

Matter coupled to quantum gravity in group field theory

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Abstract. We present an account of a new model incorporating 3d Riemannian quantum gravity and matter at the group field theory level. We outline how the Feynman diagram amplitudes of this model are spin foam amplitudes for gravity coupled to matter fields and discuss some features of the model. To conclude, we describe some related future work.

1. Introduction

I will be dealing with the coupling of matter to 3d Riemannian quantum gravity via group field theory (gft). Generically, a gft is a field theory over a group manifold which has as its Feynman amplitudes the amplitudes of a given spin foam model. For an introduction to the gft approach to quantum gravity, please refer to [1]. In this paper, however, we shall present an account of a gft model that has as its Feynman amplitudes the spin foam amplitudes for matter field theory coupled to quantum gravity. The results outlined here were obtained in collaboration with Daniele Oriti and are presented in detail in [2]. They provide a generalisation of a gft model proposed in collaboration with Laurent Freidel and Daniele Oriti [3].

Group field theory may be thought of initially as a generalisation of the matrix models of 2d quantum gravity. But it is much more than that in that it can be understood as a universal structure underlying the spin foam approach [4]. On top of this, it provides a third quantisation point of view of gravity [1], allowing us to sum over quantum gravity amplitudes associated with different topologies. In this picture, spin foams, and thus spacetime itself, appear as (higher-dimensional analogues of) Feynman diagrams of a field theory defined on a group manifold and spin foam amplitudes are simply the Feynman amplitudes weighting the different graphs in the perturbative expansion of the quantum field theory.

The spin foam structure for pure 3d Riemannian quantum gravity without cosmological constant is better known as the Ponzano-Regge (PR) model [5]. This is the spin foam quantisation of first order gravity which in 3d has a BF action and is thus a topological field theory.

The coupling of matter fields to quantum gravity in the spin foam framework is of paramount importance for various reasons, apart from the obvious one that for a consistent theory of quantum gravity to be correct, it must be able to fully incorporate the interactions of matter fields and gravity. First of all, the definition of quantum observables is notoriously difficult in pure gravity and matter insertions may well provide the best way to define observables with a

clear physical picture. Quantum gravity phenomenology also has a lot to gain from the inclusion of matter, as it is expected that quantum gravity effects will alter the usual dynamics of matter fields, for example, dispersion relations and scattering amplitudes. Furthermore, it is hoped that quantum gravity will solve problems in quantum field theory, including that of ultraviolet divergences, where it might provide an automatic covariant cut-off at the Planck scale.

In 3d, there is a duality between matter and geometry. Here the insertion of matter may be thought of as the insertion of a topological defect labelled by the Poincaré group [6, 7]. Recently, the spin foam quantisation of 3d gravity in the first order form coupled to point particles was accomplished [8]. I will refer to the resulting amplitudes as Coupled Ponzano-Regge (CPR) amplitudes. They provide the physical dynamics of states of matter and geometry combined. Furthermore, the effective theory for the particle sector, once the gravity variables are summed over, is a non-commutative field theory [9]. This encodes the complete quantum gravity effects on the particle sector and reduces to ordinary field theory in the limit where gravity becomes negligible. An interesting point is that the momenta of the field are group valued. This suggests that we should try to perform the 3rd quantisation of gravity and the 2nd quantisation of matter in one stroke, and this is the aim here.

In section 2, we will state and describe some of the properties of the CPR spin foam amplitudes occurring in [8]. While in section 3, we shall describe a group field theory that has the desired properties in that it generates a sum over CPR amplitudes [2]. Finally in section 4, we will discuss some perspectives on future work relating to this area.

2. Coupled Ponzano-Regge spin foam model

In this section, we shall describe some generic features of the spin foam model proposed in [8]. But first, let us briefly touch upon some of the characteristics of spin foam quantisation in the 3d context. Pure first order gravity in 3d is a topological theory. Spin foam quantisation is the precise implementation of a path integral sum over 3-geometries interpolating between 2d hypersurfaces. The geometric information is encoded in a combinatorial and algebraic fashion on the 2-skeleton Δ^* , dual (in the topological sense) to a triangulation Δ of a 3-manifold. The boundary information is contained in labelled spin networks. The theory is topological and so the amplitude is independent of the discretisation structure used for the 3-manifold, that is, it is background independent. In the coupled scenario, the boundary spin networks describe states of geometry and matter and the transition amplitudes depend only on the topological properties of the particle graph.

2.1. Kinematics

For pure 3d Riemannian quantum gravity, in other words the Ponzano-Regge spin foam model, the boundary states are given by closed spin networks. In other words, the geometric information is encoded on closed labelled trivalent graphs, the edges are labelled by representations of $SU(2)$ and the vertices are labelled by $SU(2)$ invariant intertwiners. $SU(2)$ is the gauge group of the 3-manifold.

In the coupled theory, the boundary states not only encode the geometric information but also the particle information. This is carried out by open spin networks. These are a generalisation of the closed case in that some of the spin network vertices have an edge which does not connect to another trivalent vertex. Instead such edges have an endpoint; they are open. This edge and endpoint encapsulate the kinematical information of the particle. The representation labelling the edge is the total angular momentum of the particle, and the endpoint is labelled by a spin s vector space.

2.2. Dynamics

The sum over intermediate configurations is given by the interpolating spin foam. This is, as we said, a labelled 2-complex. For pure gravity, it provides, for example, the transition amplitude between two 2-geometries. In the case where matter is present, it must also propagate the particle degrees of freedom. This is done by both a particle graph in the triangulation Δ , and another dual particle graph in the dual 2-skeleton Δ^* . The amplitude is given explicitly as follows:

$$Z = \int \prod_{e^*} d\alpha_{e^*} \underbrace{\prod_{e \notin \Gamma} \delta(G_e)}_A \underbrace{\prod_{e \in \Gamma} \int du_e \delta(G_e u_e^{-1} h_e u_e)}_B \underbrace{\sum_{I, I'} D^I(x) \overbrace{D_{s_e}^I(u_e^{-1}) D_{s_e}^{I'}(u_e)}^{P_{s_e}^{II'}(u_e)} D^{I'}(x')}_C, \quad (1)$$

where e^* labels dual edges and α_{e^*} is the holonomy along such an edge. Γ specifies the edges of the particle graph, $h_e \in \text{U}(1)$ encodes the mass of the particle and $u_e^{-1} h_e u_e$ denotes its momentum. G_e is the holonomy around the boundary of the face dual to the edge e and x, x' are holonomies along segments of a particle graph lying in the dual. All of G_e , x and x' are products of the α_{e^*} variables. We have not specified explicitly the boundary states.

The amplitude at first sight looks an eyeful but it can be broken down into more amenable portions. Section A is the amplitude of those edges of Δ not in the particle graph. Such an edge has the usual flatness condition for the curvature localised on it, that is, the holonomy encircling that edge is constrained to be the identity.

Section B is the amplitude for the edges with a propagating particle. The amplitude reads that for such an edge, there is a defect in the curvature, that is, the holonomy is no longer constrained to be the identity, but to be in the conjugacy class of some $\text{U}(1)$ element h_e representing the mass of the particle.

Section C is the amplitude for the dual particle graph. Once again, the I, I' and s_e variables refer to the total and spin angular momenta. The dual particle graph specifies the propagation of the particles' angular momenta. It may be described as being a representation of the holonomy along segments of a path in the dual between the projectors $P_{s_e}^{II'}(u_e)$. These projectors reside in the dual Δ^* , one for each edge of the particle graph in Δ . They account for the spin s_e of the particle on that edge of the particle graph in Δ . A more precise description of these concepts is given in [8].

There are certain consistency conditions that must hold between the particle graphs in Δ and Δ^* . For a given graph in the triangulation, there are a set of faces in the dual which completely enclose the graph, that is, form a tube around it. The dual graph must reside only on dual edges that are part of this tube. The second condition is that the two particle graphs are topologically equivalent.

Momentum conservation is automatic in this spin foam formulation. This is realised in the following sense: one could consider an amplitude of the form (1), but with no knowledge of whether the edges form a contiguous particle graph. If momentum is not conserved (at the vertices Δ) by this selection of edges, then the amplitude is automatically zero.

In the effective limit where the gravity variables are integrated out, these spin foam amplitudes reduce to the Feynman amplitudes of a non-commutative field theory. This arises because we have coupled entire Feynman histories for interacting relativistic particles to gravity, and the natural description of these particle histories is in terms of quantum field theory.

3. Group field theory for 3d quantum gravity coupled to matter fields

To recap our main aim, we wish the Feynman amplitudes generated in the perturbative expansion of our path integral to be spin foam amplitudes of the CPR model. We proposed a model in

[3] that generated the amplitudes for coupling particles with zero total angular momentum. We have since generalised this to the spinning case [2] and it is that model which we present here. The generalisation is not a direct one, however, and the earlier model has novel and interesting structure not present in the current model which we believe deserves further investigation. In conventional field theory we have a field, action and symmetries. The kinematics are taken care of by the field and its symmetries. The dynamics are codified by the action and its symmetries. The gft formalism does not deviate from this realisation. We shall see this in what follows.

3.1. Kinematics

This group field theory has two fields, one to describe the pure gravity sector and another to describe the coupled sector of the theory. The pure gravity field is that of the Boulatov model [10]. The Boulatov model is a gft that generates a sum over Feynman amplitudes that are exactly the amplitudes for the Ponzano-Regge model for pure 3d Riemannian quantum gravity. We state it as follows: The field ϕ is a map from the Cartesian product of three copies of $SU(2)$ to the real numbers,

$$\phi : SU(2) \times SU(2) \times SU(2) \rightarrow \mathbb{R} ; \quad \phi(g_1, g_2, g_3). \quad (2)$$

It has two symmetries. Lorentz symmetry which is ensured by a projection onto the $SU(2)$ invariant part of the field

$$P_\alpha(g_1, g_2, g_3) \equiv \int d\alpha \phi(g_1\alpha, g_2\alpha, g_3\alpha), \quad \text{where } \alpha \in SU(2). \quad (3)$$

Also there is a symmetry under even permutations of three arguments. We state this symmetry for completeness and we shall not delve into it more here.

Since this defines the pure gravity field, we should be able to uncover the kinematics of the pure gravity theory. Graphically, the field ϕ may be thought of as a triangle. The g variables are then the holonomies of an $su(2)$ connection from the ‘centre’ of the triangle to each of the three edges of the triangle. Preemptively, these holonomies trace out spin network edges leading from a trivalent spin network vertex. Furthermore, for a field with group-valued arguments the Fourier transform maps into representation space, one representation for each argument. These representations, in turn, label the three edges of the spin network emanating from the vertex at the ‘centre’, and the Lorentz invariance ensures that there is an invariant intertwiner labelling the vertex. Thus the field is a spin network vertex and we may generate a generic closed spin network state by multiplying fields in the appropriate manner [4].

The coupled sector of the theory is allocated a new field. It is a map on four copies of $SU(2)$

$$\begin{aligned} \psi_s : SU(2) \times SU(2) \times SU(2) \times SU(2) &\rightarrow \mathbb{R} \\ \psi_s(g_1, g_2, g_3; u) &= \sum_{I,n} \Psi_{sn}^I(g_1, g_2, g_3) D_{sn}^I(u), \end{aligned} \quad (4)$$

where D is a representation function of $SU(2)$. It has Lorentz symmetry imposed by a similar projection except this time it acts on all four arguments of the new field.

The field does not have permutation symmetry as this would ruin momentum conservation in the amplitudes. No more will be said on that here, but for more information please refer to [2].

The most noticeable property of the field is that it has four arguments. The first three refer to gravity and the final one to the momentum of the particle. The representation I of the u variable refers to the particle’s total angular momentum and the s index to its spin. Once again we have a clear relation between the field and the kinematical regime. Here the field may be

viewed diagrammatically as a 4-valent vertex. One incident edge has at its other endpoint a spin s vector space. In the kinematical arena described in section 2.1, we had only trivalent vertices, but any 4-valent $SU(2)$ invariant intertwiner can be split into two trivalent intertwiners. Thus we can produce a generic open spin network by multiplying ϕ and ψ fields. From the point of view of the triangulation, this field may be seen as a triangle with an extra degree of freedom (the would-be momentum of a particle) which in the action will be realised as a particle once this extra degree of freedom is coupled to the gravity portion of the field.

With the kinematics fully specified we shall now move on to the dynamics, that is, the action.

3.2. Dynamics

In detailing the action below we should not only recover the dynamics of the gravitational field but also those of those of the matter fields. In short, we should obtain the CPR amplitudes. The action may be stated as follows:

$$\begin{aligned}
S[\phi, \psi] = & \frac{1}{2} \int dg_i \phi(g_1, g_2, g_3) \phi(g_1, g_2, g_3) \\
& + \frac{\lambda}{4!} \int dg_i \phi(g_1, g_2, g_3) \phi(g_3, g_5, g_4) \phi(g_4, g_2, g_6) \phi(g_6, g_5, g_1) \\
& + \frac{1}{2} \int dg_i du \psi_s(g_1, g_2, g_3; u) \psi_s(g_1, g_2, g_3; u) \\
& + \mu_2 \int dg_i du_j \psi_s(g_1, g_2, g_3 u_a^{-1} h u_a; u_a) \psi_s(g_4 u_b^{-1} h^{-1} u_b, g_3, g_5; u_b) \\
& \quad \times \phi(g_4, g_2, g_6) \phi(g_6, g_5, g_1) D_{s n_a}^{I_a}(u_a) D_{s n_b}^{I_b}(u_b) \delta^{I_a I_b} \delta_{n_a n_b} \delta(u_a^{-1} h u_a u_b^{-1} h^{-1} u_b) \\
& + \mu_3 \int dg_i du_j \psi_s(g_1, g_2, g_3 u_a^{-1} h u_a; u_a) \psi_s(g_4 u_b^{-1} h^{-1} u_b, g_3, g_5; u_b) \psi_s(g_6, g_4, g_2 u_c^{-1} h u_c; u_c) \\
& \quad \times \phi(g_6, g_5, g_1) D_{s n_a}^{I_a}(u_a) D_{s n_b}^{I_b}(u_b) D_{s n_c}^{I_c}(u_c) C_{n_a n_b n_c}^{I_a I_b I_c} \delta(u_a^{-1} h u_a u_b^{-1} h^{-1} u_b u_c^{-1} h u_c), \quad (5)
\end{aligned}$$

where $C_{n_a n_b n_c}^{I_a I_b I_c}$ is an $SU(2)$ $3j$ -symbol intertwining the total angular momenta of the particles, which we sum over, and we have left out all explicit projections for simplicity. Furthermore, we have fixed the mass and spin of the particles. This action has a number of noteworthy properties which we shall discuss below.

The first two terms are those of the Boulatov model. Thus the partition function contains as a subsum of diagrams those of the pure gravity theory. The third term is the kinetic term of the coupled sector.

The fourth term, containing two ψ fields depicts a tetrahedron with particles inserted on two edges and a bivalent intertwiner for both the momentum and angular momentum degrees of freedom. To be even shorter, it outlines the propagation of a particle along (and through in the dual) a tetrahedron. This corresponds to the left picture in Figure 1. To be more precise, the insertion of the momentum $u^{-1} h u$ into a gravity argument of the field means the insertion of a defect into the holonomy around a wedge. As the wedges are dual to the edges of the tetrahedron, we have a particle associated to that edge. Furthermore, the δ -function over the momenta ensures explicit momentum conservation at the vertex shared by the two edges. In addition, the representation functions of the u variables describe the propagation of angular momentum along a dual graph, delineated by a dashed line in Figure 1. (The black dots point out dual vertices). This coincides with the functional form of the projectors in (1). The vertex amplitude resulting from the fourth term does not at first sight look likely to give us a CPR amplitude, as the momentum conservation is specified explicitly and we know that we get conservation for free in the CPR model. But we specify the vertex amplitude in this fashion for a number of reasons.

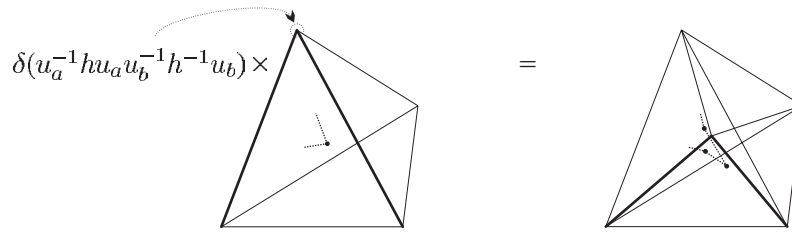


Figure 1.

First of all, we have shown that the vertex amplitude for the fourth term is exactly equal to the one where the tetrahedron is replaced by four tetrahedra, the particles reside on ‘interior’ edges and the dual particle graph is given by the dashed line once again. More importantly, explicit momentum conservation does not occur and we have an amplitude that has the required CPR form. Secondly, we have the propagation of particles along edges of the tetrahedron in the vertex term. When constructing the Feynman amplitudes we glue many tetrahedra together to form a triangulation, and thus we have the possibility of having many particles travelling along a given edge. This does not happen in the CPR amplitudes and since they already recover a field theory interpretation in the effective limit, we do not want them to occur in our theory. But by our relation above, we see that they do not really occur as we can ‘drag’ the particles inside their respective tetrahedra. And finally, in a Feynman graph we may still have many particles incident at a vertex of the triangulation. Our dual particle graph is explicitly bivalent, however. If we relied on the implicit momentum conservation of the CPR amplitude, we would have overall conservation of momentum at a vertex. It would not necessarily satisfy the consistency requirement of the particle graph and the dual particle graph. With explicit momentum conservation, on the other hand, we know that the graph in the triangulation is bivalent and furthermore we can separate the bivalent vertices on a finer triangulation.

The fifth term, the vertex with three ψ fields is of the same form as the fourth except that we have three particles on the tetrahedra and a trivalent interaction.

These are the only two vertex terms allowed as the other possibilities do not satisfy the consistency conditions for the two particle graphs [2]. A further important point is that we can generalise this model to contain particles of different mass and spin in two possible ways: either by adding appropriate terms to the action, or letting the mass and spin vary in the bulk but fixing them on the boundary.

To summarize, we now have a model that generates the spin foam amplitudes for matter fields coupled to gravity in 3d, and furthermore gives a clear view of the kinematical regime.

4. Discussion and outlook

In this paper, we have presented a group field theory formulation of 3 dimensional Riemannian quantum gravity coupled to matter fields of any mass and any spin, thus generalising the work of [3]. This model augments in a simple way the Boulatov model for pure 3d gravity and reproduces exactly the spin foam amplitudes for gravity coupled to particles constructed in [8], the CPR amplitudes. In fact, the perturbative expansion of the partition function of the group field theory produces at once a sum over 3d simplicial complexes of any topology, a sum over the corresponding geometries, and a sum over Feynman graphs for matter field interactions. This result gives further support to the view that the gft formalism represents a fundamental definition of quantum gravity in terms of spin foams and not merely an auxiliary formalism.

Most important, we believe that our results presented here may be crucial for further developments in this area. Let us give a brief outlook of possible future work. An important

achievement that was made possible by the construction of a spin foam model for 3d quantum gravity coupled to matter [8] was the identification of an effective non-commutative field theory for matter that reproduces Feynman amplitudes including the complete quantum gravity corrections [9]. Firstly, from this we have a clear realisation of the presence of non-commutative geometric structures in quantum gravity. Secondly, we now have a direct connection between the spin foams models and effective models of quantum gravity in flat spacetimes like Deformed (or Doubly) Special Relativity [11]. This signifies a good starting point for tackling issues of quantum gravity phenomenology. Therefore, having now obtained a group field theory that produces the Feynman amplitudes of [8], the first issue is to derive and understand the non-commutative field theory of [9] from the group field theory itself. It is natural to expect that it is the very action of the new gft we have constructed in this paper that, after suitable integration over quantum gravity degrees of freedom, will reduce to the effective non-commutative field theory for matter. Work on this is indeed in progress [12].

A second issue, to be tackled in the near future concerns gauge fields. The model we presented can accommodate the description of spin 1 fields with no difficulty but this is not enough to interpret them as gauge bosons; we would need a consistent coupling of an interacting (non-abelian) gauge theory as well. Work is in progress [13] on the construction of a coupled and possibly unified model of quantum gravity and Yang-Mills theory at the level of group field theory, inspired by the results obtained in [14] at the spin foam level, in four dimensions.

The ultimate goal of the work whose results we presented is the issue of matter coupling to quantum gravity in four spacetime dimensions. The results outlined in this talk can be extended formally at least to higher dimensions. What is left to understand is the physical interpretation of the ‘matter’ fields in the resulting model. To understand matter coupled to quantum gravity in 4 dimensions, one can begin as in 3 dimensions, from either classical actions for gravity coupled to matter or from Feynman diagrams of the matter quantum field theory [15], then construct the corresponding coupled spin foam models, and finally obtain the group field theory formulation of them. Our results, if suitably generalised to 4 dimensions, would allow one to proceed the other way around: starting from a group field theory that gives 4d quantum gravity as a spin foam model with extra structures that can be hoped to represent matter. This is of course a longer term programme, but it is helped on its way by the results outlined here.

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