GAUGE THEORIES OF VECTOR PARTICLES*

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1. INTRODUCTION

The general topic of this paper is the vector theory of gauge fields, but I like to think that these lectures are really concerned with the future of the relativistic field theory as an effective force in the development of fundamental physics. Two basic positions are at present under investigation as the possible organizing forces for the rapidly growing empirical data on elementary particles. To put it as extremely as possible, we might call these two positions:

(i) The particle point of view,
(ii) The field point of view.

By the particle point of view,I mean those investigations in which the physical particles, as we see them, are the basic elements. This is the whole line of development associated with the S-matrix, with the idea that the only function of the theory is to compute and to correlate the results of scattering measurements. It also underlies those further attempts intended to give a physical content to this essentially empty framework, such as dispersion relations, Regge poles, etc. And, to adopt this point of view systematically, one must necessarily accept the Orwellian philosophy that no particle is more fundamental than any other. That is the strict particle point of view; the particles are unanalysable. To our mind it is an extremely conservative position.

Opposed to this is the field point of view which supports the idea that there is a deeper dynamical level, that the empirical information we have is very complicated and that the purpose of theory is to discover simplicity - not necessarily in terms of the observed properties, but in terms of concepts, of properties which are at the moment not directly observable but which undoubtedly will become so in the course of future developments. This is the way that physics has always proceeded. The field point of view is thus the idea that there exists something more fundamental than the phenomenological particles. This is a very general statement and we should say that field theory as it now stands is based upon the tentative identification of these more fundamental entities with some localizable fields. We would almost try to make a distinction between the idea that there is something dynamically deeper than the particles, and the particular association of the deeper structure with localizable fields. Such fields may be what is required, but the important thing to our mind is the alternative between accepting the particles as they are and seeking for something more fundamental. At the

* Based on notes taken by A.P. Balachandran, B. Jaksic, I. Saavedra, J. Kvasnica, P.W. Nielsen, D. Boulware, C. Bollini and P. Rastall.

moment, the latter is identified with the idea of fields which are operator functions of the space and time coordinates. But even within this framework, there are various possible viewpoints. There is the extreme viewpoint - this is Heisenberg's attitude - that there is only one fundamental field. Everything we know must come out of this one field. This is rather hard to accept, and I myself will adopt here the intermediate position that there are several fundamental fields. As to which fundamental fields are necessary I would say that the clue must be found in the exact, or almost exact, conservation laws we know in nature. This is the line of thought that leads to the idea of vector gauge fields which I am going to explore.

It should be emphasized that in the field viewpoint the fundamental fields are not immediately correlated with observable things. This is the deeper dynamical level; out of the interplay of the dynamics that govern these entities emerges the world of particles as we know it. In other words, the important thing is to recognize that the fields we begin with in this viewpoint are not necessarily directly correlated in a simple way with the observed particles. It has become fasionable to describe field theory as "old-fashioned". I would insist upon the following: what is old-fashioned is the naive confusion of these two points of view, in which one speaks indiscriminatly of particles and fields and associates with every particle a field which is inserted in some Lagrangian for the purpose of applying perturbation theory. This is the old-fasioned, naive point of view but it is not the one I am advocating here. We must clearly understand that we are dealing with a much more sophisticated approach, in which the fundamental fields are not simply correlated with particles, although there may be an ardent relation in some individual cases. The basic physical problem, from this point of view, is to explore the possibilities of postulating various fundamental fields with their dynamics and by proving the existence of special states of definite or almost definite energy-momentum relations to identify these with physical particles.

Such are the two extreme viewpoints, and obviously it is the second one which is adopted here. I shall try to indicate some of the possibilities that are inherent within it. Now, I said that the clue to which fields are fundamental is given by the exact, or perhaps almost exact, conservation laws. And I point here, inevitably, to the example upon which the whole field theory has been built more or less by analogy, i.e. electrodynamics. The electromagnetic field has the very special feature of gauge variation; while it might be possible to advocate, as Heisenberg does, that there is no fundamental electromagnetic field, I regard this property of gauge variation to be so basic that it seems necessary to postulate a fundamental electromagnetic field.

It should be remembered that the electromagnetic field is one such that the vector potential must be allowed freedom of transformation by gradients of an arbitrary scalar function, at the moment a numerical scalar function, say:

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x). \tag{1.1}$$

Now we know that as the theory has been constructed to be invariant under such a gauge transformation, it follows automatically that the current vector $j^\mu(x)$, which is the source of this field, must be conserved:

$$\partial_\mu j^\mu(x) = 0. \tag{1.2}$$

This is the important aspect of gauge invariance: the concept of the absolute conservation of the electrical charge is not explained as the result of specific dynamical restrictions on every conceivable system, but is understood in terms of the structure of the Maxwell field itself. To put it in another way, we know that the field strength tensor $F^{\mu\nu}$ is antisymmetrical and obeys the equation:

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{1.3}$$

and in virtue of that antisymmetry, a structural property of the field, it follows automatically that the current obeys

$$\partial_\mu j^\mu \equiv 0 \,. \tag{1.4}$$

That is, the equation of local electrical charge conservation is an identity characteristic of the structure of the Maxwell equations, and, therefore, once the Maxwell field is introduced, non-conservation of the electrical charge is inconceivable. This is the perfect model of a dynamical explanation of an absolute conservation law.

One may attempt to build an explanation of another absolute conservation law along these lines. One may say that what has been explained here is, in a sense, the absolute stability of the electron. The electron, being the lightest object that carries an electrical charge, is a stable object in virtue of the conservation of the electrical charge, since there is nothing lighter for it to go into while maintaining its charge. There is an analogy between the stability of the electron and the conservation of electrical charge, on the one hand, and the stability of nuclear matter and the conservation of the nuclear charge, on the other. This nucleonic charge must be possessed by all the heavy baryons and is handed on from the cascade particle to the Λ, the Σ and the nucleon in the process of all their disintegrations. But with a nucleon, or more precisely with a proton, as the lightest object carrying this nucleonic charge, the process of decay ceases because there is nowhere else to transmit the nucleonic charge. That is, in the absolute conservation of nucleonic charge we have a description of the stability of matter and one would like to have an understanding of this most fundamental of all conservation laws on some general dynamic grounds rather than merely as a statement, since it is a rule which has to be superimposed on every possible interaction. It is natural, then, to introduce a hypothetical vector field, a gauge field analogous to the electromagnetic field and to insist that its dynamics be governed by the requirement of gauge invariance from which would follow the existence of an absolute conservation law. This dynamical explanation involves a new field and the question now is what will be the dynamic consequences of that field. Here is where the idea appears to run into immediate difficulties. If the analogy with the electromagnetic field is complete, a physical particle with zero mass, analogous to the photon should exist, and we know of no such particle. One could assume that the coupling to the new field is arbitrarily weak, there are arguments that the field must be unobservable even on a cosmological scale. That is hardly the kind of fundamental field which should be introduced to explain the conservation of the strongly interacting particles. This was the great objection: gauge invariance should imply the existence of a zero mass particle. And this is the decisive point at which we want to introduce the ideas of the new field theory.

It may be helpful to give a simple form of the argument relating gauge invariance to a massless particle. Using the notation of electrodynamics, we have the charge density equation

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{1.5}$$

and an integration over a large volume gives:

$$\int (dr)\rho = Q = \int d\vec{S} \cdot \vec{E} \tag{1.6}$$

where Q is the total charge of the system and a constant of the motion. Therefore, the electric field at large distances must fall off like

$$E \sim (Q/4\pi)(\vec{n}/r^2) \tag{1.7}$$

which is a long range field. This is a static field, but one could argue, not incorrectly, that if one finds a static field which is long range, there must be a zero-mass particle or the field would be of finite range. But it is implicitly assumed here that the total charge Q is different from zero. And it is precisely at this point that the argument fails. When a charge is inserted into the vacuum, the accompanying electric field polarizes the vacuum producing a partial compensation of the charge. That is the origin of charge renormalization. But it is conceivable that the compensation of charge is not partial, but complete is present. That is, if a charge is placed in the system, there may come into being in the course of time a vacuum polarization in which, loosely speaking, one part of the charge escapes to infinity and the compensating charge exactly balances the charge that was originally inserted. Under these conditions, the constant total charge that will be observable in any arbitrarily large volume will be zero. This is not intended as a convincing argument, but merely an indication that there is a loophole in the assertion that there must be a long-range field - or massless particle- for this depends upon the assumption that there is no complete compensation charge. The massless physical particle disappears when a non-zero total charge can no longer be maintained in the vacuum.

2. THE ONE DIMENSIONAL MODEL

Rather than indicate by general agreements that this is a very real possibility, a very simple physical model will be used to show that such a new situation can occur.

The model I want to discuss is completely physical, in the sense that no general principles of physics are violated. On the other hand, it is an unworldly one, since it is a special case of electrodynamics in one spatial dimension. Of course, all the arguments we have given until now about gauge invariance apply equally well to one spatial dimension, apart from specifically geometrical factors.

Let me write down the basic equations we shall be concerned with for electrodynamics, and then we shall specialize and solve exactly in one-dimensional space. We shall begin with the Lagrange function

$$\mathcal{L} = -(1/2)F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + (e^2/4)F^{\mu\nu}F_{\mu\nu}$$
$$+(i/2)\Psi\alpha^\mu \partial_\mu\Psi + (i\, m_0/2)\Psi\beta\Psi + (1/2)A_\mu\Psi\alpha^\mu q\Psi \qquad (2.1)$$

where we have introduced two fundamental fields, a gauge field characterized by a vector A_μ and an antisymmetric tensor $F_{\mu\nu}$, and a Fermi field Ψ. The matrices α and β are connected with the usual Dirac γ-matrices by $i\alpha^\mu = \beta\gamma^\mu$. The α^μ are all real and symmetrical, and $\alpha^0 = 1$; β is real and antisymmetrical; e is the coupling constant and m_0 the mass constant associated with the field Ψ. The antisymmetric matrix

$$q = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

is specifically associated with the charge, and is introduced here in order to work with Hermitian Ψ fields. Under the transformation $A_\mu \to e A_\mu, F_{\mu\nu} \to (i/e)F_{\mu\nu}$, the Lagrangian $\mathcal{L}$ goes back to its more familiar form (i.e. the coupling constant e appears at its usual place in the coupling term $e A^\mu j_\mu$). The Fermi field obeys the anticommutation relation

$$\left\{\Psi_\alpha(x), \Psi_\beta(x')\right\} = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}'), \quad x_0 = x_0',$$

where the indices α and β refer to spin and charge.

The one essential point to be emphasized about the distinction between three dimensions and one dimension is the question of the dimensionality of the charge, i.e. of the coupling constant. The action operator

$$w = \int (dx)\mathcal{L}$$

is dimensionless in the system of units where $\hbar = 1$. Then the dimension of the Lagrangian

$$[\mathcal{L}] = 1/(L^{n+1})$$

where L is length and n the number of spatial dimensions. From this, we obtain the dimensionality of e^2:

$$[e^2] = 1/(L^{3-n})$$

and hence for the particular case n = 3, e is dimensionless, while for n = 1,

$$[e^2] = 1/L^2,$$

i.e. the coupling constant itself carries a length, carries a mass.

We shall now show that it is possible to find an exact solution of the one-dimensional problem in the special case in which the mass constant associated with the Fermi field vanishes, $m_0 = 0$. In this one-dimensional model there are only two α-matrices, $\alpha^0 = 1$ and $\alpha^1 = \alpha_1$, which can be represented by 2×2 matrices. The spatial metric adopted is positive and the time metric negative.

3. EXACT SOLUTION OF THE ONE-DIMENSIONAL PROBLEM

What we want to do now is to solve a preliminary problem: the polarization of the vacuum of a Fermi field Ψ by an externally imposed field A_μ. We must then introduce a certain requirement of self-consistency: the charge brought in creates a field, this field polarizes the vacuum which creates a charge that polarizes the vacuum and so on. The problem can be solved exactly in our model, because of the assumption of one spatial dimension and the zero mass of the fermion field. This is familiar, for example, from the discussions that have gone on about the Thirring model which is also a one-dimensional model though not electrodynamic.

Our preliminary problem is then a Dirac field Ψ plus an external (electromagnetic) field A_μ. In terms of its solution we shall have the exact solution to our problem. We begin with a simplified Lagrange function

$$\mathcal{L} = (i/2)\Psi\alpha^\mu(\partial_\mu - iq A_\mu)\Psi. \tag{3.1}$$

We want to find the current induced in the vacuum by the external field A_μ. Let this be:

$$\langle j_\mu(x)\rangle = 1/2\langle\Psi(x)\alpha q\Psi(x)\rangle^A. \tag{3.2}$$

Since $j_\mu(x)$ is a bilinear combination of fields taken at the same point x, we construct its expectation value by first solving another problem, which is to find the expectation value of a bilinear combination of fields at arbitrary points of space and time. This is, in other words, the construction of the Green's function associated with the field. We define this Green's function as

$$G(x, x'; A) = \langle(\Psi(x)\Psi(x'))_+\rangle\epsilon(x-x') \tag{3.3}$$

which is the vacuum expectation value of the time ordered product of the fields. $\epsilon(x-x')$ is a sign function. This is the basic physical quantity in terms of which we extract physical information about the states that are created in the vacuum of the field Ψ, and in terms of which, by a limiting process with $x' \to x$, we shall construct the current operator.

The Green's function obeys an inhomogeneous differential equation which incorporates the field equations and the anti-commutation relations

$$\alpha^\mu(\partial_\mu - iq A_\mu(x))G(x, x') = \delta(x-x'). \tag{3.4}$$

Under a gauge transformation Eq. (1) this Green's function transforms according to

$$G(x, x') = \exp[iq\lambda(x)]G(x, x')\exp[-iq\lambda(x')]. \qquad (3.5)$$

The current operation $j_\mu(x)$ is given by:

$$j_\mu(x) = (1/2)\Psi(x)\alpha_\mu q\Psi(x). \qquad (3.6)$$

This is a singular expression and therefore it must be defined by a suitable limiting process as stated above. We must let $x' \to x$ along a space-like direction, since we do not want to bring dynamics into the definition of an operator. We must carry out this definition in such a way that gauge invariance is guaranteed and then we must check the covariance of the procedure.

We now rewrite the expectation value Eq. (3.2) in terms of a Green's function. We get:

$$\langle j_\mu(x)\rangle = -(1/2)\mathrm{Tr}\, \alpha_\mu\, G(x, x) \qquad (3.7)$$

where G (x, x) is defined by:

$$G(x, x) = \lim_{x' \to x} G(x, x')\exp[-iq\int_{x'}^{x} d\xi A_\mu(\xi)], \qquad (3.8)$$

the limit being taken from a spatial direction maintaining all symmetries, i.e. taking an average of the values of the limits attained from the left and from the right. The exponential factor is required in order to maintain gauge invariance for $x \neq x$.

The solution of Eq. (3.3) can be written as:

$$G(x, x') = G^0(x, x')\exp iq[\Phi(x) - \Phi(x')] \qquad (3.9)$$

where $\Phi(x)$ satisfies:

$$\alpha^\mu \partial_\mu \Phi(x) = \alpha^\mu A_\mu(x). \qquad (3.10)$$

and G^0 is the Green's function for $A_\mu = 0$

$$\alpha^\mu \partial_\mu G^0(x, x) = \delta(x - x'). \qquad (3.11)$$

The solution of this equation with the proper boundary conditions is:

$$\begin{aligned} G^0(x, x') &= (1/2\pi)\int_0^\infty dp \exp[ip\,\alpha^\mu(x_\mu - x'_\mu)\, ,\ x^0 > x^{0\prime} \\ &= -(1/2\pi)\int_0^\infty dp \exp[ip\,\alpha^\mu(x_\mu - x'_\mu)\, ,\ x^0 < x^{0\prime}. \end{aligned} \qquad (3.12)$$

To perform the limiting process (along a space-like direction), let us consider the right hand side of Eq. (3. 8) for $x_0 = x'_0$:

$$G(x, x')\{\exp iq[\Phi(x) - \Phi(x')]\}\left\{\exp[-iq\int_{x'}^{x_1} d\xi A_1]\right\}.$$

But, for equal times

$$G^0 = (1/2\pi)[\alpha_1/(x_1-x'_1)].$$

Expanding the exponentials in a Taylor series for $x'_1 \to x'_1$, we get:

$$G \cong (i/2\pi)[\alpha_1/(x_1-x'_1)][1+iq(x_1-x'_1)(\partial_1\Phi - A_1). \quad (3.13)$$

Taking now the symmetrical limit as explained above, we obtain:

$$G(x,x) = -(1/2\pi)\alpha_1\, q[\partial_1\Phi(x) - A_1(x)] = (1/2\pi)\, q(\partial_0\Phi(x) - A_0(x)). \quad (3.14)$$

Inserting this into Eq. (3.7), we obtain the covariant expression:

$$\langle j_\mu(x)\rangle = -(1/\pi)[A_\mu(x) - \partial_\mu(1/4)\,\mathrm{Tr}\,\Phi(x)]. \quad (3.15)$$

Writing explicitly Eq. (3.10)

$$(\partial_0 + \alpha^1\partial_1)\Phi(x) = A_0(x) + \alpha^1 A_1(x) \quad (3.16)$$

and multiplying it from the left by $(\partial_0 - \alpha^1\partial_1)$, we obtain the second order differential equation:

$$-\partial^2\Phi(x) = -\partial_\mu A^\mu(x) + \alpha^1[\partial_0 A_1(x) - \partial_1 A_0(x)]. \quad (3.17)$$

By taking the trace, this reduces to

$$-\partial^2(1/4)\,\mathrm{Tr}\,\Phi(x) = -\partial_\mu A^\mu(x) \quad (3.18)$$

which we can solve for $\mathrm{Tr}\,\Phi$ by means for the corresponding Green's function $D(x, x')$:

$$(1/4)\,\mathrm{Tr}\,\Phi(x) = -\int (dx')D(x,x')\,\partial_\nu A^\nu(x') \quad (3.19)$$

or, symbolically:

$$(1/4)\,\mathrm{Tr}\,\Phi(x) = -D\,\partial_\nu A^\nu(x). \quad (3.20)$$

Hence, our final result in this notation is:

$$\langle j_\mu(x)\rangle = -(1/\pi)[A_\mu(x) + \partial_\mu D\,\partial_\nu A^\nu]. \quad (3.21)$$

This is an obviously covariant expression, it is also conserved and it is gauge invariant. To show that it is conserved, let us take the divergence of Eq. (3.21). We get:

$$\partial^\mu j_\mu = -(1/\pi)\,[\partial^\mu A_\mu - \partial^\mu A_\mu] = 0.$$

Let us now indicate some of its physical implications by a simple but not wrong method. We will then justify it. Let us think of the idea of self-consistency in the simplest possible way. A_μ has been until now an external field, but suppose this field somehow is brought into existence propagating in accordance with Maxwell's equations. Then this field induces a current and this current in turn reacts back to change the nature of the field. What then is the condition of self-consistency ?

We go back to Maxwell equations

$$\partial_\nu F^{\mu\nu} = j_\mu ,$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = e^2 F_{\mu\nu} .$$

From here, $e^2 j^\mu = -\partial^2 A^\mu + \partial^\mu \partial_\lambda A^\lambda$. Adopting now the Lorentz gauge $\partial_\mu A^\mu = 0$. and using Eq. (3.21), we get the propagation equation for the vector potential:

$$(-\partial^2 + \mu^2) A_\mu(x) = 0 \tag{3.22}$$

where

$$\mu^2 = (e^2/\pi). \tag{3.23}$$

This is an equation describing non-interacting particles of finite mass $\mu = \sqrt{(e^2/\pi)}$ and shows that gauge invariance of a vector field does not necessarily require zero-mass particles.

The expression found for the vacuum expectation value of the current in the presence of an external vector potential A is

$$\langle j^\mu(x) \rangle^A = -(1/\pi)[A^\mu(x) + \int \partial^\mu D(x-x')\partial'_\nu A^\nu(x')], \tag{3.24}$$

where j^μ is the electrical current carried by the fermion field. This may be symbolically written

$$\langle j \rangle^A = -(1/\pi)(1 + \partial D \partial) A, \tag{3.25}$$

where the projection operator $(1 + \partial \Box \partial$ guarantees the conservation of charge and gauge invariance. We also found, by a simple self-consistency argument, that the condition for the vector potential to maintain itself is that it satisfy the field equation

$$[\partial^2 - (e^2/\pi)]A = 0. \tag{3.26}$$

Here $e^2/\pi = \mu^2$ plays the part of the square of a mass; so the result is - at least in a simple-minded way - that the propagation equation for A is the same as that for a particle of mass μ. We shall give a precise derivation of this result here and also show how to calculate all other properties of the system.

But before we begin the precise derivation, let me come back to another general qualitative remark that I made. The equation $\nabla . E = \rho$ implies that at great distances from the sources the electric field E satisfies $E \sim Q$

in the one-dimensional case and $E \sim Q/4\pi r^2$ in the three-dimensional case, where Q is the total charge. It is argued, quite correctly, that a long-range electrical field can only be maintained and propagated by zero-mass particles. However, the course of the argument is that the total charge should be different from zero, and this is not the case under the conditions we are talking about because the vacuum polarization effect acts to annihilate any given charge.

Suppose that we insert a static external charge density J^0 whith total charge Q_0 into the vacuum. A charge density j^0 will then be induced, whose expectation value, in the Lorentz gauge, is given by our previous equation

$$\langle j^0 \rangle = -(1/\pi) A^0 . \tag{3.27}$$

The potential A^0 has its source in the total charge density $J^0 + \langle j^0 \rangle$:

$$-\partial^2 A^0 = e (J^0 + \langle j^0 \rangle). \tag{3.28}$$

Substituting for $\langle j^0 \rangle$, and using the fact that the fields are time independent, we get

$$(d^2/dx'^2 - \mu^2) A^0 = -e^2 J^0 . \tag{3.29}$$

The solution of this is:

$$A^0 = (e^2/2\mu) \int (dx'^1) \{ \exp [-\mu (x' - x'^1)] \} J^0 (x'). \tag{3.30}$$

The total charge induced in the vacuum is therefore

$$\int \langle j^0 (x') \rangle dx' = [-e^2/(2\mu \pi)] [\int dx' dx'^1 \{ \exp [-\mu |x' - x'^1|] \} J^0 (x') = -Q_0 \tag{3.31}$$

which exactly cancels the inserted charge Q_0. Thus there is no long-range field and no longer an argument for a zero-mass particle.

4. SOLUTIONS OF THE GENERAL EQUATIONS WITH EXTERNAL SOURCES AND THE GREEN'S FUNCTIONAL.

Now we must write down the general equations of this relativistic field system and solve them exactly. The method we shall use is that of external sources and the Green's function. This is the general technique for dealing with any field problem. It depends on the idea of introducing simple excitations into the system, in terms of which all possible states can be created.

To the Lagrangian written down previously we add the source terms

$$A^\mu (x) J_\mu (x) - i\bar{\psi}(x) \eta(x). \tag{4.1}$$

The total Lagrangian must still be gauge invariant, which implies that the external current J_ν is conserved (this external current can be considered simply as an idealization of other dynamic systems which act upon our system). As for the Fermi source term η, it is a fully anticommutative quantity (just as the boson source J^μ is fully commutative). The source $\eta(x)$ anticommutes with $\eta(x')$ and $\psi(x')$ for all x and x'. Such quantities can be perfectly well realized in terms of familiar algebraic structures. The change in $\eta(x)$ under a gauge transformation must be just such as to compensate the change in ψ.

With the addition of the source terms, the field equations become inhomogeneous. That for the Fermi-field – written for the case of zero-mass constant - is

$$\alpha^\mu(\partial_\mu - i\epsilon A_\mu)\psi = \eta, \tag{4.2}$$

while that for the electromagnetic field tensor is

$$\partial_\nu F^{\mu\nu} = J^\mu + j^\mu_c. \tag{4.3}$$

Notice that this is j^μ_c, and not j^μ. The quantity $j^\mu = \frac{1}{2}\psi\alpha^\mu q\psi$ is no longer conserved in the presence of sources and therefore it would be inconsistant to write $J^\mu + j^\mu$ as the right hand side of our previous equation. A proper calculation, which takes account of the fact that there is a transfer of charge from outside the system we are considering, shows that one must extract from j^μ its conserved part j^μ.

The point of introducing these external source terms is that one can convert the Hilbert-space operator field equations by their aid into numerical functional differential operator equations. For our problem the latter equations turn out to be soluble. But how do we make the transition from one kind of equation to the other? Well, we consider that the system begins in the vacuum state and the sources are, so to speak, turned on. The system is then disturbed and by choosing the disturbance correctly one may generate any state into which the vacuum may be thrown by the action of the field operators. By watching how these states propagate in time we see their properties. Finally, we switch off the sources and return to the vacuum state. The mathematical quantity which contains all the information about this process is the transformation function which relates the vacuum state $|0_-\rangle$ before the disturbance to the vacuum state $|0_+\rangle$ after it. I shall call this transformation function the Green's functional: it is the generating functional of all the Green's functions, or propagation functions, which describe processes in our system. We write it:

$$G[\eta J] = \langle 0_+|0_-\rangle^{\eta J}. \tag{4.4}$$

We must find how the Green's functional depends on the external sources η and J. The idea is to consider its response to infinitesimal changes $\delta\eta, \delta J_\mu$. These produce a change in the Lagrangian:

$$\delta_{\eta J}\mathcal{L} = A^\mu \delta J_\mu - i\psi\, \delta\eta, \tag{4.5}$$

and a change in the Green's functional

$$\delta_{\eta J} G[\eta J] = \delta_{\eta J}\langle 0_+ | 0_- \rangle^{\eta J} = i \rangle 0_+ \Big| \int (dx) A^\mu \delta J_\mu - i\psi\,\delta\eta \,\Big| 0_- \rangle. \quad (4.6)$$

(The matrix elements are always taken between vacuum states). If one imagines a disturbance δJ_μ, $\delta\eta$ localized around the point x and if one somehow knows $\delta_{\eta J}G[\eta J]$, then immediately one gets the matrix elements of the operators $A^\mu(x)$ and $\psi(x)$. If this can be repeated at all points of space-time, one gets a general correspondance between matrix elements of the field operators and functional derivatives of $G[\eta J]$. This gives the general differential operator representation of the field operators, very much analogous to the representation of p's by differential operators with respect to q's.

Now we come to an important point. I have written down the variation δJ^μ. To find one differential operator representation we should like to make arbitrary infinitesimal variations of the J^μ. But this we cannot do: the J^μ are not independent. If we vary the m independently we shall violate charge conservation and therefore gauge invariance. The way to overcome this difficulty is to work in a specific gauge. By choosing a suitable gauge we shall be able to vary our J^μ arbitrarily while conserving charge.

How is this to be accomplished? Take an arbitrary vector J and project it by means of a projection operator Π into a vector J_c which is conserved

$$J_c = \pi J. \quad (4.7)$$

I want to make this projection so that it does not upset the temporal development of the system, so we shall choose a projection operator which is local in time. Let us introduce, in addition to the usual space-time gradient ∂_μ, the purely spatial gradient ∇_μ. The spatial components, or component since what we say applies to both one and three dimensions, of ∇_μ are the same as those of ∂_μ, but the time component is zero. The projection equation will be taken to be

$$J_c = (1 + \nabla\mathcal{D}\delta)\, J, \quad (4.8)$$

where $\mathcal{D}$ is the Green's function associated with the spatial gradient

$$\nabla^2\mathcal{D}(x, x') = -\,\delta(\underline{x}-\underline{x}'). \quad (4.9)$$

The conservation equation $\partial J_c = 0$ follows immediately.

The conservation of charge can now be ensured by replacing δJ by δJ_c in the J term of our variational integral

$$i\int (dx)\, A(1 + \nabla\mathcal{D}\partial)\delta J. \quad (4.10)$$

We can now certainly perform arbitrary variations of J, but at the cost of some awkwardness. However, if we now choose the radiation gauge, in which $\nabla.\underline{A} = 0$, and perform an integration by parts (this must be validated by appropriate restrictions, which we will not go into), then the extra term $\nabla\mathcal{D}\partial$ simply disappears. So we have exploited the gauge freedom of the theory in such a way that we can replace the variation of the conserved current by the variation of an arbitrary vector. This is true only for our special choice of gauge. Other gauges are possible, but the projection operator will no longer be local in time. However, once we have presented the whole formalism in terms of functional differentials, we shall be quite free to change the gauge as we whish. In fact, I shall immediately switch over from the radiation gauge to the Lorentz gauge, which is much more symmetrical.

Treating J as arbitrary, which is justified with our particular choice of gauge, we can write down a correspondence between the variational derivative with respect to J^μ and the vector potential A^μ

$$(1/i)[\delta/\delta J_\mu(x)] \leftrightarrows A^\mu(x). \qquad (4.11)$$

(The coefficient of δJ_μ in the variational integral is just iA^μ). In the same way, considering the $\psi\,\delta\eta$ term in the variational integral, we find the correspondence

$$-\delta_\ell/\delta\eta(x) \leftrightarrows \psi(x). \qquad (4.12)$$

The ℓ suffix indicates that this is a left derivative. In making the correspondence we must bring $\delta\eta$ to the left of ψ, which accounts for the minus sign ($\psi\,\delta\eta = -\delta\eta.\psi$).

These correspondences suggest that one can convert the field equations for ψ and A^μ into functional differential equations for $G[\eta J]$ by simply substituting $\delta/\delta J_\mu$ for $i A^\mu$ and $\delta/\delta\eta$ for ψ. First, from the Dirac equation we get:

$$[\alpha^\mu(\partial_\mu - q\frac{\delta}{\delta J^\mu(x)})\frac{\delta}{\delta\eta(x)} + \eta(x)]G[\eta J] = 0. \qquad (4.13)$$

Secondly, there is the Maxwell set and at this point we shall change over to the Lorentz gauge. The radiation gauge was described first because it is most immediate, but now let us define the conserved current J_c by

$$J_c = (1 + \partial D\partial)\,J, \qquad (4.14)$$

where D is the Green's function associated with $-\partial^2$. This equation for J_c is not local in time, but it does have the advantage of being manifestly relativistically invariant. I shall not go through the mechanics of the gauge transformation, but the result is

$$[\partial\partial\cdot\frac{1}{i}\,\frac{\delta}{\delta J} - \partial^2\frac{1}{i}\,\frac{\delta}{\delta J} - e^2(1 + \partial D\partial)(J + \tfrac{1}{2}\frac{\delta}{\delta\eta}\,\alpha q\,\frac{\delta}{\delta\eta})]G[\eta J] = 0. \qquad (4.15)$$

where J is the external current, $(\frac{1}{2})(\delta/\delta\eta)\alpha\, q(\delta/\delta\eta)$ corresponds to the physical fermion current $\frac{1}{2}\psi\,\alpha\, q\psi$ and the projection operator $(1 + \partial D\partial)$ ensures charge conservation. We still have to write the transcription for the last equation, corresponding to the choice of gauge. The final gauge equation is

$$\partial\,\frac{\delta}{\delta J}\,G[\eta J] = 0, \qquad (4.16)$$

which says that the Lorentz gauge is the chosen one.

We must now solve the functional equations for the Green's function. Let us first consider the Dirac equation. The variational derivative $\delta/\delta J$ can be treated as c-numbers (and they behave like c-numbers in the sense that they are commutative). For the moment we shall call them $i A_\mu$, i.e.

$$\frac{1}{i}\frac{\delta}{\delta J^\mu} \to A_\mu. \tag{4.17}$$

The Green's function now obeys an equation in the presence of an external field A_μ. Treating this field as a parameter we can convert the differential equation

$$\frac{\delta}{\delta\eta} G[\eta J[= -\int dx' \, G(x,x', A)\eta(x) G[\eta J], \tag{4.18}$$

where $G(x, x'. A)$ is the Green's function for the Dirac equation in the presence of an external potential A_μ.

The formal solution of the last equation can immediately be written down. It is, of course, an exponential

$$G[\eta J] = G[J] \exp\left\{-\tfrac{1}{2}\int dx\, dx' G(x,x', \frac{1}{i}\frac{\delta}{\delta J})\eta(x')\right\} \tag{4.19}$$

where $G[J[$ is a constant of integration. We can now transfer this partial solution to the Maxwell equation in order to determine also the J dependence of G:

$$[\partial\partial\frac{1}{i}\frac{\delta}{\delta J} - \partial^2\frac{\delta}{\delta J} - e^2(1+\partial D\partial)(J - \tfrac{1}{2}\,\mathrm{Tr}\,\alpha q G(x, x', \frac{1}{i}\frac{\delta}{\delta J})]G[J] = 0. \tag{4.20}$$

Taking this equation with the characteristic condition for the Lorentz gauge

$$\partial(\delta G/\delta J) = 0, \tag{4.21}$$

we see that the previous equation is equivalent to

$$\left[(-\partial \; + e^2/\pi)(1/i)\delta/\delta J - e^2(1 + \partial D\partial) J\right]G = 0 \tag{4.22}$$

in which, use has been made of the known structure of the current

$$(\tfrac{1}{2})\,\mathrm{Tr}\;\alpha q G(x, x'; A) = j(A) \tag{4.23}$$

in the case of the external potential, for which the current (in the Lorentz gauge) was proportional to A_μ. Again, the differential equation can be replaced by an integral functional equation by using the Green's function for the problem:

$$(-\partial^2 + e^2/\pi)\, G(x, x') = \delta(x, x'). \tag{4.24}$$

The Green's functional G[J] is therefore given exactly by

$$G[J] = \exp\left[\frac{i}{2}\int dx\, dx'\; J^\mu(x)\, G_{\mu\nu}(x, x')\, J^\nu(x')\right] \tag{4.25}$$

where

$$G_{\mu\nu} = (1 + \partial D \partial)_{\mu\nu} G. \tag{4.26}$$

All the physical characteristics are contained in G. The projector $(1+\partial D \partial)$ assures the fulfilment of the Lorentz gauge condition.

The expansion of the exponential in the solution of the Green's functional produces, in a sense, all the possible states of the system. The coefficients of the expansion refer to the physical propagation of the system giving the multiple Green's functions for them.

At this point it may be instructive to consider one example: This is the comparison of:

(1) The quantum electrodynamic case with its gauge invariance; and

(2) The vector field case with new zero mass and no gauge condition.

5. COMPARISON OF THE QUANTUM ELECTRODYNAMIC CASE AND THE VECTOR FIELD CASE

It has been shown that one may, under suitable physical conditions, have a gauge invariant theory, exact conservation laws and yet no zero mass vector particle.

In the two simple theories which have been considered (pure electrodynamics which is gauge invariant and a vector field which already has a mass constant and therefore is not gauge invariant) a distinct difference in the nature of the spectra has been found. In the gauge invariant case one has a particle with a non-zero mass, which depends on the coupling constant, whereas in the non-gauge invariant case one has both a vector particle with non-zero mass and a scalar particle with mass zero.

The complete set of Green's functions, which in principle contain the answers to all possible physical questions, are finite in the electrodynamic case and meet all general requirements in a perfectly reasonable way. With these Green's functions one can go on to discuss scattering and radiation properties of the Fermi particles in interaction with the Bose field.

In the non-gauge invariant case one meets "divergences", which does not mean that anything is infinite, but rather that almost everything is zero. The system does not respond to Fermi excitations in a way that is formally characterized by the vanishing of the Fermi field renormalization constant.

This does not mean that we have just two different theories. We have the choice between one field theory, where everything is finite and reasonable, and another field theory which is unphysical, even though it is "renormalizable".

The mere possibility of renormalizability is not sufficient for physical acceptability if the renormalization constants are zero. Renormalization is part of the process of physical interpretation, not a mathematical means of suppressing divergencies.

The general technique in the investigation of the simle model has been the use of the Green's functional $G[\eta, J]$ which is the response of the system to elementary disturbances.

The dependence of the Green's functional on the Fermi sources is given by:

$$G[\eta, J] = G[J]\, \exp\left\{-\tfrac{1}{2}\int \eta(x)\, G\left(x, x', \frac{1}{i}\frac{\delta}{\delta J}\right)\eta(x')\right\}. \tag{5.1}$$

This formula is completely general. In the simple model G[J] is given by:

$$G[J] = \exp\left\{\frac{ie^2}{2}\int J^{\mu}(x)\, G_{\mu\nu}(x, x')\, J^{\nu}(x')\right\}. \tag{5.2}$$

It is possible to evaluate all the Fermi Green's functions. In the electrodynamic case they will be finite, whereas in the non-gauge invariant case they all vanish.

The Green's function in the presence of an external electro-magnetic field $A_\mu(x)$ is:

$$G(x, x'; A) = G^0(x, x')\, \exp\left\{i q[(\phi(x) - \phi(x')]\right\}, \tag{5.3}$$

where

$$G^0(x, x') = \begin{cases} \displaystyle\int_0^{\infty} \frac{dp}{2\pi} \exp[ip\alpha^{\mu}(x_\mu - x_\mu')], & \text{for } x^0 > x'_0 \\ \displaystyle -\int_{-\infty}^{0} \frac{dp}{2\pi} \exp[ip\alpha^{\mu}(x_\mu - x_\mu')], & \text{for } x^0 < x'_0 . \end{cases} \tag{5.4}$$

is the Green's function for the non-interacting case. This Green's function corresponds to a Fermi particle with zero mass moving in one dimension. There is an invariant distinction between a particle moving to the right or to the left.

The function $\Phi(x)$ satisfies the differential equation

$$\alpha^{\mu}\partial_\mu \phi(x) = \alpha^{\mu} A_\mu(x) \tag{5.5}$$

from which we can construct $\phi(x)$ as a linear functional of $A_\mu(x)$. Multiplying Eq. (5.5) from the left with the operator $(\partial_0 - \alpha^1\partial_1)$ one obtains the second order differential equation:

$$-\partial^2\phi = (\partial_0 - \alpha^1\partial_1)\alpha^{\mu} A_\mu \tag{5.6}$$

This equation has the solution:

$$\Phi(x) = \int (d\xi)\, D(x-\xi)\left(\frac{\partial}{\partial\xi^0} - \alpha^1\frac{\partial}{\partial\xi^1}\right)\alpha^{\mu} A_\mu(\xi). \tag{5.7}$$

The exponential in Equ. (3.3) may now be written:

$$\exp iq[\phi(x) - \phi(x')] = \exp i\int (d\xi)\, A^{\mu}(\xi)\, J_\mu(\xi; x, x') \tag{5.8}$$

where $j_\mu(\xi; x, x')$ is a definite numerical function, which acts as a current in connection with the vector field. The following explicit expression for the current is obtained from Eqs.(5.7) and (5.8):

$$j_\mu(\xi; x, x') = q\alpha_\mu \left(\alpha^1 \frac{\partial}{\partial \xi^1} - \frac{\partial}{\partial \xi^0}\right) [D(x-\xi) - D(x'-\xi)] . \tag{5.9}$$

The divergence of the current is:

$$\partial_\mu j^\mu(\xi; x, x') = q[\delta(x-\xi) - \delta(x'-\xi)] . \tag{5.10}$$

The current is seen to have two sources with opposite signs at x and x′ respectively, corresponding to the effect of the Fermi field at these points.

The different Green's functions may be constructed by expanding the Green's functional $G[\eta, J]$ in powers of the sources η and J. Expanding the Green's functional in powers of the Fermi sources η one obtains:

$$G[\eta, J] = [1 - \tfrac{1}{2}\int \eta G \eta + \tfrac{1}{8}\int \eta\eta GG\eta\eta + \ldots] \exp \frac{ie^2}{2}\int J G J . \tag{5.11}$$

The integrands contain, apart from the Fermi sources, products of the Green's functions. The product in the third term, for example, may be written:

$$G(x_1, x_1') G(x_2, x_2') = G^0(x_1, x_1') G^0(x_2, x_2') \cdot \exp\left\{ \int (d\xi) [j(\xi; x_1, x_1') + j(\xi; x_2, x_2')] \frac{\delta}{\delta J(\xi)} \right\} \tag{5.12}$$

where $G^0(x_1, x_1')$ and $G^0(x_2, x_2')$ are the free field Green's functions. The dependence on the external potential is contained in the exponential, which is of the form

$$\left\{ \exp\left[\int (d\xi) \frac{\delta}{\delta J(\xi)} \right] \right\} f(J) = f(J + j) . \tag{5.13}$$

so that the exponentials act as simple displacement operators.

The first purely fermion Green's function is

$$G(x, x') = G^0(x, x') \left\{ \exp\left[\int j_\mu(\xi; x, x') \frac{\delta}{\delta J(\xi)} \right] \right\} \left\{ \exp\left[\frac{ie^2}{2} \int J G J \right] \right\} \Big|_{J=0} \tag{5.14}$$

Performing the variational differentiation, one obtains in the limit $J \to 0$:

$$G(x, x') = G^0(x, x') \exp\left[\frac{ie^2}{2} \int j^\mu(\xi; x, x') G_{\mu\nu}(\xi, \xi') j^\nu(\xi'; x, x') \right] . \tag{5.15}$$

This expression is still purely formal. The question of existence depends on the specific form of $G_{\mu\nu}$ which has a tensor structure and may thus be written:

$$G_{\mu\nu}(\xi,\xi') = g_{\mu\nu}G_1(\xi,\xi') + \partial_\mu \partial'_\nu G_2(\xi,\xi') \tag{5.16}$$

where $g_{\mu\nu}$ is the metric tensor and G_1 and G_2 are scalar functions. Substituting Eq. (5.16) in Eq. (5.15) one obtains

$$G(x,x') = G^0(x,x') \exp\left[\frac{ie^2}{2}\int j^\mu(\xi;x,x')\partial_\mu\partial'_\nu G_2(\xi,\xi')\, j^\nu(\xi';x,x')\right] \tag{5.16a}$$

where use has been made of

$$j^\mu j_\mu = \alpha^\mu \alpha_\mu \ldots = 0 \tag{5.17}$$

After performing partial integrations Eq. (5.16a) becomes

$$G(x,x') = G^0(x,x') \exp\left[\frac{ie^2}{2}\int \partial_\mu j^\mu(\xi;x,x')\, G_2(\xi,\xi')\, \partial_\nu j^\nu(\xi';x,x')\right] \tag{5.17a}$$

where the divergence of the current is given by Eq. (5.10). What is really involved in the calculation of the Green's function $G(x,x')$ is thus the structure of the scalar function $G_2(\xi,\xi')$.

In the electrodynamic case the function $G_{\mu\nu}$ in the Lorentz gauge is given by

$$G_{\mu\nu} = (1 + \partial D \partial)_{\mu\nu} \cdot G \tag{5.18}$$

where $(1+\partial D\partial)_{\mu\nu}$ is a projection operator and G is a scalar function corresponding to the mass $\mu = e/\sqrt{\pi}$. In the non-electrodynamic situation the function $G_{\mu\nu}$ is:

$$G_{\mu\nu} = (1 + \partial D \partial)_{\mu\nu} \cdot G - (1/\mu_0^2)\, \partial_\mu\partial_\nu D \tag{5.19}$$

where the scalar function G corresponds to the mass $\sqrt{\mu^2_0 + (e^2/\pi)}$ while the D function is associated with mass zero. In the electromagnetic case the exponential in Eq. (5.15) may be written:

$$\exp\left\{-ie^2\int \frac{dp}{(2\pi)^2}[1 - \exp ip(x-x')]\, \frac{1}{(p^2 - i\epsilon)}\, \frac{1}{(p^2+\mu^2 - i\epsilon)}\right\}. \tag{5.20}$$

The integral in the exponent is convergent, i.e. neither ultraviolet nor infrared divergences occur. The simplest Green's function has now been constructed. It is entirely finite and one would now ask for its physical interpretation.

The Green's function G^0 has a pole at mass zero. The exponential factor changes this pole into a singular branch point. This corresponds to the fact that we are dealing with particles of mass zero. When the source operates, it may produce one particle with mass zero, but in addition it may produce any number of pairs of particles, i.e. there is a continuous spectrum.

The physical particle with mass zero can be identified only to the extent that one can effectively isolate the initial point from the continuous spectrum. Thus, in ordinary quantum electrodynamics the electron can, strictly speaking, not be uniquely identified. If a charge is created, any number of photons of arbitrary small frequencies may also be created. The identification of the electron is actually the identification of a localized excitation carrying a unit charge and with a certain lattitude in the mass set by the experimental circumstances.

In the one-dimensional situation there is a zero-mass particle superimposed on a continuous background of pairs. This is the approximate physical transcription of the structure of the Green's function in which there appears, not a pole at zero mass, but a singular branch point. The physical interpretation is thus complicated by this quite irrelevant question, as far as the general picture is concerned, of the "infrared problem" which involves the identification of zero-mass particle states, despite the fact that mass zero is not separated by any finite gap from the other masses.

One can now go on to compute all the other Green's functions and to calculate how particles moving along on a line interact with each other and with vector particles of mass μ.

For the non-electromagnetic vector field, where a "bare" mass has been inserted, the exponential in Eq.(5.15) may be written:

$$\exp\left\{-ie^2\int\frac{dp}{(2\pi)^2}\left[1-\exp\left(ip(x-x')\right)\frac{1}{p^2-i\epsilon}\cdot\left(\frac{1}{p^2+\mu^2-i\epsilon}-\frac{1}{\mu_0^2}\right)\right]\right\}\cdot \qquad (5.21)$$

This integral is convergent for $-p^2\to 0$, but logarithmically divergent for $-p^2\to\infty$. From Eq.(5.15) it then follows that the Green's function $G(x,x')$ vanishes. This is also true for every Fermi Green's function, i.e. the system cannot be excited as far as Fermi responses are concerned. This contradicts the formal properties of the Green's functions as vacuum expectation values of field products, so that this theory must be rejected despite the fact that the theory would be considered renormalizable.

After this discussion of a simple model we shall turn to some general considerations of which the model can be taken as an example. The one-dimensional model is over-simplified in one essential respect, since it contains no critical dependence on the coupling constant. We have two different situations. One is electrodynamic, i.e. a vector field coupled by a gauge invariant mechanism to a charge. In this case there is a zero-mass particle. The other is a hypothetical vector field coupled to a nucleonic charge also by a gauge invariant mechanism. In this case there is no zero-mass particle. In other words, there must be a critical coupling strength such that below this the zero-mass particle remains and above this the zero-mass particle disappears.

Now I will give a general discussion of the simplest Green's function, which gives an account of the vector particle spectra. If one has a weak external current J^μ , the expectation value of A^μ (x) may be written as

$$<A^\mu(x)> = \int G^{\mu\nu}(x-x')\, J_\nu(x') + \ldots . \tag{5.22}$$

where non-linear terms have been omitted since the external current is assumed to be weak.

The Green's function $G^{\mu\nu}(x-x')$ describes free particles. The Fourier tranform of $G^{\mu\nu}$ may be written:

$$G_{\mu\nu}(p) = \Pi_{\mu\nu}(p)\, G(p) \tag{5.23}$$

where $\Pi_{\mu\nu}(p)$ is a projection operator, which is determined by the choice of gauge. The scalar function G (p) contains the specific propagation properties of the system. We are studying here the response of the system to excitation by an external current. The excitation will in general produce a spectrum of possible states. This spectrum will be represented by the spectral structure of the Green's function. The scalar function in Eq. (5.23) may be represented by

$$G(p) = \int \frac{B(m^2)\, dm^2}{p^2 + m^2 - i\epsilon} \tag{5.24}$$

where $B(m^2)\, dm^2$ is the probability that the excitation produces a transfer of energy and momentum which is characterized by the mass m. Since $B(m^2)$ is a probability density it must be non-negative

$$B(m^2) \geq 0 \, . \tag{5.25}$$

The probability density $B(m^2)$ is assumed to satisfy the sum rule

$$\int dm^2 B(m^2) = 1 \, . \tag{5.26}$$

This assumption may be justified in the following way.

In the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0 \tag{5.27}$$

and the propagation equation is

$$-\partial^2 A = J + j \tag{5.28}$$

where J is the external current and j the other physical currents. These currents would of course in turn be determined by suitable fields. I want to insist that the fundamental vector fields shall be observable for very short times (or very high frequencies). The time intervals must be so short, that the interaction effects do not have time to obscur the underlying field.

The response of the system to the current of the Green's function must then have the following asymptotic behaviour:

$$G \sim 1/p^2(1 + \ldots) \qquad \text{for } -p^2 \to \infty \tag{5.29}$$

where the omitted terms vanish for $-p^2 \to \infty$. The rate at which these terms vanish depends on the dynamics of the system and cannot be asserted in advance. It now follows from Eq. (5.24) that this asymptotic behaviour is only possible if Eq. (5.26) holds. No other sum rule can be stated in general, because that depends on an assumption of how the current behaves, i.e. of the dynamics.

It is of interest to find a representation of G (p) which incorporates the required asymptotic behaviour. Introducing the complex variable z, Eq. (5.24) may be written

$$G(z) = \int \frac{B(m^2)\, dm^2}{m^2 - z}\,. \tag{5.30}$$

This function is regular everywhere except on the positive real axis. The singularities correspond to the physical values of m. The boundary value of the function G (z) is G (p) for $z^2 \to -p^2 + i\epsilon$. If z tends to infinity, except along the real axis, one has:

$$G(z) \sim -1/z\,. \tag{5.31}$$

For the inverse function we have:

$$G^{-1}(z) \sim -z\,. \tag{5.32}$$

Since G (z) has no complex zeros, $G(z)^{-1}$ will have no complex poles or complex singularities. In addition

$$(1/z)(G^{-1} + z) \to 0\,. \tag{5.33}$$

for z tending to infinity.

The function $(1/z)(G^{-1} + z)$ has only singularities along the positive real axis, which includes a pole at z = 0. Hence

$$(1/z)(G^{-1} + z) = \lambda^2/z - \int dm^2 s(m)^2/(m^2 - z)^{-1} \tag{5.34}$$

from which we obtain the following representation, on placing $z = -p^2 + i\epsilon$:

$$G(p) = \left[p^2 - i\epsilon + \lambda^2 + (p^2 - i\epsilon)\int \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon}\right]^{-1}. \tag{5.35}$$

This representation of G (p) has the correct asymptotic behaviour. Comparing Eq. (5.24) and Eq. (5.35) one sees that $s(m^2)$ must be non-negative since $B(m^2)$ is non-negative:

$$s(m^2) \geqslant 0. \tag{5.36}$$

From Eq. (5.24) one obtains

$$G(0) = \int \frac{dm^2 B(m^2)}{m^2} > 0 \tag{5.37}$$

whereas from Eq. (5.35) it follows that

$$G(0) = 1/\lambda^2 \tag{5.38}$$

so that $\lambda^2 > 0$.

As far as the physical properties we have inserted are concerned, any non-negative λ^2 and any non-negative $s(m^2)$ for which the integral in Eq.(5.34) exists, will give a possible Green's function. If one requires zero mass to be part of the physical spectrum the parameters λ^2 and $s(m^2)$ can no longer be chosen arbitrarily. If zero mass is in the spectrum it follows from Eq. (5.37) that G(0) is infinite. Comparing with Eq. (5.38) one then obtains $\lambda = 0$ as a necessary condition. For $\lambda = 0$ Eq. (5.35) may be written:

$$G(p) = \left[1/(p^2 - i\epsilon)\right]\left[1/\left(1 + \int \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon}\right)\right] . \tag{5.39}$$

For $p^2 \sim 0$ this equation leads to:

$$G(p) = \left[1/(p^2 - i\epsilon)\right]\left[1/\left(1 + \int \frac{dm^2 s(m^2)}{m^2}\right)\right] . \tag{5.40}$$

For mass zero to be present in the physical spectrum as an isolated singularity, the residue of the pole in Eq. (40) must not vanish, i.e.

$$\int_{\to 0} \frac{dm^2 s(m^2)}{m^2} < \infty . \tag{5.41}$$

I now want to examine what dynamical changes are necessary in order to go from a situation where these conditions are satisfied to a situation where they cease to be valid. That would be the continuous change from electrodynamics where there is zero-mass particle to a theory where this particle ceases to exist.

We have found the form (see 5.35):

$$G(p) = 1/\left[p^2 - i\epsilon + \lambda^2 + (p^2 - i\epsilon)\int_0^\infty \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon}\right] \tag{5.42}$$

for the gauge independent part of the "photon Green's function", where λ^2 and $s(m^2)$ are non-negative quantities. The (-it) refers to the boundary condition of outgoing waves in time. The sum rule $\int_0^\infty dm^2\, B(m^2) = 1$ requires that $\int_0^\infty dm^2\, s(m^2) < \infty$. If we put $p^2 = 0$, we obtain $0 < 1/\lambda^2 = \int_0^\infty [B(m^2)/m^2]\, dm^2$, assuming the existence of $\int_0^\infty [s(m^2)/m^2]\, dm^2$. Now we want to find the necessary conditions for the existence of a physical particle with zero mass, so that we can imagine conditions under which such a particle would cease to exist. Then, we could suppose a continuous variation as we go from the electromagnetic field with its physical photon to the hypothetic vector field associated with nuclear charge which does not possess a zero-mass particle, and investigate how the photon ceases to exist.

If we have a photon, then it is necessary that $\lambda^2 = 0$. The resultant Green's function is then

$$G(p) = [1/(p^2 - i\epsilon)]\left[1/\left(1 + \int_0^\infty \frac{dm^2\, s(m^2)}{p^2 + m^2 - i\epsilon}\right)\right] \tag{5.43}$$

and, in the neighbourhood of $p^2 = 0$,

$$G(p) \simeq [1/(p^2 - i\epsilon)]\left[1/1 + \int_0^\infty \frac{dm^2\, s(m^2)}{m^2}\right]. \tag{5.44}$$

Then, the residue of the pole, B_0, is

$$B_0 = 1/\left[1 + \int_0^\infty \frac{dm^2\, s(m^2)}{m^2}\right] \tag{5.45}$$

which must be greater than zero for the existence of a pole, i.e.

$$\int_0^\infty dm^2 [s(m^2)/m^2] < \infty$$

is the second necessary condition for a physical particle of zero mass. The integral up to infinity certainly exists, since the representation itself was based on the existence of the integral $\int_0^\infty dm^2\, s(m^2)$. What can vary from one physical situation to another is the integration down to zero. Then we must have

$$s(m^2) \xrightarrow[m^2 \to 0]{} 0. \tag{5.46}$$

The structure of $B(m^2)$ when a photon exists is

$$B(m^2) = B_0\, \delta(m^2) + B_1(m^2) \tag{5.47}$$

where $B_1(m^2)$ is continuous and

$$1 = B_0 + \int_0^\infty dm^2\, B_1(m^2). \tag{5.48}$$

The photon exists only if $0 < B_0$. (In the usual language, $B_0 = z_3$). If we calculate the long-range Coulomb interaction, we find that the effective charge is given by

$$e^2 = B_0 e_0^2 \tag{5.49}$$

A zero renormalization constant is not to be interpreted as a mathematical problem but as a physical statement of the absence of a particle.

The function $B(m^2)$ can be interpreted as the probability that a source will produce excitations in the vacuum of mass m^2. Now, we want to find a physical interpretation for $s(m^2)$. $s(m^2)$ is the measure of excitations by an established field. Consider the vacuum state with an external current. Now, we ask for the probability that the vacuum is maintained, then the relevant quantities

$$\langle | \rangle^J \simeq \exp[(i/2)\int J^\mu G_{\mu\nu} J^\nu]$$

where we have taken J to be weak,ignoring more complicated processes. We use the exponential to take into account the possibility that many weak processes are occurring all over space. If we use a conserved current, $\partial_\mu J^\mu = 0$, then the $\pi_{\mu\nu}$ multiplying the scalar Green's function becomes effectively $g_{\mu\nu}$ and

$$\langle | \rangle^J \simeq \exp[(1/2)\, J^{*\mu}(p)\, G(p)\, J_\mu(p)\, dp], \tag{5.50}$$

writing the integrals in momentum space. Now, we introduce the vector potential

$$A^\mu(p) = G(p) J^\mu(p), \tag{5.51}$$

thus:

$$\langle | \rangle^J \simeq \exp[(1/2)\int A^{*\mu}(p)\, G^{-1*}(p)\, A_\mu(p)\, dp], \tag{5.52}$$

the probability of the vacuum's remaining unexcited is:

$$|\langle | \rangle^J|^2 \simeq \exp[-\int dp\, |A^\mu(p)|^2\, \mathrm{Im}\, G^{-1}(p)], \tag{5.53}$$

$$\mathrm{Im}\, G^{-1*}(p) = -\pi p^2 \int_0^\infty dm^2\; s(m^2)\, \delta(m^2 + p^2). \tag{5.54}$$

We have transferred our attention from the current, which may lie far outside the region of interest to the field which lies in the region. In doing so, we find that the inverse Green's function becomes the important quantity and $s(m^2)$ measures the excitations of the vacuum. The resultant expression is

$$|\langle|\rangle^J|^2 \simeq \exp[-\pi \int dp\, dm^2\, \delta(p^2+m^2)\, s(m^2)\,(-1/2)|F^{\mu\nu}(p)|^2]. \quad (5.55)$$

The expression in the exponential gives a measure of the probability of excitation of mass m by an external field $F^{\mu\nu}$.

The condition $s(0) = 0$ is characteristic of a normal threshold, i.e. (at the beginning of the excitation spectrum) there is a zero probability of exciting the vacuum by an external field. To put it another way, the condition for the existence of a photon is that nothing unusual happens at the zero-mass threshold. On the otherhand, if the photon is not to exist, then something must happen at zero mass.

If we have abnormal behaviour, we have two possibilities:

(1) $s(0)$ is finite or singular such that $g \int dm^2\, s(m^2)/(m^2-g) \underset{g\to 0}{\to} 0$

Then, we have no pole at $p^2 = 0$, but $m = 0$ is still in the spectrum, i.e. we have a branch point at $p^2 = 0$, no pole and $B(0)$ has a non-vanishing weight. Then, there is no recognizable particle of mass zero.

(2) The second possibility is that $s(m^2)$ possesses a delta function singularity at $m^2 = 0$:

$$s(m^2) = \lambda^2\, \delta(m^2) + s_1(m^2). \quad (5.56)$$

Then $m^2 = 0$ is not in the spectrum at all. The inverse Green's function is then

$$G^{-1}(p) = p^2 - i\epsilon + \lambda^2 + (p^2 - i\epsilon)\int_0^\infty \frac{dm^2\, s_1(m^2)}{p^2 - i\epsilon + m^2}, \quad (5.57)$$

our original form. Now, $B(0) = 0$.

To see this, we write:

$$B(m^2) = (1/\pi)\,\mathrm{Im}\,G(p) = (1/\pi)\,\mathrm{Im}\,G^{1*}(p)/|G(p)|^2$$

$$= m^2 s_1(m^2)/R(m^2) + [\pi m^2 s_1(m^2)]^2, \quad (5.58)$$

where

$$R(m^2) = m^2 - \lambda^2 + m^2\, P\int_2^\infty \frac{dm'^2\; s(m'^2)}{m'^2\; m^2},$$

then:

$$B(0) = \lim_{m^2\to 0} m^2 s_1(m^2)/\lambda^4 = 0.$$

Now, consider the case where $s_1(m^2)$ is zero for $m^2 < m_0^2$. In the real world, we might expect that this is never true, but it could be true as an approximation for strong interactions. If $R(m^2) = 0$, we will have a stable particle for $m < m_0$. At $-\infty$, $R(m^2) = -\infty$. Then, if $R(m_0^2) > 0$, we have a stable particle, since R must pass through zero. On the other hand, if $R(m_0^2) < 0$, there cannot be a stable particle, since R is a monotonic function for $m < m_0$. If there is no stable particle, there must be an unstable particle, since R must pass through zero in the continuum. We would only be able to recognize it as such if the width is sufficiently small. The width is given by

$$\gamma = \pi m^2 s_1 (m^2)/(dR/dm^2). \tag{5.60}$$

It is possible that R may cross through zero several times, giving more than one resonance. These would emerge from the same Green's function, reflecting the long dynamic chain from the complicated spectrum of the observed particle to the simpler underlying fields.

In the one-dimensional model, we had $s(m^2) = (e^2/\pi)\,\delta(m^2)$. This reflects the possibility of creating pairs of fermions travelling along a line. The function $s(m^2)$ is an example of the second case, and we find a single stable particle of mass e^2/π. This simplicity depends on two things: the geometry of one dimension and the fact that we only considered zero-mass fermions. The dynamics are not so simple, it is merely the elementary kinematics which allowed us to find solutions which fit the general dynamic framework. The delta function of $s(m^2)$ at $m^2 = 0$ is so because a fermion pair is still a particle of zero mass as opposed to the case in three dimensions, where the pair has a mass spectrum. In three dimensions, the probability of a photon going into three photons goes as something like the eighth power of the available energy assuring us that $s(0) = 0$ We would expect the photon to disappear as the s(0) becomes an abnormal threshold, i.e. some threshold moves down to zero mass as the strength of the interaction builds up.

A crude mathematical model might be given by a characteristic resonance function

$$s_0(m^2) = (\lambda^2/\pi^2)\, m\Gamma/[(m^2 - m_0^2 K)^2 + m^2\Gamma^2] \tag{5.61}$$

where

$$m_0 = 2m_e\,,\ K \simeq 1 - \alpha^2/2,\ \Gamma \simeq \alpha^5 . \tag{5.62}$$

In electrodynamics, s_0 certainly consists of such contributions. This could be the positronium contribution, which for $m^2 < m_0^2 K$, would be the three photon contribution (or virtual positronium). As the coupling increases, K must decrease since the binding energy of positronium increases. At some critical strength, K would become zero and p would also be zero since there is nothing into which the positronium can decay. By then the language is appropriate, since the multiple photon contribution would not be distinguishable from the "positronium", but we shall continue our terminology analytically.

Since the binding energies are so large, we would also expect to find other bound states corresponding to particles or resonances. Such particles as the spin-zero mesons would then appear as a result of the complete strongly interacting set of fields and there would be no need for a separate field.

We have not discussed the complete set of particles - there is more than one type of baryon and the list of conservation laws includes such quantities as isotopic spin. This is not an absolute conservation law, however, so we would not insist strongly on a dynamic explanation. But we can ask if a theory with a non-Abelian invariance group can be given a dynamic explanation in terms of what we might call a non-Abelian gauge field. In order to investigate such a theory, we must investigate the mathematical-physical problem of the formulation and quantization of such a theory.

In the case of electrodynamics the field is the dynamic means of manifesting an electrical charge. But the c.m. field itself does not carry a charge. On the other hand, the gravitational field interacts with all energy and momentum, including that which it carries itself. The non-Abelian gauge fields are intermediate in that they carry the quantity of which they are the dynamic manifestations, but this quantity is not a space-time property. Before we can ask physical questions about the theory, we must verify that it fits within the framework of possible quantum mechanical fields. In a theory in which the question of commutation relations is not faced, there is no difficulty in writing down a theory. Similarly, there is no difficulty in assuring appropriate three dimensional invariance properties. The difficulty arises in assuring the consistency of the commutation relations and the Lorentz invariance of the theory. There is a criterion which states in one line a sufficient and, for a certain class of theories, necessary condition for relativistic invariance.

The statement of relativistic invariance means that there exist operators, constructed from the fundamental variables of the theory, whose commutators obey the structure relations in the inhomogeneous Lorentz groups. The entire structure of the theory will then remain invariant under the unitary transformations generated by the operators. What is special about field theories is that these generators are constructed additively from contributions by small regions of space. That is:

$$P^{\mu} = \int d^3 x \, T^{0\mu} (x), \tag{5.63}$$

$$J^{\mu\nu} = \int d^3 x \, (x^{\mu} \, T^{0\nu} - x^{\nu} \, T^{0\mu}). \tag{5.64}$$

The requirement that P^{μ} and $J^{\mu\nu}$ obey the structure relations of the Lorentz-group imposes restrictions on the commutation relations of the densities. Since the three-dimensional case presents no problems, we assume that we know T^{0k} and that it gives $J^{k\ell}$ and P^k which generate the inhomogeneous rotation group in the correct way.

$$[P^k, P^{\ell}] = 0, \tag{5.65}$$

$$[P^k, J^{0\ell}] = i P^0 \delta^{k\ell} \tag{5.66}$$

are assured by the three-dimensional invariance. In order to assure such relations as

$$[P^0, J^{0k}] = i P^k, \tag{5.67}$$

$$[J^{0k}, J^{0\ell}] = iJ^{k\ell}, \tag{5.68}$$

the equal-time energy density commutator must have the following form for $x^0 = x'^0$:

$$(-i)[T^{00}(x), T^{00}(x')] = (T^{0k}(x) + T^{0k}(x')\,\partial_k \delta(x-x') + \psi(x, x') \tag{5.69}$$

where

$$\psi(x, x') = -\psi(x', x,)$$

$$\int d^3x\, \psi(x, x') = 0 = \int d^3x\, x^k\, \psi(x, x'). \tag{5.70}$$

There is a class of theories for which τ vanishes identically, which includes spin 1/2 and spin 1 fields. Within this class then, the relation

$$x^0 = x'^0 (-i)[T^{00}(x), T^{00}(x')] = -(T^{0h}(x'))\,\partial_k\, \delta(x-x')$$

is a necessary and sufficient condition for Lorentz invariance.

6. FUNDAMENTAL COMMUTATION RELATIONS

We are now going to give a general derivation of the fundamental commutation relation which relates the energy densities of a relativistic field theory at various points of space and the same time. We want to see in a general way that there exists a class of physical systems, for which a simple commutation relation relating the energy and momentum densities of a physical system is both necessary and sufficient for relativistic invariance. Much of what we shall be doing will be entirely by analogy to and in parallel with similar considerations referring to the electric current vector.

Let us start with some remarks on the analogy between the electric current vector on the one hand and the tensor of energy and momentum on the other, with regard to the question of equal-time commutation relations. Commutation relations are, of course, interpreted in quantum mechanics as statements of measurability. Measurability is fundamentally a dynamical process and therefore the underlying general dynamic properties that characterize these two sets of operators should be pointed out. We are not talking about any vector or any symmetrical tensor, but about these very special quantities with their dynamical significance. First of all, the vector j^μ and the symmetrical tensor $T^{\mu\nu}$ are locally conserved quantities:

$$\partial_\mu j^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0. \tag{6.1}$$

Secondly, these vectors are not just mathematically conserved quantities, but they are also of immediate physical significance, because we understand the electric current vector, for example, to have a dynamical meaning as the source of the electromagnetic field. The operators j^μ and $T^{\mu\nu}$ have thus in common the fact that they are the sources of important fields; j^μ is the source of the electromagnetic field and $T^{\mu\nu}$ the source of the gravitational field. That is their essential unique dynamical significance and it is upon these facts that we want to base the theory of their commutation relations. We shall understand what can be called the kinematics of special relativity - the equal-time commutation relations - in terms of the dynamics of somewhat more general systems.

Now, the fact that these physical operators (or sets of operators) are respectively sources of the two fields, electromagnetic and gravitational also gives us the general basis for understanding why they satisfy conservation laws. These are not arbitrary restrictions, they flow from the structure of the field equations, from the requirement of what we shall call generally "gauge" invariance, although this will of course take different forms. It is characteristic of both of these fields that they make use of more field components than are necessary to describe the physical information and there are corresponding freedoms of "gauge" transformations. This corresponds in the electromagnetic case to the usual gauge invariance, while for the gravitational case it is specifically the freedom of coordinate transformations. Under these general "gauge" transformations it follows that the operators which are the sources of the fields must obey certain identities; these are the laws of conservation of electrical charge and energy-momentum in ordinary flat space, respectively. We now want to exploit, not just the fact that these operators are the sources of the fields, but the reciprocal aspect, that these operators are also measures of the response of a given physical system to external fields.

Imagine a given physical system in an external electromagnetic field or an external gravitational field. How do these two basic properties enter? To answer this question, one may think of the action operator *

$$W = \int (dx)\, \mathcal{L}(x). \tag{6.2}$$

Let the external vector potential be A_μ and $G_{\mu\nu}$ be the external gravitational potential. Infinitesimal variations of these external potentials produce corresponding variations in the action operator

$$\delta_A W = \int (dx)\, \delta A_\mu j^\mu , \tag{6.3}$$

$$\delta_g W = \int (dx) \sqrt{-g}\, \tfrac{1}{2}\, \delta g_{\mu\nu} T^{\mu\nu} \tag{6.4}$$

where, as usual, $g = \det g_{\mu\nu}$. This is a way of defining the operators j^μ and $T^{\mu\nu}$. Here we are studying the responses of the system to external potentials, which must of course be such that they are consistent with the requirement of general gauge invariance. A gauge transformation is not a physical transformation; if the change of a vector potential is

* All these ideas, of course, are characteristic of the local theory of fields.

$$\delta A_\mu = \partial_\mu \delta \lambda$$

then the action integral will not change, with appropriate boundary conditions, which implies the conservation law

$$\partial_\mu j^\mu = 0. \tag{6.5}$$

Similarly, an infinitesimal co-ordinate transformation

$$\bar{x}^\mu = x^\mu - \delta\xi^\mu \tag{6.6}$$

induces an infinitesimal change in the symmetrical tensor $g_{\mu\nu}$:

$$\delta g_{\mu\nu} = \delta\xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \delta\xi^\lambda g_{\mu\nu} + \partial_\nu \delta\xi^\lambda g_{\mu\nu}, \tag{6.7}$$

and the action integral is invariant under this transformation. Then, upon inserting Eq. (6.7) into Eq. (6.4) and integrating by parts with appropriate boundary conditions, we get:

$$\partial_\mu (\sqrt{-g}\, g_{\lambda\nu} T^{\mu\nu}) = (1/2)\, \partial_\lambda g_{\mu\nu} T^{\mu\nu}. \tag{6.8}$$

If we now specialize to the ordinary space time, the left-hand side vanishes and we come back to the conservation law:

$$\partial_\mu T^{\mu\nu} = 0. \tag{6.9}$$

This expresses the fact that in an external electromagnetic field, charge conservation still has its usual form, whereas Eq. (6.9) takes a slightly different form given by Eq. (6.8) owing to the fact that the gravitational field itself carries energy and momentum. Here we see how the response of the system to an external field is the origin of these conservation laws.

Now we come back to the connection with commutation relations; we want to base the theory of commutation relations for equal time on these conservation laws (Eqs. (6.5) and (6.8)). Both of them are equations of motion of the form

$$\partial_0 A(x) = B(x) \tag{6.10}$$

which maintains its structure independently of the values of the external parameters (external potentials). The meaning of A and B will, of course, change.

I want to show now that when we have such a situation, it immediately implies an equal-time commutation relation. This is the connection between the dynamics implied in the conservation laws and the commutation relations. To do this we shall first of all use the action principle in the following way. Consider the expression

$$\partial_0 \langle t_1 | A(x) | t_2 \rangle = \langle t_1 | B(x) | t_2 \rangle \tag{6.11}$$

where $t_1 > t > t_2$ the matrix element of Eq. (6.10) between the states at times t_1 and t_2. Let us now perform an infinitesimal parameter variation. The matrix elements would change for two reasons. First, A(x) and B(x) may be explicit functions of the parameters; we shall denote by $\delta' A(x)$, say, the corresponding variation. Then, there would be a change associated with the change in dynamics of the system as a result of this parameter variation. The change in the transformation function will be given by

$$\delta\langle t_1 | t_2 \rangle = i \langle t_1 | \int (dx) \delta' \mathcal{L} | t_2 \rangle . \tag{6.12}$$

Therefore Eq. (6.11) would change into:

$$\partial_0 \langle t_1 | \delta' A(x) + i \int (dx') \, (A(x) \delta \mathcal{L}(x'))_+ \; | t_2 \rangle$$

$$= \langle t_1 | \delta' B(x) + i \int (dx') \, (B(x) \delta \mathcal{L}(x'))_+ \; | t_2 \rangle \tag{6.13}$$

where we have dropped the "dash" on δ because this is the only change in the Lagrangian we consider. Now, from Eq. (6.10) and the definition of time ordered products we have

$$\partial_0 (A(x) \delta \mathcal{L}(x'))_+ = (B(x) \delta \mathcal{L}(x'))_+$$

$$+ \delta(x^0 - x^{0'}) \, [A(x), \delta \mathcal{L}(x')]$$

and this gives us the equal-time commutation relation, written in operator form as

$$\frac{1}{i} \int (d^3 x') [A(x), \delta \mathcal{L}(x')]_{x^0 = x^{0'}} = \partial_0 \, \delta' A - \delta' B. \tag{6.14}$$

Here we have an instrument, whenever we have an equation of motion involving some parameters, to find a commutation relation at equal time between the object that obeys the equation of motion and the measure of the response of the system to the variation of parameters.

An alternative derivation (without using the action principle) can also be given. We have:

$$\partial_0 A = B$$

$$= (1/i)[A, P^0] + (\partial_0 A)_{\exp} \tag{6.15}$$

where the last term refers to any explicit time dependence.

A change in the parameters will induce the change:

$$\delta' B = (1/i)[\delta' A, P^0] + (1/i)[A, \delta' P^0] + \delta'(\partial_0 A)_{exp} \tag{6.16}$$

but

$$\delta' P^0 = -\int d^3x\, \delta\mathcal{L}(x) \tag{6.17}$$

and therefore:

$$\frac{1}{i}[A, \int (d^3x')\delta\mathcal{L}(x')]_{x^0 = x^{0\prime}} = \partial_0 \delta' A - \delta' B,$$

as established above. As an illustration, let us consider the electromagnetic field. We have

$$\partial_0 j^0 = -\partial_k j^k, \tag{6.18}$$

i.e. $A(x) = j^0(x)$ and $B(x) = -\partial_k j^k(x)$. The external parameters are the continuous set of values of the components of the external vector potential. Therefore, Eq. (6.14) gives the equal-time commutation relation

$$\frac{1}{i}\int (d\,x')\,[j^0(x), j^\mu(x')\delta A_\mu(x')]$$

$$= \partial_0\, \delta' j^0(x) + \partial_k\, \delta' j^k(x). \tag{6.19}$$

Before evaluating the right-hand side we use first Eqs. (6.12) and (6.3) to obtain

$$\delta_A \langle t_1 | t_2 \rangle = i \langle t_1 | \int (dx) j_\mu \delta A^\mu | t_2 \rangle. \tag{6.20}$$

A second variation gives:

$$\delta_A^2 \langle t_1 | t_2 \rangle = -\langle t_1 | \int\int (dx)(dx')\delta A^\mu(x)\delta A^\nu(x')$$

$$\cdot \left[(j_\mu(x) j_\nu(x'))_+ - i\,\frac{\delta' j_\mu(x)}{\delta A^\nu(x')} \right] | t_2 \rangle. \tag{6.21}$$

Now, the integral on the right-hand side is a quadratic form, symmetric in x, μ and x', ν. The first term can also be taken as such. We thus obtain the reciprocity relation:

$$[\delta' j_\mu(x)]/[\delta A^\nu(x')] = [\delta' j_\nu(x')]/[\delta A^\mu(x)]. \tag{6.22}$$

We shall consider a special class of physical systems in which $j_\mu(x)$ is local in time in its explicit dependence on the extreme potential, i.e. in which the current does not depend explicitly upon the time derivative of the external potential, $\partial_0 A^\nu(x)$. Under this assumption it follows from Eq. (6.19) that the charge density $j_0(x)$ does not depend explicitly on the potentials at all.

$$[\delta' j_0(x)]/[\delta A^\mu(x')] = 0$$

and therefore, the reciprocity relation gives:

$$[\delta' j_k(x')]/[\delta A^0(x)] = 0, \tag{6.23}$$

i.e. the spatial current density is not an explicit function of the scalar potential, A^0. In what follows we shall show that the spatial current must depend explicitly on the spatial part of the vector potential. Therefore $\delta' j^k(x)$ may be written as

$$\delta' j^k(x) = \int (d^3x')[\delta' j_k(x)]/[\delta_3 A_\ell(x')] \ \delta A_\ell(x') \tag{6.24}$$

where the sub-index 3 indicates a three-dimensional variational derivative. Inserting this into Eq. (6.19), we find the equal-time commutation relation

$$[j^0(x), j^0(x')] = 0 \tag{6.25}$$

and

$$\frac{1}{i}[j^0(x), j^\ell(x')] = \partial_k \left(\frac{\delta' j^\ell(x')}{\delta_3 A_k(x)} \right) \tag{6.26}$$

where the right-hand side has been rewritten using the reciprocity relation.

The commutator of Eq. (6.26) cannot vanish, because if it were zero, it would violate the physical requirement that there should be a vacuum state. In order to prove this, we take the three-dimensional divergence of Eq. (6.26) and use Eq. (6.18) to obtain:

$$[j^0(x), -i\partial_0 j^0(x')] = -\partial_k \partial'_l [\delta' j^1(x')]/[\delta_3 A_1(x)] . \tag{6.27}$$

Now, the commutator of an operator and its derivative is intrinsically positive, as can be seen in the following way. If $A(x^0)$ is a hermitian local operator, a spatial average of $j^0(x)$ over an arbitrary test function, then

$$[i\partial_0 A, A] = [[A, P^0], A] .$$

Taking the vacuum expectation value of this expression and using the property of the vacuum state of having zero energy, we get.

$$\langle [i\partial_0 A, A] \rangle = 2\langle AP^0 A\rangle$$

and,as the operation of A on the vacuum produces higher excited states, the right-hand side is intrinsically positive. Therefore, the commutator of Eq. (6.27) cannot vanish identically and this implies (Eg. (6.26)) that the vector current is an explicit function of the external potentials.

This result apparently contradicts what one knows about the Dirac field, where a vector current is given by

$$j_\mu \equiv (1/2)\,\psi\alpha_\mu q\psi = \bar{\psi}\,\gamma_\mu\psi$$

and does not depend explicitly on the external fields. What in fact happens is that this product is not really defined, and can be given a meaning only by separating the points spatially and defining a suitable limiting procedure which must maintain gauge invariance as in paragraph 1. In this limit the dependence on the external potentials will appear.

Let us now give a similar discussion for the case of an external gravitational field. The situation here is somewhat more complicated because the corresponding conservation law (eq. (6.8)) contains explicitly the external potential.

Eq. (6.8) can be rewritten as:

$$\partial_\mu (g_{\lambda\nu}T^{\mu\nu}) = (1/2)\,T^{\mu\nu}\,(\partial_\lambda g_{\mu\nu} - g_{\lambda\nu}g^{\alpha\beta}\,\partial_\mu\, g_{\alpha\beta})\,. \tag{6.28}$$

We now specialize to a particular gravitational field where

$$g_{k\ell} = \delta_{k\ell}\,,\; g_{ok} = 0$$

and g_{00} is an arbitrary function of x. Eq. (6.28) then reduces, for $\lambda = 0$, to

$$\partial((-g_{00})T^{00}) = -\partial_k((-g_{00})T^{ok}) + (1/2)\,T^{ok}\,\partial_k\, g_{00} \tag{6.29}$$

while, for $\lambda = k$, Eq. (6.8) gives

$$\partial_0\,(\sqrt{-g_{00}}\;T^{ok}) = -\partial_\ell(\sqrt{-g_{00}}\,T^{k\ell}) + (1/2)\,\sqrt{-g_{00}}\;T^{00}\;\partial_k\, g_{00}\,. \tag{6.30}$$

We shall use these relations to derive the commutation relation. As in the electromagnetic case we shall consider a special class: $T^{k\ell}$ may (in fact, it must) be an explicit function of g_{00} at the same time, but it does not depend on g_{00} at different times, i.e. it does not depend explicitly on the time derivatives (time locality). From this assumption it follows (Eqs. (6.29) and (6.30)) that the combinations $(g_{00})\,T^{00}$ and $\sqrt{-g_{00}}\;T^{ok}$ are not explicit functions of g_{00} at all. Performing the corresponding variation and using Eq. (6.14), we obtain, after setting $g_{00} = -1$:

$$(1/i)[T^{00}(x), T^{00}(x')] = -\partial_k\,\delta_3(x-x')\,(T^{ok}(x) + T^{ok}(x'))\,. \tag{6.31}$$

This is the fundamental commutation relation which the energy density must obey for the assumed class of physical systems . It is also a necessary and sufficient condition to guarantee the relativistic invariance for these systems. Upon integration, we obtain from it the commutation relation for the generators of the Lorentz group.

7. CONSTRUCTION OF A RELATIVISTICALLY INVARIANT, CONSISTENT THEORY OF NON-ABELIAN GAUGE FIELDS

We should mention here a few things about gauge invariance, because this will again be the motivating consideration in the construction of such a more general theory. For electromagnetic gauge invariance, we have

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x),$$
$$F_\mu(x) \to F_{\mu\nu}(x), \tag{7.1}$$
$$\chi(x) \to e^{iq\lambda(x)}\chi(x)$$

where $\chi(x)$ is the field carrying an electrical charge. This transformation forms an Abelian group.

Let us imagine a situation in which we have several charge-like properties, for instance the various components of isotopic spin, which are carried by some field but which are also carried by the gauge field itself. Let T_a, $a = 1, \ldots, n$, be the charge like matrices associated with the field χ and t the matrices associated with the gauge field $\Phi_{\mu a}$ and $G_{\mu\nu a}$. Consider the the class of infinitesimal gauge transformations

$$\chi \to [1 + i\sum_{a=1}^{n} T_a \delta\lambda_a(x)]\chi, \tag{7.2}$$

$$G_{\mu\nu} \to [1 + i\sum_{a=1}^{n} t_a \delta\lambda_a(x)]\, G_{\mu\nu}, \tag{7.3}$$

$$\Phi_\mu \to [1 + i\sum_{a=1} t_a \delta\lambda_a(x)]\Phi_\mu + \partial_\mu \delta\lambda(x). \tag{7.4}$$

Note that the field $G_{\mu\nu}$ transforms now according to Eq. (7.3), while in the electromagnetic case the corresponding field strength $F_{\mu\nu}$ remains unchanged because it does not carry an electrical charge. Also, the transformation of Φ_μ (Eq. (7.4)) expresses the fact that it carries a charge and is a gauge field. These transformations must form a group (which we assume to be compact). This requirement implies commutation relations for T_a and t_a:

$$[T_b, T_c] = \sum_{a} T_a t_{abc}, \tag{7.5}$$

and

$$[t_b, t_c] = \sum_{a} t_a t_{abc} \tag{7.6}$$

where the t_{abc} are the structure constants of the groups. Also, for the inhomogenous transformation (Eq. (7.4)) to belong to this group, we must have

$$(t_b)_{ac} \equiv t_{abc} . \tag{7.7}$$

In order to keep the fields $G_{\mu\nu}$ hermitian, the finite hermitian matrices t must be imaginary and therefore antisymmetrical. From this last property follows the antisymmetry of the structure constants in the indices a and c. Furthermore, the commutation relations imply their antisymmetry in the indices b and c. Therefore the structure constants are antisymmetrical in all indices. From here follows the important remark that for a group to be non-abelian ($t_{abc} \neq 0$), it must at least be a three-parameter group. In the three-dimensional case $t_{abc} = i\epsilon_{abc}$, ϵ_{abc} being the totally antisymmetric unit tensor, and the commutation relations become the familiar angular momentum commutation relations for isotopic spin.

The infinitesimal gauge transformations which characterize a non-abellian gauge field are:

$$\begin{aligned} \chi &\rightarrow (1 + iT\delta\lambda)\chi, \\ G_{\mu\nu} &\rightarrow (1 + it\delta\lambda)G_{\mu\nu}, \\ \Phi_\mu &\rightarrow (1 + it\delta\lambda)\Phi_\mu + \partial_\mu \delta\lambda . \end{aligned} \tag{7.8}$$

χ is a Fermi field. The T's are matrices, and in $T\delta\lambda$ we understand that there is summation over the n gauge functions:

$$T\delta\lambda = \sum_1^n T_\alpha \delta\lambda_\alpha \tag{7.9}$$

(sometimes, to avoid ambiguity, we shall use the bracket notation $T\delta\lambda$ = '$T\delta\lambda$'). In the electromagnetic case the field G is gauge invariant, but here it also undergoes gauge transformation with the characteristic n dimensional matrices t. Thus $G_{\mu\nu}$ is a vector with the n components $(G_{\mu\nu})_a$. The components of the matrices t are given by the set of structure constants t_{abc} that are characteristic of the group:

$$(t_b)_{ac} = t_{abc} . \tag{7.10}$$

The vector field Φ is on the one hand a gauge field - to this property corresponds the term $\partial_\mu \delta\lambda$ in the gauge transformation - and on the other hand carries the internal properties and so responds linearly to gauge transformations in the term $(1 + it\delta\lambda)\Phi_\mu$.

8. NOTATIONAL DEVELOPMENTS

Suppose that in an n dimensional space we have vectors A, C and matrices t_b and we form the scalar product $A\,t\,C$. This has components corresponding to the n matrices t_b and we may form its scalar product with a third vector B:

$$(A\,t\,C)B = \sum_{a,b,c} A_a t_{abc} C_c B_b. \tag{8.1}$$

t_{abc} is totally antisymmetric in the indices a, b, c so $\Sigma\, t_{abc} A_a B_b C_c$ is a totally antisymmetric function of the three vectors. The product $(A\,t\,C)\,B$ is unchanged by cyclic permutation of its factors:

$$(A\,t\,C)B = (B\,t\,A)C \text{ etc.}, \tag{8.2}$$

and is changed in sign by anticyclic permutations of its factors:

$$(A\,t\,C)B = -(B\,t\,C)A \text{ etc.}, \tag{8.3}$$

If we remove one of the vectors from these triple scalar products, say we remove A from $(A\,t\,C)B = -(A\,t\,B)C$, we get a vector equation which can be expressed in our previous notation as:

$$'t\,C'\,B = -'t\,B'C.$$

Now we return to the gauge transformation for Φ:

$$\Phi_\mu = \Phi_\mu + (\partial_\mu - i't\,\Phi_\mu')\delta\lambda. \tag{8.4}$$

We have used the last result to write $'t\delta\lambda'\Phi_\mu = -'t\Phi_\mu'\,\delta\lambda$. The gauge transformation therefore involves, not just a simple gradient, but a sort of extended gradient in which a term involving Φ_μ has to be added (cf. electromagnetism, whence one introduces the electromagnetic interactions by replacing ∂_μ by $\partial_\mu - eA_\mu$). We may call $\partial_\mu - 'i\,t\,\Phi_\mu'$ the "gauge covariant derivative".

Consider now some properties of the gauge covariant derivative. The gauge transformation can be introduced in the following algebraic way. Let us take $\partial - i't\,\Phi'$(suppressing all indices) and apply to it an orthogonal transformation in n dimensional space:

$$(1-i\,t\delta\lambda')(\partial - it\Phi)(1+it\delta\lambda) = \partial - it(\Phi - (\partial - i't\Phi')\delta\lambda). \tag{8.5}$$

We shall call this an orthogonal transformation because the matrices t are antisymmetric and imaginary. (The $\delta\lambda$ are a set of n arbitrary functions). In the derivation of this equation we have used the commutation relation

$$['t\,\Phi', 't\delta\lambda'] = -t(\Phi\; t\delta\lambda). \tag{8.6}$$

We see that the effect of the orthogonal transformation on $\partial - it\Phi$ is to maintain its structure but to replace Φ by the gauge-transformed operator corresponding to the gauge functions $-\delta\lambda$. The invariance of $\partial - it\,\Phi$ will thus be maintained under the orthogonal transformation provided that we simultaneously subject Φ to a gauge transformation corresponding to the gauge function $+\delta\lambda$.

Another important expression involving these gradients is:

$$[\partial_\mu - i't\Phi_\mu', \partial_\nu - i't\,\Phi_\nu'] = -i't\,f^2 G_{\mu\nu}' \quad \text{(definition of } f^2 G_{\mu\nu}\text{)}, \tag{8.7}$$

from which emerges

$$f^2 G_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + i(\Phi_\mu t \Phi_\nu). \qquad (8.8)$$

Note that this commutation relation refers only to the n-dimensional matrices. The commutation relations of the Φ's considered as operators will be treated later.

Let us consider the effect of an orthogonal transformation on $G_{\mu\nu}$. Multiplying $[\partial_\mu - i' t \Phi_\mu', \partial_\nu - i' t \Phi_\nu']$ from the left by $1-it\delta\lambda$ and from the right by $1+it\delta\lambda$ is equivalent to transforming the Φ's by the gauge transformation $-\delta\lambda$. What we find is

$$(1-i' t\delta\lambda') t G(1+i t' \delta\lambda') = t(1-i' t\delta\lambda')G, \qquad (8.9)$$

and the gauge transformation of G under the gauge transformation $-\delta\lambda$ is

$$G \rightarrow (1-i' t\delta\lambda')G. \qquad (8.10)$$

Replacing $-\delta\lambda$ by $\delta\lambda$, we see that the transformation law of the G introduced there is the same as that written down at the beginning of this part. Later we shall come to identify our present G with the previous one; but for the moment the result is just that when Φ undergoes an inhomogeneous gauge transformation, the structure G undergoes an homogeneous one.

Now we turn from the defining of objects with simple transformation properties and go to dynamics. The dynamics, of course, consist of our Fermi fields which carry a property we may as well call isotopic spin, interacting with the vector fields. First we consider the Fermi field by itself, treating Φ_μ effectively as an extended field. Of course, Φ_μ is not really an extended field, but we temporarily treat it as such. The Lagrange function is

$$\mathcal{L} = (i/2)\Psi\alpha^\mu(\partial_\mu - i' T\, \Phi_\mu')\Psi + (i/m)\Psi\beta\Psi \qquad (8.11)$$

which contains the gauge covariant derivative $\partial_\mu - i' T\, \Phi_\mu'$. This Lagrange function is invariant under the infinitesimal gauge transformation:

$$\begin{aligned} \Psi &\rightarrow (1+i' T\ \delta\lambda')\Psi, \\ \Psi_\mu &\rightarrow \Phi_\mu + (\partial_\mu - i' t \Phi_\mu')\delta\lambda. \end{aligned} \qquad (8.12)$$

Note the two kinds of matrices: T for the spinor field and t for the vector field. However, the charges induced involve the commutators of the matrices and the commutators of the T are given in terms of the t, so that the charge produced by the variation of Ψ can, and does, cancel that produced by the variation of Φ.

Let us consider the charge in the Lagrange function induced by an infinitesimal charge in the vector Φ. We can write it

$$\delta_\Phi \mathcal{L} = {}'\delta\, \Phi_\mu k^{\mu\prime} \qquad (8.13)$$

where each k^μ is a vector with n components k^μ_a and

$$k^\mu_a = (1/2)\Psi\alpha^\mu T_a \Psi. \tag{8.14}$$

The k^μ_a form a set of currents, since currents are always identified through the effect of a change of potential. It is a great advantage of our way of writing currents that we clearly separate the kinematic vector which is associated with flow from the object that flows. This is usually observed when talking only about an electrical charge because it can be diagonalized; but when there are n non-commuting objects they cannot all be diagonalized.

We next ask what restrictions are imposed on these currents k^μ by the requirement of gauge invariance. The action operator of the system is

$$W = \int (dx) \mathcal{L} \tag{8.15}$$

and the infinitesimal in the action operator associated with an infinitesimal charge $\delta \Phi_\mu$ in the external field is:

$$\delta W = \int (dx) k^\mu \delta \Phi_\mu. \tag{8.16}$$

If the variation $\delta \Phi_\mu$ is chosen to be that trivial charge which is associated with a gauge transformation

$$\delta \Phi_\mu \; (\partial_\mu - i't \Phi'_\mu)\delta\lambda \tag{8.17}$$

with appropriate boundary conditions at infinity, then the variation of the action must vanish locally and we find

$$(\partial_\mu - i't \Phi'_\mu)k^\mu = 0 \tag{8.18}$$

which is a kind of generalized conservation equation. Thus the current k^μ is, strictly speaking, not conserved: there is an analogy here with the stress tensor $T^{\mu\nu}$, which is not conserved in an external gravitational field because the gravitational field transports energy and momentum. So here the currents k^μ of the Fermi field are not conserved because, if you like, they transfer isotopic spin to the Bose field.

Our generalized conservation equation immediately implies commutation relations for the k^μ. We employ the same device as used in the previous sections to derive the commutation relations for the electrical charge density and for the energy density. We regard the Φ_μ as an external property which is entirely consistent for the derivation of the commutation relations for the Fermi fields alone. Proceeding as before, we write down the equation of motion for k^0:

$$\partial_0 k^0 = i't \Phi'_0 k^0 - (\partial_\ell - it \Phi_\ell)k^\ell. \tag{8.19}$$

Now we make use of two things: a parameter Φ_0 appears in this equation of motion, and the effect upon the equations of motion of a variation of Φ_0, which is coupled to k^0 in the Lagrange function, tells one the commutators at the same time between the object, k^0, which obeys the equation of motion, and the generator of those infinitesimal transformations. The commutation relations can be read off from

$$\sum_b \int (dx')(1/i)[k^0_a(x), k^0_b(x')]\delta \Phi_{0b}(x') = -i \sum_{b,c} t_{abc}\delta \Phi_{ab}(x)k^0_c(x)$$

which implies, since the $\delta\Phi$'s are arbitrary, that at equal times

$$[k^0_a(x), k^0_b(x')] = \delta(\underline{x}-\underline{x}')\, \Sigma\, t_{abc}\, k^0_c(x). \tag{8.20}$$

Thus the k^0 at different points commute and at the same point they obey something like the group commutation relations. If we define quantities K_a by integrating k^0_a over all space:

$$K_a = \int (d^3\, x) k^0_a(x) \tag{8.21}$$

then the K_a satisfy

$$[K_a, K_b] = \sum_c t_{abc} K_c = \sum_c K_c t_{cab} \tag{8.22}$$

which are just the group commutation relations. Or in other words, the K_a furnish a representation of the group. But it is important to recognize that the K_a are not constants of the motion $\dot{K}_a \neq 0$. This is because the k^0_a do not obey conservation equations or, in other words, K_a is only a part of the total isotopic spin (the vector field carries the rest).

9. DYNAMICS OF THE FULL SYSTEM

Now we turn to the dynamics of the full system. We use the notion of gauge invariance as a guide in writing down a tentative Lagrange function for the whole system. Then we attempt to find the commutation relations of the fundamental operators. Finally, we must ask whether our tentative Lagrange function is really completely satisfactory, in the sense that it produces a Lorentz invariant theory. We will find that the original Lagrange function was ambiguous within a certain class of Lagrange functions and a particular one must be selected if we are to meet the requirement of relativistic invariance. There is no guidance here to be gained from the correspondence principle: the ambiguous terms are of the order of Planck's constant squared and are simply not determined by any requirement other than that of relativistic invariance. We shall have to apply the test we developed in terms of the commutator of the energy density.

The tentative Lagrange function is constructed so as to give first order field equations. It must therefore contain first derivatives

$$\mathcal{L} = -(1/2)G^{\mu\nu}(\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + i(\Phi_\mu t\ \Phi_\nu))$$
$$+(f^2/4)G^{\mu\nu}G_{\mu\nu} + (i/2)\Psi\alpha^\mu(\partial_\mu - i'T\ \Phi'_\mu)\Psi + (i/2)\Psi\beta\Psi m_0. \tag{9.1}$$

f is a characteristic coupling constant (dimensionless in the three-dimensional case). The question of the order of multiplication of operators is of course basic, but we cannot yet usefully discuss it.

Let us now take the Lagrangian function and write down the equations of motion. If we vary $G_{\mu\nu}$ we obtain:

$$f^2 G_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + (\Phi_\mu i t \Phi_\nu). \qquad (9.2)$$

It must be said at this point that we are using the variational principle in a formal way. The purpose is to come back later to this point and criticize and rectify what we are doing. At the moment we are applying a classical action principle without imposing any particular order upon the products of operators.

If we vary Φ_μ we get:

$$(\partial_\nu - i' t \Phi_\nu') G^{\mu\nu} = k^\mu.$$

This completes the full set of the vector field equations. We also have the Dirac equation obtained by variations of Ψ:

$$(\alpha^\mu \partial_\mu - i' T \Phi_\mu' + \beta m)\Psi = 0,$$

completing the preliminary set of field equations.

We have always said that the structure of the Maxwell field equations must guarantee as an identity the conservation of charge. The same condition must be imposed here. The structure of the non-abelian vector gauge field must guarantee as an identity the extended conservation equations of the vector current. Observe that if we take the gauge covariant divergence of k^μ we will have:

$$\begin{aligned}(\partial_\mu - i' t \Phi_\mu') k^\mu &= (\partial_\mu - i' t \Phi_\mu')(\partial_\nu - i' t \Phi_\nu') G^{\mu\nu} \\ &= (1/2)[\partial_\mu - i' t \Phi_\mu'), (\partial_\nu - i' t \Phi_\nu')] G^{\mu\nu} \\ &= (1/2)(i' t G_{\mu\nu}') G^{\mu\nu} = (i/2) G^{\mu\nu} t G_{\mu\nu} = 0. \qquad (9.3)\end{aligned}$$

In the electromagnetic case the term $'t\Phi'$ is absent and the result is evident. Here it follows from the antisymmetry of $G^{\mu\nu}$ and the fact that t is totally antisymmetrical. The result obtained is so far formal, because it is necessary to take into account the possibility that the different components of Φ may not commute. In other words, the question of operator multiplication obscures the simplicity of the derivation and the simple result no longer obviously follows within the framework of operator equations, although it is true in the classical derivation. All this is preliminary to an actual derivation of the identification of the fundamental variables and their basic commutation relations.

We will now introduce source terms in the Lagrangian, to make use of a uniform technique and exploit the device we have been using so far, in which, from equations of motion in the presence of a suitably disturbed system, we infer commutation relations in such a way that we can identify the fundamental variables. We go back to the Lagrangian and introduce there the simple linear source terms

$$\frac{1}{2}\, G^{\mu\nu} M_{\mu\nu}(\Phi) + \Phi^{\mu}\, \mathcal{J}_{\mu}(\Phi)$$

where $M_{\mu\nu}$ is the external source for the field intensities $G_{\mu\nu}$ and $\mathcal{J}_\mu$ is the external current for the potential Φ^μ. However, the addition of these terms must not violate the general gauge invariance of the Lagrangian. This means that $M_{\mu\nu}$ and $\mathcal{J}_\mu$ must respond to the gauge transformations of the vector field and for that reason the sources are functions of Φ^μ. Note also that here the situation is more complicated than in the electrodynamic case because Φ undergoes an homogeneous as well as an inhomogeneous gauge transformation. But the relation between the sources and the vector field must be simple in order not to destroy the utility of this technique. We have to exhibit, for example, $M(\Phi)$ in such a way that it responds properly to a gauge transformation but also in such a form that, at least for particular calculations, the Φ dependence disappears. That means that in a particular gauge the sources are independent of Φ. In other words, we shall not insist upon full gauge invariance for M, but only explicit invariance in the neighbourhood of the specific chosen gauge. We also want the connection between the sources and the field quantities to be instantaneous, i.e. we must impose time locality. All the relations between sources and the vector field must then be local in time.

Let us again write the infinitesimal gauge transformation properties of Φ_μ:

$$\Phi_\mu \to \Phi_\mu + (\partial_\mu - i't\,\Phi'_\mu)\delta\lambda\,. \tag{9.4}$$

We see that the time component changes by the time derivative, so that if we want time locality, we must use only the space part and not the time component which carries the time derivative. The spatial part is:

$$\vec{\Phi} \to \vec{\Phi} + (\nabla - i't\,\Phi')\delta\lambda. \tag{9.5}$$

We are interested in exhibiting a function of the vector $\vec{\Phi}$ which finally will depend only on $\delta\lambda$. Isolating this dependence to counter the gauge transformation of Φ, the gauge variation of the vector $\vec{\Phi}$ contains the gradient of $\delta\lambda$, but we want to construct a scalar equation and, naturally, we take the divergence of $\vec{\Phi}$:

$$\nabla.\vec{\Phi} \to \vec{\nabla}.\vec{\Phi} + \nabla.(\nabla - i't\,\Phi')\delta. \tag{9.6}$$

Having once done that, the natural gauge about which we have to perform the infinitesimal variation of gauge appears to be the one in which $\nabla.\Phi = 0$, i.e. the radiation gauge. In an infinitesimal neighbourhood of this gauge we have:

$$\nabla.\Phi = \nabla.(\nabla - i't\,\Phi')\delta\lambda. \tag{9.7}$$

The characteristic Green's function for this differential equation satisfies:

$$-\nabla.(\nabla - i^{\prime}t\,\Phi^{\prime})\mathcal{D}(x, x^{\prime}) = \delta_3(x - x^{\prime}) \quad (9.8)$$

where $\mathcal{D}$ is real and symmetric

$$\mathcal{D}_{ab}(x, x^{\prime}) = \mathcal{D}_{ba}(x^{\prime}, x) \quad (9.10)$$

and replaces the Coulomb-gauge function for the electromagnetic case. The solution for $\delta\lambda$ is

$$\delta\lambda = -\mathcal{D}_{\Phi}\nabla.\,\Phi. \quad (9.11)$$

This solution gives the gauge variation $\delta\lambda$ for an infinitesimal neighbourhood of the radiation gauge $\nabla.\,\Phi = 0$.

We can now write down explicitly what $M_{\mu\nu}(\Phi)$ must be, not for any gauge but specifically for the consideration of infinitesimal variations about the radiation gauge. It must be such a function of Φ that it responds by the counter transformation

$$M_{\mu\nu}(\Phi) = (1 - it\,\mathcal{D}_{\Phi}\nabla.\,\Phi)M_{\mu\nu} \quad (9.12)$$

where the $M_{\mu\nu}$ are just simple numbers. This form for the source satisfies all our three requirements. First of all, these gauge variant sources are related to arbitrary numerical quantities only at the same time. Secondly, in the radiation gauge that dependence disappears and the sources are arbitrary numbers. Thirdly, for infinitesimal variations about the radiation gauge, they vary by the factor $(1+it\delta\lambda)$ which just compensates the gauge variation of $G^{\mu\nu}$. We have achieved gauge invariance in an effective computational form for infinitesimal variations about the radiation gauge. And when we actually work *in* the radiation gauge the Φ dependence disappears.

Let us now construct $\mathcal{J}_{\mu}(\Phi)$ by the same procedure:

$$\mathcal{J}_{\mu}(\Phi) = (1 - it\,\mathcal{D}_{\Phi}\nabla.\,\Phi)\mathcal{J}_{\mu}. \quad (9.13)$$

$\mathcal{J}_{\mu}$ is independent of Φ but, since there are two parts to the gauge variation of Φ, one is inhomogeneous and for the gauge invariance of this part it must be $\partial_{\mu}\,\mathcal{J}^{\mu} = 0$. Thus we have invariance under infinitesimal gauge transformation about the radiation gauge.

Returning now to the equations of motion and adding the extracontributions from the external sources, we can read all the equal time commutation relations by merely inspecting the structure of the field equations.

The new equations are, first for Φ:

$$\partial_{\mu}\Phi_{\nu} - \partial_{\nu}\Phi_{\mu} + i\Phi_{\mu}t\,\Phi_{\nu} = f^2G_{\mu\nu} + M_{\mu\nu}. \quad (9.14)$$

The $M_{\mu\nu}$ are here numbers independent of Φ because we are now working in the radiation gauge. Secondly, the equation for $G_{\mu\nu}$:

$$(\partial_{\nu} - it\,\Phi_{\nu})G^{\mu\nu} = k^{\mu} + \mathcal{J}^{\mu} + \nabla^{\mu}\mathcal{D}_{\Phi}[(1/2)G^{\lambda\nu}it\,M_{\lambda\nu} + \Phi^{\nu}it\,\mathcal{J}_{\nu}] \quad (9.15)$$

where ∇^μ is the four component vector of which the time component is zero. Let us now take the two terms involving $\mathcal{J}_\mu$:

$$J+\nabla\mathcal{D}\,(-i\,t\,\Phi)\mathcal{J} = [1+\nabla\mathcal{D}(\partial-i\,t\,\Phi)]\,\mathcal{J}. \tag{9.16}$$

the derivative ∂ has been added to obtain the "covariant gradient" $\partial - it\Phi$. Its contribution is zero because $\mathcal{J}$ is conserved. This last equation interests us because the operator that acts upon $\mathcal{J}$ is a projector operator that picks up exactly the right properties of the vector $\mathcal{J}^\mu$, in the following sense: we observe that the right side, as a current, should be conserved in the extended sens that, applying the gauge covariant divergence, one must obtain zero and this is in fact the case:

$$(\partial-i\,t\,\Phi)\cdot[1+\nabla\,\mathcal{D}(\partial-i\,t\,\Phi)] = (\partial-i\,t\,\Phi)-(\partial-i\,t\,\Phi) = 0, \tag{9.17}$$

because

$$(\partial-i\,t\,\Phi).\,\nabla = \nabla.(\nabla-i\,t\,\Phi)+i\,t\,\nabla.\,\Phi. \tag{9.18}$$

The first term on the right-hand side is the differential operator defining $\mathcal{D}$ and in the radiation gauge $\nabla.\,\Phi = 0$. This is the importance of the projection operator that guarantees charge conservation in the extended sense. The derivative ∂_μ acting upon $\mathcal{J}^\mu$ may be said to be optional, but if we use the projector in the form we wrote it, then for all variations of $\mathcal{J}^\mu$ the constraint equation $\partial_\mu\,\mathcal{J}^\mu = 0$ needs no longer be considered. Obviously, since the divergence of $\mathcal{J}^\mu$ is equal to zero, not all variations of $\mathcal{J}^\mu$ can be independent. In particular:

$$\partial_0\,\delta J^0 = -\nabla.\,\delta J \tag{9.19}$$

and the variation of the longitudinal part $\nabla.\,\mathcal{J}$ is completely determined by the constraint. But now, the structure:

$$[1+\nabla\mathcal{D}\,(\partial-i\,t\,\Phi)]\,\mathcal{J} \tag{9.20}$$

does not depend at all upon the longitudinal part of $\mathcal{J}$.

$$[1+\nabla\mathcal{D}(\partial-i\,t\,\Phi)].\,\nabla = \nabla-\nabla = 0 \tag{9.21}$$

(an integration by parts is involved in the proof of this equality). So it is not necessary to make use of the constraint equation and we can vary $\mathcal{J}^\mu$ freely.

We will now examine the field equations to see which of them are equations of motion and which of them are only equations of constraint. Let us first write down the field equations which are equations of motion, i.e. equations having time derivatives in it. They are:

$$\partial_0\,\Phi_k = (\partial_k - i\,t\,\Phi_k)\Phi_0 + f^2 G_{0k} + M_{0k}, \tag{9.22}$$

$$(\partial_0 - it\Phi_0)G^{k0} = -(\partial - it\Phi)G^{\ell k} + k^\ell + [1+\nabla\mathcal{D}(\partial - it\Phi)]\mathcal{J} + \partial^\ell \quad (1/2)G^{\lambda\nu}itM_{\lambda\nu}.$$

On the other hand, the constraint equations are:

$$f^2G_{k\ell} + M_{k\ell} = \partial_k\Phi_\ell - \partial_\ell\Phi_k + (\Phi_k it\Phi_\ell)$$
$$(\partial_k - it\Phi_k)G^{0k} = k^0 + \mathcal{J}^0. \tag{9.23}$$

They tell us of course that neither the divergence of the "electric field" G^{ok} nor the components of the "magnetic field" $G_p^{k\ell}$ can be treated as independent variables.

We will now look at the equations of motion and vary the parameters. We then automatically get a commutation relation with the operators associated with the parameters in the Lagrange function. And since we have equations of motion for the fields Φ_k and G_{0k}, we will get commutation relations between these operators and the operators that appear in the action integral.

It should be mentioned that the first equation of motion contains a hidden constraint, because in the radiation gauge $\nabla.\Phi = 0$ and so, taking the divergence of that equation, the time derivative disappears and we are left with

$$-\nabla.(\nabla - it\Phi)\Phi_0 = \partial^k(f^2G_{0k} + M_{0k}) \tag{9.24}$$

which eliminates Φ_0 as an independent variable. This is an indication that only the transverse part of Φ and the transverse part of G^{0k} can be considered to be the fundamental variables, and all this occurs exactly as in the electromagnetic case.

We will now look at the structure of the equations of motion and simply read off the commutation relations. Let us vary the chosen set of parameters starting with $\mathcal{J}_k$. Looking at the equation of motion for Φ_k we see that $\mathcal{J}_k$ does not appear nor is it even hidden in the dependent variable so that the coefficient of the variation of $\mathcal{J}_k$ is zero:

$$\delta(\partial_0\varphi_k) = (0)\delta J_k \tag{9.25}$$

and from this follows the equal-times commutation relation

$$[\Phi_k(x), \Phi_\ell(x')] = 0. \tag{9.26}$$

Next, looking at the equation of motion for G^{0k} and considering the effect of the variation of $\mathcal{J}_\ell$, which only appears explicitly and multiplied by a projection operator, we see that

$$\delta\,\partial_0\,G^{0k} = -\delta(\text{Projector})\mathcal{J} \tag{9.27}$$

from which we can read off the commutation relation

$$i[G^{0k}(x), \Phi_\ell(x)] = {}^k(1+\nabla\mathcal{D}(\partial - it\Phi'))_\ell(x, x'), \tag{9.28}$$

or, showing explicitly the different components of G^{0k} and Φ'_ℓ,

$$i[G_a^{ab}(x), \Phi_{\ell b}(x')] = \delta_\ell^k \delta_{ab} \delta(x-x') + \partial^k[\mathcal{D}_\Phi(x,x')(-\overleftarrow{\partial}'_\ell - i't\,\Phi'_\ell(x'))]_{ab}. \quad (9.30)$$

Last of all, we should find the commutation relations between the G^{0k} themselves. Once we know these, the other commutation relations can be computed, knowing the way in which the other fields depend upon the fundamental ones. The last commutation relations can be obtained considering the variations of M^{0k}. The first equation of motion does not give anything new, it merely repeats the commutation relation just found (showing, of course, that the procedure is consistent). The other equation of motion gives the information we require. It contains M^{0k} explicitly in the last term and also implicitly in the dependent variable Φ_0. Taking into account both dependences we arrive at:

$$i[G^{0k}(x), G^{0\ell}(x')] = \partial^k \mathcal{D}(x,x')it\,G^{0\ell}(x') + it\,G^{0k}(x)\mathcal{D}(x,x')\overleftarrow{\partial}^{\ell'}. \quad (9.31)$$

These commutation relations seem to be complicated but it must be realized that we have derived them for the full operator G^{0k} which consists of a dependent longitudinal part and the independent transverse part G^{0kT} which is the fundamental variable. It is possible to extract the commutation relation for G^{0kT} only. We can see that the right-hand side of the commutation relation does not contain any purely transverse part and therefore

$$[G^{0kT}(x), G^{0\ell}(x)^T] = 0 \quad (9.32)$$

which, together with

$$[\Phi_k(x), \Phi_\ell(x')] = 0 \quad (9.33)$$

and

$$i[G^{0kT}(x), \Phi_\ell(x')] = \delta_\ell^k[\delta(x-x')]^T, \quad (9.34)$$

form the canonical commutation relations between the fundamental field quantities. By comparison with the electromagnetic case, we see that it still contains the essential simplicity which consists of the fact that the fundamental variables are exactly the same transverse parts of the potential and the electric field. The commutation relations have the same appearance except of course for the fact that in the electromagnetic field we have only one such equation and here we have n X n such equations. In other words, the equations just found are really matrix equations in the "internal" vector space.