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r-modes in stratified neutron stars with entrainment

E M Kantor, V A Dommes, and M E Gusakov

Ioffe Institute, 26 Politekhnicheskaya Street, St. Petersburg 194021, Russia

E-mail: kantor.elena.m@gmail.com

Abstract. We calculate the temperature-dependent r-mode spectrum of superfluid neutron stars with $npe\mu$ (neutron, proton, electron, muon) core composition. This study is an extension of the previous work by Kantor, Gusakov [MNRAS 469, 3928 (2017)], where such spectrum was calculated under simplifying assumption of vanishing entrainment between superfluid neutrons and superconducting protons. We show that accounting for the entrainment leads to non-analytic behavior of the spectrum at small rotation rates. Namely, we find that in the leading order in rotation accounting for any non-zero value of entrainment eliminates superfluid r-modes. We show that next-to-leading order in rotation restores the superfluid r-modes in the spectrum. We calculate this spectrum and show that for certain neutron star models normal r-mode experiences stabilizing resonances with superfluid r-modes. This confirms the scenario of Gusakov, Chugunov, and Kantor [PRL 112, 151101 (2014)] that explains neutron stars in low-mass X-ray binaries.

1. Introduction

Theoretical models predict that warm and rapidly rotating neutron stars (NSs) should be unstable with respect to excitation of r-mode and radiation of gravitational waves [1, 2]. Modeling shows that such NSs quickly lose their angular momentum and leave the "instability window" (i.e. the region of parameters, where the star is unstable) [3], which means that we should not observe NSs in the instability window. However, there are a lot of NSs in low-mass X-ray binaries (LMXBs) which fall well inside the classical instability window [4]. To reconcile theory with observations, there was proposed a phenomenological model of r-mode stabilization by resonance interaction with superfluid modes at certain temperatures [5]. To put this model on more solid ground, one has to calculate temperature-dependent oscillation spectrum of rotating superfluid NS. First such calculation [6] was done under simplifying assumption of vanishing entrainment. Recently [7] entrainment was taken into account to calculate r-mode spectrum of an NS composed of neutrons, protons and electrons. Here we extend the formalism developed in [7] to account for muons in the NS core, and show that entrainment affects oscillation spectrum qualitatively at slow rotation rate.

2. Oscillation equations

We consider non-dissipative oscillations of a slowly rotating (with the spin frequency Ω) NS, adopting Cowling approximation and working in the Newtonian framework. We allow for muons in the inner layers of NS ($npe\mu$-composition) and take into account possible superfluidity of baryons (neutrons and protons) in the core. Let all the quantities depend on time t as $e^{\imath\sigma t}$ in the coordinate frame rotating with the star. Then the linearized equations governing small oscillations of superfluid (SFL) NSs in that frame are [6]:

(i) Euler equation

$$-\sigma^2 \boldsymbol{\xi}_b + 2\imath\sigma \boldsymbol{\Omega} \times \boldsymbol{\xi}_b = \frac{\delta w}{w_0^2} \boldsymbol{\nabla} P_0 - \frac{\boldsymbol{\nabla} \delta P}{w_0}, \quad (1)$$

where $w = (P + \epsilon)/c^2$, P is the pressure, ϵ is the energy density, c is the speed of light. Here and hereafter, the subscript 0 denotes the equilibrium value of some thermodynamic parameter (e.g., P_0) and δ stands for its Euler perturbation (e.g., δP). The Lagrangian displacement of baryons in equation (1) is defined as $\boldsymbol{\xi}_b \equiv \boldsymbol{j}_b/(\imath\sigma n_b)$, where $n_b \equiv n_n + n_p$ and $\boldsymbol{j}_b \equiv \boldsymbol{j}_n + \boldsymbol{j}_p$ are the baryon number density and baryon current density, respectively. Here and hereafter, subscripts n, p, e, μ, and l refer to neutrons, protons, electrons, muons, and leptons, respectively.

(ii) Continuity equations for baryons and leptons (electrons and muons)

$$\delta n_b + \mathrm{div}(n_b \boldsymbol{\xi}_b) = 0, \; \delta n_l + \mathrm{div}(n_l \boldsymbol{\xi}) = 0. \quad (2)$$

Here $\boldsymbol{\xi} \equiv \boldsymbol{j}_e/(\imath\sigma n_e)$ is the Lagrangian displacement of the normal liquid component. If neutrons are non-superfluid, then $\boldsymbol{\xi} = \boldsymbol{\xi}_b$ and hydrodynamic equations become essentially the same as in the normal matter (even if protons are SFL, see e.g. [8]). Thus, for brevity we shall call 'normal' (or 'non-superfluid') the liquid with non-superfluid neutrons, irrespectively of the state of protons.

(iii) The 'superfluid' equation in the weak-drag regime (a typical situation in NSs, see, e.g., [9, 10])

$$h\sigma^2 \boldsymbol{z} - 2\imath h_1 \sigma \boldsymbol{\Omega} \times \boldsymbol{z} = c^2 n_e \boldsymbol{\nabla} \Delta\mu_e + c^2 n_\mu \boldsymbol{\nabla} \Delta\mu_\mu, \quad (3)$$

where $\boldsymbol{z} \equiv \boldsymbol{\xi}_b - \boldsymbol{\xi}$; $\Delta\mu_l \equiv \mu_n - \mu_p - \mu_l$ is the chemical potential imbalance (note that in equilibrium $\Delta\mu_l = 0$ and thus $\delta\Delta\mu_l = \Delta\mu_l$), $h = n_b \mu_n y$, $h_1 = \mu_n n_b \left[n_b/(Y_{nn}\mu_n + Y_{np}\mu_p) - 1 \right]$, $y = n_b Y_{pp}/[\mu_n (Y_{nn}Y_{pp} - Y_{np}^2)] - 1$. Y_{ik} is the relativistic entrainment matrix [8, 11, 12, 13], which is the analogue of the superfluid mass-density matrix in the non-relativistic theory [14]. The equations (i)-(iii) should be supplemented by the 'equation of state' (EOS), $\delta n_i = \frac{\partial n_i}{\partial P}\delta P + \frac{\partial n_i}{\partial \Delta\mu_e}\Delta\mu_e + \frac{\partial n_i}{\partial \Delta\mu_\mu}\Delta\mu_\mu$.

In order to solve the oscillation equations, we express non-radial displacements as a sum of toroidal (T, T_z) and poloidal (Q, Q_z) components [15],

$$\xi_{b\theta} = \frac{\partial}{\partial\theta} Q(r,\theta) + \frac{\imath m T(r,\theta)}{\sin\theta}, \quad \xi_{b\phi} = \frac{\imath m Q(r,\theta)}{\sin\theta} - \frac{\partial}{\partial\theta} T(r,\theta), \quad (4)$$

$$z_\theta = \frac{\partial}{\partial\theta} Q_z(r,\theta) + \frac{\imath m T_z(r,\theta)}{\sin\theta}, \quad z_\phi = \frac{\imath m Q_z(r,\theta)}{\sin\theta} - \frac{\partial}{\partial\theta} T_z(r,\theta), \quad (5)$$

and expand all the unknown functions into Legendre polynomials with fixed m [16].

One can see that superfluid matter of $npe\mu$ NSs supports two velocity fields. As a result in addition to usual normal modes oscillation spectrum acquires superfluid modes, which in contrast to normal modes are driven by counter-motion of normal matter and paired neutrons.

We consider a slowly rotating NS, and expand all the quantities in a power series in small parameter Ω (in what follows we denote by Ω the spin rate normalized to Kepler frequency). We are interested in the oscillations, which have the eigenfrequencies σ vanishing at $\Omega \to 0$. Thus, σ can be presented as (e.g., [15, 17, 16]) $\sigma = \sigma_0 \Omega + O(\Omega^2)$. In [6] we found that, for zero entrainment and in the lowest order in rotation, the purely toroidal modes are possible only with $l = m$. For a given m there exist one normal r-mode and an infinite set of superfluid r-modes, all having the same frequency $\sigma_0 = 2/(m+1)$, see [18, 19, 20, 6].

3. Superfluid r-mode in the limit of small entrainment

Assuming that the entrainment effect is small, we tried to develop a perturbation theory in $\Delta h \equiv h_1/h - 1$ in the leading order in rotation for $npe\mu$ NS. This method is analogous to that of [7], where r-modes in npe matter were calculated analytically in the first order in Δh. We found that any non-zero entrainment eliminates all r-modes (i.e., purely toroidal oscillations in zero order in Δh) except for the normal one. This unphysical result takes place because in stratified star [where $d(n_e/n_\mu)/dr \neq 0$] the continuity equations for electrons and muons in the leading order in rotation imply an additional condition $\xi_{br} = z_r$, which makes the system overdetermined.

To avoid this restriction, in what follows we account for the next-to-the-leading-order corrections in rotation and in entrainment simultaneously, and adopt the following expansions:

$$\sigma = (\sigma_0 + \sigma_1)\Omega + O(\Delta h^2, \Omega^4) = \left(\frac{2}{m+1} + \sigma_1\right)\Omega + O(\Delta h^2, \Omega^4), \tag{6}$$

$$d = d^0 + d^1 + O(\Delta h^2, \Omega^3), \quad \delta A = \delta A^0 + O(\Delta h, \Omega^3), \tag{7}$$

where A stands for a thermodynamic function and d for displacement. The leading order quantities in both rotation and entrainment are labeled with the index 0, while the index 1 denotes next-to-the-leading-order corrections (both in entrainment and in rotation). Since in the absence of entrainment the superfluid r-modes are purely toroidal in the leading order in rotation, one has $\xi_{br}^0 = z_r^0 = Q^0 = Q_z^0 = 0$. Substituting the above expansions into oscillation equations [1], and adopting appropriate boundary conditions, one can calculate the r-mode spectrum, and we find that such expansions restore superfluid r-modes in the spectrum.

Let us analyze the solution at $\Omega \to 0$. In this limit we have a singularity in oscillation equations, which, as we demonstrate below, leads to a finite value of σ_1 and the following ordering of the eigenfunctions: $\xi_{br}^1 \sim z_r^1 \sim \Omega T^0 \sim \Omega T_z^0$. To demonstrate it we rewrite oscillation equations, assuming the above ordering. Then excluding the small terms, and combining equations, we arrive at

$$\frac{dT}{dr} - \frac{K}{\Omega^2}\xi = 0, \quad \frac{d\xi}{dr} - T = 0, \tag{8}$$

where we have defined $\xi \equiv \xi_{br}^1 - z_r^1$, $T \equiv \sigma_1(1+m)^2(3+2m)/[(2+4m)r]\,T^0 - (1+m)(3+2m)(\sigma_1 + m\sigma_1 - 2\Delta h)/[(2+4m)r]T_z^0$, $K \equiv \sigma_1 C_1(r) - (\sigma_1 + m\sigma_1 - 2\Delta h)C_2(r)$. Here $C_1(r)$ and $C_2(r)$ are known functions of radius. Eqns. (8) can be rewritten in a form

$$\frac{d^2\xi}{dr^2} - \frac{K}{\Omega^2}\xi = 0. \tag{9}$$

This equation has an oscillating solution for the case $K < 0$, while for $K > 0$ it describes exponential growth at $\Omega \to 0$. To illustrate the behavior of the solution at $\Omega \to 0$ we shall consider a two-layer star composed of superfluid $npe\mu$ core and the normal crust. Let us integrate Eq. (9) from the center to the core-crust interface, assuming such value for σ_1 that $K > 0$ in the internal layers. Then the solution exponentially grows from the center, but, at some radius K changes sign and eigenfunctions start to oscillate with vanishing (at $\Omega \to 0$) wavelength. At this moment we can easily match eigenfunctions in the crust and in the core. Thus at $\Omega \to 0$ σ_1 can be found from the condition $K(R_{cc}) = 0$, where R_{cc} is the radius of the core-crust interface. Moreover, all overtones of superfluid r-modes have the same σ_1 (since vanishing variation of σ_1 allows us to increase number of nodes at the infinitely small region with $K < 0$).

[1] Since we adopt the expantion upto the next-to-leading order in rotation we account for the oblateness of the star due to rotation in oscillation equations.

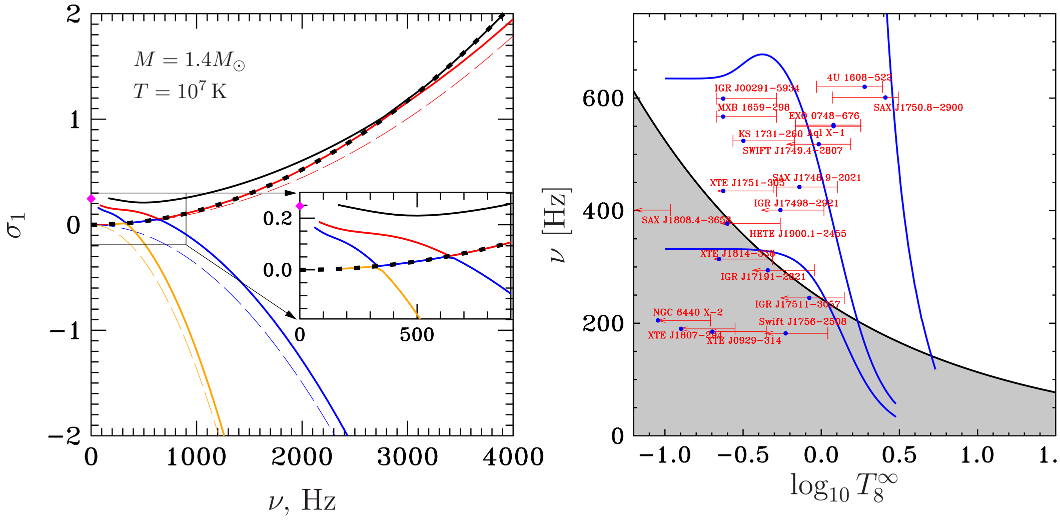

Figure 1. Left panel: σ_1 versus ν at $T = 10^7$ K, $M = 1.4M_\odot$. See the text for details. Right panel: Instability window for $l = m = 2$ normal r-mode. Region filled gray is classical stable region defined by the shear viscosity only. Points with error bars correspond to available observational data on NSs in LMXBs [21]. Solid lines show values of temperatures and rotation rates at which normal r-mode experiences resonance with (top-down) main harmonic, first overtone, and second overtone of superfluid r-modes for $M = 1.4M_\odot$ NS.

On the other hand, writing down the ratio of Eqns. (8), $TdT = K/\Omega^2\, \xi d\xi$, we find that $\xi \sim \Omega T$, or $\xi^1_{br} \sim z^1_r \sim \Omega T^0 \sim \Omega T^0_z$. This ordering differs from the standard one, which takes place at zero entrainment, when we account for rotational corrections only. In this case $\Delta h = 0$, $\sigma_1 \propto \Omega^2$ and $\xi^1_{br} \sim z^1_r \sim \Omega^2 T^0 \sim \Omega^2 T^0_z$.

4. Results

In our numerical calculations we adopt essentially the same physics input as in [6]. All results obtained below are for $l = m = 2$ r-modes. Left panel of Fig. 1 shows how σ_1 depends on rotation frequency $\nu \equiv \Omega/2\pi$ [2] for NS with $M = 1.4M_\odot$, $T = 10^7$ K, and realistic critical temperature profiles from [6]. Solid lines correspond to four different r-modes experiencing avoided-crossings with each other, dots indicate the normal r-mode. The normal r-mode is not affected by entrainment while superfluid modes deviate strongly from their "zero-entrainment" behavior (shown by dashes), especially at slow rotation frequencies. Diamond at $\nu = 0$ shows theoretically predicted limit [defined by $K(R_\mu) = 0$] for σ_1 at $\nu \to 0$ for superfluid modes. One can see that calculated curves tend to approach this limit. In the limit of rapid rotation $\sigma_1 \propto \Omega^2$, as expected.

Right panel of Fig. 1 illustrates how the values of spin frequencies, corresponding to avoided-crossings of r-modes, depend on the stellar temperature [3] (solid lines). In the vicinity of these curves normal r-mode experiences stabilizing resonance interaction with superfluid r-modes

[2] Although rotation rates higher than $\nu \sim 1000$ Hz are not relevant for neutron stars, and low-frequency approximation is invalid at such high rotation rates, we plot the spectrum upto $\nu = 4000$ Hz for the sake of completeness.

[3] Notice, that while EOS of NS matter is practically temperature independent, oscillation spectrum of superfluid NS depends on temperature. This happens because the fraction of paired neutrons strongly depends on temperature at temperatures relevant for NSs in LMXBs.

[5, 21]. This means that 'stability peaks' appear in the 'rotation frequency-stellar temperature' plane. Note that our results imply that stability peaks are not vertical, as simple model of [5, 21] suggested. Nevertheless, the star in the course of its evolution in LMXB will spend most of its life climbing up the left edge of the 'peak'. This confirms the scenario of [5, 21], and, moreover, allows to explain the existing observations within the simple model proposed here.

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