A note on Faddeev-Popov action for doubled-yet-gauged particle and graded Poisson geometry

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Abstract: The section condition of Double Field Theory has been argued to mean that doubled coordinates are gauged: a gauge orbit represents a single physical point. In this note, we consider a doubled and at the same time gauged particle action, and show that its BRST formulation including Faddeev-Popov ghosts matches with the graded Poisson geometry that has been recently used to describe the symmetries of Double Field Theory. Besides, by requiring target spacetime diffeomorphisms at the quantum level, we derive quantum corrections to the classical action involving dilaton, which might be comparable with the Fradkin-Tseytlin term on string worldsheet.

Contents

1 Introduction

This note is about Double Field Theory (DFT) which was initiated with the goal of manifesting the hidden $\mathbf{O}(D, D)$ symmetry of supergravity [1–6]. Through subsequent further developments, identifying the relevant connections (Christoffel/spin) and curvatures (scalar/Ricci/Einstein) [7–11], it has evolved into a stringy 'pure' gravitational theory, or the $\mathbf{O}(D, D)$ completion of General Relativity. DFT assumes the entire closed-string massless NS-NS sector as the gravitational multiplet and interacts with other sectors [12–16] or generic matter contents [17–19]. The Euler-Lagrange equations of the whole NS-NS sector are unified into a single formula, $G_{AB} = 8\pi G T_{AB}$, which may well be regarded as the $\mathbf{O}(D, D)$ completion of the Einstein field equations [20, 21]. The theory has been shown to admit full order (i.e. quartic in fermions) supersymmetrizations [22, 23], and turned out to contain not only supergravity but also various non-Riemannian gravities, e.g. Newton-Cartan, as different solution sectors [24–27].

Despite the nomenclature, DFT is not truly doubled: a prescription called *section condition* should be imposed on all the variables appearing in the theory, such as physical fields and local parameters. The section condition has been argued to imply that the doubled coordinates are actually gauged: a gauge orbit or an equivalence class in the doubled coordinate space corresponds to a single physical point [28]. This idea of 'coordinate gauge symmetry' is naturally realized in sigma models where the doubled target spacetime coordinates are dynamical fields and thus can be genuinely gauged [29–39].

Over the years, DFT has shown interesting and deep connections to various subfields of geometry, such as Generalized Geometry [40–45], Courant algebroid (including extensions

thereof) [46–48], and para-Hermitian/Born geometry [49–58]. More recently, graded geometry has been also used to describe the symmetries of DFT [59–66] (making, in particular, use of derived brackets introduced in [67] and further studied in [68–71]).

It is the purpose of the present note to revisit the coordinate gauge symmetry from the viewpoint of a constrained system, and along the way establish a connection with the aforementioned graded geometric approach. Specifically we shall show that the doubled-yet-gauged particle action constructed in [33] can be formulated as a simple constrained system whose BRST phase space matches with the graded manifold adopted in [62, 63]. In particular, Faddeev-Popov ghosts carrying an $\mathbf{O}(D, D)$ vector index are mapped to the Grassmann odd coordinates of the graded manifold. On top of that, the Poisson bracket and the BRST charge agree with [62, 63].

The organization of the manuscript is as follows. In the remaining of this Introduction we review the section condition, the coordinate gauge symmetry, and certain elements of the graded geometric approach to DFT. Section 2 contains our main results which split into three parts. Firstly, we introduce a constant projector into a section, and using this we reformulate the section condition as well as the coordinate gauge symmetry. Secondly, we consider the BRST formulation of the doubled-yet-gauged particle action [33], and point out its connection to the graded Poisson geometry [62, 63]. Thirdly, by requiring target spacetime diffeomorphisms at the quantum level, we derive quantum corrections to the classical action involving DFT-dilaton, which are analogous to the Fradkin-Tseytlin term on string worldsheet. We conclude with comments in section 3.

Note added. After the first version of our note, a preprint [72] appeared on arXiv which discusses the coordinate gauge symmetry within the context of higher geometry.

1.1 Section condition

DFT postulates $\mathbf{O}(D, D)$ symmetry as the first principle with an invariant metric,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & \delta^\mu{}_\nu \\ \delta_\rho{}^\sigma & 0 \end{pmatrix}. \tag{1.1}$$

Along with its inverse $\mathcal{J}^{AB}$, it can freely lower and raise $\mathbf{O}(D, D)$ vector indices (capital Roman letters). It decomposes the doubled coordinates into two parts,

$$x^A = (\tilde{x}_\mu, x^\nu), \qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu), \tag{1.2}$$

where the Greek letters are (usual) D-dimensional vector indices.

In DFT, it is necessary to impose the section condition:

$$\partial_A \partial^A = 0. \tag{1.3}$$

Acting on arbitrary functions in the theory, say Φ_r, as well as their products like $\Phi_s \Phi_t$, the $\mathbf{O}(D, D)$ invariant D'Alembertian should vanish, leading to the notion of weak and strong constraints,

$$\partial_A \partial^A \Phi_r = 0 \quad \text{(weak)}, \qquad \partial_A \Phi_s \partial^A \Phi_t = 0 \quad \text{(strong)}. \tag{1.4}$$

Here we are considering a set, $\{\Phi_r, \Phi_s, \Phi_t, \cdots\}$, formed by *all* the functions in DFT including physical fields, gauge parameters, and their derivatives (as well as numerical constants). The section condition is easily solved by letting

$$\tilde{\partial}^\mu \equiv 0 \,, \tag{1.5}$$

such that in this case the untilde coordinates, x^μ, define a section. The general solutions to the section condition are then generated by its $\mathbf{O}(D,D)$ rotations [2, 3]. Throughout the present manuscript, the symbol, '$\equiv$', denotes the equality up to $\mathbf{O}(D,D)$ duality rotations, referring to the particular choice of the section 1.5.

Diffeomorphisms in DFT consist in the transformations of the doubled coordinates,

$$\delta x^A = \xi^A \,, \quad \delta\partial_A = -\partial_A\xi^B\partial_B = (\partial^B\xi_A - \partial_A\xi^B)\partial_B \,, \tag{1.6}$$

which induce the following transformation rule for tensors (or tensor density with weight ω),

$$\delta T_{A_1\cdots A_n} = -\omega\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_B\xi_{A_i} - \partial_{A_i}\xi_B)T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n} \,. \tag{1.7}$$

The active version of this passive transformation of tensors is the 'generalized Lie derivative' [2, 6],

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} = \xi^B\partial_B T_{A_1\cdots A_n} + \omega\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n} 2\partial_{[A_i}\xi_{B]}T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n} \,. \tag{1.8}$$

Thanks to the section condition, the generalized Lie derivatives are closed under commutator:

$$\left[\hat{\mathcal{L}}_\zeta, \hat{\mathcal{L}}_\xi\right] = \hat{\mathcal{L}}_{[\zeta,\xi]_{\mathrm{C}}} \,, \qquad [\zeta,\xi]_{\mathrm{C}}^M = \zeta^N\partial_N\xi^M - \xi^N\partial_N\zeta^M + \frac{1}{2}\xi^N\partial^M\zeta_N - \frac{1}{2}\zeta^N\partial^M\xi_N \,. \tag{1.9}$$

1.2 Coordinate gauge symmetry

The section condition has been shown to be equivalent to a certain translational invariance [28, 31],

$$\Phi_r(x + \Delta) = \Phi_r(x) \,, \qquad \Delta^A\partial_A = 0 \,, \tag{1.10}$$

where the shift parameter Δ^A is 'derivative-index-valued', meaning that its superscript index should be identifiable as that of derivative:

$$\Delta^A = \Phi_s\partial^A\Phi_t \,. \tag{1.11}$$

Indeed, if the parameter Δ^A takes the form (1.11), then its contraction with a derivative vanishes by virtue of the section condition (1.4). We stress that the notion of the 'derivative-index-valuedness' is possible because DFT postulates $\mathbf{O}(D,D)$ symmetry and one raises (and lowers) indices using the $\mathbf{O}(D,D)$-invariant metric: $\partial^A = \mathcal{J}^{AB}\partial_B$. The invariance of every function in DFT (1.10) may suggest that the doubled coordinates are actually gauged by the shift [28]:

$$x^A \sim x^A + \Delta^A \,. \tag{1.12}$$

That is to say, each gauge orbit — or equivalence class — corresponds to a single physical point. This idea of 'coordinate gauge symmetry' has been applied and tested in various contexts. The finite transformation of tensors *à la* Hohm and Zwiebach [73] is equivalent to the exponentiation of the generalized Lie derivative, only up to the equivalence relation (1.12) [28] (cf. [74–76]). The usual coordinate basis of one-forms, $\mathrm{d}x^A$, is not DFT-diffeomorphism covariant,

$$\delta(\mathrm{d}x^A) = \mathrm{d}(\delta x^A) = \mathrm{d}\xi^A = \mathrm{d}x^B \partial_B \xi^A \neq \mathrm{d}x^B(\partial_B \xi^A - \partial^A \xi_B)\,. \tag{1.13}$$

However, if we literally gauge the one-form by introducing a derivative-index-valued connection, it becomes DFT-diffeomorphism covariant,

$$\begin{aligned}
\mathrm{D}x^A &:= \mathrm{d}x^A - \mathcal{A}^A\,, & \mathcal{A}^A \partial_A &= 0\,, \\
\delta(\mathrm{D}x^A) &= \mathrm{D}x^B(\partial_B \xi^A - \partial^A \xi_B)\,, & \delta\mathcal{A}^A &= \mathrm{D}x^B \partial^A \xi_B\,.
\end{aligned} \tag{1.14}$$

Further, it is invariant under the coordinate gauge symmetry and thus qualifies as a physical quantity,

$$\delta x^A = \Delta^A\,, \qquad \delta\mathcal{A}^A = \mathrm{d}\Delta^A\,, \qquad \delta(\mathrm{D}x^A) = 0\,. \tag{1.15}$$

Using this gauged one-form, we can define a gauge invariant and (arguably) physically meaningful proper length in doubled spacetime as a path integral over the gauge connection [77], recover the doubled (and gauged) string action by Hull [29] [31], and extend to Green-Schwarz superstring [34], U-duality covariant exceptional string actions [35, 36] as well as point-like particle actions [33, 37–39] (see (2.12) later).

1.3 Graded geometric approach

Symmetries of DFT, which are encompassed by the generalized Lie derivative (1.8) with C-bracket (1.9), have been revisited using graded geometry in [61] and further studied in [62, 63]. The point of this approach is to reproduce (among other things) the generalized Lie derivative, using tools from graded geometry. In the following, we shall introduce a few elements of this approach which are relevant to our work. Of particular interest for us is the appearance of a graded manifold with coordinates, $\{x^A, p_B, \theta^C\}$, where x^A and p_B are usual Grassmann even variables and θ^A's are odd (i.e. θ^A's anti-commute). This graded manifold is endowed with the graded Poisson bracket,

$$[F, G\} := \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\deg(F)}\frac{\partial F}{\partial \theta^A}\frac{\partial G}{\partial \theta_A}\,, \tag{1.16}$$

where $\deg(F)$ is zero or one for even or odd F respectively.[1] The graded bracket is graded anti-symmetric and satisfies the graded Jacobi identity,

$$\begin{aligned}
[F, G\} &= -(-1)^{\deg(F)\deg(G)}[G, F\}\,, \\
[[F, G\}, H\} &= [F, [G, H\}\} - (-1)^{\deg(F)\deg(G)}[G, [F, H\}\}\,.
\end{aligned} \tag{1.17}$$

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[1]All the derivatives are *a priori* set to act from left, $\frac{\partial F}{\partial \theta^A} = \frac{\overrightarrow{\partial}F}{\partial \theta^A} = -(-1)^{\deg(F)}\frac{\overleftarrow{\partial}F}{\partial \theta^A}$, although for the even derivatives, $\frac{\partial}{\partial x^A}$ and $\frac{\partial}{\partial p_A}$, the ordering does not matter.

A p-form (with trivial weight $\omega = 0$) in 'doubled space', $T_{A_1 A_2 \cdots A_p} = T_{[A_1 A_2 \cdots A_p]}$, can be identified with a function in the graded manifold:

$$T(x, \theta) := \frac{1}{p!} T_{A_1 A_2 \cdots A_p}(x) \theta^{A_1} \theta^{A_2} \cdots \theta^{A_p} . \tag{1.18}$$

The graded Poisson bracket then provides

i) an inner product,

$$\left[\xi_A(x) \theta^A, T(x, \theta) \right\} = \frac{1}{(p-1)!} T_{B A_1 A_2 \cdots A_{p-1}} \xi^B \theta^{A_1} \theta^{A_2} \cdots \theta^{A_{p-1}} , \tag{1.19}$$

ii) an expression very similar to an exterior derivative,

$$\begin{aligned}
\left[p_A \theta^A, T(x, \theta) \right\} = & -\frac{1}{p!} \partial_{[A_1} T_{A_2 \cdots A_{p+1}]} \theta^{A_1} \theta^{A_2} \cdots \theta^{A_{p+1}} \\
& + \frac{1}{(p-1)!} T_{B A_1 A_2 \cdots A_{p-1}} p^B \theta^{A_1} \theta^{A_2} \cdots \theta^{A_{p-1}} ,
\end{aligned} \tag{1.20}$$

iii) the generalized Lie derivative,

$$\begin{aligned}
\left[p_A \theta^A, \left[\xi_B \theta^B, T \right\} \right\} + \left[\xi_A \theta^A, \left[p_B \theta^B, T \right\} \right\} &= \left[\left[p_A \theta^A, \xi_B \theta^B \right\}, T \right\} \\
&= -\frac{1}{p!} \hat{\mathcal{L}}_\xi T_{A_1 A_2 \cdots A_p} \theta^{A_1} \theta^{A_2} \cdots \theta^{A_p} ,
\end{aligned} \tag{1.21}$$

iv) the C-bracket as a derived bracket,

$$\left[\left[p_A \theta^A, \xi_B \theta^B \right\}, \zeta_C \theta^C \right\} - \left[\left[p_A \theta^A, \zeta_B \theta^B \right\}, \xi_C \theta^C \right\} = [\zeta, \xi]_C^A \theta_A . \tag{1.22}$$

The first equality in (1.21) is due to the Jacobi identity (1.17), the second holds from $\left[p_A \theta^A, \xi_B \theta^B \right\} = p_A \xi^A - \partial_{[A} \xi_{B]} \theta^A \theta^B$, and the resulting expression is analogous to the well-known (undoubled) "Cartan's magic formula", $\mathbf{i}_\xi \, \mathrm{d} + \mathrm{d} \, \mathbf{i}_\xi = \mathcal{L}_\xi$.

2 Main results

2.1 Formulation of the section condition by a projector

Here we develop some formalism which will make the somewhat colloquial notion, 'derivative-index-valued' (1.11), more concrete, and enable us to analyze the constrained system of the doubled-yet-gauged sigma model powerfully later. Specifically, we describe the section condition by a constant projection matrix, $\mathcal{P}_A{}^B$, along the line of the earlier works [29, 30, 49] and the more recent para-Hermitian approach [50–58], as

$$\mathcal{P}_A{}^B \mathcal{P}_B{}^C = \mathcal{P}_A{}^C , \qquad \mathcal{P}_{AB} + \mathcal{P}_{BA} = \mathcal{J}_{AB} , \qquad \mathcal{P}^A{}_B \partial_A = 0 . \tag{2.1}$$

These relations imply

$$\mathcal{P}_A{}^B \partial_B = \partial_A , \tag{2.2}$$

and the section condition is now fulfilled as[2]

$$\partial^A \partial_A = \partial^A \left(\mathcal{P}_A{}^B \partial_B \right) = \left(\mathcal{P}^{AB} \partial_A \right) \partial_B = 0 \,. \tag{2.3}$$

Its orthogonal complementary projection matrix follows

$$\bar{\mathcal{P}}_A{}^B := \delta_A{}^B - \mathcal{P}_A{}^B = \mathcal{P}^B{}_A \,, \quad \bar{\mathcal{P}}_A{}^B \bar{\mathcal{P}}_B{}^C = \bar{\mathcal{P}}_A{}^C \,, \quad \mathcal{P}_A{}^B \bar{\mathcal{P}}_B{}^C = 0 \,, \quad \bar{\mathcal{P}}_A{}^B \partial_B = 0 \,. \tag{2.4}$$

The middle relation in (2.1) implies that the rank of the projection is D as $\mathcal{J}^{AB} \mathcal{P}_{AB} = \mathcal{P}_A{}^A = \mathcal{P}^A{}_A = D$. In other words, the section is D-dimensional. Specifically, $\bar{\mathcal{P}}^A{}_B$ projects the doubled coordinates into a section $\bar{\mathcal{P}}^A{}_B x^B$, as it satisfies the section condition of the form (2.1): with $\partial^A x^B = \mathcal{J}^{AB}$,

$$\bar{\mathcal{P}}_A{}^B \partial_B \left(\bar{\mathcal{P}}^C{}_D x^D \right) = \bar{\mathcal{P}}_A{}^B \partial_B \left(\mathcal{P}_D{}^C x^D \right) = \bar{\mathcal{P}}_A{}^B \mathcal{P}_B{}^C = 0 \,. \tag{2.5}$$

Accordingly, all the variables in DFT are functions of $\bar{\mathcal{P}}^A{}_B x^B$ only, independent of $\mathcal{P}^A{}_B x^B$, fulfilling the section condition, $\bar{\mathcal{P}}_A{}^B \partial_B = 0$.

Crucially, any derivative-index-valued vector is $\mathcal{P}$-projected, from (2.2),

$$\Phi \partial^A \Psi = \mathcal{P}^A{}_B \left(\Phi \partial^B \Psi \right) \,. \tag{2.6}$$

Conversely, any $\mathcal{P}$-projected vector is derivative-index-valued,

$$\mathcal{P}^A{}_B V^B = \left(\mathcal{P}_{BC} V^C \right) \partial^A \left(\bar{\mathcal{P}}^B{}_D x^D \right) \,. \tag{2.7}$$

That is to say, being $\mathcal{P}$-projected is equivalent to being derivative-index-valued. From now on, we shall always make use of the projector whenever it is necessary to consider the notion of derivative-index-valuedness. First of all, we reformulate the coordinate gauge symmetry (1.12) and the translational invariance (1.10) equivalently as

$$x^A \sim x^A + \mathcal{P}^A{}_B V^B \,, \tag{2.8}$$

$$\Phi(x + \mathcal{P}V) = \Phi(x) \quad \Longleftrightarrow \quad \mathcal{P}^A{}_B \partial_A = 0 \,. \tag{2.9}$$

Once again, $\mathcal{P}^A{}_B$ is the constant projector of (2.1) and V^B is an arbitrary variable carrying an $\mathbf{O}(D,D)$ index, cf. [30, 58].

Explicitly for the choice of the section as $\tilde{\partial}^\mu \equiv 0$ (1.5), or up to $\mathbf{O}(D,D)$ duality rotations, we have[3]

$$\mathcal{P}_A{}^B \equiv \begin{pmatrix} 0 & 0 \\ 0 & \delta_\mu{}^\nu \end{pmatrix} \,, \quad \mathcal{P}^A{}_B \equiv \begin{pmatrix} \delta_\mu{}^\nu & 0 \\ 0 & 0 \end{pmatrix} \,, \quad \bar{\mathcal{P}}_A{}^B \equiv \begin{pmatrix} \delta^\mu{}_\nu & 0 \\ 0 & 0 \end{pmatrix} \,, \quad \bar{\mathcal{P}}^A{}_B \equiv \begin{pmatrix} 0 & 0 \\ 0 & \delta^\mu{}_\nu \end{pmatrix} \,, \tag{2.10}$$

such that, with (1.2), $\mathcal{P}^{AB} \partial_B \equiv (\partial_\mu, 0)$, $\bar{\mathcal{P}}^{AB} \partial_B = \mathcal{P}^{BA} \partial_B \equiv (0, \tilde{\partial}^\mu) \equiv (0, 0)$, and only the tilde coordinates are gauged,

$$(\tilde{x}_\mu \,, x^\nu) \sim (\tilde{x}_\mu + V_\mu \,, x^\nu) \,. \tag{2.11}$$

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[2]Alternatively from (2.1), (2.4), $\partial^A \partial_A = \delta_A{}^B \partial^A \partial_B = \left(\mathcal{P}_A{}^B + \bar{\mathcal{P}}_A{}^B \right) \partial^A \partial_B = \left(\mathcal{P}_A{}^B \partial^A \right) \partial_B + \partial^A \left(\bar{\mathcal{P}}_A{}^B \partial_B \right) = 0 + 0 = 0$.

[3]Although constant, the skew-symmetrization of the projection matrix may be identified with the symplectic structure in para-Hermitian or Born geometries [49–58], as $\omega_{AB} := 2\mathcal{P}_{[AB]} = \mathcal{P}_{AB} - \bar{\mathcal{P}}_{AB}$, $\omega_A{}^B \omega_B{}^C = \delta_A{}^C$.

2.2 Doubled-yet-gauged particle action

Now we focus on the doubled-yet-gauged particle action constructed in [33],

$$S = \frac{1}{l_s} \int d\tau \, \frac{1}{2} e^{-1} D_\tau x^A D_\tau x^B \mathcal{H}_{AB}(x) - \frac{1}{2}(m \, l_s)^2 e \,. \tag{2.12}$$

Here m is the particle mass, l_s is a fundamental length scale, e is the einbein, and

$$D_\tau x^A := \dot{x}^A - \mathcal{P}^A{}_B A^B \,. \tag{2.13}$$

Compared to (1.14), the derivative-index-valued gauge connection is now equivalently set to be $\mathcal{P}$-projected as $\mathcal{A}^A = \mathcal{P}^A{}_B A^B$. Further, $\mathcal{H}_{AB}$ is the DFT-metric satisfying two defining properties,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA} \,, \qquad \mathcal{H}_A{}^C \mathcal{H}_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB} \,, \tag{2.14}$$

to which the most general solutions and thus all the possible DFT geometries have been classified in [24].

The action is invariant under worldline diffeomorphisms,

$$\delta_\epsilon x^A = \epsilon e^{-1} \dot{x}^A \,, \qquad \delta_\epsilon e = \dot{\epsilon} \,, \qquad \delta_\epsilon A^A = \frac{d}{d\tau}\left(\epsilon e^{-1} A^A\right) \,, \tag{2.15}$$

and the coordinate gauge symmetry,

$$\delta_V x^A = \mathcal{P}^A{}_B V^B \,, \qquad \delta_V e = 0 \,, \qquad \delta_V A^A = \dot{V}^A \,. \tag{2.16}$$

We combine these two local symmetries, with the shift of the parameter, $V^A \to V^A - \epsilon e^{-1} A^A$,

$$\delta x^A = \epsilon e^{-1} D_\tau x^A + \mathcal{P}^A{}_B V^B \,, \qquad \delta e = \dot{\epsilon} \,, \qquad \delta A^A = \dot{V}^A \,. \tag{2.17}$$

Thanks to the shift, δx^A now assumes a covariant form, while δe and δA^A are separately given by the time derivative of each gauge parameter.

2.3 Hamiltonian action

In order to obtain more insights into the coordinate gauge symmetry from the view point of a constrained system, e.g. [78], we reformulate the doubled-yet-gauged particle action (2.12) into the Hamiltonian form,

$$S_H = \frac{1}{l_s} \int d\tau \, p_A \dot{x}^A - A^A \bar{\mathcal{P}}_A{}^B p_B - e H(x, p) \,, \tag{2.18}$$

where the Hamiltonian is given by

$$H(x, p) = \frac{1}{2} p_A p_B \mathcal{H}^{AB}(x) + \frac{1}{2}(m \, l_s)^2 \,. \tag{2.19}$$

Now, A^A and e are Lagrange multipliers and generate two first-class constraints,

$$H_A := \bar{\mathcal{P}}_A{}^B p_B = p_B \mathcal{P}^B{}_A \approx 0 \,, \qquad H \approx 0 \,, \tag{2.20}$$

which Poisson-commute, $[H_A H]_{\text{P.B.}} = 0$, upon imposing the section condition, $\mathcal{P}^C{}_D \partial_C \mathcal{H}_{AB} = 0$. The dynamics is governed by the total Hamiltonian, $H_{\text{total}} = A^A H_A + eH$,

$$\dot{x}^A = \left[x^A, H_{\text{total}}\right]_{\text{P.B.}} = e\mathcal{H}^{AB}p_B + \mathcal{P}^A{}_B A^B \,,$$
$$\dot{p}_A = [p_A, H_{\text{total}}]_{\text{P.B.}} = -\frac{1}{2}ep_B p_C \,\partial_A \mathcal{H}^{BC} \,. \tag{2.21}$$

Integrating out the auxiliary momenta, p_A, in (2.18) one recovers (2.12). Surely, the two first-class constraints reflect the underlying two gauge symmetries of the Hamiltonian action (2.18), the coordinate gauge symmetry and the worldline diffeomorphisms:[4]

$$\delta_H x^A = \epsilon \mathcal{H}^{AB}p_B + \mathcal{P}^A{}_B V^B \,, \quad \delta_H p_A = -\frac{1}{2}\epsilon \partial_A \mathcal{H}^{BC}p_B p_C \,, \quad \delta_H e = \dot{\epsilon} \,, \quad \delta_H A^A = \dot{V}^A \,. \tag{2.22}$$

The difference between (2.17) and (2.22) amounts to the so-called trivial gauge symmetry [78]. It is important to remark that the former constraint in (2.20) is projected,

$$\bar{\mathcal{P}}_A{}^B H_B = H_B \mathcal{P}^B{}_A = H_A \,. \tag{2.23}$$

2.4 BRST formulation

Finally, let us extend the classical action, S in (2.12), to the Faddeev-Popov gauge-fixed action,

$$S_{\text{F.P.}} = \frac{1}{l_s}\int \mathrm{d}\tau \, \frac{1}{2}e^{-1}\mathrm{D}_\tau x^A \mathrm{D}_\tau x^B \mathcal{H}_{AB}(x) - \frac{1}{2}(ml_s)^2 e + k_A \mathcal{P}^A{}_B A^B + k(e-1)$$
$$+ \frac{1}{2}\theta_A \dot{\theta}^A + \sum_{\alpha=1}^2 \frac{1}{2}\vartheta_\alpha \dot{\vartheta}^\alpha \,, \tag{2.24}$$

where θ^A and ϑ^α with $\alpha = 1, 2$ are Grassmann odd variables. Readers might not immediately recognize standard ghost terms, but the above action contains them precisely and we have a (good) reason to spell the action like above, which we explain shortly.

First of all, integrating out the auxiliary variables, k_A, k, we are fixing the gauge,

$$\mathcal{P}^A{}_B A^B = 0 \,, \qquad e = 1 \,. \tag{2.25}$$

Decomposing the odd variable, θ^A, into two parts,

$$\theta^A = C^A + B^A \,, \quad C^A := \mathcal{P}^A{}_B \theta^B \,, \quad B^A := \bar{\mathcal{P}}^A{}_B \theta^B \,, \tag{2.26}$$

we may identify the standard BC ghost term, from (2.4), (2.17), up to total derivative,[5]

$$\frac{1}{2}\theta_A \dot{\theta}^A = \frac{1}{2}B_A \dot{C}^A + \frac{1}{2}C^A \dot{B}_A = B_A \dot{C}^A + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(C^A B_A\right) \,. \tag{2.27}$$

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[4]Recall that any Hamiltonian action, $S[x^a, p_b, \lambda^i] = \int \mathrm{d}t \, p_a \dot{x}^a - \lambda^i \chi_i(x, p)$, with first-class constraints, $\chi_i \approx 0$ obeying $\{\chi_i, \chi_j\}_{\text{P.B.}} = f^k{}_{ij}\chi_k$, has the gauge symmetry,

$$\delta x^a = \{x^a, \chi_i\}_{\text{P.B.}}\epsilon^i \,, \quad \delta p_a = \{p_a, \chi_i\}_{\text{P.B.}}\epsilon^i \,, \quad \delta \lambda^i = \dot{\epsilon}^i - f^i{}_{jk}\lambda^j \epsilon^k \,.$$

In our case, $f^i{}_{jk} = 0$ as it is Abelian (2.34).

[5]If we fix the section, $\tilde{\partial}^\mu \equiv 0$, with (2.10), we note

$$\mathcal{P}^A{}_B A^B \equiv (A_\mu \,, \, 0) \,, \quad \theta^A \equiv (C_\mu \,, \, B^\nu) \,, \quad C^A \equiv (C_\mu \,, \, 0) \,, \quad B^A \equiv (0 \,, \, B^\nu) \,, \quad B_A \dot{C}^A \equiv B^\mu \dot{C}_\mu \,, \quad U \equiv C_\mu B^\mu \,.$$

The ghost number U is defined as

$$U := \mathcal{P}_{AB}\theta^B\theta^A = C^A B_A \,, \tag{2.28}$$

which ranges from $-D$ to $+D$.

Similarly for the worldline diffeomorphisms, we identify

$$\vartheta_1 = \vartheta^2 = b \,, \qquad \vartheta_2 = \vartheta^1 = c \,, \tag{2.29}$$

and the corresponding bc ghost term, along with the ghost number, u,

$$\sum_{\alpha=1}^{2} \frac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha = b\dot{c} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(cb) \,, \qquad u := cb \,. \tag{2.30}$$

Intriguingly, an $\mathbf{O}(1,1)$ structure has appeared with the invariant metric given by the second Pauli matrix, σ_2, which might hint at the 'doubling' of the worldline, cf. [79]. It is also amusing to observe that the total derivatives in (2.27), (2.30) actually contribute to the action (2.24) through the ghost number changes at boundaries as $\frac{1}{2l_s}(U+u)\big|_{\tau=-\infty}^{\tau=+\infty}$.

The BRST differential, $\delta_{\mathcal{Q}}$, with the nilpotency, $\delta_{\mathcal{Q}}^2 = 0$, is given by

$$\begin{aligned}
\delta_{\mathcal{Q}} x^A &= ce^{-1}\mathrm{D}_\tau x^A + C^A \,, & \delta_{\mathcal{Q}} e &= \dot{c} \,, & \delta_{\mathcal{Q}} A^A &= \dot{C}^A \,, \\
\delta_{\mathcal{Q}} B_A &= -\bar{\mathcal{P}}_A{}^B k_B \,, & \delta_{\mathcal{Q}} C_A &= 0 \,, & \delta_{\mathcal{Q}} k_A &= 0 \,, \\
\delta_{\mathcal{Q}} b &= -k \,, & \delta_{\mathcal{Q}} c &= 0 \,, & \delta_{\mathcal{Q}} k &= 0 \,.
\end{aligned} \tag{2.31}$$

On-shell we have $\dot{\theta}^A = 0$, $\dot{\vartheta}^\alpha = 0$, and these make $\delta_{\mathcal{Q}} e$, $\delta_{\mathcal{Q}} A^A$ trivial and thus consistent with the gauge fixing (2.25).

Now, we proceed to the Hamiltonian formulation of the Faddeev-Popov action (2.24). We denote the canonical momenta of x^A, θ^B, ϑ^i by p_A, Π_B, π_i respectively, and write the Poisson bracket,

$$\begin{aligned}
[F, G]_{\text{P.B.}} &= \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} \\
&\quad + (-1)^{\deg(F)}\left(\frac{\partial F}{\partial \theta^A}\frac{\partial G}{\partial \Pi_A} + \frac{\partial F}{\partial \Pi_A}\frac{\partial G}{\partial \theta^A} + \frac{\partial F}{\partial \vartheta^\alpha}\frac{\partial G}{\partial \pi_\alpha} + \frac{\partial F}{\partial \pi_\alpha}\frac{\partial G}{\partial \vartheta^\alpha}\right) \,.
\end{aligned} \tag{2.32}$$

The dynamics after the gauge fixing (2.25) is governed by the Hamiltonian (2.19), subject to the two first-class constraints (2.20) and further two additional second-class constraints,

$$\begin{aligned}
\phi_A &:= \Pi_A + \frac{1}{2}\theta_A \approx 0 \,, & [\phi_A, \phi_B]_{\text{P.B.}} &= -\mathcal{J}_{AB} \,, \\
\varphi_\alpha &:= \pi_\alpha + \frac{1}{2}\vartheta_\alpha \approx 0 \,, & [\varphi_\alpha, \varphi_\beta]_{\text{P.B.}} &= -(\sigma_2)_{\alpha\beta} \,.
\end{aligned} \tag{2.33}$$

The first-class constraints form an Abelian algebra,

$$\begin{aligned}
[H_A, H_B]_{\text{P.B.}} &= 0 \,, & [H_A, H]_{\text{P.B.}} &= 0 \,, & [H_A, \phi_B]_{\text{P.B.}} &= 0 \,, \\
[H_A, \varphi_B]_{\text{P.B.}} &= 0 \,, & [H, \phi_A]_{\text{P.B.}} &= 0 \,, & [H, \varphi_A]_{\text{P.B.}} &= 0 \,,
\end{aligned} \tag{2.34}$$

and the second-class constraints originate directly from the Faddeev-Popov action (2.24). If we had assumed the conventional BC ghost terms as in (2.27) and (2.30), the identification of the second-class constraints would have been obscure. This justifies the precise form of our Faddeev-Popov action (2.24).

The relevant Dirac bracket reads

$$[F,G\}_{\text{D.B.}} = [F,G\}_{\text{P.B.}} + [F,\phi_A\}_{\text{P.B.}} \left[\phi^A,G\}_{\text{P.B.}} + [F,\varphi_\alpha\}_{\text{P.B.}} [\varphi^\alpha,G\}_{\text{P.B.}} \right. . \qquad (2.35)$$

which satisfies

$$
\begin{aligned}
&\left[x^A,p_B\}_{\text{D.B.}} = \delta^A{}_B \,, \qquad [\theta_A,\theta_B\}_{\text{D.B.}} = \mathcal{J}_{AB} \,, \qquad [\vartheta_\alpha,\vartheta_\beta\}_{\text{D.B.}} = (\sigma_2)_{\alpha\beta} \,, \\
&[\theta_A,\vartheta_\alpha\}_{\text{D.B.}} = 0 \,, \qquad\quad [\phi_A,F\}_{\text{D.B.}} = 0 \,, \qquad\quad [\varphi_\alpha,F\}_{\text{D.B.}} = 0 \,,
\end{aligned}
\qquad (2.36)
$$

recovering more familiar bracket structure of the ghost system,[6]

$$
\begin{aligned}
&[B_A,B_B\}_{\text{D.B.}} = 0 \,, \qquad [C_A,C_B\}_{\text{D.B.}} = 0 \,, \qquad [B_A,C_B\}_{\text{D.B.}} = [C_B,B_A\}_{\text{D.B.}} = \bar{\mathcal{P}}_{AB} = \mathcal{P}_{BA} \,, \\
&[b,b\}_{\text{D.B.}} = 0 \,, \qquad\quad [c,c\}_{\text{D.B.}} = 0 \,, \qquad\quad [b,c\}_{\text{D.B.}} = [c,b\}_{\text{D.B.}} = 1 \,.
\end{aligned}
$$
$$(2.37)$$

Thus, on the surface of the second-class constraints, the Poisson bracket reduces to Dirac bracket given by

$$[F,G\}_{\text{D.B.}} = \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\deg(F)}\left(\frac{\partial F}{\partial \theta^A}\frac{\partial G}{\partial \theta_A} + \frac{\partial F}{\partial \vartheta^\alpha}\frac{\partial G}{\partial \vartheta_\alpha}\right) . \qquad (2.38)$$

The first-class constraints (2.20) make up the BRST charge,

$$\mathcal{Q} := Q + q \,, \qquad Q := C^A H_A \,, \qquad q := cH \,, \qquad [\mathcal{Q},\mathcal{Q}\}_{\text{D.B.}} = 0 \,, \qquad (2.39)$$

where the nilpotency is ensured, with $C^A\partial_A = 0$ (the section condition), by

$$[Q,Q\}_{\text{D.B.}} = 0 \,, \qquad [Q,q\}_{\text{D.B.}} = 0 \,, \qquad [q,q\}_{\text{D.B.}} = 0 \,. \qquad (2.40)$$

The charge, $\mathcal{Q}$, generates the BRST symmetry (2.31) through the Dirac bracket,

$$
\begin{aligned}
&\left[x^A,\mathcal{Q}\}_{\text{D.B.}} = C^A + c\mathcal{H}^{AB}p_B \,, \qquad [p_A,\mathcal{Q}\}_{\text{D.B.}} = -\frac{1}{2}c\partial_A\mathcal{H}^{BC}p_B p_C \,, \\
&[B_A,\mathcal{Q}\}_{\text{D.B.}} = H_A \,, \qquad\qquad\qquad [C^A,\mathcal{Q}\}_{\text{D.B.}} = 0 \,, \\
&\quad [b,\mathcal{Q}\}_{\text{D.B.}} = H \,, \qquad\qquad\qquad\quad [c,\mathcal{Q}\}_{\text{D.B.}} = 0 \,,
\end{aligned}
\qquad (2.41)
$$

and commutes with the Hamiltonian, $[H,\mathcal{Q}\}_{\text{D.B.}} = 0$, as should be.

Crucially, restricted on the phase space of $\{x^A,p_B,\theta_C\}$ with the trivial bc ghost number, the Dirac bracket (2.38) reduces precisely to the graded bracket *à la* Deser and Sämann (1.16) [62].[7]

---

[6]For the fixed section of $\tilde{\partial}^\mu \equiv 0$, with footnote 5, $[B^\mu,C_\nu\}_{\text{D.B.}} = \delta^\mu{}_\nu$.

[7]Notice however the $\mathbb{Z}$-grading used in [62] is different from the ghost number grading of our particle action. Consequently, the bracket (2.38) has ghost number zero, but degree -2 in the grading of [62].

2.5 Quantum correction

We consider quantizing the Dirac bracket (2.36). We set a vacuum state, $|0\rangle$, which is annihilated by $\hat{p}_A$, $\hat{B}_A$, and $\hat{b}$. Any physical state, $|\Psi\rangle = \Psi(\hat{x})|0\rangle$, having trivial ghost numbers should satisfy

$$\hat{H}_A|\Psi\rangle = \bar{\mathcal{P}}_A{}^B \hat{p}_B \Psi(\hat{x})|0\rangle = -i\hbar\bar{\mathcal{P}}_A{}^B \partial_B\Psi(\hat{x})|0\rangle = 0 \,, \tag{2.42}$$

$$\hat{H}|\Psi\rangle = \frac{1}{2}\left(\hat{p}_A\mathcal{H}^{AB}(\hat{x})\hat{p}_B + (m\,l_s)^2\right)\Psi(\hat{x})|0\rangle$$
$$= \frac{1}{2}\left[-\hbar^2\partial_A\left(\mathcal{H}^{AB}(\hat{x})\partial_B\Psi(\hat{x})\right) + (m\,l_s)^2\Psi(\hat{x})\right]|0\rangle = 0 \,, \tag{2.43}$$

where the former and the latter would correspond to the section condition and the (doubled) Klein-Gordon equation respectively, while $\hat{p}_A\mathcal{H}^{AB}(\hat{x})\hat{p}_B$ is 'ordered' such that it becomes a Hermitian operator. However, (2.43) is not invariant under target spacetime DFT-diffeomorphisms [7]: the correct one should contain the DFT-dilaton, d and read

$$\left[-\hbar^2\mathcal{H}^{AB}\nabla_A\nabla_B + (m\,l_s)^2\right]\Psi(\hat{x})|0\rangle = \left[-\hbar^2\left\{\partial_A\left(\mathcal{H}^{AB}\partial_B\Psi\right) - 2\mathcal{H}^{AB}\partial_A d\partial_B\Psi\right\} + (m\,l_s)^2\Psi\right]|0\rangle$$
$$= 0 \,, \tag{2.44}$$

where the covariant derivative, ∇_A, was defined in [8] and gives the expression after the first equality. This new dilaton contribution might be comparable with the Fradkin-Tseytlin term on string worldsheet, cf. [80]. The modification (2.44) amounts to quantum corrections to the Hamiltonian constraint,

$$H_\hbar := \frac{1}{2}p_A\mathcal{H}^{AB}p_B + \frac{1}{2}(m\,l_s)^2 + i\hbar\mathcal{H}^{AB}\partial_A d\, p_B \,, \tag{2.45}$$

and accordingly to the classical action,

$$S_\hbar = \frac{1}{l_s}\int d\tau\, \frac{1}{2}e^{-1}\left(D_\tau x^A - i\hbar e\mathcal{H}^{AC}\partial_C d\right)\left(D_\tau x^B - i\hbar e\mathcal{H}^{BD}\partial_D d\right)\mathcal{H}_{AB} - \frac{1}{2}(m\,l_s)^2 e \,. \tag{2.46}$$

It is worth while to note that (2.44) corresponds to the Euler-Lagrange equation of the following action of the scalar field Ψ which is $\mathbf{O}(D,D)$ symmetric and diffeomorphism invarant,

$$S[\Psi] = \int e^{-2d}\left[\mathcal{H}^{AB}\partial_A\Psi\partial_B\Psi + (ml_s/\hbar)^2\Psi^2\right] \,. \tag{2.47}$$

The (non-constant) dilaton appears in the equation (2.44) as a dissipative term. This, in turn, is the reason for the imaginary part in the Hamiltonian (2.45). Nevertheless, the scalar field action (2.47) is real while the quantum particle action (2.46) is complex-valued.

3 Conclusion

In this note, we have shown that the BRST formulation of the doubled-yet-gauged particle action (2.24) naturally produces the graded Poisson geometry of [62, 63]. The Grassmann odd variable, θ^A, of the graded Poisson bracket carries an $\mathbf{O}(D,D)$ vector indices. Thus, if it is to be identified as a ghost of any BRST system, the underlying gauge symmetry

should be about the doubled spacetime itself, which we have shown to be the 'coordinate gauge symmetry', $x^A \sim x^A + \mathcal{P}^A{}_B V^B$ (2.8). One message our work may convey is that, the investigation of "spaces" can be performed by studying not only the functions defined on them but also the coordinate systems adopted for them, such as the doubled-yet-gauged coordinate system.

A few comments are in order. With $B^A = \bar{\mathcal{P}}^A{}_B \theta^B$ (2.26), if we set, instead of (1.18),

$$\mathcal{T}(x, B) := \frac{1}{p!} T_{A_1 A_2 \cdots A_p}(x) B^{A_1} B^{A_2} \cdots B^{A_p} , \qquad (3.1)$$

we may realize an exterior derivative precisely, cf. (1.20),

$$\big[p_A B^A , \mathcal{T}(x, B) \big\} = -\frac{1}{p!} \partial_{[A_1} T_{A_2 \cdots B_{p+1}]} B^{A_1} B^{A_2} \cdots B^{A_{p+1}} . \qquad (3.2)$$

Now, allowing $\mathbf{O}(D, D)$ duality rotations as well as coordinate gauge symmetry in addition to the DFT-diffeomorphisms for the gluing of overlapping patches, cf. [49, 81], the notion of de Rham cohomology should differ from the usual undoubled one.

It would be of interest for future work to generalize our analysis to sigma-models where the constant projector (2.1) is promoted to a local object, cf. [52, 58, 82]. Another interesting direction to pursue would be to promote the Abelian coordinate gauge symmetry to a non-Abelian version, as well as to extend this research to the doubled-yet-gauged string actions [29, 31, 34–36].

Acknowledgments

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