Weak matrix elements from lattice QCD

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I review the latest lattice determination of kaon weak matrix elements, I focus on $K \to \pi\pi$ decay amplitudes and neutral kaon mixing (within and beyond the standard model). These matrix elements are needed for the theoretical computation of the CP violation parameters ε and ε' and for a quantitative understanding of the $\Delta I = 1/2$ rule.

1 Introduction

Lattice QCD plays a central role in our understanding of particle physics, it allows for non-perturbative and model independent determinations of hadronic quantities (masses, matrix elements, form factor, etc.) in a regime where perturbation theory is non reliable. In this review, I focus on the kaon sector (for more general recent reviews on lattice flavour physics, see [1,2]). Nowadays, lattice simulations are performed at the physical value of the quark masses with three or four dynamical flavour in the sea. Recently it also became possible to compute hadronic decay amplitudes, the first applications being $K \to \pi\pi$, allowing for a theoretical determination of ε'/ε. This will be topic of the main topic of this paper (sections 2,3 and 4). In section 5, I present some new results on the $\Delta I = 1/2$ rule. Several collaborations are computing the $\Delta S = 2$ matrix elements which occur in neutral kaon mixing, within and beyond the Standard Model. Combined with the experimental value of ε, these quantities can be used to provide important constraints on New Physics (NP) models and to estimate the scale of NP. I will present the status in section 6 and 7.

2 $K \to \pi\pi$ decays and Lattice QCD

Various nice reviews are available on the subject, see for example [3]. I just recollect here some basic facts about $K \to \pi\pi$ phenomenology. Assuming isospin symmetry, the decays $K \to \pi\pi$ can be written in terms of the amplitudes

$$A\left[K \to (\pi\pi)_I\right] = A_I \mathrm{e}^{i\delta_I} \,, \tag{1}$$

where I denotes the isospin of the two-pion state, either 0 or 2, and δ_I is the corresponding strong phase. The parameters of indirect (resp. direct) CP violation, ε (resp. ε') are given by

$$\varepsilon = \frac{A\left[K_L \to (\pi\pi)_0\right]}{A\left[K_S \to (\pi\pi)_0\right]} \,, \tag{2}$$

$$\frac{\varepsilon'}{\varepsilon} = \frac{1}{\sqrt{2}}\left(\frac{A\left[K_L \to (\pi\pi)_2\right]}{A\left[K_L \to (\pi\pi)_0\right]} - \frac{A\left[K_S \to (\pi\pi)_2\right]}{A\left[K_S \to (\pi\pi)_0\right]}\right) \,. \tag{3}$$

The first measurement of ε is the well-known discovery of indirect CP violation due to Christenson, Cronin, Fitch and Turlay [4] in 1964, for which Cronin and Fitch were awarded a Nobel prize. It took tremendous efforts to measure direct CP violation, The final measurements of ε' are due to KTeV at Fermilab and NA48 at CERN [5,6], the averages read

$$\begin{aligned} |\varepsilon| &= 2.228(11) \times 10^{-3} \,, \qquad (4) \\ Re\left(\frac{\varepsilon'}{\varepsilon}\right) &= 16.6(2.3) \times 10^{-4} \,. \qquad (5) \end{aligned}$$

Theoretically, the standard framework to study $K \to \pi\pi$ decay is the $\Delta S = 1$ effective Hamiltonian obtained after integrating out the heavy degrees of freedom. In the three-flavour theory, it reads (see for example [7,8])

$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)] Q_i(\mu) \,, \qquad (6)$$

where G_F is the Fermi constant. The short-distance effects, which can be computed in perturbation theory are factorised into the so-called Wilson coefficients, y_i, z_i whose expression can be found in [7]. V_{ij} are CKM matrix elements, $\tau = V_{ts}^* V_{td} / V_{us}^* V_{ud}$ and μ is an energy scale which can be thought as a cut-off. Traditionally the four-quark operators Q_i are given by (see for example [7]):

Current-Current:

$$Q_{1,2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)(\bar{u}\gamma_\mu(1-\gamma_5)u)_{\text{unm,mix}} \,, \qquad (7)$$

QCD Penguins:

$$\begin{aligned} Q_{3,4} &= (\bar{s}\gamma_\mu(1-\gamma_5)d) \sum_{q=u,d,s} (\bar{q}\gamma_\mu(1-\gamma_5)q)_{\text{unm,mix}} \,, \qquad (8) \\ Q_{5,6} &= (\bar{s}\gamma_\mu(1-\gamma_5)d) \sum_{q=u,d,s} (\bar{q}\gamma_\mu(1+\gamma_5)q)_{\text{unm,mix}} \,, \qquad (9) \end{aligned}$$

EW Penguins:

$$\begin{aligned} Q_{7,8} &= (\bar{s}\gamma_\mu(1-\gamma_5)d) \sum_{q=u,d,s} e_q(\bar{q}\gamma_\mu(1+\gamma_5)q)_{\text{unm,mix}} \,, \qquad (10) \\ Q_{9,10} &= (\bar{s}\gamma_\mu(1-\gamma_5)d) \sum_{q=u,d,s} e_q(\bar{q}\gamma_\mu(1-\gamma_5)q)_{\text{unm,mix}} \,; \qquad (11) \end{aligned}$$

where the subscript $()_{\text{unm,mix}}$ refers to the colour structure (unmixed or mixed). The matrix elements of these four-quark operators capture the strong dynamics of the theory. We have neglected the operators which emerge from the electric and magnetic dipole part of the electromagnetic and QCD penguins. (See the talk by V.Lubicz at Lattice'14 and [9] for a recent lattice study by the ETM collaboration.) These 10 operators do not form a basis of the $\Delta S = 1$ four-quark operators in four dimensions, as they are not linearly independent.

Obtaining a reliable evaluation of the matrix elements $\langle \pi\pi | Q_i' | K \rangle$ is the most difficult part of the computation. Since one needs a non-perturbative framework, lattice QCD is a natural candidate. In the last thirty years, many attempts have been made to evaluate these matrix elements, using either effective theories or lattice simulations (or combinations of both), see for example [11,12,13,14,15,16,17] and reference therein.

From the lattice point of view, the first difficulty is to simulate the kinematic situation, in particular the final state made of two hadrons with non-vanishing momenta. This problem was formalised in 1990 by Maiani and Testa who showed that the physical amplitudes could not be extracted from "standard" euclidean lattice simulations [18]. An alternative based on χPT was proposed in [19]: the matrix elements of interests can be obtained from those of $K \to \pi$ and $K \to$ vacuum, which are numerically much simpler. This indirect approach was first used for while, see for example [20,21,10]. However the conclusion of the extensive quenched studies [21,10] is rather negative: extracting the matrix elements with a fully controlled error turned out to be very hard. One problem comes from the fact that $SU(3)$ χPT converges poorly at the kaon scale (see also [22]). What is now known as the Maiani-Testa no-go theorem was circumvented in a very elegant way by Lellouch and Lüscher in [23]. The crucial point is that in finite volume the spectrum is discrete, and the size of the box can be fine-tuned such that the pions will take the desired momentum.

3 The $\Delta I = 3/2$ channel

We first consider the amplitude of $K \to (\pi\pi)_{I=2}$ decays, there are several simplifications in this channel, most notably:

1. there is no disconnected diagram (in which no quark line connects the initial kaon and the final two-pion states; these diagrams are numerically hard to compute), and

2. only three operators contribute.

The first realistic computation (with dynamical quarks, physical kinematics and nearly-physical pion mass) was performed by the RBC-UKQCD collaborations [24,25] with Domain-Wall fermions, a discretisation of the QCD Lagrangian which preserves chiral-flavour symmetry almost exactly.

Although the method used in [24,25] is based on the Lellouch-Lüscher approach, an important ingredient is the Wigner-Eckart theorem, which tells us that the matrix elements of interest are related to those of the unphysical process $K^+ \to \pi^+\pi^+$ (in the isospin limit). Using a peculiar choice of boundary conditions, these matrix elements (with physical momenta) can be extracted using standard lattice methods. The first simulation was done at a single value of the lattice spacing ($a^{-1} \sim 1.375$ GeV, ie $a \sim 0.1435$ fm) on the so-called IDSDR lattice (ID) [26] with a pion mass of 140 MeV. (Strictly speaking this "physical pion" is partially quenched, the unitary pion mass was somewhat heavier: 170 MeV). In this work, the matrix elements are renormalised non-perturbatively with the Rome-Southampton method [27], at low energy and run to 3 GeV using the (universal) continuum scale-evolution matrix to 3 GeV [28].

More recently, the RBC-UKQCD collaborations have reported on $2+1$ lattice QCD simulations with physical pion masses [26], which have been possible thanks to a new formulation of the Domain-Wall disctretisation [29]. These lattices have been used to improve on the determination of A_2: the main source of error was the discretisation effects, the new computation [30] involves two lattice spacings of $a \sim 0.011$ and $a \sim 0.084$ fm, reducing the systematic error by roughly a factor 2 for the real part and a factor 1.5 for the imaginary part. Thanks to these new lattice determinations, the current errors on the theoretical determination of A_2 are of the order of 10%. The results are shown in figure 1.

4 Including the $\Delta I = 1/2$ channel

A complete determination of $A\left[K \to \pi\pi\right]_{I=0}$ has been a long-standing challenge for the lattice community. A first "pilot" computation with dynamical fermions was reported by RBC-UKQCD in 2011 [31]. This computation was unphysical in the sense that the amplitudes were computed at threshold and the quark masses were heavier than the physical ones, however all the required diagrams were determined (including the disconnected ones) showing the numerical feasibility of the approach. The main remaining difficulty was to implement the physical kinematics, ie the ability to extract the matrix element of interests, with the pion states having the right momenta. The Wigner-Eckart/boundary condition trick used in the $\Delta = 3/2$ channel does not work for the full computation, as it violates isospin [32]. Instead, the RBC-UKQCD collaboration have generated new ensembles with $G-$ parity boundary conditions [33,34], as reported by Christopher Kelly in a plenary session of Lattice 2015 [35], see also the plenary review given by Andreas Jüttner at the same conference [36]. From a more technical point of view, this computation requires the

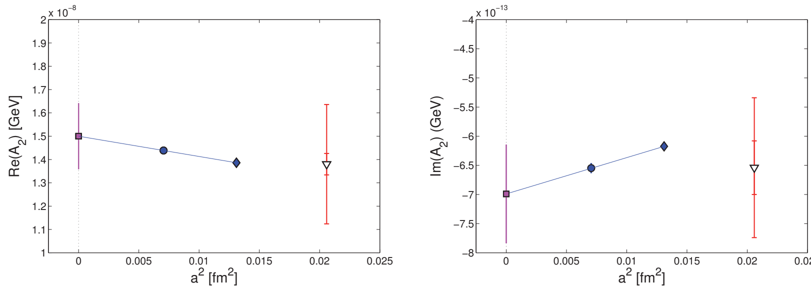

Figure 1 – Real and Imaginary part of $A_2 = A\left[K \to (\pi\pi)\right]_{I=2}$. The triangle represents the 2012 computation on the IDSDR and the blue points the 2014 determinations on the new ensembles (statistical error only), from which a continuum limit is extracted and shown in magenta (statistical and systematic errors combined). For the IDSDR points, we show both the statistical and the systematic error, largely dominated by the discretisation artefacts.

evaluation of all-to-all propagators and noise reduction techniques. The results read

$$Re(A_0) = 4.66(1.00)(1.21) \times 10^{-7}\text{GeV} \tag{12}$$
$$Im(A_0) = -1.90(1.23)(1.04) \times 10^{-11}\text{GeV} \tag{13}$$

and the corresponding theoretical value for ε'/ε

$$Re(\varepsilon'/\varepsilon) = 1.38(5.15)(4.43) \times 10^{-4} \,, \tag{14}$$

which is an approximate agreement($\sim 2.1\sigma$) with the experimental value $16.6(2.3)\times10^{-4}$. Rather than concluding that a significant deviation of the Standard Model prediction has been found, we note that the error is much larger than the experimental one. From a phenomenological point of view, at this level of precision, these results do not invalidate the Standard Model, neither do they rule out the need for new-physics in $K \to \pi\pi$ decays. The important point is that for the first time ε'/ε has been computed with a full error budget, all the different contributions of the seven linearly independent operators are computed with controlled errors and a precision which can be systematically improved. Now that the technology has been developed, reaching a precision of, say, 10% should be possible in the close future. In addition to reducing the statistical error, the simulation can be done on finer lattices and extrapolated to the continuum limit. Another systematic error is due to the truncation of the perturbation series (needed to compute the Wilson coefficients). The renormalisation was performed at a scale of $\mu \sim 1.5$ GeV in order to keep the discretisation effects under control. Clearly this can be improved by running non-perturbatively to a higher scale, as done for the $\Delta I = 3/2$ channel. Reducing the theoretical error on the matrix elements of O_i (and therefore on ε'/ε) will provide a crucial test of the Standard Model, indeed we might actually see signs of new physics. It is also worth noting that another computation (done at threshold) has been done with Wilson fermions [37].

5 The $\Delta I = 1/2$ rule

The "$\Delta I = 1/2$ rule" refers to the fact that the $I = 0$ channel is favoured over the $I = 2$ channel by the factor $1/\omega$ defined by

$$\omega = \frac{A\left[K_S \to (\pi\pi)_2\right]}{A\left[K_S \to (\pi\pi)_0\right]} \,. \tag{15}$$

Experimentally this number is around $\omega \sim 1/22$ whereas one would naively expect 1/2 [38,39]. The question whether or not the remaining factor of ~ 10 can be explained entirely by some

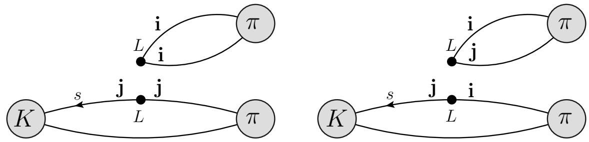

Figure 2 – The dominating contribution to the real part of the amplitude A_2 of $K \to \pi\pi$ is proportional to sum of the two contractions ① and ②. The two contractions differ by their colour structure, as indicated by the colour indices i and j. The label L stands for the left-handed structure $\gamma_\mu(1-\gamma_5)$.

surprisingly large QCD effects has been a very-long standing puzzle. It also shows the need for a better understanding of the non-perturbative regime. Several attempts to study the $\Delta I = 1/2$ rule on the lattice have been made. For example, an ongoing project based on the role of the charm quark has been developed in [40,41,42,43], see also [44].

In 2013, the RBC-UKQCD collaborations reported on a study of the origin of this enhancement [45]. The amplitude A_2 was computed with physical kinematics whereas A_0 was computed at threshold. The real part of A_2 is largely dominated by a single four-quark operator (the contributions of the electroweak penguins are negligible with respect to the tree-level diagram). This operator has a $(V-A)\times(V-A)$ Dirac structure and transforms as a $(27,1)$. Two contractions ① and ② contribute, they differ by their colour structure, as shown in figure 2. The conventions are such that $\omega = Re(A_2)/Re(A_0) \sim (①+②)/(2①-②)$, where ① and ② are the contractions shown in figure 2. The naive expectation is that $② \sim \frac{1}{3}①$. However the observation made in [45] is that $② \sim -0.7①$. Therefore, there is an important cancellation in the numerator of ω which is completely unexpected from the naive factorisation framework. Similarly the main contribution to $Re(A_0)$ is proportional to $2① - ②$. Hence, the aforementioned relative sign between ① and ② also contributes to enhancement in the denominator of ω (compared to the naive expectation).

The two recent lattice computations, the threshold one [37] and the one with physical kinematics [46] also observe this sign difference, which seems to be at the origin of the $\Delta I = 1/2$ rule. In table 1, we collect the lattice results obtained by RBC-UKQCD. The first two values of ReA_0 were obtained with unphysical kinematics and unphysical values of the quark masses. For the last lattice value, both ReA_2 and ReA_0 were obtained at physical values of the quark masses and with physical kinematics. We combine the continuum value of ReA_2 with the new value of ReA_0 (obtained at a single value of the lattice spacing), we find

$$\omega^{-1} = \frac{ReA_0}{ReA_2} = \frac{1.66(0.96)(0.27)\times 10^{-7}}{0.150(4)(14)\times 10^{-7}} \sim 31.1(6.5)\,. \qquad (16)$$

Clearly we observe and importance enhancement, which seems to be very sensitive to the quark mass. However, in order to confirm that the $\Delta I = 1/2$ effect is a pure non-perturbative QCD effect, a little bit of patience is required as the precision on A_0 has to be improved. The theoretical error affecting the amplitudes is expected to decrease by a factor of two in the next couple of years, it is very likely that we will then have the answer to this question. Note that this sign also discussed in [3,47], see also [16,7,8].

6 Neutral kaon mixing and indirect CP violation in the Standard Model

In the Standard Model picture, neutral kaon mixing is dominated by W-exchange box diagrams. By performing an operator product expansion, one can factorise the long-distance effects into

Table 1: $\Delta I = 1/2$ Rule: comparison of the lattice results with the experimental value. See text for details.

	$\omega^{-1} = ReA_0/ReA_2$	m_K (MeV)	m_π (MeV)	Kinematics
	9.1 (2.1)	878	422	Threshold
	12.0 (1.7)	662	329	Threshold
	31.1(6.5)	491(2)	143(2)	Physical
Experimental	22.5	494-493	135-140	

the matrix element of a four quark operator $O_1^{\Delta S=2}$:

$$\langle \bar{K}^0 | O_1^{\Delta S=2} | K^0 \rangle = \langle \bar{K}^0 | (\bar{s}_i \gamma_\mu (1-\gamma_5) d_i) \, (\bar{s}_j \gamma_\mu (1-\gamma_5) d_j) \, | K^0 \rangle \, . \tag{17}$$

Clearly, because of the W-exchange, the Dirac structure is (Vector-Axial) $\times$ (Vector-Axial). Only one four-quark operator contributes: the operator given in eq. 17 is invariant under Fierz re-arrangement and in a (continuum) massless renormalisation scheme, it does not mix with other four-quark operators, nor with lower dimensional operators. It is also the case on the lattice if chiral symmetry is preserved. Once the considered matrix element has been computed non-perturbatively using lattice techniques, its result is combined with the value of the Wilson coefficient $C(\mu)$ of continuum perturbation theory and experimental observables, such as the mass difference $\Delta_{M_K} = m_{K_L} - m_{K_S}$ and ε_K to obtain important constraints on the CKM matrix elements. Schematically, one obtains

$$\varepsilon_K = C(\mu) \times \langle \bar{K}^0 | O_1^{\Delta S=2} | K^0 \rangle (\mu) \times \mathcal{F}(V_{ij}^{CKM}, m_K, f_K, \Delta M_K, \ldots) \, , \tag{18}$$

where $\mathcal{F}$ is a known function of the CKM factors and of well-measured quantities.

A convenient parametrisation of this operator is the well-known bag parameter B_K,

$$B_K(\mu) \equiv \frac{\langle \bar{K}^0 | O_1^{\Delta S=2} | K^0 \rangle (\mu)}{\frac{8}{3} f_K^2 m_K^2} . \tag{19}$$

where $f_{K^-} = 156.1$ MeV and μ is a renormalisation scale, usually 2 or 3 GeV.

B_K is a standard lattice quantity, nowadays it is computed with an accuracy of a few percents [48,49,50]. The FLAG 2013 average for $N_f = 2+1$ is

$$\hat{B}_K \;=\; 0.7661(99) \, , \qquad N_f = 2+1 \, , \tag{20}$$

it is largely dominated by the BMWc result $\hat{B}_K = 0.773(8)_{stat}(3)_{syst}(8)_{PT}$. The other references are [51,52,53,54,49,55,56,57,58].

Let us consider the dominant sources of error: FLAG 2013 explains that the total error of 1.3% can be roughly seen as the combination of 0.4% statistical and 1.2% systematic, mainly due to perturbation theory: $(33)_{stat} + (93)_{syst}$. Although B_K is extracted and (in most cases) renormalised non-perturbatively on the lattice, perturbation theory is used to convert the result to the renormalisation-group-invariant (RGI) quantity $\hat{B}_K$, or alternatively to $\overline{\mathrm{MS}}$. Naturally, different collaborations estimate the perturbative error in different ways, and this estimation is of course affected by some subjective judgement. Indeed this error changes by a factor two or three depending on the estimation. In its 2013 review, it seems that FLAG chose an uncertainty very close to the one quoted by BMW (1%), whereas RBC-UKQCD quoted an error of $\sim 2\%$, based on a multiple-scheme evaluation. Actually by changing the intermediate schemes, RBC-UKQCD find that the results change by 8% if the matching is done at $\mu = 3$GeV and by 12% if $\mu = 2$GeV. The current situation is illustrated in table 2, where we show the the most recent determinations of B_K. The importance of this perturbative error can be made clear by looking at, for example, the result obtained by RBC-UKQCD [26]

$$\hat{B}_K \;=\; 0.7499(24)_{stat}(150)_{PT} \, , \tag{21}$$

$$B_K^{(\not{q},\not{q})}(3\mathrm{GeV}) \;=\; 0.5341(18)_{stat} \, , \tag{22}$$

Table 2: Collection of recent results for $B_K^{\overline{\rm MS}}(3{\rm GeV})$.

Collaboration	N_f	Discretisation	Result
RBC-UKQCD [26]	$2+1$	Domain-Wall	$0.5293(17)_{stat}(106)_{PT}$
SWME [61]	$2+1$	Staggered	$0.518(3)_{stat}(26)_{syst}$
ETM [62]	$2+1+1$	Twisted Mass	$0.506(17)_{stat+syst}(3)_{PT}$

where the first error is statistical (however it is much larger than the other errors on B_K^{bare}) and the second error is the systematic error on the renormalisation, largely dominated by the perturbative matching. This contrasts with $B_K^{(\not q,\not q)}$ which is fully non perturbative, renormalised in the SMOM $(\not q,\not q)$-scheme at the scale $\mu = 3$ GeV. Without this perturbative error, the error would be of around 0.3%. In the future, the lattice community will probably have to find an agreement on how to estimate this uncertainty as it is the dominant one. Obviously, one could reduce this perturbative error, by computing explicitly the next order in perturbation theory. The matching coefficient is currently known at next-to-leading order. Going further requires to determine the matching coefficient at the two-loop level (three-loop anomalous dimension). Alternatively, on could perform the matching at a higher scale; this could be achieved by computing the running non-perturbatively, for example using the Schrödinger functional [59] or a (S)MOM-scheme [60].

7 Neutral kaon mixing Beyond the Standard Model

Beyond the Standard Model, we have to include new Dirac-colour structures, as for example both left-handed and right-handed currents can contribute to K^0-$\bar{K}^0$ mixing (and therefore to ε_K). Hence, in addition to O_1 introduced in eq.17, one introduces new $\Delta S = 2$ four-quark operators. A possibility (the so-called SUSY-basis) is [63,64 a]

$$
\begin{aligned}
O_2^{\Delta S=2} &= (\bar{s}_i(1-\gamma_5)d_i)\,(\bar{s}_j(1-\gamma_5)d_j), &(23)\\
O_3^{\Delta S=2} &= (\bar{s}_i(1-\gamma_5)d_j)\,(\bar{s}_j(1-\gamma_5)d_i), &(24)\\
O_4^{\Delta S=2} &= (\bar{s}_i(1-\gamma_5)d_i)\,(\bar{s}_j(1+\gamma_5)d_j), &(25)\\
O_5^{\Delta S=2} &= (\bar{s}_i(1-\gamma_5)d_j)\,(\bar{s}_j(1+\gamma_5)d_i). &(26)
\end{aligned}
$$

These four-quark operators appear in the generic effective $\Delta S = 2$ Hamiltonian

$$
H^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu)\, O_i^{\Delta S=2}(\mu) + \sum_{i=1}^{3} \tilde{C}_i(\mu)\ , \qquad (27)
$$

where the Wilson coefficient $C_i(\mu), \tilde{C}_i(\mu)$ depend on the details of the new-physics model under consideration but the matrix elements $\langle \bar{K}^0|O_i^{\Delta S=2}|K^0\rangle$ are model independent. The operators $\tilde{O}_{1,2,3}$ are obtained from $O_{1,2,3}$ by replacing $(1-\gamma_5)$ by $(1+\gamma_5)$. In QCD with parity conserved, these operators are redundant and therefore discarded in the following.

A priori, one would expect that the relevant matrix elements $\langle \bar{K}^0|O_1^{\Delta S=2}|K^0\rangle$ can be obtained with an accuracy comparable to the one of the Standard Model one. However only few studies of the full set of BSM operators have been published and the history is quite interesting. First of all, in the quenched approximation, the results from [66] obtained with Ginsparg-Wilson fermions (which exhibit an exact chiral-flavour symmetry) and non-perturbative renormalisation were very different from the previous study, done with tree-level $O(a)$-improved Wilson fermions [67]. The difference was attributed to the renormalisation.

The first computation performed with dynamical fermions was reported by RBC-UKQCD [68] in 2012 and was done with $N_f = 2+1$ Domain-Wall fermions. It was followed shortly by a

[a] An alternative basis is given in [65] .

Table 3: Comparison of the BSM bag parameters B_i at 3 GeV in the SUSY basis in $\overline{\mathrm{MS}}$

	ETM 15	RBC − UKQCD 12	SWME 15	RBC − UKQCD 15(prelim.)	
interm. scheme	$RI-MOM$	$RI-MOM$	$1-loop$	$RI-SMOM$	$RI-MOM$
B_2	0.46(3)	0.43(5)	0.525(1)(23)	0.488(7)(17)(2)	0.417(6)(2)(2)
B_3	0.79(5)	0.75(9)	0.772(5)(35)	0.743(14)(64)(3)	0.655(12)(44)(2)
B_4	0.78(5)	0.69(7)	0.981(3)(61)	0.920(12)(12)(4)	0.745(9)(28)(3)
B_5	0.49(4)	0.47(6)	0.751(8)(68)	0.707(8)(34)(3)	0.555(6)(53)(2)

$N_f = 2$ twisted-mass computation of the ETM collaboration, done with several lattice spacings [69] and these two first computations are in reasonable agreement (slightly more than2% in the worse case) In 2013, the SWME collaboration reported their results, obtained with $N_f = 2+1$ flavours of improved staggered fermions [70]. A noticeable disagreement with the previous studies was found for two of the matrix elements (O_4 and O_5 of the SUSY basis). Very recently, the ETM collaboration have repeated their computation with $N_f = 2+1+1$ flavours (using again twisted mass QCD), and essentially confirmed their $N_f = 2$ results [62] (although for B_5 the agreement is only within $\sim 3\sigma$). Even more recently, SWME have extended their study by adding more ensembles, improving the extrapolation to the physical point, and they confirmed the disagreement with the other collaboration [61]. Since the results have been extrapolated to the continuum limit, one does not expect the discretisation used (Domain-Wall, Twisted-Mass, or Staggered) to be responsible of the discrepancy.

The matrix elements of these four-quark operators are usually given in terms of the so-called Bag-parameters $B_i(\mu) = \langle \bar{K}^0|O_i(\mu)|K^0\rangle / \langle \bar{K}^0|O_i(\mu)|K^0\rangle_{VS}$ where VS is the vacuum-saturation approximation. In the case of the Standard Model operator, the denominator is known in terms of physical quantities f_K and m_K, as shown in eq. (19). This normalisation is convenient because the bag parameters are dimensionless, the numerator and the denominator are very similar, therefore systematic errors are likely to cancel out in the ratio, and because the denominator is known in terms of physical quantities. However for the BSM operators, the corresponding vacuum saturation approximations involve matrix elements of the pseudo-scalar density. In particular, the renormalisation of the matrix elements in a RI-MOM scheme [27] requires a pole subtraction which could potentially be problematic.

A preliminary study of RBC-UKQCD indicates that the source of the disagreement comes from the renormalisation [73,74]. A comparison of the results for the bag parameters can be found in Table 3. Although our error budget is not complete yet, we find that if we use the standard RI-MOM scheme proposed in [27] and match to the $\overline{\mathrm{MS}}$ scheme defined in [65], our results are in a decent agreement with ETMc. Surprisingly enough, if we use a SMOM scheme (in the spirit of the schemes introduced in [75]), our results are much closer to the results quoted by SWME, for which the renormalisation is performed perturbatively. The SMOM schemes are known to be superior to standard RI-MOM schemes: they behave better non-perturbatively in the infrared (the pion pole contamination is suppressed because of the absence of exceptional channel) and perturbatively [75,48,76,77]. We suspect that the procedure employed to remove the pion pole contamination (needed in the RI-MOM case but absent for the SMOM schemes) could also affect the ultraviolet behaviour, see for example [77]. The systematic errors associated with this procedure are very hard to estimate and could have been underestimated.

Several alternatives normalisation were proposed in the literature, see for example [67]. In [66], some ratios R_i have been introduced, they are designed to be equal to the ratio of a BSM contribution to the SM one at the physical point $R_i(\mu) = \langle \bar{K}^0|O_i(\mu)|K^0\rangle / \langle \bar{K}^0|O_1(\mu)|K^0\rangle$. Another possibility, is to define products and ratios of bag parameters (called G) such that the leading chiral logarithms cancel out [71,72,70]. The problem of the normalisation ambiguity is absent for the ratios, but still there for the products. However the advantage is that the chiral extrapolations are hugely simplified.

8 Conclusions and outlook

This is an exciting time for Kaon physics, to a great extent this is due to the impressive progress achieved recently by the lattice community. The computation of $K \to (\pi\pi)_{I=2}$ is reaching a mature stage and a first computation $K \to (\pi\pi)_{I=0}$ with physical kinematic and complete error budget has recently been reported by the RBC-UKQCD collaboration. The results of these computations have a important role to play in particle physics phenomenology . The $\Delta I = 1/2$ puzzle seems to be explained by the non-perturbative effects [45]. Regarding indirect CP violation, B_K is now known with an impressive precision. The various investigations of the $\Delta S = 2$ BSM operators are converging, the discrepancies observed by several collaborations are likely to be due to systematic errors affecting the non-perturbative renormalisation procedure in RI-MOM schemes. Although a careful study is required, the solution could be provided by the SMOM schemes, which have a much better behaviour. Future improvements will also require to match the lattice computation to phenomenology at a much higher scale in order to decrease the error due to perturbation theory. There are other new interesting developments that I have not mentioned here, such as rare kaon decays and the $K_L - K_s$ mass difference (see [79] and [80]).

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