Liouville differential equation was introduced in the 19th century in the context of the uniformization theorem for Riemann surfaces [74, 75, 76]. This is a classical problem of mathematics that can be rephrased as the following question: In a two-dimensional space equipped with a metric $g_{\mu\nu}$, does it exist a function Ω such that a new metric $\tilde{g}_{\mu\nu} \equiv \Omega g_{\mu\nu}$ has a constant scalar curvature $\tilde{R}$? The answer turns out to be yes. To see this, one first defines $\Omega = e^{2\phi}$ and, then, finds that the curvature associated to the $\tilde{g}$ metric is given by

$$
\tilde{R} = e^{-2\phi}(R - 2\Box \phi).
\tag{2.120}
$$

Setting the curvature $\tilde{R}$ to an arbitrary constant $-\lambda$ (the sign is chosen for latter convenience) leads to the nonlinear equation

$$
R - 2\Box \phi + \lambda e^{2\phi} = 0,
\tag{2.121}
$$

which is the so-called *Liouville equation*. Finding a solution ϕ which satisfies (2.121) is actually giving an answer to the uniformization problem.

Later, equation (2.121) was interpreted by Polyakov as the equation of motion of the quantum field theory that appears in string theory when one studies how the path integral measure transforms under Weyl rescaling. The classical equation of motion of Liouville field theory would be (2.121), which can be derived from the *Liouville action*

$$
S_{\mathrm{Liouville}} = \int d^2 x \sqrt{|g|} \left(g^{ab} \partial_a \phi \partial_b \phi + \phi R + \frac{\lambda}{2} e^{2\phi} \right).
\tag{2.122}
$$

On the cylinder (or on the torus), one can always set the second term to zero. In this case, the Liouville action coincides, up to field redefinition, with the one obtained in (2.115), which is consistent with the fact that the metric at infinity is the flat metric on the cylinder.

2.8.2   At quantum level: stress tensor and central charge

Before closing this section, let us present some quantum properties of Liouville action. As an exact conformal field theory, the quantum Liouville action is

$$S_q = \frac{1}{4\pi} \int d^2x \sqrt{|g|} \left(g^{ab}\partial_a\varphi\partial_b\varphi + (b+1/b)R\varphi + 4\pi\mu e^{2b\varphi} \right), \qquad (2.123)$$

where μ is an arbitrary positive constant and b is a dimensionless positive parameter which controls the quantum effects. Action (2.123) corresponds to a non-free scalar field theory formulated on a two-dimensional manifold doted with a metric g_{ab}. The curvature scalar of this two-dimensional space, R, couples non-minimally (linearly) with the field φ.

The value of μ can be set to 1 without loss of generality by shifting the field as follows $\varphi \to \varphi - (2b)^{-1}\log\mu$. This shifting is a symmetry of the classical theory as it merely generates a total derivative term $\propto \int d^2x \sqrt{|g|}R$ in the Lagrangian.

One can consistently recover the classical Liouville action by setting $\varphi = \phi/b$ and $8\pi\mu b^2 \equiv \lambda$; the action above then becomes

$$S_{cl} \equiv 4\pi b^2 S_q = \int_{\mathcal{C}_2} d^2x \sqrt{g} \left(g^{ab}\partial_a\phi\partial_b\phi + R\phi(1+b^2) + \frac{\lambda}{2}e^{2\phi} \right), \qquad (2.124)$$

which reduces to (2.122) in the limit $b^2 \to 0$ (indeed, $\hbar$ corrections correspond here to corrections of order b^2).

Defining $(z, \bar{z})$ the complex coordinates as usual in $2D$ CFT, one can show that the holomorphic and antiholomorphic components of the stress tensor, T_{zz} and $T_{\bar{z}\bar{z}}$, are given by [77]

$$\begin{aligned} T &\equiv T_{zz} = -(\partial\varphi)^2 + (b+1/b)\partial^2\varphi, \\ \overline{T} &\equiv T_{\bar{z}\bar{z}} = -(\overline{\partial}\varphi)^2 + (b+1/b)\overline{\partial}^2\varphi, \end{aligned} \qquad (2.125)$$

with $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$. One can compute the central charge of Liouville by computing the operator product expansion of two stress-tensor operators and, from this, one can read the central charge. More precisely, one has

$$T(z_1)T(z_2) = \frac{c_L/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} + ... \qquad (2.126)$$

where the ellipses stand for terms that are regular at $z_1 = z_2$; which can be easily computed starting from (2.125) and using the free field correlator $\langle\varphi(z_1)\varphi(z_2)\rangle = -2\log|z_1 - z_2|$. One then reads from (2.126) the value of the central charge

$$c_L = 1 + 6(b+1/b)^2. \qquad (2.127)$$

Remarkably, one notices from (2.127) that there is a $\mathcal{O}(1/\hbar)$ contribution to c_L, namely that the theory presents a classical contribution $\mathcal{O}(1/b^2)$ to the conformal anomaly (see [78] for more details).

The spectrum of primary operators in Liouville field theory is represented by the exponential vertex operators

$$V_\alpha(z) = e^{2\alpha\phi(z)}, \tag{2.128}$$

which create primary states of the theory with momentum α. The operator product expansion between the stress-tensor and these vertex operators is

$$T(z_1)V_\alpha(z_2) = \frac{\Delta}{(z_1 - z_2)^2}V_\alpha(z_2) + \frac{1}{(z_1 - z_2)}\partial V_\alpha(z_2) + ..., \tag{2.129}$$

where the conformal dimension Δ is given in terms of the momentum by

$$\Delta = \alpha\left(b + \frac{1}{b} - \alpha\right), \tag{2.130}$$

and analogously for the antiholomorphic component. On the other hand, normalizable states in the theory correspond to momenta

$$\alpha = \frac{b}{2} + \frac{1}{2b} + is \, , \quad \text{with } s \in \mathbb{R}. \tag{2.131}$$

Therefore, the spectrum of normalizable states of Liouville field theory is *continuous* and satisfies

$$\Delta = \frac{1}{4}\left(b + \frac{1}{b}\right)^2 + s^2 \geq \frac{1}{4}\left(b + \frac{1}{b}\right)^2. \tag{2.132}$$

This means that the theory has a *gap* between the value $\Delta = 0$ and the minimum eigenvalue

$$\Delta_0 = \frac{1}{4}\left(b + \frac{1}{b}\right)^2 = \frac{c_L - 1}{24}, \tag{2.133}$$

where the continuum starts. An observation that will be relevant later is that, in the semi-classical limit, where the central charge becomes large, this gaps reads

$$\Delta_0 \approx \frac{c_L}{24}. \tag{2.134}$$

Before concluding this section, let us make the following comment: As mentioned in the previous sections, through the reduction from Einstein gravity to Liouville theory we did not consider the contribution of holonomies. This can be seen as a limitation since holonomies are important to describe, for instance, the BTZ solution. However, it turns out that Liouville theory *knows* about the holonomies. This is because the holonomies of the Chern-Simons gauge connections that correspond to the different BTZ geometries can be classified in terms of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ conjugacy classes, and the latter are closely related to the classical solutions of Liouville field theory. While the BTZ black holes (namely $|J/\ell| \leq M > 0$) correspond to the *hyperbolic conjugacy class* of $SL(2,\mathbb{R})$, the particle-like solutions (for example, $-1/(8G) \neq M < 0$) correspond to the *elliptic conjugacy class* of $SL(2,\mathbb{R})$; the massless BTZ black hole ($M = J = 0$) belonging to the *parabolic conjugacy class*. It happens that all these solutions can actually be gathered in Liouville theory by studying the monodromies of the classical solutions of the field equation[1] around singularities. We will not discuss the details of this in these notes.

---

[1] More precisely, they are related to the solutions f of the so-called Hill equation $(\partial^2 + T(z))f = 0$.

2.9 Accounting for the entropy of the BTZ black hole

In this section, we will discuss how the conformal field theory description appearing in the boundary can be used to reproduce the BTZ black hole entropy. More precisely, we will begin by reviewing how, by means of a Cardy formula, the conformal field theory structure appearing through the AdS_3 asymptotic symmetry manages to account for three-dimensional the black hole entropy. Then, we will discuss some issues about Liouville field theory and the BTZ black hole spectra.

2.9.1 Cardy formula and effective central charge

In a CFT_2, the degeneracy of states in the limit of large conformal dimension Δ, and under certain assumptions, is given by the Cardy formula [79]. Namely, let a (chiral part of a) CFT with central charge c, such that its partition function on the torus is modular invariant; then, the degeneracy of states of conformal dimension Δ, denoted $\rho(\Delta)$, at large Δ is given by

$$\rho(\Delta) \approx \exp\left[2\pi\sqrt{\frac{c_{\text{eff}}\Delta^{\text{cyl}}}{6}}\right]\rho(\Delta_0), \tag{2.135}$$

where Δ_0 is the lowest eigenvalue of L_0 on the sphere, $\Delta^{\text{cyl}} = \Delta - c/24$ is the conformal dimension on the cylinder[1], and c_{eff} is an effective central charge, defined by

$$c_{\text{eff}} = c - 24\Delta_0. \tag{2.136}$$

This result is notably relevant for the applications to three-dimensional gravity. In fact, one can show how the Cardy formula can be used to compute the entropy of BTZ black holes. This starts with the observation of Brown and Henneaux that, as we reviewed in section 2.4, the asymptotic symmetry group of $(2+1)-$dimensional gravity with $\Lambda = -1/\ell^2$ is given by two copies of Virasoro algebra with central charges

$$c = \bar{c} = \frac{3\ell}{2G}. \tag{2.137}$$

In [14], Strominger made use of this result to reproduce the entropy of the BTZ black hole from the CFT_2 point of view. Indeed, applying Cardy formula (2.135), which can be written

$$S_{\text{CFT}} \equiv \log\rho(\Delta, \bar{\Delta}) = 2\pi\sqrt{\frac{c_{\text{eff}}\Delta^{\text{cyl}}}{6}} + 2\pi\sqrt{\frac{\bar{c}_{\text{eff}}\bar{\Delta}^{\text{cyl}}}{6}}, \tag{2.138}$$

using (2.137) as the effective central charge, and the fact that the BTZ conserved charges are given by

$$\Delta^{\text{cyl}} = \frac{1}{2}(\ell M + J) \ , \quad \bar{\Delta}^{\text{cyl}} = \frac{1}{2}(\ell M - J), \tag{2.139}$$

---

[1]The shifting $-c/24$ is related to the conformal mapping between the Riemann sphere and the cylinder, which produces a non-vanishing Schwarzian derivative in the stress-tensor transformation law.

one finds, using relations (2.8),

$$S_{\text{CFT}} = \frac{2\pi r_+}{4G} = S_{\text{BH}}. \tag{2.140}$$

This is a remarkable result! It manifestly shows that the Cardy formula of the boundary CFT_2 exactly reproduces the entropy of the AdS_3 black hole (2.12).

In the derivation of the Cardy formula (2.135) one assumes that the conformal dimension Δ is large and the central charge c finite. Notice that this is not in contradiction with the semi-classical limit (large c) since one can always consider large mass black holes while considering the AdS_3 radius much larger than the Planck length G. In other words, we have $\Delta \gg c \gg 1$. For different limits in relation to the Cardy formula, in which c is large but not necessarily much smaller than Δ, see [80]; see also [13] for a computation with small Δ (and small c) black holes.

2.9.2   A caveat of the CFT spectrum and Liouville theory

Let us now discuss a subtlety in relation to the Liouville field theory spectrum. Let us begin by noticing that what actually enters in the Cardy formula (2.138) is the *effective* central charge c_{eff} rather than c. This is related to the fact that, when deriving the asymptotic growth of states at large Δ, one resorts to the saddle point approximation, which selects the state of lower conformal dimension. In other words, the possibility of $c_{\text{eff}} \neq c$ is associated to whether the theory has a gap or not. This is exactly what occurs in Liouville theory which, as we mentioned in (2.132), when formulated on the sphere has a minimum eigenvalue of Δ_0 different from zero. Therefore, according to (2.133), Liouville effective central charge would be $c_{\text{eff}} = c_L - 24\Delta_0 = 1$, and this would be in conflict with the derivation of (2.140), which requires c_{eff} to be the Brown-Henneaux central charge in order to reproduce the black hole entropy.

This issue has been discussed in the literature [81, 82, 83], where it has been proposed that this is an indication that the description of thermodynamics in terms of Liouville field theory should be considered only as an effective description. In other words, the different steps connecting Einstein theory and Liouville theory, which were proven to hold at the level of the actions, should not be taken as a proof that the theories involved are equivalent beyond the semi-classical limit. In fact, the quantum regimes of both theories can be notably different. Liouville theory exhibits indeed many issues that make difficult to believe that it could describe three-dimensional gravity beyond the classical approximation. Nevertheless, one can still insist with the Liouville effective description and see how far it brings us. With this aim, let us discuss the Liouville theory spectrum in relation to the one of AdS_3 gravity in more details.

The reduction from a WZW model to Liouville can be performed at the quantum level: this is the so-called (Drinfeld-Sokolov) Hamiltonian reduction [70, 84, 69]. Through this procedure, one finds that the level $k = \ell/(4G)$ of the WZW action and the parameter b of Liouville action are related as follows:

$$b^2 = \frac{1}{k - 2}, \tag{2.141}$$

which in the semi-classical approximation $\ell \gg G$, reads $b^2 \approx 1/k \ll 1$.

Since the central charge of Liouville theory is given by (2.127), one finds that, in the semi-classical approximation,

$$c_L \approx 6k = \frac{3\ell}{2G}. \tag{2.142}$$

That is, the central charge of Liouville coincides exactly with the one of Brown-Henneaux. In addition, one observes that, in the same limit, the gap in the spectrum (2.134) also agrees with the gap in the spectrum of BTZ black holes. More precisely, from (2.139) one finds that for the BTZ black holes one has

$$\Delta^{\mathrm{cyl}} + \bar{\Delta}^{\mathrm{cyl}} = \ell M \ , \quad \Delta^{\mathrm{cyl}} - \bar{\Delta}^{\mathrm{cyl}} = J \ ; \tag{2.143}$$

on the other hand, AdS$_3$ space, which is the natural vacuum of the theory, corresponds to $\ell M = -1/(8G)$ and $J = 0$, which reads

$$\Delta^{\mathrm{cyl}} = \bar{\Delta}^{\mathrm{cyl}} = -\frac{\ell}{16G} \approx -\frac{c_L}{24} \ . \tag{2.144}$$

We see therefore that the mass gap of the BTZ spectrum (namely the gap Δ_0 between AdS$_3$ geometry and the massless BTZ black hole) coincides with the gap in the spectrum of Liouville theory (2.134). This suggests to include, apart from the normalizable states that account for the black hole spectrum, other states in Liouville theory, such as (2.144). In fact, apart from the normalizable states (2.131), in Liouville field theory there exists another set of interesting operators to be explored [81, 85]. These are the so-called degenerate operators, which correspond to values of momenta [77]

$$\alpha = \frac{1-n}{2b} + \frac{1-m}{2}b, \tag{2.145}$$

with m and n two positive integer numbers. Notice that the states (2.145) with $m > 1$ become irrelevant in the semi-classical limit $b \to 0$. Notice also that the state with $m = n = 1$ corresponds exactly to the vacuum $\Delta = \bar{\Delta} = 0$, namely $\Delta^{\mathrm{cyl}} = \bar{\Delta}^{\mathrm{cyl}} \approx -c/24$ as in (2.144). A natural question is to find what geometries correspond to the other degenerate states, namely those with $m = 1 < n$ in (2.145). According to (2.130), in the semi-classical limit theses states must correspond to gravity solutions with charges $\Delta = \bar{\Delta} \approx (1-n^2)/(4b^2)$; that is,

$$\Delta^{\mathrm{cyl}} + \bar{\Delta}^{\mathrm{cyl}} \approx -n^2 \frac{c_L}{12} \ , \quad \Delta^{\mathrm{cyl}} + \bar{\Delta}^{\mathrm{cyl}} = 0, \tag{2.146}$$

or equivalently,

$$\ell M = -\frac{n^2}{8G} \ , \quad J = 0, \tag{2.147}$$

with $n \in \mathbb{Z}_{>0}$. These states play an important role in the discussion [69] and can be shown to represent the special cases of negative mass geometries whose angular excesses around $r = 0$ are integer multiples of 2π. These *exact angular excesses* geometries also exhibit interesting supersymmetric properties as we will see in the next chapter (see section 3.3), and their relevance in higher-spin theories in AdS$_3$ has been discussed recently [86].

2.10 Other directions and recent advances

2.10.1 The choice of boundary conditions

The asymptotic symmetries and dynamics of three-dimensional gravity is highly sensitive to the choice of boundary conditions. In this chapter, we have focused on the seminal Brown-Henneaux boundary conditions, where the metric at the boundary has no dynamics. In the last years, there have been many works generalizing or modifying these boundary conditions. In [87], the boundary metric is allowed to be dynamical; in [88], chiral boundary conditions were given, while in [89, 90] the boundary metric is in a conformal gauge and light-cone gauge, respectively. These new notions of asymptotically AdS_3 spacetimes provide new potential CFT_2s living at the boundary [91, 92, 93].

2.10.2 Towards flat holography: the $\mathfrak{bms}_3$ algebra

Before concluding this chapter, let us make a crucial connection with an important direction that will be taken in the next chapter of the thesis: So far, we have reviewed in details the asymptotic analysis of Anti-de Sitter spacetimes, motivated by the power of AdS/CFT to learn more about the dual field theory that would describe these spaces at the boundary. However, over the last years, significant effort has been made to extend holographic tools to the case of non-AdS spacetimes. In particular, holographic properties of gravitational theories with a vanishing cosmological constant have been investigated [94, 95, 96, 97]. In this flat holography perspective, a rich asymptotic dynamics can be found at null infinity: the symmetry algebra of asymptotically flat spacetimes is the so-called $\mathfrak{bms}_3$ algebra [98, 99]

$$
\begin{aligned}
i\{J_m, J_n\} &= (m-n)J_{m+n} + \frac{c_1}{12}m^3\delta_{m+n,0}, \\
i\{P_m, P_n\} &= 0, \\
i\{J_m, P_n\} &= (m-n)P_{m+n} + \frac{c_2}{12}m^3\delta_{m+n,0},
\end{aligned}
\tag{2.148}
$$

with $c_1 = 0$, $c_2 = 3/G$[1]. Algebra (2.148) is an infinite-dimensional algebra made out of supertranslations and superrotations generated by P_m and J_m, respectively (we will introduce these terms and the algebra in a broader context in the next chapter). This algebra can be obtained from the (two copies of the) Virasoro algebra (2.63) by writing the latter in terms of the generators $P_m = (L_m + \bar{L}_{-m})/\ell$, $J_m = (L_m - \bar{L}_{-m})$ and then taking the limit $\ell \to \infty$. Taking the AdS radius to infinity consists indeed of a natural limit in order to describe flat spacetimes. Notice that the $\mathfrak{bms}$ central charges are related to the Virasoro ones accordingly to $c_1 = c - \bar{c}$, $c_2 = (c + \bar{c})/\ell$.

One can also connect through a well-defined flat-space limit [100] the phase space of asymptotically AdS and flat spacetimes: the limit of the BTZ black holes are cosmological

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[1]Notice that one can always set the value of c_2 to 1 by an appropriate rescaling of generators P_n. This amounts to set the energy with respect to which one measures the charge P_0. Here, though, we prefer to keep the value $c_2 = 3/G$, which basically sets the reference energy to the Planck mass $M_P \sim 1/G$.

solutions [101] whose thermodynamical properties can be understood from a holographic perspective [102, 103]. Finally, let us mention the fact that the dual dynamics of three-dimensional asymptotically flat spacetimes at null infinity has been shown to be a $\mathfrak{bms}_3$ invariant Liouville theory, through a reduction very similar to the one presented here; one goes first through a chiral iso(2, 1) WZW model and the Hamiltonian reduction reduces further the theory to a flat chiral boson action which can be finally related to a Liouville theory [104, 68, 105]. In the next chapter, we will be precisely interested in the asymptotic dynamics that govern flat spacetimes, our aim will be to find what is the supersymmetric extension of algebra (2.148) and what theory at the boundary of the spacetime classical describes the dual dynamics.

CHAPTER 3

Asymptotic dynamics of three-dimensional flat supergravity

The study of asymptotically flat spacetimes brings us back to the early 60's with the work of Bondi, van der Burg, Metzner and Sachs [18, 19, 20]. At that time, an important question arising in general relativity was the study of gravitational radiation, and the authors wanted to investigate the relation between the mass loss of the source and its radiation by means of the study of the asymptotic symmetry group of four-dimensional spacetimes at null infinity. What they found out is that this group does not merely consists of the Poincaré group, but rather an infinite-dimensional extension of it: the so-called BMS[2] group. It contains, on top of the Lorentz boosts, an extension of the usual translations to arbitrary angle-dependent translations, called *supertranslations*[3]. As such, this result can be seen as the first example of a symmetry enhancement phenomenon, where the asymptotic symmetry group is different (in this case considerably larger) with respect to the isometry group of the background metric.

The BMS group has recently attracted a lot of attention since Strominger and collaborations showed that this group allowed for connecting *a priori* disconnected subjects in theoretical physics. In [25, 24], it was shown that the supertranslation subgroup of BMS_4 is a symmetry of both the classical gravitational scattering problem and the quantum gravitational S-matrix. This was used to show that Ward identities associated to the supertranslation symmetry is equivalent to Weinberg's soft graviton theorem, a universal formula relating scattering amplitudes with and without soft gravitons insertions and that is valid for any theory of gravity. Moreover, supertranslations turned out be related to physical displacements known as memory effects [26].

Remarkably, the $\mathfrak{bms}$ algebra can be further extended, as Barnich and Troessaert showed [21, 22, 23]: if one allows for local singularities in the asymptotic Killing vector, then the Lorentz part of the algebra gets enhanced to two copies of the Virasoro algebra. The generators of this Virasoro algebra are called *superrotations*, and now both factors (supertranslations and superrotations) are infinite-dimensional. In opposition to the globally well-defined $\mathfrak{bms}$ algebra, the extended version of Barnich and Troessaert lead to a

---

[2]BBMS if we want to give some credits to van der Burg.

[3]Notice that the prefix *super* has nothing to do with supersymmetry here.

non-vanishing central extension in the charge algebra at null infinity. This central extension turns out to be field-dependent and, when evaluating for a Kerr black hole, some of the supertranslation charges, as well as the non-vanishing extension, involve divergent integrals on the 2-sphere. However, as noticed in [106], these divergences related to poles are no longer a problem when one uses a formulation in terms of currents (and thus there is no need to integrate anymore).

A good strategy to better understand the putative conformal dual to the extended 𝔟𝔪𝔰 algebra in four dimensions is to have a total control on the three-dimensional case. Not only the asymptotic analysis is considerably simpler on technical aspects with respect to the four-dimensional case, but also there exists a direct connection between asymptotically flat and asymptotically anti-de Sitter spaces through the well known flat limit [100, 107, 95, 108, 109, 110] that we have just encountered in section 2.10.2.

The purpose of this chapter is to show that it is possible to extend the asymptotic analysis of three-dimensional flat spaces to the simplest $\mathcal{N} = 1$ flat supergravity in three dimensions. In other words, we provide a supersymmetric extension of the $\mathfrak{bms}_3$ algebra, dubbed super-$\mathfrak{bms}_3$[1]. In fact, the existence of a rich asymptotic structure in $2 + 1$-dimensional supergravity is not obvious: It has been shown in [111, 112] that, when restricting the gravitational phase-space to conical spacetimes [36, 37], one can define neither linear momentum nor supercharge but only energy and angular momentum because the asymptotic dynamics does not allow for the associated symmetries. The absence of unbroken supercharge in this context has important physical implications as it can serve as a mechanism to ensure vanishing cosmological constant for the vacuum while at the same time boson and fermion masses need no longer be degenerate [113]. However, we have seen that in the case of a negative cosmological constant, when consistently relaxing the boundary conditions, the asymptotic symmetry group is enlarged and contains not only $SO(2,2)$ but the full local conformal group in two dimensions. At the same time, the phase-space now includes, besides the angular defects, further zero mode solutions, such as the BTZ black black hole (see section 2.2) and more generally, two arbitrary functions that make up the coadjoint representation of two copies of the Virasoro algebra at central charge $c = 3\ell/2G$ (see section 2.4). These results on an enhanced asymptotic structure have been extended to AdS$_3$ supergravity for which the boundary dynamics is governed by the superconformal algebra [50, 66]. In this chapter, we will extend the asymptotic analysis of supergravity in three dimensions to the case of a vanishing cosmological constant.

This chapter in organized as follows: In section 3.1, we briefly describe $\mathcal{N} = 1$ flat supergravity in three dimensions together with its Chern-Simons formulation. Additional conventions are given in the appendix A.

The rest of the chapter contains the original contributions, based on [114, 115]. The main result is contained in section 3.2, where we provide suitable fall-off conditions and work out the asymptotic symmetry algebra, the general solution to the supergravity equations of motion consistent with the boundary conditions, the transformation laws of the functions parametrizing solution space and the Poisson bracket algebra of the canonical

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[1]Notice that now the prefix *super* does have to do with supersymmetry.

symmetry generators together with the associated central charge.

In section 3.3, we discuss energy bounds and the Killing spinor equation, while section 3.4 is devoted to rederiving the flat space results from the corresponding ones for asymptotically AdS$_3$ supergravity by rephrasing the latter in a suitable gauge that allows one to perform the vanishing cosmological constant limit in a straightforward way. Section 3.5 is devoted to the minimal locally supersymmetric extension of the most general three-dimensional gravity theory without cosmological constant that leads to first order field equations for the dreibein and the spin connection. Due to additional parity odd terms, the Poisson algebra of canonical generators is given again by the centrally extended super-$\mathfrak{bms}_3$ algebra, but now with an additional central charge for the superrotation sub-algebra.

Finally, in section 3.6, we discuss field theoretic realizations of the super-$\mathfrak{bms}_3$ algebra. The first consists of a chiral Wess-Zumino-Witten theory based on the three-dimensional super-Poincaré algebra. It is obtained from the Chern-Simons formulation of at three-dimensional supergravity, including both the parity odd terms and appropriate boundary terms, through the standard correspondence with a chiral Wess-Zumino-Witten theory on its boundary. The effect of the remaining gravitational boundary conditions is then to add additional first class constraints. The second realization consists of a flat limit of super-Liouville theory that arises when performing the Hamiltonian reduction of the constrained chiral Wess-Zumino-Witten theory. Besides the extension to the supersymmetric case, these results also generalize previous results in the purely bosonic sector because the inclusion of the parity odd terms suitably modifies the Poincaré current algebra. Finally, we provide a Lagrangian formulation in terms of a gauged chiral WZW theory.

3.1 $\mathcal{N}=1$ flat supergravity in $3D$

The minimal locally supersymmetric extension of General Relativity in three dimensions with $\mathcal{N}=1$ gravitino was constructed in [116, 117, 118]. Nowadays, it is well-known that the theory can be described in terms of a Chern-Simons action in the cases of negative [41] or vanishing [119] cosmological constant, as we reviewed it in details in section 2.3. In the latter case, different extensions of the theory have been developed in e.g., [120, 121, 122, 123, 124, 125].

Let us consider here the simplest case which corresponds to $\mathcal{N}=1$ supergravity theory with vanishing cosmological constant. In this case, the gauge field $A = A_\mu dx^\mu$ is given by

$$A = e^a P_a + \omega^a J_a + \psi^\alpha Q_\alpha \,, \tag{3.1}$$

where e^a, ω^a and ψ^α stand for the dreibein, the dualized spin connection $\omega_a = \frac{1}{2}\epsilon_{abc}\omega^{bc}$, and the (Majorana) gravitino, respectively; while the set $\{P_a, J_a, Q_\alpha\}$ spans the super-Poincaré algebra, given by

$$[J_a, J_b] = \epsilon_{abc}J^c \quad ; \quad [J_a, P_b] = \epsilon_{abc}P^c \quad ; \quad [P_a, P_b] = 0 \,,$$

$$[J_a, Q_\alpha] = \frac{1}{2}\left(\Gamma_a\right)^\beta{}_\alpha Q_\beta \quad ; \quad [P_a, Q_\alpha] = 0 \quad ; \quad \{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(C\Gamma^a\right)_{\alpha\beta} P_a, \tag{3.2}$$

where C is the charge conjugation matrix. The Chern-Simons action that is equivalent to $\mathcal{N} = 1$ flat supergravity reads[1]

$$I[A] = \frac{k}{4\pi} \int \langle A, dA + \frac{2}{3}A^2 \rangle, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ stands for an invariant nondegenerate bilinear form, whose only nonvanishing components are given by

$$\langle P_a, J_b \rangle = \eta_{ab}, \quad \langle Q_\alpha, Q_\beta \rangle = C_{\alpha\beta} \ , \tag{3.4}$$

and the level is related to the Newton constant according to $k = 1/(4G)$. One can check that, up to a boundary term, the action reduces to

$$I = \frac{k}{4\pi} \int \left(2R^a e_a - \bar{\psi} D\psi \right) \ , \tag{3.5}$$

where $\bar{\psi}_\alpha = C_{\alpha\beta}\psi^\beta$ is the Majorana conjugate, while the curvature two-form and the covariant derivative of the gravitino are defined as

$$R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \omega_c \quad ; \quad D\psi = d\psi + \frac{1}{2}\omega^a \Gamma_a \psi \ . \tag{3.6}$$

By construction, the action is invariant, up to a surface term, under the local supersymmetry transformations spanned by $\delta A = d\lambda + [A, \lambda]$, with $\lambda = \epsilon^\alpha Q_\alpha$, whose components read

$$\delta e^a = \frac{1}{2}\bar{\epsilon}\Gamma^a \psi \ ; \ \delta\omega^a = 0 \ ; \ \delta\psi = D\epsilon \ . \tag{3.7}$$

Analogously, the field equations $F = dA + A^2 = 0$, whose general solution is locally given by $A = G^{-1}dG$, reduce to

$$R^a = 0 \ ; \ T^a = -\frac{1}{4}\bar{\psi}\Gamma^a \psi \ ; \ D\psi = 0 \ , \tag{3.8}$$

where $T^a = de^a + \epsilon^{abc}\omega_b e_c$ is the torsion two-form.

Defining $\omega = \frac{1}{2}\omega^a \Gamma_a$, $e = \frac{1}{2}e^a \Gamma_a$ and contracting the first two equations with $\frac{1}{2}\Gamma_a$ gives the matrix form $d\omega + \omega^2 = 0$, $de + [\omega, e] = -\frac{1}{4}\psi\bar{\psi}$. The local solution then decomposes as

$$\omega = \Lambda^{-1}d\Lambda \ \ , \ \ \psi = \Lambda^{-1}d\eta \ \ , \ \ e = \Lambda^{-1}(-\frac{1}{4}\eta d\bar{\eta} - \frac{1}{8}d\bar{\eta}\eta \mathbf{1} + db)\Lambda \ , \tag{3.9}$$

where Λ is a SL$(2, \mathbb{R})$ group element, η a Grassmann-valued spinor and b a traceless 2 by 2 matrix.

---

[1]From now on, the wedge products will be omitted.

3.2 Asymptotic behavior and the super-$\mathfrak{bms}_3$ algebra

The aim is now to provide a suitable set of fall-off conditions for the gauge fields at infinity that (i) extends the one of the purely gravitational sector so as to include the bosonic solutions of interest, and (ii) is relaxed enough so as to enlarge the set of asymptotic symmetries from $\mathfrak{bms}_3$ to a minimal supersymmetric extension thereof. In order to fulfill these requirements, the behavior of the gauge fields at the boundary is taken to be of the form

$$A = h^{-1}ah + h^{-1}dh \ , \tag{3.10}$$

where the radial dependence is completely captured by the group element $h = e^{-rP_0}$, and[1]

$$a = \left(\frac{\mathcal{M}}{2}du + \frac{\mathcal{N}}{2}d\phi \right)P_0 + du\,P_1 + \frac{\mathcal{M}}{2}d\phi J_0 + d\phi\,J_1 + \frac{\psi}{2^{1/4}}d\phi\,Q_+ \ , \tag{3.11}$$

where the functions $\mathcal{M}$, $\mathcal{N}$, and the Grassmann-valued spinor component ψ are assumed to depend on the remaining coordinates u, ϕ.

The asymptotic symmetries correspond to the set of gauge transformations, $\delta A = d\lambda + [A, \lambda]$, that preserves this behavior. When applied to the dynamical gauge fields A_ϕ, one finds that the Lie-algebra-valued parameter λ must be of the form

$$\lambda = \xi^a(u,\phi)P_a + \chi^a(u,\phi)J_a + 2^{1/4}\epsilon^+(u,\phi)Q_+ + 2^{1/4}\epsilon^-(u,\phi)Q_- \ , \tag{3.12}$$

with

$$
\begin{aligned}
\xi^0(u,\phi) &= \frac{1}{2}\mathcal{N}(u,\phi)\chi^1(u,\phi) + \frac{1}{2}\mathcal{M}(u,\phi)\xi^1(u,\phi) - \xi^{1\prime\prime}(u,\phi) + \frac{1}{2}\epsilon^-(u,\phi)\psi(u,\phi) \\
\xi^2(u,\phi) &= -\xi^{1\prime}(u,\phi) \\
\chi^0(u,\phi) &= \frac{1}{2}\mathcal{M}(u,\phi)\chi^1(u,\phi) - \chi^{1\prime\prime}(u,\phi) \\
\chi^2(u,\phi) &= -\chi^{1\prime}(u,\phi) \\
\epsilon^+(u,\phi) &= \frac{1}{\sqrt{2}}\left(\chi^1(u,\phi)\psi(u,\phi) - 2\epsilon^{-\prime}(u,\phi)\right) \ ,
\end{aligned}
\tag{3.13}
$$

in terms of functions $\chi^1, \xi^1, \epsilon^-$ of u, ϕ and prime denotes a derivative with respect to ϕ. When applied to the Lagrange multipliers A_u, λ is restricted further to depend only on three arbitrary functions of the angular coordinate, two bosonic ones $Y(\phi)$, $T(\phi)$, and one fermionic $\mathcal{E}(\phi)$,

$$\chi^1(u,\phi) = Y(\phi) \quad , \quad \epsilon^-(u,\phi) = \mathcal{E}(\phi) \quad , \quad \xi^1(u,\phi) = T(\phi) + uY'(\phi) \ , \tag{3.14}$$

and, at the same time, the field equations are required to hold in the asymptotic region: the fields $\mathcal{M}, \mathcal{N}, \psi$ become subject to the conditions

$$\partial_u\mathcal{M} = 0 \quad , \quad \partial_u\mathcal{N} = \partial_\phi\mathcal{M} \quad , \quad \partial_u\psi = 0 \ , \tag{3.15}$$

---

[1]Hereafter we assume light-cone coordinates in tangent space. See appendix A for the Γ-matrices representation and further conventions.

which are trivially solved by

$$\mathcal{M} = \mathcal{M}(\phi) \quad , \quad \mathcal{N} = \mathcal{J}(\phi) + u\mathcal{M}'(\phi) \quad , \quad \psi = \Psi(\phi) \ . \tag{3.16}$$

The phase space is thus reduced to three arbitrary functions of the angular coordinate, $\mathcal{M}, \mathcal{J}, \Psi$, whose transformation laws under the asymptotic symmetries are given by

$$\begin{aligned}
\delta\mathcal{M} &= Y\mathcal{M}' + 2Y'\mathcal{M} - 2Y''' \ , \\
\delta\mathcal{J} &= Y\mathcal{J}' + 2Y'\mathcal{J} + T\mathcal{M}' + 2T'\mathcal{M} + \mathcal{E}\Psi' + 3\mathcal{E}'\Psi - 2T''' \ , \\
\delta\Psi &= Y\Psi' + \frac{3}{2}Y'\Psi + \frac{1}{2}\mathcal{M}\mathcal{E} - 2\mathcal{E}'' \ .
\end{aligned} \tag{3.17}$$

The would-be variation of the canonical generators that corresponds to the asymptotic symmetries spanned by $\lambda(T, Y, \mathcal{E})$ can be readily found in the canonical approach [126]. In the case of a Chern-Simons theory in three dimensions, we have already encountered their simple expressions in section 2.4:

$$\delta\!\!\!/\, Q[\lambda] = -\frac{k}{2\pi}\int_0^{2\pi}\langle\lambda, \delta A_\phi\rangle d\phi \ . \tag{3.18}$$

For the asymptotic behavior described here, it is straightforward to verify that this expression becomes linear in the deviation of the fields with respect to the reference background, so that it can be directly integrated as

$$Q[T, Y, \mathcal{E}] = -\frac{k}{4\pi}\int_0^{2\pi}[T\mathcal{M} + Y\mathcal{J} - 2\mathcal{E}\Psi]\,d\phi \ . \tag{3.19}$$

Therefore, since the Poisson brackets fulfill $\delta_{\lambda_1}Q[\lambda_2] = \{Q[\lambda_2], Q[\lambda_1]\}$, the algebra of the canonical generators can be directly read from the transformation laws in (4.24). When expanded in Fourier modes,

$$\mathcal{P}_m = \frac{k}{4\pi}\int_0^{2\pi}e^{im\phi}\mathcal{M}\,d\phi, \ \mathcal{J}_m = \frac{k}{4\pi}\int_0^{2\pi}e^{im\phi}\mathcal{J}\,d\phi, \ \mathcal{Q}_m = \frac{k}{4\pi}\int_0^{2\pi}e^{im\phi}\Psi\,d\phi, \tag{3.20}$$

the (non-vanishing) Poisson brackets read explicitly

$$\begin{aligned}
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m - n)\mathcal{J}_{m+n} + \frac{c_1}{12}m^3\delta_{m+n,0} \ , \\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m - n)\mathcal{P}_{m+n} + \frac{c_2}{12}m^3\delta_{m+n,0} \ , \\
i\{\mathcal{J}_m, \mathcal{Q}_n\} &= \left(\frac{m}{2} - n\right)\mathcal{Q}_{m+n} \ , \\
\{\mathcal{Q}_m, \mathcal{Q}_n\} &= \mathcal{P}_{m+n} + \frac{c_2}{6}m^2\delta_{m+n,0} \ ,
\end{aligned} \tag{3.21}$$

where the central charges are given by $c_1 = 0$ and $c_2 = \frac{3}{G}$. Note that the standard redefinitions $\mathcal{J}_0 \to \mathcal{J}_0 + \frac{c_1}{24}$, $\mathcal{P}_0 \to \mathcal{P}_0 + \frac{c_2}{24}$ change the central terms in the algebra to $\frac{c_1}{12}m(m^2 - 1)\delta_{m+n,0}$, $\frac{c_2}{12}m(m^2 - 1)\delta_{m+n,0}$ and $\frac{c_2}{6}(m^2 - \frac{1}{4})$.

Algebra (3.21) constitutes the minimal supersymmetric extension of the $\mathfrak{bms}_3$ algebra with central extensions, and we call it super-$\mathfrak{bms}_3$ algebra. The classification of unitary representations of the super-BMS$_3$ group was given in [127], where it was also shown that the corresponding vacuum character coincides (in the Neveu-Schwarz sector) with the one-loop partition function of $\mathcal{N} = 1$ supergravity.

Notice that in the context of Galilean conformal algebras, superalgebras isomorphic to the super-$\mathfrak{bms}_3$ algebra, but with a different physical interpretation for the generators, have been constructed previously [128, 129] by taking a non-relativistic limit of the superconformal algebra (see also [130, 131] for finite-dimensional versions).

3.3 Energy bounds and Killing spinors

Energy bounds from quantum superalgebra

If the gravitino fulfills antiperiodic (Neveu-Schwarz) boundary conditions, the modes $\mathcal{Q}_p$ involve half-integer p. The wedge subalgebra is then spanned by the subset $\mathcal{P}_m$, $\mathcal{J}_m$, $\mathcal{Q}_p$, with $m = \pm 1, 0$, and $p = \pm 1/2$, which corresponds to the super-Poincaré algebra. Indeed, this can be explicitly seen once the modes in (3.21) are identified with the generators in (3.2) according to $\mathcal{J}_{-1} = -\sqrt{2}J_0$, $\mathcal{J}_1 = \sqrt{2}J_1$, $\mathcal{J}_0 = J_2$, $\mathcal{P}_{-1} = -\sqrt{2}P_0$, $\mathcal{P}_1 = \sqrt{2}P_1$, $\mathcal{P}_0 = P_2 - \frac{1}{8G}$, $\mathcal{Q}_{1/2} = \sqrt{2}Q_-$ and $\mathcal{Q}_{-1/2} = \sqrt{2}Q_+$.

In the quantum theory, one can then use arguments similar to those of [132, 133, 46]: the last of the brackets in (3.21) becomes an anticommutator to lowest order in $\hbar$ and the quantum generator $\mathcal{P}_0$ is bounded according to

$$\mathcal{P}_0 = \mathcal{Q}_{1/2}\mathcal{Q}_{-1/2} + \mathcal{Q}_{-1/2}\mathcal{Q}_{1/2} - \frac{1}{8G} \geq -\frac{1}{8G} \ . \tag{3.22}$$

In classical supergravity, the simplest solution that saturates the bound is Minkowski spacetime with $\mathcal{P}_0 = -\frac{1}{8G}$ and all other modes of $\mathcal{M}, \mathcal{J}, \Psi$ vanishing.

For the case of periodic (Ramond) boundary conditions for the gravitino, the modes $\mathcal{Q}_p$ involve integer p and the bound on the quantum generator becomes

$$\mathcal{P}_0 = \mathcal{Q}_0^2 \geq 0 \ . \tag{3.23}$$

The simplest classical supergravity solution that saturates this bound is the null orbifold [134] with all modes vanishing.

Asymptotic Killing spinors

Starting from transformations (3.17), one can systematically discuss the isotropy subalgebras of various solutions. A particular case of this problem is the "asymptotic Killing spinor equation", i.e., the question which asymptotic supersymmetry transformations leave purely bosonic solutions invariant,

$$\delta_{\mathcal{E}}\Psi = -2\mathcal{E}'' + \frac{1}{2}\mathcal{M}\mathcal{E} = 0 \ . \tag{3.24}$$

Asymptotic Killing spinors of solutions with constant $\mathcal{M} \neq 0$, are given by

$$\mathcal{E} = A e^{\frac{\sqrt{\mathcal{M}}}{2}\phi} + B e^{-\frac{\sqrt{\mathcal{M}}}{2}\phi} \ , \tag{3.25}$$

with A, B constants. They are globally well-defined provided $\mathcal{M} = -n^2$, with $n > 0$ a strictly positive integer,

$$\mathcal{E} = \mathcal{E}_{\frac{n}{2}} e^{in\frac{\phi}{2}} + \mathcal{E}_{-\frac{n}{2}} e^{-in\frac{\phi}{2}} \ . \tag{3.26}$$

Solutions with $n > 1$ are below the bounds (3.22) or (3.23). This singles out $n = 1$, Minkowski spacetime for $\mathcal{J} = 0$, in which case there are two independent antiperiodic solutions.

In the remaining case, $\mathcal{M} = 0$, the solution of (3.24) is given by

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{F}_0 \, \phi \ , \tag{3.27}$$

with $\mathcal{E}_0$, $\mathcal{F}_0$ constants, which is single-valued provided $\mathcal{F}_0 = 0$. This means in particular that there is a single periodic solution for the null orbifold at $\mathcal{J} = 0$.

Exact Killing spinors of bosonic zero mode solutions

Purely bosonic solutions ($\psi = 0$) to the field equations (3.8) in the asymptotic region are described in outgoing Eddington-Finkelstein coordinates by metrics

$$ds^2 = \mathcal{M} du^2 - 2 du dr + \mathcal{N} du d\phi + r^2 d\phi^2 \ , \tag{3.28}$$

with $\mathcal{M}, \mathcal{N}$ as in (3.16). The "zero mode solutions"

$$\mathcal{M} = 8GM \quad , \quad \mathcal{N} = 8GJ \ , \tag{3.29}$$

with M, J constants, describe cosmological solutions for nonnegative mass ($M \geq 0$) and arbitrary values of the angular momentum J, while for $-\frac{1}{8G} < M < 0$, the geometry corresponds to stationary conical defects. For $M = -\frac{1}{8G}$, the curvature is no longer singular at the origin, but the torsion is unless $J = 0$, which corresponds to Minkowski spacetime. Below this value of the mass, the geometry describes angular excesses (see, e.g., [36, 100]).

Such solutions admit global supersymmetries when they are invariant under supersymmetry transformations of the form (3.7), provided the spinorial parameter ϵ is globally defined. The Killing spinor equation to be solved is then given by

$$D\varepsilon = (d + \omega)\varepsilon = 0 \ , \tag{3.30}$$

with $\omega = \frac{1}{2}\omega^a \Gamma_a$.

This equation can be solved directly through $\varepsilon = \Lambda^{-1}\varepsilon_0$ with ε_0 a constant spinor and Λ the Lorentz group element associated to the flat spin connection, $\omega = \Lambda^{-1}d\Lambda$, whose form can be read off (3.11),

$$\Lambda = \exp\left[\frac{1}{2}\left(\Gamma_1 + \frac{1}{2}\mathcal{M}\Gamma_0\right)\phi\right] = \begin{pmatrix} \cosh\left(\frac{\sqrt{\mathcal{M}}}{2}\phi\right) & \sqrt{\frac{\mathcal{M}}{2}}\sinh\left(\frac{\sqrt{\mathcal{M}}}{2}\phi\right) \\ \sqrt{\frac{2}{\mathcal{M}}}\sinh\left(\frac{\sqrt{\mathcal{M}}}{2}\phi\right) & \cosh\left(\frac{\sqrt{\mathcal{M}}}{2}\phi\right) \end{pmatrix} .$$
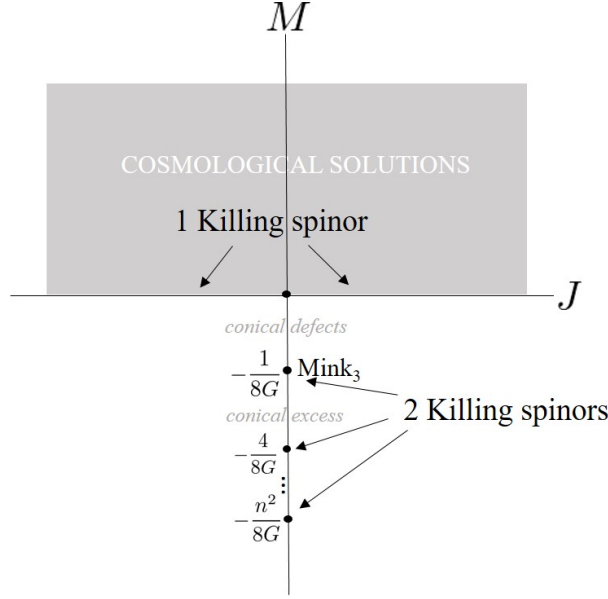
Figure 3.1: Zero mode solutions of $2 + 1$ dimensional flat gravity and their supersymmetries. Zero-mass cosmological solutions possess one Killing spinor, while the geometries with $J = 0$, $M = -n^2/8G$ ($n \in \mathbb{N}^*$) have the maximal number of supersymmetries, the case $n = 1$ beeing Minkowski space-time.

Alternatively, one can first solve the Killing spinor equation for the upper component. According to (3.13), this amounts to $\epsilon^+ = -\sqrt{2}\epsilon^{-\prime}$. The equation for the lower component then reduces to the asymptotic Killing spinor equation (3.24).

When suitably identifying the constants $\epsilon_0^+, \epsilon_0^-$, one finds in both cases that the Killing spinor ε is globally defined provided $\mathcal{M} = -n^2$ with n a positive integer. For $n > 0$, one finds two independent Killing spinors which can be periodic (even n) or antiperiodic (odd n) given explicitly by $\epsilon = (-\sqrt{2}\mathcal{E}', \mathcal{E})$, with $\mathcal{E}$ as in (3.26). For $n = 0$, one finds a single independent periodic solution given explicitly by $\epsilon = (0, \mathcal{E}_0)$.

In summary, massive cosmological solutions ($\mathcal{M} > 0$) do not admit global supersymmetries, while the massless case admits only one periodic Killing spinor. For $\mathcal{M} = -n^2$, the geometries possess two (the maximum number of) global supersymmetries, which includes, for $n = 1$, the case of Minkowski spacetime. The different cases are summarized in Figure 3.1.

Note that the geometries with $\mathcal{M} = -n^2$, $n > 1$ can be interpreted as suitable unwrappings of those for $n = 1$ with n playing the role of the winding number. Indeed, the rescalings

$$\phi' = n\phi\,, \qquad r' = n^{-1}r\,, \qquad u' = nu,$$

amount to the change $M \to n^2 M$, $J \to n^2 J$ in (3.29). As we have argued in section 3.3, these geometries actually become excluded when one insists on fulfilling the energy bounds in eqs. (3.22) and (3.23), for the periodic and antiperiodic boundary conditions, respectively.

It is worth pointing out that geometries endowed with angular deficit or excess actually possess a curvature singularity on top of the source at the origin, so that they do not fulfill the integrability condition of (3.30), i.e., $DD\varepsilon \neq 0$. Minkowski spacetime is obviously devoid of this problem, while a detailed discussion of the singularity of the null orbifold $\mathcal{M} = 0 = \mathcal{J}$ at $r = 0$ can be found in section 2.3 of [135].

3.4   Flat limit of asymptotically AdS$_3$ supergravity

The standard $\mathcal{N} = 1$ supergravity action (3.5) can be directly recovered either from the $(1, 0)$ or the $(0, 1)$ AdS supergravity theory in the vanishing cosmological constant limit. However, when one deals with the asymptotic behavior of the fields, even in the case of pure gravity, the limiting process turns out to be much more subtle [100]. In this section we show how the results obtained in section 3.2 can be recovered from the corresponding ones in the case of asymptotically AdS$_3$ supergravity. Here we follow a similar strategy as the one carried out in [136] for the vanishing cosmological constant limit of higher spin gravity, which consists in finding a particularly suitable gauge choice that allows to perform the limit in a straightforward way.

Asymptotic behavior of minimal AdS$_3$ supergravity, canonical generators and superconformal symmetry

There are two inequivalent minimal locally supersymmetric extensions of General Relativity with negative cosmological constant in three spacetime dimensions, known as the (1,0) and (0,1) theories. Since both possess the same vanishing cosmological limit, without loss of generality we will choose the (1,0) one, which can be formulated as a Chern-Simons theory whose gauge group is given by $OSp(2|1) \otimes Sp(2)$ [41]. The action depends on two independent connections A^+ and A^-, for $OSp(2|1)$ and $Sp(2)$, respectively, and is given by

$$I_{\text{SAdS}} = I[A^+] - I[A^-] \ ,$$

where $I[A]$ is defined in (3.3).

The asymptotic behavior of the fields has been previously discussed in [50], [66]. The fall-off of the fields can be written as

$$A^{\pm} = b_{\pm}^{-1} a^{\pm} b_{\pm} + b_{\pm}^{-1} db_{\pm} \tag{3.31}$$

with $b_{\pm} = e^{\pm \log(r/\ell) L_0}$, and

$$\begin{aligned}
a^+ &= \left(L_1^+ - \mathcal{L}_+ L_{-1}^+ + \psi Q_+ \right) dx^+ \ , \\
a^- &= \left(L_{-1}^- - \mathcal{L}_- L_1^- \right) dx^- \ ,
\end{aligned} \tag{3.32}$$

where $x^{\pm} = \frac{t}{\ell} \pm \phi$. Here the generators $L_i^{\pm}$, with $i = -1, 0, 1$, span the left and right copies of $sp(2)$, and Q_{α}, with $\alpha = 1, -1$, correspond to the (left) fermionic generators

of $osp(2|1)$. On-shell, the functions $\mathcal{L}_\pm$ and the Grassmann-valued ψ, are required to be chiral, i.e.,

$$\partial_\mp \mathcal{L}_\pm = 0, \quad \partial_- \psi = 0 , \tag{3.33}$$

so that they depend only on x^+ or x^-.

The asymptotic symmetries are given by the gauge transformations $\delta a^\pm = d\lambda^\pm + [a^\pm, \lambda^\pm]$ that maintain the form of (3.32), so that $\lambda^\pm$ are given by

$$\lambda^+ = \chi^+ L_1 - \chi^{+\prime} L_0 + \frac{1}{2}\left(-2\mathcal{L}_+\chi^+ - \epsilon\Psi + \chi^{+\prime\prime}\right) L_{-1} + (\chi^+\Psi + \epsilon')Q_+ + \epsilon Q_- ,$$

and

$$\lambda^- = \chi^- L_{-1} + \chi^{-\prime} L_0 + \frac{1}{2}\left(-2\mathcal{L}_-\chi^- + \chi^{-\prime\prime}\right) L_1,$$

which depend on three arbitrary chiral functions, fulfilling

$$\partial_\pm \chi^\mp = 0 \quad , \quad \partial_- \epsilon = 0 . \tag{3.34}$$

The on-shell transformation law of the fields $\mathcal{L}_\pm$, ψ reads

$$\delta\mathcal{L}_+ = \chi^+ \mathcal{L}'_+ + 2\mathcal{L}_+\chi'_+ - \frac{1}{2}\chi^{+\prime\prime\prime} + \frac{3}{2}\psi\epsilon^{-\prime} + \frac{1}{2}\epsilon^-\psi',$$

$$\delta\psi = -\mathcal{L}_+\epsilon^- + \epsilon^{-\prime\prime} + \frac{3}{2}\psi\chi^{+\prime} + \chi^+\psi', \tag{3.35}$$

$$\delta\mathcal{L}_- = \chi^-\mathcal{L}'_- + 2\mathcal{L}_-\chi'_- - \frac{1}{2}\chi^{-\prime\prime\prime}.$$

The canonical generators associated to the asymptotic symmetries spanned by $\lambda^+ = \lambda^+(\chi^+, \epsilon)$ and $\lambda^- = \lambda^-(\chi^-)$, are given by

$$Q^+[\chi^+, \epsilon] = -\frac{\kappa}{2\pi} \int_0^{2\pi} \left[\chi^+\mathcal{L}_+ - \epsilon\psi\right] d\phi ,$$

$$Q^-[\chi^-] = -\frac{\kappa}{2\pi} \int_0^{2\pi} \left[\chi^-\mathcal{L}_-\right] d\phi , \tag{3.36}$$

where $\kappa := \ell k$, which by virtue of (3.35), can be readily shown to fulfill the (super) Virasoro algebra. Expanding in Fourier modes $\mathcal{L}_m^\pm = \frac{k\ell}{4\pi}\int \mathcal{L}_\pm e^{\pm im\phi}\, d\phi$ and $\mathcal{Q}_m = \frac{k\ell}{4\pi}\int \psi e^{im\phi}\, d\phi$, the nonvanishing Poisson brackets read

$$i\{\mathcal{L}_m^\pm, \mathcal{L}_n^\pm\} = (m-n)\mathcal{L}_{n+m}^\pm + \frac{c}{12}m^3\delta_{m+n,0} ,$$

$$i\{\mathcal{L}_m^+, \mathcal{Q}_n^+\} = \left(\frac{m}{2} - n\right)\mathcal{Q}_{m+n}^+ , \tag{3.37}$$

$$\{\mathcal{Q}_m^+, \mathcal{Q}_n^+\} = 2\mathcal{L}_{m+n}^+ + \frac{c}{3}m^2\delta_{m+n,0} .$$

where the central charge is given by $c = \frac{3\ell}{2G}$.

Vanishing cosmological constant limit

In order to recover the results of section 3.2 from the ones described in the previous subsection once the vanishing cosmological constant limit is taken, it turns out to be useful to express the asymptotic behavior of the gauge fields of the (1,0) AdS supergravity theory in a different gauge. We then choose different group elements $g_\pm$, so that the fall-off of the connections now read

$$A^\pm = g_\pm^{-1} a^\pm g_\pm + g_\pm^{-1} dg_\pm \; , \tag{3.38}$$

where $a^\pm$ are given by (3.32), and

$$
\begin{aligned}
g_+ &= b_+ e^{-\log(\frac{r}{\ell})L_0} e^{\frac{r}{2l}L_{-1}} \; , \\
g_- &= b_- e^{-\log(\frac{r}{4\ell})L_0} e^{\frac{r}{2\ell}L_{-1}} e^{\frac{2\ell}{r}L_1} \; .
\end{aligned}
\tag{3.39}
$$

In this gauge, the asymptotic form of the super-AdS gauge field is explicitly given by

$$
\begin{aligned}
A^+ &= \frac{r}{\ell} dx^+ L_0^+ + \frac{1}{2}\left[\frac{dr}{\ell} + \left(\frac{r^2}{2\ell^2} - 2\mathcal{L}_+\right)dx^+\right]L_{-1}^+ + dx^+ L_1^+ + \psi^+ Q_+ dx^+ \; , \\
A^- &= \frac{r}{\ell} dx^- L_0^- - \frac{1}{2}\left[\frac{dr}{\ell} + \left(\frac{r^2}{2\ell^2} - 2\mathcal{L}_-\right)dx^-\right]L_{-1}^- - dx^- L_1^- \; .
\end{aligned}
\tag{3.40}
$$

It is now convenient to make the change $t = u$ and to perform the change of basis

$$L_{-1}^{(\pm)} = -2J_0^\pm, \quad L_0^\pm = J_2^\pm, \quad L_1^{(\pm)} = J_1^\pm \; , \quad Q_+ = \frac{1}{2^{1/4}}\tilde{Q}_+, \tag{3.41}$$

followed by

$$J_a^\pm = \frac{J_a \pm \ell P_a}{2}, \quad Q_+ = \sqrt{\ell}\,\tilde{Q}_+ \; , \tag{3.42}$$

so that the full gauge field reads

$$
\begin{aligned}
A &= \left(-dr + \frac{\mathcal{M}}{2}du + \frac{\mathcal{N}}{2}d\phi - \frac{r^2}{2\ell^2}du\right)P_0 + d\phi J_1 + du P_1 + rd\phi P_2 \\
&\quad + \left(\frac{\mathcal{M}}{2}d\phi + \frac{\mathcal{N}}{2l^2}du - \frac{r^2}{2l^2}d\phi\right)J_0 + \frac{r}{\ell^2}du J_2 + \frac{\Psi}{2^{1/4}}\tilde{Q}_+ d\phi + \frac{1}{\ell}\frac{\Psi}{2^{1/4}}\tilde{Q}_+ du,
\end{aligned}
\tag{3.43}
$$

where the arbitrary functions $\mathcal{L}_\pm$, ψ have been redefined according to

$$\mathcal{M} = (\mathcal{L}_+ + \mathcal{L}_-), \quad \mathcal{N} = \ell(\mathcal{L}_+ - \mathcal{L}_-), \quad \Psi = \sqrt{\ell}\,\psi \; . \tag{3.44}$$

The chirality conditions (3.33) now read

$$\partial_u \mathcal{M} = \frac{1}{\ell^2}\partial_\phi \mathcal{N} \quad , \quad \partial_u \mathcal{N} = \partial_\phi \mathcal{M} \quad , \quad \partial_u \Psi = \frac{1}{\ell}\partial_\phi \Psi. \tag{3.45}$$

The vanishing cosmological constant limit can now be directly performed in a transparent way. Indeed, for the full gauge field $A = A^+ + A^-$, one just takes $\ell \to \infty$, so that it reduces to

$$A = \left(-dr + \frac{\mathcal{M}}{2}du + \frac{\mathcal{N}}{2}d\phi\right)P_0 + du P_1 + rd\phi P_2 + \frac{\mathcal{M}}{2}d\phi J_0 + d\phi J_1 + \frac{\Psi}{2^{1/4}}Q_+ d\phi \; ,$$

which coincides with the asymptotic form of the connection in the asymptotically flat case, eqs. (3.10), (3.11). Analogously, in the limit, the chirality conditions (3.45) take the flat space form (3.15), whose solution is given by (3.16).

It is simple to verify that the expression for the global charges for the gauge choice (3.39) remains the same as in the gauge (3.31) and is still given by (3.36). After making use of the redefinition for the fields in (3.44), they acquire the form

$$Q[f, h, \mathcal{E}] = -\frac{k}{4\pi} \int d\phi \left(f\mathcal{M} + h\mathcal{N} - 2\mathcal{E}\Psi \right), \tag{3.46}$$

where the parameters that characterize the asymptotic symmetries have been conveniently redefined as

$$f = \ell(\chi^+ + \chi^-), \quad h = \chi^+ - \chi^-, \quad \mathcal{E} = \sqrt{\ell}\varepsilon.$$

The chirality conditions (3.34) on the gauge parameters then read

$$\partial_u f = \partial_\phi h, \quad \partial_u h = \frac{1}{\ell^2}\partial_\phi f, \quad \partial_u \mathcal{E} = \frac{1}{\ell}\partial_\phi \mathcal{E}, \tag{3.47}$$

and, in the limit $\ell \to \infty$, they imply that

$$h = Y(\phi), \quad f = T(\phi) + uY', \quad \mathcal{E} = \mathcal{E}(\phi),$$

and hence, by virtue of (3.16), the global charges (3.46) reduce to the ones of the asymptotically flat case given in (3.19).

As explained in section 3.4, the canonical generators of (1,0) AdS supergravity satisfy the centrally-extended superconformal algebra in two dimensions given by (3.37). In order to take the flat limit, it is useful to change the basis according to

$$P_m \equiv \frac{1}{\ell}(L_m^+ + L_{-m}^-), \qquad J_m \equiv L_m^+ - L_{-m}^-,$$

as well as rescaling the supercharges as

$$Q_m \equiv \frac{1}{\sqrt{\ell}}Q_m^+.$$

After this has been done, in the limit $\ell \to \infty$, algebra (3.37) readily reduces to the minimal supersymmetric extension of the $\mathfrak{bms}_3$ algebra (3.21), where the central charges are given by $\ell c_1 = c^+ - c^-$ and $\ell c_2 = c^+ + c^-$. In particular, it also follows that the bounds for the generators that are obtained from the superconformal algebra, reduce to the ones in eqs. (3.22) and (3.23) in the limit of vanishing cosmological constant.

3.5 Asymptotic structure of $\mathcal{N} = 1$ flat supergravity with parity odd terms

3.5.1 The most general first order action for 3D gravity

There exists a more general action for gravity in three dimensions which, apart from the Einstein-Hilbert term with cosmological constant, contains the Lorentz-Chern-Simons

form and a term involving the torsion each with arbitrary couplings [137, 138]:

$$I_G(e^a, \omega^a) = \frac{k}{4\pi} \int \left(2\alpha_1 R^a e_a - \frac{\alpha_2}{3} \epsilon_{abc} e^a e^b e^c + \alpha_3 L(\omega) + \alpha_4 T^a e_a \right), \tag{3.48}$$

where $L(\omega) = \omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c$ is the Lorentz-Chern-Simons form. This model, also known as the Mielke-Baekler model, can be thought of as a generalization of General Relativity ($\alpha_3 = 0 = \alpha_4$) to a topological gravity theory in Riemann-Cartan spacetime[1]. This more general theory of gravity is interesting also because it admits a black hole solution with negative Riemannian curvature [139] which is essentially a BTZ black hole with torsion. The field equations obtained from the variation of (3.48) with respect to the dreibein and the spin connection read respectively

$$\begin{aligned} 2\alpha_1 R_a - \alpha_2 \epsilon_{abc} e^b e^c + 2\alpha_4 T^a &= 0, \\ 2\alpha_1 T_a + 2\alpha_3 R_a + \alpha_4 \epsilon_{abc} e^b e^c &= 0, \end{aligned} \tag{3.49}$$

which imply that the associated geometry has a constant curvature and torsion given by

$$\begin{aligned} R_a &= \frac{1}{2}(\gamma^2 + \frac{\sigma}{\ell^2})\epsilon_{abc} e^b e^c, \\ T_a &= -\gamma \epsilon_{abc} e^b e^c, \end{aligned} \tag{3.50}$$

where we have introduced the constants γ, ℓ which are expressed in terms of the couplings as

$$\gamma = \frac{\alpha_2 \alpha_3 + \alpha_1 \alpha_4}{2(\alpha_1^2 - \alpha_3 \alpha_4)}, \qquad \gamma^2 + \frac{\sigma}{\ell^2} = \frac{\alpha_1 \alpha_2 + \alpha_4^2}{\alpha_1^2 - \alpha_3 \alpha_4}, \tag{3.51}$$

as well as $\sigma = +1, -1, 0$. From (3.50), one can see that the parameter γ parametrizes the torsion, while ℓ parametrizes the radius curvature.

One can re express action (3.48) in terms of the parameter $\alpha_3 \equiv \mu$ and γ; it reads, in the case of a vanishing cosmological constant ($\ell \to \infty$)

$$I_G = \frac{k}{4\pi} \int 2\left(1 + \mu\gamma\right) R^a e_a + \gamma^2 \left(1 + \mu\frac{\gamma}{3}\right) \epsilon_{abc} e^a e^b e^c + \mu L(\omega) + \gamma\left(2 + \mu\gamma\right) T^a e_a. \tag{3.52}$$

3.5.2 $\mathcal{N} = 1$ supersymmetric extension

The locally supersymmetric extension of the Mielke-Baekler model was constructed in [123]. In the vanishing cosmological constant limit, the action with $\mathcal{N} = 1$ supersymmetry is given by

$$I_{(\mu,\gamma)} = I_G - \frac{k}{4\pi} \int \bar\psi \left(D + \frac{\gamma}{2} e^a \Gamma_a \right) \psi, \tag{3.53}$$

where I_G is the bosonic part given in (3.52). This action is invariant, up to a surface term, under the following local supersymmetry transformations

$$\delta e^a = \frac{1}{2}\bar\epsilon \Gamma^a \psi \;, \quad \delta \omega^a = \frac{1}{2}\gamma \bar\psi \Gamma^a \epsilon \;, \quad \delta \psi = D\epsilon + \frac{1}{2}\gamma e^a \Gamma_a \epsilon \;. \tag{3.54}$$

---

[1]Notice that this model differs from Topologically Massive Gravity (TMG) which, apart from the Lorentz-Chern-Simons form, requires the inclusion of a constraint term that fixes the torsion T^a to zero.

Note that in the case of $\mu = \gamma = 0$, the action (3.53) and the supersymmetry transformations (3.54) reduce to the standard ones, given by (3.5) and (3.7), respectively.

Remarkably, in spite of the presence of a volume term in (3.53), the theory can also be formulated in terms of a Chern-Simons action for the super-Poincaré group. This can be seen as follows. In terms of the shifted spin connection

$$\hat{\omega}^a \equiv \omega^a + \gamma e^a, \tag{3.55}$$

action (3.53) reads

$$I_{(\mu,\gamma)} = \frac{k}{4\pi} \int 2\hat{R}^a e_a + \mu L(\hat{\omega}) - \psi_\alpha \hat{D} \psi^\alpha , \tag{3.56}$$

where $\hat{D}$, $\hat{R}^a$, and $L(\hat{\omega})$ stand for the covariant derivative, the curvature two-form, and the Lorentz-Chern-Simons form constructed out from $\hat{\omega}^a$, respectively. Hence, up to a boundary term, the action can be written as

$$I[A] = \frac{k}{4\pi} \int \langle A, dA + \frac{2}{3} A^2 \rangle , \tag{3.57}$$

where now the gauge field is given by

$$A = e^a P_a + \hat{\omega}^a J_a + \psi^\alpha Q_\alpha , \tag{3.58}$$

and the nonvanishing components of the invariant nondegenerate bilinear form read

$$\langle P_a, J_b \rangle = \eta_{ab}, \quad \langle J_a, J_b \rangle = \mu \eta_{ab}, \quad \langle Q_\alpha, Q_\beta \rangle = C_{\alpha\beta} , \tag{3.59}$$

so that it reduces to the standard bracket in (3.4) in the case of $\mu = 0$. We will see now that the presence of the parity odd term turns on an additional central charge in the (super-)$\mathfrak{bms}_3$ algebra.

3.5.3 A new central charge in the (super-)$\mathfrak{bms}_3$ algebra

The asymptotic behavior of the gauge fields in this case is then proposed to be exactly of the same form as in eqs. (3.10), (3.11), which by virtue of (3.58), amounts just to modify the fall-off of the spin connection ω^a in the asymptotic region. This has to be so because the field equations now imply a nonvanishing torsion even in vacuum.

Therefore, the asymptotic symmetries are spanned by the same Lie-algebra valued parameter $\lambda = \lambda(T, Y, \mathcal{E})$ as in section 3.2 but, since the invariant form has been modified according to (3.59), the global charges acquire a correction, so that they now read

$$\mathcal{Q}[T, Y, \mathcal{E}] = -\frac{k}{4\pi} \int_0^{2\pi} [T\mathcal{M} + Y (\mathcal{J} + \mu\mathcal{M}) - 2\mathcal{E}\Psi] \, d\phi . \tag{3.60}$$

Consequently, once expanded in modes, the Poisson bracket algebra of the canonical generators are given by the minimal supersymmetric extension of the $\mathfrak{bms}_3$ algebra (3.21), but with an additional central charge[1],

$$c_1 = \mu \frac{3}{G}, \qquad c_2 = \frac{3}{G}. \tag{3.61}$$

---

[1]Notice that the μ here should not be confused with the coupling of TMG, usually also denoted μ; the constant μ here has dimensions of length.

3.6   Super-$\mathfrak{bms}_3$ invariant models

In this section, we want to complete the asymptotic analysis by constructing the two-dimensional super-$\mathfrak{bms}_3$ invariant action that is dual to the three-dimensional asymptotically flat supergravity. As we have seen in chapter 2, a prime example of duality between a three-dimensional and a two-dimensional theory is the relation between a Chern-Simons theory in the presence of a boundary and the associated chiral Wess-Zumino-Witten (WZW) model: on the classical level, the variational principles are equivalent as the latter is obtained from the former by solving the constraints in the action [42, 65, 64].

In Chern-Simons formulation, the role of the boundary is played by the non trivial fall-off conditions for the gauge fields. A suitable boundary term is required in order to make solutions with the prescribed asymptotics true extrema of the variational principle. Furthermore, the fall-off conditions lead to additional constraints that correspond to fixing a subset of the conserved currents of the WZW model [43, 66]. We detailed all the steps of this reduction in chapter 2 for the case of a negative cosmological constant. We saw that the associated reduced phase space description is given by a Liouville theory. For the (bosonic) case of vanishing cosmological constant, it was shown that the dual theory was given by a suitable flat limit of Liouville theory, which possesses $\mathfrak{bms}_3$ invariance [104, 68].

The purpose of this section is to extend the construction to $D = 3$ asymptotically *flat* $\mathcal{N} = 1$ supergravity, whose algebra of surface charges has been shown in the previous section to realize the centrally extended super-$\mathfrak{bms}_3$ algebra. As we will see, the resulting two-dimensional field theory admits a global super-$\mathfrak{bms}_3$ invariance. By construction, the associated algebra of Noether charges realizes (3.21) with the same values of the central charges. We will provide three equivalent descriptions of this theory: (i) a Hamiltonian description in terms of a constrained chiral WZW theory based on the three-dimensional super-Poincaré algebra, (ii) a Lagrangian formulation in terms of a gauged chiral WZW theory and (iii) a reduced phase space description that corresponds to a supersymmetric extension of flat Liouville theory.

In order to extend previous results not only to the supersymmetric case, but also in the purely bosonic sector, we will consider the more generic action of section 3.5, namely we will consider the inclusion of parity-odd terms. Indeed, we saw that this action suitably modifies the Poincaré current subalgebra, and consequently, turns on the additional central charge c_1 in the (super-)$\mathfrak{bms}$ algebra.

3.6.1   Chiral constrained super-Poincaré WZW theory

Solving the constraints in the action

We adopt here the Hamiltonian form of the Chern-Simons action (3.57), which is given, up to boundary terms and an overall sign which we change for later convenience, by

$$I_H[A] = -\frac{k}{4\pi} \int \langle \tilde{A}, du\dot{\tilde{A}} \rangle + 2\langle duA_u, \tilde{d}\tilde{A} + \tilde{A}^2 \rangle \,, \tag{3.62}$$

where $A = duA_u + \tilde{A}$.

One of the advantages of the gauge choice in (3.10), for which the dependence in the radial coordinate is completely absorbed by the group element h, is that the boundary can be assumed to be unique and located at an arbitrary fixed value of $r = r_0$. Hence, the boundary $\partial\mathcal{M}$ generically describes a two-dimensional timelike surface with the topology of a cylinder ($\mathbb{R} \times S^1$). We will also discard all holonomy terms. As a consequence, the resulting action principle at the boundary only captures the asymptotic symmetries of the original gravitational theory. Note also that positive orientation in the bulk is taken as $dud\phi dr$.

The boundary term in the variation of the Hamiltonian action is given by

$$-\frac{k}{2\pi}dud\tilde{}\langle A_u, \delta\tilde{A}\rangle. \tag{3.63}$$

By virtue of the boundary conditions (3.11), the components of the gauge field at the boundary fulfill[1]

$$\omega_\phi^a = e_u^a \quad , \quad \omega_u^a = 0 \quad , \quad \psi_u^+ = 0 = \psi_u^- \,, \tag{3.64}$$

which consist of our first set of boundary conditions. With these relations, we see that they are such that the boundary term (3.63) becomes integrable. Consequently, one finds that the improved action principle that has a true extremum when the equations of motion are satisfied is given by

$$I_I[A] = I_H[A] - \frac{k}{4\pi}\int_{\partial\mathcal{M}} dud\phi\, \omega_\phi^a \omega_{a\phi}\,. \tag{3.65}$$

In this action principle A_u are Lagrange multipliers, whose associated constraints are locally solved by $\tilde{A} = G^{-1}\tilde{d}G$ for some group element $G(u, r, \phi)$. Solving the constraints in the action yields

$$I = \frac{k}{4\pi}\left(\int_{\partial\mathcal{M}} dud\phi\,[\langle G^{-1}\partial_\phi G, G^{-1}\partial_u G\rangle - \omega_\phi^a \omega_{a\phi}] + \Gamma[G]\right)\,, \tag{3.66}$$

where

$$\Gamma[G] = \frac{1}{3}\int\langle G^{-1}dG, (G^{-1}dG)^2\rangle\,. \tag{3.67}$$

Equivalently, in terms of the gauge field components, the action can be conveniently written as

$$I = \frac{k}{4\pi}\left(\int_{\partial\mathcal{M}} dud\phi\,[\omega_\phi^a e_{au} + e_\phi^a \omega_{au} - \omega_\phi^a \omega_{a\phi} + \mu\omega_\phi^a \omega_{au} - \bar{\psi}_u\psi_\phi] + \Gamma[G]\right)\,, \tag{3.68}$$

with

$$\Gamma[G] = \frac{1}{6}\int(3\epsilon_{abc}e^a\omega^b\omega^c + \mu\epsilon_{abc}\omega^a\omega^b\omega^c - \frac{3}{2}\omega^a(C\Gamma_a)_{\alpha\beta}\psi^\alpha\psi^\beta)\,, \tag{3.69}$$

---

[1]From now on, we drop the hat on the shifted spin connection (3.55).

and the understanding that $A_\mu = G^{-1}\partial_\mu G$. Decomposing this connection according to eq. (3.9) allows one to rewrite this expression in terms of a 2 by 2 matrix trace, so that integrating by parts the first term in $\Gamma[G]$ gives

$$
\begin{aligned}
I = &\frac{k}{2\pi}\int_{\partial\mathcal{M}} du d\phi \operatorname{Tr}\Big[2\dot{\Lambda}\Lambda^{-1}(-\frac{\eta\bar{\eta}'}{4}+b') - (\Lambda'\Lambda^{-1})^2 + \mu\Lambda'\Lambda^{-1}\dot{\Lambda}\Lambda^{-1} + \frac{\eta'\dot{\bar{\eta}}}{2}\Big] \\
&+ \frac{\mu}{3}\int \operatorname{Tr}(d\Lambda\Lambda^{-1})^3 \,.
\end{aligned}
\tag{3.70}
$$

Furthermore, the boundary conditions (3.10), (3.11) imply that $\partial_\phi A_r = 0$, and hence $G = g(u,\phi)h(u,r)$. More precisely, since in the asymptotic region $h = e^{-rP_0}$, one obtains in particular that $\dot{h}(u,r_0) = 0$. The decomposition in (3.9) is then refined as

$$
\begin{aligned}
\Lambda &= \lambda(u,\phi)\,\varsigma(u,r)\,, \\
\eta &= \nu(u,\phi) + \lambda\varrho(u,r)\,, \\
b &= \alpha(u,\phi) + \frac{1}{4}\nu\bar{\varrho}\lambda^{-1} + \frac{1}{8}\bar{\varrho}\lambda^{-1}\nu\mathbf{1} + \lambda\beta(u,r)\lambda^{-1}\,,
\end{aligned}
\tag{3.71}
$$

where $\dot{\varsigma}(u,r_0) = \dot{\varrho}(u,r_0) = \dot{\beta}(u,r_0) = 0$. Therefore, up to a total derivative in u and ϕ, one finds that the action reduces to that of a chiral super-Poincaré Wess-Zumino-Witten theory,

$$
\begin{aligned}
I[\lambda,\alpha,\nu] = &\frac{k}{2\pi}\int du d\phi \operatorname{Tr}\Big[2\dot{\lambda}\lambda^{-1}\alpha' - (\lambda'\lambda^{-1})^2 + \mu\lambda'\lambda^{-1}\dot{\lambda}\lambda^{-1} + \frac{1}{2}\nu'\dot{\bar{\nu}} - \frac{1}{2}\dot{\lambda}\lambda^{-1}\nu\bar{\nu}'\Big] \\
&+ \frac{\mu}{3}\int \operatorname{Tr}(d\Lambda\Lambda^{-1})^3 \,.
\end{aligned}
\tag{3.72}
$$

The field equations are then obtained by varying (3.72) with respect to α, ν, λ, which gives

$$
\begin{aligned}
&(\dot{\lambda}\lambda^{-1})' = 0\,, \\
&D_u^{-\dot{\lambda}\lambda^{-1}}\nu' = 0\,, \\
&D_u^{-\dot{\lambda}\lambda^{-1}}\alpha' + (\mu\partial_u - \partial_\phi)(\lambda'\lambda^{-1}) - \frac{1}{4}\dot{\nu}\bar{\nu}' - \frac{1}{8}\bar{\nu}'\dot{\nu}\mathbf{1} + \frac{1}{4}\dot{\lambda}\lambda^{-1}\nu\bar{\nu}' + \frac{1}{8}\bar{\nu}'\dot{\lambda}\lambda^{-1}\nu\mathbf{1} = 0\,,
\end{aligned}
\tag{3.73}
$$

respectively. The general solution of these equations is given by

$$
\begin{aligned}
\lambda &= \tau(u)\kappa(\phi)\,, \\
\nu &= \tau(\zeta_1(u) + \zeta_2(\phi))\,, \\
\alpha &= \tau\Big(a(\phi) + \delta(u) + u\kappa'\kappa^{-1} - \mu[\ln\tau,\ln\kappa] + \frac{1}{4}\zeta_1\bar{\zeta}_2 + \frac{1}{8}\bar{\zeta}_2\zeta_1\mathbf{1}\Big)\tau^{-1}\,.
\end{aligned}
\tag{3.74}
$$

Symmetries of the chiral WZW model

By using the Polyakov-Wiegmann identities, the action (3.72) can be shown to be invariant under the gauge transformations

$$
\lambda \to \Xi(u)\lambda \quad,\quad \nu \to \Xi\nu \quad,\quad \alpha \to \Xi\alpha\Xi^{-1}\,.
\tag{3.75}
$$

Moreover, it is also invariant under the following global symmetries

$$
\begin{aligned}
\lambda \to \lambda \quad &, \quad \nu \to \nu \quad , \quad \alpha \to \alpha + \lambda \Sigma(\phi) \lambda^{-1} \,, \\
\lambda \to \lambda \Theta^{-1}(\phi) \quad &, \quad \nu \to \nu \quad , \alpha \to \alpha - u \lambda \Theta^{-1} \Theta' \lambda^{-1} \,, \\
\lambda \to \lambda \quad &, \nu \to \nu + \lambda \Upsilon(\phi) \quad , \quad \alpha \to \alpha + \frac{1}{4} \nu \bar{\Upsilon} \lambda^{-1} + \frac{1}{8} \bar{\Upsilon} \lambda^{-1} \nu \mathbf{1} \,,
\end{aligned}
\tag{3.76}
$$

whose associated infinitesimal transformations read

$$
\begin{aligned}
\delta_\sigma \lambda = 0 \quad &, \quad \delta_\sigma \nu = 0 \quad , \quad \delta_\sigma \alpha = \lambda \sigma(\phi) \lambda^{-1} \,, \\
\delta_\vartheta \lambda = -\lambda \vartheta(\phi) \quad &, \quad \delta_\vartheta \nu = 0 \quad , \quad \delta_\vartheta \alpha = -u \lambda \vartheta' \lambda^{-1} \,, \\
\delta_\gamma \lambda = 0 \quad &, \quad \delta_\gamma \nu = \lambda \gamma(\phi) \quad , \quad \delta_\gamma \alpha = \frac{1}{4} \nu \bar{\gamma} \lambda^{-1} + \frac{1}{8} \bar{\gamma} \lambda^{-1} \nu \mathbf{1} \,.
\end{aligned}
\tag{3.77}
$$

The Noether currents associated to a global symmetry, whose parameters are collectively denoted by X_1, generically read

$$
J_{X_1}^\mu = -k_{X_1}^\mu + \frac{\partial \mathcal{L}}{\partial_\mu \phi^i} \delta_{X_1} \phi^i,
\tag{3.78}
$$

with $\delta_{X_1} \mathcal{L} = \partial_\mu k_{X_1}^\mu$. Hence, in the case of global symmetries spanned by (3.77), the corresponding currents are given by

$$
J_\sigma^\mu = 2 \delta_0^\mu \mathrm{Tr}[\sigma P] \,, \qquad J_\vartheta^\mu = 2 \delta_0^\mu \mathrm{Tr}[\vartheta J] \,, \qquad J_\gamma^\mu = 2 \delta_0^\mu \mathrm{Tr}[\gamma Q],
\tag{3.79}
$$

where

$$
\begin{aligned}
P &= \frac{k}{2\pi} \lambda^{-1} \lambda' \,, \\
J &= -\frac{k}{2\pi} \left(\lambda^{-1} \alpha' \lambda - u (\lambda^{-1} \lambda')' + \mu \lambda^{-1} \lambda' - \frac{1}{4} \lambda^{-1} \nu \bar{\nu}' \lambda - \frac{1}{8} \bar{\nu}' \nu \mathbf{1} \right) \,, \\
Q &= \frac{k}{4\pi} \bar{\nu}' \lambda \,.
\end{aligned}
\tag{3.80}
$$

For the Noether $n-1$-forms $j_{X_1} = J_{X_1}^\mu (d^{n-1}x)_\mu$, the current algebra can then be worked out by applying a subsequent symmetry transformation δ_{X_2}, so that

$$
\delta_{X_2} j_{X_1} = j_{[X_1, X_2]} + K_{X_1, X_2} + \text{``trivial''} \,,
\tag{3.81}
$$

where $[\delta_{X_1}, \delta_{X_2}] = \delta_{[X_2, X_1]}$, and K_{X_1, X_2} denotes a possible field dependent central extension, and "trivial" stands for exact $n-1$ forms plus terms that vanish on-shell. Furthermore, general results guarantee that, in the Hamiltonian formalism, this computation corresponds to the Dirac bracket algebra of the canonical generators of the symmetries, i. e., $\delta_{X_2} J_{X_1}^0 = \{J_{X_1}^0, J_{X_2}^0\}^*$, see e.g. [140, 141, 142, 106]. Once applied to the components of the currents, given by

$$
P_a(\phi) = \mathrm{Tr}[\Gamma_a P] \quad , \quad J_a(\phi) = \mathrm{Tr}[\Gamma_a J] \quad , \quad Q_\alpha(\phi) = -\frac{k}{2\pi} \bar{\nu}'_\beta \lambda^\beta_\alpha \,,
\tag{3.82}
$$

this yields

$$
\begin{aligned}
&\{P_a(\phi), P_b(\phi')\}^* = 0\,, \\
&\{J_a(\phi), J_b(\phi')\}^* = \epsilon_{abc}J^c\delta(\phi - \phi') - \mu\frac{k}{2\pi}\eta_{ab}\partial_\phi\delta(\phi - \phi')\,, \\
&\{J_a(\phi), P_b(\phi')\}^* = \epsilon_{abc}P^c\delta(\phi - \phi') - \frac{k}{2\pi}\eta_{ab}\partial_\phi\delta(\phi - \phi')\,, \\
&\{P_a(\phi), Q_\alpha(\phi')\}^* = 0\,, \\
&\{J_a(\phi), Q_\alpha(\phi')\}^* = \frac{1}{2}(Q\Gamma_a)_\alpha\delta(\phi - \phi')\,, \\
&\{Q_\alpha(\phi), Q_\beta(\phi')\}^* = -\frac{1}{2}(C\Gamma^a)_{\alpha\beta}P_a\delta(\phi - \phi') - \frac{k}{2\pi}C_{\alpha\beta}\partial_\phi\delta(\phi - \phi')\,,
\end{aligned}
\tag{3.83}
$$

which is the affine extension of the super-Poincaré algebra (3.2).

3.6.2   Super-$\mathfrak{bms}_3$ algebra from a modified Sugawara construction

In order to recover the super-$\mathfrak{bms}_3$ algebra (3.21) from the affine extension of the super-Poincaré algebra in (3.83), it can be seen that the standard Sugawara construction has to be slightly improved. Indeed, let us consider bilinears made out of the currents components P_a, J_a, Q_α, given by

$$
\begin{aligned}
&\mathcal{H} = \frac{\pi}{k}P^a P_a\,, \qquad \mathcal{P} = -\frac{2\pi}{k}J^a P_a + \mu\mathcal{H} + \frac{\pi}{k}Q_\alpha C^{\alpha\beta}Q_\beta, \\
&\mathcal{G} = 2^{3/4}\frac{\pi}{k}\left(P_2\,Q_+ + \sqrt{2}\,P_0\,Q_-\right),
\end{aligned}
\tag{3.84}
$$

for which the current algebra (3.83) implies

$$
\begin{aligned}
&\{\mathcal{H}(\phi), P_a(\phi')\}^* = 0\,, \qquad \{\mathcal{P}(\phi), P_a(\phi')\}^* = P_a(\phi)\delta'(\phi - \phi')\,, \\
&\{\mathcal{H}(\phi), J_a(\phi')\}^* = -P_a(\phi)\delta'(\phi - \phi')\,, \qquad \{\mathcal{P}(\phi), J_a(\phi')\}^* = J_a(\phi)\delta'(\phi - \phi')\,, \\
&\{\mathcal{H}(\phi), Q_\alpha(\phi')\}^* = 0\,, \qquad \{\mathcal{P}(\phi), Q_\alpha(\phi')\}^* = Q_\alpha(\phi)\delta'(\phi - \phi')\,, \\
&\{\mathcal{G}(\phi), P_a(\phi')\}^* = 0\,, \\
&\{\mathcal{G}(\phi), J_a(\phi')\}^* = -\frac{\pi}{2^{1/4}k}(\epsilon_{abc}(Q\Gamma^b)_+P^c + P_aQ_+)\delta(\phi - \phi') \\
&\qquad\qquad\qquad - \delta'(\phi - \phi')\frac{1}{2^{1/4}}(Q\Gamma_a)_+(\phi')\,, \\
&\{\mathcal{G}(\phi'), Q_\alpha(\phi')\}^* = -\frac{1}{2^{1/4}}\mathcal{H}C_{\alpha+}\delta(\phi - \phi') + \delta'(\phi - \phi')\frac{1}{2^{1/4}}(C\Gamma_a)_{\alpha+}P^a(\phi)\,.
\end{aligned}
\tag{3.85}
$$

When expressed in terms of modes, the algebra of generators $\mathcal{H}$, $\mathcal{P}$ is found to correspond to the pure $\mathfrak{bms}_3$ algebra without central extensions, i.e., the bosonic part of (3.21) with $c_1 = 0 = c_2$. This does however not hold for the mode expansion of the full set $\mathcal{H}$, $\mathcal{P}$, $\mathcal{G}$ whose algebra disagrees with the non-centrally extended super-$\mathfrak{bms}_3$ algebra given in (3.21). It reflects the fact that the non-constrained super-WZW model (3.72) is invariant under global $\mathfrak{bms}_3$ transformations, but not under the full super-$\mathfrak{bms}_3$ symmetries, in

the sense that there are no (obvious) superpartners to $\mathcal{H}$, $\mathcal{P}$ that would close with them according to the (non centrally extended) super-𝔟𝔪𝔰 algebra (see [66] for an analogous discussion in the case of the superconformal algebra).

According to the fall-off of the gauge field in (3.11), the remaining boundary conditions that have to be taken into account are

$$\omega_\phi^1 = 1 \quad , \quad e_\phi^1 = 0 \quad , \quad \psi_\phi^- = 0 \,. \tag{3.86}$$

This second set implies, using the decomposition (3.9),

$$[\lambda^{-1}\lambda']^1 = 1 \,, \qquad [\lambda^{-1}\nu']^- = 0 \,, \qquad [\lambda^{-1}(-\frac{1}{4}\nu\bar{\nu}' - \frac{1}{8}\bar{\nu}'\nu\mathbf{1} + \alpha')\lambda]^1 = 0. \tag{3.87}$$

In terms of the currents, this amounts to imposing the following first class constraints

$$P_0 = \frac{k}{2\pi} \quad , \quad J_0 = -\frac{\mu k}{2\pi} \quad , \quad Q_+ = 0 \,. \tag{3.88}$$

The super-𝔟𝔪𝔰₃ invariance of our model with the correct values of the central charges is recovered only once the constraints (3.88) are imposed. The generators of super-𝔟𝔪𝔰₃ symmetry in the constrained theory are given by

$$
\begin{aligned}
\tilde{\mathcal{H}} &= \mathcal{H} + \partial_\phi P_2, \\
\tilde{\mathcal{P}} &= \mathcal{P} - \partial_\phi J_2, \\
\tilde{\mathcal{G}} &= \mathcal{G} + 2^{3/4}\partial_\phi Q_+(\phi),
\end{aligned}
\tag{3.89}
$$

which are representatives that commute with the first class constraints (3.88), on the surface defined by these constraints. Furthermore, on this surface, the Dirac brackets of the generators are given by

$$
\begin{aligned}
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{H}}(\phi')\}^* &= 0 \,, \\
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{P}}(\phi')\}^* &= (\tilde{\mathcal{H}}(\phi) + \tilde{\mathcal{H}}(\phi'))\partial_\phi\delta(\phi - \phi') - \frac{k}{2\pi}\partial_\phi^3\delta(\phi - \phi') \,, \\
\{\tilde{\mathcal{P}}(\phi), \tilde{\mathcal{P}}(\phi')\}^* &= (\tilde{\mathcal{P}}(\phi) + \tilde{\mathcal{P}}(\phi'))\partial_\phi\delta(\phi - \phi') - \frac{\mu k}{2\pi}\partial_\phi^3\delta(\phi - \phi') \,, \\
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{G}}(\phi')\}^* &= 0 \,, \\
\{\tilde{\mathcal{P}}(\phi), \tilde{\mathcal{G}}(\phi')\}^* &= (\tilde{\mathcal{G}}(\phi) + \frac{1}{2}\tilde{\mathcal{G}}(\phi'))\partial_\phi\delta(\phi - \phi') \,, \\
\{\tilde{\mathcal{G}}(\phi), \tilde{\mathcal{G}}(\phi')\}^* &= \tilde{\mathcal{H}}(\phi)\delta(\phi - \phi') - \frac{k}{\pi}\partial_\phi^2\delta(\phi - \phi') \,,
\end{aligned}
\tag{3.90}
$$

so that, once expanded in modes according to

$$\mathcal{P}_m = \int_0^{2\pi} d\phi\, e^{im\phi}\tilde{\mathcal{H}} \quad , \quad \mathcal{J}_m = \int_0^{2\pi} d\phi\, e^{im\phi}\tilde{\mathcal{P}} \quad , \quad \mathcal{Q}_m = \int_0^{2\pi} d\phi\, e^{im\phi}\tilde{\mathcal{G}} \,, \tag{3.91}$$

the super-𝔟𝔪𝔰₃ algebra (3.21) with central charges given in (3.61) is recovered.

3.6.3   Reduced super-Liouville-like theory

In order to obtain the reduced phase space description of the action (3.72) on the constraint surface defined by (3.88), it is useful to decompose the fields according to

$$\lambda = e^{\sigma\Gamma_1/2}e^{-\varphi\Gamma_2/2}e^{\tau\Gamma_0} \quad , \quad \alpha = \frac{\eta}{2}\Gamma_0 + \frac{\theta}{2}\Gamma_2 + \frac{\zeta}{2}\Gamma_1 \, , \tag{3.92}$$

where $\sigma, \varphi, \tau, \eta, \theta, \zeta$ stand for functions of u, ϕ. The constraints (3.88) then become

$$\begin{aligned}
\sigma' &= e^{\varphi} \, , \\
\zeta' &= \mu(e^{\varphi} - \sigma') + \frac{1}{2}\sigma^2\eta' + \sigma\theta' \, , \\
\nu^{-\prime} &= \frac{1}{\sqrt{2}}\sigma\nu^{+\prime} \, ,
\end{aligned} \tag{3.93}$$

and hence, by virtue of (3.92) and (3.93), the reduced chiral super-WZW action (3.72) is given by

$$I_R = \frac{k}{4\pi}\int dud\phi \left[\xi'\dot{\varphi} - \varphi'^2 + \mu\varphi'\dot{\varphi} + \frac{1}{\sqrt{2}}\chi\dot{\chi} \right] \, , \tag{3.94}$$

where $\xi := -2(\theta + \eta\sigma) + \frac{1}{2}(\nu^-\nu^+)$, and $\chi := e^{\varphi/2}\nu^+$.

It is worth noting that, in the case of $\mu = 0$, and turning off the supersymmetric field, one consistently recovers the (centrally extended) $\mathfrak{bms}_3$ invariant action

$$I = \frac{k}{4\pi}\int dud\phi \left(\xi'\dot{\varphi} - \varphi'^2 \right) \tag{3.95}$$

obtained in [104]. The latter is related to a flat limit of Liouville theory in the following way: One starts with the Hamiltonian form of the Liouville action

$$S_H = \int dud\phi \left(\pi\dot{\varphi} - \frac{1}{2}\pi^2 - \frac{1}{2\ell^2}\varphi'^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi} \right) \, , \tag{3.96}$$

and then rescale the fields as $\varphi = \ell\Phi$, $\pi = \Pi/\ell$. Taking the limit $\ell \to \infty$, while keeping $\beta = \gamma\ell$, $\nu = \mu\ell^2$ fixed, one obtains the $\mathfrak{bms}$ Liouville action [68]

$$S = \int dud\phi \left(\Pi\dot{\Phi} - \frac{1}{2}\Phi'^2 - \frac{\nu}{2\beta^2}e^{\beta\Phi} \right) \, . \tag{3.97}$$

Finally, one goes from the flat Liouville action (3.97) to (3.95) by performing the Bäcklund transformation

$$\begin{aligned}
\beta\Phi &= 2\varphi - 2\ln\sigma + \ln\frac{4}{\nu} \, , \\
\beta\Pi &= \xi' - (\ln\sigma)'\xi \, ,
\end{aligned} \tag{3.98}$$

with $\beta^2 = 32\pi G$ and $\sigma' = e^{\varphi}$.

For completeness, let us now give the expressions for the super-$\mathfrak{bms}_3$ generators (3.89) which now reduce to

$$\tilde{\mathcal{H}} = \frac{k}{4\pi} \left(\varphi'^2 - 2\varphi'' \right)$$

$$\tilde{\mathcal{P}} = \frac{k}{4\pi} \left(\xi'\varphi' - \xi'' + \frac{1}{\sqrt{2}} \chi\chi' \right) + \mu\tilde{\mathcal{H}}, \tag{3.99}$$

$$\tilde{\mathcal{G}} = 2^{1/4} \frac{k}{4\pi} \left(\frac{1}{2}\varphi'\chi - \chi' \right),$$

and generate the following transformations

$$\delta\varphi = Y\varphi' + Y',$$

$$\delta\xi = 2f\varphi' + \xi'Y + 2f' - 2^{1/4}\epsilon\chi, \tag{3.100}$$

$$\delta\chi = Y\chi' + \frac{1}{2}Y'\chi + 2^{-1/4}\epsilon\varphi' + 2^{3/4}\epsilon',$$

with $f = T(\phi) + uY'$, $Y = Y(\phi)$, and $\epsilon = \epsilon(\phi)$. One can check, and this is the case by construction, that the super-Liouville-like theory (3.94) is invariant under (3.100), and that the mode expansion of the algebra of Noether charges is again given by (3.21) and (3.61). Also, one can check that applying transformations (3.100) to the generators (3.89) reproduces the transformation laws (3.17), where $\tilde{\mathcal{H}} = \frac{k}{4\pi}\mathcal{M}$, $\tilde{\mathcal{P}} = \frac{k}{4\pi}\mathcal{J}$, $\tilde{\mathcal{G}} = \frac{k}{4\pi}\Psi$, as it should.

3.6.4   Gauged chiral super-WZW model

The super-Liouville-like action (3.94), that has been shown to be equivalent to the chiral super-WZW model (3.72) on the constraint surface given by (3.88), can also be described through a gauged chiral super-WZW model. Here we follow the procedure given in [73], recalled in Appendix B, where it is shown that Toda theories (and Liouville theories as a particular case) can be written as gauged WZW models based on a Lie group G. The action is endowed with additional terms involving the currents linearly coupled to some gauge fields that belong to the adjoint representation of the subgroups of G generated by the step operators associated to the positive and negative roots.

Generalizing the analysis detailed in Appendix B for the bosonic case (see section B.3), we consider the following action principle

$$I[\lambda, \alpha, \nu, A_\mu, \bar{\Psi}] = I[\lambda, \alpha, \nu]$$

$$+ \frac{k}{\pi} \int du d\phi \, \text{Tr} \big[A_u (\lambda^{-1}\alpha'\lambda - u(\lambda^{-1}\lambda')' - \frac{1}{4}\lambda^{-1}\nu\bar{\nu}'\lambda - \frac{1}{8}\bar{\nu}'\nu\mathbf{1})$$

$$+ \tilde{A}_u (\lambda^{-1}\lambda') - \mu_M \tilde{A}_u + (\frac{1}{4}\lambda^{-1}\nu')\bar{\Psi} \big], \tag{3.101}$$

where $I[\lambda, \alpha, \nu]$ is the flat chiral super-Poincaré WZW action (3.72). Here $A_u, \tilde{A}_u$ are along Γ_0, and $\mu_M := \mu\Gamma_1$ with μ an arbitrary constant, while the fermionic gauge field $\bar{\Psi}$ fulfills $[\bar{\Psi}]_+ = 0$.

One can then show that the action (3.101) is invariant (up to boundary terms) under the transformations given in (3.77), where a subset of the symmetries has been gauged by allowing for an arbitrary u dependence of the part of σ, ϑ that belongs to the subspace generated by Γ_0, of the fermionic parameters that belong to the subspace defined by $[\bar{\gamma}]_+ = 0$, $[\lambda\gamma]^- = 0$ and the non-trivial transformations for the gauge fields are

$$\delta_\sigma \tilde{A}_u = - \left(\dot{\sigma} + [A_u, \sigma] \right) , \ \delta_\gamma \bar{\Psi} = -\partial_u \bar{\gamma} \, .$$
$$\delta_\vartheta A_u = -(\dot{\vartheta} + [A_u, \vartheta]) , \ \delta_\vartheta \tilde{A}_u = -[\tilde{A}_u, \vartheta].$$

(3.102)

Therefore, the reduced theory described by the action in (3.94) is equivalent to the one in (3.101), which corresponds to a WZW model in which the subgroup generated by the first class constraints has been gauged. Indeed, the gauge fields A_u, $\tilde{A}_u$ and Ψ act as Lagrange multipliers for these currents, so that the variation of the action with respect to these non-propagating fields sets them to constants. In other words, solving the algebraic field equations for the gauge fields into the action amounts to imposing the first class constraints (3.88), which shows the equivalence of both descriptions.

3.7   Conclusion and outlook

In this chapter, we have studied the asymptotic dynamics of $\mathcal{N} = 1$ three-dimensional flat supergravity by imposing a consistent set of asymptotic conditions and shown that they are governed by the super-$\mathfrak{bms}_3$ algebra, the minimal supersymmetric extension of the $\mathfrak{bms}_3$ algebra. We proved that the energy was manifestly bounded from below, with the ground state given by the null orbifold or Minkowski spacetime for periodic, respectively antiperiodic boundary conditions on the gravitino. These results were then related to the corresponding ones in AdS$_3$ supergravity by a suitable flat limit. We then generalized our analysis to inclusion of parity odd terms for which the Poisson algebra of canonical generators was shown to form a representation of the same algebra but with an additional central charge. Finally, we constructed two-dimensional super-$\mathfrak{bms}_3$ invariant theories that describe the dual dynamics of three-dimensional asymptotically flat $\mathcal{N} = 1$ supergravity in three different ways: first a Hamiltonian description in terms of a constrained chiral WZW theory based on the three-dimensional super-Poincaré algebra, secondly a reduced phase space description that corresponds to a supersymmetric extension of flat Liouville theory, and finally a Lagrangian formulation in terms of a gauged chiral WZW theory.

A natural generalization of this work would be to consider the extended case of $\mathcal{N}$ supersymmetries. This analysis was already carried out in the case of AdS supergravities in [66], and the asymptotic symmetries were shown to be described by an extended version of the superconformal algebra. One expects that a similar treatment would be applicable in the case of a vanishing cosmological constant.

Finally, it would be interesting to extend this work to the four-dimensional case. In four dimensions, the Chern-Simons formulation is no longer available, and thus we will have to reformulate the problem in the first order formalism. Having control on the three-dimensional case, one should be able to suitably generalize the boundary conditions to the four-dimensional case and find the asymptotic symmetry algebra formed by the asymptotic

vector fields. Obtaining the super-$\mathfrak{bms}_4$ algebra could be relevant in the context of a Weinberg's soft gravitino theorem. Indeed, in the bosonic case, Ward identities associated to the BMS supertranslation symmetry were shown to be equivalent to Weinberg's soft graviton theorem [24].

CHAPTER 4

Asymptotic symmetries on the black hole horizon

Since the discovery that black holes are thermal objects, an outstanding open question has been whether black hole thermodynamics could be explained by means of a microscopical description of states. Not only we still ignore what are these degrees of freedom that account for the macroscopic entropy of the black hole, but neither do we know where these degrees of freedom would be located. For the three-dimensional black hole, we have seen in section 2.9.1 that the number of the microscopic states in the corresponding quantum theory is expressed in terms of the central charge, and that the Cardy formula correctly reproduces the Bekenstein–Hawking formula for the entropy. This method is based on the asymptotic symmetries for Anti-de Sitter spacetimes that arise at spatial infinity; a natural and old question is instead to consider asymptotic symmetries *on* black hole horizons (see for instance [143, 144]), raising the hope to establish a universal method to reproduce the Bekenstein-Hawking formula. Notice that it may seem unappropriated to use the word "asymptotics" symmetries in this context since a black hole horizon is generically located at a finite distance from the exterior region; however, we will keep this nomenclature in light of their similarity with the usual asymptotic symmetries.

The main motivation of this chapter to study the asymptotic symmetries on the black hole horizon comes from a recent claim by Hawking, Perry, and Strominger according to which non-extremal stationary black holes exhibit supertranslations symmetries in the near-horizon region, and it was proposed in [27] that this observation could contribute to solve the information paradox for black holes. Let us remind here that supertranslations arise in the study of asymptotically flat space-times at null infinity, whose algebra is the Bondi-Metzner-Sachs ($\mathfrak{bms}$) algebra we have introduced previously; supertranslations extend the usual translations to a infinite-dimensional algebra. Furthermore, in addition to these supertranslations, we have seen that the $\mathfrak{bms}$ algebra was further extended in Barnich and Troessaert to include superrotations and central extensions [21, 22, 23, 145]. In presence of black holes, besides the null infinity region, there exists a second co-dimension 1 null hypersurface near which the geometry is flat: the black hole event horizon. Therefore, a natural question is whether the features associated to holography, such as the enhanced $\mathfrak{bms}$ symmetry, also appear in the near-horizon geometry of black holes.

In this chapter, we will show that, for an adequate choice of boundary conditions, the nearby region to the horizon of a stationary non-extremal black hole exhibits a generalization of supertranslations, including a semi-direct sum with superrotations, represented by Virasoro algebra. In this sense, we will see that both supertranslations and superrotations arise close to the horizon. Interestingly enough, this particular extension differs from the extended 𝔟𝔪𝔰 obtained Barnich and Troessaert at null infinity.

This chapter is based on the publication [146] and contains the following original results: We start the asymptotic analysis in section 4.1 with the simplified case of three spacetime dimensions. This allows us to identify the appropriate boundary conditions at the horizon and construct a family of exact solutions satisfying them. This family includes, as a particular case, the Bañados-Teitelboim-Zanelli (BTZ) black hole. We compute the algebra obeyed by the asymptotic Killing vectors and show that they expand supertranslations in semi-direct sum with superrotations. The charges associated to such asymptotic symmetries are shown to expand the same algebra, and by evaluating them on the BTZ solution, we verify that they correspond to the angular momentum and the entropy of the black hole. We follow the same strategy in section 4.2, where we address the four-dimensional case. We demonstrate that the symmetry group generated by these charges correspond to two copies of Virasoro algebra and two sets of supertranslations. Finally, we show that the zero mode conserved quantities of Kerr black hole coincide with the entropy and the angular momentum.

4.1   The near-horizon geometry of three-dimensional black holes

We are interested in studying the symmetries preserved by stationary non-extremal black hole metrics close to an event horizon, first in three dimensions and then we move to the four dimensional case.

The near-horizon geometry of three-dimensional black holes can be expressed using Gaussian null coordinates

$$ds^2 = f dv^2 + 2k dv d\rho + 2h dv d\phi + R^2 d\phi^2, \tag{4.1}$$

where $v \in \mathbb{R}$ represents the retarded time, $\rho \geq 0$ is the radial distance to the horizon and ϕ is the angular coordinate of period 2π. Functions f, k, h, and R are demanded to obey the following fall-off conditions close to $\rho = 0$

$$\begin{aligned}
f &= -2\kappa\rho + \mathcal{O}(\rho^2), \\
k &= 1 + \mathcal{O}(\rho^2), \\
h &= \theta(\phi)\rho + \mathcal{O}(\rho^2), \\
R^2 &= \gamma(\phi)^2 + \lambda(v,\phi)\rho + \mathcal{O}(\rho^2),
\end{aligned} \tag{4.2}$$

where $\mathcal{O}(\rho^2)$ stands for functions of v and ϕ that vanish at short ρ equally or faster than ρ^2, consistent with the near-horizon approximation. The metric components $g_{\rho\rho}$ and $g_{\rho\phi}$,

which do not appear in (4.1), are supposed to be $\mathcal{O}(\rho^2)$. Functions θ, λ and γ are arbitrary, the latter describing the shape of the horizon. As we will see, boundary conditions (4.2) break Poincaré symmetry.

The constant κ corresponds to the black hole surface gravity. Our boundary conditions assume that κ is a *fixed constant without variation*, i.e., they describe the spectrum of black holes at fixed Hawking temperature $T = \kappa/(2\pi)$. In the case of non-extremal BTZ black hole, this is given by

$$\kappa = \frac{r_+^2 - r_-^2}{\ell^2 r_+}, \tag{4.3}$$

where r_+ and r_- are the outer and inner horizons.

The asymptotic Killing vectors χ preserving the above asymptotic boundary conditions are (see Appendix C for the derivation)

$$\begin{aligned}
\chi^v &= T(\phi) + \mathcal{O}(\rho^3), \\
\chi^\rho &= \frac{\theta}{2\gamma^2} T'(\phi)\rho^2 + \mathcal{O}(\rho^3), \\
\chi^\phi &= Y(\phi) - \frac{1}{\gamma^2} T'(\phi)\rho + \frac{\lambda}{2\gamma^4} T'(\phi)\rho^2 + \mathcal{O}(\rho^3),
\end{aligned} \tag{4.4}$$

where $T(\phi)$ and $Y(\phi)$ are arbitrary functions and the prime stands for derivative with respect to ϕ. Under such transformation, the arbitrary functions $\gamma(\phi)$ and $\theta(\phi)$ transform as

$$\delta_\chi \theta = (\theta Y)' - 2\kappa T', \quad \delta_\chi \gamma = (\gamma Y)'. \tag{4.5}$$

From (4.4), we notice that the asymptotic Killing vectors depend on fields defined on the metric. Accordingly, the algebra spanned by Lie brackets does not close. However, following [22, 147], if we introduce a modified version of Lie brackets

$$[\chi_1, \chi_2]_M = [\chi_1, \chi_2] - \delta_{\chi_1}\chi_2(g) + \delta_{\chi_2}\chi_1(g), \tag{4.6}$$

which takes into account the metric dependence of the asymptotic Killing vectors, the algebra does close. Indeed, the rationale for definition (4.6) is as follows: the ∂_μ derivative in the commutator acts on the fields appearing in the symmetry parameters. These contributions are then canceled by the two additional terms[1]. Equipped with this modified bracket, we find that the algebra of the asymptotic Killing vectors is given by

$$[\chi(T_1, Y_1), \chi(T_2, Y_2)]_M = \chi(T_{12}, Y_{12}), \tag{4.7}$$

where

$$\begin{aligned}
T_{12} &= Y_1 T_2' - Y_2 T_1', \\
Y_{12} &= Y_1 Y_2' - Y_2 Y_1'.
\end{aligned} \tag{4.8}$$

---

[1]Notice that the signs of this terms, namely $[\chi_1, \chi_2]_M = [\chi_1, \chi_2] \mp \delta_{\chi_1}\chi_2(g) \pm \delta_{\chi_2}\chi_1(g)$, are fixed with the convention $\delta_\xi g_{\mu\nu} = \pm \mathcal{L}_\xi g_{\mu\nu}$.

By defining Fourier modes, $T_n = \chi(e^{in\phi}, 0)$ and $Y_n = \chi(0, e^{in\phi})$ we find

$$
\begin{aligned}
i[Y_m, Y_n] &= (m - n)Y_{m+n}, \\
i[Y_m, T_n] &= -nT_{m+n}, \\
i[T_m, T_n] &= 0.
\end{aligned}
\tag{4.9}
$$

This is a semi-direct sum of a Witt algebra generated by Y_n with an abelian current T_n. The set of generators Y_{-1}, Y_0, Y_1 and T_0 form a $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ subalgebra.

The T_n generator is a supertranslation associated to the symmetry,

$$
v \to v + T(\phi),
\tag{4.10}
$$

already observed by Hawking [148] in four dimensions. In the current analysis, we have extended this symmetry by adding a vector field Y_n which is the responsible of generating superrotations

$$
\phi \to \phi + Y(\phi),
\tag{4.11}
$$

on the circle of the horizon geometry.

Transformations (4.4) have associated conserved charges at the horizon $\rho = 0$. When considering three-dimensional Einstein gravity, these can be calculated in the covariant approach (see section 5.1.2 for more details about this method to compute surface charges), yielding the charges

$$
Q(\chi) = \frac{1}{16\pi G} \int_0^{2\pi} d\phi \left[2\kappa T(\phi)\gamma(\phi) - Y(\phi)\theta(\phi)\gamma(\phi) \right].
\tag{4.12}
$$

Their Poisson bracket algebra can be computed by noticing that, canonically, these charges generate the transformations (4.5), i.e., $\{Q(\chi_1), Q(\chi_2)\} = \delta_{\chi_2} Q(\chi_1)$. In Fourier modes, $\mathcal{T}_n = Q(T = e^{in\phi}, Y = 0)$ and $\mathcal{Y}_n = Q(T = 0, Y = e^{in\phi})$, the algebra spanned by $\mathcal{T}_n$ and $\mathcal{Y}_n$ is isomorphic to (4.9), with no central extensions:

$$
\begin{aligned}
i\{\mathcal{Y}_m, \mathcal{Y}_n\} &= (m - n)\mathcal{Y}_{m+n}, \\
i\{\mathcal{Y}_m, \mathcal{T}_n\} &= -n\mathcal{T}_{m+n}, \\
i\{\mathcal{T}_m, \mathcal{T}_n\} &= 0.
\end{aligned}
\tag{4.13}
$$

It is worthwhile noticing that by defining the generator

$$
\mathcal{P}_n = \sum_{k \in \mathbb{Z}} : \mathcal{T}_k \mathcal{T}_{n-k} :,
\tag{4.14}
$$

with : : the normal ordering, the algebra (4.13) becomes $\mathfrak{bms}_3$[1]

$$
\begin{aligned}
i\{\mathcal{Y}_m, \mathcal{Y}_n\} &= (m - n)\mathcal{Y}_{m+n}, \\
i\{\mathcal{Y}_m, \mathcal{P}_n\} &= (m - n)\mathcal{P}_{m+n}, \\
i\{\mathcal{P}_m, \mathcal{P}_n\} &= 0.
\end{aligned}
\tag{4.15}
$$

Therefore, although our asymptotic symmetries does not contain a Poincaré subgroup, the full $\mathfrak{bms}$ symmetry is recovered by means of the above Sugawara construction.

---

[1]Notice that this relation between $\mathfrak{bms}_3$ algebra and the enveloping algebra of $\hat{u}(1)$ current algebra has been independently found in [149].

Exact solution

Three-dimensional Einstein gravity in presence of a negative cosmological term allows us to find an exact solution satisfying the above asymptotic boundary conditions, including BTZ black hole as a particular case. Its line element is (4.1), where the functions read

$$
\begin{aligned}
f &= -2\kappa\rho + \rho^2 \left(\frac{\theta(\phi)^2}{4\gamma(\phi)^2} - \frac{1}{\ell^2} \right), \\
k &= 1, \\
h &= \theta(\phi)\rho + \rho^2 \frac{\theta(\phi)}{4\gamma(\phi)^2}\lambda(\phi), \\
R &= \gamma(\phi) + \rho\frac{\lambda(\phi)}{2\gamma(\phi)},
\end{aligned}
\tag{4.16}
$$

and where λ is defined by

$$
\kappa\lambda(\phi) = \theta'(\phi) - \frac{1}{2}\theta(\phi)^2 + \frac{2}{\ell^2}\gamma(\phi)^2 - \theta(\phi)\frac{\gamma'(\phi)}{\gamma(\phi)}.
\tag{4.17}
$$

$\theta(\phi)$ and $\gamma(\phi)$ are arbitrary functions, and ℓ stands for the AdS radius. The BTZ black hole (see section (2.2)) is obtained by making the choice $\theta(\phi) = 2r_-/\ell$ and $\gamma(\phi) = r_+$, while choosing κ as (4.3). Notice that a solution similar to (4.16) was presented in [150], although with a different boundary condition on the function R^2.

It is interesting to study the special case $\kappa = 0$ and $\theta = 2\gamma/\ell$. For these values, the metric acquires the form

$$
ds^2 = 2dvd\rho + \frac{4}{\ell}\rho\gamma(\phi)dvd\phi + \gamma(\phi)^2 d\phi^2,
\tag{4.18}
$$

which has been found recently in the context of near-horizon geometries of three-dimensional extremal black holes [151]. Note that the remaining symmetry algebra is just one copy of Virasoro.

When taking the flat limit $\ell \to \infty$, solution (4.16) also solves Einstein equations without cosmological constant. After choosing $\kappa = -J^2/2r_H^3$, its zero mode solution, i.e. $\theta = J/r_H$ and $\gamma = r_H$, corresponds to a flat cosmology with horizon radius r_H [152].

The charges associated to solution (4.16) are given by (4.12). Evaluating for the case of the BTZ black hole, they read

$$
\mathcal{T}_n = \frac{\kappa r_+}{4G}\delta_{n,0}, \quad \mathcal{Y}_n = -\frac{r_+ r_-}{4G\ell}\delta_{n,0}.
\tag{4.19}
$$

Hence, the charge associated to time translations $T_0 = \partial_v$ is the product of the black hole entropy $S = \pi r_+/(2G)$ and its temperature $T = \kappa/(2\pi)$. This means that the particular charge $\mathcal{T}_0$, when varying the configuration space by fixing the temperature, corresponds to the entropy of the black hole. On the other hand, the charge associated to rotations along $Y_0 = \partial_\phi$ coincides exactly with the angular momentum.

4.2 Four-dimensional analysis

It is possible to extend the analysis of the first section to four dimensions. A suitable generalization of (4.1) is given by

$$ds^2 = f dv^2 + 2k dv d\rho + 2g_{vA} dv dx^A + g_{AB} dx^A dx^B, \tag{4.20}$$

where coordinates x^A parameterize the induced surface at the horizon. The fall-off conditions on the fields as $\rho \to 0$ are

$$\begin{aligned}
f &= -2\kappa\rho + \mathcal{O}(\rho^2), \\
k &= 1 + \mathcal{O}(\rho^2), \\
g_{vA} &= \rho\theta_A + \mathcal{O}(\rho^2), \\
g_{AB} &= \Omega\gamma_{AB} + \rho\lambda_{AB} + \mathcal{O}(\rho^2),
\end{aligned} \tag{4.21}$$

while components $g_{\rho A}$ and $g_{\rho\rho}$ decay as $\mathcal{O}(\rho^2)$ close to the horizon. Here, θ_A and Ω are functions of the coordinates x^A, $\lambda^{AB} = \lambda^{AB}(v, x^A)$ and γ_{AB} is chosen to be the metric of the two-sphere. It is convenient to use stereographic coordinates $x^A = (\zeta, \bar{\zeta})$ on γ_{AB}, in such a way that

$$\gamma_{AB} dx^A dx^B = \frac{4}{(1 + \zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}. \tag{4.22}$$

By a computation similar to the three-dimensional case, we find that the set of asymptotic conditions is preserved by the following vector fields

$$\begin{aligned}
\chi^v &= T(\zeta, \bar{\zeta}) + \mathcal{O}(\rho^3), \\
\chi^\rho &= \frac{\rho^2}{2\Omega}\theta_A \partial^A T + \mathcal{O}(\rho^3), \\
\chi^A &= Y^A - \frac{\rho}{\Omega}\partial^A T + \frac{\rho^2}{2\Omega^2}\lambda^{AB}\partial_B T + \mathcal{O}(\rho^3),
\end{aligned} \tag{4.23}$$

where we have used γ^{AB} to raise indices and Y^A is a function of x^A only, i.e. $Y^\zeta = Y(\zeta)$ and $Y^{\bar{\zeta}} = \bar{Y}(\bar{\zeta})$. Under these transformations, the fields transform as

$$\begin{aligned}
\delta_\chi \theta_A &= Y^B \partial_B \theta_A + \partial_A Y^B \theta_B - 2\kappa \partial_A T, \\
\delta_\chi \Omega &= \nabla_B(Y^B \Omega),
\end{aligned} \tag{4.24}$$

∇ standing for the covariant derivative on γ_{AB}. Under modified Lie brackets (4.6), transformations (4.23) satisfy

$$[\chi(T_1, Y_1^A), \chi(T_2, Y_2^A)] = \chi(T_{12}, Y_{12}^A), \tag{4.25}$$

where

$$\begin{aligned}
T_{12} &= Y_1^A \partial_A T_2 - Y_2^A \partial_A T_1, \\
Y_{12}^A &= Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A.
\end{aligned} \tag{4.26}$$

Notice that the transformations generated by Y^A, in general, are not globally well defined on the 2-sphere. The only invertible transformations are those spanning the global conformal group, which is isomorphic to the proper, orthochronous Lorentz group. However, if we focus only on the local properties, all functions are allowed. This was first proposed in [21, 22] in the context of asymptotically flat spacetimes, where the integration constants were allowed to be meromorphic functions (i.e. to admit poles singularities). In that case, the algebra found is the local, or extended, $\mathfrak{bms}_4$ algebra[1], which consists of the semi-direct sum between the supertranslations and the local conformal transformations, the so-called superrotations [23].

Following this idea, we develop the functions T, Y^A in Laurent series,

$$
\begin{aligned}
T_{(n,m)} &= \chi(\zeta^n \bar{\zeta}^m, 0, 0), \\
Y_n &= \chi(0, -\zeta^{n+1}, 0), \\
\bar{Y}_n &= \chi(0, 0, -\bar{\zeta}^{n+1}),
\end{aligned}
\tag{4.27}
$$

and the non-vanishing commutation relations are found to be

$$
\begin{aligned}
[Y_n, Y_m] &= (n - m)Y_{n+m}, \\
[\bar{Y}_n, \bar{Y}_m] &= (n - m)\bar{Y}_{n+m}, \\
[Y_k, T_{(n,m)}] &= -nT_{(n+k,m)}, \\
[\bar{Y}_k, T_{(n,m)}] &= -mT_{(n,m+k)}.
\end{aligned}
\tag{4.28}
$$

The exact isometry algebra corresponds to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{R}$ whose elements correspond to the globally well-defined transformations on the sphere plus $T_{(0,0)}$. At this stage, it is interesting to compare algebra (4.28) with the $\mathfrak{bms}_4$ obtained in [21, 22]; they look very similar, but are not exactly the same. Understanding the precise relationship between the two way would be worthwhile.

To complete the analysis, we compute the conserved charges at the horizon; they read

$$
Q(T, Y^A) = \frac{1}{16\pi G} \int d\zeta d\bar{\zeta} \sqrt{\gamma} \, \Omega \left[2\kappa T - Y^A \theta_A \right],
\tag{4.29}
$$

and close under Poisson bracket

$$
\{Q(T_1, Y_1^A), Q(T_2, Y_2^A)\} = Q(T_{12}, Y_{12}^A).
\tag{4.30}
$$

By defining $\mathcal{T}_{(m,n)} = Q(\zeta^n \bar{\zeta}^m, 0, 0)$, $\mathcal{Y}_n = Q(0, -\zeta^{n+1}, 0)$ and $\bar{\mathcal{Y}}_n = Q(0, 0, -\bar{\zeta}^{n+1})$, we find that these quantities satisfy the same algebra (4.28).

We can perform the Sugawara construction as we did in the previous section. Defining

$$
\mathcal{P}_{(n,l)} = \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathcal{T}_{(m,t)} \mathcal{T}_{(n-m,l-t)}
\tag{4.31}
$$

and using (4.28), one finds

$$
\begin{aligned}
[\mathcal{P}_{(n,l)}, \mathcal{Y}_m] &= (n - m)\mathcal{P}_{(n+m,l)} \\
[\mathcal{P}_{(n,l)}, \bar{\mathcal{Y}}_m] &= (l - m)\mathcal{P}_{(n,l+m)}.
\end{aligned}
\tag{4.32}
$$

---

[1] By opposition to the global $\mathfrak{bms}_4$ algebra obtained in the 60's, which consists of the semi-direct sum between smooth functions on the two-sphere (supertranslations) with global conformal Killing vectors.

Finally, let us note that the Kerr black hole fits in our boundary conditions (4.21). An explicit construction of this solution in term of Gaussian normal coordinates can be found in Appendix D, following the results of [153]. One can verify that

$$\mathcal{T}_{(0,0)} = \frac{\kappa}{2\pi}\frac{\mathcal{A}}{4G}, \quad \mathcal{Y}_0 = \frac{iaM}{2}, \quad \bar{\mathcal{Y}}_0 = -\frac{iaM}{2}, \tag{4.33}$$

where $\mathcal{A}$ is the area of the horizon, while M and a are the usual parameters of the Kerr solution. That is, the zero mode of the supertranslation is the product of the black hole entropy with its temperature. Since our boundary conditions are defined by fixing κ, we can associate this charge with Wald entropy. On the other hand, the charge $Q(0, \partial_\phi) = -i(\mathcal{Y}_0 - \bar{\mathcal{Y}}_0) = aM$ is the angular momentum.

In the case where m and n are different from zero, $\mathcal{Y}_n$, $\bar{\mathcal{Y}}_n$ and $\mathcal{T}_{(m,n)}$ with $m \neq n$ vanish. In contrast, charges $\mathcal{T}_{(m,m)}$ with $m \neq 0$ diverge. This phenomenon was first notice in [145] and has been explained in reference [106]. Let us explain the origin of this divergence for the case of Schwarzschild black hole. In this case, the supertranslation charge reads

$$\mathcal{T}_{(m,n)} = \frac{\kappa r_+^2}{4G}\delta_{m,n}I(m), \tag{4.34}$$

where $I(m) = \int_0^\pi d\theta \sin(\theta) \cot^{2m}(\theta/2)$ is divergent for $m \neq 0$, with the divergence coming from the poles of the sphere. If instead of Laurent modes, the supertranslation $T(\zeta, \bar{\zeta})$ is expanded in spherical harmonics, the charges can be seen to vanish.

4.3   Discussion

We have shown that the near-horizon geometry of non-extremal black holes exhibits an infinite-dimensional extension of supertranslation algebra, which in particular contains superrotations. This phenomenon is similar to what happens in the asymptotically flat spacetimes at null infinity, although the algebra obtained differs from the standard extended 𝔟𝔪𝔰. We have explicitly worked out the cases of three-dimensional and four-dimensional stationary black holes, for which the zero modes of the charges associated to the infinite-dimensional symmetries were shown to exactly reproduce the entropy and the angular momentum of the solutions.

In this chapter, we used the Barnich-Brandt formalism to compute the surface charges. However, when one computes charges deep in the bulk (in this case on the horizon), one should take into account the full interacting theory, and not only in the linearized regime. Still, in [154], the closed $n-2$ forms of the full interacting theory were shown, under suitable assumptions, to reduce asymptotically to the conserved $n-2$ forms of the linearized theory used in the definition of the boundary charges. Even if a general proof for asymptotic charges in the deep bulk is still missing, in the case at hand, the charges were explicitly shown to be integrable, conserved and to close the asymptotic algebra.

In the three-dimensional case, we have presented a family of explicit solutions that obey the proposed boundary conditions at the horizon and, therefore, realize the infinite-dimensional symmetry generated by the semi-direct sum of Virasoro algebra and supertranslations. Although this family of solutions represent locally AdS_3 spacetimes, they

do not satisfy the standard Brown-Henneaux asymptotic conditions at $\rho \to \infty$, as we are imposing boundary conditions at the horizon $\rho \to 0$. In [88], a set of asymptotically AdS$_3$ boundary conditions were found whose associated charges yield a centrally extended version of algebra (4.9). It would be interesting to study the relation between such boundary conditions and (4.2); in particular, to clarify the precise connection between the family of solutions (4.16) and those presented in [88]. The latter also includes the BTZ black hole as a particular example; however, in contrast to (4.2), which fixes the black hole surface gravity κ, the boundary conditions considered in [88] are defined by fixing the value of $\Delta = M\ell + J$. Very recently, new boundary conditions close to the horizon appeared in the literature [155], and the symmetry algebra obtained is a Heisenberg algebra whose associated charges are examples of soft hairs on the horizon. It would be interesting to see how these results are related to the ones presented here. In particular, our algebra was used in [155] to reproduce the Bekenstein-Hawking law by means of a warped version of the Cardy formula.

Another question is whether it is possible to modify our boundary conditions in such a way of getting non-vanishing central extensions. In this regard, it is worthwhile mentioning that boundary conditions we have considered allow for exponentially decaying modes $e^{-\kappa v}X(\phi)$ which yield extra infinite-dimensional symmetry also associated to an extension of supertranslations. On the other hand, an important point to address is the study of the extremal limit, for which the boundary conditions at the horizon need to be reconsidered since the leading term in g_{vv} vanishes.

CHAPTER 5

Liouville theory beyond the cosmological horizon

In the fist chapter, we have seen in details how three-dimensional Einstein gravity with negative cosmological constant can be rewritten as Lorentzian Liouville theory defined on the conformal boundary cylinder of AdS, upon imposing suitable Dirichlet-type boundary conditions. The Hamiltonian reduction procedure is achieved in two steps, with the non-chiral WZW model as an intermediate theory. In retrospect, this provided a first toy model of a conformal field theory that is classically equivalent to gravity in AdS, before string proposals [9] and higher spin proposals [156] were made.

Since then, there has been the hope that a similar analysis in the case of a positive cosmological constant could give insights to gravity in de Sitter (dS) spacetimes. The motivation for this comes in great part from recent astrophysical data indicating that we live in a universe with $\Lambda > 0$ [157]. Another motivation is to understand what is the role of de Sitter spacetimes in string theory and, in particular, clarifying the microscopic origin of entropy for these spaces remains an outstanding challenge.

Given the analytic continuation relating anti-de Sitter to de Sitter spacetime, it comes as no surprise that one can, similarly to the $\Lambda < 0$ case, rewrite Einstein gravity with positive cosmological constant (with similar Dirichlet-type boundary conditions) in terms of *Euclidean* Liouville theory, as Cacciatori and Klemm showed in [158]. More precisely, the Einstein-Hilbert action reduces to two copies of Euclidean Liouville theory, the first defined on the future boundary $\mathcal{I}^+$ and the second on the past boundary $\mathcal{I}^-$, since these boundaries border the complete spacetime bulk. However, bulk null geodesics connect any point on the sphere $\mathcal{I}^-$ to the antipodal point on the sphere $\mathcal{I}^+$. It has been argued, then, that the formulation of a full-fledged dual quantum theory, a "dS/CFT correspondence", would only require one boundary [28]. No UV complete string embedding of such a dS/CFT correspondence has been formulated so far but proposals using higher spins have been made [159].

In the dS/CFT proposal [28], the holographic screen where the CFT would be best defined is the future (or past) conformal boundary. There, one can define the asymptotic symmetries, whose complexification consist of two copies of the Virasoro algebra. One can also define the conformal dimensions and correlation functions of the operators dual

to bulk fields. The presence of the cosmological horizon of a thermal and entropic nature [160] between the static observer and the conformal boundary however raises questions on whether the holographic description extends all the way to the static observer. In addition, even though one can define the Virasoro central charges to be positive, the semi-classical spectrum of zero modes, which corresponds to spinning conical defects [37], is complex, which challenges the existence of a Hilbert space with a unitarity inner product. Such issues were further discussed in the literature [161, 162, 163, 164, 165, 166]. Other holographic scenarios were also proposed [167, 168, 169].

In this chapter, we present the original results obtained in [170] : We show that Euclidean Liouville theory is also dual to Einstein gravity with Dirichlet boundary conditions on a fixed timelike slice in the static patch. Intriguingly, the spacetime interpretation of Euclidean Liouville time is the physical time of the static observer. As a prerequisite of this correspondence, we show that the asymptotic symmetry algebra which consists of two copies of the Virasoro algebra extends everywhere into the bulk.

On the technical side, we use the reformulation of Einstein gravity with positive cosmological constant as two copies of $SL(2, \mathbb{C})$ Chern-Simons theory with a reality constraint [41, 42]. We note that the Fefferman-Graham gauge for the metric naturally leads to the highest weight gauge for the first Chern-Simons gauge field and lowest weight gauge for the second. Instead, Eddington-Finkelstein coordinates for the metric, which cover both the conformal boundary and the static observer, lead to a highest weight gauge for both Chern-Simons gauge fields. This distinction leads to some new features of the Hamiltonian reduction to Liouville theory with respect to previous treatments [43, 50, 158, 71]. Usually, one performs a Gauss decomposition of an $SL(2, \mathbb{C})$ element around the identity in order to reduce the non-chiral WZW model to Liouville theory. Here, it turns out that a natural Gauss decomposition involves particular coordinates far from the identity, in order to parameterize the Liouville field without otherwise intricate field redefinitions.

The chapter is organized as follows. In section 5.1, we derive the symmetry algebra of pure Einstein gravity in the bulk spacetime, both at the level of asymptotic Killing vector fields and associated conserved charges. In section 5.2, we review the Chern-Simons formalism for asymptotically dS$_3$ spacetimes and present the classical phase space of spinning conical defects equipped with Virasoro gravitons in two sets of coordinates of interest. We perform the reduction to the WZW model and then to Liouville theory in section 5.3. The last section contains our conclusions.

5.1   Asymptotic symmetries everywhere

Instead of the Fefferman-Graham system of coordinates, we adopt here the Gaussian null or Eddington-Finkelstein coordinate system[1],

$$g_{rr} = 0 \,, \qquad g_{ru} = -1 \,, \qquad g_{r\phi} = 0, \tag{5.1}$$

---

[1]It may happen that we will use later the word *gauge* to refer to the system of coordinates, though it may not be the most appropriate word.

which has, among others, the advantage that the limit $\ell \to \infty$ can be made well-defined. The phase space of Einstein gravity with positive cosmological constant in three dimensions can be written in such coordinate system as

$$ds^2 = \left(\frac{r^2}{\ell^2} + 8GM(u,\phi) \right) du^2 - 2dudr + 8GJ(u,\phi)dud\phi + r^2 d\phi^2, \qquad (5.2)$$

where the functions $M(u,\phi)$, $J(u,\phi)$ satisfy $\partial_u J = \partial_\phi M$ and $\partial_u M = -\frac{1}{\ell^2}\partial_\phi J$. Note that we will keep all factors of ℓ explicit in order to also discuss the AdS analytic continuation $\ell \to i\ell$ and the flat spacetime limit $\ell \to \infty$.

5.1.1 Symmetry algebra

The phase space (5.2) is preserved under the action of the vector field

$$\xi = f\partial_u + \left(-r\partial_\phi Y + \partial_\phi^2 f - \frac{8GJ}{2r}\partial_\phi f \right) \partial_r + \left(Y - \frac{\partial_\phi f}{r} \right) \partial_\phi, \qquad (5.3)$$

where the functions $f(u,\phi)$ and $Y(u,\phi)$ satisfy $\partial_u f = \partial_\phi Y$, $\partial_u Y = -\frac{1}{\ell^2}\partial_\phi f$. Interestingly, the perturbative expansion in r of the symmetry generator in this gauge stops at next-to-next-to-leading order.

At leading order close to future infinity $\mathcal{I}^+$ (defined as the limit $r \to \infty$), the vector field (5.3) reduces to

$$\bar{\xi} = f\partial_u - r\partial_\phi Y \partial_r + Y\partial_\phi, \qquad (5.4)$$

and its algebra is found to be

$$[\bar{\xi}_1, \bar{\xi}_2] \equiv \hat{f}\partial_u - r\partial_\phi \hat{Y}\partial_r + \hat{Y}\partial_\phi, \qquad (5.5)$$

where

$$\begin{aligned}
\hat{f} &= Y_1\partial_\phi f_2 - Y_2\partial_\phi f_1 + f_1\partial_\phi Y_2 - f_2\partial_\phi Y_1, \\
\hat{Y} &= Y_1\partial_\phi Y_2 - Y_2\partial_\phi Y_1 - \frac{1}{\ell^2}\left(f_1\partial_\phi f_2 - f_2\partial_\phi f_1 \right).
\end{aligned} \qquad (5.6)$$

These relations define the symmetry algebra and can be written more compactly as

$$[(f_1, Y_1), (f_2, Y_2)] = (\hat{f}, \hat{Y}). \qquad (5.7)$$

When ℓ is finite, it is convenient to define the coordinates $t^\pm = u \pm i\ell\phi$. One has $\partial_+\partial_- f = 0 = \partial_+\partial_- Y$, which can be integrated for f, Y in terms of two arbitrary functions $l^+(t^+)$, $l^-(t^-)$:

$$f = \frac{1}{2}\left(l^+ + l^- \right), \qquad Y = \frac{-i}{2\ell}\left(l^+ - l^- \right). \qquad (5.8)$$

The leading-order symmetry vector (5.4) therefore becomes

$$\bar{\xi} = l^+\partial_+ \ + \ l^-\partial_- - \frac{r}{2}(\partial_+ l^+ + \partial_- l^-)\partial_r, \qquad (5.9)$$

and, expanding the generators as

$$
\begin{aligned}
l_m^+ &= \{\bar{\xi}: \quad l^+ = \ell e^{-m\frac{t^+}{\ell}}, l^- = 0\} = \ell e^{-m\frac{t^+}{\ell}}\left(\partial_+ + \frac{m}{2\ell}r\partial_r\right), \\
l_m^- &= \{\bar{\xi}: \quad l^+ = 0, l^- = \ell e^{-m\frac{t^-}{\ell}}\} = \ell e^{-m\frac{t^-}{\ell}}\left(\partial_- + \frac{m}{2\ell}r\partial_r\right),
\end{aligned}
\tag{5.10}
$$

one finds that the algebra of the vector fields consists of two copies of the Witt algebra

$$
[l_m^\pm, l_m^\pm] = (m - n)l_{m+n}^\pm.
\tag{5.11}
$$

Note the relations $(l_m^\pm)^* = l_m^\mp$.

Modified Lie bracket and symmetry realization in the bulk

The bulk symmetry parameter (5.3) is field dependent (through the metric function $\mathcal{J}$) and therefore, as we saw already in the section 4.1, its algebra is in this case given by the modified bracket (4.6)

$$
[\xi_1, \xi_2]_M = [\xi_1, \xi_2] - \delta_{\xi_1}\xi_2(g) + \delta_{\xi_2}\xi_1(g).
\tag{5.12}
$$

By means of this modified bracket, one can show that the bulk field (5.3) forms a representation of the symmetry algebra (5.7):

$$
[\xi_1, \xi_2]_M = \hat{f}\partial_u + \left(-r\partial_\phi\hat{Y} + \partial_\phi^2\hat{f} - \frac{8GJ}{2r}\partial_\phi\hat{f}\right)\partial_r + \left(\hat{Y} - \frac{\partial_\phi\hat{f}}{r}\right)\partial_\phi.
\tag{5.13}
$$

The symmetry algebra is thus represented everywhere in the bulk of the spacetime even though it has been defined at infinity. This has been first pointed out in [22] for the AdS and flat case, in three and four spacetime dimensions (see also [171] for Einstein-Yang-Mills in three and higher dimensions), the key point being the introduction of the modified bracket.

5.1.2 Surface charge algebra

So far, we have seen how to compute surface charges in the Chern-Simons formalism, and their expressions turned out to be very simple (see for instance expression (3.18)). However, the Chern-Simons formalism is not always accessible, for instance when working in four spacetime dimensions (as in section 4.2) or for theories that possess propagating degrees of freedom (as we will be dealing with in the next chapter, see section 6.3.2). In that case, it is useful to know how to compute surface charges in the metric formulation. This is the reason why we introduce here the covariant (or Barnich-Brandt) formalism [172, 142]: A 1-form $\oint \mathcal{Q}_\xi$ which depends on a solution $g_{\mu\nu}$ and its variation $\delta g_{\mu\nu} \equiv h_{\mu\nu}$ is associated to a vector field ξ. $\oint \mathcal{Q}_\xi$ is defined in n spacetime dimension by[1]

$$
\oint \mathcal{Q}_\xi[h, g] = \frac{1}{16\pi G}\int_{\partial\Sigma}(d^{n-2}x)_{\mu\nu}\sqrt{-g}\,k^{\mu\nu}[h, g],
\tag{5.14}
$$

---

[1]The charge may be non integrable, hence the $\oint$ notation.

where $h = g^{\mu\nu} h_{\mu\nu}$ and $(d^{n-2}x)_{\mu\nu} \equiv \dfrac{1}{2!(n-2)!}\epsilon_{\mu\nu\sigma_1\cdots\sigma_{n-2}} dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_{n-2}}$ denotes the dual of a 2-form in n dimensions. The surface one-form $k^{\mu\nu}$ was shown to be given, for pure Einstein gravity (GR) with or without cosmological constant, by the explicit expression

$$k_{\text{GR}}^{\mu\nu} = \xi_\alpha D^{[\mu}h^{\nu]\alpha} - \xi^{[\mu}D_\alpha h^{\nu]\alpha} - h^{\alpha[\mu}D_\alpha\xi^{\nu]} + \xi^{[\mu}D^{\nu]}h + \frac{1}{2}hD^{[\mu}\xi^{\nu]}. \tag{5.15}$$

In three dimensions, we have $n = 3$ and the surface integration $\partial\Sigma$ is taken to be the circle (u and r fixed). Furthermore, with $g_{\mu\nu}$ given by (5.2), one finds that the only non-vanishing $h_{\mu\nu}$ are $h_{uu} = \delta M$ and $h_{u\phi} = \delta J/2$. Having this at hand, one evaluates (5.15) for the symmetry generator (5.3) and one finds that the expression for the variation of the surface charges reads

$$\oint\!\mathcal{Q}_\xi[h,g] = \frac{1}{2\pi}\int_0^{2\pi}\left(f\delta M + Y\delta J - \frac{1}{2r}(f\partial_\phi\delta J + \delta J\partial_\phi f)\right)d\phi. \tag{5.16}$$

Crucially, the $1/r$ term vanishes due to an integration by parts with respect to the ϕ coordinate. Because the remaining right-hand side of (5.16) is made of δ-exact terms, the associated charge is integrable and reads

$$\mathcal{Q}_\xi = \frac{1}{2\pi}\int_0^{2\pi}\left(fM + YJ\right)d\phi. \tag{5.17}$$

Here, we fixed the normalization such that $\mathcal{Q}_\xi$ is zero for $M = J = 0$. The charge is r independent. Therefore, this expression for the charge is the same everywhere in the bulk of the spacetime.

One could have also used the Iyer-Wald formula for the charges [173], which is equal to the expression (5.14) with $k^{\mu\nu}$ given by (5.15) with the last term removed. The final term might in general be non-zero for non-Killing vectors fields, such as the symmetries that we are using. However, the term evaluates to zero, and the Iyer-Wald charges are identical to (5.17).

The charge formula (5.17) makes explicit the relationship between the integration functions of the symmetries (f, Y) and the integration functions of the solution to the equations of motion (M, J). More precisely, the charge $\mathcal{Q}_\xi$ in (5.17) provides an inner product between the space of solutions and the asymptotic symmetries.

Upon defining $M = \mathcal{L}_+ + \mathcal{L}_-$, $J = i\ell(\mathcal{L}_+ - \mathcal{L}_-)$, the charge is given by

$$\mathcal{Q}_\xi = \frac{1}{2\pi}\int_0^{2\pi}\left(l^+\mathcal{L}_+ + l^-\mathcal{L}_-\right)d\phi, \tag{5.18}$$

which also makes manifest the relationship between the functions (l^+, l^-) and the integrations functions of the solution $(\mathcal{L}_+, \mathcal{L}_-)$. Note that the semi-classical spectrum of $\mathcal{L}_+$ and $\mathcal{L}_-$ is complex.

It is worth pointing out that the result (5.17) is valid for asymptotically flat, anti-de Sitter and de Sitter cases. The anti-de Sitter case is simply obtained by analytic continuation $\ell \to i\ell$. The asymptotically flat case is then obtained by taking the limit

$\ell \to \infty$. Since all quantities $f, Y, \mathcal{M}, \mathcal{J}$ are finite in the flat limit, one readily obtains the result. (One cannot however use $l_m^\pm$ which are not well-defined in the flat limit).

More conceptually, the fact that the charges are independent of the radius follows from the vanishing of the symplectic structure of the theory. Indeed, the symplectic structure evaluated on the Lie derivative of the metric is a boundary term, $\omega(\mathcal{L}_\xi g_{\mu\nu}, \delta g_{\mu\nu}, g) = dk_\xi(\delta g, g)$ where $\oint Q_\xi[\delta g, g] = \int k_\xi(\delta g, g)$ is precisely the charge (5.14). The vanishing of the symplectic structure implies that the difference of charge $\oint Q_\xi$ evaluated on two surfaces $r = r_1$ and $r = r_2$ constant is zero. Therefore, the charge is independent of the radius.

Algebra of surface charges: two Virasoro in the bulk

The transformation laws of the functions M, J under the symmetry transformation generated by (5.3) are given by[1]

$$\delta M = Y\partial_\phi M + 2M\partial_\phi Y - \frac{1}{4G}\partial_\phi^3 Y - \frac{1}{\ell^2}(2J\partial_\phi f + f\partial_\phi J),$$
$$\delta J = Y\partial_\phi J + 2J\partial_\phi Y - \frac{1}{4G}\partial_\phi^3 f + 2M\partial_\phi f + f\partial_\phi M. \tag{5.19}$$

One can rewrite the transformation laws as

$$\delta\mathcal{L}_\pm = l_\pm\partial_\pm\mathcal{L}_\pm + 2\mathcal{L}_\pm\partial_\pm l_\pm + \frac{\ell^2}{8G}\partial_\pm^3 l_\pm. \tag{5.20}$$

The algebra of surface charges (5.17) can then be computed with the Poisson bracket defined by

$$\{\mathcal{Q}_{\xi_1}, \mathcal{Q}_{\xi_2}\} = -\delta_{\xi_1}\mathcal{Q}_{\xi_2}. \tag{5.21}$$

One finds

$$\delta_{\xi_1}\mathcal{Q}_{\xi_2} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\left(\hat{f}M + \hat{Y}J\right) - \frac{1}{8\pi G}\int_0^{2\pi} d\phi(f_1\partial_\phi^3 Y_2 + Y_1\partial_\phi^3 f_2), \tag{5.22}$$

where $\hat{f}, \hat{Y}$ are given in (5.6). Therefore, one has

$$\{\mathcal{Q}_{\xi_1}, \mathcal{Q}_{\xi_2}\} = \mathcal{Q}_{[\xi_1,\xi_2]} + \mathcal{K}_{\xi_1,\xi_2}, \tag{5.23}$$

where $\mathcal{K}_{\xi_1,\xi_2}$ is by definition the second term of (5.22).

Introducing $L_m^\pm = \mathcal{Q}_{l_m^\pm}$, we find that the charge algebra consists of two copies of the Virasoro algebra

$$\{L_m^\pm, L_n^\pm\} = (m-n)L_{m+n}^\pm + \frac{c^\pm}{12}m^3\delta_{m+n,0}, \tag{5.24}$$

---

[1]Notice that one recovers the bosonic transformation laws (3.17) in the limit $\ell \to \infty$, the relation between the notations being $\mathcal{M} = 8GM$, $\mathcal{J} = 8GJ$.

everywhere in the bulk, with central charge $c^{\pm} = \frac{3\ell}{2G}$. The charges obey $(L_m^+)^* = L_m^-$. Note that with our definitions there is no i on the left-hand side of the above relation (5.24). This is in contrast to the AdS result [12].

In the AdS case obtained by analytical continuation, it similarly follows that the Brown-Henneaux realization of asymptotic symmetries (2.63) can be extended everywhere in the bulk. In the case of the asymptotically flat limit, we have shown that the bms_3 charge algebra (2.148) is defined everywhere into the bulk. All these results are valid for three-dimensional Einstein gravity without matter. The generalization with propagating modes is far from obvious.

Notice that we derived here the asymptotic structure in Gaussian null coordinate system and found exactly the same two holomorphic functions, with same symmetry algebra and central extensions as in the Fefferman-Graham coordinates used in section 2.4, showing therefore the one-to-one correspondence between these two phase-spaces.

Recently, these asymptotic symmetries everywhere were dubbed "symplectic symmetries" [151, 60]. The reason for this name is, as we already mentioned in the previous section, the presymplectic form vanishes on-shell: $\omega(\mathcal{L}_\xi g_{\mu\nu}, \delta g_{\mu\nu}, g) \approx 0$, while $\mathcal{L}_\xi g_{\mu\nu} \neq 0$[1]. Symplectic symmetries are large gauge transformations that are defined everywhere in spacetime, not only in an asymptotic region. Their appearance is most likely conditioned by the absence of propagating degrees of freedom in the bulk. As an application, let us mention that this symplectic structure was recently used in the study of near-horizon region of d-dimensional extremal black holes [60].

5.2 Chern-Simons formulation

It is now natural to perform the Hamiltonian reduction in the static patch, along the lines of the Coussaert-Henneaux-van Driel method that we have recalled in details in the chapter 2. The computations that follow are very similar to the case of negative cosmological constant, however, since one has to be careful with all the i factors that might appear in the de Sitter case, and that the interpretation is rather different, we find it more careful to write all the details.

Three-dimensional Einstein gravity with positive cosmological constant is given by action (2.1), where $\Lambda = 1/\ell^2$. It can be formulated as two copies of $SL(2, \mathbb{C})$ Chern-Simons theory with a reality constraint [41, 42], with the action (up to a boundary term)

$$S_E[A, \bar{A}] = -iS_k[A] + iS_k[\bar{A}], \tag{5.25}$$

where $k = \ell/(4G)$ and

$$S_k[A] = \frac{k}{4\pi} \int_{Bulk} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{5.26}$$

The equations of motion derived from 5.25 read

$$F \equiv dA + A \wedge A = 0, \qquad \bar{F} \equiv d\bar{A} + \bar{A} \wedge \bar{A} = 0, \tag{5.27}$$

---

[1]By opposition to Killing symmetries, for which $\mathcal{L}_\xi g_{\mu\nu} = 0$.

where

$$A = A^a \tau_a = \left(\omega^a + \frac{i}{\ell} e^a \right) \tau_a, \qquad \bar{A} = \bar{A}^a \tau_a = \left(\omega^a - \frac{i}{\ell} e^a, \right) \tau_a \qquad (5.28)$$

and here τ_a are $SL(2, \mathbb{C})$ generators which are normalized as $\text{Tr}(\tau_a \tau_b) = \frac{1}{2} \eta_{ab}$.

We consider Dirichlet boundary conditions, which we will present in section 5.2.3. The resulting classical phase space contains spinning conical defects studied in 1984 by Deser and Jackiw [37]. It also contains the "boundary gravitons" or Virasoro descendants which were derived in [28]. We will here present the space of solutions in two distinct coordinate systems which have distinct features. Fefferman-Graham coordinates are adapted to the conformal boundary and its holographic interpretation in terms of a CFT. However, already for the vacuum, these coordinates do not cover the static patch since they break down at the cosmological horizon. In contrast, Eddington-Finkelstein coordinates cover both the future diamond and static patch of global de Sitter, as we will see.

5.2.1   Fefferman-Graham slicing

We consider asymptotically de Sitter metrics of the form

$$ds^2 = -\ell^2 \frac{d\tau^2}{\tau^2} + \left(\tau^2 + \frac{16 G^2 \mathcal{L}_+ (t^+) \mathcal{L}_-(t^-)}{\tau^2} \right) dt^+ dt^-$$
$$- 4G \, \mathcal{L}_+(t^+) \, (dt^+)^2 - 4G \, \mathcal{L}_-(t^-) \, (dt^-)^2 , \quad (5.29)$$

where $t^\pm = t \pm i\ell\phi$, $\phi \sim \phi + 2\pi$[1]. The complex functions $\mathcal{L}_\pm$ parametrize the phase space of such metrics. They are constrained by the relation $\mathcal{L}_+^* = \mathcal{L}_-$. It is convenient to define the real functions

$$M(t^+, t^-) = \mathcal{L}_+(t^+) + \mathcal{L}_-(t^-) , \qquad J(t^+, t^-) = i\ell(\mathcal{L}_+(t^+) - \mathcal{L}_-(t^-)), \qquad (5.30)$$

whose zero modes are the mass and angular momentum, respectively. The coordinate system breaks down at $\tau = 0$ or even at the larger $\tau = 2G^{1/2}(\mathcal{L}_+\mathcal{L}_-)^{1/4}$ if $\mathcal{L}_+\mathcal{L}_- > 0$.

This coordinate system is not suitable to describe the coordinate patch of the static observer at the south pole beyond his cosmological horizon. To see this, let us consider the case of the dS$_3$ vacuum, with $M = \frac{1}{8G}$, $J = 0$:

$$ds^2 = -\ell^2 \frac{d\tau^2}{\tau^2} + \left(\tau - \frac{1}{4\tau} \right)^2 dt^2 + \ell^2 \left(\tau + \frac{1}{4\tau} \right)^2 d\phi^2 \qquad (5.31)$$

which is valid when $\frac{1}{2} \leq \tau \leq \infty$. One recognizes the static patch coordinates after defining $r = \tau + \frac{1}{4\tau}$, $1 \leq r \leq \infty$ such that

$$ds^2 = -\ell^2 \frac{dr^2}{r^2 - 1} + (r^2 - 1)dt^2 + \ell^2 r^2 d\phi^2 . \qquad (5.32)$$

This coordinate system only covers the upper diamond of global de Sitter, see Figures 5.1 and 5.2.

---

[1]For book-keeping purposes, we have explicitly $t = \frac{1}{2}(t^+ + t^-)$, $\phi = \frac{i}{2\ell}(-t^+ + t^-)$, $\partial_t = \partial_+ + \partial_-$, $\partial_\phi = i\ell(\partial_+ - \partial_-)$, $\partial_+ = \frac{1}{2}(\partial_t - \frac{i}{\ell}\partial_\phi)$, $\partial_- = \frac{1}{2}(\partial_t + \frac{i}{\ell}\partial_\phi)$.
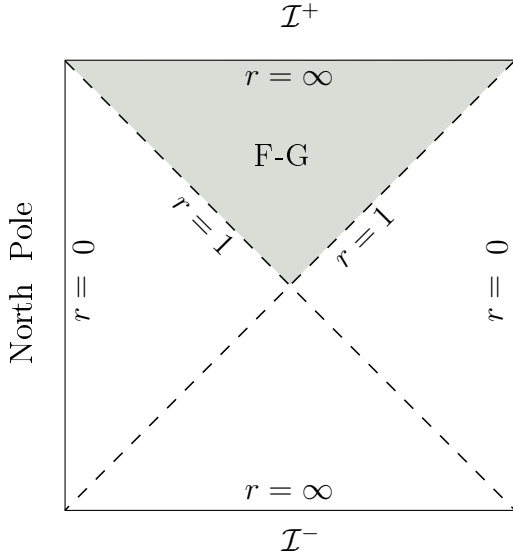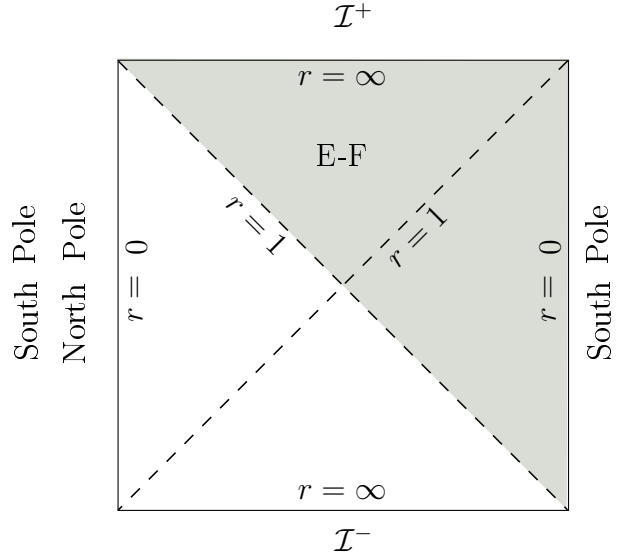
Figure 5.1: Fefferman-Graham coordinates

Figure 5.2: Eddington-Finkelstein coordinates

To obtain the gauge field, we have to specify the vielbein and the $SL(2,\mathbb{C})$ generators. The choice of the $SL(2,\mathbb{C})$ generators, τ_a, should be consistent with $ds^2 = \tilde{\eta}_{ab}e^a e^b$ and $\operatorname{Tr}\tau_a \tau_b = \frac{1}{2}\tilde{\eta}_{ab}$. We choose

$$
\begin{aligned}
e^0 &= -\frac{\ell}{r}dr, \\
e^1 &= -r\,dt + \frac{2G}{r}\left(M\,dt + J\,d\phi\right), \\
e^2 &= -\ell r\,d\phi - \frac{2\ell G}{r}\left(M\,d\phi - \frac{J}{\ell}dt\right),
\end{aligned}
\tag{5.33}
$$

and the generators as

$$
\tau_0^{FG} = -iL_0, \qquad \tau_1^{FG} = \frac{1}{2}(L_1 - L_{-1}), \qquad \tau_2^{FG} = \frac{i}{2}(L_1 + L_{-1})
\tag{5.34}
$$

where $L_{\pm 1}, L_0$ are defined in Appendix A. We then have $\tilde{\eta}_{ab} = \operatorname{diag}(-1,1,1)$. The gauge fields A and $\bar{A}$ are then

$$
\begin{aligned}
A^{FG} &= \frac{1}{2r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}dr + \begin{pmatrix} 0 & \frac{4iG\mathcal{L}_+(t^+)}{\ell r} \\ -\frac{ir}{\ell} & 0 \end{pmatrix}dt^+, \\
\bar{A}^{FG} &= \frac{1}{2r}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}dr + \begin{pmatrix} 0 & \frac{ir}{\ell} \\ -\frac{4iG\mathcal{L}_-(t^-)}{\ell r} & 0 \end{pmatrix}dt^-.
\end{aligned}
\tag{5.35}
$$

In the approach of [43], the boundary conditions are specified at future infinity for the gauge field after the r-dependence is factorized out. An interesting feature of de Sitter

space time in the Fefferman-Graham coordinates is that the r-dependence factorizes out not only at the future infinity but in the whole upper diamond of the Penrose diagram. We call the r-independent factor of the gauge field the reduced gauge connection a^{FG}:

$$
\begin{aligned}
a^{FG} &= -\frac{i}{\ell} \begin{pmatrix} 0 & -4G\mathcal{L}_+(t^+) \\ 1 & 0 \end{pmatrix} dt^+ = -i \left(\frac{1}{\ell}L_1 + \frac{1}{k}\mathcal{L}_+(t^+)L_{-1} \right) dt^+, \\
\bar{a}^{FG} &= \frac{i}{\ell} \begin{pmatrix} 0 & 1 \\ -4G\mathcal{L}_-(t^-) & 0 \end{pmatrix} dt^- = -i \left(\frac{1}{\ell}L_{-1} + \frac{1}{k}\mathcal{L}_-(t^-)L_1 \right) dt^-, \quad (5.36)
\end{aligned}
$$

where $A^{FG}, \bar{A}^{FG}$ and $a^{FG}, \bar{a}^{FG}$ are related by the gauge transformation

$$
a^{FG} = K^{-1}A^{FG}K + K^{-1}dK, \qquad \bar{a}^{FG} = K\bar{A}^{FG}K^{-1} + KdK^{-1}, \quad (5.37)
$$

with $K = \mathrm{diag}(r^{-1/2}, r^{1/2})$.

A useful property of this basis is

$$
(\tau_a^{FG})^\dagger = \sigma\tau_a^{FG}\sigma, \qquad \text{with} \qquad \sigma \equiv 2iL_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (5.38)
$$

The reduced gauge connection a^{FG} is in lowest weight form while $\bar{a}^{FG}$ is in highest weight form. As a consequence of (5.38) they are related by

$$
a_{FG}^\dagger = \sigma\bar{a}_{FG}\sigma = \bar{a}_{FG}. \quad (5.39)
$$

5.2.2 Eddington-Finkelstein slicing

Since Fefferman-Graham coordinates do break at the cosmological horizon, it is necessary to consider another coordinate system in order to impose boundary conditions beyond the horizon. We will now repeat all steps from the previous subsection in Eddington-Finkelstein type coordinates.

The phase space of asymptotically de Sitter spacetimes is now given by

$$
ds^2 = \left(\frac{r^2}{\ell^2} - 8G(\mathcal{L}_+ + \mathcal{L}_-) \right) du^2 - 2dudr - 8i\ell\, G(\mathcal{L}_+ - \mathcal{L}_-)dud\phi + r^2d\phi^2, \quad (5.40)
$$

where $u \in \mathbb{R}$, $\phi \sim \phi + 2\pi$ and $0 \leq r$.

We choose

$$
\begin{aligned}
e^0 &= \left(\frac{r^2}{2\ell^2} - 4G\left(\mathcal{L}_+ + \mathcal{L}_-\right) \right) du - dr - 4i\ell G\left(\mathcal{L}_+ - \mathcal{L}_-\right)d\phi, \\
e^1 &= -2du, \qquad e^2 = rd\phi, \\
\tau_0^{EF} &= -\frac{1}{2}L_1, \qquad \tau_1^{EF} = -\frac{1}{2}L_{-1}, \qquad \tau_2^{EF} = L_0,
\end{aligned} \quad (5.41)
$$

such that $ds^2 = -e^0e^1 + (e^2)^2 \equiv \bar{\eta}_{ab}e^a e^b$.

From the choice of the generators and dreibein, we obtain the gauge fields

$$
A^{EF} = \frac{i}{2\ell} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dr + \begin{pmatrix} \frac{r}{2\ell^2} & -\frac{i}{\ell} \\ -\frac{ir^2}{4\ell^3} + \frac{i\mathcal{L}_+}{k} & -\frac{r}{2\ell^2} \end{pmatrix} dt^+ \,,
$$

$$
\bar{A}^{EF} = -\frac{i}{2\ell} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dr + \begin{pmatrix} \frac{r}{2\ell^2} & \frac{i}{\ell} \\ \frac{ir^2}{4\ell^3} - \frac{i\mathcal{L}_-}{k} & -\frac{r}{2\ell^2} \end{pmatrix} dt^- \,, \tag{5.42}
$$

where we defined $t^\pm = u \pm i\ell\phi$.

As in the case of the Fefferman-Graham coordinates the A_- and $\bar{A}_+$ contributions are zero. Again, it is possible to factorize out the r-dependence. We define the reduced gauge connection as

$$
a^{EF} = K^{-1}A^{EF}K + K^{-1}dK, \qquad \bar{a}^{EF} = K\bar{A}^{EF}K^{-1} + KdK^{-1} \tag{5.43}
$$

where $K = \begin{pmatrix} 1 & 0 \\ -\frac{i}{2\ell}r & 1 \end{pmatrix}$. Note that the form of the matrix K differs from the one in Fefferman-Graham coordinates (5.37).

On-shell we find for the reduced gauge field

$$
a^{EF} = \begin{pmatrix} 0 & -\frac{i}{\ell} \\ \frac{i\mathcal{L}_+(t^+)}{k} & 0 \end{pmatrix} dt^+ = \frac{i}{\ell}\left(L_{-1} + \frac{\ell}{k}\mathcal{L}_+(t^+)L_1\right)dt^+, \tag{5.44}
$$

$$
\bar{a}^{EF} = \begin{pmatrix} 0 & \frac{i}{\ell} \\ \frac{-i\mathcal{L}_-(t^-)}{k} & 0 \end{pmatrix} dt^- = -\frac{i}{\ell}\left(L_{-1} + \frac{\ell}{k}\mathcal{L}_-(t^-)L_1\right)dt^-. \tag{5.45}
$$

Since the basis of generators τ_a^{EF} is real, it implies $\bar{a}_{EF} = a^*_{EF}$. A useful property of this basis is

$$
(\tau_a^{EF})^\dagger = -\hat{\sigma}\tau_a^{EF}\hat{\sigma}, \qquad \text{with} \qquad \hat{\sigma} \equiv i(L_1 + L_{-1}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.46}
$$

5.2.3 Boundary conditions

Let us define a slicing of (a part of) spacetime into fixed radial slices Σ_r, such that in the limit $r \to \infty$, Σ_∞ coincides with the future conformal boundary $\mathcal{I}^+$. There is an infinite number of such slicings. Two examples (Fefferman-Graham and Eddington-Finkelstein slicings) were provided above. We then define the reduced gauge connections a and $\bar{a}$ as

$$
a = K^{-1}AK + K^{-1}dK, \qquad \bar{a} = \bar{K}^{-1}\bar{A}\bar{K} + \bar{K}^{-1}d\bar{K} \tag{5.47}
$$

such that $a_r = 0 = \bar{a}_r$. This fixes K, $\bar{K} \in SL(2,\mathbb{C})$ up to an $SL(2,\mathbb{C})$ element on Σ_r which corresponds to (t^+, t^-)-dependent diffeomorphisms tangent to the slices. For simplicity, we will assume that K and $\bar{K}$ only depend on r.

We are now ready to state our boundary conditions. They come in two sets:

1. $A_- = \bar{A}_+ = 0$ on Σ_r.

2. $a_+ = \frac{i}{\ell}L_{-1} + 0\,L_0 + \mathcal{O}(1)L_1$ and $\bar{a}_- = -\frac{i}{\ell}L_{-1} + 0\,L_0 + \mathcal{O}(1)L_1$ on Σ_r, where L_{-1}, L_0, L_1 form the canonical $SL(2, \mathbb{R})$ algebra given in Appendix A.

The phase space in Eddington-Finkelstein coordinates clearly obeys the boundary conditions, with $K = \bar{K}^{-1}$ given above. In fact, the phase space in Fefferman-Graham coordinates also obeys the boundary conditions, once we realize that the definition of $SL(2, \mathbb{R})$ generators in the boundary conditions is related to the choice of generators in (5.44) via the inner automorphism $\hat{\sigma}$ of the algebra defined in Appendix A. More precisely, we have the following relationship between the reduced connections obtained from Fefferman-Graham and Eddington-Finkelstein coordinates (and our choice of basis and dreibein):

$$a^{EF} = \hat{\sigma}^{-1} a^{FG} \hat{\sigma}, \qquad \bar{a}^{EF} = \bar{a}^{FG}. \tag{5.48}$$

Therefore, for Σ_∞, $K = \bar{K}^{-1} = \text{diag}(r^{-1/2}, r^{1/2})$ and after applying the automorphism on the a sector, the boundary conditions exactly coincide with the ones of [158].

Note that in the two phase spaces that we considered, one has $\partial_- A_+ = 0$ and $\partial_+ \bar{A}_+ = 0$. These conditions are not part of the boundary conditions but are only on-shell conditions.

5.3 Hamiltonian reduction

The Hamiltonian reduction in Fefferman-Graham gauge on the conformal boundary Σ_∞ is well known to lead to Liouville theory [158]. More precisely, the reduction of the entire bulk has two boundaries, one at the future and one at the past boundary. Here, we generalize this result to a Hamiltonian reduction performed over an arbitrary bulk region. We distinguish a piece of bulk bounded by two spacelike surfaces Σ_r^+ and Σ_r^- in the upper and lower diamond, and a piece of bulk bounded by one timelike surface Σ_r in either the northern or southern patch, see Figures 5.3 and 5.4.

We will carefully derive all steps in the reduction procedure in the upcoming sections. In section 5.3.2 we will emphasize the new features arising from the reality conditions occurring in the Eddington-Finkelstein gauge instead of the Fefferman-Graham gauge. We will see that it is then convenient to perform a Gauss decomposition of the $SL(2, \mathbb{C})$ element far from the identity.

5.3.1 Reduction to the non-chiral $SL(2, \mathbb{C})$ WZW model

The first set of boundary conditions allows us to reduce the two Chern-Simons theories to the non-chiral $SL(2, \mathbb{C})$ WZW model. Let us start by specifying the boundary terms in the action. We denote the coordinates[1] as (t, ϕ, r), $\phi \sim \phi + 2\pi$ and define $t^\pm = t \pm i\ell\phi$.

---

[1]The following derivation does not assume any choice of gauge. The t coordinate might as well be denoted as u in Eddington-Finkelstein coordinates.
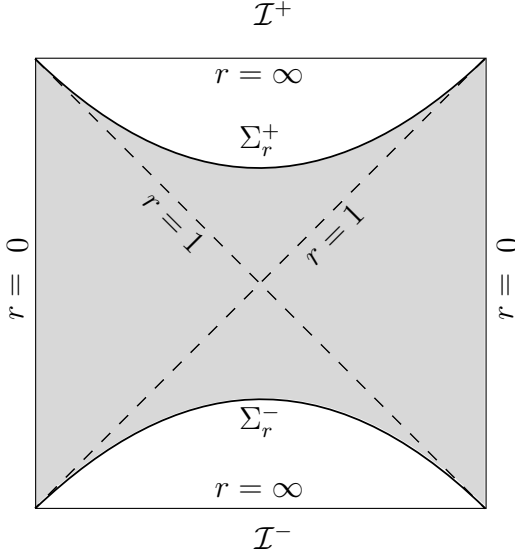
Figure 5.3: Bulk region bounded by Σ_r^+ and Σ_r^-
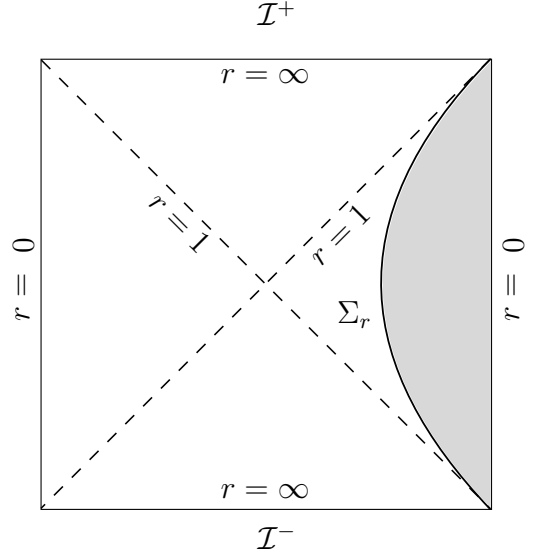
Figure 5.4: Bulk region bounded by Σ_r

We define

$$
\begin{aligned}
S_k[A, \bar{A}] &= \frac{k}{4\pi} \int_{Bulk} \operatorname{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{k}{4\pi} \int_{\partial Bulk} dt\, d\phi \, \operatorname{Tr}(A_t A_\phi), \\
&= \frac{k}{4\pi} \int_{Bulk} d^3 x \, \operatorname{Tr}\left(2 A_t F_{\phi r} - \partial_t A_r A_\phi + \partial_t A_\phi A_r \right).
\end{aligned}
$$

Here $\partial Bulk$ is the boundary of the bulk region under consideration at fixed radius r (with one connected component Σ_r or two connected components $\Sigma_r^\pm$). Then, the variation of the full action $S_E[A, \bar{A}]$ given in (5.25) is

$$
\delta S_E[A, \bar{A}] = \frac{ik}{2\pi} \int_{\partial Bulk} dt\, d\phi \, \operatorname{Tr}\left(A_t \delta A_\phi - \bar{A}_t \delta \bar{A}_\phi \right). \tag{5.49}
$$

From the first boundary condition, we deduce that a consistent variational principle is given by

$$
S_{total} = S_E[A, \bar{A}] - \frac{k}{4\pi \ell} \int_{\partial Bulk} dt\, d\phi \, \operatorname{Tr}\left(A_\phi^2 + \bar{A}_\phi^2 \right). \tag{5.50}
$$

We observe that A_t is the Lagrange multiplier for the constraint $F_{r\phi} = 0$. This constraint on the initial data implies that locally A_r and A_ϕ are pure gauge locally,

$$
A_i = G^{-1} \partial_i G, \qquad i = r, \phi, \tag{5.51}
$$

where $G \in SL(2, \mathbb{C})$. It is convenient to choose $A_t = G^{-1} \partial_t G$ as a gauge condition on the Lagrange multiplier A_t. We then have $A = G^{-1} dG$ and similarly $\bar{A} = \bar{G}^{-1} d\bar{G}$ with $\bar{G} \in SL(2, \mathbb{C})$. We will assume that the decomposition holds globally (no holonomies).

Using the orientation $\epsilon^{tr\phi} = 1$, we have after imposing the constraint,

$$S_k[A] = -\frac{k}{4\pi}\int_{Bulk}\frac{1}{3}\text{Tr}(G^{-1}dG)^3 - \frac{k}{4\pi}\int_{\partial Bulk} dt\, d\phi\,\text{Tr}\left(g^{-1}\partial_t g g^{-1}\partial_\phi g\right). \qquad (5.52)$$

where we defined $g = (GK)|_{\Sigma_r}$ as the pull-back of G times K defined in (5.47) on Σ_r. We also define $\bar{g} = (\bar{G}\bar{K})|_{\Sigma_r}$.

Therefore, the action is the sum of two chiral WZW models,

$$S_{total} = \frac{ki}{4\pi}S_{WZW}[g] - \frac{ki}{4\pi}\bar{S}_{WZW}[\bar{g}] \qquad (5.53)$$

with

$$\begin{aligned} S_{WZW}[g] &= \frac{1}{3}\int_{Bulk}\text{Tr}\left(G^{-1}dG\right)^3 + 2\int_{\partial Bulk} dt\, d\phi\,\text{Tr}\left(g^{-1}\partial_- g g^{-1}\partial_\phi g\right), \\ \bar{S}_{WZW}[\bar{g}] &= \frac{1}{3}\int_{Bulk}\text{Tr}\left(\bar{G}^{-1}d\bar{G}\right)^3 + 2\int_{\partial Bulk} dt\, d\phi\,\text{Tr}\left(\bar{g}^{-1}\partial_+ \bar{g}\bar{g}^{-1}\partial_\phi \bar{g}\right). \end{aligned} \qquad (5.54)$$

These first order actions describe respectively a right-moving group element $g(t^+)$ and a left-moving group element $g(t^-)$. One thus has $A_- = \bar{A}_+ = 0$ on-shell. The first set of boundary conditions is therefore compatible with the equations of motion of the WZW action.

Additionally, we could reformulate the combination of two chiral WZW models as one non-chiral WZW model. To perform this rewriting, one defines $h \equiv g^{-1}\bar{g}$ and $H \equiv G^{-1}\bar{G} = Kh\bar{K}^{-1}$.

We are allowed to trade the variables from g and $\bar{g}$ to h and $\Pi \equiv -\bar{g}^{-1}\partial_\phi g g^{-1}\bar{g} - \bar{g}^{-1}\partial_\phi \bar{g}$. The action then reads (using the analog of the formula (2.102))

$$S_{total} = \frac{ik}{4\pi}\left(\int dt\, d\phi\,\text{Tr}\,(\frac{i}{2\ell}\Pi^2 + \frac{i}{2\ell}h^{-1}\partial_\phi h\, h^{-1}\partial_\phi h + \Pi h^{-1}\partial_t h) - \Gamma[H]\right). \qquad (5.55)$$

Eliminating the auxiliary variable Π by its equation of motion, one finally gets

$$S_{total} = -\frac{k\ell}{2\pi}\int_{\partial Bulk} dt\, d\phi\,\text{Tr}\left(h^{-1}\partial_+ h h^{-1}\partial_- h\right) - \frac{ik}{12\pi}\int_{Bulk}\text{Tr}\left(H^{-1}dH\right)^3 \qquad (5.56)$$

which is the standard non-chiral $SL(2,\mathbb{C})$ WZW action for h. It agrees with [158].

One can express the action in local form upon performing a Gauss decomposition of the form

$$H = \begin{pmatrix} 1 & \hat{X} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\frac{1}{2}\hat{\Phi}} & 0 \\ 0 & e^{-\frac{1}{2}\hat{\Phi}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \hat{Y} & 1 \end{pmatrix}, \qquad (5.57)$$

where $\hat{X}, \hat{Y}, \hat{\Phi}$ depend not only on u, ϕ but also on r. We assume that the decomposition holds globally. For subtleties in the presence of global obstructions, see [69]. The latter Gauss decomposition allows to rewrite the 3-dimensional integral in (5.56) as 2-dimensional integrals using the relation

$$\frac{1}{3}\text{Tr}(H^{-1}dH)^3 = d^3x\;\epsilon^{\alpha\beta\gamma}\,\partial_\alpha\left(e^{-\hat{\Phi}}\partial_\beta\hat{X}\,\partial_\gamma\hat{Y}\right). \qquad (5.58)$$

The 2-dimensional integral in (5.56) can be rewritten equivalently by replacing h by $H|_\Sigma$ since all factors of $K, \bar{K}$ exactly cancel in the trace. We can then combine all terms (keeping only the radial boundary term) and we find

$$S_{total} = -\frac{kl}{2\pi} \int_{\partial Bulk} dt \, d\phi \left(2e^{-\hat{\Phi}} \partial_- \hat{X} \, \partial_+ \hat{Y} + \frac{1}{2} \partial_- \hat{\Phi} \, \partial_+ \hat{\Phi} \right) , \qquad (5.59)$$

where all fields $\hat{X}, \hat{Y}, \hat{\Phi}$ have been pull-backed on $\partial Bulk$ which is either Σ_r or $\Sigma_r^+ \cup \Sigma_r^-$.

5.3.2 Reality condition and Gauss decomposition

Even though the Chern-Simons connection is complex, it describes a real metric and spin connection. Therefore, there is a reality condition on the connection components, whose precise form depends upon the basis of $SL(2,\mathbb{C})$ generators used to express the connection in components. Moreover, there is also a reality condition on the $SL(2,\mathbb{C})$ elements $K, \bar{K}$ used to define the reduced gauge connection. It reflects the fact that the submanifold spanned by (t^+, t^-) is a real submanifold.

In Eddington-Finkelstein coordinates, we encountered the reality condition

$$(EF) \qquad A^\dagger = -\hat{\sigma} \bar{A} \hat{\sigma}, \qquad \hat{\sigma}^2 = \mathbb{I}, \qquad \hat{\sigma}^\dagger = \hat{\sigma} \qquad (5.60)$$

together with $\left(\bar{K}^{-1}\right)^\dagger \hat{\sigma} K = \hat{\sigma} = \bar{K}^{-1} \hat{\sigma} \left(K^{-1}\right)^\dagger$, see section 5.2.2.

In Fefferman-Graham coordinates , we encountered the different reality condition

$$(FG) \qquad A^\dagger = \sigma \bar{A} \sigma, \qquad \sigma^2 = -\mathbb{I}, \qquad \sigma^\dagger = -\sigma \qquad (5.61)$$

together with $\left(\bar{K}^{-1}\right)^\dagger \sigma K = \sigma = \bar{K}^{-1} \sigma \left(K^{-1}\right)^\dagger$, see section 5.2.1. The matrices $\hat{\sigma}$ and σ were defined in (5.46) and (5.38) respectively. They are defined up to an irrelevant overall sign.

We expect that there might be other reality conditions in other gauges but we will limit our discussion to two cases above.

In the case (EF), one finds $\bar{G}^{-1} = \hat{\sigma} G^\dagger \tau$ where $\tau \in SL(2,\mathbb{C})$ and upon choosing $\tau^\dagger = -\tau$, one has $H^\dagger = -\hat{\sigma} H \hat{\sigma}$. This then implies $h^\dagger = -\hat{\sigma} h \hat{\sigma}$. In the case (FG), one finds $\bar{G}^{-1} = \sigma G^\dagger \tau$ where $\tau \in SL(2,\mathbb{C})$ and again upon choosing $\tau^\dagger = -\tau$, one has $H^\dagger = -\sigma H \sigma$. This then implies $h^\dagger = -\sigma h \sigma$.

In case (FG), as discussed in [158], the matrix h takes the form

$$h_{(FG)} = \begin{pmatrix} u & w \\ -\bar{w} & v \end{pmatrix} \qquad (5.62)$$

with $u, v \in \mathbb{R}$, $w \in \mathbb{C}$ and $uv + w\bar{w} = 1$ while in case (EF), the matrix h takes the form

$$h_{(EF)} = \begin{pmatrix} z & ir_1 \\ ir_2 & \bar{z} \end{pmatrix} \qquad (5.63)$$

with $z \in \mathbb{C}$, $r_1, r_2 \in \mathbb{R}$ and $\bar{z}z + r_1 r_2 = 1$.

We observe that one can relate these $SL(2, \mathbb{C})$ elements as

$$h_{(EF)} = \hat{\sigma} h_{(FG)} \sigma \tag{5.64}$$

which reads in components as

$$u = \text{Re}z + \frac{1}{2}(r_1 - r_2), \quad v = \text{Re}z - \frac{1}{2}(r_1 - r_2), \quad w = \text{Im}z + \frac{i}{2}(r_1 + r_2). \tag{5.65}$$

The group manifold $SL(2, \mathbb{R})$ can be completely covered with the help of 4 coordinate patches. It is natural to use the coordinate patch close to the identity in the case (FG), as done in [158], using the Gauss decomposition

$$h_{(FG)} = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\Phi} & 0 \\ 0 & e^{-\frac{1}{2}\Phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}, \tag{5.66}$$

where X, Y, and Φ are function of the coordinates on the slice, t^+, t^-. Then, the reality conditions imply $Y = -\bar{X}$ and Φ to be real. After imposing the second set of boundary conditions as discussed in the next section, Φ will turn out to be the real Liouville field.

In case (EF) it is then convenient to use the relation (5.64) with the Gauss decomposition (5.66). It is easy to see that this coordinate patch for $h_{(EF)}$ does not cover the identity.

On the one hand, in the (FG) case, comparing the Gauss decompositions (5.66) and (5.57) and evaluating $K = \bar{K}^{-1} = \exp(-\log r L_0)$ at fixed $r = r_\Sigma$ we obtain

$$\hat{X} = \frac{1}{r_\Sigma} X, \qquad \hat{Y} = \frac{1}{r_\Sigma} Y, \qquad e^{\hat{\Phi}} = \frac{1}{r_\Sigma^2} e^{\Phi}. \tag{5.67}$$

On the other hand, in the (EF) case, comparing the Gauss decompositions (5.64)-(5.66) and (5.57) and using the values of $K = \bar{K}^{-1} = \exp(-\frac{i}{2\ell} r L_1)$ at fixed $r = r_\Sigma$ we obtain

$$\hat{X} = X + \frac{i r_\Sigma}{2\ell}, \qquad \hat{Y} = Y + \frac{i r_\Sigma}{2\ell}, \qquad e^{\hat{\Phi}} = e^{\Phi}. \tag{5.68}$$

In both cases, the action (5.59) reduces to

$$S_{total} = -\frac{k\ell}{2\pi} \int_{\partial Bulk} dt d\phi \left(2e^{-\Phi} \partial_- X \, \partial_+ Y + \frac{1}{2} \partial_- \Phi \, \partial_+ \Phi \right) \tag{5.69}$$

which is the standard action for the WZW theory. All radial dependence in the action has disappeared. The only possible difference between the Fefferman-Graham and Eddington-Finkelstein cases is the definition of the boundary $\partial Bulk$.

5.3.3 Further reduction to Liouville theory

The second set of boundary conditions on the gauge fields further reduces the WZW model to a Liouville action.

The boundary conditions were written down in the language of the gauge field components. Let us first rewrite these boundary conditions in terms of the $SL(2, \mathbb{C})$ element

h. One way to proceed is to consider the left and right moving WZW currents. They are given by

$$j_a = h^{-1}\partial_a h, \qquad \bar{j}_a = -\partial_a h h^{-1}. \tag{5.70}$$

Using the definition of $h = g^{-1}\bar{g}$ we deduce

$$j_- = -h^{-1}a_- h + \bar{a}_-, \qquad \bar{j}_+ = a_+ - h\bar{a}_+ h^{-1}. \tag{5.71}$$

Using the first set of boundary conditions $a_- = \bar{a}_+ = 0$, we obtain a simple relation between h, the WZW currents, and the gauge fields: $j_- = h^{-1}\partial_- h = \bar{a}_-, \bar{j}_+ = -\partial_+ h h^{-1} = a_+$.

For the Fefferman-Graham and Eddington-Finkelstein choices of the $SL(2,\mathbb{C})$ generators we have

$$(FG) \qquad j_-^1 - i\,j_-^2 = \frac{2i}{\ell}, \qquad \bar{j}_+^1 + i\,\bar{j}_+^2 = -\frac{2i}{\ell}, \qquad j_-^0 = \bar{j}_+^0 = 0,$$

$$(EF) \qquad j_-^1 = \frac{2i}{\ell}, \qquad\qquad \bar{j}_+^1 = -\frac{2i}{\ell}, \qquad j_-^2 = \bar{j}_+^2 = 0 \,. \tag{5.72}$$

In either case, the first pair of conditions are first class among themselves. The second pair of conditions can be understood as a gauge condition for the symmetry generated by the first pair, as discussed in [66, 158].

Using the appropriate Gauss decomposition discussed in the last section, one can rewrite those constraints in terms of the Φ, X, Y coordinates, with $Y = -\bar{X}$ and Φ real. In both cases, (EF) or (FG), the first two constraints are exactly

$$e^{-\Phi}\partial_- X = \frac{i}{\ell}, \qquad e^{-\Phi}\partial_+ Y = \frac{i}{\ell} \,. \tag{5.73}$$

and the second set of constraints, once combined with the first, becomes

$$X = \frac{i\ell}{2}\partial_+\Phi, \qquad Y = \frac{i\ell}{2}\partial_-\Phi \,. \tag{5.74}$$

The constraints are independent of the radius r_Σ and independent of the choice of (EF) or (FG) slicing.

Before inserting the constraints we have to make sure that the action obeys the variational principle. This is the case once we add an improvement term to the action (5.69):

$$S_{\text{impr}} = S_{total} + \frac{k\ell}{2\pi}\int_0^{2\pi} d\phi \left(e^{-\Phi}\left(X\partial_+ Y + Y\partial_- X\right)\right)\Big|_{t_1}^{t_2} \,. \tag{5.75}$$

After inserting the constraints we are left with the Liouville action

$$S_{\text{impr}} = -\frac{k\ell}{2\pi}\int_{\partial Bulk} dt\,d\phi \left(\frac{1}{2}\partial_+\Phi\,\partial_-\Phi + \frac{2}{\ell^2}\exp\Phi\right) \,. \tag{5.76}$$

Note that the boundary term in (5.75) contributes as $-2\frac{k\ell}{2\pi}\int_{\partial Bulk} dt d\phi \frac{2}{\ell^2}e^\Phi$.

The final action is therefore the Liouville action evaluated on the boundary of the bulk region, which can be either two connected components $\Sigma_r^\pm$ or one connected component Σ_r, see figures 5.3 and 5.4. One can write the Liouville action in covariant form upon coupling it to a metric of Euclidean signature. It is bizarre that when one chooses the radial slice Σ_r in the static patch, t is a time coordinate in spacetime, while it is still a Euclidean coordinate of the boundary action.

5.4   Conclusion and discussion

In this chapter, we first recalled the important fact that, by introducing a modified version of the Lie bracket, one sees that the asymptotic symmetry group is not limited to act at the conformal boundary [22]. Instead, one can extend the notion of the "asymptotic" symmetries anywhere into the bulk. Following this idea, we then showed that the associated surface charges for three-dimensional Einstein gravity could be as well extended in the bulk of spacetime since they turn out the be r-independent, and this fact holds independently of the sign of the cosmological constant. A convenient way to define the generators everywhere in the bulk makes use of Eddington-Finkelstein type coordinates which were thoroughly used e.g. in [100]. As a result, in the de Sitter case, the conformal group acts naturally in the static patch beyond the cosmological horizon. This provided consistent boundary conditions (which are compatible with conformal symmetry) on any fixed radial slice and in particular close to the horizon.

It was then natural to perform the Hamiltonian reduction of Einstein gravity in the static patch, taking as a boundary a Lorentzian signature fixed radial slice Σ_r with boundary conditions preserving the conformal group. Naively, one might expect to find Lorentzian Liouville theory. This turned out not to be the case. The Hamiltonian reduction is *in fine* independent of the chosen radial slice. Since a fixed radial slice close to $\mathcal{I}^+$ leads to the Euclidean Liouville theory, the same theory was found on a fixed radial slice inside the static patch, namely

$$S_{EH} = -\frac{\ell^2}{64\pi G} \int d\phi \, dt \left((\partial_t \Phi)^2 + \frac{1}{\ell^2} (\partial_\phi \Phi)^2 + \frac{16}{\ell^2} e^\Phi \right) , \qquad (5.77)$$

where the boundary terms of the Einstein-Hilbert action S_{EH} were chosen to enforce the boundary conditions. The awkward feature is now that t, the Euclidean time in the boundary field theory, is a timelike coordinate of the boundary Σ_r. Overall, our result is consistent with the dS/CFT conjecture [28]: we find a Euclidean CFT, even when the holographic boundary is a timelike cylinder in the static patch. Note that there is no holographic RG flow in the sense of [174] since no bulk fields are integrated out upon displacing the holographic boundary into the bulk.

Our derivation can be extended in a straightforward manner to higher spin fields as long as no propagating degrees of freedom are involved. We expect that the notion of asymptotic symmetry can be realized everywhere in the bulk and we similarly expect that the Hamiltonian reduction can be done on any slice in the bulk without any dependence on the choice of slice. The addition of propagating modes on the other hand is non-trivial and further analysis would be required.

CHAPTER 6

Holographic entropy of Warped-AdS$_3$ black holes

In this chapter, we will study the asymptotic symmetries of spaces that are not asymptotically (A)dS, but rather a *warped* deformation of it that is particularly relevant because of different reasons: On the one hand, these spaces provide a new example of the so-called non-AdS holography, that is of the proposal to generalize AdS/CFT holographic duality to cases in which the gravity side is not given by an asymptotically Anti-de Sitter spaces (AdS) space, and this is interesting on its own right. On the other hand, these Warped AdS$_3$ spaces (WAdS$_3$) are related to other problems in physics, such as Kerr/CFT correspondence, Schrödinger spaces in non-relativistic holography, lower-spin gravity, among others.

WAdS$_3$ spaces are squashed or stretched deformations of AdS$_3$ [33] and have very interesting applications [175, 176, 177]. One of the most salient properties of these spaces is the fact that they admit black holes [34]. This permits to explore the black hole physics from the holographic point of view in a setup that goes beyond the asymptotically AdS examples.

In the recent years, different proposals for a WAdS$_3$/CFT$_2$ correspondence have been explored [29, 178, 179]. One of such proposals, dubbed WAdS/WCFT, states that asymptotically WAdS$_3$ geometries, including black holes, are dual to what has been called a *warped conformal field theory* (WCFT), i.e. a peculiar type of scale invariant two-dimensional theory that lacks of Lorentz invariance. In [179], this realization was studied in the case of Topologically Massive Gravity (TMG) and String Theory. Here, we will discuss WAdS$_3$/CFT$_2$ correspondence in a new setup, namely in the context of the parity-even three-dimensional massive gravity known as New Massive Gravity (NMG). We will give strong evidence supporting the dual description of quantum gravity about WAdS$_3$ spaces in terms of the WCFT$_2$ description.

We will study the asymptotic symmetries of WAdS$_3$ in NMG and we will find that the asymptotic symmetry algebra is infinite-dimensional and coincides with the semidirect sum of Virasoro algebra with non-vanishing central charge and an affine $\hat{u}(1)_k$ Kac-Moody algebra. We will identify the Virasoro generators that organize the states associated to the WAdS$_3$ black hole configurations, and by applying the Cardy formula, we will prove that

the microscopic computation exactly reproduces the entropy of the WAdS$_3$ black holes. In addition, we will explain why the WCFT$_2$ version of the Cardy formula proposed in [179] also reproduces the right result.

The chapter is organized as follows: In section 6.1, we briefly review the theory of massive gravity considered. In section 6.2, we move to the geometry of WAdS$_3$ space: the first part 6.2.1 focuses on the timelike case and presents the original results obtained in [180], namely the computation of conserved charges in NMG. The second part of the section (6.2.2) reviews the properties of the spacelike WAdS$_3$ black hole. The rest of the chapter contains original contributions based on [181, 182]. In section 6.3, we study the asymptotic isometries in WAdS$_3$ spaces and compute the algebra of charges associated to the asymptotic Killing vectors, which is found to be the semidirect sum of Virasoro algebra and the affine $\hat{u}(1)_k$ Kac-Moody algebra. We also study the representations of this conformal algebra and identify the states that correspond to the black hole configurations in the bulk. In section 6.4, we show how the black hole entropy is reproduced by the (W)CFT dual computation. We also make some remarks about the inner black hole mechanics; namely, about the relation between thermodynamics properties that one can formally attribute to the inner black hole horizon. We explicitly check that this is in perfect agreement with previous conjectures [183]. Finally, in section 6.5, we present an extension of our results to a more general set of boundary conditions in WAdS$_3$ space, which manifestly shows that the holographic computation of the WAdS$_3$ black hole entropy is robust in the sense that it still holds when configurations with more relaxed asymptotic are considered. We give explicit examples of solutions fulfilling such relaxed boundary conditions.

6.1   Three-dimensional massive gravity

A feature that makes WAdS$_3$ spaces of particular interest is that these geometries appear as exact solutions of a large variety of models, including string theory [184, 185, 186], topologically massive gauge theories [187, 188, 189, 190, 191], higher-derivative theories [192], bi-gravity theories [193], and Einstein gravity non-minimally coupled to matter fields [194]. A minimal setup in which WAdS$_3$ spaces appear is three-dimensional gravity with no matter fields. Indeed, spacelike and timelike WAdS$_3$ geometries are exact solutions of pure three-dimensional gravity provided one gives a small mass to the graviton. In three-dimensions, there are different manners to give mass to the graviton in a consistent way. Here, we will adopt the particular parity-even theory of massive gravity proposed in Ref. [195], usually called New Massive Gravity (NMG), which we will review in this section.

New Massive Gravity is a parity-even theory of gravity in three dimensions which, when linearized around maximally symmetric backgrounds, coincides with massive spin-2 Fierz-Pauli action. Therefore, at a generic point of the space of parameters, it propagates two local degrees of freedom.

NMG is defined by the action

$$\mathcal{I} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(R - 2\Lambda + \frac{1}{m^2}(R_{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2) \right), \tag{6.1}$$

where m represents the mass of the graviton.

The equations of motion derived from (6.1) take the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{m^2}K_{\mu\nu} = 0, \tag{6.2}$$

which, apart from the Einstein tensor, involve the tensor

$$\begin{aligned} K_{\mu\nu} =& 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} \\ & - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}. \end{aligned} \tag{6.3}$$

The relative coefficient 3/8 between the two quadratic terms in action (6.1) is crucial: it is such that the trace of the equations of motion (6.2) does not involve the mode $\Box R$. This is why, for instance, NMG is free of ghost about flat space.

Let us recall the other main properties of theory (6.1): as we said, it propagates two degrees of freedom corresponding to a spin 2 particle (unlike General Relativy, which has no none, or Topologically Massive Gravity which has only one), which makes of this theory a very good toy model for four-dimensional gravity. In addition, NMG admits a rich set of solutions, such as Schrödinger invariant spaces [196], Lifshitz spaces and Lifshitz black holes [197], logarithmic deformation of the Bañados-Teitelboim-Zanelli geometry [198], hairy (A)dS₃ black holes [199], WAdS₃ black holes, and others [192, 200].

6.2   Warped AdS₃ Spaces

As said, WAdS₃ spaces are solutions of NMG [201]. These geometries are squashed or stretched deformations of AdS₃ space [33]. More precisely, one starts with AdS₃ metric in coordinates $x, y, \tau \in \mathbb{R}$

$$ds^2 = \frac{\ell^2}{4} \left(-\cosh^2 x \, d\tau^2 + dx^2 + (dy + \sinh x \, d\tau)^2 \right), \tag{6.4}$$

and then introduces a deformation parameter $K \in \mathbb{R}$:

$$ds^2 = \frac{\ell_K^2}{4} \left(-\cosh^2 x \, d\tau^2 + dx^2 + K(dy + \sinh x \, d\tau)^2 \right). \tag{6.5}$$

This deformation amounts to writing AdS₃ as a Hopf fibration of $\mathbb{R}$ over AdS₂ and then multiplying the fiber by the constant warp factor K. Usually, one parameterizes the deformation by a positive constant ν defined by $K = 4\nu^2/(\nu^2 + 3)$, such that the case $\nu = 1$ corresponds to undeformed, or unwarped, AdS₃. Through the deformation, the AdS₃ radius ℓ gets also rescaled as $\ell^2 \to \ell_K^2 = 4\ell^2/(\nu^2 + 3)$. Spaces (6.5) with $\nu^2 > 1$ describe *stretched* AdS₃ spaces, while those with $\nu^2 < 1$ describe *squashed* deformations of AdS₃. Through a double Wick rotation $(x, \tau) \to (ix, i\tau)$ one goes from the *spacelike* WAdS₃ metric (6.5) to a *timelike* analog of it. We will analyze these two cases separately.

6.2.1 Timelike WAdS$_3$ space

To organize the discussion in a convenient way, let us begin by considering the timelike WAdS$_3$ space. This geometry is important for our discussion as it will be ultimately associated to the vacuum of the WAdS$_3$ black hole spectrum we are interested in.

Timelike WAdS$_3$ from Gödel metric

The four-dimensional Gödel cosmological solution is the direct product of the real line, $\mathbb{R}$, and a three-dimensional manifold Σ equipped with a metric [202, 203]

$$ds^2 = -\left(d\hat{t} + e^{\sqrt{2}\omega x}dy\right)^2 + dx^2 + \frac{1}{2}e^{2\sqrt{2}\omega x}dy^2, \qquad (6.6)$$

with coordinates $x, y, \hat{t} \in \mathbb{R}$, and ω being a real parameter that represents the *vorticity* of the Gödel solution. This coordinate system gives a complete chart of the space, and the four-dimensional solution is then homeomorphic to $\mathbb{R}^4$. The space is geodesically complete, and hence singularity free; it is spatially homogeneous, though non-isotropic.

In a convenient system of coordinates, metric (6.6) above takes the form

$$ds^2 = -\left(dt + \frac{2}{\omega}\sinh^2\left(\frac{\omega\rho}{\sqrt{2}}\right)d\phi\right)^2 + \frac{1}{2\omega^2}\sinh^2(\sqrt{2}\omega\rho)d\phi^2 + d\rho^2, \qquad (6.7)$$

where the three-dimensional metric is now written as a Hopf fiber over the hyperbolic plane. This space exhibits closed timelike curves (CTCs), as it can be seen from the role played by coordinates t and ϕ in the first term of (6.7).

The prominent properties of the Gödel space persist if one considers a particular one-parameter deformation of the metric (6.7) which, in particular, permits to interpolate between the three-dimensional section of Gödel space and AdS$_3$ [204]. This deformation is given by the metric

$$ds^2 = -\left(dt + \frac{4\omega}{\lambda^2}\sinh^2\left(\frac{\lambda\rho}{2}\right)d\phi\right)^2 + \frac{\sinh^2(\lambda\rho)}{\lambda^2}d\phi^2 + d\rho^2, \qquad (6.8)$$

which, apart from the vorticity ω, includes an additional real parameter λ that controls the deformation. For the particular value $\lambda^2 = 2\omega^2$, metric (6.8) corresponds to the three-dimensional section of Gödel solution (6.7); when $\lambda^2 = 4\omega^2$ it corresponds to the universal covering of AdS$_3$. For generic values of λ and ω within the range $0 \leq \lambda^2 \leq 4\omega^2$, metric (6.8) describes the timelike stretched WAdS$_3$ spaces we will be concerned with.

It is convenient to consider a slightly different parameterization: Define the parameter

$$\ell^2 = \frac{2}{\lambda^2 - 2\omega^2}, \qquad (6.9)$$

and then use ω and ℓ^2 (instead of λ) to describe the family of WAdS$_3$ metrics. For instance, in terms of ω and ℓ^2, the Gödel solution corresponds to $\ell^2 = \infty$, while AdS$_3$ space corresponds to $\ell^2 = \omega^{-2}$. The range $0 \leq \lambda^2 \leq 4\omega^2$, in terms of these parameters, translates into $|\omega^2\ell^2| \geq 1$. Notice that $\omega^2\ell^2$ may take values between -1 and $-\infty$. Spaces

with $|\omega^2\ell^2| < 1$ are also interesting, although present a different causal structure; they correspond to the timelike *squashed* WAdS₃ spaces.

Now, continuing with the convenient changes of coordinates, define the new radial variable $r = 2\lambda^{-2}\sinh^2(\lambda\rho/2)$, such that $r \in \mathbb{R}_{\geq 0}$. Metric (6.8) now reads

$$ds^2 = -dt^2 - 4\omega r dt d\phi + 2\left(r + (\ell^{-2} - \omega^2)r^2\right)d\phi^2 + \frac{dr^2}{2\left(r + (\ell^{-2} + \omega^2)r^2\right)}. \qquad (6.10)$$

This is one of the standard ways of representing timelike WAdS₃ space. The curvature invariants associated to this metric are constant, and take the remarkably succinct form

$$R_{\mu_n}^{\ \mu_1}R_{\mu_1}^{\ \mu_2}R_{\mu_2}^{\ \mu_3}...R_{\mu_{n-1}}^{\ \mu_n} = (-1)^n\frac{2^n}{\ell^{2n}}(\omega^{2n}\ell^{2n} + 2). \qquad (6.11)$$

Another interesting property of metric (6.7) is that it is spatially homogeneous. As it happens with the universal covering of AdS, the WAdS spaces are not globally hyperbolic.

The isometry group of WAdS₃ spaces (6.10) is $SL(2,\mathbb{R}) \times U(1)$, which is generated by four out of the five Killing vectors that Gödel solution admits. This isometry is the remnant piece of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry group of AdS₃ that survives through the stretched/squashed deformation.

From (6.10), it is easy to verify that in the special point $\omega^2\ell^2 = 1$, the solution coincides with AdS₃ space. Indeed, defining the new coordinates $\theta = t - \phi$ and $\rho^2 = 2r$ and replacing $\omega = \ell = 1$ in (6.10), gives

$$ds^2_{\text{AdS}_3} = -(\rho^2 + 1)dt^2 + \frac{d\rho^2}{(\rho^2 + 1)} + \rho^2 d\theta^2. \qquad (6.12)$$

Introducing a defect

Let us now introduce a pointlike defect in spacetime (6.10). This is achieved by performing the change

$$\phi \to (1 - \mu)\varphi, \quad \text{with} \quad 0 \leq \mu < 1, \qquad (6.13)$$

while keeping the same periodicity for the φ coordinate, namely $\varphi \in [0, 2\pi)$. This certainly changes the global properties of the space in a way that is equivalent to introducing an angular deficit $\delta\phi = \mu/(2\pi)$ in the original angular coordinate. By doing (6.13) and rescaling the radial coordinate as $r \to r/(1 - \mu)$ one finds the metric

$$\begin{aligned} ds^2 = &-dt^2 - 4\omega r dt d\varphi + 2r\left((\ell^{-2} - \omega^2)r + (1 - \mu)\right)d\varphi^2 \\ &+ \frac{dr^2}{2r\left((\omega^2 + \ell^{-2})r + (1 - \mu)\right)}, \end{aligned} \qquad (6.14)$$

where $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, and $\varphi \in [0, 2\pi)$. This metric shares the asymptotic behavior with (6.10); namely both have the large r behavior

$$ds^2 = -dt^2 - 4\omega r dt d\varphi + 2(\ell^{-2} - \omega^2)r^2 d\varphi^2 + \frac{dr^2}{2r^2(\ell^{-2} + \omega^2)} + h_{\mu\nu}dx^\mu dx^\nu, \qquad (6.15)$$

with, in particular, $\delta g_{\varphi\varphi} \equiv h_{\varphi\varphi} \simeq \mathcal{O}(r)$ and $\delta g_{rr} \equiv h_{rr} \simeq \mathcal{O}(r^{-3})$.

Metric (6.14) represents a particle-like object located at $r = 0$, in the bulk of Gödel universe. The object disappears when μ tends to zero, which permits to anticipate that μ is somehow related to the mass of the defect. More general defects will be introduced later (see (6.16) below), which will represent spinning point particles in Gödel spacetime.

Spinning point particles in Gödel spacetime

Let us now consider more general defects which represent spinning point particles in Gödel spacetime. The metric of Gödel spacetime with both mass and angular momentum now reads[1]

$$ds^2 = -dt^2 - 4\omega r dt d\varphi + \frac{dr^2}{(2r^2\omega^2 + \lambda_{\mu,j}(r))} - \left(2r^2\omega^2 - \lambda_{\mu,j}(r)\right) d\varphi^2, \tag{6.16}$$

where

$$\lambda_{\mu,j}(r) = \frac{2r^2}{\ell^2} + 2(1-\mu)r - j\ell^2, \tag{6.17}$$

and where $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, $0 \leq \mu \leq 1$, and $\phi \in [0, 2\pi)$. Metric (6.16) involves a new parameter $j \in \mathbb{R}$, and reduces to (6.14) when $j = 0$. Notice also that only $\xi^t \sim \partial_t$ and $\xi^\varphi \sim \partial_\varphi$ out of the four generators of $SL(2,\mathbb{R}) \times U(1)$ survive as exact Killing vectors of the metric (6.16).

Such as in the case of the parameter μ, the introduction of j is achieved by means of a (improper, i.e. not globally well-defined) diffeomorphism from metric (6.10).

Conserved charges in NMG

As we said earlier, timelike WAdS₃ geometries are exact solutions of massive gravity; the graviton mass is what ultimately induces the *vorticity* required to support the Gödel universe or, more precisely, the three-dimensional non-trivial part of it.

The metric (6.16) solves NMG equations of motion (6.2) for the particular choice of parameters

$$m^2 = -\frac{(19\omega^2\ell^2 - 2)}{2\ell^2}, \qquad \Lambda = -\frac{(11\omega^4\ell^4 + 28\omega^2\ell^2 - 4)}{2\ell^2(19\omega^2\ell^2 - 2)}. \tag{6.18}$$

Recall that AdS₃ space corresponds to $\omega^2\ell^2 = 1$, for which $\Lambda = -35/(34\ell^2)$ and $m^2 = -17/(2\ell^2)$.

The mass and angular momentum of the spinning defect (6.16) in NMG can be computed in the covariant formalism (see below); it gives[2][180]

$$\mathcal{M}_{\text{Göd}} = \frac{4(\mu-1)\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)}, \tag{6.19}$$

---

[1]Notice that we can assign dimensions to the parameters and coordinates as follows: $[t] = l^1, [r] = l^2, [\phi] = l^0, [\ell] = l^1, [\omega] = l^{-1}, [\mu] = l^0, [j] = l^0$, where l has dimension of length.

[2]This result is up to μ-independent and j-independent terms, which can not be gathered in the integration.

and

$$\mathcal{J}_{\text{Göd}} = -\frac{4j\ell^4\omega^3}{G(19\ell^2\omega^2 - 2)}, \tag{6.20}$$

for the mass and the angular momentum of the solution (6.16), respectively. Notice that, as expected, the angular momentum changes its sign when ω does so.

A special case to consider is the actual Gödel spacetime, which corresponds to the limit $\ell \to \infty$. In this case, the mass formula (6.19) yields

$$\mathcal{M}_{\text{Göd}} = \frac{4(\mu - 1)}{19G}, \tag{6.21}$$

which is independent of ω. For $\mu = 0$ the result is negative and will be of crucial importance in the study of the spacelike WAdS$_3$ black hole spectrum in section 6.4.

Another special case to analyze is the AdS$_2 \times \mathbb{R}$ space. This corresponds to the limit $\omega \to 0$. To see this explicitly, we define coordinate $\tilde{\rho}^2 = 1 + 4(r^2/\ell^4 + r/\ell^2)$, in which the metric for $\omega = 0$ takes the form

$$ds^2_{|\omega=0} = -dt^2 + ds^2_{\text{AdS}_2} = -dt^2 + \frac{\ell^2}{2}(\tilde{\rho}^2 - 1)d\phi^2 + \frac{\ell^2}{2}\frac{d\tilde{\rho}^2}{(\tilde{\rho}^2 - 1)}. \tag{6.22}$$

In this case, the mass also tends to zero,

$$\mathcal{M}_{\mathbb{R}\times\text{AdS}_2} = 0. \tag{6.23}$$

Locally AdS$_2 \times \mathbb{R}$ spaces appear in the limit in which (6.18) yields $\Lambda = -m^2$ [199].

In [180], another method was used to compute the quasi-local gravitational energy, where a boundary stress-tensor for NMG was defined, which is the generalization of the Brown-York quasi-local stress-tensor. For NMG theory, such a tensor exists and has been defined in Ref. [205], and can be used to compute the mass of the defect in timelike WAdS$_3$ as seen from infinity, i.e. from the region that is beyond the radius where CTCs appear. Intriguingly, the Brown-York quasi-local energy obtained through this approach gives only one half of the mass (6.19). In addition, the definition of charges in terms of the quasi-local stress-tensor was shown not to be suitable to compute the angular momentum of spinning defects, the failure being associated to the impossibility of regularizing the boundary stress-tensor by means of local counterterms. The question remains as to whether it is possible to formulate a holographic renormalization recipe in WAdS$_3$ spaces. Indeed, this phenomenon had also been observed both in TMG and in NMG for the case of spacelike WAdS$_3$ [206, 207], suggesting this is a general feature of this type of backgrounds. Whether or not this problem is related to the lack of Lorentz invariance in the dual theory is still to be understood.

6.2.2   WAdS$_3$ black holes

Let us move to the analysis of the spacelike WAdS$_3$ spaces. In particular, we will be interested in the black hole geometries found in [34, 201], which at large distance

asymptote squashed spacelike WAdS$_3$ space. The metric of these black holes is

$$
\begin{aligned}
\frac{ds^2}{l^2} = {}& dt^2 + \frac{dr^2}{(\nu^2+3)(r-r_+)(r-r_-)} + (2\nu r - \sqrt{r_+ r_-(\nu^2+3)})dt d\varphi \\
& + \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)} \right] d\varphi^2,
\end{aligned}
\tag{6.24}
$$

and solves NMG equations of motion for the values of parameters

$$
m^2 = -\frac{(20\nu^2-3)}{2l^2}, \quad \Lambda = -\frac{m^2(4\nu^4 - 48\nu^2 + 9)}{(400\nu^4 - 120\nu^2 + 9)}.
\tag{6.25}
$$

These black holes can be obtained from the timelike solution by means of a change of coordinates involving a double Wick rotation, detailed in Appendix E.

The conserved charges of WAdS$_3$ black holes have been computed by different methods [201, 206, 208]. The mass is given by

$$
\mathcal{M} = Q_{\partial_t} = \frac{\nu(\nu^2+3)}{Gl(20\nu^2-3)}\left((r_- + r_+)\nu - \sqrt{r_+ r_-(\nu^2+3)} \right),
\tag{6.26}
$$

while the angular momentum is given by

$$
\mathcal{J} = Q_{\partial_\varphi} = \frac{\nu(\nu^2+3)}{4Gl(20\nu^2-3)}\left((5\nu^2+3)r_+ r_- - 2\nu\sqrt{r_+ r_-(\nu^2+3)}(r_+ + r_-) \right).
\tag{6.27}
$$

Black holes (6.24) include extremal configurations, corresponding to $r_+ = r_-$. In those cases, the angular momentum saturates the condition

$$
\mathcal{J} \leq \frac{Gl(20\nu^2-3)}{4\nu(\nu^2+3)}\mathcal{M}^2,
\tag{6.28}
$$

which is the necessary condition for the existence of horizons. Condition (6.28) is supplemented with $\mathcal{M} \geq 0$.

Black hole solutions (6.24) are obtained from the timelike WAdS$_3$ space (6.16) by means of global identifications [29], in the same way as BTZ black holes [38] are obtained from AdS$_3$ as discrete quotients [39]. This orbifold construction preserves a $U(1) \times U(1)$ subgroup of $SL(2,\mathbb{R}) \times U(1)$ isometries, which is generated by the two Killing vectors

$$
\xi^{(1)} = \partial_t \,, \quad \xi^{(2)} = -\partial_\varphi \,.
\tag{6.29}
$$

The global identifications, generated by to Killing vectors (6.29), generate two periods β_R and β_L. The inverse of these periods yield the two *geometrical* temperatures

$$
\begin{aligned}
T_R &= \beta_R^{-1} = \frac{(\nu^2+3)}{8\pi l^2}(r_+ - r_-), \\
T_L &= \beta_L^{-1} = \frac{(\nu^2+3)}{8\pi l^2}\left(r_+ + r_- - \frac{1}{\nu}\sqrt{(\nu^2+3)r_- r_+}\right).
\end{aligned}
\tag{6.30}
$$

It is important to point out that these temperatures are not the Hawking temperature of the black hole, which can be computed and found to be

$$T_H = \frac{(\nu^2 + 3)}{4\pi l \nu} \frac{T_R}{T_R + T_L} = \frac{(\nu^2 + 3)}{4\pi l} \frac{(r_+ - r_-)}{2\nu r_+ - \sqrt{(\nu^2 + 3)r_- r_+}}. \tag{6.31}$$

The entropy, which can be computed with the Wald formula, is given by

$$S_{\text{BH}} = \frac{8\pi \nu^3}{(20\nu^2 - 3)G}(r_+ - \frac{1}{2\nu}\sqrt{(\nu^2 + 3)r_- r_+}), \tag{6.32}$$

and reads, in terms of the charges (6.26)-(6.27),

$$S_{\text{BH}} = \frac{4\pi l \nu}{(\nu^2 + 3)}(\mathcal{M} + \sqrt{\mathcal{M}^2 - k\mathcal{J}}), \tag{6.33}$$

where $k = 4\nu(3 + \nu^2)/(Gl(20\nu^2 - 3))$. This way of writing the entropy will be important for our purpose.

Let us also mention that the quantities above satisfy the first law of the black hole thermodynamics

$$d\mathcal{M} = T_H dS_{\text{BH}} + \Omega_H \, d\mathcal{J} \tag{6.34}$$

where Ω_H is the angular velocity associated to the horizon; namely

$$\Omega_H = \frac{2}{2\nu r_+ - \sqrt{(\nu^2 + 3)r_- r_+}}. \tag{6.35}$$

6.3 Asymptotic symmetries

6.3.1 Asymptotic isometry algebra

In this section, we will study the notion of asymptotically WAdS$_3$ spaces. To do this, first we choose as a background metric, g, the solution (6.24) with $r_+ = 0 = r_-$; and then we impose the Compère-Detournay boundary conditions proposed in [209], namely[1]

$$\begin{aligned}
g_{tt} &= l^2 + \mathcal{O}(r^{-1}), & g_{tr} &= \mathcal{O}(r^{-2}), \\
g_{t\varphi} &= l^2 \nu r + \mathcal{O}(1), & g_{rr} &= \frac{l^2}{(\nu^2 + 3)r^2} + \mathcal{O}(r^{-3}), \\
g_{\varphi\varphi} &= \frac{3}{4}r^2 l^2(\nu^2 - 1) + \mathcal{O}(r), & g_{r\varphi} &= \mathcal{O}(r^{-1}),
\end{aligned} \tag{6.36}$$

which include in particular the black hole solutions (6.24). Notice that, although NMG is parity even, unlike TMG, the boundary conditions (6.36) distinguish between right and left. This is because of the term linear in ν in the leading part of the component $g_{t\varphi}$. In fact, the property of NMG of being parity even is associated to the fact that analogous

---

[1] Originally, these boundary conditions were proposed in the context of Topologically Massive Gravity.

boundary conditions for $-\nu$ are possible, and not to the fact that boundary conditions are themselves parity even. In other words, imposing boundary conditions demands, in particular, to specify the sign of ν. Here, with no loss of generality, we consider ν positive.

The set of asymptotic diffeomorphisms allowed by these boundary conditions are

$$
\begin{aligned}
\ell_n &= (inre^{-in\varphi} + \mathcal{O}(1))\partial_r - (e^{-in\varphi} + \mathcal{O}(r^{-2}))\partial_\varphi, \\
t_n &= (e^{in\varphi} + \mathcal{O}(r^{-1}))\partial_t,
\end{aligned}
\tag{6.37}
$$

where $n \in \mathbb{Z}$. Indeed, acting with ℓ_n, t_n on a metric obeying (6.36) leads to a perturbation obeying the same falling-off conditions.

The generators (6.37) satisfy the algebra

$$
\begin{aligned}
i[\ell_m, \ell_n] &= (m - n)\ell_{m+n}, \\
i[\ell_m, t_n] &= -nt_{m+n}, \\
i[t_m, t_n] &= 0.
\end{aligned}
\tag{6.38}
$$

This is the semidirect sum of Witt algebra and the loop algebra of $u(1)$.

6.3.2 Algebra of charges

We use here the covariant formalism presented in section 5.1.2 to compute conserved charges associated to an asymptotic Killing vector ξ. Expression (5.14) for the charges reads, in three-dimensions,

$$
\delta\!\!\!/ Q_\xi[h, g] = \frac{1}{16\pi G} \int_0^{2\pi} \sqrt{-g}\, \epsilon_{\mu\nu\varphi}\, k_\xi^{\mu\nu}[h, g]d\varphi,
\tag{6.39}
$$

Here, the potential $k_\xi^{\mu\nu}[h, g]$ of the linearized theory is not merely the one of pure Einstein gravity but rather the one corresponding to New Massive Gravity: it has the form

$$
k_{\text{NMG}}^{\mu\nu} = k_{\text{GR}}^{\mu\nu} + \frac{1}{2m^2}k_K^{\mu\nu},
\tag{6.40}
$$

where the first contribution comes from the pure GR part of the equations of motion, given by (5.15), while the piece $k_K^{\mu\nu} = k_{R_2}^{\mu\nu} - \frac{3}{8}k_{R^2}^{\mu\nu}$ takes into account terms arising from the $K_{\mu\nu}$ tensor. The latter was explicitly computed in [208], with the following expressions:

$$
\begin{aligned}
k_{R^2}^{\mu\nu} &= 2R\, k_{\text{GR}}^{\mu\nu} + 4\xi^{[\mu}D^{\nu]}\delta R + 2\delta R D^{[\mu}\xi^{\nu]} - 2\xi^{[\mu}h^{\nu]\alpha}D_\alpha R, \\
k_{R_2}^{\mu\nu} &= D^2 k_{\text{GR}}^{\mu\nu} + \frac{1}{2}k_{R^2}^{\mu\nu} - 2k_{\text{GR}}^{\alpha[\mu}R_\alpha^{\nu]} - 2D^\alpha\xi^\beta D_\alpha D^{[\mu}h_\beta^{\nu]} - 4\xi^\alpha R_{\alpha\beta}D^{[\mu}h^{\nu]\beta} \\
&\quad - Rh_\alpha^{[\mu}D^{\nu]}\xi^\alpha + 2\xi^{[\mu}R_\alpha^{\nu]}D_\beta h^{\alpha\beta} + 2\xi_\alpha R^{\alpha[\mu}D_\beta h^{\nu]\beta} + 2\xi^\alpha h^{\beta[\mu}D_\beta R_\alpha^{\nu]} \\
&\quad + 2h^{\alpha\beta}\xi^{[\mu}D_\alpha R_\beta^{\nu]} - (\delta R + 2R^{\alpha\beta}h_{\alpha\beta})D^{[\mu}\xi^{\nu]} - 3\xi^\alpha R_\alpha^{[\mu}D^{\nu]}h - \xi^{[\mu}R^{\nu]\alpha}D_\alpha h,
\end{aligned}
\tag{6.41}
$$

and where $\delta R = (-R^{\alpha\beta}h_{\alpha\beta} + D^\alpha D^\beta h_{\alpha\beta} - D^2 h)$.

As a direct application[1] of formula (6.39), the black hole mass and angular momentum (6.26)-(6.27) can be recovered by computing the charges associated to the Killing vectors ∂_t and ∂_φ respectively.

Before moving to the algebra of charges, a comment is in order: In the case of asymptotic Killing vectors, the one-form potential to be considered is given by $k_\xi[\delta g, g] + k_\xi^S[\delta g, \mathcal{L}_\xi g]$, where the second term is a supplementary contribution linear in the Killing equation and its derivatives (for the pure gravity case, the supplementary term is the last term in (5.15)). Let us recall that this term is at the origin of the difference between the conserved charges in the Barnich-Brandt-Compère formalism [172, 142] and in covariant phase space methods à la Iyer-Wald [173]. However, in most of the cases, this term does not contribute to any charge. For instance, in the case of the WAdS black hole solution (6.24) and with the asymptotic Killing vectors (6.37), this term has been shown to be of order $\mathcal{O}(r^{-1})$ in Topologically Massive Gravity (TMG) [211]. An easy way to see that it will not contribute to the asymptotic charge neither here is to notice that the piece coming from the $K_{\mu\nu}$ tensor of NMG can only contain terms proportional to the second derivative of the Killing equation, and therefore will be at most of order $\mathcal{O}(r^{-1})$, as it happens in TMG.

If we denote the charges differences between the black hole solution (6.24) and the background g by $L_n = Q_{\ell_n}$, $P_n = Q_{t_n}$, we find the following charge algebra

$$
\begin{aligned}
i\{L_m, L_n\} &= (m - n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}, \\
i\{L_m, P_n\} &= -nP_{m+n}, \\
i\{P_m, P_n\} &= \frac{k}{2}m\delta_{m+n,0},
\end{aligned}
\tag{6.42}
$$

where

$$
c = \frac{96l\nu^3}{G(20\nu^4 + 57\nu^2 - 9)},
\tag{6.43}
$$

and

$$
k = \frac{4\nu(3 + \nu^2)}{Gl(20\nu^2 - 3)}.
\tag{6.44}
$$

Algebra (6.42) is equivalent to the semidirect sum of Virasoro algebra with central charge c and the affine $\hat{u}(1)_k$ Kac-Moody algebra of level k.

Note that the value (6.43) obtained coincides with the value of the central charge conjectured in [206], which leads to reproduce the entropy of WAdS$_3$ black holes (6.24); namely

$$
S_{\text{BH}} = \frac{\pi^2 l}{3}c(T_R + T_L).
\tag{6.45}
$$

We will come back to this matching later.

---

[1]These expressions for the potential in NMG can be implemented in a Mathematica code, using the Package SurfaceCharges [210].

Before concluding this section, let us mention that the computation of asymptotic charges can also be carried out in the case of timelike WAdS$_3$ spaces. The algebra obtained in that case is the same, with the central charge c and the level k given by

$$c = \frac{48\ell^4\omega^3}{G(19\ell^4\omega^4 + 17\ell^2\omega^2 - 2)} \,, \qquad k = \frac{8\omega(1 + \ell^2\omega^2)}{G(19\ell^2\omega^2 - 2)}. \tag{6.46}$$

respectively, where $\omega = \nu/l$ and $\omega^2\ell^2 + 2 = 3\ell^2/l^2$. As a consistency check of this result, one can observe that the value of c tends to the Brown-Henneaux central charge in NMG, namely $c_{\text{AdS}} = 3\ell/(2G)(1 + 1/(2m^2\ell^2))$ in the limit $\omega^2\ell^2 = 1$.

6.3.3   Unitary highest-weight representations

Algebra (6.42) admits a simple automorphism, given by the spectral flow transformation

$$P_n \to \tilde{P}_n = P_n + p_0\delta_{n,0}, \tag{6.47}$$

with p_0 being an arbitrary complex number. This one-parameter transformation, which in the case of $\hat{u}(1)_k$ algebra merely amounts to shift the zero-mode of P_n, has to be taken into account when building up the highest-weight representations.

We can now play the standard game and promote charges L_n and P_n to the rank of operators acting on a vector space whose elements are represented by quantum states $|v\rangle$. This amounts to replace the Poisson brackets in (6.42) by commutators, namely $i\{,\} \to [,]$. In addition, for these operators we have the hermiticity relations[1]

$$P_n^\dagger = P_{-n}, \qquad L_n^\dagger = L_{-n}. \tag{6.48}$$

Since, in particular, $[L_0, \tilde{P}_0] = 0$, then one can construct the highest-weight representations starting with the primary states $|v\rangle = |h, p, p_0\rangle$, labeled by three complex parameters h, p, p_0 corresponding to the eigenvalues

$$L_0|h, p, p_0\rangle = h \ |h, p, p_0\rangle, \qquad \tilde{P}_0|h, p, p_0\rangle = p \ |h, p, p_0\rangle, \tag{6.49}$$

and imposing

$$L_n|h, p, p_0\rangle = 0, \qquad \tilde{P}_n|h, p, p_0\rangle = 0, \qquad \forall n > 0 \tag{6.50}$$

where p_0 refers to which spectrally flowed sector the state corresponds to. For instance, the state of the $p_0 = 0$ sector obeys $P_0|h, p, 0\rangle = p|h, p, 0\rangle$ and $\tilde{P}_0|h, p, 0\rangle = (p+p_0)|h, p, 0\rangle$, where $\tilde{P}_0$ is defined as in (6.47). This invites to identify states $|h, p - p_0, 0\rangle$ with states $|h, p, p_0\rangle$ for all p_0. This seems trivial in the case of $\hat{u}(1)_k$ affine algebra, but spectral flow acts in a non-trivial way on algebras such as $\hat{su}(2)_k$ or $\hat{sl}(2)_k$, of which $\hat{u}(1)_k$ is a subalgebra, mapping in the former cases Kac-Moody primary states to descendents and, in the case $\hat{sl}(2)_k$, generating new representations.

Descendent states are then defined by acting on primaries $|h, p, p_0\rangle$ with arrays of positive modes P_{-n} and L_{-n} with $n \geq 0$.

---

[1]We omit hats.

Unitarity constraints are derived from algebra (6.42) in the usual way. In particular, this yields the conditions on c and k, together with the dimension h and momentum p_0 of the states. More precisely, demanding $\|L_{-1}|h, p, p_0\rangle\|^2 \geq 0$ yields $h \geq 0$; analogously, $\|P_0|h, p, p_0\rangle\|^2 \geq 0$ implies $p \in \mathbb{R}$. Spectral flow symmetry (6.47) also implies $p_0 \in \mathbb{R}$. On the other hand, positivity of $\|L_{-n}|h, p, p_0\rangle\|^2$ (for large n) and $\|P_{-1}|h, p, p_0\rangle\|^2$ yields

$$c > 0 \quad \text{and} \quad k \geq 0, \tag{6.51}$$

respectively. To be able to identify the the Virasoro operators associated to the black hole spectrum that are bounded from below, we first define

$$L_n^- \equiv \frac{1}{k} \sum_m : P_{-n-m} P_m :, \tag{6.52}$$

where $:\ :$ stands for *normal ordering*. Operators L_n^- obey Virasoro algebra[1] with $c_- = 1$, and satisfy

$$[L_m^-, P_n] = -n P_{-m+n}. \tag{6.53}$$

This is nothing but the Sugawara construction in the case of $\hat{u}(1)_k$; see also [212].

Secondly, we define operators

$$L_n^+ \equiv L_n^- + L_n, \tag{6.54}$$

which also generate a Virasoro algebra.

Notice from (6.42) and (6.53) that operators L_n^+ commute with P_m and, consequently, one finds two commuting Virasoro algebras; namely

$$\begin{aligned}
[L_m^\pm, L_n^\pm] &= (m - n) L_{m+n}^\pm + \frac{c^\pm}{12} m^3 \delta_{m+n,0}, \\
[L_n^+, L_m^-] &= 0.
\end{aligned} \tag{6.55}$$

where $c^+ = c + 1$, $c^- = 1$.

In addition, unlike what happens with Virasoro algebra generated by L_n, operators $L_n^\pm$ evaluated on the black hole spectrum turn out to be bounded from below. To see this explicitly, notice that the energy spectrum $L_0^\pm$ is given by

$$h^+ = \frac{1}{k} \mathcal{M}^2 - \mathcal{J}, \quad h^- = \frac{1}{k} \mathcal{M}^2, \tag{6.56}$$

where $h^\pm$ refer to the eigenvalue of $L_0^\pm$. Therefore, we observe that the black hole spectrum is such that both L_0^+ and L_0^- are bounded from below. Indeed, from (6.28) we have

$$\frac{\mathcal{M}^2}{k} - \mathcal{J} = \frac{\nu^2 k}{16} (r_+ - r_-)^2 \geq 0, \tag{6.57}$$

which implies the bounds

$$L_0^\pm \geq 0. \tag{6.58}$$

In the next subsection we will see how the microstates representing black hole configurations (6.24) seem to organize themselves in representations of Virasoro algebras generated by $L_n^\pm$. Strong evidence of that is the CFT$_2$ rederivation of the black hole entropy (6.32).

---

[1]Notice also that if one applies spectral flow transformation (6.47), one verifies that the zero mode changes as follows $L_0^- \to L_0^- - \frac{2p_0}{k} P_0^2 - \frac{p_0^2}{k}$.

6.4   (W)CFT$_2$ and microscopic entropy

In Ref. [179], a Cardy-like formula for Warped conformal field theories (WCFTs) has been proposed. This formula is supposed to give the asymptotic growth of states in the dual theory, which would lack of full Lorentz invariance. We will describe below how such a formula actually comes from the usual Cardy formula for the two Virasoro algebras generated by $L_n^\pm$.

First, we write the standard CFT Cardy formula

$$S_{\text{CFT}} = 2\pi\sqrt{-4L_0^{-(\text{vac})}L_0^-} + 2\pi\sqrt{-4L_0^{+(\text{vac})}L_0^+}, \tag{6.59}$$

where $L_0^{\pm(\text{vac})}$ correspond to the minimum values of $L_0^\pm$, i.e. the value of the *vacuum geometry*. It is important to remark that the way of writing Cardy formula in (6.59) admits the possibility of the spectrum of $L_0^\pm$ to exhibit a gap with respect to the value $-c^\pm/24$. More precisely, it takes into account that in theories with such a gap, the saddle point approximation involved in the derivation of the Cardy formula yields an effective central charge c_{eff} given by $c_{\text{eff}}/6 = -4L_0^{(\text{vac})}$.

Recalling that $L_0^- = \tilde{P}_0^2/k$, formula (6.59) reads

$$S_{\text{WCFT}} = \frac{4\pi i}{k}\tilde{P}_0^{(\text{vac})}\tilde{P}_0 + 4\pi\sqrt{-L_0^{+(\text{vac})}L_0^+}. \tag{6.60}$$

In turn, the only remaining ingredient needed to apply this formula is to find out which is the right *vacuum geometry*. Because the theory is parity even, we naturally expect the vacuum geometry to be[1] $\mathcal{J}^{(\text{vac})} = 0$; that is to say,

$$\tilde{P}_0^{(\text{vac})} = \mathcal{M}^{(\text{vac})}, \qquad L_0^{+(\text{vac})} = \frac{1}{k}(\mathcal{M}^{(\text{vac})})^2. \tag{6.61}$$

This yields

$$S_{\text{CFT}} = \frac{4\pi i}{k}\mathcal{M}^{(\text{vac})}(\mathcal{M} + \sqrt{\mathcal{M}^2 - k\mathcal{J}}). \tag{6.62}$$

And, indeed, we verify that entropy (6.32) exactly matches formula (6.60) if one identifies the vacuum geometry with the Gödel geometry (6.16); namely

$$\mathcal{M}^{(\text{vac})} = i\mathcal{M}_{\text{Göd}} = -i\frac{4\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)}. \tag{6.63}$$

The identification of timelike WAdS$_3$ (6.63) spacetime as the vacuum geometry of the spacelike WAdS$_3$ black hole spectrum is something that had been observed in [179] for the case of TMG and String Theory. Here we obtain the similar result for the case of NMG. It is natural, on the other hand, that the vacuum geometry preserves the full $SL(2,\mathbb{R}) \times U(1)$. See Figure 6.1.

In conclusion, we have shown that

$$S_{\text{BH}} = S_{\text{CFT}}. \tag{6.64}$$

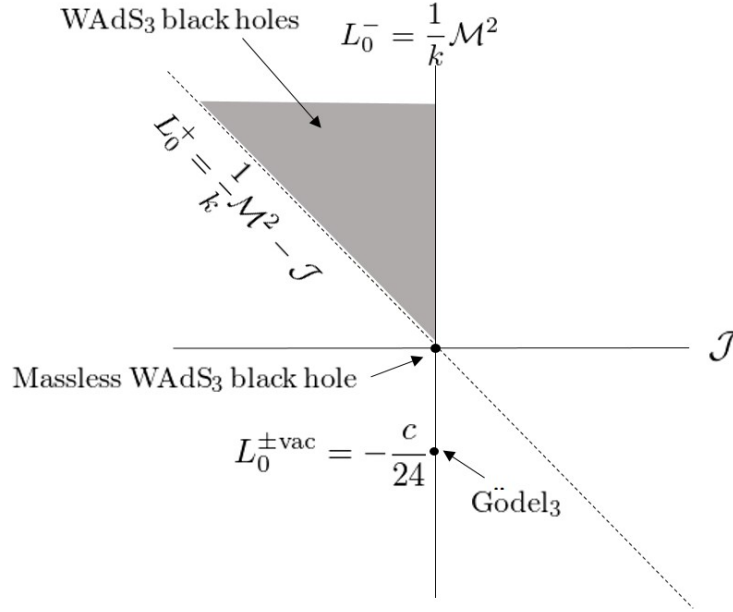[1]This is different from the case of TMG.

Figure 6.1: Spectrum of the WAdS black hole. The relative asymmetry is due to the choice of a given sign of ν: similar boundary conditions can be imposed for the opposite sign, resulting in the mirror image of this diagram.

This proves that, for the particular case of massive gravity, the WAdS$_3$ black hole entropy can be microscopically described in terms of a dual CFT$_2$. More precisely, we showed that the states of the dual theory that corresponds to the black hole configurations organize in highest weight representations of two mutually commuting Virasoro algebras, and that the Cardy formula for the asymptotic growth of such states exactly reproduces the black hole entropy.

Before concluding this section, let us make some remarks about the central charges involved in the Cardy formula. Notice that while the central charge c entering in (6.45) is associated to the Virasoro algebra generated by L_n, the Cardy formula (6.59) corresponds to the CFT$_2$ whose symmetries are generated by the two copies of Virasoro algebras generated by $L_n^\pm$. Therefore, the question that arises is how these two alternative ways of expressing the black hole entropy are related. To understand this, it is necessary to notice first that the minimum value of the operators $L_0^\pm$ are given by

$$L_0^{\pm(\text{vac})} = -\frac{c}{24}. \tag{6.65}$$

Then, one notices that the *effective* central charges $c_{\text{eff}}^\pm = -24 L_0^{\pm(\text{vac})}$, which are the actual quantities that enter in the Cardy formula, are indeed given by

$$c_{\text{eff}}^\pm = c. \tag{6.66}$$

In other words, (6.45) can be also written as

$$S_{\text{BH}} = \frac{\pi^2 l}{3}(c_{\text{eff}}^+ T_L + c_{\text{eff}}^- T_R), \tag{6.67}$$

explaining why the matching oberved in [206] actually works.

6.4.1 Inner black hole mechanics

Let us now comment on the so-called black hole inner mechanics, which has been proposed in [183] and which can be regarded as a further consistency check of the conjecture of a dual holographic description of the WAdS₃ black holes in terms of a CFT.

Inner black hole mechanics establishes the following two facts:

- The thermodynamical quantities associated to the inner Killing horizon of a black hole satisfy a inner version of the first law of black hole entropy, as it happens with the thermodynamical quantities of the outer event horizon.

- The product of the entropies associated to all the horizons of a black hole is independent of the black hole mass and depends only on quantities such as angular momentum and charges.

The second of these statements, i.e. the one regarding the product of the entropies, can be thought of as a consistency condition of the dual description of the black hole thermodynamics in terms of a CFT₂. This is because of the following: In the same manner as that the entropy of the outer horizon is given in terms of the Cardy formula (6.59), it turns out that the entropy obtained by the Wald formula evaluated on the inner horizon r_-, which can be found to be

$$S_{\text{inner}} = \frac{8\pi\nu^3}{(20\nu^2 - 3)G}(r_- - \frac{1}{2\nu}\sqrt{(\nu^2 + 3)r_- r_+}), \tag{6.68}$$

is given by

$$S_{\text{inner}} = 2\pi\sqrt{-4L_0^{-(\text{vac})}L_0^-} - 2\pi\sqrt{-4L_0^{+(\text{vac})}L_0^+}. \tag{6.69}$$

This means that the product of the entropies associated to both horizons is

$$S_{\text{BH}}S_{\text{inner}} = 8\pi c(L_0^+ - L_0^-), \tag{6.70}$$

where we have used (6.65). Then, one concludes that $S_{\text{BH}}S_{\text{inner}}$ has to be quantized because in a CFT₂ the level matching condition demands

$$L_0^+ - L_0^- \in \mathbb{Z}. \tag{6.71}$$

Therefore, for this to be consistent, one expects the product of entropies $S_{\text{BH}}S_{\text{inner}}$ to depend only on conserved charges that, at quantum level, are supposed to be quantized as well. This is the case for the angular momentum, but not necessarily for the mass. Then, one has to verify that, indeed, the product (6.70) is independent of the mass. A direct computation shows that, actually,

$$S_{\text{BH}}S_{\text{inner}} = -8\pi c\mathcal{J}, \tag{6.72}$$

and does not involve $\mathcal{M}$. This represents a further non-trivial consistency check of the CFT$_2$ dual proposal, specially because, as pointed out in [213], the inner black hole mechanics does not necessarily hold in a general higher-curvature model such as the one we are considering here.

About the first affirmation of the inner black hole mechanics, i.e. the one about the first principle holding on the inner Killing horizon as well, it is not difficult to check that the WAdS$_3$ black holes we considered here does obey such a relation. Indeed, computing the temperature associated to the inner horizon, one finds

$$T_{\text{inner}} = \frac{(\nu^2 + 3)}{4\pi l \nu} \frac{T_R}{T_R - T_L} = \frac{(\nu^2 + 3)}{4\pi l} \frac{(r_+ - r_-)}{2\nu r_- - \sqrt{(\nu^2 + 3)r_- r_+}}, \qquad (6.73)$$

and this can be seen to satisfy the relation

$$d\mathcal{M} = T_{\text{inner}} dS_{\text{inner}} + \Omega_{\text{inner}} \, d\mathcal{J}, \qquad (6.74)$$

where Ω_{inner} is the angular velocity associated to the inner horizon,

$$\Omega_{\text{inner}} = \frac{2}{2\nu r_- - \sqrt{(\nu^2 + 3)r_- r_+}}. \qquad (6.75)$$

6.5 WAdS$_3$/CFT$_2$ correspondence in presence of bulk massive gravitons

A crucial ingredient in the analysis done in the previous section is the asymptotic boundary conditions considered. In this section, we want to address the question whether a similar WAdS/CFT computation can still be carried out for a set of boundary conditions which, on top of the black hole solutions, admit also the presence of bulk gravitons[1]. That is, we will consider a one-parameter family of boundary conditions which, while accommodating the WAdS$_3$ black holes configurations, also gather non-locally WAdS$_3$ solutions.

The asymptotic boundary conditions we consider here are defined as follows: We first consider as reference background the metric (6.24) with $r_- = r_+ = 0$; we denote the metric of such geometry by ds_0^2. Then, the set of geometries to be considered are defined as those deformations of the form $ds^2 = ds_0^2 + \delta g_{\mu\nu} dx^\nu dx^\mu$ that respect the following fall-off conditions

$$\begin{aligned} \delta g_{tt} = \mathcal{O}(r^{-3})\,, \qquad \delta g_{tr} = \mathcal{O}(r^{\alpha-4})\,, \qquad \delta g_{t\varphi} = \mathcal{O}(r^{\alpha-1}), \\ \delta g_{rr} = \mathcal{O}(r^{\alpha-4})\,, \qquad \delta g_{r\varphi} = \mathcal{O}(r^{\alpha-2})\,, \qquad \delta g_{\varphi\varphi} = \mathcal{O}(r^{\alpha}). \end{aligned} \qquad (6.76)$$

where α is a real parameter that here we will assumed greater that 1. For $\alpha > 1$, asymptotic conditions (6.76) exhibit components that fall off slower that the ones considered in the previous section.

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[1]The term *bulk graviton* is used in opposition to the term *boundary graviton*, which is usually employed to refer to degrees of freedom associated to the representations of the asymptotic isometries; the latter being also present in theories with no local degrees of freedom.

Boundary conditions (6.76) are similar to those considered in the literature for the parity-odd theory [214] and it is worthwhile comparing such definition with the asymptotic boundary conditions of Ref. [209]. Fall-off conditions (6.76) accommodate, in particular, the black hole solutions (6.24), but also include other solutions: An example of such a solution is given by the Kerr-Schild ansatz

$$ds^2 = ds_0^2 + e^{\omega t} r^{-\frac{2\nu\omega}{3+\nu^2}} h(u) \; k_\mu k_\nu dx^\mu dx^\mu, \tag{6.77}$$

with the null vector k_μ

$$k_\mu dx^\mu = \frac{2}{3+\nu^2}\frac{dr}{r^2} - d\varphi \; , \tag{6.78}$$

$h(u)$ being an arbitrary periodic function of the variable $u \equiv \varphi + 2/((\nu^2+3)r)$ (hereafter, we set $l = 1$) and ω being a solution of the polynomial equation

$$P(\nu, \omega) \equiv -4\nu\omega^2 - 6\omega + 10\omega\nu^2 + 16\nu^3 - \omega^3 = 0. \tag{6.79}$$

Solutions (6.77)-(6.79) obeys the boundary conditions (6.76) with $\alpha = -2\nu\omega/(\nu^2+3)$ and are not locally equivalent to WAdS$_3$.

Asymptotic Killing vectors respecting the set of new boundary conditions (6.76) are given by

$$\ell_n = \frac{4\nu n^2}{(3+\nu^2)}\frac{1}{r}e^{-in\varphi}\partial_t + \frac{2\nu n^2 - r^2(3+\nu^2)}{(3+\nu^2)}\frac{1}{r^2}e^{-in\varphi}\partial_\varphi - inre^{-in\varphi}\partial_r + ...$$
$$t_n = (e^{in\varphi} + \mathcal{O}(r^{-1}))\partial_t + ... \tag{6.80}$$

where $n \in \mathbb{Z}$, and where the ellipses stand for sub-leading terms. In particular, these asymptotic Killing vectors include the exact Killing vectors $\ell_0 = -\partial_\varphi$ and $t_0 = \partial_t$. The full set of asymptotic Killing vectors satisfy the algebra

$$i[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad i[\ell_m, t_n] = -nt_{m+n}, \quad i[t_m, t_n] = 0, \tag{6.81}$$

which is a semi-direct sum of Witt algebra and the loop algebra of $u(1)$.

The next step is to compute the charges associated to vectors ℓ_n and t_n. This computation can be performed as explained in the previous section that is, by resorting to the covariant formalism, using the metric ds_0^2 as reference background, and the expression (5.14) for the variation of the charges in NMG. If we denote by L_n and P_n the charges associated to ℓ_n and t_n respectively, we recover the charge algebra (6.42) and where c and k coincides with the value appearing in (6.43), (6.44). Therefore, once again, one finds (6.64). This result thus strongly suggests that WAdS/CFT correspondence is still valid in presence of bulk gravitons.

6.6   Conclusions

In this chapter, we have computed the asymptotic symmetry algebra corresponding to Warped Anti-de Sitter (WAdS) spaces in three-dimensional massive gravity. We

have shown that this is given by the semi-direct sum of one Virasoro algebra (with non-vanishing central charge) and one affine $\hat{u}(1)_k$ Kac-Moody algebra. We have identified the precise Virasoro generators that organize the states associated to the WAdS$_3$ black hole configurations, which led us to rederive the WCFT entropy formula (6.60). Our result then can be thought of as a consistency check supporting the WCFT proposal of Ref. [179].

By applying the WCFT entropy formula, we have proved that the microscopic computation in the dual WCFT exactly reproduces the entropy of the WAdS$_3$ black holes. Essential ingredients for the matching (6.64) to hold are: on one hand, the definition of Virasoro algebras generated by $L_n^{\pm}$ as in (6.52) and (6.54) and, on the other hand, the identification of the vacuum geometry of the black hole spectrum as in (6.63). These ingredients agree with the recipe proposed in [179] for the cases of Topologically Massive Gravity and String Theory.

We then extended this computation to a set of asymptotic boundary conditions that, while still gathering the WAdS$_3$ black holes, also allow for new solutions that are not locally equivalent to WAdS$_3$ space, and therefore are associated to the local degrees of freedom of the theory (bulk massive gravitons).

As further direction, we can mention the study of the applications of WAdS$_3$/WCFT$_2$ to Kerr$_4$/CFT$_2$ correspondence. In Kerr/CFT in four (and higher) dimensions, the so-called Near Horizon Extremal Kerr (NHEK) geometry is closely related to the WAdS$_3$ spaces studied here [33, 29], and, therefore, the application of the holographic ideas developed for
WAdS$_3$/WCFT$_2$ to the more realistic case of four-dimensional spinning black holes is of principal interest.

CHAPTER 7

Conclusions

In this thesis, we have applied asymptotic symmetry techniques to investigate diverse extensions of holographic dualities appearing in gravitational scenarios that involve non-AdS backgrounds.

The first part of the thesis has focused on the case of asymptotically flat spacetimes, being the first example of a symmetry enhancement phenomenon, where the symmetry algebra at the boundary of the spacetime (in this case at null infinity) is infinitely enlarged compared to the rigid symmetry algebra of the bulk theory. For three-dimensional flat supergravity, we have shown that, with an appropriate choice of boundary conditions, the canonical generators associated to the asymptotic symmetries span a supersymmetric extension of the $\mathfrak{bms}_3$ algebra with a non-vanishing central charge [114]. We have also shown that the inclusion of parity odd terms in the gravitational action has the effect of turning on the central extension in the superrotation subalgebra, vanishing in the case of parity-preserving models. We have shown that our results could be equivalently obtained through a well-defined flat limit of the ones of AdS_3 supergravity, where one takes the AdS radius to infinity after having rescaled the generators in a suitable way. We then used two fundamental facts: the first is that three-dimensional gravity can be written as a Chern-Simons theory; the second is that the latter induces at the boundary a two-dimensional Wess-Zumino-Witten (WZW) model. In our case (but this also holds in the case of a non-vanishing cosmological constant), the role of the boundary is played by the non trivial fall-off conditions for the gauge field. In order to make solutions with the prescribed asymptotics be true extrema of the variational principle, we have added a suitable boundary term. By solving the constraints in the improved action, we were left with a chiral WZW theory based on the three-dimensional super-Poincaré algebra. Furthermore, the fall-off conditions lead to additional constraints that correspond to fixing a subset of the conserved currents of the WZW model. We have shown that the associated reduced phase space description is given by a supersymmetric extension of flat Liouville theory [115]. We have also provided a Lagrangian formulation of the two-dimensional field theory admitting a global super-$\mathfrak{bms}_3$ invariance in terms of a gauged chiral WZW theory.

We have then considered another region of physical relevance that is null and asymptotically flat: the near-horizon region of non-extremal black holes. We have constructed

sensible boundary conditions describing the nearby region of a three-dimensional black hole event horizon, where the surface gravity is kept fixed. We have constructed an explicit solution satisfying these fall-offs which has been shown to include the BTZ black hole as a particular case. Interestingly, both supertranslations and superrotations (different than the ones obtained at null infinity by Barnich and Troessaert) have been shown to arise close to the horizon [146].

The second part of the thesis has been dealing with two other non-AdS backgrounds that present a great interest as well.

We have discussed the issue of holographic scenarios in the case of a positive cosmological constant: In the dS/CFT proposal, the holographic screen where the CFT would be best defined is the future (or past) conformal boundary. However, a static observer is separated from this conformal boundary by a cosmological horizon of a thermal and entropic nature, raising questions on whether the holographic description extends all the way to the static observer. We have shown that an Euclidean Liouville theory also describes the dual dynamics of Einstein gravity with Dirichlet boundary conditions on a fixed timelike slice in the static patch [170]. As a prerequisite of this correspondence, we have demonstrated that the surface charge algebra which consists of two copies of the Virasoro algebra extends everywhere into the bulk of spacetime.

The last example studied was the case of stretched deformations of Anti-de Sitter spacetimes: the Warped AdS spaces. The latter exhibit less symmetry than the undeformed AdS spaces, admit black holes and are solutions of three-dimensional gravity, provided one gives a mass to the graviton. We have studied two sets of asymptotic symmetries of WAdS$_3$ in parity-preserving model of massive gravity (with and without boundary gravitons) and we have found that, in both cases, the asymptotic symmetry algebra coincides with the semidirect sum of Virasoro algebra with non-vanishing central charge and an affine Kac-Moody algebra [181, 182]. We have shown that a Cardy formula reproduces the entropy of the Warped black holes, the key point being to identify the vacuum geometry as the timelike spacetime, the Gödel geometry, whose conserved charges we had previously computed in [180].

We hope that the results presented in this thesis have given further evidence that the infinite-dimensional nature of asymptotic algebras is not a mere curiosity of AdS gravity models, but instead a fundamental feature of holographic scenarios in diverse backgrounds and at different boundaries (spatial and null infinity, but also at the event horizon of a black hole).

Recently, new connections emerged between BMS symmetries and the physics of gravity and matter in asymptotically flat spacetimes. First, in [24, 25], Strominger and collaborators have shown that the supertranslation subgroup of BMS$_4$ is a symmetry of both the classical gravitational scattering problem and the quantum gravitational S-matrix, where the BMS symmetry acts simultaneously on the future and past null infinity. This was then used to show that Ward identities associated to the supertranslation symmetry is equivalent to Weinberg's soft graviton theorem, a universal formula relating scattering amplitudes with and without soft graviton insertions and that is valid for any theory of gravity. Besides, this construction has been argued to be valid also in the case of abelian

and non-abelian gauge fields, whereby the classical asymptotic symmetry gives rise to soft photon and soft gluon theorems respectively [215, 216]. In the case of extended BMS symmetries [21], the new superrotation symmetry was claimed to give rise to new universal soft gravitons theorems [217]. Finally, the third connection relates BMS symmetries and gravitational memory effects following from the existence of a supertranslation field [26, 218]; in [219], it was shown that memory effects lead as final state of gravitational collapse a Schwarzschild black hole with supertranslation hair.

These interplays between various concepts such as asymptotic symmetries, gravitational memories, soft theorems and holography have profound implications, and suggest that many more fascinating features of gravity are still to be discovered.

APPENDICES

APPENDIX A

Conventions

Tangent space metric

Our conventions are such that the Levi-Civita symbol fulfills $\epsilon_{012} = 1$, and the tangent space metric η_{ab}, with $a = 0, 1, 2$, is off-diagonal, given by

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.1}$$

This definition leads to $ds^2 = \eta_{ab}e^a e^b = 2e^0 e^1 + (e^2)^2$.

$sl(2, \mathbb{R})$ generators

We define the generators j_a $(a, b = 0, 1, 2)$ as

$$j_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad j_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad j_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

which satisfy $[j_a, j_b] = \epsilon_{abc}j^c$, $\mathrm{Tr}(j_a j_b) = \frac{1}{2}\eta_{ab}$.

It is also useful to introduce the usual (Chevalley-Serre) generators

$$E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

which satisfy

$$[E_+, E_-] = H , \qquad [H, E_+] = 2E_+ , \qquad [H, E_-] = -2E_-. \tag{A.2}$$

Dirac matrices and spinors

The Dirac matrices in three spacetime dimensions satisfy the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\,\eta_{ab}$, and have been chosen as

$$\Gamma_0 = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \Gamma_1 = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.3}$$

The matrices fulfill the following useful properties:

$$\Gamma_a\Gamma_b = \epsilon_{abc}\Gamma^c + \eta_{ab}\mathbf{1} \quad , \quad (\Gamma^a)^\alpha{}_\beta(\Gamma_a)^\gamma{}_\delta = 2\delta^\alpha_\delta\delta^\gamma_\beta - \delta^\alpha_\beta\delta^\gamma_\delta \,, \tag{A.4}$$

where $\alpha = +1, -1$. The Majorana conjugate is defined as $\bar\psi_\alpha = C_{\alpha\beta}\psi^\beta$, where

$$C_{\alpha\beta} = \varepsilon_{\alpha\beta} = C^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.5}$$

stands for the charge conjugation matrix, which satisfies $C^T = -C$ and $C\Gamma_a C^{-1} = -(\Gamma_a)^T$. Note that this implies that $\overline{\Lambda^{-1}\psi} = \bar\psi\Lambda$, for any $\Lambda \in \mathrm{SL}(2,\mathbb{R})$. The conjugate of the product of real Grassmann variables is assumed to fulfill $(\theta_1\theta_2)^* = \theta_1\theta_2$.

$SL(2,\mathbb{C})$ basis

In the main text of chapter 5, we use real $SL(2,\mathbb{C})$ generators:

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tag{A.6}$$

with the commutation relations given by

$$[L_0, L_1] = -L_1 \,, \qquad [L_0, L_{-1}] = L_{-1} \,, \qquad [L_1, L_{-1}] = 2L_0 \,. \tag{A.7}$$

Furthermore, we define the automorphism of the algebra $\hat\sigma$ as

$$\hat\sigma(L_{-1}) = -L_1, \qquad \hat\sigma(L_1) = -L_{-1}, \qquad \hat\sigma(L_0) = -L_0, \tag{A.8}$$

where $\hat\sigma(a) = \hat\sigma^{-1}a\hat\sigma$. This automorphism exchanges the raising and lowering Lie algebra elements. We also refer to the $SL(2,\mathbb{C})$ element $\hat\sigma = i(L_1 + L_{-1})$ with the same notation.

APPENDIX B

Note on gauged Wess-Zumino-Witten models

B.1   Introduction and standard gauged WZW models

The non-chiral WZW action based on a connected real Lie group G is given by

$$S[g] = \frac{k}{2} \int d^2x \, \mathrm{Tr}\left[\eta^{\mu\nu} g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g\right] + k\Gamma[G], \tag{B.1}$$

where $g \in G$ and

$$\Gamma[G] = \frac{1}{3} \int_{\mathcal{M}} \mathrm{Tr}(G^{-1}dG)^3. \tag{B.2}$$

The conserved Noether currents are given by $J = g^{-1}\partial_- g$, $\bar{J} = -\partial_+ g \, g^{-1}$; $\partial_+ J = 0 = \partial_- \bar{J}$.

We want to consider first what is called in [73] the "standard" gauged WZW model, namely the case where one gauges by a diagonal subgroup H of G. We are looking for an action invariant under

$$g \to \gamma g \gamma^{-1}\,, \qquad \gamma(x^+, x^-) \in H. \tag{B.3}$$

It is obvious to see that the action

$$I[g, h, \tilde{h}] = S[hg\tilde{h}^{-1}] - S[h\tilde{h}^{-1}]\,, \qquad h, \tilde{h} \in H, \tag{B.4}$$

is gauge invariant, provided h and $\tilde{h}$ transform as

$$h \to h\gamma^{-1}\,, \qquad \tilde{h} \to \tilde{h}\gamma^{-1}. \tag{B.5}$$

Notice that the two terms in (B.4) are invariant separately; the second term is simply there to have an action with three independent fields. Also, notice that the minus sign between the two terms is chosen for latter convenience to obtain (B.7).
Using the Polyakov-Wiegmann identity (with A, B, C three matrices)

$$\begin{aligned}
S[ABC] &= S[A] + S[B] + S[C] \\
&+ \frac{k}{\pi} \int d^2x \mathrm{Tr}\left[A^{-1}\partial_- A \partial_+ B B^{-1} + B^{-1}\partial_- B \partial_+ C C^{-1} + A^{-1}\partial_- A B \partial_+ C C^{-1} B^{-1}\right],
\end{aligned} \tag{B.6}$$

one can rewrite $I[g, h, \tilde{h}]$ in (B.4) as

$$I[g, A_-, A_+] = S[g] + \frac{k}{\pi} \int d^2x \text{Tr}[g^{-1}\partial_-g A_+ - A_-\partial_+g g^{-1}$$
$$- A_-gA_+g^{-1} + A_-A_+], \tag{B.7}$$

where $S[g]$ is the (non-chiral) WZW action (B.1) and where we have defined

$$A_- = -h^{-1}\partial_-h\,, \qquad A_+ = (\partial_+\tilde{h}^{-1})\tilde{h}. \tag{B.8}$$

One can see from (B.7) that A_+ and A_- play the role of gauge fields with no dynamics; the variation of the action with respect to them sets the currents of H (we note T_α the generators of $\mathcal{H}$)

$$J_\alpha = [g^{-1}(\partial_-g)]_\alpha\,, \qquad \bar{J}_\alpha = [-(\partial_+g)g^{-1}]_\alpha \tag{B.9}$$

to zero. Note that the currents in G/H are still conserved, but not zero.

B.2 Toda theories and Gauged WZW models

If one wants to obtain Toda theories from WZW models, we know that, rather than set all currents to zero, one has to set some currents to *constants*. Indeed, we know that imposing the constraints (see the last section for the notations and conventions)

$$\begin{aligned} J(E_\alpha) &= \mu^\alpha\,, & \bar{J}(E_{-\alpha}) &= -\nu^\alpha\,, & \alpha &\in \Delta, \\ J(E_\phi) &= 0\,, & \bar{J}(E_{-\phi}) &= 0\,, & \phi &\in \Phi^+ \setminus \Delta, \end{aligned} \tag{B.10}$$

reduces the WZW model to a Toda theory. For instance, for the group $SL(2, \mathbb{R})$, one simply sets the two currents $J(E_+)$ and $\bar{J}(E_-)$ to constants to reduce the action to Liouville. To see that this reduction is equivalent to gauging the WZW action, we have to look for an action invariant under

$$g \to \alpha g \beta^{-1}\,, \qquad \alpha = \alpha(x^+, x^-) \in H, \beta = \beta(x^+, x^-) \in \tilde{H}, \tag{B.11}$$

where $H, \tilde{H}$ are two different isomorphic subgroups of G. It seems impossible to generalize the standard procedure to this more general transformation with $H \neq \tilde{H}$, since now the only obvious candidate for an invariant action is $S[hg\tilde{h}^{-1}]$, with

$$h \to h\alpha^{-1}\,, \qquad \tilde{h} \to \tilde{h}\beta^{-1}, \tag{B.12}$$

which is a non-local action in the gauge fields. (Note that $S(h\tilde{h}^{-1})$ used in the previous section does not make sense any longer here.) However, if we take H and $\tilde{H}$ to be the subgroups of G generated by the step operators associated to the positive and negative roots, denoted by N and $\tilde{N}$ respectively, we have the crucial property that[1]

$$S[h] = 0 = S[\tilde{h}]. \tag{B.13}$$

---

[1] One way to see that is to realize that the only traces appearing in the WZW action $S(h)$, for instance, will be of the type $\text{Tr}(E_\alpha E_\beta)$, which vanishes.

Therefore, taking inspiration from the standard gauged WZW, one could use the action

$$S[hg\tilde{h}^{-1}] = S[g] + \frac{k}{\pi} \int d^2x \, \mathrm{Tr} \left[g^{-1}\partial_- g A_+ - A_- \partial_+ g g^{-1} \right]. \tag{B.14}$$

However, without modification, this would be nothing than the mere WZW action for G, while what we really want to do is to set some currents to constants. This is why we consider instead the following action:

$$
\begin{aligned}
I[g, A_-, A_+] = S[g] + \frac{k}{\pi} \int d^2x \, \mathrm{Tr}[& g^{-1}\partial_- g A_+ - A_- \partial_+ g g^{-1} - A_- g A_+ g^{-1} \\
& + A_- \mu_M + A_- \nu_M],
\end{aligned}
\tag{B.15}
$$

with μ_M, ν_M constant matrices given by

$$\nu_M = \frac{1}{2}|\alpha|^2 \nu^\alpha E_\alpha \,, \qquad \mu_M = \frac{1}{2}|\alpha|^2 \mu^\alpha E_{-\alpha}, \tag{B.16}$$

where the sum runs over the simple roots ($\alpha \in \Delta$).

One can check that the action (B.15) is invariant (up to a total derivative) under

$$
\begin{aligned}
g \to \alpha g \beta^{-1} \,, \qquad & \alpha = \alpha(x^+, x^-) \in N, \beta = \beta(x^+, x^-) \in \tilde{N}, \\
A_- \to \alpha A_- \alpha^{-1} + \alpha \partial_- \alpha^{-1} \,, \qquad & A_+ \to -\beta A_+ \beta^{-1} - (\partial_+ \beta)\beta^{-1}.
\end{aligned}
\tag{B.17}
$$

The equations of motion derived from action (B.15) are (with $\varphi \in \Phi^+$)

$$
\begin{aligned}
& \partial_+(g^{-1}\partial_- g - g^{-1}A_- g) - [A_+, g^{-1}\partial_- g - g^{-1}A_- g] + \partial_- A_+ = 0, \\
& \partial_-(\partial_+ g g^{-1} + g A_+ g^{-1}) - [A_-, \partial_+ g g^{-1} + g A_+ g^{-1}] - \partial_+ A_- = 0, \\
& \mathrm{Tr}[E_{-\varphi}(g^{-1}\partial_- g - g^{-1}A_- g - \nu_M)] = 0, \\
& \mathrm{Tr}[E_\varphi(\partial_+ g g^{-1} + g A_+ g^{-1} - \mu_M)] = 0.
\end{aligned}
\tag{B.18}
$$

The last two equations are constraints that indeed set to non-vanishing constants only the currents along α (see (B.16)).

B.3 $ISO(2,1)$ Gauged WZW

Let us describe here a way to construct a gauged chiral $\mathfrak{iso}(2,1)$ WZW model associated to (3.72) for the purely bosonic case and $\mu = 0$. The action is given by

$$I[\lambda, \alpha] = \frac{k}{\pi} \int du d\phi \, \mathrm{Tr} \left[\dot{\lambda}\lambda^{-1}\alpha' - \frac{1}{2}(\lambda'\lambda^{-1})^2 \right], \tag{B.19}$$

and it has the following Noether symmetries

$$
\begin{aligned}
\delta_\sigma \lambda = 0 \,, \qquad & \delta_\sigma \alpha = \lambda \sigma(\phi) \lambda^{-1} \,, \\
\delta_\vartheta \lambda = -\lambda \vartheta(\phi) \,, \qquad & \delta_\vartheta \alpha = -u\lambda \vartheta' \lambda^{-1} \,.
\end{aligned}
\tag{B.20}
$$

According to (3.88), we are interested in gauging the subset of these symmetries involving the parts of σ and ϑ along Γ_0. These parameters are promoted to depend on both u and ϕ.

One can check that the action

$$I(\lambda, \alpha, A_\mu) = I(\lambda, \alpha) + \frac{k}{\pi} \int dud\phi\, \mathrm{Tr} \left[-A_u(\lambda^{-1}\alpha'\lambda - u(\lambda^{-1}\lambda')') + \tilde{A}_u \lambda^{-1}\lambda' \right] \quad \text{(B.21)}$$

is invariant under

$$\begin{aligned}
&\delta_\sigma \lambda = 0\,, \ \delta_\sigma \alpha = \lambda\sigma(u,\phi)\lambda^{-1}\,, \ \delta_\sigma A_u = 0\,, \ \delta_\sigma \tilde{A}_u = -\left(\dot{\sigma} + [A_u, \sigma]\right), \\
&\delta_\vartheta \lambda = -\lambda\vartheta(u,\phi)\,, \ \delta_\vartheta \alpha = -u\lambda\vartheta'\lambda^{-1}\,, \ \delta_\vartheta A_u = -(\dot{\vartheta} + [A_u, \vartheta])\,, \ \delta_\vartheta \tilde{A}_u = -[\tilde{A}_u, \vartheta],
\end{aligned} \quad \text{(B.22)}$$

with σ and ϑ along Γ_0.

Since the constraints we want to implement set some current components to a constant, the suitable final action is

$$\begin{aligned}
I(\lambda, \alpha, A_\mu) &= I(\lambda, \alpha) \\
&\quad + \frac{k}{\pi} \int dud\phi\, \mathrm{Tr} \left[-A_u(\lambda^{-1}\alpha'\lambda - u(\lambda^{-1}\lambda')') + \tilde{A}_u \lambda^{-1}\lambda' - \mu_M \tilde{A}_u \right],
\end{aligned} \quad \text{(B.23)}$$

where $\mu_M := \mu\Gamma_1$, with μ an arbitrary constant, and $A_u, \tilde{A}_u$ are along Γ_0. The action (B.23) is indeed still gauge invariant since, as noticed in [73], the variation of $\mathrm{Tr}[\mu_M \tilde{A}_u]$ under a gauge transformation is a boundary term.

Finally, in order to see how the constraints are explicitly implemented, it is useful to parametrize the fields according to

$$\lambda = e^{\sigma\Gamma_1/2}e^{-\varphi\Gamma_2/2}e^{\tau\Gamma_0} \quad , \quad \alpha = \frac{\eta}{2}\Gamma_0 + \frac{\theta}{2}\Gamma_2 + \frac{\zeta}{2}\Gamma_1\,. \quad \text{(B.24)}$$

The field equations for the gauge fields imply that $\sigma'e^{-\varphi} = \mu$ and $\eta'\sigma^2 + 2\theta'\sigma - 2\zeta' = 0$, so that, taking $\mu = 1$, the reduced action is

$$I = \frac{k}{4\pi} \int dud\phi \left[\xi'\dot{\varphi} - \varphi'^2 \right], \quad \text{(B.25)}$$

where $\xi := -2(\theta + \eta\sigma)$, in full agreement with the centrally extended $\mathfrak{bms}_3$ invariant action found in [104].

Conventions

Let Φ the set of roots with respect to some Cartan subalgebra, Φ^+ a set of positive roots, and Δ a set of simple roots. There is a Cartan generator H_ϕ associated to every root $\phi \in \Phi$ and the Cartan matrix $K_{\alpha\beta}$ is given by

$$K_{\alpha\beta} = \frac{2\alpha.\beta}{|\beta|^2} = \frac{|\alpha|^2}{2}\mathrm{Tr}(H_\alpha.H_\beta)\,, \qquad \alpha, \beta \in \Delta, \quad \text{(B.26)}$$

where Tr is the usual matrix trace up to an appropriate normalization constant.
For any positive root $\alpha \in \Phi^+$, we choose step operators $E_{\pm\alpha}$ such that

$$
\begin{aligned}
H_\phi &= [E_\phi, E_{-\phi}], \\
\mathrm{Tr}(E_\phi . E_\varphi) &= \frac{2}{|\phi|^2} \delta_{\phi,-\varphi}, \qquad \mathrm{Tr}(E_\phi . H_\varphi) = 0,
\end{aligned}
\tag{B.27}
$$

for $\phi, \varphi \in \Phi$, and

$$
[H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta,
\tag{B.28}
$$

for $\alpha, \beta \in \Delta$.

APPENDIX C

Asymptotic Killing vectors on the horizon

We want to study the symmetries preserved in the vicinity of the horizon, located at $\rho = 0$, of metrics whose components are given by

$$g_{\rho\rho} = 0 \,, \quad g_{\rho\phi} = 0 \,, \quad g_{v\rho} = 1, \tag{C.1}$$

$$\begin{aligned}
g_{vv} &= -2\kappa\rho + g_{vv}^{(2)}\rho^2 + O(\rho^3), \\
g_{v\phi} &= \theta(\phi)\rho + \mathcal{O}(\rho^2), \\
g_{\phi\phi} &= \gamma(\phi)^2 + \lambda(v,\phi)\rho + O(\rho^2).
\end{aligned} \tag{C.2}$$

To do this, we consider the most general set of Killing vectors χ that keep conditions (C.1) and (C.2) invariant, namely that satisfy[2]

$$\mathcal{L}_\chi g_{\rho\rho} = \mathcal{L}_\chi g_{\rho\phi} = \mathcal{L}_\chi g_{v\rho} = 0, \tag{C.3}$$

$$\mathcal{L}_\chi g_{vv} = O(\rho^2), \quad \mathcal{L}_\chi g_{v\phi} = O(\rho), \quad \mathcal{L}_\chi g_{\phi\phi} = O(1). \tag{C.4}$$

The exact relations (C.3) correspond to the requirement that the gauge conditions remain fixed and imply that

$$\begin{aligned}
\chi^v &= f, \\
\chi^\phi &= Y - \frac{1}{\gamma^2}\partial_\phi f \rho + \frac{\lambda}{2\gamma^4}\partial_\phi f \rho^2 + O(\rho^3), \\
\chi^\rho &= Z - \partial_v f \rho + \frac{\theta}{2\gamma^2}\partial_\phi f \rho^2 + O(\rho^3),
\end{aligned} \tag{C.5}$$

where $f = f(v,\phi)$, $Y = Y(v,\phi)$ and $Z = Z(v,\phi)$ are arbitrary functions of v and ϕ.

Equations (C.4) constrain the form of these functions. In fact, the first equation of (C.4) gives

$$\mathcal{L}_\chi g_{vv} = -2\kappa Z + 2\partial_v Z + (2g_{vv}^{(2)}Z - 2\partial_v^2 f - 2\kappa\partial_v f + 2\theta(\phi)\partial_v Y)\rho + O(\rho^2). \tag{C.6}$$

---

[2]Recall that we fix κ, therefore we do not allow fluctuations of $\mathcal{O}(\rho)$ in g_{vv}.

Looking at order ρ and since $g_{vv}^{(2)}$ is an arbitrary fluctuation, the only possibility is to demand $Z = 0$. Doing so, we obtain

$$\kappa \partial_v f + \theta(\phi) \partial_v Y + \partial_v^2 f = 0. \tag{C.7}$$

The rest of the components gives

$$\mathcal{L}_\chi g_{v\phi} = \gamma^2 \partial_v Y + O(\rho), \tag{C.8}$$
$$\mathcal{L}_\chi g_{\phi\phi} = Y 2\gamma \partial_\phi \gamma + 2\partial_\phi Y \gamma^2 + O(\rho), \tag{C.9}$$

which imply $Y = Y(\phi)$ only.

We can now easily solve equation (C.7); we obtain

$$f(v, \phi) = T(\phi) + e^{-\kappa v} X(\phi). \tag{C.10}$$

So, we have a Killing vector with three arbitrary parameters $\chi(T, Y, X)$.

Taking $X(\phi) = 0$, we therefore obtain

$$\begin{aligned}
\chi^v &= T(\phi) + \mathcal{O}(\rho^3), \\
\chi^\rho &= \frac{\theta}{2\gamma^2} T'(\phi)\rho^2 + \mathcal{O}(\rho^3), \\
\chi^\phi &= Y(\phi) - \frac{1}{\gamma^2} T'(\phi)\rho + \frac{\lambda}{2\gamma^4} T'(\phi)\rho^2 + \mathcal{O}(\rho^3),
\end{aligned} \tag{C.11}$$

where the prime denotes the derivative with respect to ϕ. Also, looking at $\mathcal{L}_\chi g_{\phi\phi}$ at $\mathcal{O}(1)$ and $\mathcal{L}_\chi g_{v\phi}$ at $\mathcal{O}(\rho)$ we obtain the transformation laws for $\gamma(\phi)$ and $\theta(\phi)$, respectively:

$$\delta_\chi \theta = (\theta Y)' - 2\kappa T', \quad \delta_\chi \gamma = (\gamma Y)'. \tag{C.12}$$

APPENDIX D

Kerr metric in Gaussian coordinates

Let us consider the Kerr metric written in the Eddington-Finkelstein coordinates; namely

$$ds^2 = \left(\frac{\Delta - \Xi}{\Sigma} - 1 \right) dv^2 + 2 \, dv \, dr - \frac{2a(\Xi - \Delta)\sin^2\theta}{\Sigma} dv \, d\phi -$$

$$-2a\sin^2\theta \, dr \, d\phi + \Sigma d\theta^2 + \frac{(\Xi^2 - a^2\Delta\sin^2\theta)\sin^2\theta}{\Sigma} d\phi^2 \qquad \text{(D.1)}$$

where the functions Δ, Ξ, and Σ are given by

$$\Delta(r) = r^2 - 2GMr + a^2 \ , \quad \Xi(r) = r^2 + a^2 \ , \quad \Sigma(r) = r^2 + a^2\cos^2\theta, \qquad \text{(D.2)}$$

where M is the mass and a is the angular momentum per unit of mass. The outer horizon of the Kerr black hole is located at $r_+ = GM + \sqrt{G^2M^2 - a^2}$.

Kerr metric can be written in the form (4.21), namely

$$
\begin{aligned}
g_{vv} &= -2\kappa\rho + \mathcal{O}(\rho^2), \\
g_{v\rho} &= 1 + \mathcal{O}(\rho^2), \\
g_{vA} &= \rho\theta_A + \mathcal{O}(\rho^2), \\
g_{\rho\rho} &= \mathcal{O}(\rho^2), \\
g_{\rho A} &= \mathcal{O}(\rho^2) \\
g_{AB} &= \Omega\gamma_{AB} + \lambda_{AB}\rho + \mathcal{O}(\rho^2).
\end{aligned}
\qquad \text{(D.3)}
$$

The explicit change of coordinates can be found, for instance, in Ref. [153]. It leads to

$$g_{\rho v} = 1, \quad g_{\rho\varphi} = 0, \quad g_{\rho\theta} = 0, \quad g_{\rho\rho} = 0, \qquad \text{(D.4)}$$

and

$$g_{vv} = -\frac{\Delta'(r_+)}{\Xi(r_+)}\rho + \mathcal{O}(\rho^2),$$

$$g_{v\varphi} = \frac{2a^2 \sin\theta \cos\theta}{\Sigma(r_+)}\rho + \mathcal{O}(\rho^2),$$

$$g_{v\theta} = \left(\frac{a\Delta'(r_+)\sin^2\theta}{\Sigma(r_+)} + \frac{2ar_+\Xi(r_+)\sin^2\theta}{\Sigma^2(r_+)}\right)\rho + \mathcal{O}(\rho^2),$$

$$g_{\theta\theta} = \Sigma(r_+) + \frac{2r_+\Xi(r_+)}{\Sigma(r_+)}\rho + \mathcal{O}(\rho^2), \tag{D.5}$$

$$g_{\varphi\theta} = -\frac{2a^3\Xi(r_+)\sin^3\theta\cos\theta}{\Sigma^2(r_+)}\rho + \mathcal{O}(\rho^2),$$

$$g_{\varphi\varphi} = \frac{\Xi^2(r_+)\sin^2\theta}{\Sigma(r_+)}$$
$$- \left(\frac{a^2\Xi(r_+)\Delta'(r_+)\sin^4\theta}{\Sigma^2(r_+)} - \frac{2r_+\Xi^2(r_+)\sin^2\theta(\Sigma(r_+) - a^2\sin^2\theta)}{\Sigma^3(r_+)}\right)\rho + \mathcal{O}(\rho^2),$$

where $\Delta'(r_+) = 2(r_+ - GM) = (r_+^2 - a^2)/r_+$. Notice that, since

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{\Delta'(r_+)}{\Sigma(r_+)}, \tag{D.6}$$

then indeed one finds

$$g_{vv}^{(1)} = -\frac{\Delta'(r_+)}{\Sigma(r_+)} = -2\kappa , \tag{D.7}$$

in accordance with the near horizon expansion (D.3). After writing the metric of the horizon 2-sphere in terms of complex coordinates $z = e^{i\varphi}\cot(\theta/2)$, one finally finds the Kerr metric in its near horizon region written in the coordinates (D.3).

From timelike WAdS space to the spacelike black hole

Warped black holes (WBHs) are black hole solutions that asymptote stretched spacelike WAdS$_3$ space. As we will describe below, these black holes can be obtained from the timelike solution by means of a complex change of coordinates: Consider first the double Wick rotation

$$t \to i\tau \,, \qquad \varphi \to -i\Theta \,, \qquad \omega \to -\omega \,, \qquad r \to -r \,, \qquad j \to -j, \qquad \text{(E.1)}$$

and, secondly, $\tau = t' - \ell\sqrt{j}\,\Theta$. Finally, in order to compare with the coordinates used in the literature, let us rescale time as $t' \to LT$.

The change of coordinates above maps the timelike metric (6.16) into the WBH solution

$$
\begin{aligned}
ds^2 = L^2 dT^2 &+ \frac{L^2\,dR^2}{(\nu^2+3)(R-r_+)(R-r_-)} + L^2(2\nu R - \sqrt{r_+ r_-(\nu^2+3)})dT d\Theta \\
&+ \frac{RL^2}{4}\left[3(\nu^2-1)R + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right]d\Theta^2,
\end{aligned}
\qquad \text{(E.2)}
$$

with $R = -2r/L^2$ and provided one identifies the parameters as follows

$$
\begin{aligned}
\nu &= \omega L \,; \qquad L^2 = \frac{3}{\omega^2 + 2\ell^{-2}} \,; \\
r_\pm &= \frac{\ell^2}{L^2}\left[\frac{-(1-\mu) \pm \sqrt{(1-\mu)^2 - 2(\omega^2\ell^2+1)j}}{(\omega^2\ell^2+1)}\right].
\end{aligned}
\qquad \text{(E.3)}
$$

Notice the useful relations

$$r_+ + r_- = \frac{2\ell^2(\mu-1)}{L^2(1+\ell^2\omega^2)} \,; \qquad r_+ r_- = \frac{2j\ell^4}{L^4(1+\ell^2\omega^2)}. \qquad \text{(E.4)}$$

The timelike and spacelike Killing vectors are related in the following way

$$\partial_t = \frac{i}{L}\partial_T \,, \qquad \partial_\varphi = \frac{\ell}{L}\sqrt{j}\,\partial_T + i\partial_\Theta. \qquad \text{(E.5)}$$

This charge dependent change of coordinates makes the relation between timelike and spacelike charges more involved than a mere analytic continuation.

Changing in (6.24) $LT \to t$, $R \to r$ and $L\Theta \to \varphi$, we can assign the dimensions as $[t] = l^1, [r] = l^1, [\varphi] = l^0, [L] = l^1, [\nu] = l^0, [r_\pm] = l^1$ and the expression of the mass of the WBH then becomes

$$\mathcal{M}_{\text{WBH}} = Q_{\partial_T} = \frac{\nu(\nu^2 + 3)}{GL(20\nu^2 - 3)} \left((r_- + r_+)\nu - \sqrt{r_+ r_- (\nu^2 + 3)} \right), \qquad \text{(E.6)}$$

while the expression for the angular momentum is

$$\mathcal{J}_{\text{WBH}} = Q_{\partial_\Theta} = \frac{\nu(\nu^2 + 3)}{4GL(20\nu^2 - 3)} \left((5\nu^2 + 3)r_+ r_- - 2\nu\sqrt{r_+ r_- (\nu^2 + 3)}(r_+ + r_-) \right). \quad \text{(E.7)}$$

Using the relations (E.3) between the spacelike and timelike parameters, one observes that going from the timelike to the spacelike metric involves a charge-dependent and globally not-well defined change of coordinates, namely the definition $\tau = t' - \ell\sqrt{j}\Theta$ above. This implies that the spacelike and timelike charges do not coincide. Only in the case $j = 0$, one sees that the masses are related according to $\partial_t \sim L^{-1}\partial_T$,

$$\mathcal{M}_{\text{WBH}}|_{j=0} = L^{-1}\mathcal{M}. \qquad \text{(E.8)}$$

It is important to remark that, in the case of spinning defects in timelike WAdS$_3$, and due to the j-dependent change of coordinates, the conserved charges can not be simply obtained from the mass and angular momentum of spacelike solutions.

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